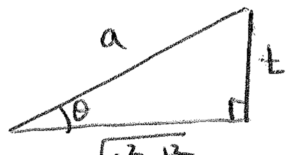
HW §7.3 Solution

#39 (a) $\displaystyle\int_0^x \sqrt{a^2-t^2}\,dt = \int_{\theta=0}^{\theta=\sin^{-1}(x/a)} a\cos\theta \; a\cos\theta \, d\theta = a^2 \int_0^{\sin^{-1}(x/a)} \tfrac{1}{2}\left(1+\cos(2\theta)\right) d\theta$

$t = a\sin\theta$

$\sqrt{a^2-t^2} = a\cos\theta$

$dt = a\cos\theta\, d\theta$



$$= \frac{a^2}{2}\left(\theta + \tfrac{1}{2}\sin(2\theta)\right)\Big|_0^{\sin^{-1}(x/a)}$$

$$= \frac{a^2}{2}\left(\theta + \sin\theta\cos\theta\right)\Big|_0^{\sin^{-1}(x/a)}$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\tfrac{x}{a}\right) + \sin\left(\sin^{-1}\left(\tfrac{x}{a}\right)\right)\cos\left(\sin^{-1}\left(\tfrac{x}{a}\right)\right)\right)$$

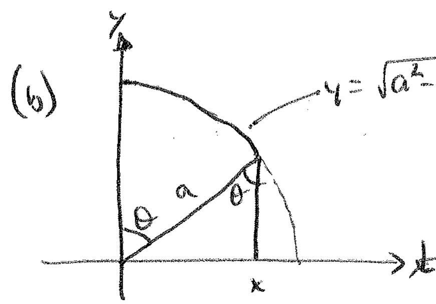
$\sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \sin\theta = \frac{x}{a}$

$\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \cos\theta = \frac{1}{a}\sqrt{a^2-x^2}$

$$= \frac{a^2}{2}\sin^{-1}\left(\tfrac{x}{a}\right) + \frac{a^2}{2}\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}$$

$$= \frac{a^2}{2}\sin^{-1}\left(\tfrac{x}{a}\right) + \tfrac{1}{2}x\sqrt{a^2-x^2}.$$

(b)



The definite integral $\int_0^x \sqrt{a^2-t^2}\,dt$ represents the area under the top half of the circle $t^2+y^2=a^2$ that is above $y=0$ and between $t=0$ and $t=x$.

This area can be computed as the sum of the areas of a sector of the circle with radius a and angle measure $\theta = \sin^{-1}\left(\frac{x}{a}\right)$ and a triangle with base x and height $\sqrt{a^2-x^2}$.

The area of the triangle is $\frac{1}{2}x\sqrt{a^2-x^2}$.

The area of the sector is $\frac{\theta}{2}a^2 = \frac{a^2}{2}\sin^{-1}(x/a)$.

Hence the total area is

$$\frac{a^2}{2}\sin^{-1}\left(\tfrac{x}{a}\right) + \tfrac{1}{2}x\sqrt{a^2-x^2}.$$

Note: The general formula for the area of a sector of a circle with radius a angle angle θ (radians) is

$$\underbrace{\frac{\theta}{2\pi}}_{\text{fraction of circle}} \cdot \underbrace{\pi a^2}_{\text{area of full circle}} = \frac{\theta}{2}a^2.$$