```
> restart;
```

The Football Chase Problem
with Proportional Leading

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Modified for Maple V, Release 3 by Doug Meade

Here's the original problem I set for the students.

It was the day of the big game, NU against Enormous State University, on national TV, the fictional collegiate championship, the works! The Huskers led 24-21, very late in the 4th quarter, the ESU Bruisers had the ball on their own 25. Bart Simpson, star running back for the Bruisers, burst through the Husker line. Looking up as he crossed the 25, he saw 75 yards of open field between him and the goal line. ''Here comes six points, the game, the title, a first round draft choice, and a lucrative career in the NFL!'' thought Bart.

Thirty yards to Bart's left, also on the 25 yard line, Husker linebacker Matt E. Mattics saw Bart break through. Due to excellent scouting reports, Matt knew he was faster than Bart. In fact, Bart could run 8 yards//second while Matt could do 8.4 yards/second. Hence, given enough time, Matt knew he could run Bart down. Matt immediately set out in pursuit of Bart. He figured the best way to catch his opponent and to save the game was to constantly adjust his path so that he was always running right at Bart; that is, run in such a path that at each instant his velocity vector was pointing directly at Bart. (These football players think fast on their feet!)

As Matt started out on his pursuit, a groan immediately rose from the special block of seats reserved for the 221H class. Why did the students groan?

The Parameters and the Strategy

Set the parameters:
See figure below for schematic diagram:

a is the initial cross-field distance,
b is the initial down-field distance,
c = b/a,
v1 is the velocity of the defending runner (Matt E. Mattics),
v2 is the velocity of the offensive runner (Bart Simpson),
v = v2/v1 (< 1 for the sake of an interesting problem),
e is the leading factor or strategy variable, see the modeling below,
(0, 0) is the initial position of the defending runner,
(x(t), y(t)) is the position of the defending runner,
(a, v2*t) is the position of the offensive runner,
the defender always "leads" the offensive runner by aiming at a point ahead of the offensive runner,
The "leading point" of the defender is e*b + (1-e)*v2*t:
for e = 0, the defender always aims at the offensive runner,
for e = 1, the defender always aims at the goal-line,
for 0 < e < 1, the defender "leads" the offensive runner.

The velocity vector (dx/dt, dy/dy) of the defender then satisfies
(dy/dt)/(dx/dt) = ((e*b + (1-e)*v2*t) - y(t))/(a-x(t)).
Then
(dy/dx)*(a - x) - ((e*b + (1-e)*v2*t) - y) = 0.
Differentiating with respect to x
(a-x) * d^2 y/ dx^2 - dy/dx - (1-e)*v2*(dt/dx) + dy/dx = 0.
From the velocity relationship
(dx/dt)^2 + (dy/dt)^2 = v1^2,
or
dx^2 + dy^2 = v1^2 dt^2
or
1 + (dy/dx)^2 = v1^2* (dt/dx)^2
so
(a-x) * d^2 y/dx^2 = (1-e)*(v2/v1)*sqrt(1 + (dy/dx)^2).
The initial conditions are:
y(0) = 0
dy/dx|(x = 0) = e*b/a = e*c
Note that for a > 0, and e = 1, we get the differential equation
y''(x) = 0, y(0) = 0, y'(0) = b/a
indicating the straight line race to the goal-line.

Set p = dy/dx, so the differential equation becomes:
(dp/dx)/sqrt(1+p^2) = (1-e)*v/(a-x), p(0) = e*b/a

```
> SUBS := { b=c*a, v2=v*v1 };
```

$$SUBS := \{ b = c\,a,\ v2 = v\,v1 \}$$

```
> PARAM := { a=30, b=75, v1=8.4, v2=8, c=75/30, v=8/8.4 };
```

$$PARAM := \left\{ a = 30,\ b = 75,\ v1 = 8.4,\ v2 = 8,\ c = \frac{5}{2},\ b = .9523809524 \right\}$$

Solve the problem of the straight line race to the goal first:

The time for the offensive runner to reach the goal is

```
> time_offense := b/v2;
```

$$time_offense := \frac{b}{v2}$$

The time for the defensive runner to reach the goal along the hypoteneuse to the goal is

```
> time_defense := sqrt(a^2 + b^2)/ v1;
```

$$time_defense := \frac{\sqrt{a^2 + b^2}}{v1}$$

Matt will beat Bart to the goalline if

```
> WIN := time_defense < time_offense;
```

$$WIN := \frac{\sqrt{a^2 + b^2}}{v1} < \frac{b}{v2}$$

or, equivalently

```
> WIN2 := simplify( subs( SUBS, WIN*(v2/sqrt(a^2+b^2)) ) );
```

$$WIN2 := v < \frac{c\,a}{\sqrt{a^2\,(1 + c^2)}}$$

Therefore, using the fact that a>0, this shows the strategy e = 1 requires a velocity ratio of c/sqrt(1 + c^2) to succeed in (barely) intercepting the offensive runner. It seems clear that as the velocity ratio decreases (the defender is faster than the offensive runner), a strategy 0 <= e < 1 is possible. So the question becomes, for a given velocity ratio less than the absolutely necessary, what should be the strategy? I.e., find e as a function of v.

The Solution for the Path

```
> dsolve({diff(p(x),x)/(sqrt(1+p(x)^2)) = (1-e)*v/(a-x), p(0)=e*c}, p(x) );
```

$$p(x) = \sinh\left(-\ln(-a+x)\,v + \ln(-a+x)\,v\,e + \ln\left(e\,c + \sqrt{e^2\,c^2+1}\right) + \ln(-a)\,v - \ln(-a)\,v\,e\right)$$

OK, that's a little too much to ask. Let's see if we can obtain a better form for this solution "by hand".

```
> implicit_eqn_1 := int( 1/sqrt(1+p^2), p) = int( (1-e)*v/(a-x), x) + C1;
```

$$implicit_eqn_1 := \operatorname{arcsinh}(p) = -\ln(a-x)\,v + \ln(a-x)\,v\,e + C1$$

```
> const := solve( subs(p=e*c, x = 0, implicit_eqn_1), C1);
```

$$const := \operatorname{arcsinh}(e\,c) + \ln(a)\,v - \ln(a)\,v\,e$$

```
> implicit_eqn_2 := subs( C1 = const, implicit_eqn_1);
```

$$implicit_eqn_2 := \operatorname{arcsinh}(p) = -\ln(a-x)\,v + \ln(a-x)\,v\,e + \operatorname{arcsinh}(e\,c) + \ln(a)\,v - \ln(a)\,v\,e$$

Let k = e*c + sqrt(1+ e^2*c^2))
For e = 1, get k = c + sqrt(1 + c^2)
For e = 0, get k = 1

Exponentiate implicit_eqn_2 on both sides to remove logarithms (look up the definition of arcsinh). Do this "in your head", since Maple simplification needs too much coercion to get right.

```
> implicit_eqn_3 := (p + sqrt(1 + p^2)) = k*(a/(a-x))^((1-e)*v);
```

$$implicit_eqn_3 := p + \sqrt{1+p^2} = k\left(\frac{a}{a-x}\right)^{(1-e)\,v}$$

Now remember that p = dy/dx, so need to solve for p, and then integrate and apply second initial condition to get the explicit path y(x).

```
> ivp2 := p  = simplify(solve(implicit_eqn_3, p));
```

$$ivp2 := p = \frac{1}{2}\,\frac{-(-1)^{(-v+ve)}\,a^{(-v+ve)}\,(-a+x)^{(v-ve)} + (-1)^{(v-ve)}\,a^{(v-ve)}\,(-a+x)^{(-v+ve)}\,k^2}{k}$$

The powers of (-1) are a real nuisance -- they lead to complex-valued solutions -- and I don't see an easy way of removing them. Once again, a little human intervention is very helpful.

```
> FORM := p+sqrt(1+p^2)=A;
```

$$FORM := p + \sqrt{1+p^2} = A$$

```
> ivp2 := p = subs( A=rhs(implicit_eqn_3), solve(FORM,p) );
```

$$ivp2 := p = \frac{1}{2}\,\frac{-1 + k^2\left(\left(\frac{a}{a-x}\right)^{(1-e)\,v}\right)^2}{k\left(\frac{a}{a-x}\right)^{(1-e)\,v}}$$

```
> explicit_soln := y = int(rhs(ivp2), x) + c2;
```

$$explicit_soln := y = -\frac{1}{2}\frac{a^{(-v+ve+1)}(a-x)^{(v-ve)}}{k(ve-v-1)} + \frac{1}{2}\frac{a^{(v-ve)}(a-x)^{(-v+ve)}kx}{-v+ve+1} + \frac{1}{2}\frac{a^{(-v+ve)}(a-x)^{(v-ve)}x}{k(ve-v-1)}$$

$$-\frac{1}{2}\frac{a^{(-ve+v+1)}(a-x)^{(-v+ve)}k}{-v+ve+1} + c2$$

```
> const2 := simplify(solve( subs( y=0, x= 0, explicit_soln), c2));
```

$$const2 := \frac{1}{2}\frac{a(-v+ve+1+k^2ve-k^2v-k^2)}{k(ve-v-1)(-v+ve+1)}$$

This gives us the path of the defender, if we care to graph it or use it.

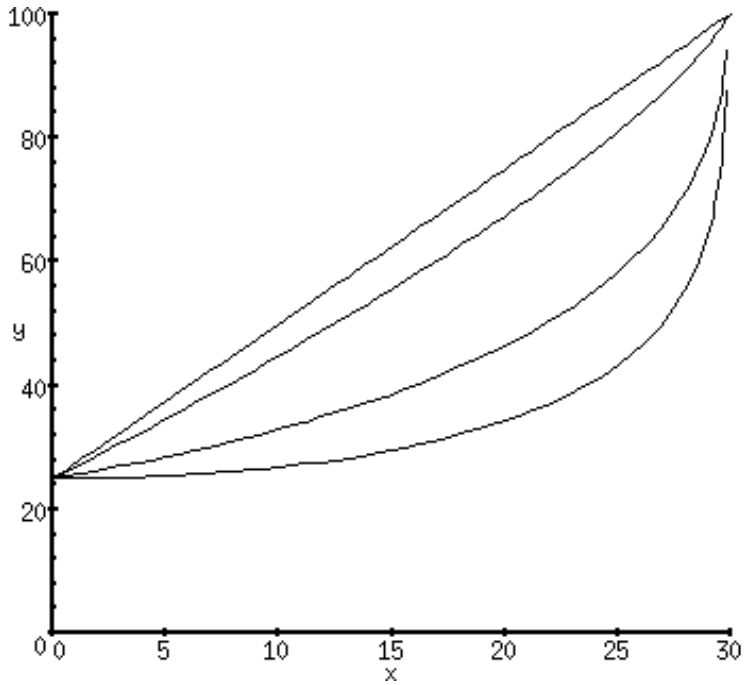
```
> y := subs( c2  = const2, k = e*c + sqrt(1 + e^2*c^2), rhs(explicit_soln));
```

$$y := -\frac{1}{2}\frac{a^{(-v+ve+1)}(a-x)^{(v-ve)}}{\%1(ve-v-1)} + \frac{1}{2}\frac{a^{(v-ve)}(a-x)^{(-v+ve)}\%1x}{-v+ve+1} + \frac{1}{2}\frac{a^{(-v+ve)}(a-x)^{(v-ve)}x}{\%1(ve-v-1)}$$

$$-\frac{1}{2}\frac{a^{(-ve+v+1)}(a-x)^{(-v+ve)}\%1}{-v+ve+1} + \frac{1}{2}\frac{a(-v+ve+1+\%1^2ve-\%1^2v-\%1^2)}{\%1(ve-v-1)(-v+ve+1)}$$

$$\%1 := ec + \sqrt{e^2c^2+1}$$

```
> plot({subs( PARAM union { e = 0   }, y+25),
        subs( PARAM union { e = .25 }, y+25),
        subs( PARAM union { e = .75 }, y+25),
        subs( PARAM union { e = 1.0 }, y+25)}, x=0..30, 'y'=0..100);
```



The Distance Along the Path

Now we really need to compute times along the paths to determine when the defender and the offensive runner collide. To get the time required to go completely across field for the defender along the defender's path need the length of the defender's path across field . For that need the differential element of arc for the defender which uses p = dy/dx already calculated above.

```
> diff_elt_arc := sqrt( factor( expand( 1 + rhs(ivp2)^2 ) ), symbo
  lic );
```

$$diff_elt_arc := \frac{1}{2}\,\frac{(a^{(v\,e)})^2\left((a-x)^{v}\right)^2 + k^2\,(a^{v})^2\left((a-x)^{(v\,e)}\right)^2}{k\,a^{v}\,(a-x)^{(v\,e)}\,a^{(v\,e)}\,(a-x)^{v}}$$

```
> arc_length := subs(h=x, simplify(int( diff_elt_arc, x=0..h)));
```

$$\begin{aligned} arc_length := \frac{1}{2}\Bigg(& x\,v\left(-\frac{-a+x}{a}\right)^{(-v\,(-1+e))} - a^{(-v+v\,e+1)}\,(a-x)^{(-v\,(-1+e))}\,v - x\,v\,e\left(-\frac{-a+x}{a}\right)^{(-v\,(-1+e))} \\ & + a^{(-v+v\,e+1)}\,(a-x)^{(-v\,(-1+e))}\,v\,e - x\left(-\frac{-a+x}{a}\right)^{(-v\,(-1+e))} + a^{(-v+v\,e+1)}\,(a-x)^{(-v\,(-1+e))} \\ & + a^{(-v\,e+v+1)}\,(a-x)^{(v\,(-1+e))}\,k^2 - a^{(-v\,e+v+1)}\,(a-x)^{(v\,(-1+e))}\,k^2\,v\,e + k^2\,x\,v\,e\left(-\frac{-a+x}{a}\right)^{(v\,(-1+e))} \\ & + a^{(-v\,e+v+1)}\,(a-x)^{(v\,(-1+e))}\,k^2\,v - k^2\,x\,v\left(-\frac{-a+x}{a}\right)^{(v\,(-1+e))} - k^2\,x\left(-\frac{-a+x}{a}\right)^{(v\,(-1+e))} - a\,k^2 - a\,v\,e + a\,v \\ & + a\,k^2\,v\,e - a - a\,k^2\,v\Bigg) \Big/ (k\,(v\,e-v-1)\,(-v+v\,e+1)) \end{aligned}$$

Here's a simple test to see if the arclength came out correctly:

```
> simplify(subs(x = 0, arc_length));
```

$$0$$

So at least that much is OK! Now figure out the time it takes the defender to cross the field:

```
> intersect_time := simplify(subs(x = a, arc_length)/v1);
```

$$intersect_time := \frac{1}{2}\,\frac{a\,(-k^2 - v\,e + v + k^2\,v\,e - 1 - k^2\,v)}{k\,(v\,e-v-1)\,(-v+v\,e+1)\,v1}$$

```
> new_capture_eqn := simplify(subs(x=a, arc_length))/v1 = b/v2;
```

$$new_capture_eqn := \frac{1}{2}\,\frac{a\,(-k^2 - v\,e + v + k^2\,v\,e - 1 - k^2\,v)}{k\,(v\,e-v-1)\,(-v+v\,e+1)\,v1} = \frac{b}{v2}$$

Render the new_capture_eqn in dimensionless form:

```
> new_capture_eqn := lhs(new_capture_eqn)*v1/a = rhs(new_capture_eqn)*v2/b*c/
  v;
```

$$new_capture_eqn := \frac{1}{2}\,\frac{-k^2 - v\,e + v + k^2\,v\,e - 1 - k^2\,v}{k\,(v\,e-v-1)\,(-v+v\,e+1)} = \frac{c}{v}$$

```
> eqn_for_e := subs( k = e*c + sqrt(1+e^2*c^2), new_capture_eqn);
```

$$eqn_for_e := \frac{1}{2}\,\frac{-\%1^2 - v\,e + v + \%1^2\,v\,e - 1 - \%1^2\,v}{\%1\,(v\,e-v-1)\,(-v+v\,e+1)} = \frac{c}{v}$$

$$\%1 := e\,c + \sqrt{e^2\,c^2 + 1}$$

Check this equation by substituting in e =1 and the known solution v = c/sqrt(1 + c^2) to check for consistency with the physical reasoning before:

```
> subs( e = 1, v = c/sqrt(1+ c^2), eqn_for_e);
```

$$-\frac{1}{2}\,\frac{-\left(c + \sqrt{1 + c^2}\right)^2 - 1}{c + \sqrt{1 + c^2}} = \sqrt{1 + c^2}$$

```
> simplify( lhs(")-rhs(") );
```

$$0$$

By inspection, this is valid, so must be OK to this point.

Now solve for v in terms of the strategy e, and the shape parameter c.

```
> eqn_for_e;
```

$$\frac{1}{2}\,\frac{-\%1^2 - v\,e + v + \%1^2\,v\,e - 1 - \%1^2\,v}{\%1\,(v\,e - v - 1)\,(-v + v\,e + 1)} = \frac{c}{v}$$

$$\%1 := e\,c + \sqrt{e^2\,c^2 + 1}$$

```
> solve( eqn_for_e, v);
```

$$\mathrm{RootOf}\left((e\,c - c)\,_Z^2 - _Z\sqrt{e^2\,c^2 + 1} + c\right)$$

```
> allvalues(");
```

$$\frac{1}{2}\,\frac{\sqrt{e^2\,c^2 + 1} + \sqrt{e^2\,c^2 + 1 - 4\,e\,c^2 + 4\,c^2}}{e\,c - c}, \frac{1}{2}\,\frac{\sqrt{e^2\,c^2 + 1} - \sqrt{e^2\,c^2 + 1 - 4\,e\,c^2 + 4\,c^2}}{e\,c - c}$$

```
> vvse1 := "[1]; vvse2 := ""[2];
```

$$vvse1 := \frac{1}{2}\,\frac{\sqrt{e^2\,c^2 + 1} + \sqrt{e^2\,c^2 + 1 - 4\,e\,c^2 + 4\,c^2}}{e\,c - c}$$

$$vvse2 := \frac{1}{2}\,\frac{\sqrt{e^2\,c^2 + 1} - \sqrt{e^2\,c^2 + 1 - 4\,e\,c^2 + 4\,c^2}}{e\,c - c}$$

Again test vvse2 to see if the solution is valid!

```
> subs(e =1, vvse2);
```

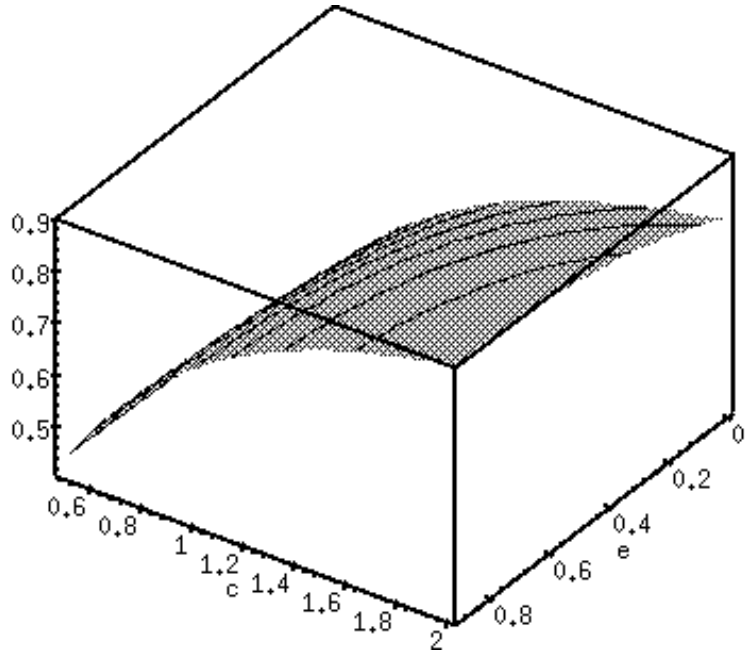
Error, division by zero

Hmm,.... didn't work, seems to be some kind of singularity here.

```
> subs( c = sqrt(3), vvse2);
```

$$\frac{1}{2}\,\frac{\sqrt{3\,e^2 + 1} - \sqrt{3\,e^2 + 13 - 12\,e}}{e\sqrt{3} - \sqrt{3}}$$

```
> plot3d(vvse2, e=0..0.9, c = (1/2)..2);
```



The plot reveals that the velocity ratio is much more dependent on the shape parameter than on the proportional leading strategy chosen.

The Separation of the Runners

What I am going to do in this section is to compare the various strategies for the football chase problem. This will be done in the following way. Realisitically, the chase will last for a maximum of 9.375 seconds (= 75 yards/ (8 yards/sec)). I am going to graph the downfield separation of the two runners, which will be v_q t - y, as a functionof t.. The problem with this is that I don't have y as a function of t . Instead, I have the following quantities

y(x) = downfield position of Matt as a functionof cross-field position x
s(x) = distance along the path followed by Matt as a functionof cross-field position x
t(x) = time used by Matt along the path as a functionof cross-field position x

The downfield position occupied by Bart is then vq t(x) as a function of Matt's cross-field position.

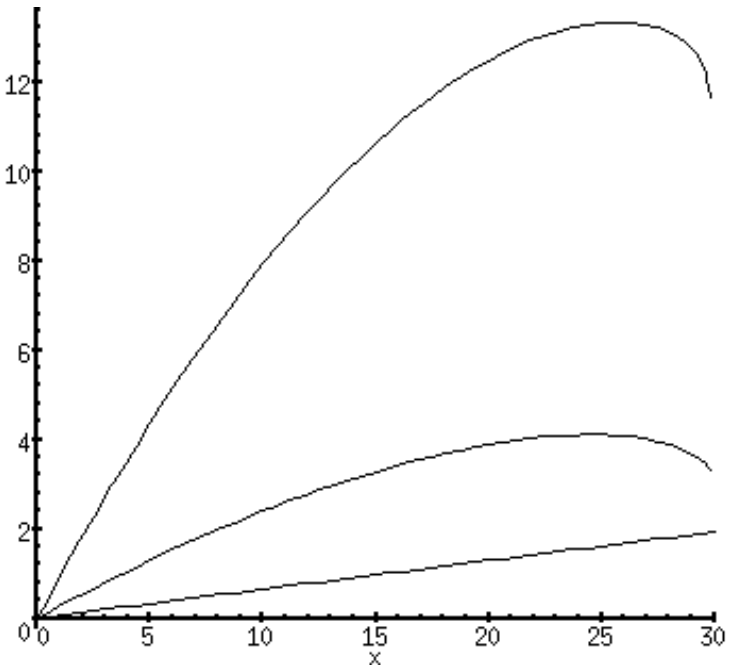
Then I will consider the differences v2 t(x) - y(x) = v2*s(x)/v1 - y(x) = v*s(x) - y(x) as a function of x, ans x goes from 0 to 30.

```
> down_field_sep := v*arc_length - y;
```

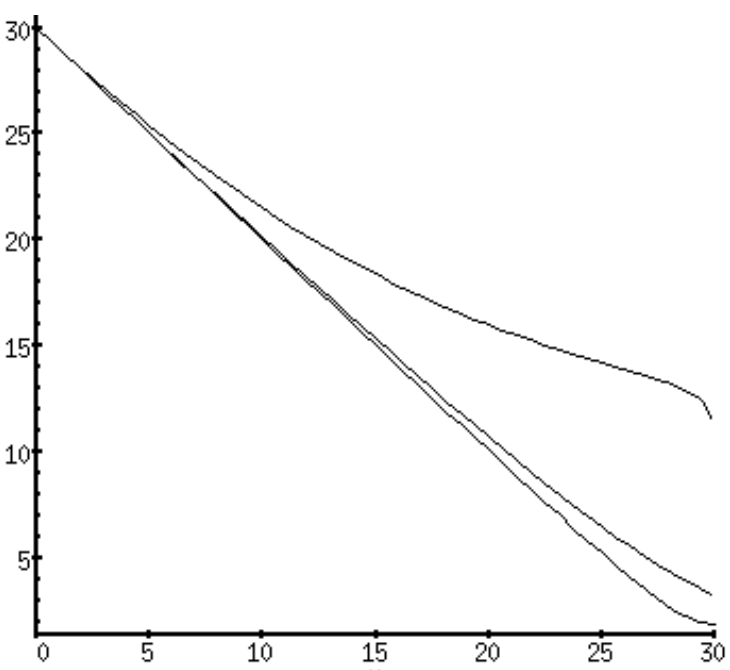
$$down_field_sep := \frac{1}{2} v \left(x\, v \left(-\frac{-a+x}{a} \right)^{(-v(-1+e))} - a^{(-v+ve+1)} (a-x)^{(-v(-1+e))} v - x\, v\, e \left(-\frac{-a+x}{a} \right)^{(-v(-1+e))} + a^{(-v+ve+1)} (a-x)^{(-v(-1+e))} v\, e - x \left(-\frac{-a+x}{a} \right)^{(-v(-1+e))} + a^{(-v+ve+1)} (a-x)^{(-v(-1+e))} + a^{(-ve+v+1)} (a-x)^{(v(-1+e))} k^2 - a^{(-ve+v+1)} (a-x)^{(v(-1+e))} k^2 v\, e + k^2 x\, v\, e \left(-\frac{-a+x}{a} \right)^{(v(-1+e))} + a^{(-ve+v+1)} (a-x)^{(v(-1+e))} k^2 v - k^2 x\, v \left(-\frac{-a+x}{a} \right)^{(v(-1+e))} - k^2 x \left(-\frac{-a+x}{a} \right)^{(v(-1+e))} - a\, k^2 - a\, v\, e + a\, v + a\, k^2 v\, e - a - a\, k^2 v \right) \Big/ (k (v\, e - v - 1) (-v + v\, e + 1)) + \frac{1}{2} \frac{a^{(-v+ve+1)} (a-x)^{(v-ve)}}{\%1 (v\, e - v - 1)} - \frac{1}{2} \frac{a^{(v-ve)} (a-x)^{(-v+ve)} \%1\, x}{-v + v\, e + 1} - \frac{1}{2} \frac{a^{(-v+ve)} (a-x)^{(v-ve)} x}{\%1 (v\, e - v - 1)} + \frac{1}{2} \frac{a^{(-ve+v+1)} (a-x)^{(-v+ve)} \%1}{-v + v\, e + 1} - \frac{1}{2} \frac{a (-v + v\, e + 1 + \%1^2\, v\, e - \%1^2\, v - \%1^2)}{\%1 (v\, e - v - 1) (-v + v\, e + 1)}$$

$$\%1 := e\, c + \sqrt{e^2 c^2 + 1}$$

```
> plot({subs( k = e*c + sqrt( 1 + e^2 * c^2),
              a = 30, v = 8/8.4, e = 0.0, c= 75/30, down_field_sep),
  subs( k = e*c + sqrt( 1 + e^2 * c^2),
       a = 30, v = 8/8.4, e = 0.5, c= 75/30, down_field_sep),
  subs( k = e*c + sqrt( 1 + e^2 * c^2),
       a = 30, v = 8/8.4, e = 1.0, c= 75/30, down_field_sep)},
   x = 0..30);
```




```
> totsep := sqrt( down_field_sep^2 + (a-x)^2):
> plot({subs( k = e*c + sqrt( 1 + e^2 * c^2),
       a = 30, v = 8/8.4, e = 0.0, c= 75/30, totsep),
  subs( k = e*c + sqrt( 1 + e^2 * c^2),
       a = 30, v = 8/8.4, e = 0.5, c= 75/30,
       totsep),
  subs( k = e*c + sqrt( 1 + e^2 * c^2),
       a = 30, v = 8/8.4, e = 1.0, c= 75/30, totsep)},
   x = 0..30);
```



Notice that the runners are always separated by at least a yard. This occurs because the speed parameters are rigged in such a way that Matt can never quite capture Bart, given the current speeds, even with the best possible strategy.

END OF WORKSHEET

>