MATH 142 (Section 501)
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University of South Carolina
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Exam 2
October 7, 2003

Name: Key
SS #: ______________________

Instructions:

1. There are a total of 4 problems (including the Extra Credit problem) on 7 pages. Check that your copy of the exam has all of the problems.
2. Except for Question 1, *all work must be shown* to receive credit for a correct answer.
3. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
4. *No calculators!* If you believe you need to use a calculator you are doing something wrong!!

Problem	Points	Score
1	18	
2	16	
3	50	
4	16	
Extra Credit	10	
Total	100	

Good Luck!

1. (18 points) [3 points each] Concepts Review. Determine if each statement is true or false.

(a) F To evaluate $\int x \sin x \, dx$, use a trigonometric substitution.

Use integration by parts.

(b) F To evaluate $\int \frac{x+1}{x^2+2x+2} \, dx$, first complete the square in the denominator.

Use the substitution

$u = x^2+2x+2$

(c) T To evaluate $\int \frac{x-1}{x^2+2x+2} \, dx$, first complete the square in the denominator.

The denominator can be written as $x^2+2x+2 = (x+1)^2+1$

Next, use the substitution $u = x+1$.

(d) T To evaluate $\int \frac{1}{\sqrt{4-5x^2}} \, dx$, use the trigonometric substitution $\sqrt{5}x = 2\sin t$.

This substitution works because, then,

$\sqrt{4-5x^2} = \sqrt{4-(2\sin t)^2} = 2\cos t.$

(e) F To evaluate $\int \sin^{14} x \, dx$, factor one power of $\sin x$ and use a half-angle formula.

For an even power of $\sin x$, the first step is to use the half-angle formula: $\sin x = \frac{1}{2}(1-\cos(2x))$.

(f) T To evaluate $\int x^2 \ln x \, dx$, use integration by parts.

In fact, use $u = \ln x$ $dv = x^2 dx$.

2. (16 points) [4 points each] Express the partial fraction decomposition of each proper rational function *without computing the exact coefficients.* If an integrand is not a proper rational function, explain what must be done before a partial fraction decomposition can be found.

(a) $\dfrac{7x^5 + x^2 - 8}{(x-1)(2-x)(x+3)}$

This function is not a proper rational function. Before finding the partial fraction decomposition, long division must be used.

(b) $\dfrac{x^2 - 8}{(x-1)(2-x)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{2-x} + \dfrac{C}{x+3}.$

(c) $\dfrac{x+3}{(x-2)(x^2+2x+1)} = \dfrac{A}{x-2} + \cancel{\dfrac{Bx+C}{x^2+2x+1}} \quad \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

Note: $x^2+2x+1 = (x+1)^2$

(d) $\dfrac{x+3}{(x-2)(x^2+2x+3)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{(x+1)^2+2}$

3. (50 points) [10 points each] Evaluate each integral.

(a) $\int x\sqrt{x-3}\,dx = \int (u+3)\sqrt{u}\,du = \int u^{3/2}+3u^{1/2}\,du = \frac{2}{5}u^{5/2}+3\cdot\frac{2}{3}u^{3/2}+C$

Rationalizing substitution: $u=x-3$, $x=u+3$, $dx=du$

$= u^{3/2}\left(\frac{2}{5}u+2\right)+C$

$= (x-3)^{3/2}\left(\frac{2}{5}(x-3)+2\right)+C$

$= (x-3)^{3/2}\left(\frac{2}{5}x+\frac{4}{5}\right)+C$

Or, with $u=\sqrt{x-3}$, $u^2=x-3$, $x=u^2+3$, $dx=2u\,du$

so $\int x\sqrt{x-3}\,dx = \int (u^2+3)u(2u)\,du$

$= 2\int u^4+3u^2\,du$

$= 2\left(\frac{1}{5}u^5+u^3\right)+C$

$= 2u^3\left(\frac{1}{5}u^2+1\right)+C$

$= 2(x-3)^{3/2}\left(\frac{1}{5}(x-3)+1\right)+C$

$= 2(x-3)^{3/2}\left(\frac{1}{5}x+\frac{2}{5}\right)+C.$

(b) $\int \frac{(\ln y)^5}{y}\,dy = \int u^5\,du = \frac{1}{6}u^6+C = \frac{1}{6}(\ln y)^6+C.$

Substitution: $u=\ln y$, $du=\frac{1}{y}dy$

Or, by int. by parts: $u=x$, $dv=(x-3)^{1/2}dx$, $du=dx$, $v=\frac{2}{3}(x-3)^{3/2}$

so $\int x\sqrt{x-3}\,dx = \frac{2x}{3}(x-3)^{3/2}-\frac{2}{3}\int(x-3)^{3/2}dx = \frac{2x}{3}(x-3)^{3/2}-\frac{2}{3}\cdot\frac{2}{5}(x-3)^{5/2}+C$

$= \frac{2}{3}(x-3)^{3/2}\left(x-\frac{2}{5}(x-3)\right)+C$

$= \frac{2}{3}(x-3)^{3/2}\left(\frac{3}{5}x+\frac{6}{5}\right)+C = (x-3)^{3/2}\left(\frac{2}{5}x+\frac{4}{5}\right)+C$

(c) $\int \arctan y\,dy = y\arctan y - \int\frac{y}{1+y^2}\,dy = y\arctan y - \frac{1}{2}\int\frac{du}{u}$

Integration by Parts: $u=\arctan y$, $dv=dy$, $du=\frac{1}{1+y^2}dy$, $v=y$

Substitution: $u=y^2+1$, $du=2y\,dy$

$= y\arctan y - \frac{1}{2}\ln|u|+C$

$= y\arctan y - \frac{1}{2}\ln(y^2+1)+C$

$= \cancel{y\arctan y - \ln y + C}$

Check: $\frac{d}{dy}\left(y\arctan y - \frac{1}{2}\ln(y^2+1)+C\right) = (1)\arctan y + \frac{y}{1+y^2} - \frac{1}{2}\frac{2y}{1+y^2}$

$= \arctan y + \frac{y}{1+y^2} - \frac{y}{1+y^2}$

$= \arctan y$

(d) $\int \sin^3(2t)\, dt = \int \sin^2(2t) \sin(2t)\, dt$

$$= \int (1-\cos^2(2t)) \sin(2t)\, dt$$

$u = \cos(2t)$

$du = -2\sin(2t)dt$

$$= \int (1-u^2)(-\tfrac{1}{2}du)$$

$$= -\tfrac{1}{2}\left(u - \tfrac{1}{3}u^3\right) + C$$

$$= -\tfrac{1}{2}\left(\cos(2t) - \tfrac{1}{3}\cos^3(2t)\right) + C$$

$$= -\tfrac{1}{2}\cos(2t) + \tfrac{1}{6}\cos^3(2t) + C.$$

(e) $\int x^2 e^x\, dx = x^2e^x - \int 2xe^x\,dx = x^2e^x - \left(2xe^x - \int 2e^x dx\right)$

Integration by Parts:

$u = x^2 \quad dv = e^x dx$

$du = 2x dx \quad v = e^x$

Again: $u = 2x \quad dv = e^x dx$

$du = 2dx \quad v = e^x$

$$= x^2e^x - 2xe^x + 2e^x + C$$

$$= (x^2 - 2x + 2)e^x + C.$$

4. (16 points) Consider the region bounded by the x-axis, the curve $y = \dfrac{18}{x^2\sqrt{x^2+9}}$, and the lines $x = \sqrt{3}$ and $x = 3\sqrt{3}$. The area of this region is given by the definite integral

$$A = \int_{\sqrt{3}}^{3\sqrt{3}} \frac{18}{x^2\sqrt{x^2+9}}\, dx.$$

(a) [4 points] Setup the definite integral for the <u>volume</u> of the solid formed by revolving this region about the x-axis.
Do not evaluate the integral!

$$V = \int_{\sqrt{3}}^{3\sqrt{3}} \pi \left(\frac{18}{x^2\sqrt{x^2+9}}\right)^2 dx = \int_{\sqrt{3}}^{3\sqrt{3}} \frac{324\pi}{x^4(x^2+9)}\, dx$$

(b) [2 points] What technique would you use to evaluate the integral in (a)?

Partial fractions: $\dfrac{324\pi}{x^4(x^2+9)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x^4} + \dfrac{Ex+F}{x^2+9}$.

(c) [10 points] Evaluate the definite integral for the <u>area</u> of the region.

$$A = \int_{\sqrt{3}}^{3\sqrt{3}} \frac{18}{x^2\sqrt{x^2+9}}\, dx = \int_{x=\sqrt{3}}^{3\sqrt{3}} \frac{18}{9\tan^2 t\, 3\sec t} \cdot 3\sec^2 t\, dt = \int_{x=\sqrt{3}}^{3\sqrt{3}} \frac{6}{3}\frac{\sec t}{\tan^2 t}\, dt$$

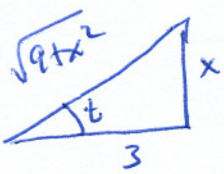
let $x = 3\tan t$
$x^2+9 = 9(\tan^2 t + 1) = 9\sec^2 t$
$dx = 3\sec^2 t$

$t = \arctan\left(\frac{x}{3}\right)$

$$= \frac{6}{3}\int_{x=\sqrt{3}}^{3\sqrt{3}} \frac{1/\cos t}{\sin^2 t/\cos^2 t}\, dt = \frac{6}{3}\int_{x=\sqrt{3}}^{3\sqrt{3}} \frac{\cos t}{\sin^2 t}\, dt$$

let $u = \sin t$
$du = \cos t\, dt$

$$= \frac{6}{3}\int_{x=\sqrt{3}}^{3\sqrt{3}} u^{-2}\, du = -6u^{-1}\Big|_{x=\sqrt{3}}^{3\sqrt{3}}$$



$$= -\frac{6}{3}\Big/\sin t\,\Big|_{x=\sqrt{3}}^{3\sqrt{3}} = \frac{-6/3}{\sin\left(\arctan\frac{x}{3}\right)}\Bigg|_{x=\sqrt{3}}^{3\sqrt{3}}$$

$$= \frac{-6/3}{x/\sqrt{9+x^2}}\Bigg|_{\sqrt{3}}^{3\sqrt{3}} = -\frac{6\sqrt{9+x^2}}{3x}\Bigg|_{\sqrt{3}}^{3\sqrt{3}=\sqrt{27}}$$

$$= \frac{-6\sqrt{36}}{3\,3\sqrt{3}} + \frac{6\sqrt{12}}{3\sqrt{3}}$$

$$= \left(-12/\sqrt{3} + 12\right)1/3$$

$$= \frac{12}{3}\left(1 - 1/\sqrt{3}\right) = 4\left(1 - 1/\sqrt{3}\right)$$

Extra Credit (10 points) Show all steps in the derivation of the formula

$$\int 2x \arctan x \, dx = (x^2+1)\arctan x - x + C.$$

$$\int 2x \arctan x \, dx = x^2 \arctan x - \int \frac{x^2}{1+x^2}\, dx$$

$u = \arctan x \qquad dv = 2x\,dx$

$du = \dfrac{dx}{1+x^2} \qquad v = x^2$

$$= x^2 \arctan x - \int \frac{1+x^2 \ -1}{1+x^2}\, dx$$

$$= x^2 \arctan x - \int 1 - \frac{1}{1+x^2}\, dx$$

$$= x^2 \arctan x - x + \arctan x + C$$

$$= (x^2+1)\arctan x - x + C.$$