S E I R Model with Nonconstant Total Population and Vaccination

$$
\begin{aligned}
\dot{S} &= \Lambda \quad + (1-\rho_c)\mu_0 N \quad - \mu_1 S \quad - \beta S I \\
&\qquad - \textstyle\sum_{n=0}^{\infty} \rho_p S(nT^-)\delta(t-nT) \\
\dot{E} &= \qquad\qquad\qquad - \mu_1 E \quad + \beta S I \quad - \delta E \\
\dot{I} &= \qquad\qquad\qquad - \mu_1 I \qquad\qquad + \delta E \quad - \gamma I \quad - \epsilon I \\
\dot{R} &= \qquad \rho_c \mu_0 N \quad - \mu_1 R \qquad\qquad\qquad + \gamma I \\
&\qquad + \textstyle\sum_{n=0}^{\infty} \rho_p S(nT^-)\delta(t-nT) \\
\dot{N} &= \ \Lambda - (\mu_1 - \mu_0) N - \ \epsilon I
\end{aligned}
\tag{1}
$$

Analysis: No Vaccination

Theorem

Let $\rho_c = \rho_p = 0$. The reproduction number for (1) is

$$R_0 = \frac{\Lambda}{\mu_1 - \mu_0} \frac{\delta}{\mu_1 + \delta} \frac{\beta}{\mu_1 + \gamma + \epsilon}.$$

1. If $R_0 > 1$, then the unique endemic equilibrium is globally stable and the infection-free equilibrium is unstable.
2. If $0 < R_0 \leq 1$, then the infection-free equilibrium is globally stable (and the endemic equilibrium is not physically realistic).

Analysis: Constant Vaccination

Theorem

Let $\rho_c \in (0, 1]$ and $\rho_p = 0$. The reproduction number for (1) is $R_0(\rho_c) = (1 - \frac{\mu 0}{\mu 1}\rho_c)R_0$ where R_0 is the reproduction number with $\rho_c = 0$. Define $\rho_c^* = \frac{\mu_1}{\mu_0}\left(1 - \frac{1}{R_0}\right)$.

1. If $0 < \rho_c < \rho_c^*$, then, since $R_0(\rho_c) > 1$, the unique endemic equilibrium is globally stable and the infection-free equilibrium is unstable.

2. If $\rho_c \geq \rho_c^*$, then, since $0 < R_0(\rho_c) \leq 1$, the infection-free equilibrium is globally stable (and the endemic equilibrium is not physically realistic).

Analysis: Pulse Vaccination

Conjecture

Let $\rho_c = 0$. For each $\rho_p \in (0, 1]$ and $T > 0$, there is a unique infection-free periodic equilibrium solution (with period T). This solution is locally asymptotically stable when the mean susceptible population over each period is below a threshold (that depends on R_0).