```
> restart;
```

The Football Chase Problem
with Proportional Leading

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It was the day of the big game, NU against Enormous State University, on national TV, the fictional collegiate championship, the works! The Huskers led 24-21, very late in the 4th quarter, the ESU Bruisers had the ball on their own 25. Bart Simpson, star running back for the Bruisers, burst through the Husker line. Looking up as he crossed the 25, he saw 75 yards of open field between him and the goal line. ‘‘Here comes six points, the game, the title, a first round draft choice, and a lucrative career in the NFL!’’ thought Bart.

Thirty yards to Bart’s left, also on the 25 yard line, Husker linebacker Matt E. Mattics saw Bart break through. Due to excellent scouting reports, Matt knew he was faster than Bart. In fact, Bart could run 8 yards//second while Matt could do 8.4 yards/second. Hence, given enough time, Matt knew he could run Bart down. Matt immediately set out in pursuit of Bart. He figured the best way to catch his opponent and to save the game was to constantly adjust his path so that he was always running right at Bart; that is, run in such a path that at each instant his velocity vector was pointing directly at Bart. (These football players think fast on their feet!)

As Matt started out on his pursuit, a groan immediately rose from the special block of seats reserved for the 221H class. Why did the students groan?

The Parameters and the Strategy

Set the parameters:
See figure below for schematic diagram:

a is the initial cross-field distance,
b is the initial down-field distance,
c = b/a,
v1 is the velocity of the defending runner (Matt E. Mattics),
v2 is the velocity of the offensive runner (Bart Simpson),
v = v2/v1 (< 1 for the sake of an interesting problem),
(0, 0) is the initial position of the defending runner,
(x(t), y(t)) is the position of the defending runner,
(a, v2*t) is the position of the offensive runner,
the defender always "leads" the offensive runner by aiming at a point ahead of the offensive runner,
The "leading point" of the defender is v2*t:

The velocity vector (dx/dt, dy/dy) of the defender then satisfies
(dy/dt)/(dx/dt) = (v2*t - y(t))/(a-x(t)).
Then
(dy/dx)*(a - x) - (v2*t - y) = 0.
Differentiating with respect to x
(a-x) * d^2 y/ dx^2 - dy/dx - v2*(dt/dx) + dy/dx = 0.
From the velocity relationship
(dx/dt)^2 + (dy/dt)^2 = v1^2,
or
dx^2 + dy^2 = v1^2 dt^2
or
1 + (dy/dx)^2 = v1^2* (dt/dx)^2
so
(a-x) * d^2 y/dx^2 = (v2/v1)*sqrt(1 + (dy/dx)^2).
The initial conditions are:
y(0) = 0
dy/dx|(x = 0) = 0

Set p = dy/dx, so the differential equation becomes:
(dp/dx)/sqrt(1+p^2) = v/(a-x), p(0) = 0

```
> PARAM := { a=30, b=75, v1=8.4, v2=8 }:
> PARAM2 := PARAM union { v=8/8.4 }:
```

The Solution for the Path

```
> dsolve({diff(p(x),x)/(sqrt(1+p(x)^2)) = v/(a-x), p(0)=0}, p(x) );
```

$$\mathrm{p}(x) = \sinh(-\ln(-a + x)\,v + \ln(-a)\,v)$$

Doesn't work -- note the logarithm of a negative quantity -- so do by hand!

```
> implicit_eqn_1 := int( 1/sqrt(1+p^2), p) = int( v/(a-x), x) + C1;
```

$$\mathit{implicit\_eqn\_1} := \mathrm{arcsinh}(p) = -\ln(a - x)\,v + C1$$

```
> const := solve( subs(p=0, x = 0, implicit_eqn_1), C1);
```

$$\mathit{const} := \mathrm{arcsinh}(0) + \ln(a)\,v$$

```
> implicit_eqn_2 := subs( C1 = const, implicit_eqn_1);
```

$$\mathit{implicit\_eqn\_2} := \mathrm{arcsinh}(p) = -\ln(a - x)\,v + \ln(a)\,v$$

```
> solve( implicit_eqn_2, p );
```

$$\sinh(-\ln(a - x)\,v + \ln(a)\,v)$$

If you don't care for sinh, this can be converted to exponentials.

```
> convert( ", exp );
```

$$\frac{1}{2}\,e^{(-\ln(a-x)\,v + \ln(a)\,v)} - \frac{1}{2}\,\frac{1}{e^{(-\ln(a-x)\,v + \ln(a)\,v)}}$$

```
> combine( ", ln, anything );
```

$$\frac{1}{2}\,(a - x)^{(-v)}\,a^{v} - \frac{1}{2}\,\frac{1}{(a - x)^{(-v)}\,a^{v}}$$

```
> ivp2 := simplify( p = " );
```

$$\mathit{ivp2} := p = \frac{1}{2}\left(-\frac{a}{-a + x}\right)^{v} - \frac{1}{2}\left(-\frac{a}{-a + x}\right)^{(-v)}$$

Recall that p is the velocity, the position (of the defender) is found by integrating and applying the initial condition

```
> explicit_soln := y = int( rhs(ivp2), x) + c2;
```

$$\mathit{explicit\_soln} := y = \frac{1}{2}\,\frac{a^{(-v+1)}\,(a - x)^{v}}{v + 1} + \frac{1}{2}\,\frac{a^{(v+1)}\,(a - x)^{(-v)}}{v - 1} - \frac{1}{2}\,\frac{a^{v}\,(a - x)^{(-v)}\,x}{v - 1} - \frac{1}{2}\,\frac{a^{(-v)}\,(a - x)^{v}\,x}{v + 1} + c2$$

```
> const2 := simplify(solve( subs( y=0, x= 0, explicit_soln), c2));
```

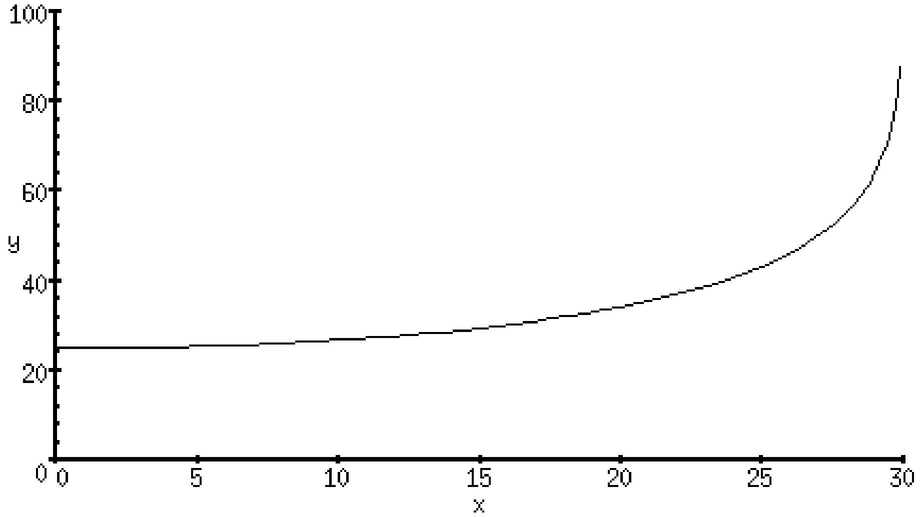
$$\mathit{const2} := -\frac{a\,v}{(v + 1)\,(v - 1)}$$

This gives us the path of the defender, if we care to graph it or use it.

```
> y := subs( c2  = const2, rhs(explicit_soln));
```

$$y := \frac{1}{2}\,\frac{a^{(-v+1)}\,(a - x)^{v}}{v + 1} + \frac{1}{2}\,\frac{a^{(v+1)}\,(a - x)^{(-v)}}{v - 1} - \frac{1}{2}\,\frac{a^{v}\,(a - x)^{(-v)}\,x}{v - 1} - \frac{1}{2}\,\frac{a^{(-v)}\,(a - x)^{v}\,x}{v + 1} - \frac{a\,v}{(v + 1)\,(v - 1)}$$

```
> plot(subs( PARAM2, y+25), x = 0..30, 'y' = 0..100);
```



The Distance Along the Path

Now we really need to compute times along the paths to determine when the defender and the offensive runner collide. To get the time required to go completely across field for the defender along the defender's path the length of the defender's path across field is needed. For that we need the differential element of arc for the defender which uses p = dy/dx already calculated above.

```
> # diff_elt_arc := simplify( sqrt( factor( expand(1 + rhs(ivp2)^2))));
> diff_elt_arc := simplify( sqrt(1 + rhs(ivp2)^2) );
```

$$diff\_elt\_arc := \frac{1}{2}\sqrt{2 + \left(-\frac{a}{-a+x}\right)^{(2\,v)} + \left(-\frac{a}{-a+x}\right)^{(-2\,v)}}$$

```
> #arc_length := subs(h=x, simplify(int( diff_elt_arc, x=0..h)));
> arc_length := 1/2*(-(a-x)^(v+1)*a^(-v)*v+(a-x)^(v+1)*
  a^(-v)+(a-x)^(1-v)*a^v*v+(a-x)^(1-v)*a^
  v-2*a)/(-v-1)/(1-v);
```

$$arc\_length := \frac{1}{2}\,\frac{-(a-x)^{(v+1)}\,a^{(-v)}\,v + (a-x)^{(v+1)}\,a^{(-v)} + (a-x)^{(-v+1)}\,a^{v}\,v + (a-x)^{(-v+1)}\,a^{v} - 2\,a}{(-1-v)\,(-v+1)}$$

Here's a simple test to see if the arclength came out correctly:

```
> simplify(subs(x = 0, arc_length));
```

$$0$$

So at least that much is OK! Now figure out the time it takes the defender to cross the field:

```
> defense_time := simplify(subs(x = a, arc_length)/v1);
```

$$defense\_time := -\frac{a}{(v+1)\,(v-1)\,v1}$$

```
> offense_time := b/v2;
```

$$offense\_time := \frac{b}{v2}$$

The defender will save the day if he arrives at the goalline in advance of the running back.

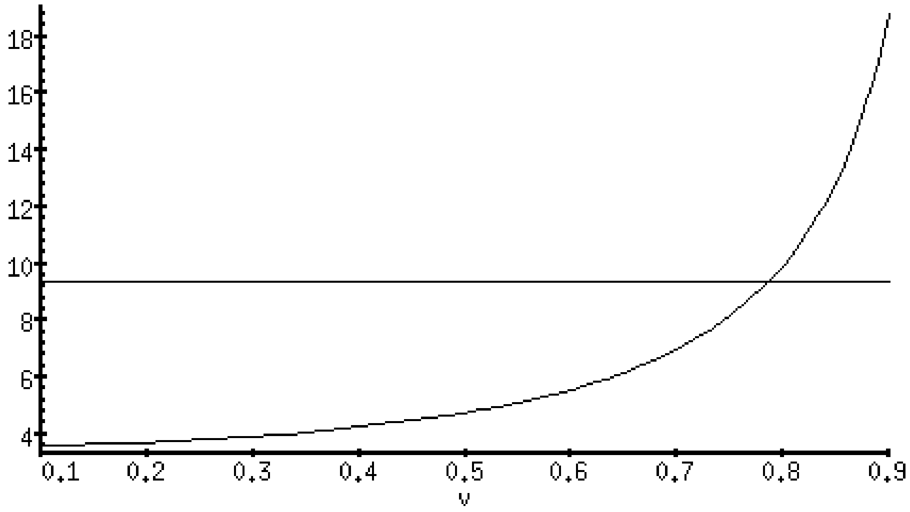
```
> evalf( subs( PARAM2, [ defense_time, offense_time ] ) );
```

$$[38.41463417, 9.375000000]$$

AHA! So that is why the DE section of the stadium instantly knew the game was lost!

How fast does the defender need to be to beat Bart to the goalline?

```
> plot( subs( PARAM, { defense_time, offense_time } ), v=0.1..0.9
  );
```




```
> new_capture_eqn := defense_time = offense_time;
```

$$new\_capture\_eqn := -\frac{a}{(v+1)\,(v-1)\,v1} = \frac{b}{v2}$$

```
> subs( {v2=v1*v }, new_capture_eqn*v2 );
```

$$-\frac{a\,v}{(v+1)\,(v-1)} = b$$

```
> solve( ", {v} );
```

$$\left\{v = -\frac{1}{2}\,\frac{a+\sqrt{a^2+4\,b^2}}{b}\right\}, \left\{v = -\frac{1}{2}\,\frac{a-\sqrt{a^2+4\,b^2}}{b}\right\}$$

```
> critical_velocity := "[2];
```

$$critical\_velocity := \left\{v = -\frac{1}{2}\,\frac{a-\sqrt{a^2+4\,b^2}}{b}\right\}$$

```
> subs( PARAM, op(critical_velocity) );
```

$$v = -\frac{1}{5} + \frac{1}{150}\sqrt{23400}$$

```
> evalf( " );
```

$$v = .8198039030$$

Thus, the defender's velocity needs to be

```
> subs( ", v1=8/v );
```

$$v1 = 9.758431216$$

Thus, the defender’s velocity needs to be at least 9.75 yd/sec to have any chance of beating Bart to the goalline. In terms of common football statistics, this corresponds to a 40 yard time of

```
> 40/rhs(");
```

$$4.099019516$$

That’s fast! In fact, is this even humanly possible?

Another variation that you might consider is, using the original values for v1 and v2, how close (i.e. what value of b) must the defender be to the running back in order to beat Bart to the goalline?

END OF WORKSHEET

```
>
```