Mathematical Models with Boundary Value Problems

Douglas B. Meade
Department of Mathematics
University of South Carolina
Columbia, SC 29208

(803) 777-6183
meade@math.sc.edu
http://www.math.sc.edu/~meade/

Allan A. Struthers
Department of Mathematical Sciences
Michigan Technical University
Houghton, MI 49931

(906) 487-3541
struther@math.mtu.edu
http://www.math.mtu.edu/~struther/

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Outline

- Past and Present
- Computer Algebra Tools for Solving BVPs
- Application — Infinite Rotating Disk (Chemical Engineering)
- Future

Past and Present

Boundary Value Problems. When the values of a solution to a differential equation are specified at two different points, these conditions are called **boundary conditions**. (In contrast, initial conditions specify the values of a function and its derivative at the same point.) Given that every solution to $y'' + y = 0$ is of the form

$$y(x) = c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 are arbitrary constants, show that

(a) there is a unique solution that satisfies the boundary conditions $y(0) = 2$ and $y(\pi/2) = 0$;

(b) there is no solution that satisfies $y(0) = 2$ and $y(\pi) = 0$;

(c) there are infinitely many solutions that satisfy $y(0) = 2$ and $y(\pi) = -2$.

— *Fundamentals of Differential Equations, Nagle & Saff (# 30, p. 159)*

Computer Algebra Tools for Solving BVPs

```
> ODE := diff( y(x), x$2 ) + y(x) = 0:
> BCa := y(0)=2, y(Pi/2)=0:
> SOLa := dsolve( { ODE, BCa }, y(x) );
```

$$SOLa := \mathrm{y}(x) = 2\cos(x)$$

```
> BCb := y(0)=2, y(Pi)=0:
> SOLb := dsolve( { ODE, BCb }, y(x) );
```

$$SOLb :=$$

```
> BCc := y(0)=2, y(Pi)=-2:
> SOLc := dsolve( { ODE, BCc }, y(x) );
```

$$SOLc := \mathrm{y}(x) = \_C1\sin(x) + 2\cos(x)$$

Customized symbolic BVP solver: `BVPsolve`

```
> BVPsolve := proc( BVP::set(equation), FN::function )
>   local ODE, BC, gSOL, Y, DVAR, IVAR, CONST, cSOL;
>   ODE := select( has, BVP, diff );
>   BC := BVP minus ODE;
>   DVAR, IVAR := op(0,FN), op(1,FN);
>   gSOL := dsolve( ODE, FN );
>   if gSOL=NULL then
>     RETURN( "No general solution found" )
>   fi;
>   if nops([gSOL])>1 then
>     RETURN( "Nonunique general solution found" )
>   fi;
>   CONST := indets(gSOL,name) minus {IVAR};
>   Y := unapply( rhs( gSOL ), IVAR );
>   cSOL := solve( eval( subs( DVAR=Y, BC ) ), CONST );
>   if cSOL=NULL then
>     RETURN( "No particular solution satisfies all BCs" )
>   fi;
>   subs( cSOL, gSOL );
> end:
```

Simple Examples

```
> BVPsolve( { ODE, BCa }, y(x) );
```

$$\mathrm{y}(x) = 2\cos(x)$$

```
> BVPsolve( { ODE, BCb }, y(x) );
```

No particular solution satisfies all BCs

```
> BVPsolve( { ODE, BCc }, y(x) );
```

$$\mathrm{y}(x) = 2\cos(x) + \_C2\sin(x)$$

More Challenging Problems

- multi-point boundary conditions

$$y'' + y = 0, \quad y(\pi/4) + y(\pi/2) = 1, \quad y'(\pi/4) + y'(\pi/2) = -1$$

- nonlinear boundary conditions

$$y'' + y = 0, \quad y(0) = 2, \quad y(\pi/2)^2 + y'(\pi) = 1$$

- nonlinear differential equation

$$y'' + y^2 = 0, \quad y(0) = 2, \quad y(\pi/2) = 0$$

Multi-point Boundary Conditions

```
> ODE := diff( y(x), x$2 ) + y(x) = 0:
> BC2p := D(y)(Pi/2)+D(y)(Pi/4)=-1, y(Pi/4)+y(Pi/2)=1;
```

$$BC2p := \mathrm{D}(y)(\frac{1}{2}\,\pi) + \mathrm{D}(y)(\frac{1}{4}\,\pi) = -1,\ \mathrm{y}(\frac{1}{4}\,\pi) + \mathrm{y}(\frac{1}{2}\,\pi) = 1$$

```
> dsolve( { ODE, BC2p }, y(x) );
```

```
> BVPsolve( { ODE, BC2p }, y(x) );
```

$$\mathrm{y}(x) = \frac{1}{2}\,\sqrt{2}\cos(x) + (1 - \frac{1}{2}\,\sqrt{2})\sin(x)$$

Nonlinear Boundary Conditions

```
> ODE := diff( y(x), x$2 ) + y(x) = 0:
> BCnl := y(0)=2, y(Pi/2)^2 + D(y)(Pi)=1;
```

$$BCnl := \mathrm{y}(0) = 2,\ \mathrm{y}(\frac{1}{2}\,\pi)^2 + \mathrm{D}(y)(\pi) = 1$$

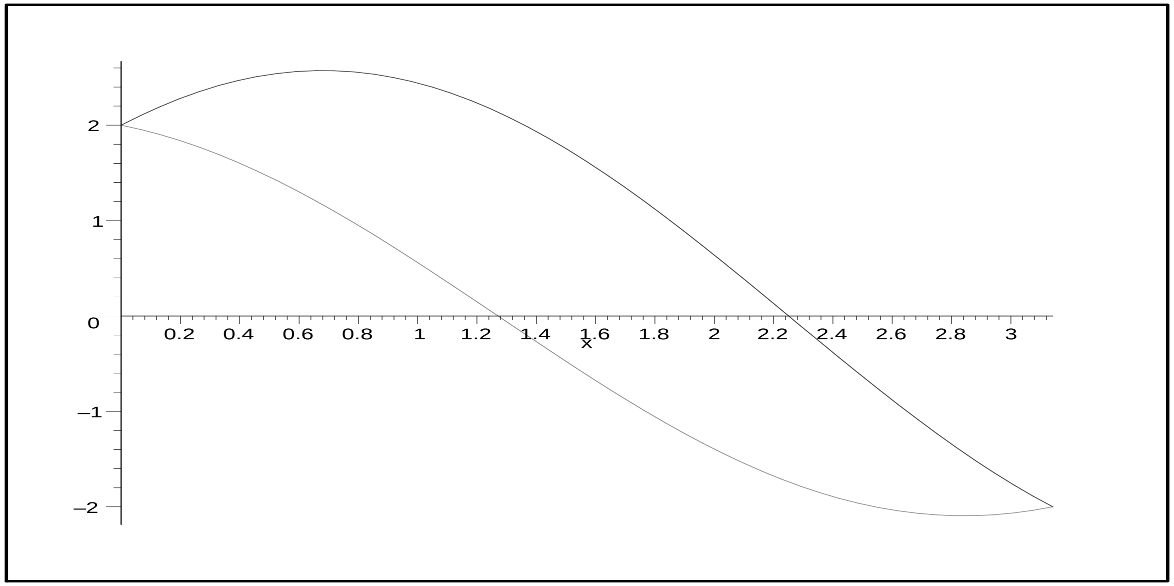
```
> dsolve( { ODE, BCnl }, y(x) );
> BVPsolve( { ODE, BCnl }, y(x) );
```

$$\mathrm{y}(x) = 2\cos(x) + \mathrm{RootOf}(\_Z^2 - \_Z - 1)\sin(x)$$

```
> SS := allvalues( % );
```

$$SS := \mathrm{y}(x) = 2\cos(x) + (\frac{1}{2}\sqrt{5} + \frac{1}{2})\sin(x),\ \mathrm{y}(x) = 2\cos(x) + (\frac{1}{2} - \frac{1}{2}\sqrt{5})\sin(x)$$

```
> plot( map( rhs, [SS] ), x=0..Pi );
```



Nonlinear ODE I

```
> ODE2 := diff( y(x), x$2 ) + y(x)^2 = 0;
```

$$ODE2 := (\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) + \mathrm{y}(x)^2 = 0$$

```
> dsolve( { ODE2, BCa }, y(x) );
> BVPsolve( { ODE2, BCa }, y(x) );
```

Nonunique general solution found

```
> dsolve( ODE2, y(x) );
```

$$x = \int_0^{\mathrm{y}(x)} -\frac{3}{\sqrt{-6\,y1^3 + 9\,\_C1}}\,dy1 - \_C2,\; x = \int_0^{\mathrm{y}(x)} \frac{3}{\sqrt{-6\,y2^3 + 9\,\_C1}}\,dy2 - \_C2$$

Nonlinear ODE II

```
> ODE3 := diff( y(x),x$2 ) + y(x)^2 = 7*x^(-4);
```

$$ODE3 := (\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) + \mathrm{y}(x)^2 = \frac{7}{x^4}$$

```
> IC := y(1)=1; BC := y(3)=1/3;
```

$$IC := \mathrm{y}(1) = 1$$

$$BC := \mathrm{y}(3) = \frac{1}{3}$$

```
> dsolve( { ODE3, IC, BC }, y(x) );
> BVPsolve( { ODE3, IC, BC }, y(x) );
```

No general solution found

Numerical Solutions to Second-Order BVPs

```
> fBVPsolve := proc( BVP::set(equation),
>                    FN::function,
>                    RANGE::function=range(numeric) )
>   local i, auxIC, ODE, BC, IC, IVP, IVAR, RANGE2,
>         RANGE3, PARAM, POINTS, RESULT, X0;
>   global n, BCERR;
>
>   IVAR   := op(1,FN);
>   ODE    := select( has, BVP, IVAR );
>   BC     := BVP minus ODE;
>   X0     := op(1,lhs(RANGE));
>
>   IC     := select( (l,r) -> evalb(op(1,lhs(l))=r), BC, X0 );
>   BC     := convert(map(e->lhs(e)-rhs(e), BC minus IC ),list);
>
>   POINTS := convert(map2( op, 1, indets(BC,function) ),list);
>
>   RANGE2 := [RANGE];
>   auxIC  := {seq(lhs(RANGE2[i])=_alpha[i],i=1..nops(RANGE2))};
>   RANGE3 := {seq(_alpha[i]=rhs(RANGE2[i]),i=1..nops(RANGE2))};
>
>   IVP    := ODE union IC union auxIC;
```

Numerical Solutions to Second-Order BVPs

```
>  BCERR  := subs( _IVP=IVP, _FN=FN, _POINTS=POINTS, _BC=BC,
>    proc( )
>     local i, j, ARGS, FF, PTS, SOL, SYS, VALS, bcerr;
>     global n;
>     n := n+1;
>
>     ARGS := seq( _alpha[i]=args[i], i=1..nargs );
>     SYS := subs( ARGS, _IVP );
>     PTS := convert(map2( op, 1, indets(_BC,function) ),list);
>
>     SOL := dsolve( SYS, _FN, type=numeric,
>                    value=array(evalf(PTS)) );
>
>     FF := convert(map( convert, SOL[1,1], D ),list);
>     VALS := [ seq( seq( op(0,FF[j])(PTS[i]) = SOL[2,1][i,j],
>                         i=1..nops(PTS) ), j=2..nops(FF) ) ];
>
>     bcerr := evalf( op( subs( VALS, _BC ) ) );
>     userinfo( 3, BCERR,
>       sprintf(`iteration %2d: alpha=%10.6f bcerr=%10.6f`,
>               n, rhs(ARGS), bcerr) );
>     RETURN( bcerr );
>   end );
```

Numerical Solutions to Second-Order BVPs

```
>   PARAM  := op( map( lhs, RANGE3 ) );
>
>   Digits := 5:
>   n      := 0:
>   RESULT := fsolve( 'BCERR'(PARAM)=0, {PARAM}, op(RANGE3) );
>   Digits := 10:
>
>   RETURN( [RESULT,dsolve(subs(RESULT,IVP),FN,type=numeric)] );
> end:
>
> infolevel[BCERR]:=3:
```

Example 1: Linear Two-Point BVP

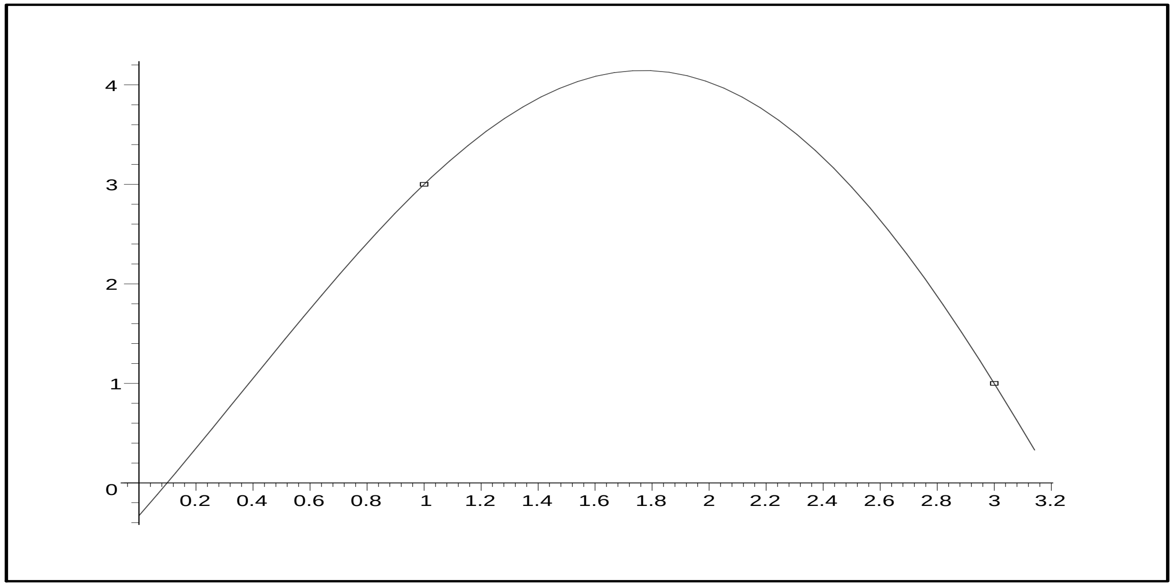
```
> ODE := diff( y(x),x$2 ) + y(x) = cos(x);
> IC := y(1)=3;
> BC := y(3)=1;
> PARAM, SOL := op(fBVPsolve( {ODE,IC,BC}, y(x), D(y)(1)=0..5)):

BCERR:   "iteration  1: alpha=  2.500000 bcerr=  -.213300"
BCERR:   "iteration  2: alpha=  0.000000 bcerr= -2.489100"
BCERR:   "iteration  3: alpha=   .100000 bcerr= -2.398200"
BCERR:   "iteration  4: alpha= -.010000 bcerr= -2.498800"
BCERR:   "iteration  5: alpha=  2.580837 bcerr=  -.139900"
BCERR:   "iteration  6: alpha=  2.734313 bcerr=  -.000500"
BCERR:   "iteration  7: alpha=  2.734863 bcerr=  -.001400"
BCERR:   "iteration  8: alpha=  2.734012 bcerr=  -.001800"
BCERR:   "iteration  9: alpha=  2.734292 bcerr=  -.000500"
BCERR:   "iteration 10: alpha=  2.734485 bcerr=  0.000000"
BCERR:   "iteration 11: alpha=  2.734485 bcerr=  -.003434"
BCERR:   "iteration 12: alpha=  2.734485 bcerr=  -.003434"
BCERR:   "iteration 13: alpha=  2.738262 bcerr=   .000000"

> PARAM;
```

$$\{\_\alpha_1 = 2.7383\}$$

```
> P1 := plots[odeplot]( SOL, [x,y(x)], 0..Pi):
> P2 := plot( [[1,3],[3,1]], style=POINT,
>              color=BLUE, symbol=BOX ):
> plots[display]( [P1,P2] );
```



```
> dsolve( { ODE, IC, BC }, y(x) );
```

$$\mathrm{y}(x) = \frac{1}{2}\sin(x)\,x + \frac{1}{2}\,\frac{(-3\sin(1) - 8 + 4\sin(1)^3 + 12\sin(1)^2)\cos(1)\cos(x)}{\sin(1)^2 - 1} + \frac{(2\sin(1)^3 + 6\sin(1)^2 - 2\sin(1) - 1)\sin(x)}{\sin(1)}$$

```
> BVPsolve( { ODE, IC, BC }, y(x));
```

$$\mathrm{y}(x) = \frac{1}{2}\sin(x)\,x + \frac{1}{2}\,\frac{(-3\sin(1) - 8 + 4\sin(1)^3 + 12\sin(1)^2)\cos(1)\cos(x)}{\sin(1)^2 - 1} + \frac{(2\sin(1)^3 + 6\sin(1)^2 - 2\sin(1) - 1)\sin(x)}{\sin(1)}$$

```
> evalf( subs( x=1, diff( rhs(%), x ) ));
```

$$2.738261577$$

Example 2: Linear BVP with Robin BC

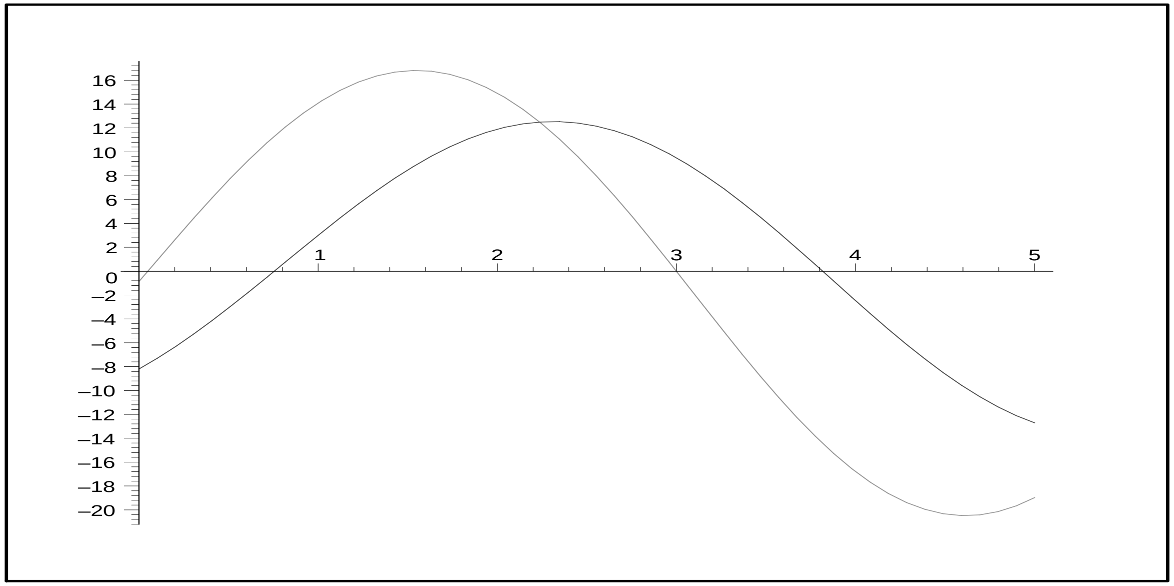
```
> ODE := diff( y(x),x$2 ) + y(x) = cos(x);
> IC := y(1)=3;
> BC := y(3)+D(y)(3)=1;
> PARAM, SOL := op(fBVPsolve({ODE,IC,BC}, y(x), D(y)(1)=0..15)):

BCERR: "iteration  1: alpha=  7.500000 bcerr= -2.262600"
BCERR: "iteration  2: alpha=  0.000000 bcerr= -5.962100"
BCERR: "iteration  3: alpha=   .100000 bcerr= -5.912900"
BCERR: "iteration  4: alpha=  -.010000 bcerr= -5.967200"
BCERR: "iteration  5: alpha= 11.728022 bcerr=  -.177000"
BCERR: "iteration  6: alpha= 12.085920 bcerr=   .000500"
BCERR: "iteration  7: alpha= 12.084912 bcerr=  -.002450"
BCERR: "iteration  8: alpha= 12.084914 bcerr=  -.002449"
BCERR: "iteration  9: alpha= 12.089880 bcerr=  -.000000"

> PARAM;
```

$$\{\_\alpha_1 = 12.090\}$$

```
> plots[odeplot]( SOL,
>                 [ [x,y(x)], [x,y(x)+diff(y(x),x)-1] ],
>                 0..5 );
```



```
> dsolve( { ODE, IC, BC }, y(x) );
> exsol := BVPsolve( { ODE, IC, BC }, y(x));
```

$$exsol := \mathrm{y}(x) = \frac{1}{2}\sin(x)\,x + \frac{1}{2}(12\sin(1)^4 + 24\sin(1)^3 + 8\cos(1)\sin(1)^3 + 24\cos(1)\sin(1)^2 - 2\cos(1)\sin(1) - 6\cos(1) - 9\sin(1)^2 - 16\sin(1)) \cos(x)\Big/(2\sin(1)^2 - 1 - 2\cos(1)\sin(1)) + \frac{1}{2}(-12\sin(1)^2 + 3 + 11\cos(1)\sin(1) - 12\cos(1)\sin(1)^3 - 24\cos(1)\sin(1)^2 + 8\sin(1)^4 - 18\sin(1) + 24\sin(1)^3 + 4\cos(1))\sin(x)\Big/(2\sin(1)^2 - 1 - 2\cos(1)\sin(1))$$

```
> evalf( subs( x=1, diff(rhs(exSOL),x) ) );
```

$$12.08987956$$

Example 3: Nonlinear Two-Point BVP

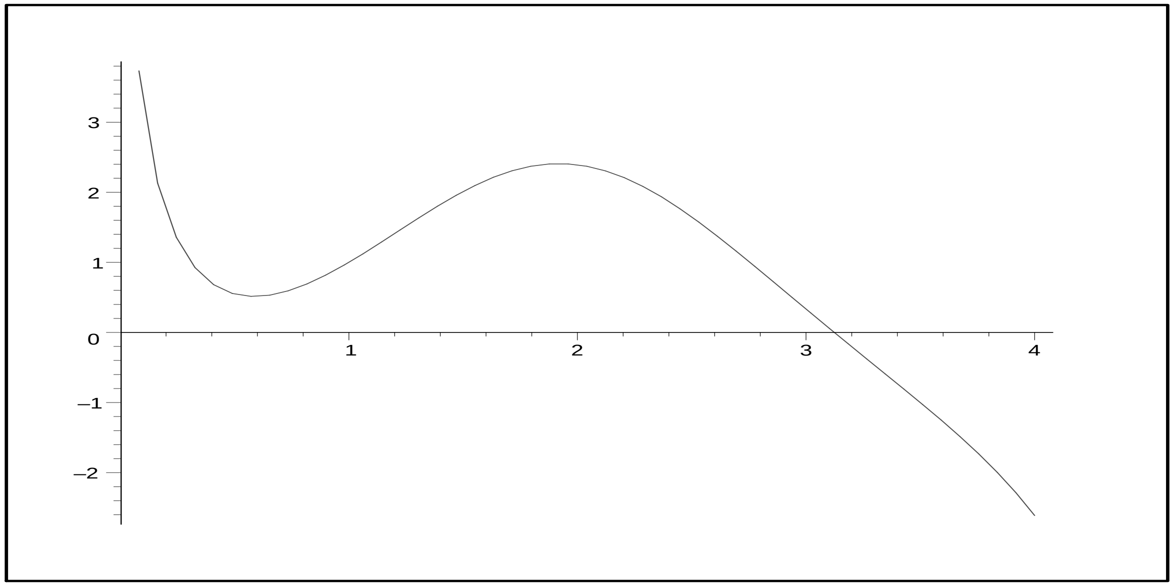
```
> ODE := diff( y(x),x$2 ) + y(x)^2 = 3*x^(-2);
> IC := y(1)=1;
> BC := y(3)=1/3;
> PARAM, SOL := op(fBVPsolve({ODE,IC,BC}, y(x), D(y)(1)=0..5 )):

BCERR: "iteration  1: alpha=  2.500000 bcerr=  -.313240"
BCERR: "iteration  2: alpha=  0.000000 bcerr=   .609290"
BCERR: "iteration  3: alpha=   .100000 bcerr=   .605030"
BCERR: "iteration  4: alpha=  -.010000 bcerr=   .609630"
BCERR: "iteration  5: alpha=  1.250000 bcerr=   .313420"
BCERR: "iteration  6: alpha=  1.375000 bcerr=   .262170"
BCERR: "iteration  7: alpha=  2.044988 bcerr=  -.059830"
BCERR: "iteration  8: alpha=  1.934214 bcerr=  -.002030"
BCERR: "iteration  9: alpha=  1.930259 bcerr=  -.000030"
BCERR: "iteration 10: alpha=  1.930200 bcerr=  -.005350"
BCERR: "iteration 11: alpha=  1.930200 bcerr=  -.005351"
BCERR: "iteration 12: alpha=  1.919758 bcerr=  -.000008"
BCERR: "iteration 13: alpha=  1.919743 bcerr=  -.000000"

> PARAM;
```

$$\{\_\alpha_1 = 12.090\}$$

```
> plots[odeplot]( SOL, [x,y(x)], 0..4 );
```



Application — Infinite Rotating Disk

- Physical Problem

  determine the steady fluid flow generated when the infinite plane $z = 0$, immersed in a Newtonian viscous fluid, rotates about the axis $r = 0$ with constant angular velocity ω

Application — Infinite Rotating Disk

- Notation

p	pressure
(v^r, v^θ, v^z)	velocity
ρ	fluid density
ν	kinematic viscosity

- dimensionless variables

$$\begin{aligned} z^* &= \sqrt{\frac{\omega}{\nu}}z \\ v^r &= r\omega F(z^*) \\ v^\theta &= r\omega G(z^*) \\ v^z &= \sqrt{\omega\nu}H(z^*) \\ p &= \rho\omega\nu P(z^*) \end{aligned}$$

Application — Infinite Rotating Disk

- boundary value problem

$$
\begin{aligned}
H' &= -2F \\
F'' &= F^2 - G^2 + F'H \\
G'' &= 2FG + HG' \\
P' &= -HH' + H'' \\
F(0) = H(0) = P(0) &= 0 \\
G(0) &= 1 \\
\lim_{z^* \to \infty} F(z^*) = \lim_{z^* \to \infty} G(z^*) &= 0
\end{aligned}
$$

Maple Solution via Shooting Method

```
> restart;
> read shoot;
> with(plots):
> FNS:={ F(Z), G(Z), H(Z), Fp(Z), Gp(Z) }:
> ODE:={diff(H(Z),Z)  = -2*F(Z),
>       diff(F(Z),Z)  = Fp(Z),
>       diff(Fp(Z),Z) = -G(Z)^2+F(Z)^2+Fp(Z)*H(Z),
>       diff(G(Z),Z)  = Gp(Z),
>       diff(Gp(Z),Z) = 2*F(Z)*G(Z)+H(Z)*Gp(Z)}:
> IC:={ F(0)=0, G(0)=1, H(0)=0, Fp(0)=alpha, Gp(0)=beta }:
> BC:={ F(10)=0, G(10)=0 }:
```

Maple Solution via Shooting Method

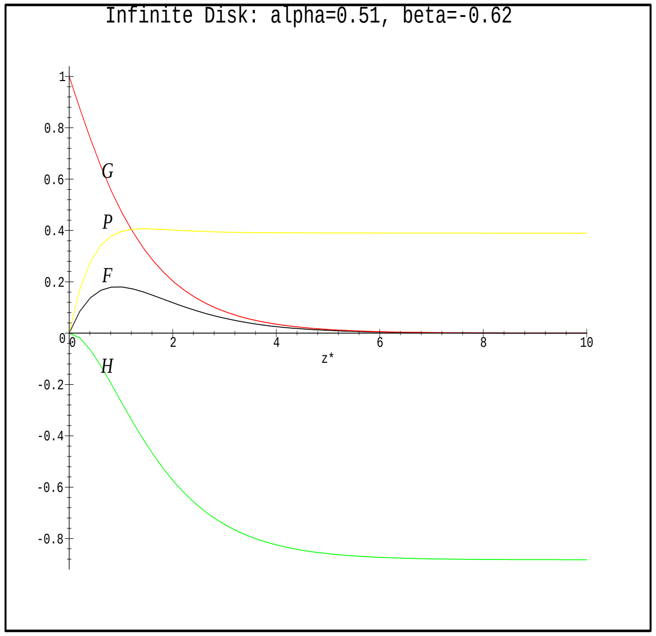
```
> infolevel[shoot]:=1:
> S:=shoot( ODE union IC, FNS, BC, [alpha=0.51, beta=-0.62] ):

shoot:   alpha = .51, beta = -.62
shoot:   alpha = .5100033193, beta = -.6161487475
shoot:   alpha = .5102117355, beta = -.6159070527
shoot:   alpha = .5102135910, beta = -.6159096496
```

Maple Solution via Shooting Method

```
> P:=Z->-H(Z)^2/2-2*F(Z):
> C:=odeplot(S,[ [Z,F(Z)], [Z,G(Z)],
>                [Z,H(Z)], [Z,-P(Z)] ], 0..10 ):
>
> subs( [seq( op(0,lhs(f))=rhs(f(3/4)),
>             f=select(has,S,{F,G,H}) )] ):
> [op(%),P=subs(%, H^2+2*F)]:
>
> L:=textplot( subs(z=3/4, map(f->[z,rhs(f)+1/45,lhs(f)],%)),
>              font=[TIMES,ITALIC,18], align=ABOVE ):
> display( {C,L}, labels=[`z*`,``],
>          title=`Infinite Disk: alpha=0.51, beta=-0.62` );
```

Maple Solution via Shooting Method



Maple Solution via Shooting Method

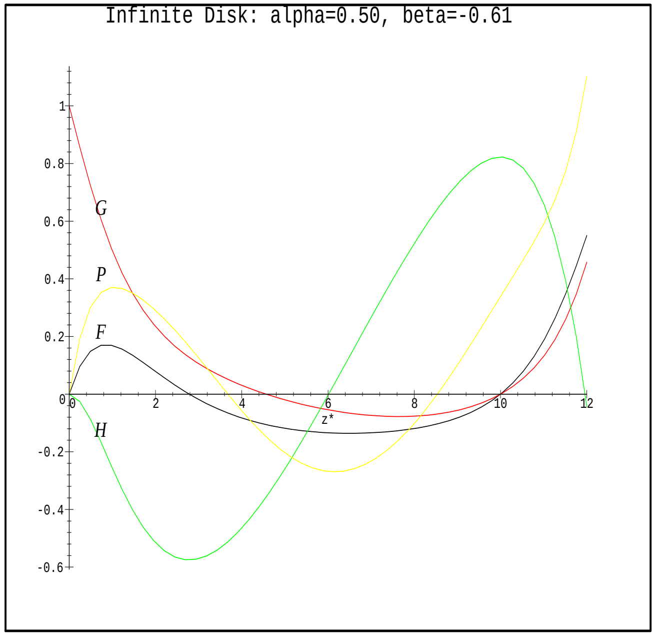
```
> S2:=shoot( ODE union IC, FNS, BC, [alpha=0.5, beta=-0.61] ):

shoot:   alpha = .5, beta = -.61
shoot:   alpha = .5181506133, beta = -.5834922887
shoot:   alpha = .5178653168, beta = -.5900697906
shoot:   alpha = .5019889119, beta = -.5940268791
shoot:   alpha = .5040798081, beta = -.5927957583
shoot:   alpha = .5042719718, beta = -.5927086067
shoot:   alpha = .5042737786, beta = -.5927079110
```

Maple Solution via Shooting Method

```
> C:=odeplot(S2,[ [Z,F(Z)], [Z,G(Z)],
>                 [Z,H(Z)], [Z,-P(Z)] ], 0..12 ):
>
> subs( [seq( op(0,lhs(f))=rhs(f(3/4)),
>             f=select(has,S2,{F,G,H}) )] ):
> [op(%),P=subs(%, H^2+2*F)]:
>
> L:=textplot( subs(z=3/4, map(f->[z,rhs(f)+1/45,lhs(f)],%)),
>              font=[TIMES,ITALIC,18], align=ABOVE ):
> display( {C,L}, labels=[`z*`,``],
>          title=`Infinite Disk: alpha=0.50, beta=-0.61` );
```

Maple Solution via Shooting Method



Ongoing Efforts

- Implementation
  - continued development of specialized and general CAS tools for symbolic, numeric, and graphic analysis of BVPs
  - collect realistic and interesting examples of BVPs for use as homework, in-class examples, and projects
  - generalize and extend `BVPsolve` and `fBVPsolve` (and `shoot`)
- General Goals
  - BVPs deserve more attention than they currently receive
  - should some material be omitted from traditional courses to allow discussion of BVPs?
  - how do BVPs fit into newer courses that emphasize dynamical systems?


FoxTrot
BILL AMEND
JABBA THE HUTT... HUTT... HIKE!
I SAID HIKE.
I CAN'T GET THE BALL TO FIT BETWEEN MY LEGS.
PETER FOX, WIDE RECEIVER, CUTS THROUGH THE DEFENSE!
HE JUKES THE SAFETY! HE'S WIDE OPEN IN THE END ZONE!
HE ZIG-ZAGS LEFT AND RIGHT! HE'S STILL WIDE OPEN!
REALLY WIDE OPEN!
NO-ONE-FOR-MILES-A-TOUCH-DOWN-IS-A-GIVEN-THROW-THE-BALL-ALREADY WIDE OPEN!
OK, SO IF WE GO WITH A TRAJECTORY OF 30 DEGREES AND NEGLECT THE EFFECT OF AIR RESISTANCE...
OH, WAIT. HE STOPPED RUNNING. CHANGE V_{TARGET} TO ZERO IN THAT SECOND EQUATION.
GUYS, ALLOW ME TO DEMONSTRATE WHY QUARTERBACKS WEAR HELMETS...