MATH 141 (Section 11 & 12)
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University of South Carolina
Fall 2007

Exam 3
October 26, 2007

Name: Key
Section: 011 / 012 (circle one)

Instructions:

1. There are a total of 5 problems on 7 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	40	
2	32	
3	10	
4	8	
5	10	
Total	100	

Do Your Best!

1. (40 points) Find $\frac{dy}{dx}$ in each of the following. Indicate each time you use Implicit Differentiation (ID) or Logarithmic Differentiation (LD).

   (a) $y = \arctan(\sqrt{(x)}) + 2^x$

$$\frac{dy}{dx} = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}\left(\sqrt{x}\right) + 2^x \ln(2)$$

$$= \frac{1}{1+x}\left(\frac{1}{2}x^{-1/2}\right) + 2^x \ln(2)$$

$$= \frac{1}{2(1+x)\sqrt{x}} + 2^x \ln(2)$$

   (b) $5y^2 + \sin(y) = x^2$

ID

$$5y(x)^2 + \sin(y(x)) = x^2$$

$$\frac{d}{dx}\left(5y(x)^2\right) + \frac{d}{dx}\left(\sin(y(x))\right) = \frac{d}{dx}(x^2).$$

$$5 \cdot 2y \cdot \frac{dy}{dx} + \cos(y)\frac{dy}{dx} = 2x$$

$$\left(10y + \cos(y)\right)\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{10y+\cos(y)}.$$

(c) $y = x^{\sin(x)}$

LD

$$\ln(y) = \ln\left(x^{\sin(x)}\right) = \sin(x) \ln(x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin(x) \cdot \frac{1}{x} + \cos(x) \ln(x).$$

$$\frac{dy}{dx} = y\left(\frac{\sin(x)}{x} + \cos(x) \ln(x)\right)$$

$$= x^{\sin(x)}\left(\frac{\sin(x)}{x} + \cos(x) \ln(x)\right).$$

$$= \sin(x)\, x^{\sin(x)-1} + x^{\sin(x)} \ln(x) \cos(x).$$

(d) $y = \dfrac{\sin(x)\cos(x)}{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{\sqrt{x}\left(\frac{d}{dx}(\sin(x)\cos(x))\right) - \sin(x)\cos(x) \cdot \left(\frac{d}{dx}\sqrt{x}\right)}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}\left(\cos(x)\cos(x) + \sin(x)(-\sin(x))\right) - \sin(x)\cos(x)\left(\frac{1}{2}x^{-1/2}\right)}{x}$$

$$= \frac{\sqrt{x}\left(\cos^2(x) - \sin^2(x)\right) - \frac{\sin(x)\cos(x)}{2\sqrt{x}}}{x}$$

$$= \frac{\cos^2(x) - \sin^2(x)}{\sqrt{x}} - \frac{\sin(x)\cos(x)}{2x\sqrt{x}}.$$

OR: LD

$$\ln(y) = \ln\left(\frac{\sin(x)\cos(x)}{\sqrt{x}}\right) = \ln(\sin(x)) + \ln(\cos(x)) - \frac{1}{2}\ln(x).$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sin(x)} \cdot \cos(x) + \frac{1}{\cos(x)} \cdot (-\sin(x)) - \frac{1}{2x}$$

$$\frac{dy}{dx} = y\left(\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)} - \frac{1}{2x}\right) = \frac{\sin(x)\cos(x)}{\sqrt{x}}\left(\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)} - \frac{1}{2x}\right)$$

$$= \frac{\cos^2(x)}{\sqrt{x}} - \frac{\sin^2(x)}{\sqrt{x}} - \frac{\sin(x)\cos(x)}{2x\sqrt{x}}.$$

2. (32 points) Evaluate each of the following limits. Identify each indeterminate form that you encounter and indicate each time you use l'Hôpital's Rule.

(a) $\lim_{x \to 0^+} \frac{\cos(x)}{x}$ $= \frac{1}{0^+} = +\infty$

(b) $\lim_{x \to \infty} x \sin\left(\frac{\pi}{x}\right)$ $= \lim_{x\to\infty} \frac{\sin(\pi/x)}{1/x} \underset{l'H\left(\frac{0}{0}\right)}{=} \lim_{x\to\infty} \frac{\cos(\pi/x)\,(-\pi/x^2)}{-1/x^2}$

$\infty \cdot 0$

$= \lim_{x\to\infty} \cos\left(\frac{\pi}{x}\right)\left(-\frac{\pi}{x^2}\right)\cdot\left(-\frac{x^2}{1}\right)$

$= \lim_{x\to\infty} \cos\left(\frac{\pi}{x}\right)\cdot\pi$

$= \cos(0)\cdot\pi$

$= 1\cdot\pi$

$= \pi.$

(c) $\lim_{t\to 0} \frac{te^t}{1-e^t} \underset{l'H(\frac{0}{0})}{=} \lim_{t\to 0} \frac{te^t + e^t}{-e^t}$

$= \lim_{t\to 0} \frac{(t+1)e^t}{-e^t}$

$= \lim_{t\to 0} \frac{t+1}{-1}$

$= -1$

(d) $\lim_{x\to 0} (e^x + x)^{1/x} = L$

1^∞

$\ln(L) = \ln\left(\lim_{x\to 0} (e^x+x)^{1/x}\right)$

$= \lim_{x\to 0} \ln\left((e^x+x)^{1/x}\right)$

$= \lim_{x\to 0} \frac{1}{x}\ln(e^x+x)$

$= \lim_{x\to 0} \frac{\ln(e^x+x)}{x}$

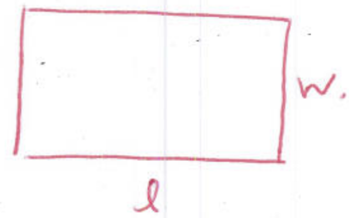
$\underset{l'H(\frac{0}{0})}{=} \lim_{x\to 0} \frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1}$

$= \lim_{x\to 0} \frac{e^x+1}{e^x+x} = \frac{1+1}{1+0} = 2.$ so $L = e^{\ln(L)} = e^2.$

3. (10 points) The length of a rectangle is increasing at a rate of 8 cm/s and its width is decreasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, answer the following two questions:

   (a) is the area of the rectangle increasing or decreasing?

   (b) how fast is the area of the rectangle changing?



$A = lw.$

$A(t) = l(t)\,w(t).$

$l = 20\text{cm}$ $\quad \frac{dl}{dt} = 8\ \text{cm}/\text{s}$

$w = 10\text{cm}$ $\quad \frac{dw}{dt} = -3\ \text{cm}/\text{s}.$

$$\frac{dA}{dt} = \frac{d}{dt}\left(l(t)\,w(t)\right) = \frac{dl}{dt}w + l\frac{dw}{dt}.$$

$$\frac{dA}{dt} = \frac{dl}{dt}w + l\frac{dw}{dt}$$

$$= 8(10) + 20(-3)$$

$$= 80 - 60$$

$$= 20\ \text{cm}^2/\text{s}.$$

Because $\frac{dA}{dt} > 0$, the area is <u>increasing</u>.

The area is changing at the rate of <u>$20\ \text{cm}^2/\text{sec}$.</u>

4. (8 points) Find the local linear approximation of $f(x) = \dfrac{1}{(1+2x)^5}$ at $x_0 = 0$.

$f(0) = 1$

$= (1+2x)^{-5}$

$L(x) = f(0) + f'(0)(x-0)$

$f'(x) = -5(1+2x)^{-6}(2)$ $f'(0) = -10$

$= \dfrac{-10}{(1+2x)^6}$

$= 1 + (-10)(x-0)$

$= 1 - 10x.$

5. (10 points) The area of a circle is to be computed from a measured value of its radius. Use differentials to estimate the maximum permissible percentage error in the measurement if the percentage error in the area must be kept with $\pm 1\%$.

$A = \pi r^2$

$\left|\dfrac{dA}{A}\right| \le 0.01.$

$dA = 2\pi r\, dr$

$\dfrac{dA}{A} = \dfrac{2\pi r\, dr}{\pi r^2} = 2\dfrac{dr}{r}.$

$\left|\dfrac{dA}{A}\right| = \left|2\dfrac{dr}{r}\right| = 2\left|\dfrac{dr}{r}\right| \le 0.01.$

So $\left|\dfrac{dr}{r}\right| \le \dfrac{0.01}{2} = 0.005.$

The maximum permissible percentage error in the measurement of the radius is 0.5%.