Jerks Don't Jump — Mathematical Modelling with an Impact

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Overview of Presentation

- Introduction
- Example – Parachute Problem
  - Standard Problem
  - Interpret Solution
  - Improved Model
  - Related Questions
- Additional Examples
- Advertisements

Questions to Ponder

- What do we want our students to learn?
  - mechanics?
  - concepts?
  - theory?
  - application?

Example: The Parachute Problem

Nagle and Saff (Addison–Wesley, 1996, p. 112)

A parachutist whose mass is 75 kg drops from a helicopter hovering 4000 m above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant and that the force due to air resistance is proportional to the velocity of the parachutist, with a proportionality constant $k_1 = 15$ kg/sec when the chute is closed and with a constant $k_2 = 105$ kg/sec when the chute is open. If the chute does not open until 1 min after the parachutist leaves the helicopter, after how many seconds will she hit the ground?

Example: The Parachute Problem

Edwards and Penney (Prentice Hall, 1996, p. 95)

A woman bails out of an airplane at an altitude of 10,000 ft, fall freely for 20 s, then opens her parachute. How long will it take her to reach the ground? Assume that $\rho = 0.15$ without the parachute and that $\rho = 1.5$ with the parachute. (*Suggestion:* First determine the height and velocity when the parachute opens.)

NOTE: $\rho = \frac{k}{m}$

- Suppose that $k/m = 0.2$ before the skydiver opens the parachute. What is the terminal velocity for the free-falling skydiver?
- Suppose the airplane is 2000 m above the ground when the skydiver jumps out. What is the skydiver's altitude [when the skydiver reaches 95 percent of the terminal velocity]?
- The coefficient of air resistance changes after the parachute opens. Suppose that the skydiver, who has reached a new terminal velocity, hits the ground at a rate which is equal to that attained by jumping off a wall which is 2 m high. (Call this rate v_{ground}.) Determine k/m. *Hint*: Calculate the velocity for a free fall from 2 m and equate it to the terminal velocity of the parachute.
- In order to avoid serious injury, suppose that the skydiver must hit the ground at a velocity that is within 5 percent of v_{ground}. By what altitude must the parachute be opened to satisfy this restriction? Discuss whether your answer is reasonable and how other restrictions might yield a different answer.

Example: The Parachute Problem

Abell and Braselton (Saunders, 1996, p. 140)

A parachutist weighing 60 kg falls from a plane (that is, $v_0 = 0$) and is subjected to air resistance equivalent to $F_R = 10v$. After one minute, the parachute is opened so that the parachutist is subjected to air resistance equivalent to $F_R = 100v$.

1. What is the parachutist's velocity when the parachute is opened?
2. What is the parachutist's velocity $v(t)$ after the parachute is opened?
3. What is the parachutist's limiting velocity? How does this compare with the limiting velocity if the parachute does not open?

Example: The Parachute Problem

Kostelich and Armbruster (Addison–Wesley, 1996, p. 109)

The objective of this problem is to calculate some of the relevant quantities for the fall of a skydiver out of an airplane. We use the simplest model of air resistance, in which we suppose that the drag force through air is proportional to the velocity. The relevant differential equation is

$$v' = -g - \frac{k}{m}v.$$

We take the ground as $x = 0$ and let $x > 0$ correspond to positions above the ground. As $v = x'$, a falling skydiver has negative velocity in this coordinate system. The gravitational constant is g and the coefficient of air resistance is k.

Mathematical Formulation

- 2nd-order ODE

$$\begin{aligned} mx'' &= -mg - kx' \\ x(0) &= x_0 \\ x'(0) &= 0 \end{aligned}$$

- 1st-order system

$$\begin{aligned} x' &= v, & x(0) &= x_0 \\ v' &= -g - \tfrac{k}{m}v, & v(0) &= 0 \end{aligned}$$

- k — piecewise constant coefficient of air resistance

$$k(t) = \begin{cases} k_1, & t < t_d \\ k_2, & t \geq t_d \end{cases}$$

- t_d — time of (instantaneous) deployment

Step 1 – Free-Fall Solution

```
> SOL1 := dsolve( subs(k=k1,SYS) union IC, { x(t), v(t) } ):
> V1 := subs( SOL1, v(t) );
```

$$V1 := -\frac{m\,g}{k1} + \frac{m\,g\,\mathrm{e}^{\left(-\frac{t\,k1}{m}\right)}}{k1}$$

```
> X1 := subs( SOL1, x(t) );
```

$$X1 := \frac{-m\,g\,t\,k1 + m^2\,g + x0\,k1^2 - m^2\,g\,\mathrm{e}^{\left(-\frac{t\,k1}{m}\right)}}{k1^2}$$

Example: The Parachute Problem

Drucker (College Mathematics Journal, May 1995)

A parachutist exits an airplane 1200 ft above the ground with zero vertical velocity. He wishes to land with a velocity of 40 ft/sec. The skydiver is assumed to weigh 240 pounds, and the problem is given assuming drag force due to air resistance proportional to velocity. The proportionality constant is taken as 2 before the parachute opens and 10 afterward. The question is then, at what time after exiting the plane should the diver open the parachute in order to land with a velocity of 40 ft/sec?

Explicit Solution — via Maple V (Release 3)

```
> SYS := { diff( x(t), t ) = v(t),
>          diff( v(t), t ) = -g - k/m*v(t) };
```

$$SYS := \left\{ \frac{\partial}{\partial t}\mathrm{x}(\,t\,) = \mathrm{v}(\,t\,), \frac{\partial}{\partial t}\mathrm{v}(\,t\,) = -g - \frac{k\,\mathrm{v}(\,t\,)}{m} \right\}$$

```
> IC := { x(0)=x0, v(0)=0 };
```

$$IC := \{\mathrm{x}(\,0\,) = x0, \mathrm{v}(\,0\,) = 0\}$$

```
> K[pw_const] :=  k1+(k2-k1)*Heaviside(t-td);
```

$$K_{pw\_const} := k1 + (\,k2 - k1\,)\,\mathrm{Heaviside}(\,t - td\,)$$

Step 3 – Composite Solution

```
> V := V1 + (V2-V1)*Heaviside(t-td):
> X := X1 + (X2-X1)*Heaviside(t-td):
```

$$v(t) = \begin{cases} -\frac{mg}{k_1}\left(1-e^{-\frac{tk_1}{m}}\right), & t < t_d \\ -\frac{mg}{k_2}+\frac{mg}{k_1k_2}\left(k_1-k_2\left(1-e^{-\frac{t_dk_1}{m}}\right)\right)e^{-\frac{(t-t_d)k_2}{m}}, & t \geq t_d \end{cases}$$

$$x(t) = \begin{cases} x_0-\frac{mg}{k_1}\left(t-\frac{m}{k_1}\left(1-e^{-\frac{tk_1}{m}}\right)\right), & t < t_d \\ \text{what would you do with this if you had it?}, & t \geq t_d \end{cases}$$

Interpretation of Solution

- terminal velocities:
  - free-fall: $v_{\text{ff}} \approx 49$ m/s
  - final descent: $v_{\text{impact}} \approx 7$m/s
- position:
  - deployment at 1300 m
  - land after 241 s
- acceleration:
  - jump discontinuity at t_d
  - magnitude: 6 G

(all values approximate)

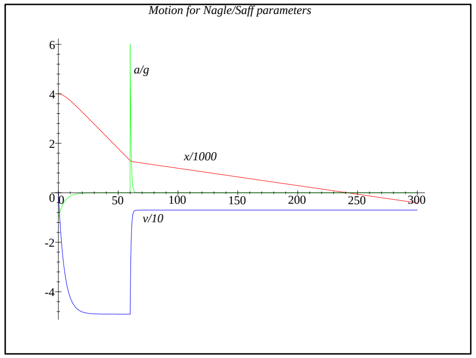
Step 2 – Solution with Deployed Parachute

```
> IC2 := { x(td) = subs( t=td, X1 ),
>          v(td) = subs( t=td, V1 ) };
```

$$IC2 := \left\{ \mathrm{v}(\,td\,) = -\frac{m\,g}{k1}+\frac{m\,g\,\mathrm{e}^{\left(-\frac{td\,k1}{m}\right)}}{k1},\right.$$
$$\left.\mathrm{x}(\,td\,) = \frac{-m\,g\,td\,k1+m^2\,g+x0\,k1^2-m^2\,g\,\mathrm{e}^{\left(-\frac{td\,k1}{m}\right)}}{k1^2}\right\}$$

```
> SOL2 := dsolve( subs(k=k2,SYS) union IC2, { x(t), v(t) } ):
> V2 := subs( SOL2, v(t) ):
> X2 := subs( SOL2, x(t) ):
```


Graphical Solution



Parameters (Nagle & Saff):

- $k_1 = 15$ kg/s, $k_2 = 105$ kg/s, $m = 75$ kg
- $x_0 = 4000$ m, $t_d = 60$ s, $g = 9.81$ m/s^2

Evaluation of Solution

- v_{ff} is reasonable; within 10% of actual data
- v_{impact} in not reasonable; 33 – 50% too fast
- snatch force would be "uncomfortable" (or worse)
- opening shock not "smooth"

Questions:

- Can these problems be rectified with different parameters?
- Would a modified model do better?
- How should the model be modified?

Objectives

Physical

- physically realistic free-fall and impact velocities
- survivable snatch force
- smooth opening shock

Mathematical

- $v_{\text{ff}} \approx 53$ m/s
- $v_{\text{impact}} \in (4.6, 5.2)$ m/s
- $a \in C^1(0, t_{\text{impact}})$

Real-Life Expectations

- $x_0 = 1219$ m, $t_d = 10$ s, $v_{\text{ff}} = 53.6$ m/s
- v_{impact} comparable to jump from 5 foot high wall
  $v_{\text{impact}} \in (4.6, 5.2)$ m/s
- full deployment of parachute requires 3.2 s
- "opening shock" is a heavy, but smooth, tug; < 500 lbs ≈ 3 G

References:

- Air Force Academy Parachute Team Training Guide, 1990.
- Zim, *Parachutes*, 1942.

Example: The Parachute Problem

Melka and Farrior (C-ODE-E Newsletter, Summer-Fall 1995)

Extensions to Drucker

1. stepwise change in the drag coefficient:

$$k = \begin{cases} 2 & t \le t_d \\ 3 & t_d < t \le t_d + 1 \\ 7 & t_d + 1 < t \le t_d + 2 \\ 10 & t_d + 2 < t \end{cases}$$

2. linear increase in the drag coefficient:

$$k = \begin{cases} 2 & t \le t_d \\ 2 + \frac{8}{3}(t - t_d) & t_d < t \end{cases}$$

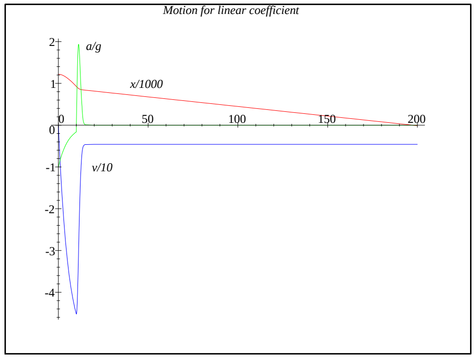
Analysis of Acceleration and Jerk

$$\begin{aligned} v &\in C(0, t_{\text{impact}}) \\ a(t) &= -g - \frac{k(t)}{m} v(t) \\ j(t) &= \frac{k(t)}{m} g + \left(\frac{k^2(t)}{m^2} - \frac{k'(t)}{m} \right) v(t) \end{aligned}$$

$$k \in C^1(0, t_{\text{impact}}) \implies a \in C^1(0, t_{\text{impact}}) \text{ and } j \in C(0, t_{\text{impact}})$$

Graphical Analysis of Solution

k_d linear interpolant between k_1 and k_2



Note:

- $k \in C^0(0, t_{\text{impact}}) \implies a \in C^0(0, t_{\text{impact}})$

Possible Modifications

- nonlinear air resistance
  — increases snatch force and opening shock
- correct parameter values
  — does not change smoothness of solution ($k_1 \neq k_2$)
- eliminate assumption of instantaneous deployment
  — how? what properties are needed?

A Generalized Model

$$\begin{aligned} x' &= v, & x(0) &= x_0 \\ v' &= -g - \tfrac{k}{m} v, & v(0) &= 0 \end{aligned}$$

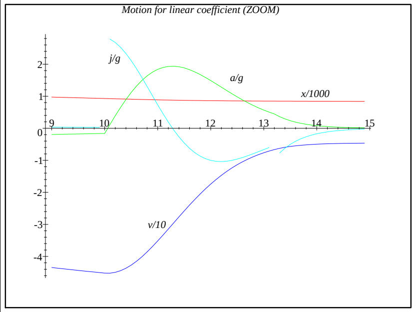
where

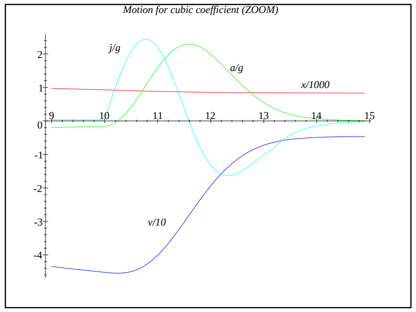
$$k(t) = \begin{cases} k_1, & t < t_d \\ k_d, & t_d \le t < t_d + \tau \\ k_2, & t \ge t_d + \tau \end{cases}$$

with parameter values

$x_0 = 1219.2$ m	$m = 75$ kg	$g = 9.81$ m/s^2
$k_1 = 13.63$ kg/s	$k_2 = 160$ kg/s	
$t_d = 10$ s	$\tau = 3.2$ s	
$k_d = ?$		

Graphical Analysis of Solution – A Closer Look





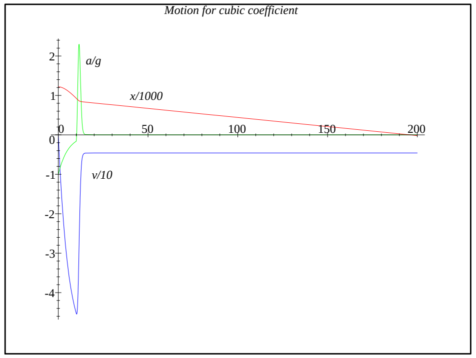
Interpretation of Solution

- terminal velocities:
  - free-fall: $v_{\rm ff} \approx 53.5$ m/s
  - final descent: $v_{\rm impact} \approx 4.6$m/s $= 15.1$ ft/s
- position:
  - deployment at 928 m
  - land after 196 s
- acceleration and jerk:
  - no discontinuities
  - maximum acceleration: ≈ 2.2 G
  - maximum jerk: ≈ 2.4 G/s

(all values approximate)

Graphical Analysis of Solution

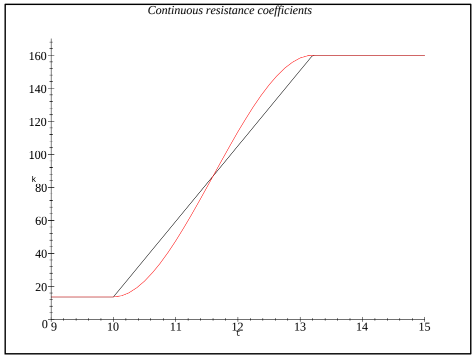
k_d smooth cubic interpolant between k_1 and k_2



Note:

- $k \in C^1(0, t_{\rm impact}) \implies a \in C^1(0, t_{\rm impact})$

Comparison of Resistance Coefficients



General Remarks on Modelling

- good problems are everywhere, you just have to look
- be skeptical of what you read (particularly in textbooks)
- understand the problem
  - physically
  - mathematically
- seek expert advice
- experiment
- keep problems realistic
  — and interesting!

Questions to Ponder — Revisited

- What do we want our students to learn?
  - mechanics!
  - concepts!
  - theory!
  - application!
- How well are we currently meeting these goals?
- What can be done to improve our effectiveness?
- Can "effectiveness" be modelled as a local function of space and time?

Related Questions

- How does motion depend on deployment criterion?
  - choose t_d such that $v(t_d) = v^*$
  - choose t_d such that $x(t_d) = x^*$
  - choose t_d to minimize t_{impact} subject to $v(t_{\text{impact}}) \leq v^*$
- What limitations should be imposed to minimize safety risks?
  - reserve chute performance characteristics
    $\tau = 1.5$ s, $v_{\text{impact}} = 17.5$ ft/s
  - maximum free fall which leaves enough time to safely activate reserve?

Interesting Applications

- T-cells, HIV, and AIDS
  (Scientific American, MapleTech, Mathematical Biosciences, Notices AMS)

- hydraulics
  (C-ODE-E Newsletter, Spring-Summer 1996)

- sprints and human-powered vehicles
  (American J. of Physics, C-ODE-E Newsletter, Physics Today, SIAM Review)

Advertisements

- Using Maple to Explore Mathematical Models in Biology
  Matt Miller (USC), Friday, 9AM, Crystal 1
- Computer Algebra and the WWW in the Math Curriculum
  Murphy, Miller, Meade, Friday, 12PM, Crystal 2
- Maple Supplement to Nagle/Saff, Addison–Wesley, 1996
  Meade
- Maple V for Engineers, Addison–Wesley, 1997
  Meade and Bourkoff, (a module the Engineer's Toolkit series)