MATH 142 (Section H01)
Prof. Meade

Exam 4
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University of South Carolina
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Name: Key
Section H01

Instructions:

1. There are a total of 4 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	32	
2	30	
3	24	
4	14	
Total	100	

Happy Thanksgiving!

There is no exam content on this page.

1. (32 points) [8 points each] Determine if each series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ Note $0 \le \cos^2 n \le 1$ for all n,

$$0 \le \frac{\cos^2 n}{n^2} \le \frac{1}{n^2} \text{ for all } n$$

By the Comparison Test, because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2>1$),

so does $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$.

(b) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ Trying the Limit Comparison Test with the divergent p-series ($p = 1/2 < 1$) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$:

$$c = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n-1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{n-1}\frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{n}{n-1} = 1 > 0 \text{ (\& finite)}$$

we conclude that $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

(c) $\sum_{n=0}^{\infty} \frac{1+2n^2+3n^3}{\sqrt{1+n^2+n^4}}$

$$\lim_{n\to\infty} \frac{1+2n^2+3n^3}{\sqrt{1+n^2+n^4}} = \lim_{n\to\infty} \frac{1+2n^2+3n^3}{n^2\sqrt{\frac{1}{n^4}+\frac{1}{n^2}+1}} = \lim_{n\to\infty} \frac{\overset{\to 0}{1/n^2} + \overset{2}{2} + \overset{\to\infty}{3n}}{\underbrace{\sqrt{1/n^4 + 1/n^2 + 1}}_{\to 1}} = +\infty$$

$\sum_{n=1}^{\infty} \frac{1+2n^2+3n^3}{\sqrt{1+n^2+n^4}}$ diverges by the n^{th} Term Test.

(d) $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ Trying the Limit Comparison Test with the convergent p-series ($p = 4/3 > 1$):

$$c = \lim_{n\to\infty} \frac{\frac{n+5}{\sqrt[3]{n^7+n^2}}}{1/n^{4/3}} = \lim_{n\to\infty} \frac{n^{4/3}(n+5)}{n^{7/3}(1+1/n^5)^{1/3}} = \lim_{n\to\infty} \frac{\cancel{n^{7/3}}(1+5/n)}{\cancel{n^{7/3}}(1+1/n^5)^{1/3}} = 1$$

so $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$.

2. (30 points) [10 points each] Determine if each series converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}}$ $= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ is a p-series with $p = 3/4$, so it diverges.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$ is an alternating series with terms that both approach zero ($\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$) and are decreasing ($n+1 > n \Rightarrow (n+1)^{3/4} > n^{3/4} \Rightarrow \frac{1}{(n+1)^{3/4}} < \frac{1}{n^{3/4}}$)

so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$ converges

$\therefore \sum_{n=1}^{\infty} (-1)^n/n^{3/4}$ converges conditionally

(b) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$

Note that while this is an alternating series,

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1 \neq 0$$

so the terms do not approach zero

and so $\sum_{n=1}^{\infty} (-1)^n \cos(\pi/n)$ diverges by the n^{th} Term Test.

(c) $\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$

Ratio Test: $\lim_{n\to\infty} \left| \frac{(-10)^{n+1}/(n+1)!}{(-10)^n/n!} \right| = \lim_{n\to\infty} \left| \frac{(-10)^{n+1}}{(-10)^n} \frac{n!}{(n+1)!} \right|$

$$= \lim_{n\to\infty} \frac{10}{n}$$

$$= 0$$

So $\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$ converges absolutely by the Absolute Ratio Test.

3. (24 points) [8 points each] Determine if each series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1.1^n}{n^4}$

Ratio Test: $L = \lim_{n\to\infty} \frac{1.1^{n+1}/(n+1)^4}{1.1^n/n^4} = \lim_{n\to\infty} \frac{1.1^{n+1}}{1.1^n} \frac{n^4}{(n+1)^4} = \lim_{n\to\infty} 1.1 \left(\frac{n}{n+1}\right)^4 = 1.1 > 0$

since $L = 1.1 > 1$, $\sum_{n=1}^{\infty} \frac{1.1^n}{n^4}$ diverges by the Ratio Test.

or

n^{th} Term Test: $\lim_{n\to\infty} \frac{1.1^n}{n^4} \overset{l'H}{=} \lim_{n\to\infty} \frac{1.1^n \ln(1.1)}{4n^3} \overset{l'H}{=} \lim_{n\to\infty} \frac{1.1^n (\ln 1.1)^2}{12n^2}$

$\overset{l'H}{=} \lim_{n\to\infty} \frac{1.1^n (\ln 1.1)^3}{24n} \overset{l'H}{=} \lim_{n\to\infty} \frac{1.1^n (\ln 1.1)^4}{24} = \infty$

so $\sum_{n=1}^{\infty} \frac{1.1^n}{n^4}$ diverges by the n^{th} Term Test.

(b) $\sum_{n=0}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

Ratio Test: $L = \lim_{n\to\infty} \frac{10^{n+1}/(n+2)4^{2n+3}}{10^n/(n+1)4^{2n+1}} = \lim_{n\to\infty} \frac{10^{n+1}}{10^n} \frac{n+1}{n+2} \frac{4^{2n+1}}{4^{2n+3}}$

$= \lim_{n\to\infty} \frac{10}{16} \frac{n+1}{n+2}$

$= \frac{5}{8} < 1$ so $\sum_{n=0}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ converges by the Ratio Test.

(c) $\sum_{n=0}^{\infty} \frac{3+\sin(n)}{n+3^n}$

Note that since $-1 \le \sin(n) \le 1$, we know $2 \le 3 + \sin(n) \le 4$.

Also, since $n \ge 0$, $3^n + n \ge 3^n$ so that $\frac{1}{3^n+n} \le \frac{1}{3^n}$.

Therefore, $\frac{3+\sin(n)}{n+3^n} \le \frac{4}{n+3^n} \le \frac{4}{3^n}$ for all $n \ge 0$.

Now, because $\sum_{n=0}^{\infty} \frac{4}{3^n}$ is a ~~p-series with p~~ geometric series with $r = \frac{1}{3} < 1$, we know that $\sum_{n=0}^{\infty} \frac{4}{3^n}$ converges. Thus, by the Comparison Test, we also know that $\sum_{n=0}^{\infty} \frac{3+\sin(n)}{n+3^n}$ converges.

4. (14 points) Consider the convergent series

$$\frac{1}{1\cdot 2} - \frac{1}{2!\cdot 2^2} + \frac{1}{3!\cdot 2^3} - \frac{1}{4!\cdot 2^4} + \cdots.$$

(a) Write this series in summation notation.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!\,2^n}$$

(b) Explain why we know this is a convergent series.

The Alternating Series Test can be used to show this series converges: $\lim_{n\to\infty} \frac{1}{n!\,2^n} = 0$ and $\frac{1}{(n+1)!\,2^{n+1}} < \frac{1}{n!\,2^n}$.

(c) Estimate the error in using the sixth partial sum to approximate the total sum.
NOTE: Leave your answer as a rational number (fraction).

$$|R_6| \le |a_7| = \left| \frac{(-1)^8\, 1}{7!\, 2^7} \right| = \frac{1}{5040(128)} = \frac{1}{645{,}120}.$$

(d) How many terms are needed to approximate the sum of this series with an error that does not exceed $0.0005 = 1/2000$? (Do not approximate the sum!)
HINT: $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$ and $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$, $8! = 40320$.

$$|R_n| \le |a_{n+1}| \le 0.0005 = \frac{1}{2000}$$

$$\left| \frac{(-1)^{n+1}}{(n+1)!\,2^{n+1}} \right| < \frac{1}{2000}$$

$$2000 < (n+1)!\,2^{n+1}$$

n	$(n+1)!$	2^{n+1}	$(n+1)!\,2^{n+1}$
1	2	4	8
2	6	8	48
3	24	16	384
4	120	32	3840

The partial sum S_4 meets this error bound; this sum involves 4 terms.