MATH 142 (Section 501)
Prof. Meade

University of South Carolina
Fall 2003

Exam 1
September 11, 2003

Name: Key
SS #: __________

Instructions:

1. There are a total of 5 problems (including the Extra Credit problem) on 7 pages. Check that your copy of the exam has all of the problems.
2. You must show all of your work to receive credit for a correct answer.
3. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
4. *No calculators!* If you believe you need to use a calculator you are doing something wrong!!

Problem	Points	Score
1	18	
2	35	
3	15	
4	16	
5	16	
Extra Credit	5	
Total	100	

Go Cocks! Beat the Dawgs!!

1. (18 points) [2 points each] Concepts Review. Determine if each statement is true or false.

(a) F $\ln(2x^2 - 18) - \ln(x-3) - \ln(x+3) = \ln 2$ for all real numbers x.

(not counted!)

While $\ln(2x^2-18) - \ln(x-3) - \ln(x+3) = \ln\left(\frac{2x^2-18}{x^2-9}\right) = \ln 2$

this ignores the fact that $2x^2-18>0 \iff x^2>9 \iff |x|>3$.

$x-3>0 \iff x>3$

$x+3>0 \iff x>-3$

so, in fact, this identity is true only when $\underline{x>3}$.

(b) T If $x > y$, then $\ln x > \ln y$.

ln is an increasing function

(c) F If $a < b$ and $0 < x < e$, then $a \ln x < b \ln x$.

if $0<x<1$ then $\ln x < 0$ and $a<b \Rightarrow a \ln x > b \ln x$

(d) F $\ln(3^{100}) < 100$.

$100 \ln 3 < 100$

$\ln 3 < 1$ ← false, because $\ln 3 > 1$.

(e) T $\frac{d}{dx}x^e = ex^{e-1}$.

(f) F $\tan^{-1} x = \frac{\sin^{-1} x}{\cos^{-1} x}$.

(g) F $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{3}$.

$\lim_{x\to\infty} \tan^{-1} x = \pi/2$

(h) T Both $y = \sin x$ and $y = \cos x$ satisfy the differential equation $y'' + y = 0$.

if $y = \sin x$, then $y' = \cos x$, $y'' = -\sin x$ and so $y''+y = \sin x + (-\sin x) = 0$

if $y = \cos x$, then $y' = -\sin x$, $y'' = -\cos x$ $\quad y''+y = \cos x + (-\cos x) = 0$.

(i) T An integrating factor for $y' - \frac{y}{x} = 3x^3$ is $\mu(x) = \frac{1}{x}$.

$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{-1}{x}dx} = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}$.

2. (35 points) [7 points each] Evaluate each limit, derivative, and integral.

(a) $\lim_{u\to 0}(1+2u)^{\frac{2}{u}}$ $= \lim_{h\to 0}(1+h)^{2/(h/2)} = \lim_{h\to 0}(1+h)^{4/h} = \left(\lim_{h\to 0}(1+h)^{1/h}\right)^4 = e^4.$

let $h = 2u$

then $u = \frac{h}{2}$

(b) $\frac{d}{dx}x^{\frac{1}{x}}$ $= \frac{d}{dx}\, e^{\ln(x^{1/x})} = \frac{d}{dx}\, e^{\frac{1}{x}\ln x}$

$= e^{\frac{1}{x}\ln x}\,\frac{d}{dx}\left(\frac{1}{x}\ln x\right)$

$= e^{\frac{1}{x}\ln x}\left(\frac{1}{x}\cdot\frac{1}{x} + \left(-\frac{1}{x^2}\right)\ln x\right)$

$= x^{1/x}\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right) = \frac{1-\ln x}{x^2}\, x^{1/x} = (1-\ln x)\, x^{(1/x - 2)}.$

(c) $\int \frac{e^{\frac{-1}{t}}}{t^2}\, dt$ $= \int e^u\, du = e^u + C = e^{-\frac{1}{t}} + C.$

$u = -\frac{1}{t}$

$du = \frac{1}{t^2}\, dt$

(d) $\int \frac{\sin\theta}{1+\cos^2\theta}\, d\theta$ $= -\int \frac{-\sin\theta}{1+(\cos\theta)^2}\, d\theta = -\int \frac{du}{1+u^2}$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$= -\tan^{-1} u + C$

$= -\tan^{-1}(\cos\theta) + C$

(e) $\int \frac{\cosh\sqrt{z}}{\sqrt{z}}\, dz$ $= 2\int \left(\frac{1}{2} z^{-1/2}\right)\cosh\left(z^{1/2}\right) dz$

$u = z^{1/2}$

$= 2\int \cosh u\, du$

$du = \frac{1}{2} z^{-1/2}\, dz$

$= 2\sinh u + C$

$= 2\sinh(\sqrt{z}) + C.$

3. (15 points) The number of bacteria in a rapidly growing culture was estimated to be 10,000 at noon and 40,000 after 2 hours. Assuming the bacteria grow according to an exponential model,

(a) [8 points] Find C and k so that the number of bacteria, B, at time t is written as

$$B = Ce^{kt}.$$

$B(0) = 10{,}000 \Rightarrow 10{,}000 = Ce^0 = C \Rightarrow \boxed{C = 10{,}000}$

$B(2) = 40{,}000 \Rightarrow 40{,}000 = 10{,}000\, e^{2k}$

$4 = e^{2k}$

$\ln 4 = 2k$

$k = \frac{1}{2}\ln 4 = \ln 4^{1/2} = \ln 2$

so $\boxed{k = \ln 2}$

Note: $B = 10{,}000\, e^{t \ln 2}$

$= 10{,}000\, e^{\ln(2^t)}$

$= 10{,}000\,(2^t)$.

(b) [3 points] Explain, in words, why the doubling time for this bacteria is 1 hour.

The population grows by a factor of 4 in 2 hours. Because we are using an exponential model, this means the population doubles each hour.

(c) [4 points] Predict how many bacteria there will be at 5P.M.

noon	10,000
1	20,000
2	40,000
3	80,000
4	160,000
5	320,000

$B(5) = 320{,}000$

4. (16 points) Define $f(x) = \int_0^x \sqrt{4+\cos^2 t}\, dt$ and $A = f(\frac{\pi}{2})$.

(a) [4 points] Explain, in words, why f has an inverse.

$f'(x) = \sqrt{4+\cos^2 x} > 0$ for all x.

Because f is an increasing function, this function is invertible.

(b) Find each of the following quantities:

i. [4 points] $f(0)$

$f(0) = \int_0^0 \sqrt{4+\cos^2 t}\, dt = 0$

ii. [4 points] $f'(\frac{\pi}{2})$

$$f'\left(\frac{\pi}{2}\right) = \sqrt{4+\cos^2(\pi/2)} = \sqrt{4} = 2.$$

iii. [4 points] $(f^{-1})'(A)$

$(f^{-1})'(A) = \frac{1}{f'(x)}$ where $A = f(x)$ i.e. $x = \pi/2$.

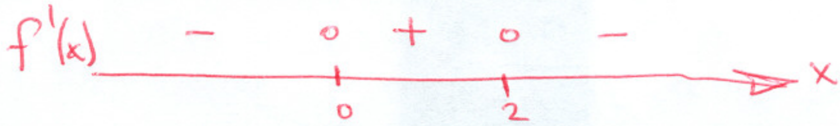
$$= \frac{1}{f'(\pi/2)} = \frac{1}{2}.$$

5. (16 points) Define $f(x) = \frac{x^2}{e^x}$. $= x^2 e^{-x}$

(a) [3 points] Find $f'(x)$.

$$f'(x) = x^2(-e^{-x}) + 2xe^{-x}$$
$$= (2x - x^2)e^{-x}$$
$$= x(2-x)e^{-x}$$

(b) [3 points] Find the intervals on which f is increasing and decreasing.



f is increasing on $(0,2)$

decreasing $(-\infty,0) \cup (2,\infty)$

(c) [4 points] Find, and classify, all local extreme values. HINT: $\lim_{x\to\infty} f(x) = 0$.

From (b):

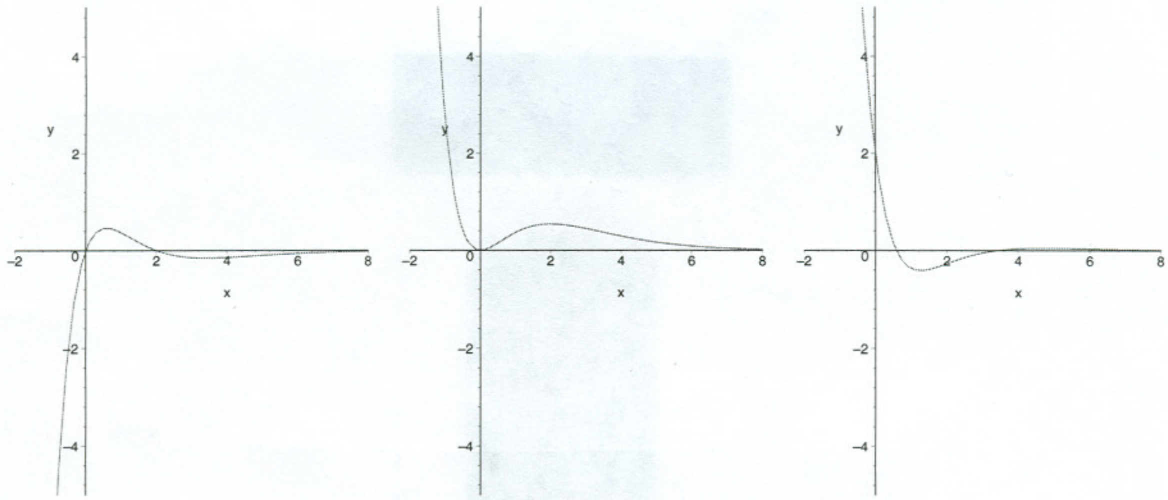
$x=0$ is a local minimum

$x=2$ is a local maximum

not really needed for local extreme values; is important for global extreme values.

problem continues on next page

(d) [6 points] The following figure displays the graphs of the function and its first two derivatives. Match each curve to the appropriate function by writing $f(x)$, $f'(x)$, or $f''(x)$ on the blank below the appropriate plot.



i) $y =$ $f'(x)$ ii) $y =$ $f(x)$ iii) $y =$ $f''(x)$

Extra Credit (5 points) I provided two derivations of the fact that $\frac{d}{dx}\sinh^{-1} y = \frac{1}{\sqrt{y^2+1}}$. In the derivation based on the fact that $\sinh^{-1}$ is the inverse of sinh, the algebraic manipulations were relatively complex. The first steps in the derivation are

$$\frac{d}{dy}\sinh^{-1} y = \frac{1}{\cosh(\sinh^{-1}(y))} = \frac{2}{y+\sqrt{y^2+1}+\frac{1}{y+\sqrt{y^2+1}}} = \frac{y+\sqrt{y^2+1}}{y^2+y\sqrt{y^2+1}+1}$$

Supply algebraic manipulations to show that the last expression shown above simplifies to

$$\frac{1}{\sqrt{y^2+1}}.$$

$$\frac{y+\sqrt{y^2+1}}{y^2+y\sqrt{y^2+1}+1} = \frac{y+\sqrt{y^2+1}}{y^2+1+y\sqrt{y^2+1}}$$

$$= \frac{y+\sqrt{y^2+1}}{(\sqrt{y^2+1})^2+y\sqrt{y^2+1}}$$

$$= \frac{y+\sqrt{y^2+1}}{\sqrt{y^2+1}\,(\sqrt{y^2+1}+y)}$$

$$= \frac{1}{\sqrt{y^2+1}}.$$