MATH 141 (Section 11 & 12)
Prof. Meade

Exam 4 (corrected)
November 19, 2007

University of South Carolina
Fall 2007

Name: Key
Section: 011 / 012 (circle one)

Instructions:

1. There are a total of 6 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	25	
2	18	
3	15	
4	15	
5	15	
6	12	
Total	100	

Study Smart!

1. (25 points) The sign chart for the first and second derivatives of a function f is shown below.

interval	sign of $f'(x)$	sign of $f''(x)$
$x < 1$	-	+
$1 < x < 2$	+	+
$2 < x < 3$	+	-
$3 < x < 4$	-	+
$x > 4$	-	+

Assuming that f is continuous everywhere, find

(a) the interval(s) on which f is increasing

f is increasing when $f'(x) > 0$

so $(1,2) \cup (2,3)$ or $(1,3)$.

(b) the interval(s) on which f is decreasing

f is decreasing when $f'(x) < 0$

so $(-\infty, 1) \cup (3,4) \cup (4,\infty)$ or $(-\infty, 1) \cup (3, \infty)$.

(c) the open intervals on which f is concave up

f is concave up when $f''(x) > 0$

so $(-\infty, 1) \cup (1,2) \cup (3,4) \cup (4,\infty)$ or $(-\infty, 2) \cup (3, \infty)$.

(d) the open interval(s) on which f is concave down

f is concave down when $f''(x) < 0$.

so $(2,3)$.

(e) the x-coordinates of all inflection points

inflection points occur when $f''(x)$ changes sign.
This occurs at $x=2$ and at $x=3$.

2. (18 points) Let $f'(x) = \dfrac{e^{4-x^2} - 1}{x}$ be the first derivative of a continuous function f. Find all critical points of f and determine whether each is a relative maximum, relative minimum, or neither.

NOTE: Show enough work to justify your answers.

critical points exist when $f'(x) = 0$ or $f'(x)$ dne.

$$f'(x) = 0 \iff \frac{e^{4-x^2} - 1}{x} = 0$$

$$\iff e^{4-x^2} - 1 = 0$$

$$\iff e^{4-x^2} = 1 = e^0$$

$$\iff 4 - x^2 = 0 \quad \text{so} \quad \underline{\underline{x = \pm 2}}.$$

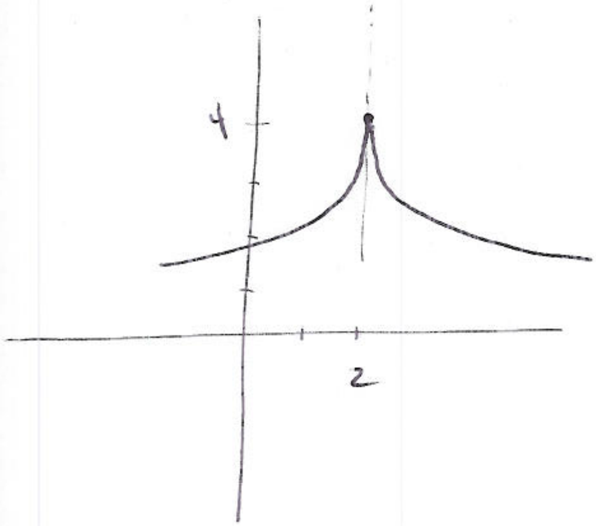
$$f'(x) \text{ dne} \iff \underline{\underline{x = 0}}$$

$f'(x)$	$+$	0	$-$	dne	$+$	0	$-$
x		-2		0		2	

$x = 0$ is neither a max nor a min.

$x = +2$ & $x = -2$ are both local maximums.

3. (15 points) Sketch the graph of a continuous curve $y = f(x)$ with the following properties: $f(2) = 4$, $f''(x) > 0$ for $x < 2$, $f''(x) > 0$ for $x > 2$, $\lim_{x\to 2^+} f'(x) = -\infty$, $\lim_{x\to 2^-} f'(x) = \infty$.



4. (15 points) Consider the following applied optimization problem:

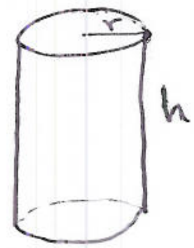
A closed cylindrical can is to have a surface area of 600 square units. Find the can that has the largest volume.

Find

(a) the function to be maximized or minimized (indicate which it is)

(b) the interval of possible values for the variable in this problem

Do not solve the optimization problem.



$$S = \underbrace{2\pi r h}_{\text{sides}} + \underbrace{2\pi r^2}_{\text{top + bottom}} = 600$$

$$2\pi r h = 600 - 2\pi r^2$$

$$h = \frac{600 - 2\pi r^2}{2\pi r} = \frac{600}{2\pi r} - 2r = \frac{300}{\pi r} - 2r$$

$$V = \pi r^2 h = \pi r^2\left(\frac{300}{\pi r} - 2r\right)$$

maximize $V = \pi r^2\left(\frac{300}{\pi r} - 2r\right) = 300r - 2\pi r^3$

for $0 < r < \sqrt{\frac{150}{\pi}}$

To have $h > 0$:

$$\frac{300}{\pi r} - 2r > 0$$

$$\frac{300}{\pi r} > 2r$$

$\frac{150}{\pi} > r^2$ so $r < \sqrt{\frac{150}{\pi}}$

5. (15 points) Find the absolute maximum and absolute minimum values of $f(x) = \frac{x}{x^2+4}$ on $[-4, 1]$.

$$f'(x) = \frac{(x^2+4)1 - x(2x)}{(x^2+4)^2} = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$$

Because $x^2+4 > 0$, $f'(x)$ exists everywhere

Solving $f'(x) = 0$: $\frac{4-x^2}{(x^2+4)^2} = 0 \iff 4 - x^2 = 0$

$\iff x^2 = 4$

$\iff x = \pm 2$.

$x = 2$ is not in the interval.

x	$f(x)$
-4	$-1/5 = -0.2$
1	$1/5 = +0.2$
-2	$-1/4 = -0.25$

$f(-4) = \frac{-4}{16+4} = -\frac{4}{20} = -\frac{1}{5}$

$f(1) = \frac{1}{1+4} = \frac{1}{5}$

$f(-2) = \frac{-2}{4+4} = -\frac{2}{8} = -\frac{1}{4}$

The absolute maximum of $f(x) = \frac{x}{x^2+4}$ on $[-4,1]$ is 0.2 which occurs for $x = 1$.

The absolute minimum of $f(x) = \frac{x}{x^2+4}$ on $[-4,1]$ is -0.25 which occurs for $x = -2$.

6. (12 points) The position function for a particle moving along a coordinate line is

$$s = \ln(3t^2 - 12t + 13).$$

(a) Find the velocity of the particle at time $t = 1$.

$$v = s' = \frac{6t-12}{3t^2-12t+13}$$

$$\text{at } t=1: \quad v = \frac{6-12}{3-12+13} = \frac{-6}{4} = -\frac{3}{2}.$$

(b) Find the acceleration of the particle at time $t = 1$.

$$a = v' = \frac{(3t^2-12t+13)\,6 - (6t-12)(6t-12)}{(3t^2-12t+13)^2}$$

$$= \frac{18t^2-72t+78 - (36t^2-144t+144)}{(3t^2-12t+13)^2}$$

$$= \frac{-18t^2+72t-66}{(3t^2-12t+13)^2}$$

$$\text{at } t=1: \quad a = \frac{-18+72-66}{(3-12+13)^2} = \frac{-12}{16} = -\frac{3}{4}.$$

(c) Find the time when the particle is stopped.

Stopped when $v=0$:

$$\frac{6t-12}{3t^2-12t+13} = 0 \iff 6t-12=0$$

$$\iff 6t = 12$$

$$\iff \underline{\underline{t=2}}$$

(d) At this time, is the particle's acceleration positive or negative?

$$\text{at } t=2: \quad a = \frac{-18(2)^2+72(2)-66}{(3(2)^2-12(2)+13)^2} = \frac{6}{(\;)^2} > 0$$

← the exact value does not matter, but it would be $(-3)^2 = 9$

So the acceleration is positive

(the particle is changing direction here; for $t>2$ it will be moving to the right)