MATH 142 (Section 502)
Prof. Meade

Exam 2
September 11, 2008

University of South Carolina
Fall 2008

Name: Key
Section 502

Instructions:

1. There are a total of 5 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	20	
2	32	
3	28	
4	10	
5	10	
Total	100	

Beat the Dawgs!

1. (20 points) Evaluate the following expressions.

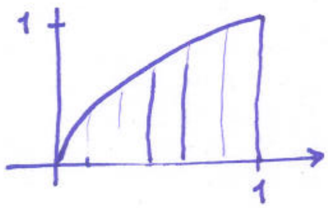
(a) $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec(t)\, dt$ $= \sec(x^4) \cdot \frac{d}{dx}(x^4)$

$= \sec(x^4) \cdot 4x^3$

$= 4x^3 \sec(x^4).$

(b) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \cdots + \sqrt{\frac{n}{n}}\right)$ $= \displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\frac{k}{n}} \cdot \frac{1}{n}$

HINT: Try to recognize the sum as a Riemann sum for a function defined on $[0, 1]$.



In the limit as $n \to \infty$ these sums give the area under $y = \sqrt{x}$ between $x = 0$ and $x = 1$.

$$A = \int_0^1 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3} - 0 = \frac{2}{3}.$$

(c) $\displaystyle\lim_{x\to\infty}\left(1 + \frac{1}{3x}\right)^{2x}$ $= \displaystyle\lim_{u\to\infty} \left(1 + \frac{1}{u}\right)^{\frac{2}{3}u} = \left(\lim_{u\to\infty} \left(1 + \frac{1}{u}\right)^{u}\right)^{2/3} = e^{2/3}.$

$\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$

$u = 3x$

$x = \frac{u}{3}$

$2x = \frac{2}{3}u$

(d) Given that $\ln(a) = 9$, find $\displaystyle\int_1^{\sqrt{a}} \frac{dt}{t}$ and $\displaystyle\int_1^{2a} \frac{dt}{t}$.

NOTE: Clearly label your results.

$$\int_1^{\sqrt{a}} \frac{dt}{t} = \ln(\sqrt{a}) = \ln(a^{1/2}) = \frac{1}{2}\ln(a) = \frac{9}{2}.$$

$$\int_1^{2a} \frac{dt}{t} = \ln(2a) = \ln 2 + \ln a = 9 + \ln(2).$$

2. (32 points) Evaluate the following definite and indefinite integrals.

(a) $\int_0^{\sqrt{\pi/3}} \theta \cos\left(\theta^2\right)\, d\theta$ $= \frac{1}{2} \int_0^{\pi/3} \cos(u)\, du$

let $u = \theta^2$
$du = 2\theta\, d\theta$
$\frac{1}{2} du = \theta\, d\theta$
$\theta = 0 \Rightarrow u = 0$
$\theta = \sqrt{\pi/3} \Rightarrow u = \pi/3$

$= \frac{1}{2} \sin(u) \Big|_0^{\pi/3}$
$= \frac{1}{2} \sin(\pi/3) - \frac{1}{2} \sin(0)$
$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - 0$
$= \sqrt{3}/4.$

(b) $\int \frac{\sec^2(1+\ln(x))}{x}\, dx$ $= \int \sec^2(u)\, du = \tan(u) + C$
$= \tan(1 + \ln(x)) + C.$

let $u = 1 + \ln(x)$
$du = \frac{1}{x} dx$

(c) $\int_{\pi/6}^{\pi/2} \frac{\cos(\theta)}{\sin^2(\theta)} d\theta$ $= \int_{1/2}^{1} \frac{1}{u^2} du = -\frac{1}{u}\Big|_{1/2}^{1} = -1 - \frac{-1}{1/2} = -1 + 2 = 1.$

$\left(\int u^{-2} du = \frac{1}{-1} u^{-1} + c\right)$

let $u = \sin(\theta)$
$du = \cos(\theta) d\theta$
$\theta = \pi/6 \Rightarrow u = \sin(\pi/6) = 1/2$
$\theta = \pi/2 \Rightarrow u = \sin(\pi/2) = 1$

(d) $\int x^{-1/2} e^{\sqrt{x}}\, dx$ $= 2\int e^u\, du = 2e^u + c = 2e^{\sqrt{x}} + C.$

let $u = \sqrt{x} = x^{1/2}$
$du = \frac{1}{2} x^{-1/2} dx$
$2\, du = x^{-1/2} dx$

3. (28 points) Let R be the region enclosed by the curves $y = x^2$ and $y = x + 6$.

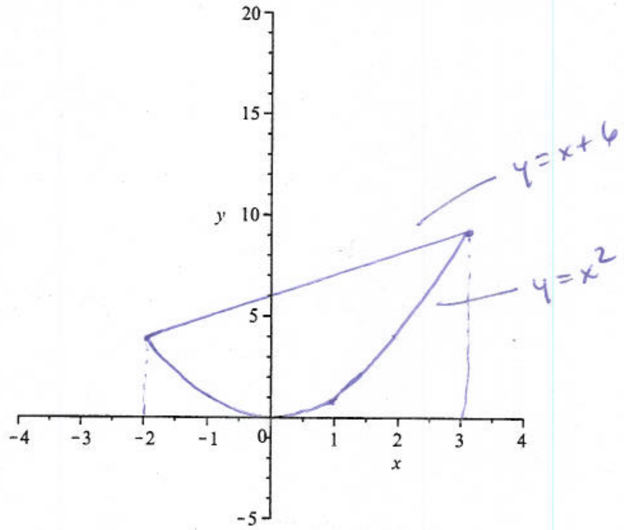
(a) [8 points] Sketch the region R on the axes provided.
NOTE: Be sure your sketch clearly shows all points of intersection between the two curves.

$x^2 = x+6$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3$ or $x = -2$.



(b) [10 points] Setup a definite integral for the area of R.

$$A = \int_{-2}^{3} (\text{top} - \text{bottom})\, dx = \int_{-2}^{3} (x+6) - x^2 \, dx$$

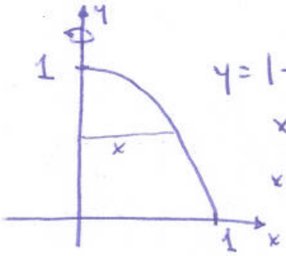
(c) [10 points] Setup a definite integral for the volume of the solid formed by revolving R about the line $y = -5$.

$$V = \int_{-2}^{3} \pi (\text{outer radius})^2 - \pi (\text{inner radius})^2 \, dx$$

$$= \int_{-2}^{3} \pi (x+6-(-5))^2 - \pi (x^2 - (-5))^2 \, dx$$

$$= \pi \int_{-2}^{3} (x+11)^2 - (x^2+5)^2 \, dx.$$

4. (10 points) Find the volume of the solid whose base is the region enclosed in the first quadrant between the curve $y = 1 - x^2$, the x-axis, and the y-axis and whose cross-sections taken perpendicular to the y-axis are squares.



$y = 1 - x^2$

$x^2 = 1 - y$

$x = \sqrt{1-y}$.

$$V = \int_0^1 x^2 \, dy$$

$$= \int_0^1 1 - y \, dy$$

$$= \left(y - \tfrac{1}{2}y^2\right)\Big|_0^1$$

$$= \left(1 - \tfrac{1}{2}\right) - (0 - 0)$$

$$= \tfrac{1}{2}.$$

5. (10 points) Express the exact arclength of the curve $y = \ln(\cos(x))$ over the interval from $x = 0$ to $x = \frac{\pi}{4}$ as an integral that has been simplified to eliminate the radical.

NOTE: Do not evaluate the integral.

$$L = \int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx$$

$$y = \ln(\cos(x))$$

$$y' = \frac{1}{\cos(x)} \cdot (-\sin(x))$$

$$= -\frac{\sin(x)}{\cos(x)}$$

$$= \int_0^{\pi/4} \sqrt{1 + \left(-\frac{\sin(x)}{\cos(x)}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\frac{1}{\cos^2(x)}}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx$$

$$= \int_0^{\pi/4} \sec(x)\, dx.$$