Theory and Applications of Boundary Value Problems for ODEs

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Outline

- Introduction
- BVPs in Introductory ODE Courses
  - Past and Present
  - Computer Algebra Tools for Solving BVPs
  - Future

Play at the Plate

Bottom of the ninth; one out; tying run on third. A fly ball is hit to an outfielder positioned 300 feet from the plate. The catch is made on the run; 0.6 seconds later the ball starts a perfect flight to the catcher (*i.e.*, on target with no bounces). The speedy baserunner runs from third to home in 4.0 seconds. How hard must the ball be thrown to prevent the runner from scoring?

— *based on "The Physics of Baseball", Robert K. Adair, 1994.*

Play at the Plate

- Assumptions
  - throw arrives at home plate without bouncing
  - catcher needs 0.3 seconds to apply the tag
  - no wind
  - quadratic air resistance (coefficient k)
  - the ball has mass 5 oz.
  - the ball is released from a height of 8 feet above the ground
  - $\mathbf{g} = 9.8\mathbf{j}$ m/sec^2
  - the umpire is in a good position to make the call

Play at the Plate

- Problem Formulation:

determine $\mathbf{x}'(0)$ and $\|\mathbf{x}'(0)\|$ where $\mathbf{x}$ is a solution to

$$\begin{aligned} m\mathbf{x}'' &= -k\|\mathbf{x}'\|\mathbf{x}' - \mathbf{g} \\ \mathbf{x}(0) &= 300\mathbf{i} + 8\mathbf{j} \\ \mathbf{x}(3.1) &= \mathbf{0} \end{aligned}$$

Play at the Plate

- Further Discussion
  - Why does the fielder try to make the catch while running towards home plate?
  - Are there advantages of bouncing the ball to the catcher?
  - Given the maximum initial velocity for a player, where should the fielder be positioned?

Past and Present

Boundary Value Problems. When the values of a solution to a differential equation are specified at two different points, these conditions are called **boundary conditions**. (In contrast, initial conditions specify the values of a function and its derivative at the same point.) Given that every solution to $y'' + y = 0$ is of the form

$$y(x) = c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 are arbitrary constants, show that

(a) there is a unique solution that satisfies the boundary conditions $y(0) = 2$ and $y(\pi/2) = 0$;

(b) there is no solution that satisfies $y(0) = 2$ and $y(\pi) = 0$;

(c) there are infinitely many solutions that satisfy $y(0) = 2$ and $y(\pi) = -2$.

— *Fundamentals of Differential Equations, Nagle & Saff (# 30, p. 159)*

Computer Algebra Tools for Solving BVPs

```
> ODE := diff( y(x), x$2 ) + y(x) = 0:
> BCa := y(0)=2, y(Pi/2)=0:
> SOLa := dsolve( { ODE, BCa }, y(x) );
```

$$SOLa := \mathrm{y}(x) = 2\cos(x)$$

```
> BCb := y(0)=2, y(Pi)=0:
> SOLb := dsolve( { ODE, BCb }, y(x) );
```

$$SOLb :=$$

```
> BCc := y(0)=2, y(Pi)=-2:
> SOLc := dsolve( { ODE, BCc }, y(x) );
```

$$SOLc := \mathrm{y}(x) = \_C1\,\sin(x) + 2\cos(x)$$

Customized symbolic BVP solver: `bvpsolve`

```
> bvpsolve := proc( BVP::set(equation), FN )
>   local ODE, BC, gSOL, Y, DVAR, IVAR, CONST, cSOL;
>   ODE := select( has, BVP, diff );
>   BC := BVP minus ODE;
>   DVAR := op(0,FN); IVAR := op(1,FN);
>   gSOL := dsolve( ODE, FN );
>   if gSOL=NULL then
>     RETURN( `No general solution found` )
>   fi;
>   if nops([gSOL])>1 then
>     RETURN( `Nonunique general solution found` )
>   fi;
>   CONST := indets(gSOL,name) minus {IVAR};
>   Y := unapply( rhs( gSOL ), IVAR );
>   cSOL := solve( eval( subs( DVAR=Y, BC ) ), CONST );
>   if cSOL=NULL then
>     RETURN( `No particular solution satisfies all BCs` )
>   fi;
>   subs( cSOL, gSOL );
> end:
```

Simple Examples

```
> bvpsolve( { ODE, BCa }, y(x) );
```

$$\mathrm{y}(x) = 2\cos(x)$$

```
> bvpsolve( { ODE, BCb }, y(x) );
```

No particular solution satisfies all BCs

```
> bvpsolve( { ODE, BCc }, y(x) );
```

$$\mathrm{y}(x) = 2\cos(x) + \_C2\sin(x)$$

More Challenging Problems

- multi-point boundary conditions

$$y'' + y = 0, \quad y(\pi/4) + y(\pi/2) = 1, \quad y'(\pi/4) + y'(\pi/2) = -1$$

- nonlinear boundary conditions

$$y'' + y = 0, \quad y(0) = 2, \quad y(\pi/2)^2 + y'(\pi) = 1$$

- nonlinear differential equation

$$y'' + y^2 = 0, \quad y(0) = 2, \quad y(\pi/2) = 0$$

Multi-point Boundary Conditions

```
> ODE := diff( y(x), x$2 ) + y(x) = 0:
> BC2p := D(y)(Pi/2)+D(y)(Pi/4)=-1, y(Pi/4)+y(Pi/2)=1;
```

$$BC2p := \mathrm{D}(y)(\frac{1}{2}\,\pi) + \mathrm{D}(y)(\frac{1}{4}\,\pi) = -1,\ \mathrm{y}(\frac{1}{4}\,\pi) + \mathrm{y}(\frac{1}{2}\,\pi) = 1$$

```
> dsolve( { ODE, BC2p }, y(x) );
```

```
> bvpsolve( { ODE, BC2p }, y(x) );
```

$$\mathrm{y}(x) = \frac{1}{2}\,\sqrt{2}\cos(x) + (1 - \frac{1}{2}\,\sqrt{2})\sin(x)$$

Nonlinear Boundary Conditions

```
> ODE := diff( y(x), x$2 ) + y(x) = 0:
> BCnl := y(0)=2, y(Pi/2)^2 + D(y)(Pi)=1;
```

$$BCnl := \mathrm{y}(0) = 2,\ \mathrm{y}(\frac{1}{2}\,\pi)^2 + \mathrm{D}(y)(\pi) = 1$$

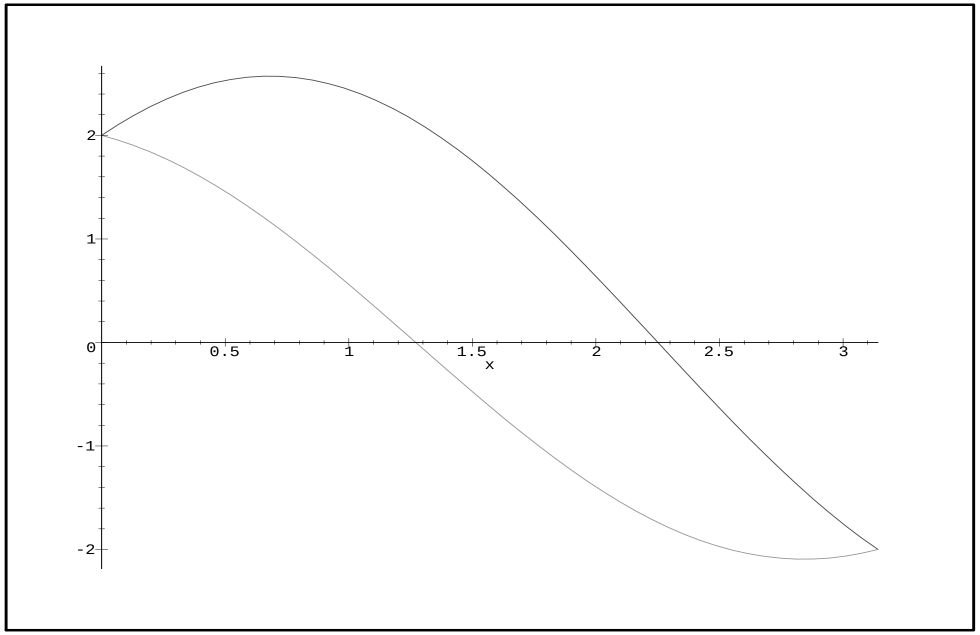
```
> dsolve( { ODE, BCnl }, y(x) );
> bvpsolve( { ODE, BCnl }, y(x) );
```

$$\mathrm{y}(x) = 2\cos(x) + \mathrm{RootOf}(\_Z^2 - \_Z - 1)\sin(x)$$

```
> SS := allvalues( " );
```

$$SS := \mathrm{y}(x) = 2\cos(x) + (\frac{1}{2}\sqrt{5} + \frac{1}{2})\sin(x),\ \mathrm{y}(x) = 2\cos(x) + (\frac{1}{2} - \frac{1}{2}\sqrt{5})\sin(x)$$

```
> plot( map( rhs, [SS] ), x=0..Pi );
```



Nonlinear ODE I

```
> ODE2 := diff( y(x), x$2 ) + y(x)^2 = 0;
```

$$ODE2 := (\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) + \mathrm{y}(x)^2 = 0$$

```
> dsolve( { ODE2, BCa }, y(x) );
> bvpsolve( { ODE2, BCa }, y(x) );
```

Nonunique general solution found

```
> dsolve( ODE2, y(x) );
```

$$x = \int_0^{\mathrm{y}(x)} -\frac{3}{\sqrt{-6\,y1^3 + 9\,\_C1}}\,dy1 - \_C2,\ x = \int_0^{\mathrm{y}(x)} \frac{3}{\sqrt{-6\,y2^3 + 9\,\_C1}}\,dy2 - \_C2$$

Nonlinear ODE II

```
> ODE3 := diff( y(x),x$2 ) + y(x)^2 = 7*x^(-4);
```

$$ODE3 := \left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)\right) + \mathrm{y}(x)^2 = \frac{7}{x^4}$$

```
> IC := y(1)=1; BC := y(3)=1/3;
```

$$IC := \mathrm{y}(1) = 1$$

$$BC := \mathrm{y}(3) = \frac{1}{3}$$

```
> dsolve( { ODE3, IC, BC }, y(x) );
> bvpsolve( { ODE3, IC, BC }, y(x) );
```

No general solution found

Numerical Solutions to Second-Order BVPs

```
> bvpsolve2 := proc( alpha )
>   local SYS, Y, dY, bcerr;
>   global n, SOL;
>   n := n+1;
>
>   SYS := { ODE, IC, D(y)(1)=alpha };
>   SOL := dsolve( SYS, y(x), type=numeric,
>                  output=listprocedure );
>
>   Y  := subs( SOL, y(x) );
>   dY := subs( SOL, diff(y(x),x) );
>   bcerr := eval( subs( D(y)=dY, y=Y, lhs(BC) - rhs(BC) ) );
>
>   print(printf( `iteration %d: alpha=%10.6f, bcerr=%10.6f\n`,
>                 n, alpha, bcerr ));
>   RETURN( bcerr )
> end:
```

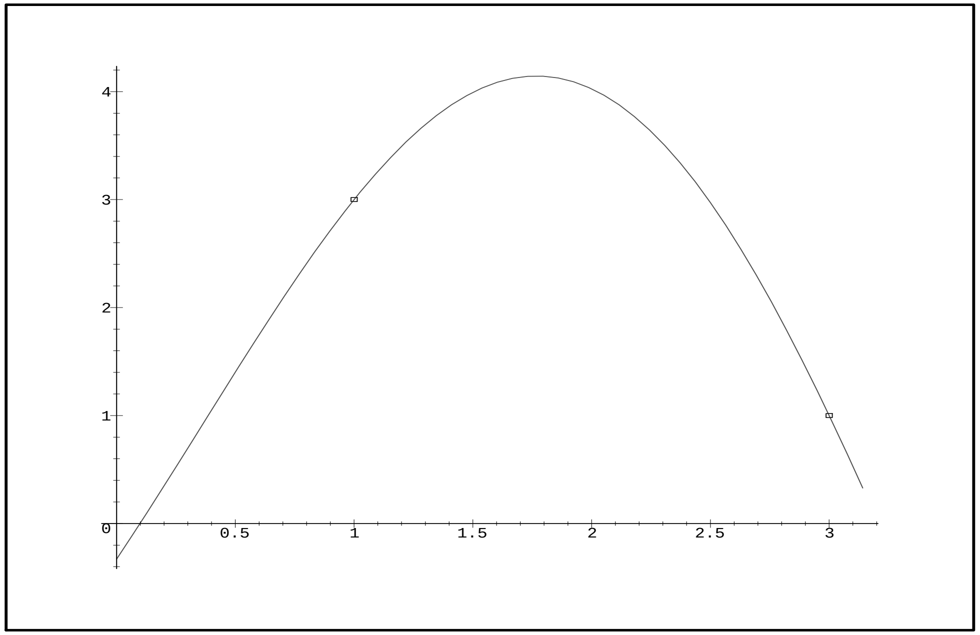
Example 1: Linear Two-Point BVP

```
> ODE := diff( y(x),x$2 ) + y(x) = cos(x);
> IC := y(1)=3;
> BC := y(3)=1;
> Digits := 5:
> n:=0:
> fsolve( 'bvpsolve2(alpha)'=0, alpha, 0..5);

iteration 1: alpha=  0.000000, bcerr= -2.489894
iteration 2: alpha=  5.000000, bcerr=  2.056592
iteration 3: alpha=  2.500000, bcerr=  -.216650
iteration 4: alpha=  2.738262, bcerr=   .000000
iteration 5: alpha=  2.738253, bcerr=  -.000007
```

$$2.7383$$

```
> Digits := 10:
```

```
> P1 := plots[odeplot]( SOL, [x,y(x)], 0..Pi):
> P2 := plot( [[1,3],[3,1]], style=POINT,
>               color=BLUE, symbol=BOX ):
> plots[display]( [P1,P2] );
```



```
> dsolve( { ODE, IC, BC }, y(x) );
```

$$
\begin{aligned}
\mathrm{y}(x) = {} & \frac{1}{2}\sin(x)\,x \\
& + \frac{1}{2}\,\frac{(-3\sin(1) - 8 + 4\sin(1)^3 + 12\sin(1)^2)\cos(1)\cos(x)}{\sin(1)^2 - 1} \\
& + \frac{(2\sin(1)^3 + 6\sin(1)^2 - 2\sin(1) - 1)\sin(x)}{\sin(1)}
\end{aligned}
$$

```
> bvpsolve( { ODE, IC, BC }, y(x));
```

$$
\begin{aligned}
\mathrm{y}(x) = {} & \frac{1}{2}\sin(x)\,x \\
& + \frac{1}{2}\,\frac{(-3\sin(1) - 8 + 4\sin(1)^3 + 12\sin(1)^2)\cos(1)\cos(x)}{\sin(1)^2 - 1} \\
& + \frac{(2\sin(1)^3 + 6\sin(1)^2 - 2\sin(1) - 1)\sin(x)}{\sin(1)}
\end{aligned}
$$

```
> evalf( subs( x=1, diff( rhs("), x ) ));
```

$$2.738261577$$

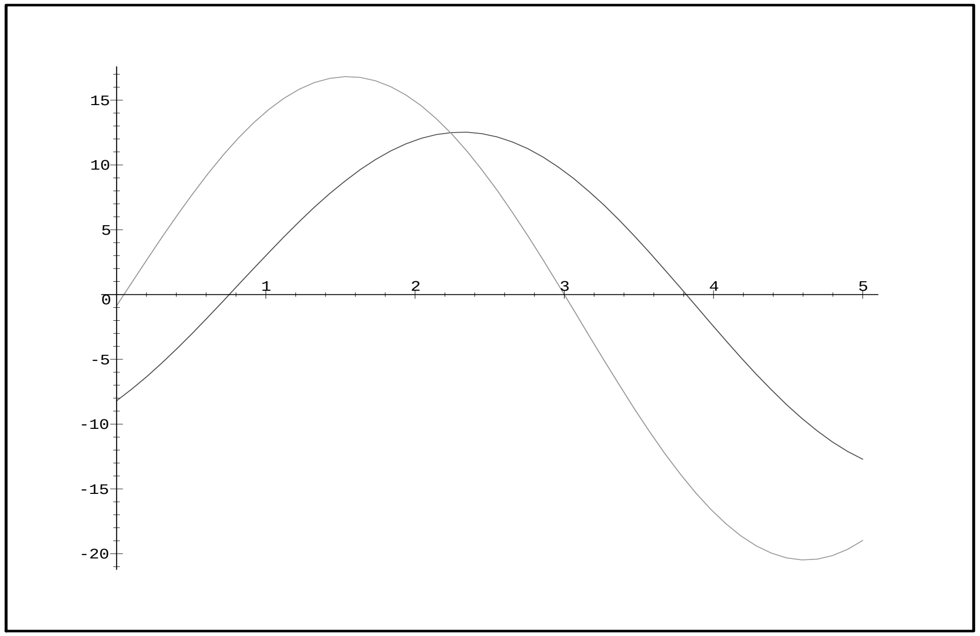
Example 2: Linear BVP with Robin BC

```
> ODE := diff( y(x),x$2 ) + y(x) = cos(x);
> IC := y(1)=3;
> BC := y(3)+D(y)(3)=1;
> Digits := 5:
> n:=0:
> fsolve( 'bvpsolve2(alpha)'=0, alpha, 0..15);

iteration 1: alpha=  0.000000, bcerr= -5.962131
iteration 2: alpha= 15.000000, bcerr=  1.435127
iteration 3: alpha=  7.500000, bcerr= -2.263501
iteration 4: alpha= 12.089880, bcerr=   .000000
iteration 5: alpha= 12.089840, bcerr=  -.000019
```

$$12.090$$

```
> Digits := 10:
```

```
> plots[odeplot]( SOL,
>                 [ [x,y(x)], [x,y(x)+diff(y(x),x)-1] ],
>                 0..5 );
```



```
> dsolve( { ODE, IC, BC }, y(x) );
> bvpsolve( { ODE, IC, BC }, y(x));
```

$$\begin{aligned}\mathrm{y}(x) = {}& \frac{1}{2}\sin(x)\,x + \frac{1}{2}(12\sin(1)^4 + 24\sin(1)^3 + 8\cos(1)\sin(1)^3 \\ & + 24\cos(1)\sin(1)^2 - 2\cos(1)\sin(1) - 6\cos(1) - 9\sin(1)^2 - 16\sin(1)) \\ & \cos(x)\Big/(2\sin(1)^2 - 1 - 2\cos(1)\sin(1)) + \frac{1}{2}(-12\sin(1)^2 + 3 \\ & + 11\cos(1)\sin(1) - 12\cos(1)\sin(1)^3 - 24\cos(1)\sin(1)^2 + 8\sin(1)^4 \\ & - 18\sin(1) + 24\sin(1)^3 + 4\cos(1))\sin(x)\Big/(\\ & 2\sin(1)^2 - 1 - 2\cos(1)\sin(1))\end{aligned}$$

```
> evalf( subs( x=1, diff(rhs(exSOL),x) ) );
```

$$12.08987956$$

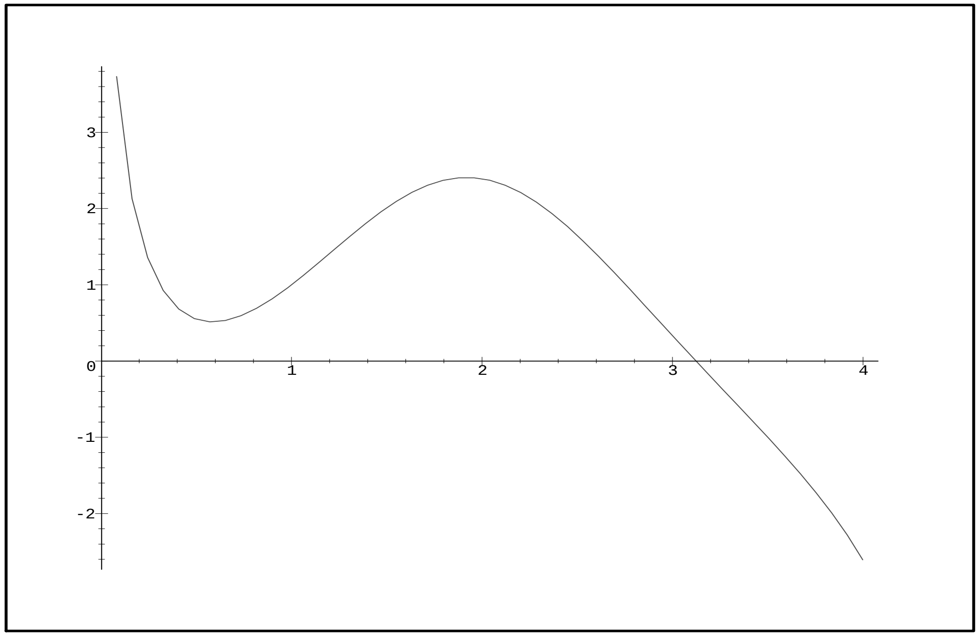
Example 3: Nonlinear Two-Point BVP

```
> ODE := diff( y(x),x$2 ) + y(x)^2 = 3*x^(-2);

> IC := y(1)=1;
> BC := y(3)=1/3;

> Digits := 5:
> n:=0:

> fsolve( 'bvpsolve2(alpha)'=0, alpha, 0..5);


iteration 1: alpha=  0.000000, bcerr=   .606250
iteration 2: alpha=  5.000000, bcerr= -2.090706
iteration 3: alpha=  2.500000, bcerr=  -.318299
iteration 4: alpha=  1.858074, bcerr=   .031241
iteration 5: alpha=  1.918843, bcerr=   .000459
iteration 6: alpha=  1.919744, bcerr=  -.000000
iteration 7: alpha=  1.919743, bcerr=   .000000
```

$$1.9197$$

```
> Digits := 10:
```

```
> plots[odeplot]( SOL, [x,y(x)], 0..4 );
```



Ongoing Efforts

- General Goals
  - BVPs deserve more attention than they currently receive
  - should some material be omitted from traditional courses to allow discussion of BVPs?
  - how do BVPs fit into newer courses that emphasize dynamical systems?
- Implementation
  - continued development of specialized and general CAS tools for symbolic, numeric, and graphic analysis of BVPs
  - collect realistic and interesting examples of BVPs for use as homework, in-class examples, and projects
  - generalize and extend `bvpsolve` and `bvpsolve2` (and `shoot`)


FoxTrot
BILL AMEND
JABBA THE HUTT... HUTT... HIKE!
I SAID HIKE.
I CAN'T GET THE BALL TO FIT BETWEEN MY LEGS.
PETER FOX, WIDE RECEIVER, CUTS THROUGH THE DEFENSE!
HE JUKES THE SAFETY! HE'S WIDE OPEN IN THE END ZONE!
HE ZIG-ZAGS LEFT AND RIGHT! HE'S STILL WIDE OPEN!
REALLY WIDE OPEN!
NO-ONE-FOR-MILES-A-TOUCH-DOWN-IS-A-GIVEN-THROW-THE-BALL-ALREADY WIDE OPEN!
OK, SO IF WE GO WITH A TRAJECTORY OF 30 DEGREES AND NEGLECT THE EFFECT OF AIR RESISTANCE...
OH, WAIT. HE STOPPED RUNNING. CHANGE V_{TARGET} TO ZERO IN THAT SECOND EQUATION.
GUYS, ALLOW ME TO DEMONSTRATE WHY QUARTERBACKS WEAR HELMETS...