Analytic, Geometric and Numeric Analysis of the Shrinking Circle and Sphere Problems

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Shrinking Circle Problem

Let

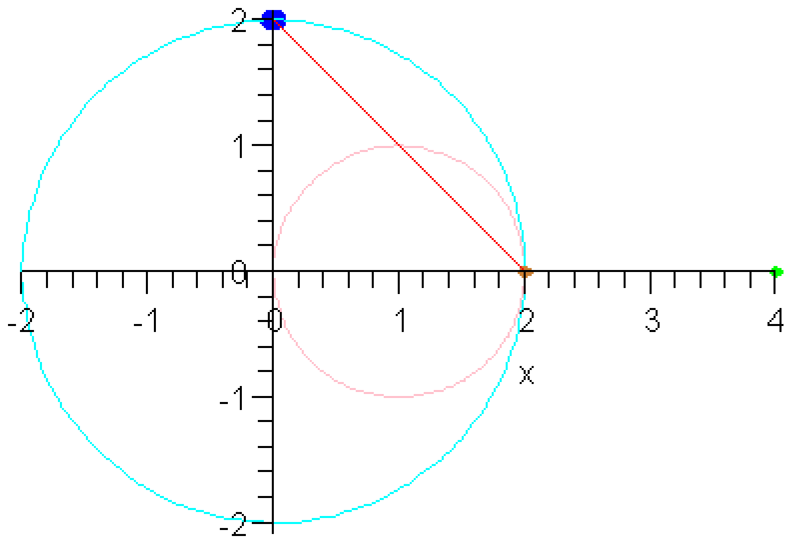
- C be the unit circle with center $(1,0)$
- C_r be the circle with radius r and center $(0,0)$
- P be the point $(0, r)$
- Q be the upper point of intersection of C and C_r
- R be the intersection of line PQ and the x-axis.

What happens to R as C_r shrinks to the origin?

Stewart, *Essential Calculus: Early Transcendentals*, Thomson Brooks/Cole, 2007, p. 45, Exercise 56.

Visualization

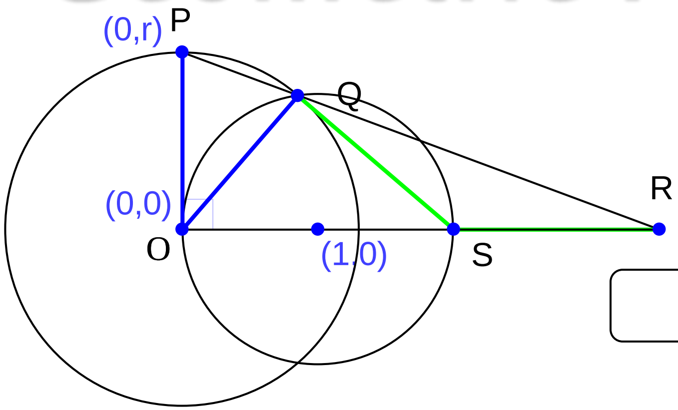
[Geometry Expressions]   [Excel]   [Maple]



Proof ?

- Numeric
  - Sensitive to floating-point cancellation
- Symbolic
  - Indeterminate form
    (l'Hopital's Rule, or simply rationalize)
- Geometric
  - Tom Banchoff (Brown University)

Geometric Proof



$OP = OQ$

$\angle OPQ = \angle OQP$

$OP \perp OR$

$\angle OPQ + \angle SRQ = \frac{\pi}{2}$

$OQ \perp QS$

$\angle OQP + \angle SQR = \frac{\pi}{2}$

$\angle OQP + \angle SRQ = \frac{\pi}{2}$

$\angle SQR = \angle SRQ$

ΔSQR is isosceles

$SR = SQ < OS$

$OR = OS + SR < 2\, OS$

$P \rightarrow O$

$Q \rightarrow O$

$QS \rightarrow OS$

$OR = OS + SR = OS + QS \rightarrow 2OS$

Generalized Shrinking Circle Problem

Let

C be a fixed curve

C_r be circle with center at origin and radius r

P is point $(0, r)$

Q is upper point of intersection of C and C_r

R is point of intersection of line PQ and x–axis

What happens to R as C_r shrinks to a point?

Lemma

Let *C* be the circle with center *(a, b)* that includes the origin, i.e.,

$$(x - a)^2 + (y - b)^2 = a^2 + b^2.$$

Define C_r, *P*, *Q*, and *R* as in the Generalized Shrinking Circle Problem.

Then, $\lim_{r\to 0} R = \begin{cases} (4a, 0) & \text{if } b = 0 \\ (0, 0) & \text{otherwise} \end{cases}$

Theorem

Let C be a curve in the plane that includes the origin and is twice continuously differentiable at the origin. Define C_r, P, Q, and R as in the Generalized Shrinking Circle Problem. If the curvature at the origin, κ, is positive, the osculating circle of C at the origin has radius $\rho=1/\kappa$ and center (a, b) where $a^2 + b^2 = \rho^2$.

Moreover, $\lim_{r\to 0} R = \begin{cases} (4\rho, 0) & \text{if } b = 0 \\ (0,0) & \text{otherwise} \end{cases}$

Theorem (Coordinate-Free)

Let O be a point on a curve C in the plane where the osculating circle to C at O exists. Let **T** and **N** be the unit tangent and normal vectors to C at O, respectively. Let κ be the curvature of C at O.

(Orient **N** so that $O + 1/\kappa\,\mathbf{N}$ is the center of the osculating circle to C at O.)

For any $r > 0$, define

- C_r to be the circle with radius r centered at O,
- $P = O + r\,\mathbf{T}$, to be the point at the "top" of C_r,
- Q to be the intersection of C and C_r, and
- R to be the point on the line through P and Q such that OR is parallel with N

Then, as r decreases to 0,
R converges to the point $R_0 = O + 4/\kappa\,\mathbf{N}$.

Shrinking Sphere Problem

Let

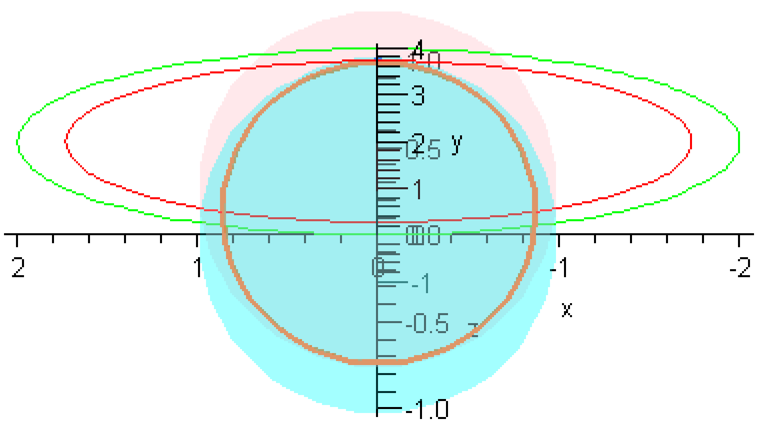
- $S =$ sphere with center (0,a,b) that includes the origin, i.e., $x^2+(y-a)^2+(z-b)^2 = a^2+b^2$
- $S_r =$ sphere with radius r and center $(0,0,0)$
- $P =$ point $(0,0,r)$, the "north pole" of S_r
- $Q =$ curve of intersection of S and S_r
- $R =$ projection of P through Q onto xy plane

What happens to R as S_r shrinks to the origin?

Note: R is now, by definition, a curve.

Shrinking Sphere Problem

$$\lim_{r\to 0} R = \begin{cases} x^2 + (y-2\rho)^2 = 4\rho^2 & \text{if } b = 0 \\ (0,0) & \text{otherwise} \end{cases}$$



Theorem (Coordinate-Free)

Let O be a point on a surface S in R^3 with a well-defined normal vector, N, at O. Let C be a curve on S such that, at O, the unit tangent vector to C on S is **T** and the principal normal vector for the curve C coincides with the normal vector to S at O, i.e., $\mathbf{N}=|dT/ds|$ (where s is arclength).

For any $r>0$, define
- S_r to be the sphere with radius r centered at O,
- $P = O + r\mathbf{T}$, to be the point at the "top" of S_r,
- Q to be the intersection of S and S_r, and
- R to be the curve that is the projection of P through Q onto the plane containing O that is orthogonal to **T**.

Then, as r decreases to 0, R converges to the circle with radius $2/\kappa$, centered at $O + 2/\kappa\,\mathbf{N}$, and lies in the plane with normal vector **T**.

Is This Original?

- These geometric results are so elegant, and seemingly simplistic.
- Is it possible they were never observed or published until now?
- How did Stewart come up with this problem?

Meusnier's Theorem

- Wikipedia (http://en.wikipedia.org/wiki/Meusnier%27s_theorem)

  In differential geometry, **Meusnier's theorem** states that all curves on a surface passing through a given point *p* and having the same tangent line at *p* also have the same normal curvature at *p* and their osculating circles form a sphere.

  First announced by Jean Baptiste Meusnier in 1776.

Meusnier's Theorem

- Answers.com (from Sci-Tech Dictionary)

  A theorem stating that the curvature of a surface curve equals the curvature of the normal section through the tangent to the curve divided by the cosine of the angle between the plane of this normal section and the osculating plane of the curve.

Summary

- Simple, routine-sounding textbook exercises led to interesting discoveries, even if they were not completely new.
- Numerical simulations were incomplete, or misleading.
- The essential ingredients for the general problems became clear only through *careful use of technology for both visual and symbolic analysis.*
- Three-dimensional visualization tools are lacking.

Further Reading and Demonstrations

D.B. Meade and W-C Yang, *Analytic, Geometric, and Numeric Analysis of the Shrinking Circle and Sphere Problems*, Electronic Journal of Mathematics and Technology, v 1, issue 1, Feb. 2007, ISSN 1993-2823

- https://php.radford.edu/~ejmt/deliveryBoy.php?paper=eJMT_v1n1p4
- http://www.math.sc.edu/~meade/eJMT-Shrink/