```
%%%
```

New Maple Commands:

laplace	apply the Laplace transform
invlaplace	apply the inverse Laplace transform
convert(..., parfrac, s)	compute the partial fraction decomposition of an expression
completesquare	complete the square of a polynomial of degree 2

The full implementation of the Laplace transform is available only after the inttrans library is loaded. Likewise, the student package is needed if it is desired to complete the square on an expression.

```
> with( inttrans );
```

[addtable, fourier, fouriercos, fouriersin, hankel, hilbert, invfourier, invhilbert, invlaplace, laplace, mellin]

```
> with( student );
```

[D, Diff, Doubleint, Int, Limit, Lineint, Product, Sum, Tripleint, changevar, combine, completesquare, distance, equate, extrema, integrand, intercept, intparts, isolate, leftbox, leftsum, makeproc, maximize, middlebox, middlesum, midpoint, minimize, powsubs, rightbox, rightsum, showtangent, simpson, slope, trapezoid, value]

```
>
```

INTRODUCTION TO LAPLACE TRANSFORMS

The Laplace transform is defined in terms of an improper integral. For example,

```
> int( 1*exp(-s*t), t=0..infinity );
Definite integration: Can't determine if the integral is convergent.
Need to know the sign of --> s
Will now try indefinite integration and then take limits.
```

$$\lim_{t \to \infty} -\frac{e^{(-s\,t)}}{s} + \frac{1}{s}$$

```
> int( exp(a*t)*exp(-s*t), t=0..infinity );
Definite integration: Can't determine if the integral is convergent.
Need to know the sign of --> -a+s
Will now try indefinite integration and then take limits.
```

$$\lim_{t \to \infty} \frac{e^{(a\,t - s\,t)}}{a - s} - \frac{1}{a - s}$$

```
> int( cos(a*t)*exp(-s*t), t=0..infinity );
```

$$\lim_{t \to \infty} \; -\frac{s\, e^{(-s\,t)}\, \cos(a\,t)}{s^2 + a^2} + \frac{a\, e^{(-s\,t)}\, \sin(a\,t)}{s^2 + a^2} + \frac{s}{s^2 + a^2}$$

>

Notice that these are not the expected results. In each case, the anticipated result is present, but so is a limit. This limit arises, of course, because the integral is improper.

Questions
- What do we know about s that Maple does not know?
- How does this knowledge allow us to conclude that each limit is zero?

The extra information, "*Need to know the sign of --> ...*" directs our attention to the appropriate part of the expression. (Note that this information is not provided in Release 3.)

The laplace command does know the necessary information about the transform parameter. Here are the same problems using laplace:

```
> laplace( 1, t, s );
```

$$\frac{1}{s}$$

```
> laplace( exp(a*t), t, s );
```

$$\frac{1}{-a + s}$$

```
> laplace( cos(a*t), t, s );
```

$$\frac{s}{s^2 + a^2}$$

>

Inverse Laplace transforms work similarly, except that there is no explicit formula. Regardless, it is still quite beneficial to use Maple to convert an expression into a form that can be inverted using the table of Laplace Transforms (inside back cover of Nagle and Saff). The convert(..., parfrac, s) and completesquare commands are particularly helpful.

For example, the function

```
> F1 := (7*s-1)/(s^3-7*s-6);
```

$$F1 := \frac{7\,s - 1}{s^3 - 7\,s - 6}$$

obviously has at least one real root. The partial fraction decomposition is

```
> convert( F1, parfrac, s );
```

$$\frac{1}{s-3} - \frac{3}{s+2} + \frac{2}{s+1}$$

The inverse Laplace transform is now easy to identify. Check your answer with the answer provided by Maple:

```
> f1 := invlaplace( F1, s, t );
```

$$f1 := e^{(3\,t)} - 3\,e^{(-2\,t)} + 2\,e^{(-t)}$$

```
>
```

The next example

```
> F2 := (s-1)/(s^2-2*s+5);
```

$$F2 := \frac{s-1}{s^2 - 2\,s + 5}$$

involves an irreducible quadratic term, which should be rewritten in a form with a complete square. This is achieved by using the completesqare command:

```
> completesquare( F2 );
```

$$\frac{s-1}{(s-1)^2 + 4}$$

The presence of (s-1) says that the inverse Laplace transform includes e^t. The inverse Laplace transform of s/(s^2+4) is cos(2t); thus, the inverse Laplace transform of F must be f(t) = e^t cos(2t). As before, this answer is easily checked using Maple

```
> f2 := invlaplace( F2, s, t );
```

$$f2 := e^{t}\,\cos(2\,t)$$

```
>
```

The third example looks innocuous enough, but the exponential term does not appear in any of the transforms first presented. Even though you may not yet possess enough knowledge to determine the inverse Laplace transform of this function, we can still ask Maple for an answer.

```
> F3 := exp(-2*s)/s^3;
```

$$F3 := \frac{e^{(-2\,s)}}{s^3}$$

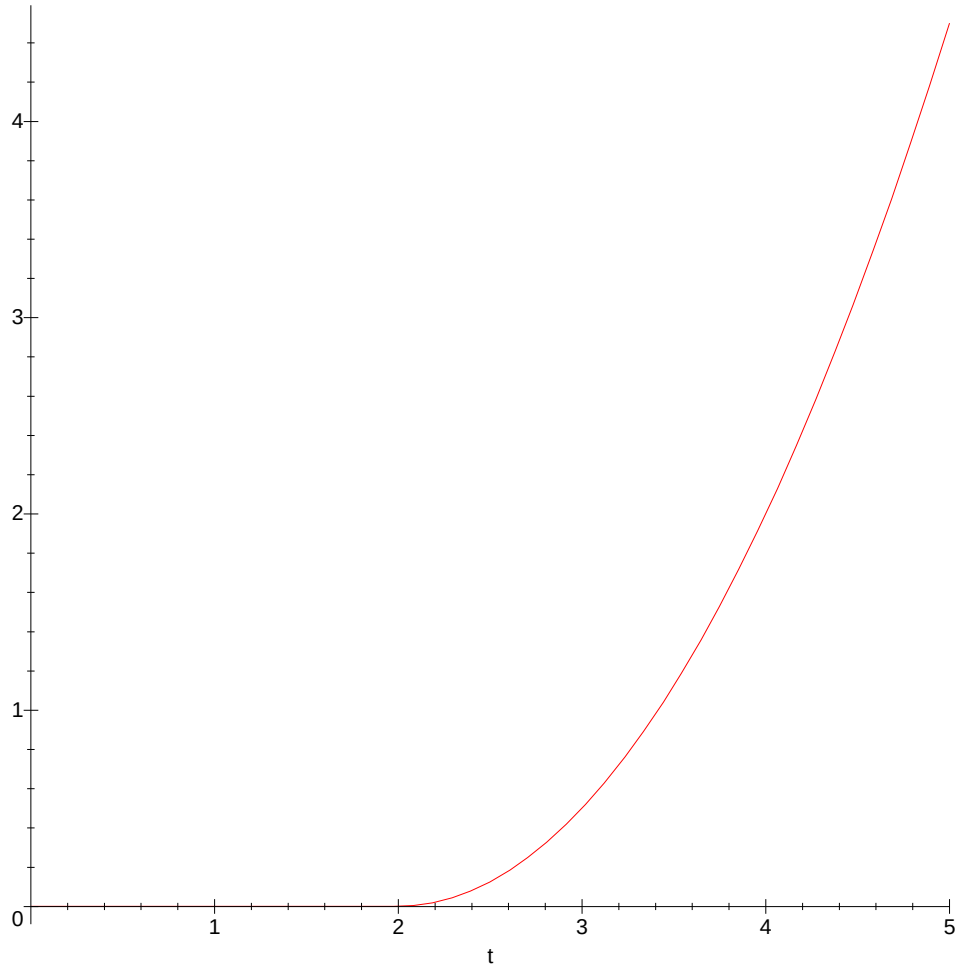
```
> f3 := invlaplace( F3, s, t );
```

$$f3 := \frac{1}{2}\,Heaviside(t-2)\,t^2 - 2\,Heaviside(t-2)\,t + 2\,Heaviside(t-2)$$

What is this function Heaviside? One of the easiest ways to learn about Heaviside functions is to plot this function, and attempt to understand how it relates to this function.

```
> plot( f3, t=0..5,
        title='Function with Heaviside' );
```

Function with Heaviside



>

Notice that the solution appears to be identically zero for t<2, and is quadratic with a vertex at t=2 for t>2. This suggests that the Heaviside function is zero when its argument is negative and is identically one when the argument is positive. (How might you use Maple to test this conjecture?)

Update for Release 4

A new feature in Release 4 is the ability to convert between Heaviside functions and piecewise definitions of functions (see ?convert,piecewise). In this case, we find

```
> convert( f3, piecewise );
```

$$\begin{cases} 0 & t \le 2 \\ \frac{1}{2} t^2 - 2\,t + 2 & 2 < t \end{cases}$$

Thus, the inverse Laplace transform of F3 is identically zero for t<=2; for t>2, the inverse is quadratic (not unexpected from the cubic numerator in F3).

>

LAPLACE TRANSFORMS FOR INITIAL VALUE PROBLEMS

Example 1

These are the problems that are of most interest. To illustrate the general procedure, consider the problem of solving the initial value problem

```
> ODE := diff( y(t), t$2 ) - 3*diff( y(t), t ) + 2*y(t)
         = cos(t) + (t^2-3)*exp(-2*t);
```

$$ODE := \left(\frac{\partial^2}{\partial t^2} y(t)\right) - 3\left(\frac{\partial}{\partial t} y(t)\right) + 2\,y(t) = \cos(t) + (t^2-3)\,e^{(-2\,t)}$$

```
> IC := y(0) = 0, D(y)(0) = 1;
```

$$IC := y(0) = 0, D(y)(0) = 1$$

using Laplace transforms.

The first step is to apply the Laplace transform to the entire equation

```
> ODE_lap := laplace( ODE, t, s );
```

$$ODE_lap :=$$

$$s\,(s\,laplace(y(t), t, s) - y(0)) - D(y)(0) - 3\,s\,laplace(y(t), t, s) + 3\,y(0) + 2\,laplace(y(t), t, s)$$

$$= \frac{s}{s^2+1} + \frac{2}{(s+2)^3} - \frac{3}{s+2}$$

Note the way in which the Laplace transform of the solution is denoted in this equation. Since specific initial conditions are known, these should be introduced at the earliest convenience. For example:

```
> ODE_lap2 := subs( IC, ODE_lap );
```

$$ODE_lap2 :=$$

$$s^2\,laplace(y(t), t, s) - 1 - 3\,s\,laplace(y(t), t, s) + 2\,laplace(y(t), t, s) = \frac{s}{s^2+1} + \frac{2}{(s+2)^3} - \frac{3}{s+2}$$

Next, it is necessary to explicitly solve for the Laplace transform of the solution:

```
> SOLN_lap := solve( ODE_lap2, { laplace( y(t), t, s ) } );
```

$$SOLN_lap := \{laplace(y(t), t, s) = \frac{-2 + 8\,s + 13\,s^2 + s^5 + 4\,s^4 + 7\,s^3}{16 + s^7 - 3\,s^5 - 13\,s^4 - 4\,s^3 + 3\,s^6}\}$$

The solution can be obtained in one step using invlaplace, but the solution is also available by inspection once the right-hand side is written in its partial fraction decomposition.

```
> SOLN_lap2 := convert( SOLN_lap, parfrac, s );
```

$$SOLN_lap2 :=$$

$$\{laplace(y(t), t, s) = -\frac{31}{54}\frac{1}{s-1} + \frac{109}{160}\frac{1}{s-2} + \frac{1}{6}\frac{1}{(s+2)^3} + \frac{7}{72}\frac{1}{(s+2)^2} - \frac{179}{864}\frac{1}{s+2} + \frac{1}{10}\frac{s-3}{s^2+1}\}$$

```
> SOLN := invlaplace( SOLN_lap2, s, t );
```

$$SOLN := \{y(t) = -\frac{31}{54}e^{t} + \frac{109}{160}e^{(2t)} + \frac{1}{12}e^{(-2t)}t^2 + \frac{7}{72}t\,e^{(-2t)} - \frac{179}{864}e^{(-2t)} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)\}$$

Compare this solution with the solutions obtained by directly inverting the Laplace transform once it is available

```
> invlaplace( SOLN_lap, s, t );
```

$$\{y(t) = -\frac{31}{54}e^{t} + \frac{109}{160}e^{(2t)} + \frac{1}{12}e^{(-2t)}t^2 + \frac{7}{72}t\,e^{(-2t)} - \frac{179}{864}e^{(-2t)} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)\}$$

or the solution obtained by explicitly telling Maple to use Laplace transforms to solve the problem.

```
> dsolve( { ODE, IC }, y(t), method=laplace );
```

$$y(t) = -\frac{31}{54}e^{t} + \frac{109}{160}e^{(2t)} + \frac{1}{12}e^{(-2t)}t^2 + \frac{7}{72}t\,e^{(-2t)} - \frac{179}{864}e^{(-2t)} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)$$

```
>
```

Example 2

For a second example consider the problem of finding the general solution to

```
> ODE := diff( y(t), t$2 )  + 2*y(t) = cos(2*t);
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}y(t)\right) + 2\,y(t) = \cos(2\,t)$$

The Laplace transform of this equation is

```
> ODE_lap := laplace( ODE, t, s );
```

$$ODE_lap := s\,(s\,laplace(y(t), t, s) - y(0)) - D(y)(0) + 2\,laplace(y(t), t, s) = \frac{s}{s^2+4}$$

```
> SOLN_lap := solve( ODE_lap, { laplace(y(t),t,s) } );
```

$$SOLN_lap := \{laplace(y(t), t, s) = \frac{y(0)\,s^3 + 4\,s\,y(0) + D(y)(0)\,s^2 + 4\,D(y)(0) + s}{s^4 + 6\,s^2 + 8}\}$$

```
> invlaplace( SOLN_lap, s, t );
```

$$\{y(t) = -\frac{1}{2}\cos(2\,t) + \frac{1}{2}\cos(\sqrt{2}\,t) + y(0)\cos(\sqrt{2}\,t) + \frac{1}{2}\sqrt{2}\,D(y)(0)\sin(\sqrt{2}\,t)\}$$

Note the explicit use of the initial conditions in this solution.

As before, there are several alternate methods to reach the same solution, including

```
> dsolve( ODE, y(t), method=laplace );
```

$$y(t) = -\frac{1}{2}\cos(2\,t) + \frac{1}{2}\cos(\sqrt{2}\,t) + y(0)\cos(\sqrt{2}\,t) + \frac{1}{2}\sqrt{2}\,D(y)(0)\sin(\sqrt{2}\,t)$$

It is particularly noteworthy to compare this solution with the one returned by dsolve when the solution method is not explicitly specified:

```
> dsolve( ODE, y(t) );
```

$$y(t) = -\frac{1}{4}\cos(\sqrt{2}\,t)\cos((2+\sqrt{2})\,t) + \frac{1}{4}\cos(\sqrt{2}\,t)\cos((2+\sqrt{2})\,t)\sqrt{2}$$

$$-\frac{1}{4}\cos(\sqrt{2}\,t)\cos((\sqrt{2}-2)\,t)\sqrt{2} - \frac{1}{4}\cos(\sqrt{2}\,t)\cos((\sqrt{2}-2)\,t)$$

$$-\frac{1}{4}\sin(\sqrt{2}\,t)\sin((\sqrt{2}-2)\,t)\sqrt{2} - \frac{1}{4}\sin(\sqrt{2}\,t)\sin((\sqrt{2}-2)\,t) - \frac{1}{4}\sin(\sqrt{2}\,t)\sin((2+\sqrt{2})\,t)$$

$$+\frac{1}{4}\sin(\sqrt{2}\,t)\sin((2+\sqrt{2})\,t)\sqrt{2} + _C1\cos(\sqrt{2}\,t) + _C2\sin(\sqrt{2}\,t)$$

This type of solution has been observed previously (see forced.mws). The key in that situation was to ask Maple to apply a variety of trigonometric identities to convert the solution to a more standard form:

```
> combine( ", trig );
```

$$y(t) = -\frac{1}{2}\cos(2\,t) + _C1\cos(\sqrt{2}\,t) + _C2\sin(\sqrt{2}\,t)$$

```
>
```

Example 3

The same techniques can be used to solve problems that involve piecewise-defined forcing functions (Heaviside) and/or impulse functions (Dirac). For example, the general solution to the equation

```
> ODE := diff( y(t), t$2 ) + 2*diff( y(t), t )
         = 3*Dirac(t-1) + Heaviside(t-Pi)*sin(t-Pi) + exp(-t);
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}y(t)\right) + 2\left(\frac{\partial}{\partial t}y(t)\right) = 3\,Dirac(t-1) - Heaviside(t-\pi)\sin(t) + e^{(-t)}$$

can be found using the Laplace transform, exactly as before (this should illustrate the general structure of these problems):

```
> ODE_lap := laplace( ODE, t, s );
```

$$ODE_lap := s\,(s\,laplace(y(t), t, s) - y(0)) - D(y)(0) + 2\,s\,laplace(y(t), t, s) - 2\,y(0) =$$

$$3\,e^{(-s)} + \frac{e^{(-s\,\pi)}}{s^2+1} + \frac{1}{s+1}$$

```
> SOLN_lap := solve( ODE_lap, { laplace( y(t), t, s ) } );
```

$$SOLN_lap := \{ laplace(y(t), t, s) = (1 + s^2 + D(y)(0) + 2\,y(0) + 3\,e^{(-s)} + e^{(-s\,\pi)} + 3\,y(0)\,s^2$$

$$+3\,y(0)\,s^3+3\,s\,y(0)+y(0)\,s^4+D(y)(0)\,s^3+D(y)(0)\,s^2+D(y)(0)\,s+3\,e^{(-s)}\,s^3+3\,e^{(-s)}\,s^2$$
$$+3\,e^{(-s)}\,s+e^{(-s\,\pi)}\,s)\Big/(s\,(2+3\,s^2+3\,s^3+3\,s+s^4))\}$$

```
> SOLN := invlaplace( SOLN_lap, s, t );
```

$$SOLN := \{y(t)=\frac{1}{2}-\frac{1}{2}\,D(y)(0)\,e^{(-2\,t)}+\frac{1}{2}\,D(y)(0)+y(0)-e^{(-t)}+\frac{1}{2}\,e^{(-2\,t)}+\frac{3}{2}\,Heaviside(t-1)$$
$$+\frac{1}{2}\,Heaviside(t-\pi)-\frac{3}{2}\,Heaviside(t-1)\,e^{(-2\,t+2)}-\frac{1}{10}\,Heaviside(t-\pi)\,e^{(-2\,t+2\,\pi)}$$
$$+\frac{2}{5}\,Heaviside(t-\pi)\,cos(t)+\frac{1}{5}\,Heaviside(t-\pi)\,sin(t)\}$$

This solution look needlessly complicated; it can be written in a format that clearly identifies the role of the initial data and the different solutions on each interval

```
> SOLN := op(collect( SOLN, { y(0), D(y)(0), Heaviside(t-1),
  Heaviside(t-Pi) } ));
```

$$SOLN := y(t)=y(0)+\left(-\frac{1}{2}\,e^{(-2\,t)}+\frac{1}{2}\right)D(y)(0)$$
$$+\left(\frac{1}{2}-\frac{1}{10}\,e^{(-2\,t+2\,\pi)}+\frac{2}{5}\,cos(t)+\frac{1}{5}\,sin(t)\right)Heaviside(t-\pi)+\left(\frac{3}{2}-\frac{3}{2}\,e^{(-2\,t+2)}\right)Heaviside(t-1)$$
$$+\frac{1}{2}+\frac{1}{2}\,e^{(-2\,t)}-e^{(-t)}$$

A plot of the solution with initial conditions

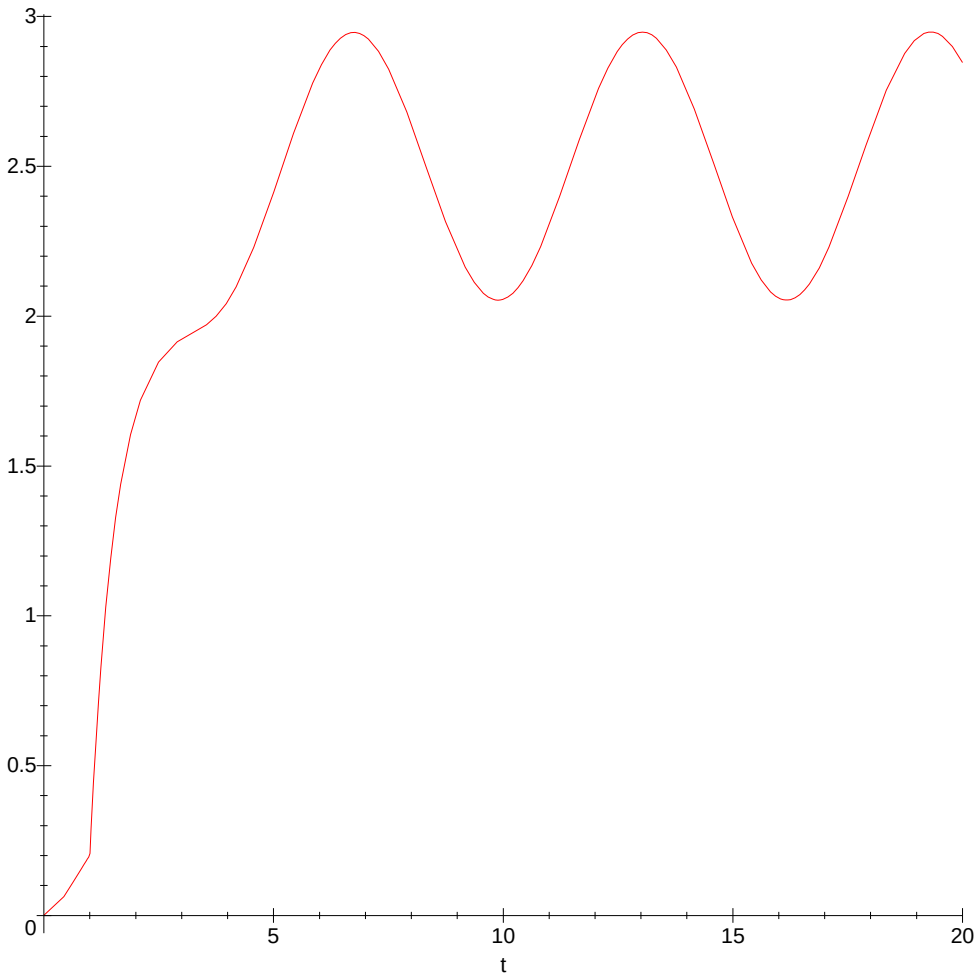
```
> IC := y(0)=0, D(y)(0)=0;
```

$$IC := y(0)=0,\ D(y)(0)=0$$

is also illuminating

```
> plot( subs( IC, rhs(SOLN) ), t=0..20,
        title='Solution to IVP with Heaviside' );
```

Solution to IVP with Heaviside



Update for Release 4

Conversion to a piecewise-defined function yields

```
> convert( subs(IC, SOLN), piecewise );
```

$$y(t) = \begin{cases} \frac{1}{2} + \frac{1}{2} e^{(-2t)} - e^{(-t)} & t \le 1 \\ 2 - \frac{3}{2} e^{(-2t+2)} + \frac{1}{2} e^{(-2t)} - e^{(-t)} & t \le \pi \\ \frac{5}{2} - \frac{1}{10} e^{(-2t+2\pi)} + \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t) - \frac{3}{2} e^{(-2t+2)} + \frac{1}{2} e^{(-2t)} - e^{(-t)} & \pi < t \end{cases}$$

The same solution can be obtained directly using dsolve with method=laplace.

>

LAPLACE TRANSFORMS FOR SYSTEMS OF LINEAR ODES

One of the great strengths of the Laplace transform is its applicability to both single equations and systems using exactly the same procedure; the only change is that now the algebraic equations will be a linear system. We illustrate with a single example.

```
> SYS := { diff( x(t), t ) = y(t) + sin(t),
           diff( y(t), t ) = x(t) + cos(t) };
```

$$SYS := \{\frac{\partial}{\partial t} y(t) = x(t) + \cos(t), \frac{\partial}{\partial t} x(t) = y(t) + \sin(t)\}$$

```
> SYS_lap := laplace( SYS, t, s );
```

$$SYS_lap := \{s\ laplace(y(t), t, s) - y(0) = laplace(x(t), t, s) + \frac{s}{s^2+1},$$

$$s\ laplace(x(t), t, s) - x(0) = laplace(y(t), t, s) + \frac{1}{s^2+1}\}$$

```
> SOLN_lap := solve( SYS_lap,
                     { laplace( x(t), t, s ), laplace( y(t), t, s )
  } );
```

$$SOLN_lap := \{$$

$$laplace(y(t), t, s) = \frac{s\,y(0) + 1 + x(0)}{s^2 - 1}, laplace(x(t), t, s) = \frac{x(0)\,s^3 + y(0)\,s^2 + 2\,s + x(0)\,s + y(0)}{s^4 - 1}$$

$$\}$$

```
> SOLN_lap2 := convert( ", parfrac, s );
```

$$SOLN_lap2 := \{laplace(x(t), t, s) = \frac{1}{2}\frac{1 + y(0) + x(0)}{s - 1} - \frac{1}{2}\frac{y(0) - 1 - x(0)}{s + 1} - \frac{s}{s^2+1},$$

$$laplace(y(t), t, s) = \frac{1}{2}\frac{1 + y(0) + x(0)}{s - 1} + \frac{1}{2}\frac{y(0) - 1 - x(0)}{s + 1}\}$$

```
> SOLN := invlaplace( SOLN_lap2, s, t );
```

$$SOLN := \{x(t) = \left(\frac{1}{2} + \frac{1}{2}y(0) + \frac{1}{2}x(0)\right)e^{t} + \left(-\frac{1}{2}y(0) + \frac{1}{2} + \frac{1}{2}x(0)\right)e^{(-t)} - \cos(t),$$

$$y(t) = \left(\frac{1}{2} + \frac{1}{2}y(0) + \frac{1}{2}x(0)\right)e^{t} + \left(\frac{1}{2}y(0) - \frac{1}{2} - \frac{1}{2}x(0)\right)e^{(-t)}\}$$

```
> SOLN2 := collect( ", { x(0), y(0) } );
```

$$SOLN2 := \{x(t) = \left(\frac{1}{2}e^{t} - \frac{1}{2}e^{(-t)}\right)y(0) + \left(\frac{1}{2}e^{t} + \frac{1}{2}e^{(-t)}\right)x(0) + \frac{1}{2}e^{t} + \frac{1}{2}e^{(-t)} - \cos(t),$$

$$y(t) = \left(\frac{1}{2}e^{t} + \frac{1}{2}e^{(-t)}\right)y(0) + \left(\frac{1}{2}e^{t} - \frac{1}{2}e^{(-t)}\right)x(0) + \frac{1}{2}e^{t} - \frac{1}{2}e^{(-t)}\}$$

>

>