```
> restart;
  with( linalg ):
Warning, new definition for norm
Warning, new definition for trace
```

Maple Worksheet for Geometric Programming

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WWW URL: http://www.math.sc.edu/~meade/math524/worksheets/geomprog.ms

Updated to Maple V, Release 4, by Douglas B. Meade (5 November 1996)

I suggest that you modify the commands to assist you with next Monday's homework problem (#26).

Background

The primal (unconstrained) geometric program is to minimize a "posynomial" by looking at the feasible set for the corresponding dual geometric program. If the feasible set contains exactly one element, duality leads directly to the unique solution to the primal problem. If the feasible set is empty, then the primal problem does not have a solution. The third possibility is that there are an infinite number of feasible vectors; it is this case that is considered in this worksheet

Case 3: Infinite number of feasible vectors for dual geometric program.

This is example 2.5.5 c) (p. 72):
  minimize g(t1,t2) = 1/(t1*t2) + t1*t2 + t1 + t2
  s.t.   t1, t2 > 0

The objective function for the (primal) geometric program is

```
> g := 1/t[1]/t[2] + t[1]*t[2] + t[1] + t[2];
```

$$g := \frac{1}{t_1\, t_2} + t_1\, t_2 + t_1 + t_2$$

The objective function for the dual geometric program is

```
> v := (1/delta[1])^delta[1] * (1/delta[2])^delta[2] *
  (1/delta[3])^delta[3] * (1/delta[4])^delta[4];
```

$$v := \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{1}{\delta_2}\right)^{\delta_2} \left(\frac{1}{\delta_3}\right)^{\delta_3} \left(\frac{1}{\delta_4}\right)^{\delta_4}$$

The feasible vectors are the solution(s) to the positivity, normality, and orthogonality conditions:

```
> eqP := { delta[1]>0, delta[2]>0, delta[3]>0, delta[4]>0 };
```

$$eqP := \{0 < \delta_1, 0 < \delta_2, 0 < \delta_3, 0 < \delta_4\}$$

```
> eqN := delta[1] + delta[2] + delta[3] + delta[4] = 1;
```

$$eqN := \delta_1 + \delta_2 + \delta_3 + \delta_4 = 1$$

```
> eqO := -delta[1] + delta[2] + delta[3]              = 0,
         -delta[1] + delta[2]               + delta[4] = 0;
```

$$eqO := -\delta_1 + \delta_2 + \delta_3 = 0, -\delta_1 + \delta_2 + \delta_4 = 0$$

```
>
> F :=solve( {eqN, eqO}, {delta[1],delta[2],delta[3],delta[4]} );
```

$$F := \{\delta_4 = -2\,\delta_1 + 1, \delta_1 = \delta_1, \delta_3 = -2\,\delta_1 + 1, \delta_2 = 3\,\delta_1 - 1\}$$

NOTE that there are an infinite number of solutions to these equations.

With the above solutions, the dual objective function becomes

```
> vv := subs( F, v );
```

$$vv := \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{1}{3\,\delta_1 - 1}\right)^{(3\,\delta_1 - 1)} \left(\left(\frac{1}{-2\,\delta_1 + 1}\right)^{(-2\,\delta_1 + 1)}\right)^2$$

To simplify matters, let s = delta[1]. (Note that the independent variable in vv may change from one run to the next; be sure that the correct substitution is made in the following command; this may be different from the text.)

```
> vv := subs( delta[1]=s, vv );
```

$$vv := \left(\frac{1}{s}\right)^{s} \left(\frac{1}{3\,s - 1}\right)^{(3\,s - 1)} \left(\left(\frac{1}{-2\,s + 1}\right)^{(-2\,s + 1)}\right)^2$$

```
>
```

We still have to enforce the positivity condition.

```
> subs( F, delta[1]=s, eqP );
```

$$\{0 < 3\,s - 1, 0 < -2\,s + 1, 0 < s\}$$

Unfortunately, Maple is not yet able to solve this system of inequalities. It’s not difficult to see that the above conditions are satisified when:

s>0
3s>1
2s<1
Thus, we restrict our search to 1/3 < s < 1/2.

Release 4 Update

These inequalities can now be treated naturally by solve

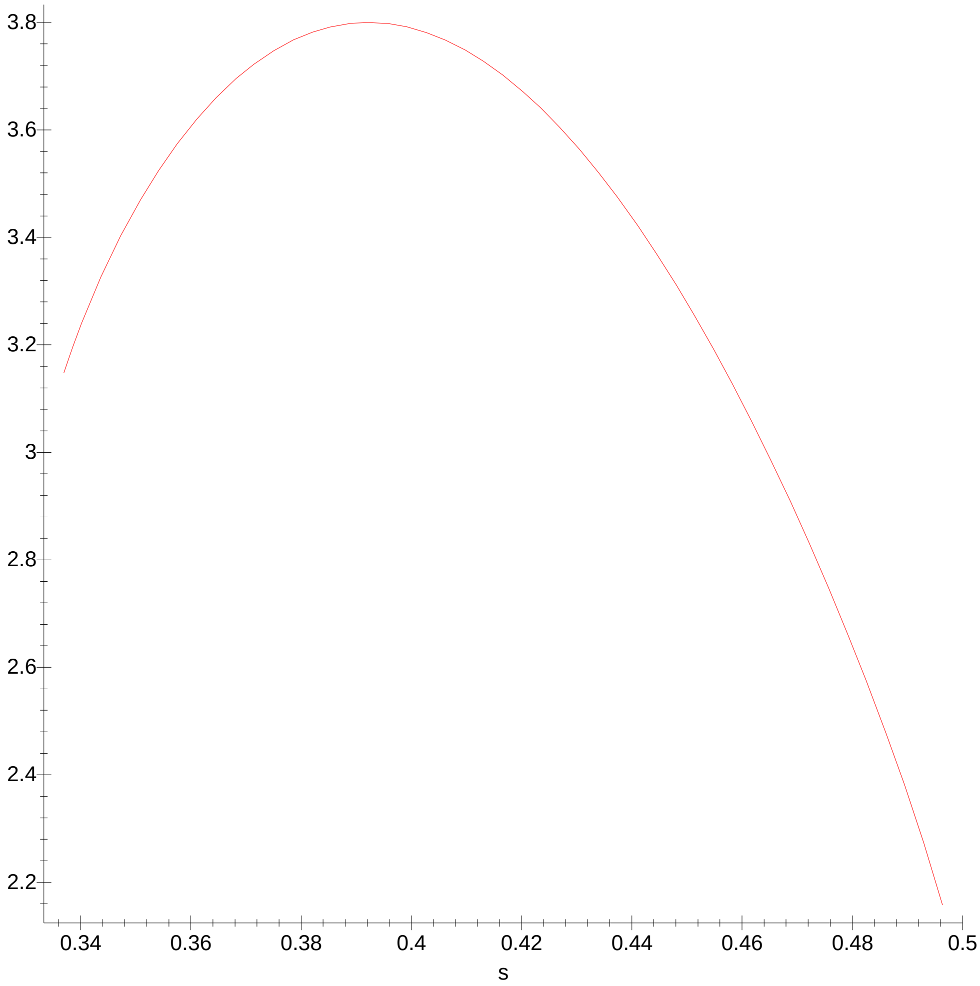
```
> solve( subs( F, delta[1]=s, eqP ), {s} );
```

$$\{\frac{1}{3} < s, s < \frac{1}{2}\}$$

So, how would you find the maximum value of vv on the interval (1/3,1/2) ?

A good way to begin is to get a picture of the function.

```
> plot( vv, s=1/3..1/2 );
```



In this case we can learn much about the solution, which will be denoted ss (for s-star). However, we can also look for the exact solution -- using Calculus.

```
> dv := diff( vv, s );
```

$$dv := \left(\frac{1}{s}\right)^s \left(\ln\left(\frac{1}{s}\right) - 1\right) \left(\frac{1}{3\,s-1}\right)^{(3\,s-1)} \left(\left(\frac{1}{-2\,s+1}\right)^{(-2\,s+1)}\right)^2 + \left(\frac{1}{s}\right)^s \left(\frac{1}{3\,s-1}\right)^{(3\,s-1)} \left(3\ln\left(\frac{1}{3\,s-1}\right) - 3\right) \left(\left(\frac{1}{-2\,s+1}\right)^{(-2\,s+1)}\right)^2 + 2\left(\frac{1}{s}\right)^s \left(\frac{1}{3\,s-1}\right)^{(3\,s-1)} \left(\left(\frac{1}{-2\,s+1}\right)^{(-2\,s+1)}\right)^2 \left(-2\ln\left(\frac{1}{-2\,s+1}\right) + 2\right)$$

The critical points of v are the zeroes of the derivative of v:

```
> cp := solve( dv=0, s );
```

$$cp := RootOf(11\_Z^4 + 5\_Z^3 - 15\_Z^2 + 7\_Z - 1, .3922029308)$$

Amazingly, the zeroes of v' are simply the roots of the above fourth-order polynomial (in _Z). As such, the exact solution can be obtained.

```
> cp := allvalues(cp);
```

$$cp := -\frac{5}{44} + \frac{1}{132}\sqrt{\%2} + \frac{1}{132} I \sqrt{6}\sqrt{\frac{-1395\,\%4 - 11\,\%3 + 528\sqrt{\%2} + 91809\,\%1}{(108+12\sqrt{849})^{1/3}\sqrt{\%2}}},$$

$$-\frac{5}{44} + \frac{1}{132}\sqrt{\%2} - \frac{1}{132} I \sqrt{6}\sqrt{\frac{-1395\,\%4 - 11\,\%3 + 528\sqrt{\%2} + 91809\,\%1}{(108+12\sqrt{849})^{1/3}\sqrt{\%2}}},$$

$$-\frac{5}{44} - \frac{1}{132}\sqrt{\%2} + \frac{1}{132}\sqrt{6}\sqrt{\frac{1395\,\%4 + 11\,\%3 - 528\sqrt{\%2} + 91809\,\%1}{(108+12\sqrt{849})^{1/3}\sqrt{\%2}}},$$

$$-\frac{5}{44} - \frac{1}{132}\sqrt{\%2} - \frac{1}{132}\sqrt{6}\sqrt{\frac{1395\,\%4 + 11\,\%3 - 528\sqrt{\%2} + 91809\,\%1}{(108+12\sqrt{849})^{1/3}\sqrt{\%2}}}$$

$$\%1 := (108+12\sqrt{849})^{1/3}$$

$$\%2 := \frac{4185\,\%1 - 66\,(108+12\sqrt{849})^{2/3} + 3168}{(108+12\sqrt{849})^{1/3}}$$

$$\%3 := \sqrt{\%2}\,(108+12\sqrt{849})^{2/3}$$

$$\%4 := (108+12\sqrt{849})^{1/3}\sqrt{\%2}$$

What can we see from this? Note that there are four roots (as expected). The last two roots appear to be complex-valued, so the maximizer should come from one of the first two roots. Which one? And, are you sure this falls in the needed range?

To find out, let's ask Maple for floating-point approximations to these roots.

```
> evalf( cp );
```

$$.3707075143 + .09233090417\,I,\ .3707075143 - .09233090417\,I,\ .3922029305,\ -1.588163414$$

Thus, the second root is our optimal choice for s (and this is consistent with our graph).

```
> ss := cp[3];
```

$$ss := -\frac{5}{44} - \frac{1}{132}\sqrt{\%2} + \frac{1}{132}\sqrt{6}$$

$$\sqrt{\frac{1395\,(108+12\sqrt{849})^{1/3}\sqrt{\%2} + 11\sqrt{\%2}\,(108+12\sqrt{849})^{2/3} - 528\sqrt{\%2} + 91809\,\%1}{(108+12\sqrt{849})^{1/3}\sqrt{\%2}}}$$

$$\%1 := (108+12\sqrt{849})^{1/3}$$

$$\%2 := \frac{4185\,\%1 - 66\,(108+12\sqrt{849})^{2/3} + 3168}{(108+12\sqrt{849})^{1/3}}$$

The maximum value of the dual objective function is

```
> vvs := subs( s=ss, vv );
  evalf( " );
```

$$vvs := \left(\frac{1}{-\frac{5}{44} - \frac{1}{132}\sqrt{\%2} + \frac{1}{132}\%3}\right)^{(-5/44 - 1/132\sqrt{\%2} + 1/132\,\%3)}$$

$$\left(\frac{1}{-\frac{59}{44} - \frac{1}{44}\sqrt{\%2} + \frac{1}{44}\%3}\right)^{\left(-\frac{59}{44} - 1/44\sqrt{\%2} + 1/44\,\%3\right)}$$

$$\left(\left(\frac{1}{\frac{27}{22} + \frac{1}{66}\sqrt{\%2} - \frac{1}{66}\%3}\right)^{\left(\frac{27}{22} + 1/66\sqrt{\%2} - 1/66\,\%3\right)}\right)^2$$

$$\%1 := (108+12\sqrt{849})^{1/3}$$

$$\%2 := \frac{4185\,\%1 - 66\,(108+12\sqrt{849})^{2/3} + 3168}{(108+12\sqrt{849})^{1/3}}$$

$$\%3 := \sqrt{6}$$

$$\sqrt{\frac{1395\,(108+12\sqrt{849})^{1/3}\sqrt{\%2} + 11\sqrt{\%2}\,(108+12\sqrt{849})^{2/3} - 528\sqrt{\%2} + 91809\,\%1}{(108+12\sqrt{849})^{1/3}\sqrt{\%2}}}$$

$$3.799604751$$

and this value is attained with delta* as follows (remember that delta*[4] = ss):

```
> ds := subs( delta[1]=ss, F );
  map( evalf, ds );
```

$$ds := \{\delta_4 = \frac{27}{22} + \frac{1}{66}\sqrt{\%2} - \frac{1}{66}\%3, -\frac{5}{44} - \frac{1}{132}\sqrt{\%2} + \frac{1}{132}\%3 = -\frac{5}{44} - \frac{1}{132}\sqrt{\%2} + \frac{1}{132}\%3,$$

$$\delta_3 = \frac{27}{22} + \frac{1}{66}\sqrt{\%2} - \frac{1}{66}\%3, \delta_2 = -\frac{59}{44} - \frac{1}{44}\sqrt{\%2} + \frac{1}{44}\%3\}$$

$$\%1 := (108 + 12\sqrt{849})^{1/3}$$

$$\%2 := \frac{4185\,\%1 - 66\,(108 + 12\sqrt{849})^{2/3} + 3168}{(108 + 12\sqrt{849})^{1/3}}$$

$$\%3 := \sqrt{6}$$

$$\sqrt{\frac{1395\,(108 + 12\sqrt{849})^{1/3}\sqrt{\%2} + 11\sqrt{\%2}\,(108 + 12\sqrt{849})^{2/3} - 528\sqrt{\%2} + 91809\,\%1}{(108 + 12\sqrt{849})^{1/3}\sqrt{\%2}}}$$

$$\{.3922029305 = .3922029305, \delta_4 = .215594139, \delta_2 = .176608791, \delta_3 = .215594139\}$$

The last step is to take the solution to the dual geometric program and compute the solution of the (primal) geometric program (which will be denoted ts[1] and ts[2]). According to Theorem 2.5.2, we know the solution to the primal and dual geometric programs are related by:

delta*[1] * v(delta*) = u[1](t*) = 1/ts[1]/ts[2]
delta*[1] * v(delta*) = u[2](t*) = ts[1] * ts[2]
delta*[3] * v(delta*) = u[3](t*) = ts[1]
delta*[4] * v(delta*) = u[4](t*) = ts[2]

While this could be a real mess, look at the last two equations -- they tell us exactly how to compute the solution to the original problem.

```
> ts[1] := subs( ds, delta[3]*vvs );
  evalf( " );
```

$$ts_1 := \left(\frac{27}{22} + \frac{1}{66}\sqrt{\%2} - \frac{1}{66}\%3\right)\left(\frac{1}{-\frac{5}{44} - \frac{1}{132}\sqrt{\%2} + \frac{1}{132}\%3}\right)^{(-5/44 - 1/132\sqrt{\%2} + 1/132\,\%3)}$$

$$\left(\frac{1}{-\frac{59}{44}-\frac{1}{44}\sqrt{\%2}+\frac{1}{44}\%3}\right)^{\left(-\frac{59}{44}-1/44\sqrt{\%2}+1/44\,\%3\right)}$$

$$\left(\left(\frac{1}{\frac{27}{22}+\frac{1}{66}\sqrt{\%2}-\frac{1}{66}\%3}\right)^{\left(\frac{27}{22}+1/66\sqrt{\%2}-1/66\,\%3\right)}\right)^{2}$$

$$\%1 := (108+12\sqrt{849})^{1/3}$$

$$\%2 := \frac{4185\,\%1-66\,(108+12\sqrt{849})^{2/3}+3168}{(108+12\sqrt{849})^{1/3}}$$

$$\%3 := \sqrt{6}$$

$$\sqrt{\frac{1395\,(108+12\sqrt{849})^{1/3}\sqrt{\%2}+11\sqrt{\%2}\,(108+12\sqrt{849})^{2/3}-528\sqrt{\%2}+91809\,\%1}{(108+12\sqrt{849})^{1/3}\sqrt{\%2}}}$$

$$.8191725148$$

```
> ts[2] := subs( ds, delta[4]*vvs );
  evalf( " );
```

$$ts_2 := \left(\frac{27}{22}+\frac{1}{66}\sqrt{\%2}-\frac{1}{66}\%3\right)\left(\frac{1}{-\frac{5}{44}-\frac{1}{132}\sqrt{\%2}+\frac{1}{132}\%3}\right)^{\left(-5/44-1/132\sqrt{\%2}+1/132\,\%3\right)}$$

$$\left(\frac{1}{-\frac{59}{44}-\frac{1}{44}\sqrt{\%2}+\frac{1}{44}\%3}\right)^{\left(-\frac{59}{44}-1/44\sqrt{\%2}+1/44\,\%3\right)}$$

$$\left(\left(\frac{1}{\frac{27}{22}+\frac{1}{66}\sqrt{\%2}-\frac{1}{66}\%3}\right)^{\left(\frac{27}{22}+1/66\sqrt{\%2}-1/66\,\%3\right)}\right)^{2}$$

$$\%1 := (108+12\sqrt{849})^{1/3}$$

$$\%2 := \frac{4185\,\%1-66\,(108+12\sqrt{849})^{2/3}+3168}{(108+12\sqrt{849})^{1/3}}$$

$$\%3 := \sqrt{6}$$

$$\sqrt{\frac{1395\,(108+12\sqrt{849})^{1/3}\sqrt{\%2}+11\sqrt{\%2}\,(108+12\sqrt{849})^{2/3}-528\sqrt{\%2}+91809\,\%1}{(108+12\sqrt{849})^{1/3}\sqrt{\%2}}}$$

$$.8191725148$$

The fact that t*[1] = t*[2] should not be surprising. (Why?)

As a final check, let's look at the minimum value of g.

```
> subs( t[1]=ts[1], t[2]=ts[2], g );
  evalf( " );
```

$$\frac{1}{\%4^2\left(\left(\frac{1}{\%6}\right)^{\%6}\right)^2\left(\left(\frac{1}{\%5}\right)^{\%5}\right)^2\left(\left(\frac{1}{\%4}\right)^{\%4}\right)^4}+\%4^2\left(\left(\frac{1}{\%6}\right)^{\%6}\right)^2\left(\left(\frac{1}{\%5}\right)^{\%5}\right)^2\left(\left(\frac{1}{\%4}\right)^{\%4}\right)^4$$

$$+2\,\%4\left(\frac{1}{\%6}\right)^{\%6}\left(\frac{1}{\%5}\right)^{\%5}\left(\left(\frac{1}{\%4}\right)^{\%4}\right)^2$$

$$\%1 := (108+12\sqrt{849})^{1/3}$$

$$\%2 := \frac{4185\,\%1 - 66\,(108+12\sqrt{849})^{2/3} + 3168}{(108+12\sqrt{849})^{1/3}}$$

$$\%3 := \sqrt{6}\sqrt{\frac{1395\,(108+12\sqrt{849})^{1/3}\sqrt{\%2} + 11\sqrt{\%2}\,(108+12\sqrt{849})^{2/3} - 528\sqrt{\%2} + 91809\,\%1}{(108+12\sqrt{849})^{1/3}\sqrt{\%2}}}$$

$$\%4 := \frac{27}{22}+\frac{1}{66}\sqrt{\%2}-\frac{1}{66}\%3$$

$$\%5 := -\frac{59}{44}-\frac{1}{44}\sqrt{\%2}+\frac{1}{44}\%3$$

$$\%6 := -\frac{5}{44}-\frac{1}{132}\sqrt{\%2}+\frac{1}{132}\%3$$

$$3.799604754$$

Does this agree with what you expect? Explain.

END OF WORKSHEET

```
>
>
```