MATH 141 (Section 3 & 4)
Prof. Meade

Exam 4
November 23, 2009

University of South Carolina
Fall 2009

Name: Key
Section: 003 / 004 (circle one)

Instructions:

1. There are a total of 8 problems (including the Extra Credit problem) on 7 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	14	
2	20	
3	9	
4	8	
5	7	
6	12	
7	15	
8	15	
Total	100	

Happy Thanksgiving!

1. (14 points) Sketch the graph of $f(x) = \dfrac{x-5}{(x-4)^2}$ on the interval $0 \le x \le 10$. Be sure to clearly identify the intervals of increase and decrease, the intervals of concavity, all local maximum and local minimum points, all inflection points, and all asymptotes.

NOTE: $f'(x) = \dfrac{6-x}{(x-4)^3}$ and $f''(x) = \dfrac{2(x-7)}{(x-4)^4}$

HINT: $2/9 \approx 0.22$

$f'(x)$	$-$	dne	$+$	0	$-$
x		4		6	

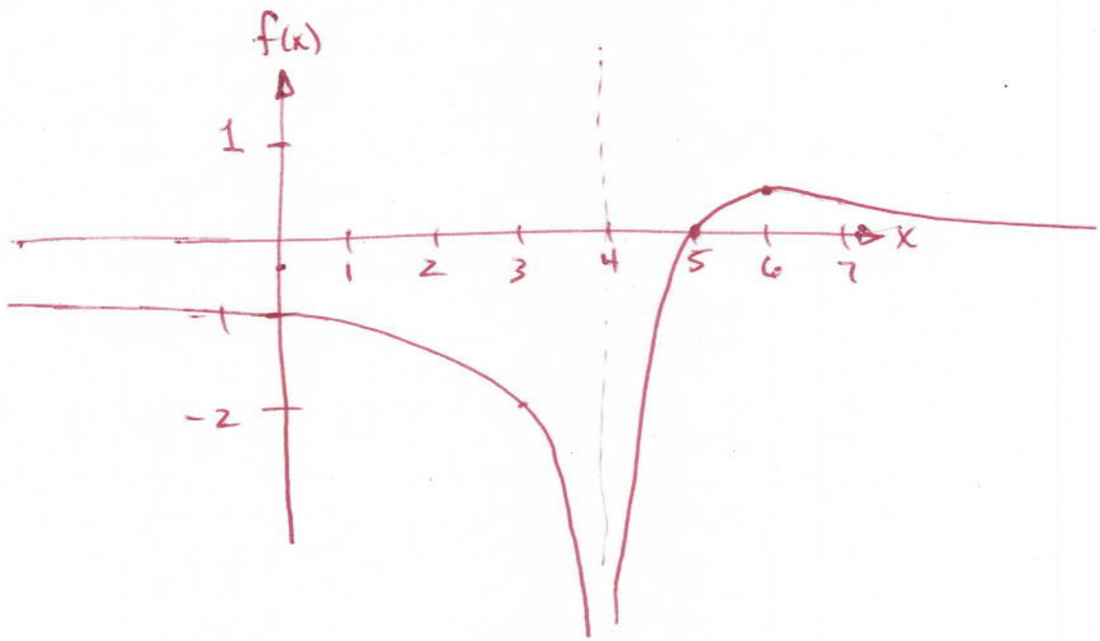
$f''(x)$	$-$	dne	$-$	0	$+$
x		4		7	

The graph of $y = f(x)$ is
increasing for $4 < x < 6$
decreasing for $x < 4$ and $x > 6$
concave up for $x > 7$
concave down for $x < 4$ and $4 < x < 7$.

The graph has a vertical asymptote of $x = 4$ (because $\lim_{x\to 4} f(x) = -\infty$) and a horizontal asymptote of $y = 0$ (because $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$).

It has a local maximum at $x = 6$ and an inflection point at $x = 7$

x	$f(x)$
6	$1/4 = 0.25$
7	$2/9 = 0.22$
5	0
4	not defined
3	-2
0	$-5/16 = -0.31$



2. (20 points) Evaluate each of the following limits. Be sure to indicate each time l'Hôpital's Rule is applied.

(a) $\lim_{x\to 0} \frac{e^{4x}-1-4x}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{4e^{4x}-4}{2x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{16e^{4x}}{2} = \frac{16}{2} = 8.$

(b) $\lim_{t\to 0} \frac{5^t+3^{-t}}{t^2} = +\infty$ because $5^t \to 1$ & $3^{-t} \to 1$ as $t\to 0$ and because $t^2 \to 0$ (and is always positive) as $t \to 0$.

(c) $\lim_{x\to\infty} x^2 e^{-x^2}$ $\infty\cdot 0$ $= \lim_{x\to\infty} \frac{x^2}{e^{x^2}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\cancel{2x}}{\cancel{2x} e^{x^2}} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0.$

(d) $\lim_{x\to 0^+} (\cos(x))^{1/x^2}$ 1^∞

$= \lim_{x\to 0^+} e^{\ln\left(\cos(x)^{1/x^2}\right)}$

$= e^{\lim_{x\to 0^+} \frac{1}{x^2}\ln(\cos(x))}$

$= e^L$

consider $L = \lim_{x\to 0^+} \frac{\ln(\cos(x))}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0^+} \frac{\frac{1}{\cos(x)}(-\sin(x))}{2x}$

$= \lim_{x\to 0^+} \frac{-\sin(x)}{2x\cos(x)} = \lim_{x\to 0^+} \frac{-1}{2\cos(x)} \cdot \frac{\sin(x)}{x}$

$= \frac{-1}{2\cos(0)} \cdot 1 = -\frac{1}{2}.$ (or, use l'H again)

so $\lim_{x\to 0^+} (\cos(x))^{1/x^2} = e^L = e^{-1/2}.$

(e) [Extra Credit (5pts)] $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 - \frac{2i}{n}\right)^3 = \int_0^2 (5-x)^3\,dx$

a Riemann sum!

HINT: Recognize this limit as a definite integral.

$u = 5-x$
$du = -dx$
$-du = dx$
$x=0 \Rightarrow u=5$
$x=2 \Rightarrow u=3$

$= \int_5^3 u^3(-du)$

$= -\int_5^3 u^3\,du$

$= -\frac{1}{4}u^4\Big|_5^3 = \frac{-3^4}{4} + \frac{5^4}{4}$

$= \frac{-81+625}{4} = \frac{544}{4} = 136.$

3. (9 points) Find the function f with $f'(t) = 2\cos(t) + \sec^2(t)$ for $-\pi/2 < t < \pi/2$ and $f(\pi/3) = 4$.

$f(t) = \int f'(t)dt = \int 2\cos t + \sec^2 t \, dt = 2\sin(t) + \tan(t) + C$

$4 = f(\pi/3) = 2\sin(\pi/3) + \tan(\pi/3) + C$

$= 2\frac{\sqrt{3}}{2} + \frac{\sqrt{3}/2}{1/2} + C$

$= \sqrt{3} + \sqrt{3} + C$

$= 2\sqrt{3} + C.$ so $C = 4 - 2\sqrt{3}.$

$f(t) = 2\sin(t) + \tan(t) + 4 - 2\sqrt{3}.$

4. (8 points) If $\int_1^5 f(x)\,dx = 12$, $\int_4^5 f(x)\,dx = 4$ and $\int_2^4 f(x)\,dx = -2$, find $\int_1^2 f(x)\,dx$.

$\int_1^5 f(x)dx = \int_1^2 f(x)dx + \int_2^4 f(x)dx + \int_4^5 f(x)dx$

$12 = \int_1^2 f(x)dx + (-2) + 4$

$\int_1^2 f(x)dx = 12 + 2 - 4 = 10.$

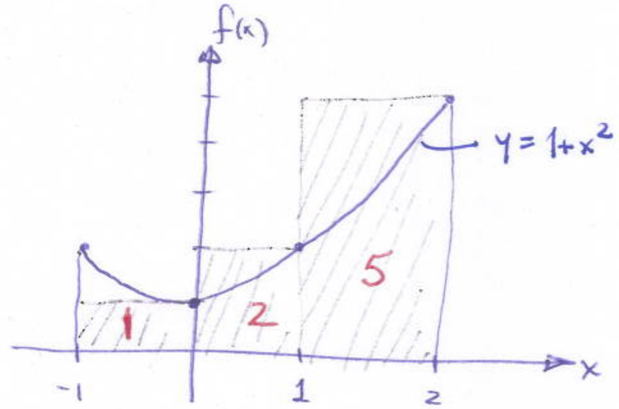
5. (7 points) Suppose $g(x) = \int_{-1}^{x} \frac{u^2-1}{u^2+1}\,du$. Find $g'(3)$.

$g'(x) = \frac{x^2-1}{x^2+1}$ by the FTC.

$g'(3) = \frac{9-1}{9+1} = \frac{8}{10} = \frac{4}{5} = 0.8$

6. (12 points) Consider the region under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$.

(a) Sketch the region.



(b) Estimate the area of this region using three rectangles and right-hand endpoints. Is this estimate smaller or larger than the exact area?

HINT: Draw the three rectangles on the sketch in (a).

$$A \approx f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1$$
$$= 1\cdot 1 + 2\cdot 1 + 5\cdot 1$$
$$= 1 + 2 + 5$$
$$= 8.$$

This approximate area should be too large.

(c) Find the exact area of this region.

$$A = \int_{-1}^{2} 1+x^2 \, dx = \left(x + \tfrac{1}{3}x^3\right)\Big|_{-1}^{2} = \left(2 + \tfrac{8}{3}\right) - \left(-1 - \tfrac{1}{3}\right)$$
$$= \frac{14}{3} + \frac{4}{3}$$
$$= \frac{18}{3}$$
$$= 6.$$

7. (15 points) Evaluate each indefinite integral.

(a) $\int (2x - e^x)\, dx$ $= x^2 - e^x + C$

(b) $\int (y-1)(2y-1)\, dy$ $= \int 2y^2 - 3y + 1 \, dy$

$= \frac{2}{3}y^3 - \frac{3}{2}y^2 + y + C$

(c) $\int \frac{x}{x^2+1}\, dx$ $= \int \frac{1}{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2+1| + C$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

8. (15 points) Evaluate each definite integral.

(a) $\int_0^1 (\sqrt{t} - t^2)\, dt$ $= \int_0^1 t^{1/2} - t^2 \, dt = \left(\frac{2}{3}t^{3/2} - \frac{1}{3}t^3\right)\Big|_0^1$

$= \frac{2}{3} - \frac{1}{3} - (0 - 0)$

$= \frac{1}{3}$

(b) $\int_0^1 \frac{4}{t^2+1}\, dt$ $= 4\arctan(t)\Big|_0^1 = 4\arctan(1) - 4\arctan(0)$

$= 4 \cdot \frac{\pi}{4} - 4 \cdot 0$

$= \pi.$

(c) $\int_{-\pi/2}^{\pi/2} \cos(x) e^{\sin(x)}\, dx$ $= \int_{-1}^{1} e^u \, du = e^u\Big|_{-1}^{1} = e^1 - e^{-1}.$

$u = \sin(x)$

$du = \cos(x)dx$

$x = -\pi/2 \Rightarrow u = -1$

$x = \pi/2 \Rightarrow u = 1$