Jerks Don't Jump — Mathematical Modelling with an Impact

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Overview of Presentation

- Introduction
- Example – Parachute Problem
  - Standard Problem
  - Interpret Solution
  - Improved Model
  - Related Questions
- Additional Examples
- Advertisements

- presentation will be more demonstration than general discussion of ideas (willing to talk details in person or remotely)

Questions to Ponder

- What do we want our students to learn?
  - mechanics?
  - concepts?
  - theory?
  - application?

Introduction

- instructor must stay within comfort zone (may need to actively work to extend comfort zone, but should not go beyond it when teaching)
- this is easier than it might seem; many good problems can be found in the textbooks
- this talk will provide an example of how this might look in a sophomore-level ODE course

Example: The Parachute Problem

Nagle and Saff (Addison–Wesley, 1996, p. 112)

A parachutist whose mass is 75 kg drops from a helicopter hovering 4000 m above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant and that the force due to air resistance is proportional to the velocity of the parachutist, with a proportionality constant $k_1 = 15$ kg/sec when the chute is closed and with a constant $k_2 = 105$ kg/sec when the chute is open. If the chute does not open until 1 min after the parachutist leaves the helicopter, after how many seconds will she hit the ground?

Example: The Parachute Problem

Abell and Braselton (Saunders, 1996, p. 140)

A parachutist weighing 60 kg falls from a plane (that is, $v_0 = 0$) and is subjected to air resistance equivalent to $F_R = 10v$. After one minute, the parachute is opened so that the parachutist is subjected to air resistance equivalent to $F_R = 100v$.

1. What is the parachutist's velocity when the parachute is opened?
2. What is the parachutist's velocity $v(t)$ after the parachute is opened?
3. What is the parachutist's limiting velocity? How does this compare with the limiting velocity if the parachute does not open?

Example: The Parachute Problem

Edwards and Penney (Prentice Hall, 1996, p. 95)

A woman bails out of an airplane at an altitude of 10,000 ft, fall freely for 20 s, then opens her parachute. How long will it take her to reach the ground? Assume that $\rho = 0.15$ without the parachute and that $\rho = 1.5$ with the parachute. (*Suggestion:* First determine the height and velocity when the parachute opens.)

NOTE: $\rho = \frac{k}{m}$

Example: The Parachute Problem

Kostelich and Armbruster (Addison–Wesley, 1996, p. 109)

The objective of this problem is to calculate some of the relevant quantities for the fall of a skydiver out of an airplane. We use the simplest model of air resistance, in which we suppose that the drag force through air is proportional to the velocity. The relevant differential equation is

$$v' = -g - \frac{k}{m}v.$$

We take the ground as $x = 0$ and let $x > 0$ correspond to positions above the ground. As $v = x'$, a falling skydiver has negative velocity in this coordinate system. The gravitational constant is g and the coefficient of air resistance is k.

- Suppose that $k/m = 0.2$ before the skydiver opens the parachute. What is the terminal velocity for the free-falling skydiver?
- Suppose the airplane is 2000 m above the ground when the skydiver jumps out. What is the skydiver's altitude [when the skydiver reaches 95 percent of the terminal velocity]?
- The coefficient of air resistance changes after the parachute opens. Suppose that the skydiver, who has reached a new terminal velocity, hits the ground at a rate which is equal to that attained by jumping off a wall which is 2 m high. (Call this rate v_{ground}.) Determine k/m. *Hint*: Calculate the velocity for a free fall from 2 m and equate it to the terminal velocity of the parachute.
- In order to avoid serious injury, suppose that the skydiver must hit the ground at a velocity that is within 5 percent of v_{ground}. By what altitude must the parachute be opened to satisfy this restriction? Discuss whether your answer is reasonable and how other restrictions might yield a different answer.

Example: The Parachute Problem

Drucker (College Mathematics Journal, May 1995)

A parachutist exits an airplane 1200 ft above the ground with zero vertical velocity. He wishes to land with a velocity of 40 ft/sec. The skydiver is assumed to weigh 240 pounds, and the problem is given assuming drag force due to air resistance proportional to velocity. The proportionality constant is taken as 2 before the parachute opens and 10 afterward. The question is then, at what time after exiting the plane should the diver open the parachute in order to land with a velocity of 40 ft/sec?

Mathematical Formulation

- 2nd-order ODE

$$\begin{aligned} mx'' &= -mg - kx' \\ x(0) &= x_0 \\ x'(0) &= 0 \end{aligned}$$

- 1st-order system

$$\begin{aligned} x' &= v, & x(0) = x_0 \\ v' &= -g - \tfrac{k}{m}v, & v(0) = 0 \end{aligned}$$

- k — piecewise constant coefficient of air resistance

$$k(t) = \begin{cases} k_1, & t < t_d \\ k_2, & t \geq t_d \end{cases}$$

- t_d — time of (instantaneous) deployment

General Comments:

- problem is not constant coefficients – it's piecewise constant coefficients
- explicit solution can be found (need to continuously match solutions)
- system, but decoupled: v equation can be solved as 1st-order linear IVP; x equation can be solved by direct integration
- *show explicit solution? Maple code for computation? YES!!*
- nondimensionalization would make problem appear even simpler
- same methods work with Release 4; `piecewise` is nice, but `dsolve` does not appear to be able to solve the general problem, *e.g.*, `K[pw_const] := piecewise( t<td, k1, k2 );`

Pedagogy:

- how many students could solve this problem?
- how long? how many tries?
- how would the solution be used?
- how many could understand the basic idea? (piecewise)
- what is really important?
- CLAIM:
  - working with mathematical models
  - structure of the problem and the solution
  - interpreting solutions; limitations of applicability
  - confirm/verify results
  - explain model; revise/extend as necessary

Explicit Solution — via Maple V (Release 3)

```
> SYS := { diff( x(t), t ) = v(t),
>          diff( v(t), t ) = -g - k/m*v(t) };
```

$$SYS := \left\{ \frac{\partial}{\partial t}\,\mathrm{x}(\,t\,) = \mathrm{v}(\,t\,), \frac{\partial}{\partial t}\,\mathrm{v}(\,t\,) = -g - \frac{k\,\mathrm{v}(\,t\,)}{m} \right\}$$

```
> IC := { x(0)=x0, v(0)=0 };
```

$$IC := \{\,\mathrm{x}(\,0\,) = \mathit{x0}, \mathrm{v}(\,0\,) = 0\,\}$$

```
> K[pw_const] :=  k1+(k2-k1)*Heaviside(t-td);
```

$$K_{pw\_const} := \mathit{k1} + (\,\mathit{k2} - \mathit{k1}\,)\,\mathrm{Heaviside}(\,t - td\,)$$

Step 1 – Free-Fall Solution

```
> SOL1 := dsolve( subs(k=k1,SYS) union IC, { x(t), v(t) } ):
> V1 := subs( SOL1, v(t) );
```

$$V1 := -\frac{m\,g}{k1} + \frac{m\,g\,\mathrm{e}^{\left(-\frac{t\,k1}{m}\right)}}{k1}$$

```
> X1 := subs( SOL1, x(t) );
```

$$X1 := \frac{-m\,g\,t\,k1 + m^2\,g + x0\,k1^2 - m^2\,g\,\mathrm{e}^{\left(-\frac{t\,k1}{m}\right)}}{k1^2}$$

Step 2 – Solution with Deployed Parachute

```
> IC2 := { x(td) = subs( t=td, X1 ),
>          v(td) = subs( t=td, V1 ) };
```

$$IC2 := \left\{ \mathrm{v}(\, td\,) = -\frac{m\,g}{k1} + \frac{m\,g\,\mathrm{e}^{\left(-\frac{td\,k1}{m}\right)}}{k1},\right.$$

$$\left.\mathrm{x}(\, td\,) = \frac{-m\,g\,td\,k1 + m^2\,g + x0\,k1^2 - m^2\,g\,\mathrm{e}^{\left(-\frac{td\,k1}{m}\right)}}{k1^2}\right\}$$

```
> SOL2 := dsolve( subs(k=k2,SYS) union IC2, { x(t), v(t) } ):
> V2 := subs( SOL2, v(t) ):
> X2 := subs( SOL2, x(t) ):
```

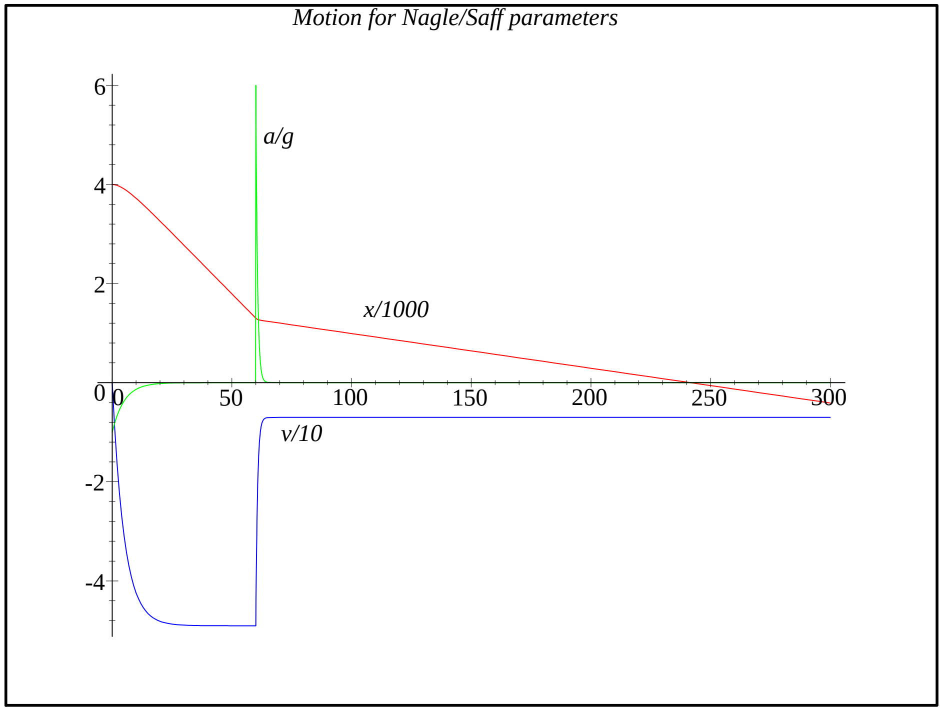
Step 3 – Composite Solution

```
> V := V1 + (V2-V1)*Heaviside(t-td):
> X := X1 + (X2-X1)*Heaviside(t-td):
```

$$v(t) = \begin{cases} -\frac{mg}{k_1}\left(1-e^{-\frac{tk_1}{m}}\right), & t < t_d \\ -\frac{mg}{k_2}+\frac{mg}{k_1k_2}\left(k_1-k_2\left(1-e^{-\frac{t_dk_1}{m}}\right)\right)e^{-\frac{(t-t_d)k_2}{m}}, & t \geq t_d \end{cases}$$

$$x(t) = \begin{cases} x_0-\frac{mg}{k_1}\left(t-\frac{m}{k_1}\left(1-e^{-\frac{tk_1}{m}}\right)\right), & t < t_d \\ \text{what would you do with this if you had it?}, & t \geq t_d \end{cases}$$

Graphical Solution



Parameters (Nagle & Saff):

- $k_1 = 15$ kg/s, $k_2 = 105$ kg/s, $m = 75$ kg
- $x_0 = 4000$ m, $t_d = 60$ s, $g = 9.81$ m/s^2

Interpretation of Solution

- terminal velocities:
  - free-fall: $v_{\text{ff}} \approx 49$ m/s
  - final descent: $v_{\text{impact}} \approx 7$m/s
- position:
  - deployment at 1300 m
  - land after 241 s
- acceleration:
  - jump discontinuity at t_d
  - magnitude: 6 G

(all values approximate)

Recall the variety of textbook problems. Some contained rather diverse sets of parameters. Are all realistic? An answer to this question requires some knowledge about the physics of falling objects and skydiving practices and procedures. This information is not difficult to find and is easily understandable – by students (and faculty)!

We need to be looking both for realistic parameter values and for indications that the model is correct.

<u>Definitions</u>:

- *snatch force* is acceleration at instant parachute deployment begins
- *opening shock* is force produced while parachute deploys
- *jerk* is rate of change of acceleration (opening shock during deployment)

Real-Life Expectations

- $x_0 = 1219$ m, $t_d = 10$ s, $v_{\text{ff}} = 53.6$ m/s
- v_{impact} comparable to jump from 5 foot high wall
  $v_{\text{impact}} \in (4.6, 5.2)$ m/s
- full deployment of parachute requires 3.2 s
- "opening shock" is a heavy, but smooth, tug; < 500 lbs ≈ 3 G

References:

- Air Force Academy Parachute Team Training Guide, 1990.
- Zim, *Parachutes*, 1942.

Evaluation of Solution

- v_{ff} is reasonable; within 10% of actual data
- v_{impact} in not reasonable; 33 – 50% too fast
- snatch force would be "uncomfortable" (or worse)
- opening shock not "smooth"

Questions:

- Can these problems be rectified with different parameters?
- Would a modified model do better?
- How should the model be modified?

Example: The Parachute Problem

Melka and Farrior (C-ODE-E Newsletter, Summer-Fall 1995)

Extensions to Drucker

1. stepwise change in the drag coefficient:

$$k = \begin{cases} 2 & t \le t_d \\ 3 & t_d < t \le t_d + 1 \\ 7 & t_d + 1 < t \le t_d + 2 \\ 10 & t_d + 2 < t \end{cases}$$

2. linear increase in the drag coefficient:

$$k = \begin{cases} 2 & t \le t_d \\ 2 + \frac{8}{3}(t - t_d) & t_d < t \end{cases}$$

Objectives

Physical

- physically realistic free-fall and impact velocities
- survivable snatch force
- smooth opening shock

Mathematical

- $v_{\text{ff}} \approx 53$ m/s
- $v_{\text{impact}} \in (4.6, 5.2)$ m/s
- $a \in C^1(0, t_{\text{impact}})$

Possible Modifications

- nonlinear air resistance

  — increases snatch force and opening shock

- correct parameter values

  — does not change smoothness of solution ($k_1 \neq k_2$)

- eliminate assumption of instantaneous deployment

  — how? what properties are needed?

Analysis of Acceleration and Jerk

$$
\begin{aligned}
v &\in C(0, t_{\text{impact}}) \\
a(t) &= -g - \frac{k(t)}{m} v(t) \\
j(t) &= \frac{k(t)}{m} g + \left(\frac{k^2(t)}{m^2} - \frac{k'(t)}{m} \right) v(t)
\end{aligned}
$$

$$k \in C^1(0, t_{\text{impact}}) \implies a \in C^1(0, t_{\text{impact}}) \text{ and } j \in C(0, t_{\text{impact}})$$

A Generalized Model

$$
\begin{aligned}
x' &= v, & x(0) &= x_0 \\
v' &= -g - \tfrac{k}{m}v, & v(0) &= 0
\end{aligned}
$$

where

$$
k(t) = \begin{cases} k_1, & t < t_d \\ k_d, & t_d \le t < t_d + \tau \\ k_2, & t \ge t_d + \tau \end{cases}
$$

with parameter values

$$
\begin{array}{lll}
x_0 = 1219.2 \text{ m} & m = 75 \text{ kg} & g = 9.81 \text{ m/s}^2 \\
k_1 = 13.63 \text{ kg/s} & k_2 = 160 \text{ kg/s} & \\
t_d = 10 \text{ s} & \tau = 3.2 \text{ s} & \\
k_d = ? & &
\end{array}
$$

Jumps:

- $[a] = -\frac{(k_2 - k_1)v(t_d)}{m}$
- $[j] = \frac{(k_2 - k_1)(mg + (k_2 + k_1)v(t_d))}{m^2}$
- in this case: $[j] \approx -8.4$ G/s

Selection of k_d:

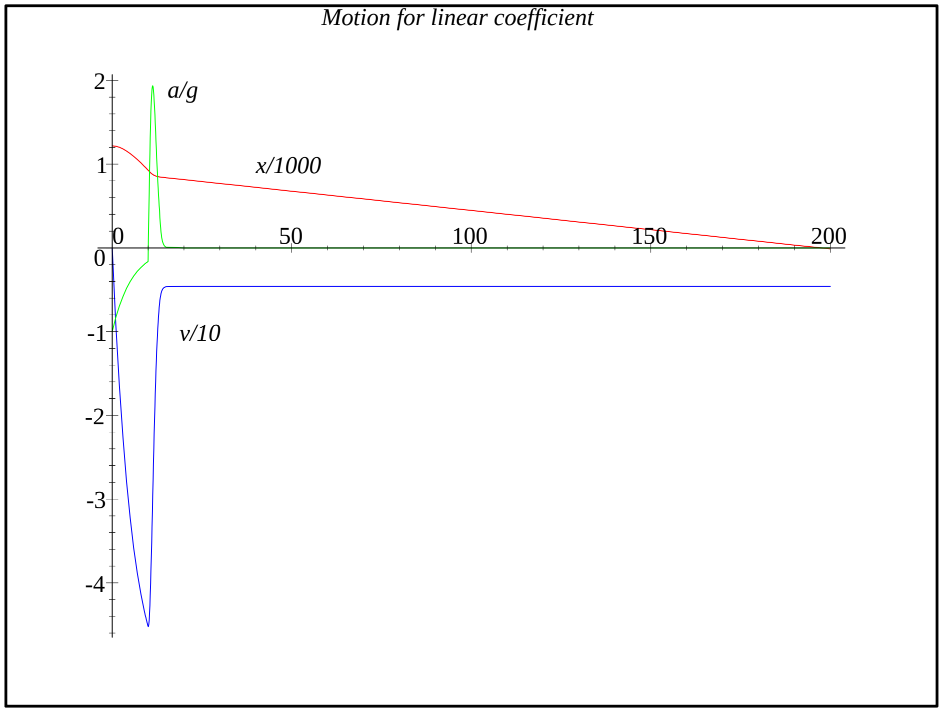
- k_d interpolate k_1 and k_2:
  $k \in C^0 \Rightarrow a \in C^0$, but $[j(t_d)], [j(t_d + \tau)] \neq 0$
- k_d interpolate with smoothness:
  $k \in C^1 \Rightarrow a \in C^1$, $j \in C^0$ (smooth tug)

Evaluation of solutions:

- are forces within reasonable limits
  (by all indications, YES!)
- do the accelerations exhibit the expected smoothness?
  (pictures can be deceiving – let's take a closer look)
- note that the zoomed view confirms that a and j exhibit the exact smoothness that is expected
- would an explicit solution be of any use?

Graphical Analysis of Solution

k_d linear interpolant between k_1 and k_2

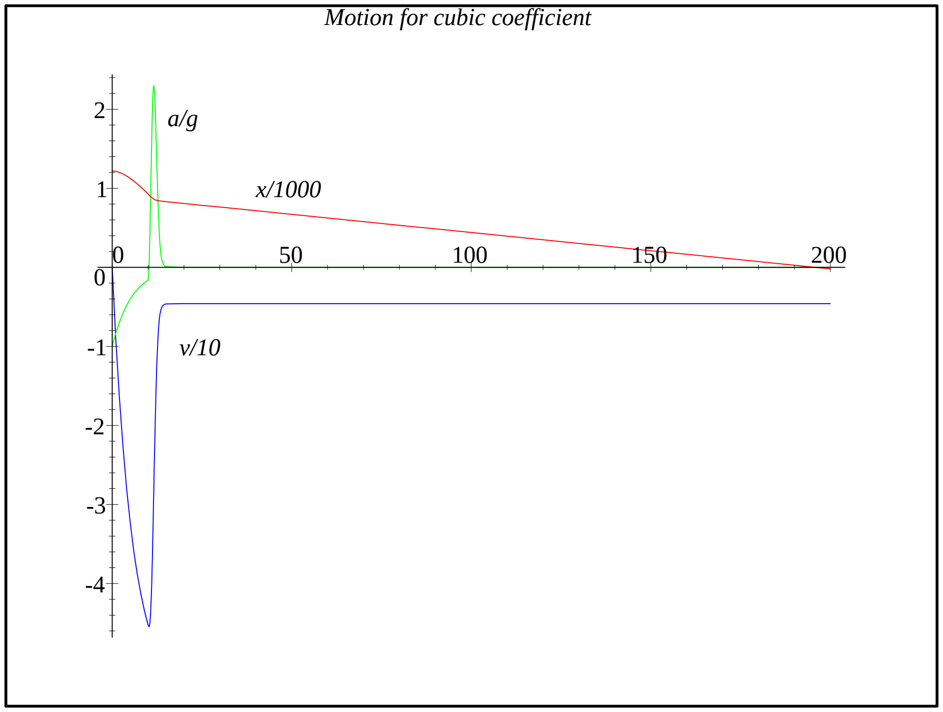


Note:

- $k \in C^0(0, t_{\text{impact}}) \implies a \in C^0(0, t_{\text{impact}})$

Graphical Analysis of Solution

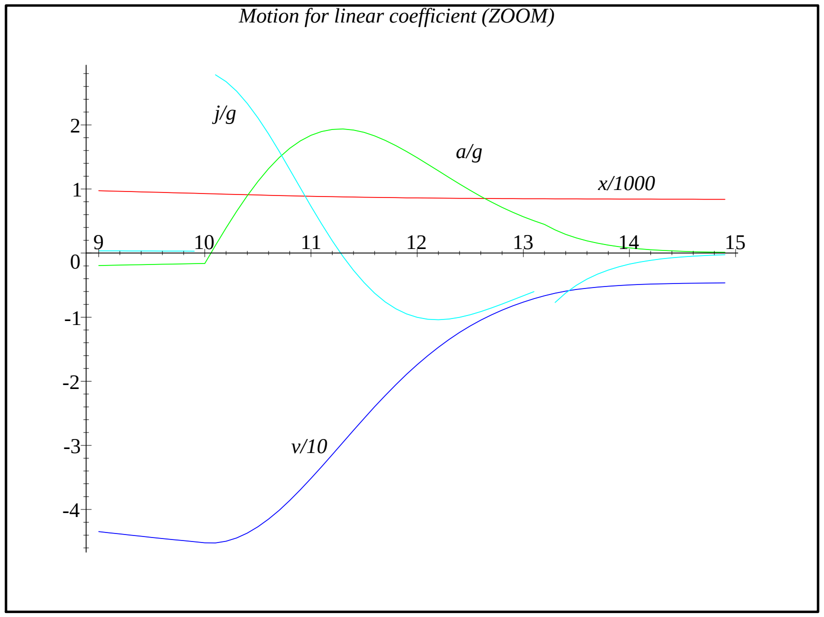
k_d smooth cubic interpolant between k_1 and k_2

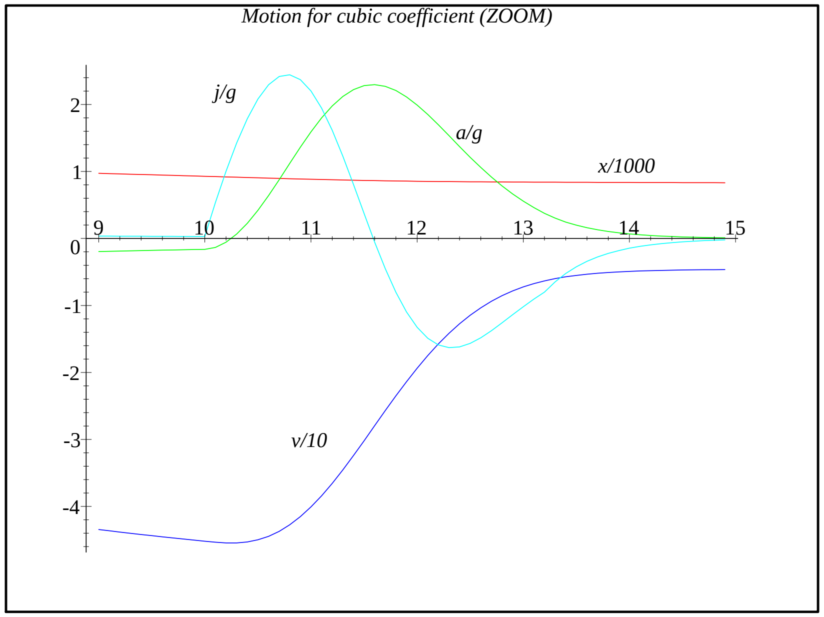


<u>Note</u>:

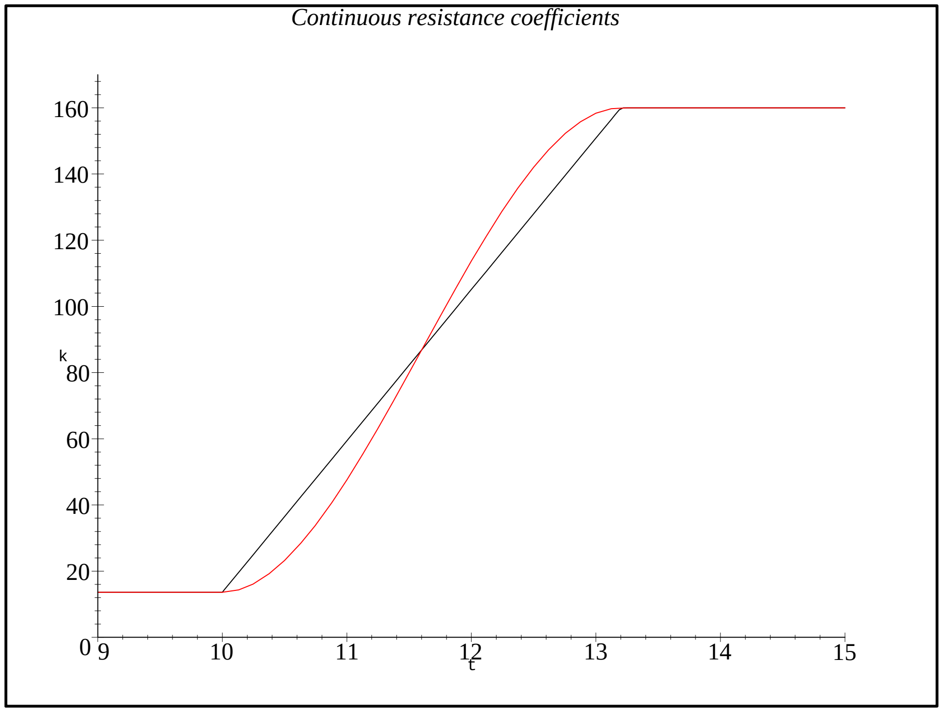
- $k \in C^1(0, t_{\text{impact}}) \implies a \in C^1(0, t_{\text{impact}})$

Graphical Analysis of Solution – A Closer Look





Comparison of Resistance Coefficients



Interpretation of Solution

- terminal velocities:
  - free-fall: $v_{\text{ff}} \approx 53.5$ m/s
  - final descent: $v_{\text{impact}} \approx 4.6$m/s $= 15.1$ ft/s
- position:
  - deployment at 928 m
  - land after 196 s
- acceleration and jerk:
  - no discontinuities
  - maximum acceleration: ≈ 2.2 G
  - maximum jerk: ≈ 2.4 G/s

(all values approximate)

Related Questions

- How does motion depend on deployment criterion?
  - choose t_d such that $v(t_d) = v^*$
  - choose t_d such that $x(t_d) = x^*$
  - choose t_d to minimize t_{impact} subject to $v(t_{\text{impact}}) \leq v^*$
- What limitations should be imposed to minimize safety risks?
  - reserve chute performance characteristics
    $\tau = 1.5$ s, $v_{\text{impact}} = 17.5$ ft/s
  - maximum free fall which leaves enough time to safely activate reserve?

Notes:

- some deployment criteria lead to non-linear problems
- AFA Training Manual contains information on safety/reserve chute
- refer to SIAM Review and MapleTech papers for details
- guidelines for warnings/termination are consistent with results from model ($t_d > 13$ s implies deployment below 2500 ft)

General Remarks on Modelling

- good problems are everywhere, you just have to look
- be skeptical of what you read (particularly in textbooks)
- understand the problem
  - physically
  - mathematically
- seek expert advice
- experiment
- keep problems realistic

  — and interesting!

- good problems don't have to be difficult
- simple problems can be effective and interesting
- ref: SIAM Review, Notices (HIV), texts, library
- mention Nagle/Saff Maple Supplement
- textbook problems sound good, but often don't hold water when studied closely
- good experiments should have clear objectives
- talk with the experts — good PR for mathematics
- opens doors for interdisciplinary research
- mention recent papers — SIAM Review and MapleTech
- mention Engineer's Toolkit — as math courses make more use of CAS, need materials to support learning to make effective and efficient use of software; modules are brief (3 week) exposures that provide good basis while giving a good view of the discipline (so far, just engineering)

Interesting Applications

- T-cells, HIV, and AIDS

  (Scientific American, MapleTech, Mathematical Biosciences, Notices AMS)

- hydraulics

  (C-ODE-E Newsletter, Spring-Summer 1996)

- sprints and human-powered vehicles

  (American J. of Physics, C-ODE-E Newsletter, Physics Today, SIAM Review)

Questions to Ponder — Revisited

- What do we want our students to learn?
  - mechanics!
  - concepts!
  - theory!
  - application!
- How well are we currently meeting these goals?
- What can be done to improve our effectiveness?
- Can "effectiveness" be modelled as a local function of space and time?

Some personal opinions:

- mechanical proficiency should not be the ultimate objective; application of concepts and theory is not the ultimate objective
- proper balance is difficult to determine

  — not to mention achieve
- teaching, learning, testing measures may need to be modified
- good modelling requires good background information
- (faculty/students) may have to do some background reading
- good to practice library skills, technical reading, ...
- interpretation of solutions is important

  — even to mathematicians (the problem is not over once the solution has been found)
- if mechanics is not of primary concern, CAS/Maple can be extremely useful (does not mean that CAS replaces need for mechanics!)
- personal opinions still contain significant transient terms

Advertisements

- Using Maple to Explore Mathematical Models in Biology
  Matt Miller (USC), Friday, 9AM, Crystal 1
- Computer Algebra and the WWW in the Math Curriculum
  Murphy, Miller, Meade, Friday, 12PM, Crystal 2
- Maple Supplement to Nagle/Saff, Addison–Wesley, 1996
  Meade
- Maple V for Engineers, Addison–Wesley, 1997
  Meade and Bourkoff, (a module the Engineer's Toolkit series)