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%
%  systems.ms  -- a Maple worksheet for use with systems of ODEs and IVPs
%
% Prepared by:  Douglas B. Meade (7 March 1996)
%                    as part of the Maple Supplement for Nagle/Saff (Addison-Wesley)
%
% Updated for Maple V, Release 4, by Douglas B. Meade (5 November 1996)
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```

SYSTEMS OF FIRST-ORDER EQUATIONS (Section 9.4--9.7, p. 533--564)

```
> restart;
  with( plots ):
  with( DEtools ):
  with( linalg ):
Warning, new definition for norm
Warning, new definition for trace
```

First-order linear systems of ordinary differential equations (and initial value problems) can be solved in a number of different ways. The most efficient technique depends on your background in linear algebra, and the specific problem being studied. The general idea, as introduced in Section 9.3, is to write the system in normal form, then find a fundamental matrix for the homogeneous system and, if the system is non-homogeneous, a particular solution to the nonhomogeneous system. This worksheet will present and demonstrate three different methods for the determination of the fundamental matrix.

Objectives:

- to demonstrate the critical role that the fundamental matrix plays in the solution of linear systems of ODEs
- to present multiple methods for the (theoretical) computation of the fundamental matrix

- to illustrate techniques that can be used to realize the different methods for computing the fundamental matrix

New Maple Commands:

matrix	creates a matrix data structure of the given size
evalm	evaluates a given matrix expression
&*	non-commutative matrix multiplication (replaces *)
diag	returns the diagonal matrix with specified diagonal entries
det	computes the determinant of a matrix
nullspace	determines a basis for the nullspace of a given matrix
transpose	returns the transpose of a matrix
inverse	computes the inverse of a matrix
eigenvects	determines the eigenvalues (with multiplicity) and, eigenvectors for a square matrix
exponential	the matrix exponential

EXAMPLE 1 (p. 550)

Find a general solution to the first-order linear system in normal form (x' = Ax + f) with

```
> x := matrix( 3, 1 ):
  evalm( x );
```

$$\begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix}$$

Note that the solution vector x, and all other vectors, will be defined to be a 3 x 1 matrix, i.e., a column vector, not a (row) vector.

```
> A := matrix( 3, 3, [[1, -2, 2], [-2, 1, 2], [2, 2, 1]] );
```

$$A := \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

```
> f := t -> t * matrix( 3, 1, [-9, 0, -18 ] ):
  f(t);
```

$$t \begin{bmatrix} -9 \\ 0 \\ -18 \end{bmatrix}$$

The solution is found in Nagle and Saff using the Method of Undetermined Coefficients. Let's repeat the process using a fundamental matrix and variation of parameters. The first step towards finding a fundamental matrix is to find the eigenvalue decomposition of the coefficient matrix A. That is, $A = T \Lambda T^{\{-1\}}$ where Λ is the diagonal matrix containing the eigenvalues of A and T is the corresponding matrix containing the eigenvectors of A. The characteristic matrix, $A - \lambda I$,, and characteristic polynomial, $det(A - \lambda I)$, are

```
> CHARm := evalm( A - lambda*diag(1,1,1) );
```

$$CHARm := \begin{bmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{bmatrix}$$

```
> CHARp := det(CHARm);
```

$$CHARp := -27 + 9\,\lambda + 3\,\lambda^2 - \lambda^3$$

The eigenvalues of A are the roots of the characteristic polynomial

```
> EIGvals := solve( CHARp=0, lambda );
```

$$EIGvals := -3, 3, 3$$

Thus,

```
> Lambda := diag( EIGvals );
```

$$\Lambda := \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The eigenspace for $\lambda = -3$ is the null space of the characteristic matrix with $\lambda = -3$, that is,

```
> EIGvects[-3] := nullspace( subs( lambda=-3, evalm(CHARm) ) );
```

$$EIGvects_{-3} := \{ [-1, -1, 1] \}$$

For $\lambda = 3$ the eigenspace could be two-dimensional:

```
> EIGvects[3] := nullspace( subs( lambda=3, evalm(CHARm) ) );
```

$$EIGvects_{3} := \{ [-1, 1, 0], [1, 0, 1] \}$$

The eigenvector matrix is, therefore,

```
> T := transpose( matrix( 3, 3, [ op(EIGvects[-3]), op(EIGvects[3])
  ] ) );
```

$$T := \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Just to check, let's see that $A = T \Lambda T^{\{-1\}}$.

```
> evalm( A = T &* Lambda &* inverse(T) );
```

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Note that the same information can be obtained in one step with the use of the eigenvects command from the linalg package.

```
> EIGSYS := eigenvects( A );
```

$$EIGSYS := [3, 2, \{ [-1, 1, 0], [1, 0, 1] \}], [-3, 1, \{ [-1, -1, 1] \}]$$

```
>
```

It can be a challenge to extract the eigenvalues and eigenvectors from the data structure returned by eigenvects (see ?eigenvects for a full description of this data structure).

A fundamental matrix for the homogeneous differential equation is now available

```
> X := t ->  evalm( T &* diag( exp(-3*t), exp(3*t), exp(3*t) ) ):
  evalm( X(t) );
```

$$\begin{bmatrix} -e^{(-3\,t)} & -e^{(3\,t)} & e^{(3\,t)} \\ -e^{(-3\,t)} & e^{(3\,t)} & 0 \\ e^{(-3\,t)} & 0 & e^{(3\,t)} \end{bmatrix}$$

```
>
```

Prior to finding a particular equation, by variation of parameters, let's check that X actually solves the homogeneous equation

```
> CHECK1 := evalm( map( diff, X(t), t ) - A &* X(t) );
```

$$CHECK1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution to the homogeneous equation is $X(t)\,c$ for any three-dimensional vector c. That is,

```
> C := matrix( 3, 1 ):
  evalm( C );
```

$$\begin{bmatrix} C_{1,\,1} \\ C_{2,\,1} \\ C_{3,\,1} \end{bmatrix}$$

```
> xHOMO := evalm( X(t) &* C );
```

$$xHOMO := \begin{bmatrix} -e^{(-3\,t)}\,C_{1,\,1} - e^{(3\,t)}\,C_{2,\,1} + e^{(3\,t)}\,C_{3,\,1} \\ -e^{(-3\,t)}\,C_{1,\,1} + e^{(3\,t)}\,C_{2,\,1} \\ e^{(-3\,t)}\,C_{1,\,1} + e^{(3\,t)}\,C_{3,\,1} \end{bmatrix}$$

A particular solution is, now, given by $x_p(t) = X(t) \int_0^t X(s)^{\{-1\}}\,f(s)\,ds$:

```
> xPART := evalm( X(t) &* map( int, multiply( inverse( X(s) ), f(s)
  ), s=0..t ) );
```

$$xPART := \begin{bmatrix} -\%1 - e^{(3\,t)}\left(\left(t+\frac{1}{3}\right)e^{(-3\,t)} - \frac{1}{3}\right) + e^{(3\,t)}\left(\left(5\,t+\frac{5}{3}\right)e^{(-3\,t)} - \frac{5}{3}\right) \\ -\%1 + e^{(3\,t)}\left(\left(t+\frac{1}{3}\right)e^{(-3\,t)} - \frac{1}{3}\right) \\ \%1 + e^{(3\,t)}\left(\left(5\,t+\frac{5}{3}\right)e^{(-3\,t)} - \frac{5}{3}\right) \end{bmatrix}$$

$$\%1 := e^{(-3\,t)}\left(-t\,e^{(3\,t)} + \frac{1}{3}e^{(3\,t)} - \frac{1}{3}\right)$$

Further simplification occurs by normalizing each term, then combining the powers:

```
> xPART := map( normal, evalm(xPART) );
```

$$xPART := \begin{bmatrix} 5\,e^{(-3\,t)}\,t\,e^{(3\,t)} + e^{(-3\,t)}\,e^{(3\,t)} + \frac{1}{3}e^{(-3\,t)} - \frac{4}{3}e^{(3\,t)} \\ 2\,e^{(-3\,t)}\,t\,e^{(3\,t)} + \frac{1}{3}e^{(-3\,t)} - \frac{1}{3}e^{(3\,t)} \\ 4\,e^{(-3\,t)}\,t\,e^{(3\,t)} + 2\,e^{(-3\,t)}\,e^{(3\,t)} - \frac{1}{3}e^{(-3\,t)} - \frac{5}{3}e^{(3\,t)} \end{bmatrix}$$

```
> xPART := map( combine, xPART, power );
```

$$xPART := \begin{bmatrix} 5\,t + 1 + \frac{1}{3}e^{(-3\,t)} - \frac{4}{3}e^{(3\,t)} \\ 2\,t + \frac{1}{3}e^{(-3\,t)} - \frac{1}{3}e^{(3\,t)} \\ 4\,t + 2 - \frac{1}{3}e^{(-3\,t)} - \frac{5}{3}e^{(3\,t)} \end{bmatrix}$$

Therefore, the general solution to the nonhomogeneous equation is

```
> x := evalm( xHOMO + xPART );
```

$$x := \begin{bmatrix} -e^{(-3\,t)}\,C_{1,\,1} - e^{(3\,t)}\,C_{2,\,1} + e^{(3\,t)}\,C_{3,\,1} + 5\,t + 1 + \frac{1}{3}e^{(-3\,t)} - \frac{4}{3}e^{(3\,t)} \\ -e^{(-3\,t)}\,C_{1,\,1} + e^{(3\,t)}\,C_{2,\,1} + 2\,t + \frac{1}{3}e^{(-3\,t)} - \frac{1}{3}e^{(3\,t)} \\ e^{(-3\,t)}\,C_{1,\,1} + e^{(3\,t)}\,C_{3,\,1} + 4\,t + 2 - \frac{1}{3}e^{(-3\,t)} - \frac{5}{3}e^{(3\,t)} \end{bmatrix}$$

The terms in this solution can be re-written in a variety of ways. Two of the most effective are to group the terms according to the type of exponential term that appears in the solution

```
> x := map( collect, x, {exp(-3*t), exp(3*t)} );
```

$$x := \begin{bmatrix} \left(-C_{1,1}+\frac{1}{3}\right)e^{(-3t)}+\left(-C_{2,1}+C_{3,1}-\frac{4}{3}\right)e^{(3t)}+1+5t \\ \left(-C_{1,1}+\frac{1}{3}\right)e^{(-3t)}+\left(-\frac{1}{3}+C_{2,1}\right)e^{(3t)}+2t \\ \left(C_{1,1}-\frac{1}{3}\right)e^{(-3t)}+\left(C_{3,1}-\frac{5}{3}\right)e^{(3t)}+4t+2 \end{bmatrix}$$

or by the unknown coefficient for each eigenvector

```
> x := map( collect, x, {C[1,1], C[2,1], C[3,1]} );
```

$$x := \begin{bmatrix} -e^{(-3t)}C_{1,1}-e^{(3t)}C_{2,1}+e^{(3t)}C_{3,1}+5t+1+\frac{1}{3}e^{(-3t)}-\frac{4}{3}e^{(3t)} \\ -e^{(-3t)}C_{1,1}+e^{(3t)}C_{2,1}+2t+\frac{1}{3}e^{(-3t)}-\frac{1}{3}e^{(3t)} \\ e^{(-3t)}C_{1,1}+e^{(3t)}C_{3,1}+4t+2-\frac{1}{3}e^{(-3t)}-\frac{5}{3}e^{(3t)} \end{bmatrix}$$

```
>
```

This solution is not the same as the one found in Nagle and Saff.

```
> map( simplify,
       subs( C[1,1]=1/3+c[3], C[2,1]=5/3+c[1], C[3,1]=1/3+c[2],
       evalm(x) ) );
```

$$\begin{bmatrix} \frac{1}{3}\left(-3e^{(-6t)}c_3-8-3c_1+3c_2+15te^{(-3t)}+3e^{(-3t)}\right)e^{(3t)} \\ \frac{1}{3}\left(-3e^{(-6t)}c_3+4+3c_1+6te^{(-3t)}\right)e^{(3t)} \\ \frac{1}{3}\left(3e^{(-6t)}c_3-4+3c_2+12te^{(-3t)}+6e^{(-3t)}\right)e^{(3t)} \end{bmatrix}$$

Note that $X(0) \neq I$. A fundamental solution with $X(0)=I$ is

```
> X1 := evalm( X(t) &* inverse( X(0) ) );
```

$$X1 := \begin{bmatrix} \frac{1}{3}e^{(-3t)}+\frac{2}{3}e^{(3t)} & \frac{1}{3}e^{(-3t)}-\frac{1}{3}e^{(3t)} & \%1 \\ \frac{1}{3}e^{(-3t)}-\frac{1}{3}e^{(3t)} & \frac{1}{3}e^{(-3t)}+\frac{2}{3}e^{(3t)} & \%1 \\ \%1 & \%1 & \frac{1}{3}e^{(-3t)}+\frac{2}{3}e^{(3t)} \end{bmatrix}$$

$$\%1 := -\frac{1}{3}e^{(-3t)}+\frac{1}{3}e^{(3t)}$$

There are two alternate methods that should lead to the same fundamental matrix. The matrix exponential, introduced in Section 9.7, produce

```
> X2 := exponential( A*t );
```

$$X2 := \begin{bmatrix} \frac{1}{3}e^{(-3\,t)}+\frac{2}{3}e^{(3\,t)} & \frac{1}{3}e^{(-3\,t)}-\frac{1}{3}e^{(3\,t)} & \%1 \\ \frac{1}{3}e^{(-3\,t)}-\frac{1}{3}e^{(3\,t)} & \frac{1}{3}e^{(-3\,t)}+\frac{2}{3}e^{(3\,t)} & \%1 \\ \%1 & \%1 & \frac{1}{3}e^{(-3\,t)}+\frac{2}{3}e^{(3\,t)} \end{bmatrix}$$

$$\%1 := -\frac{1}{3}e^{(-3\,t)}+\frac{1}{3}e^{(3\,t)}$$

Similarly, the Laplace transform yields an altogether different approach to the problem of finding a fundamental matrix. If $\frac{\partial}{\partial t}X(t)=AX(t)$, then $s\,XX(s)-X(0)=A\,XX(s)$ and so $XX(s)=-(A-sI)^{\{-1\}}\,X(0)$. The fundamental matrix is obtained by applying the inverse Laplace transform componentwise:

```
> invCHARm := inverse( CHARm );
```

$$invCHARm := \begin{bmatrix} -\frac{\lambda+1}{\lambda^2-9} & \frac{2}{\lambda^2-9} & -\frac{2}{\lambda^2-9} \\ \frac{2}{\lambda^2-9} & -\frac{\lambda+1}{\lambda^2-9} & -\frac{2}{\lambda^2-9} \\ -\frac{2}{\lambda^2-9} & -\frac{2}{\lambda^2-9} & -\frac{\lambda+1}{\lambda^2-9} \end{bmatrix}$$

The inverse Laplace transform can be determined by inspection once the negative of the inverse of the characteristic matrix is written with the partial fraction decomposition

```
> map( convert, evalm(-invCHARm), parfrac, lambda);
```

$$\begin{bmatrix} \frac{2}{3}\frac{1}{\lambda-3}+\frac{1}{3}\frac{1}{\lambda+3} & -\frac{1}{3}\frac{1}{\lambda-3}+\frac{1}{3}\frac{1}{\lambda+3} & \%1 \\ -\frac{1}{3}\frac{1}{\lambda-3}+\frac{1}{3}\frac{1}{\lambda+3} & \frac{2}{3}\frac{1}{\lambda-3}+\frac{1}{3}\frac{1}{\lambda+3} & \%1 \\ \%1 & \%1 & \frac{2}{3}\frac{1}{\lambda-3}+\frac{1}{3}\frac{1}{\lambda+3} \end{bmatrix}$$

$$\%1 := \frac{1}{3}\frac{1}{\lambda-3}-\frac{1}{3}\frac{1}{\lambda+3}$$

or, letting Maple do the work for us,

```
> with( inttrans ):
Warning, new definition for hilbert
```

```
> X3 := map( -invlaplace, evalm(-invCHARm), lambda, t );
```

$$X3 := \begin{bmatrix} -\frac{1}{3}e^{(-3t)} - \frac{2}{3}e^{(3t)} & -\frac{1}{3}e^{(-3t)} + \frac{1}{3}e^{(3t)} & \%1 \\ -\frac{1}{3}e^{(-3t)} + \frac{1}{3}e^{(3t)} & -\frac{1}{3}e^{(-3t)} - \frac{2}{3}e^{(3t)} & \%1 \\ \%1 & \%1 & -\frac{1}{3}e^{(-3t)} - \frac{2}{3}e^{(3t)} \end{bmatrix}$$

$$\%1 := \frac{1}{3}e^{(-3t)} - \frac{1}{3}e^{(3t)}$$

```
>
```

It is a simple matter to visually verify that X1 = X2 = X3. In general, any of the three methods will lead to the equivalent general solutions for the nonhomogeneous problem.

If A has an eigenvalue which does not have a full set of eigenvectors, then the explicit computation of a fundamental matrix requires special handling. Likewise, when A has complex eigenvalues (and eigenvectors), real valued solutions are obtained only after careful handling of the real and imaginary parts of the eigenvalues and eigenvectors. In either case, the matrix exponential, as computed by exponential, and the Laplace transform avoid this complication.

```
>
>
```