Maple Worksheet for Math 726 -- Singular Value Decomposition

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WWW URL: http://www.math.sc.edu/~meade/math726/maple/svd.ms

This worksheet provides an indication of some of Maple's abilities in linear algebra. Note that the symbolic computations are minimal; moreover, the manipulation of matrices is less intuitive than in Matlab. The ability to compute floating point numbers with extended precision is the feature of Maple that is used most in this worksheet.

All examples are to be worked using both Maple and Matlab (you may want to work in pairs). You should find that the efficiency, convenience, and accuracy vary greatly. Sometimes Maple is the better tool to use, other times Matlab.
The real problem is to know which tool is better at the outset.

The first step is to load all of the linear algebra routines from the linalg package. Other packages that are frequently used include the plots package (for a variety of extra plotting commands) and the DEtools package (for the graphical manipulation of ordinary differential equations).

```
> restart;
> with( linalg );
Warning, new definition for norm
Warning, new definition for trace
```

[*BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert,* *htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stack, submatrix, subvector sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian*]

All successful users of Maple (or Matlab) know how to access the on-line help.

Maple's on-line help can be accessed via the help command (abbreviated "?") or the Help Browser or Keyword Search. The later two can be accessed via the pull-down menu in the upper right corner of the Maple window.

In Matlab, the help command is used to access the on-line help.

>

Problem #1:

1. Find the Maple and Matlab commands for the singular value decomposition of a matrix.
2. Use these commands to compute the singular value decomposition of the 5x5 Hilbert matrix, *H*.
3. What is the rank of *H*?
4. What is the rank of the 10x10 Hilbert matrix? (How can you be sure/confident of your answer?)

```
> n := 10:
> H := hilbert( n );
```

$$H := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} \\ \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} \end{bmatrix}$$

This returns the singular values of H in a list (in descending order)

```
> sigma := evalf( Svd(H,U,V) );
```

$$\sigma := [1.751919670, .3429295485, .03574181628, .002530890753, .0001287496197, .4729668244\ 10^{-5}, .1229107112\ 10^{-6}, .2145213116\ 10^{-8}, .2184353478\ 10^{-10}, .5564185164\ 10^{-13}]$$

For further processing we might like to have the diagonal matrix Σ. Here is one way to create this matrix

```
> Sigma := diag( op(convert(sigma,list)) );
```

$$
\Sigma := \begin{bmatrix}
1.751919670, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, .3429295485, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, .03574181628, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, .002530890753, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, .0001287496197, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, .4729668244\ 10^{-5}, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, .1229107112\ 10^{-6}, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, .2145213116\ 10^{-8}, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, .2184353478\ 10^{-10}, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, .5564185164\ 10^{-13}
\end{bmatrix}
$$

The left and right singular vectors can also be viewed

```
> print( U );
```

[-.6995148912 , .6376408418 , .3034042060 , -.1055157957 , .02931267473 , -.006674849842 , .001247496520 , -.0001876430535 , .00002169408441 , $.2087489714\ 10^{-5}$]

[-.4259989127 , -.07043710514 , -.5995601506 , .5762717855 , -.3228791194 , .1275058055 , -.03768971584 , .008420004250 , -.001378118676 , -.0001704915398]

[-.3169769879 , -.2335566050 , -.3785715057 , -.2019336815 , .5831140063 , -.4973913682 , .2551673303 , -.08884006813 , .02116890145 , .003484357036]

[-.2555230059 , -.2821088346 , -.1443136744 , -.4120435361 , .1695528942 , .4109721318 , -.5665223095 , .3567726349 , -.1330192861 , -.03068613512]

[-.2152778395 , -.2936571654 , .02749695193 , -.3513222149 , -.2294932714 , .3632922608 , .2251841686 , -.5704938871 , .4025536017 , .1427183394]

[-.1865782383 , -.2909818234 , .1469165783 , -.1954329129 , -.3748368284 , -.04873222050 , .4263382973 , .1338756394 , -.5748570366 , -.3843364603]

[-.1649465263 , -.2824979910 , .2291388187 , -.01939900507 , -.3093620618 , -.3432164355 , .02934863201 , .4471106995 , .2039667263 , .6198648698]

[-.1479921432 , -.2717442367 , .2856315775 , .1472123988 , -.1121436073 ,

```
-.3529306242 , -.3719228973 , -.06168789064 , .4161112431 , -.5904063759 ]
[ -.1343104464 , -.2603237653 , .3242640024 , .2942089253 , .1542113423 ,
-.07576771072 , -.3264338078 , -.4947524471 , -.4979055808 , .3061296130 ]
[ -.1230167128 , -.2489884105 , .3503612268 , .4195141631 , .4482049660 ,
.4300547181 , .3664706910 , .2699453245 , .1633550524 , -.06660080165 ]
> print( V );
[ -.6995148912 , .6376408418 , .3034042060 , -.1055157957 , .02931267473 ,
-.006674849842 , .001247496520 , -.0001876430535 , .00002169408368 ,
.2087491007 10^-5 ]
[ -.4259989127 , -.07043710514 , -.5995601506 , .5762717855 , -.3228791194 ,
.1275058055 , -.03768971584 , .008420004252 , -.001378118641 ,
-.0001704916198 ]
[ -.3169769879 , -.2335566050 , -.3785715057 , -.2019336815 , .5831140063 ,
-.4973913682 , .2551673303 , -.08884006816 , .02116890101 , .003484358242 ]
[ -.2555230059 , -.2821088346 , -.1443136744 , -.4120435361 , .1695528942 ,
.4109721318 , -.5665223096 , .3567726352 , -.1330192835 , -.03068614259 ]
[ -.2152778395 , -.2936571654 , .02749695193 , -.3513222149 , -.2294932714 ,
.3632922608 , .2251841686 , -.5704938880 , .4025535924 , .1427183617 ]
[ -.1865782383 , -.2909818234 , .1469165783 , -.1954329129 , -.3748368284 ,
-.04873222050 , .4263382972 , .1338756407 , -.5748570153 , -.3843364918 ]
[ -.1649465263 , -.2824979910 , .2291388187 , -.01939900507 , -.3093620618 ,
-.3432164355 , .02934863200 , .4471106993 , .2039666936 , .6198648807 ]
[ -.1479921432 , -.2717442367 , .2856315775 , .1472123988 , -.1121436073 ,
-.3529306242 , -.3719228973 , -.06168789204 , .4161112753 , -.5904063531 ]
[ -.1343104464 , -.2603237653 , .3242640024 , .2942089253 , .1542113423 ,
-.07576771072 , -.3264338078 , -.4947524457 , -.4979055988 , .3061295860 ]
[ -.1230167128 , -.2489884105 , .3503612268 , .4195141631 , .4482049660 ,
.4300547181 , .3664706910 , .2699453240 , .1633550567 , -.06660079286 ]
>
```

Problem #2:

Complete each of the following using both Maple and Matlab

1. Construct the rank four matrix that corresponds to the largest four singular values of the 10x10 Hilbert matrix.
2. What is the rank of the matrix just computed, according to Maple and Matlab? (Explain.)
2. Find the 1-, 2-, infinity- and Frobenius norms of the 5x5 Hilbert matrix.

```
> h1 := evalm( sigma[1]*col(U,1)&*transpose(col(V,1)) ):
> h2 := evalm( sigma[2]*col(U,2)&*transpose(col(V,2)) ):
> h3 := evalm( sigma[3]*col(U,3)&*transpose(col(V,3)) ):
> h4 := evalm( sigma[4]*col(U,4)&*transpose(col(V,4)) ):
> H4 := evalm( h1+h2+h3+h4 );
```

H4 :=

[.9999998890 , .5000012224 , .3333311169 , .2499993730 , .2000008775 , .1666680798 , .1428582996 , .1250004120 , .1111105267 , .09999832188]

[.5000012224 , .3333198339 , .2500245414 , .2000067978 , .1666569085 , .1428415921 , .1249873467 , .1111066603 , .1000064549 , .09092746535]

[.3333311169 , .2500245414 , .1999550443 , .1666549221 , .1428752198 , .1250280132 , .1111335284 , .1000076007 , .09089734529 , .08330068425]

[.2499993730 , .2000067978 , .1666549221 , .1428526030 , .1250043197 , .1111194181 , .1000074222 , .09091219911 , .08333009185 , .07691248203]

[.2000008775 , .1666569085 , .1428752198 , .1250043197 , .1111036991 , .09998899672 , .09090053957 , .08333063641 , .07692777205 , .07144106583]

[.1666680798 , .1428415921 , .1250280132 , .1111194181 , .09998899672 , .09089096762 , .08331832269 , .07691760301 , .07143601346 , .06668837693]

[.1428582996 , .1249873467 , .1111335284 , .1000074222 , .09090053957 , .08331832269 , .07691019725 , .07142353318 , .06667268760 , .06251854862]

[.1250004120 , .1111066603 , .1000076007 , .09091219911 , .08333063641 , .07691760301 , .07142353318 , .06666444132 , .06250208512 , .05883073544]

[.1111105267 , .1000064549 , .09089734529 , .08333009185 , .07692777205 ,

.07143601346 , .06667268760 , .06250208512 , .05882042683 , .05554682571]

[.09999832188 , .09092746535 , .08330068425 , .07691248203 , .07144106583 , .06668837693 , .06251854862 , .05883073544 , .05554682571 , .05260482332]

```
> U4 := submatrix(U,1..n,1..4):
  S4 := submatrix(Sigma,1..4,1..4):
  V4 := submatrix(V,1..n,1..4):
> H4 := evalm( U4 &* S4 &* transpose( V4) );
```

H4 :=

[.9999998887 , .5000012222 , .3333311167 , .2499993729 , .2000008774 , .1666680796 , .1428582995 , .1250004120 , .1111105267 , .09999832188]

[.5000012224 , .3333198339 , .2500245415 , .2000067978 , .1666569085 , .1428415921 , .1249873467 , .1111066603 , .1000064549 , .09092746535]

[.3333311167 , .2500245414 , .1999550442 , .1666549221 , .1428752197 , .1250280132 , .1111335284 , .1000076007 , .09089734529 , .08330068425]

[.2499993730 , .2000067978 , .1666549221 , .1428526030 , .1250043198 , .1111194181 , .1000074222 , .09091219912 , .08333009185 , .07691248204]

[.2000008774 , .1666569084 , .1428752196 , .1250043197 , .1111036991 , .09998899670 , .09090053955 , .08333063639 , .07692777204 , .07144106582]

[.1666680797 , .1428415921 , .1250280132 , .1111194181 , .09998899673 , .09089096762 , .08331832270 , .07691760301 , .07143601348 , .06668837693]

[.1428582996 , .1249873467 , .1111335284 , .1000074222 , .09090053957 , .08331832269 , .07691019724 , .07142353318 , .06667268760 , .06251854862]

[.1250004121 , .1111066603 , .1000076007 , .09091219912 , .08333063642 , .07691760301 , .07142353319 , .06666444133 , .06250208511 , .05883073544]

[.1111105267 , .1000064549 , .09089734529 , .08333009184 , .07692777205 , .07143601346 , .06667268760 , .06250208511 , .05882042683 , .05554682571]

[.09999832188 , .09092746535 , .08330068426 , .07691248202 , .07144106584 , .06668837692 , .06251854862 , .05883073543 , .05554682571 , .05260482332]

```
>
> rank(H4);
```

10

```
> Nh4inf := norm(H4,‘infinity‘);
```

$$Nh4inf := 2.928968119$$

```
> Nh42 := norm(H4,2);
```

$$Nh42 := 1.751919669$$

```
> Nhinf := norm(H,‘infinity‘);
```

$$Nhinf := \frac{7381}{2520}$$

```
> Nh4inf - Nhinf;
```

$$-.135\ 10^{-6}$$

```
> Nh2 := norm(H,2);
```

$$\begin{aligned}
&Nh2 := \max\Big(\frac{1}{100}\Big|\mathrm{RootOf}(2135077048313835328102464569534154759766272826\backslash\\
&\quad 25109487193632424272672659639135098634 24\ _Z^{10} - 680685430899431405111\backslash\\
&\quad 5578378314726151740235030431930578449614771452574092533331069501440\\
&\quad _Z^{9} + 77937616530354812941996151698017277282842020856319781849727154\backslash\\
&\quad 4375390370513697429153382 40\ _Z^{8} - 98948053426298970155275260383 08024\backslash\\
&\quad 5475869117284119869322970539536969664506775043938713 60\ _Z^{7} + 6322392\backslash\\
&\quad 341627254714661142124723676543355404888448753972703987422103727 8167\backslash\\
&\quad 43695482880\ _Z^{6} - 10467217832189191770315032908463380806748444422724\backslash\\
&\quad 83740757083928776526263545036 8\ _Z^{5} + 23399279901749170284229545051 95\backslash\\
&\quad 807870730626668950179770364308501582003 20\ _Z^{4}\\
&\quad - 3532828299357813620817270561821933722439113252346974282656 0\ _Z^{3}\\
&\quad + 162884532929181733297498175763535726679561 85\ _Z^{2}\\
&\quad - 836849861238523073945290\ _Z + 1)\Big|\Big)^{1/2}
\end{aligned}$$

```
> Nh2 := max( evalf( allvalues( Nh2 ) ) );
```

$$Nh2 := 1.751919670$$

```
> Nh42-Nh2;
```

$$-.1\ 10^{-8}$$

```
>
```

```
>
```

Problem #3:

One use of the singular value decomposition is the computation of the inverse of a matrix.

1. Given the SVD of a non-singular matrix A, what is the inverse of A?
2. Use this result to compute the inverse of the 5x5 and 10x10 Hilbert matrices.
3. What is the largest Hilbert matrix for which a reasonable floating-point inverse can be computed?

Compare these last results with the exact inverse of the Hilbert matrix. What does this tell you about Hilbert matrices and floating point arithmetic?

```
> n := 10:
> H := hilbert( n ):
> sigma := evalf( Svd(H,U,V) ):
> Sigma := diag( op(convert(sigma,list)) );
```

$$\Sigma := \begin{bmatrix} 1.751919670, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, .3429295485, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, .03574181628, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, .002530890753, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, .0001287496197, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, .4729668244\ 10^{-5}, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, .1229107112\ 10^{-6}, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, .2145213116\ 10^{-8}, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, .2184353478\ 10^{-10}, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, .5564185164\ 10^{-13} \end{bmatrix}$$

```
> iSigma := inverse( Sigma );
```

$$iSigma := \begin{bmatrix} .5708024273, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 2.916050846, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 27.97843266, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 395.1178054, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 7767.013233, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 211431.3200, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & .8135987419\ 10^{7}, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & .4661541516\ 10^{9}, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & .4578013632\ 10^{11}, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & .1797208343\ 10^{14} \end{bmatrix}$$

```
> Hinv := inverse( hilbert(n) );
```

Hinv :=

[100 , -4950 , 79200 , -600600 , 2522520 , -6306300 , 9609600 , -8751600 , 4375800 , -923780]

[-4950 , 326700 , -5880600 , 47567520 , -208107900 , 535134600 , -832431600 , 770140800 , -389883780 , 83140200]

[79200 , -5880600 , 112907520 , -951350400 , 4281076800 , -11237826600 , 17758540800 , -16635041280 , 8506555200 , -1829084400]

[-600600 , 47567520 , -951350400 , 8245036800 , -37875637800 , 101001700800 , -161602721280 , 152907955200 , -78843164400 , 17071454400]

[2522520 , -208107900 , 4281076800 , -37875637800 , 176752976400 , -477233036280 , 771285715200 , -735869534400 , 382086104400 , -83223340200]

[-6306300 , 535134600 , -11237826600 , 101001700800 , -477233036280 , 1301544644400 , -2121035716800 , 2037792556800 , -1064382719400 , 233025352560]

[9609600 , -832431600 , 17758540800 , -161602721280 , 771285715200 , -2121035716800 , 3480673996800 , -3363975014400 , 1766086882560 , -388375587600]

[-8751600 , 770140800 , -16635041280 , 152907955200 , -735869534400 , 2037792556800 , -3363975014400 , 3267861442560 , -1723286307600 , 380449555200]

[4375800, -389883780, 8506555200, -78843164400, 382086104400, -1064382719400, 1766086882560, -1723286307600, 912328045200, -202113826200]

[-923780, 83140200, -1829084400, 17071454400, -83223340200, 233025352560, -388375587600, 380449555200, -202113826200, 44914183600]

```
> iH := evalm( U &* iSigma &* transpose(V) );
```

$iH :=$

$[153.4691946, -9166.515949, 162945.4836, -.1320830015\ 10^{7}, .5805734840\ 10^{7}, -.1499728967\ 10^{8}, .2341936142\ 10^{8}, -.2173464946\ 10^{8}, .1103051885\ 10^{8}, -.2356806408\ 10^{7}]$

$[-9166.516897, 658341.9724, -.1245379181\ 10^{8}, .1040019172\ 10^{9}, -.4649968983\ 10^{9}, .1214303205\ 10^{10}, -.1910451571\ 10^{10}, .1782667191\ 10^{10}, -.9084391337\ 10^{9}, .1947222807\ 10^{9}]$

$[162945.5191, -.1245379313\ 10^{8}, .2429734971\ 10^{9}, -.2066506334\ 10^{10}, .9351363734\ 10^{10}, -.2462935351\ 10^{11}, .3899590156\ 10^{11}, -.3656684904\ 10^{11}, .1870748070\ 10^{11}, -.4022770445\ 10^{10}]$

$[-.1320830446\ 10^{7}, .1040019384\ 10^{9}, -.2066506531\ 10^{10}, .1779523894\ 10^{11}, -.8125558176\ 10^{11}, .2154801529\ 10^{12}, -.3430195165\ 10^{12}, .3230628875\ 10^{12}, -.1658772809\ 10^{12}, .3577839302\ 10^{11}]$

$[.5805737315\ 10^{7}, -.4649970353\ 10^{9}, .9351365433\ 10^{10}, -.8125558865\ 10^{11}, .3736356776\ 10^{12}, -.9964311508\ 10^{12}, .1593559904\ 10^{13}, -.1506676162\ 10^{13}, .7761609150\ 10^{12}, -.1678880524\ 10^{12}]$

$[-.1499729742\ 10^{8}, .1214303662\ 10^{10}, -.2462935988\ 10^{11}, .2154801873\ 10^{12}, -.9964312242\ 10^{12}, .2669876167\ 10^{13}, -.4286949172\ 10^{13}, .4067173752\ 10^{13}, -.2101465743\ 10^{13}, .4557525414\ 10^{12}]$

$[.2341937544\ 10^{8}, -.1910452432\ 10^{10}, .3899591436\ 10^{11}, -.3430195940\ 10^{12}, .1593560125\ 10^{13}, -.4286949451\ 10^{13}, .6907455680\ 10^{13}, -.6573409574\ 10^{13},$

$.3405611703\ 10^{13}$, $-.7403686930\ 10^{12}$]

[$-.2173466408\ 10^{8}$, $.1782668113\ 10^{10}$, $-.3656686332\ 10^{11}$, $.3230629797\ 10^{12}$, $-.1506676458\ 10^{13}$, $.4067174247\ 10^{13}$, $-.6573409945\ 10^{13}$, $.6272632699\ 10^{13}$, $-.3257759503\ 10^{13}$, $.7097928168\ 10^{12}$]

[$.1103052701\ 10^{8}$, $-.9084396585\ 10^{9}$, $.1870748905\ 10^{11}$, $-.1658773371\ 10^{12}$, $.7761611073\ 10^{12}$, $-.2101466103\ 10^{13}$, $.3405612063\ 10^{13}$, $-.3257759663\ 10^{13}$, $.1695724086\ 10^{13}$, $-.3702101555\ 10^{12}$]

[$-.2356808296\ 10^{7}$, $.1947224039\ 10^{9}$, $-.4022772448\ 10^{10}$, $.3577840686\ 10^{11}$, $-.1678881017\ 10^{12}$, $.4557526402\ 10^{12}$, $-.7403688041\ 10^{12}$, $.7097928827\ 10^{12}$, $-.3702101716\ 10^{12}$, $.8097490238\ 10^{11}$]

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