Jerks Don't Jump: Mathematical Modelling with an Impact

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Overview of Presentation

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The Parachute Problem

A parachutist whose mass is 75 kg drops from a helicopter hovering 4000 m above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant and that the force due to air resistance is proportional to the velocity of the parachutist, with a proportionality constant $k_1 = 15$ kg/sec when the chute is closed and with a constant $k_2 = 105$ kg/sec when the chute is open. If the chute does not open until 1 min after the parachutist leaves the helicopter, after how many seconds will she hit the ground?

Source: Nagle and Saff (4th ed., p. 112)

Mathematical Formulation

- 2nd-order ODE

$$
\begin{aligned}
mx'' &= -mg - kx' \\
x(0) &= x_0 \\
x'(0) &= 0
\end{aligned}
$$

- 1st-order system

$$
\begin{aligned}
x' &= v, & x(0) = x_0 \\
v' &= -g - \tfrac{k}{m}v, & v(0) = 0
\end{aligned}
$$

- k —coefficient of air resistance

$$
k(t) = \begin{cases} k_1, & t < t_d \\ k_2, & t \geq t_d \end{cases}
$$

- t_d — time of (instantaneous) deployment

Explicit Solution

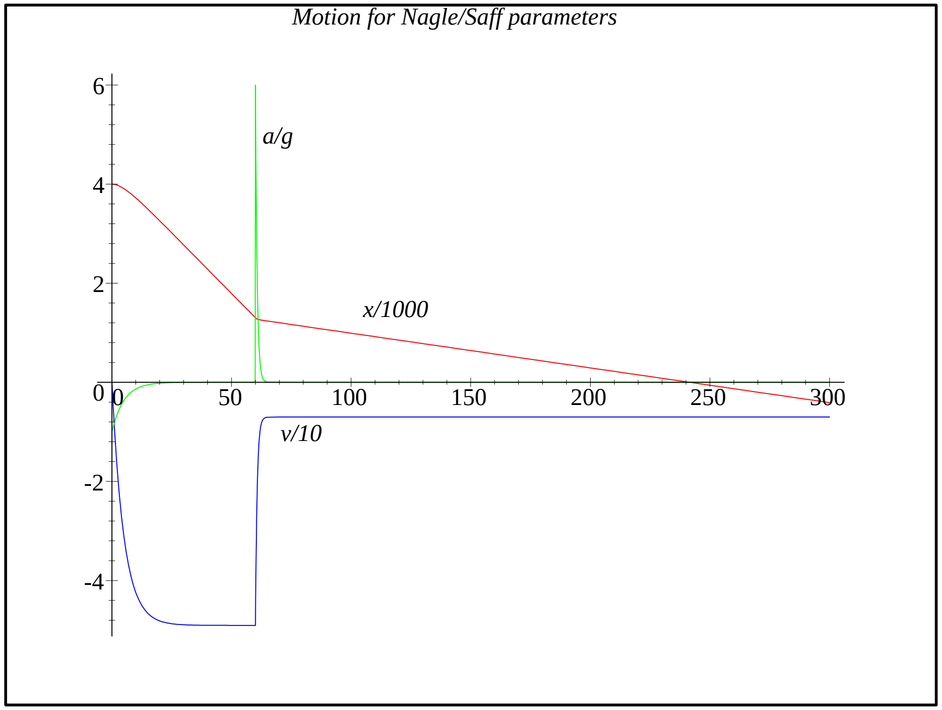
```
> SYS := { diff( x(t), t ) = v(t),
>          diff( v(t), t ) = -g - k/m*v(t) };
> IC := { x(0)=x0, v(0)=0 };
> K[pw_const] :=  k1+(k2-k1)*Heaviside(t-td):
> SOL1 := dsolve( eval(SYS,k=k1) union IC, { x(t), v(t) } ):
> X1 := eval( x(t), SOL1 );
> V1 := eval( v(t), SOL1 );
> IC2 := { x(td) = eval( X1, t=td ),
>          v(td) = eval( V1, t=td ) };
> SOL2 := dsolve( eval(SYS,k=k2) union IC2, { x(t), v(t) } ):
> X2 := eval( x(t), SOL2 );
> V2 := eval( v(t), SOL2 );
> X := X1 + (X2-X1)*Heaviside(t-td):
> V := V1 + (V2-V1)*Heaviside(t-td):
```

$$v(t) = \begin{cases} -\frac{mg}{k_1}\left(1 - e^{-\frac{tk_1}{m}}\right), & t < t_d \\ -\frac{mg}{k_2} + \frac{mg}{k_1 k_2}\left(k_1 - k_2\left(1 - e^{-\frac{t_d k_1}{m}}\right)\right) e^{-\frac{(t-t_d)k_2}{m}}, & t \geq t_d \end{cases}$$

Graphical Solution

Parameters (Nagle/Saff):

- $k_1 = 15$ kg/s, $k_2 = 105$ kg/s, $m = 75$ kg
- $x_0 = 4000$ m, $t_d = 60$ s, $g = 9.81$ m/s^2



Interpretation of Solution

- terminal velocities:
  - free-fall: $v_{\text{ff}} \approx 49$ m/s
  - final descent: $v_{\text{impact}} \approx 7$m/s
- position:
  - deployment at 1300 m
  - land after 241 s
- acceleration:
  - jump discontinuity at t_d
  - magnitude: 6 G

(all values approximate)

Real-Life Expectations

USAF Academy Parachute Team Training Jumps

- $x_0 = 1219$ m, $t_d = 10$ s, $v_{\text{ff}} = 53.6$ m/s
- $v_{\text{impact}} \in (4.6, 5.2)$ m/s
- parachute requires 3.2 s to fully deploy
- "snatch force" is a heavy, but smooth, tug; < 500 lbs$\approx$ 3 G)

An Improved Model

$$\begin{aligned} x' &= v, & x(0) &= x_0 \\ v' &= -g - \tfrac{k}{m} v, & v(0) &= 0 \end{aligned}$$

where

$$k(t) = \begin{cases} k_1, & t < t_d \\ k_d, & t_d \le t < t_d + \tau \\ k_2, & t \ge t_d + \tau \end{cases}$$

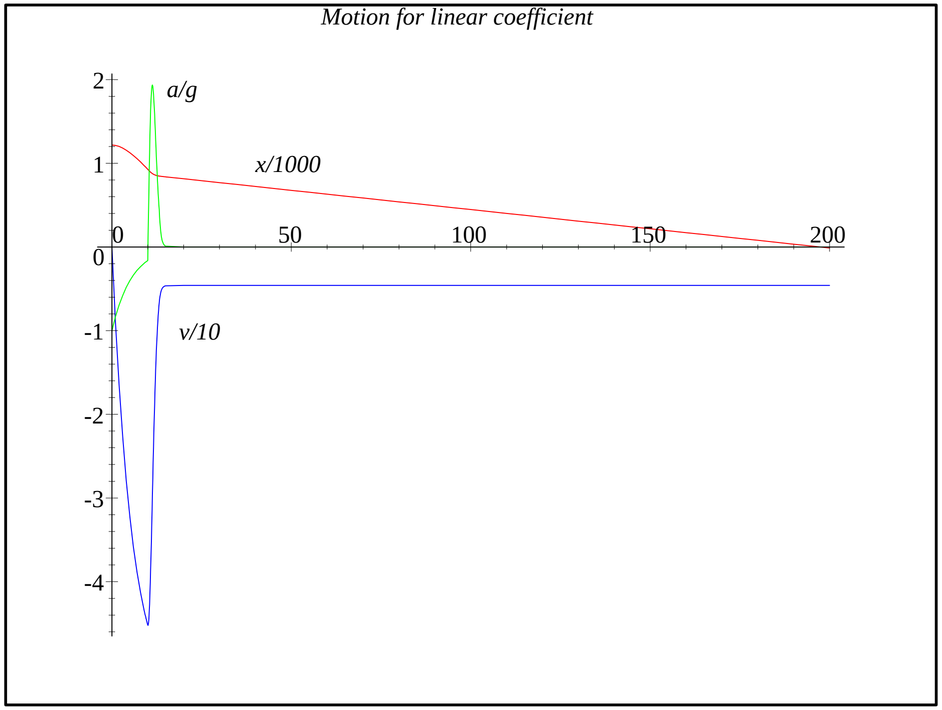
with parameter values

$k_1 = 13.63$ kg/s $\quad k_2 = 160$ kg/s $\quad t_d = 10$ s $\quad \tau = 3.2$ s

$x_0 = 1219.2$ m $\quad m = 75$ kg $\quad g = 9.81$ m/s^2

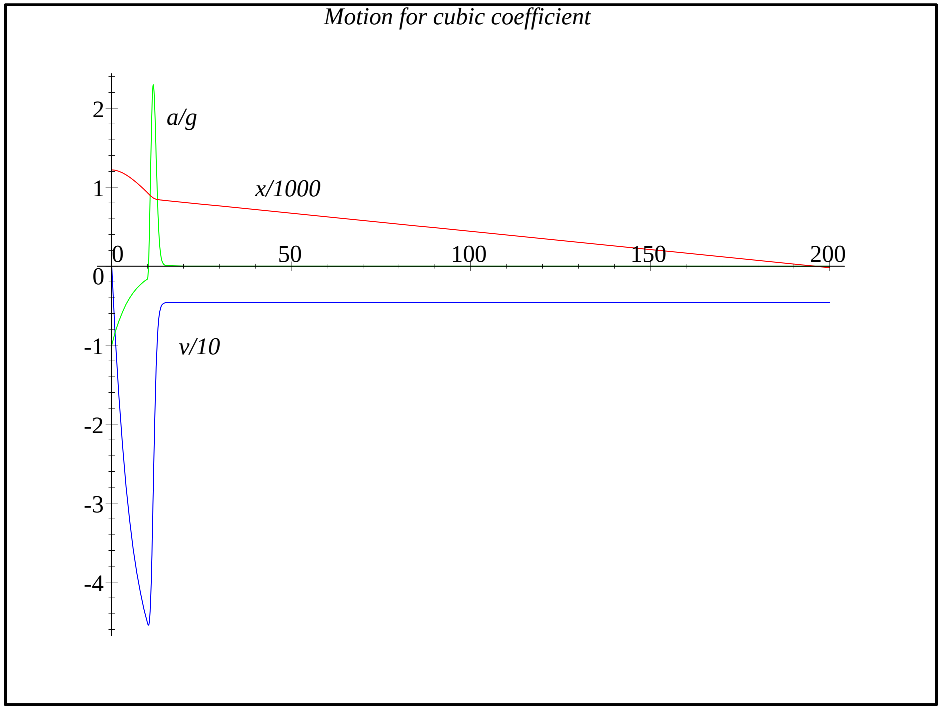
Graphical Analysis of Solution

- $k \in C^0$; k_d linear interpolant
- $a \in C^0$

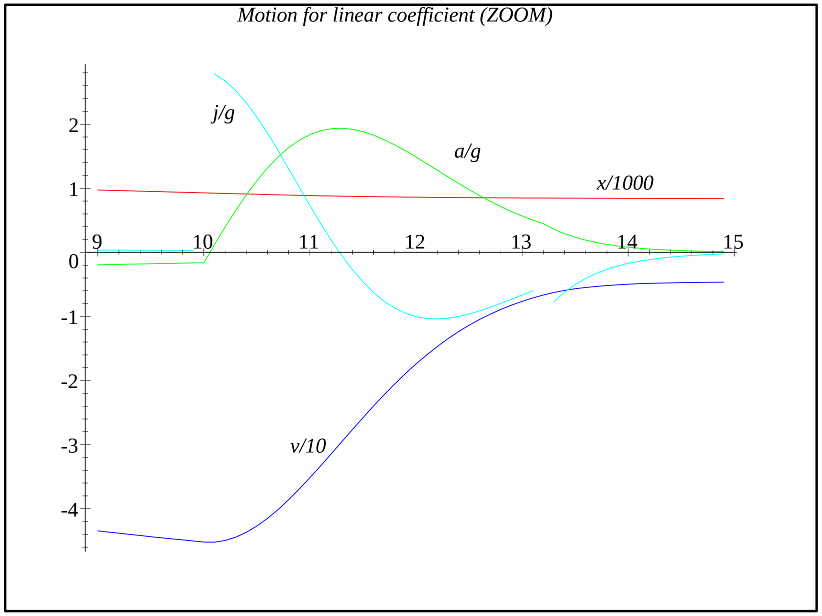


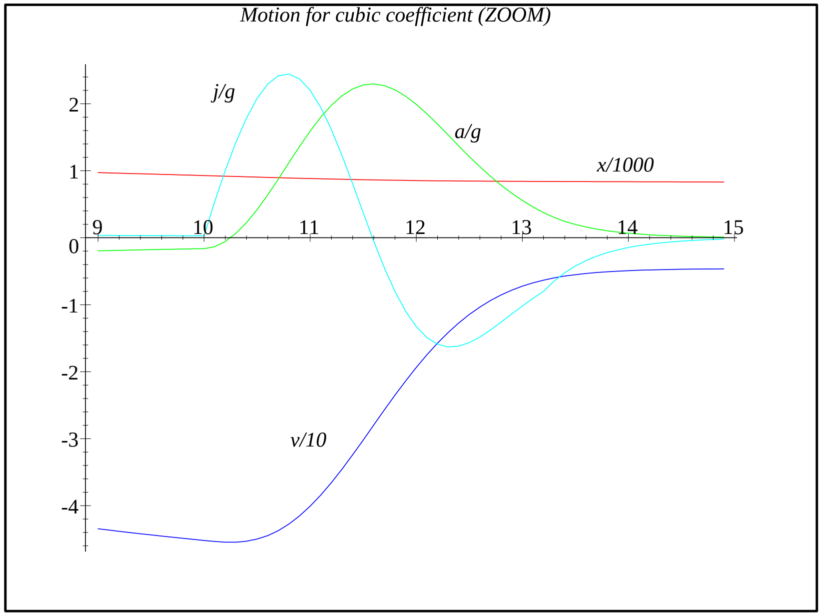
Graphical Analysis of Solution

- $k \in C^1$; k_d smooth cubic interpolant
- $a \in C^1$



Graphical Analysis of Solution – A Closer Look





Interpretation of Solution

- terminal velocities:
  - free-fall: $v_{\text{ff}} \approx 53.5$ m/s
  - final descent: $v_{\text{impact}} \approx 4.6\text{m/s} = 15.1$ ft/s
- position:
  - deployment at 928 m
  - land after 196 s
- acceleration and jerk:
  - no discontinuities
  - maximum acceleration: ≈ 2.2 G
  - maximum jerk: ≈ 2.4 G/s

(all values approximate)

Related Questions

- How does motion depend on deployment criterion?
  - choose t_d such that $v(t_d) = v^*$
  - choose t_d such that $x(t_d) = x^*$
  - choose t_d to minimize t_{impact} subject to $v(t_{\text{impact}}) \leq v^*$
- What limitations should be imposed to minimize safety risks?
  - consider parameters for reserve chute
  - maximum free fall which leaves enough time to safely activate reserve

General Remarks on Modelling

- good problems are everywhere, you just have to look
- be skeptical of what you read in textbooks
- understand the problem
  - physically
  - mathematically
- seek expert advice
- experiment
- keep problems realistic

  — and interesting!

References

- Air Force Academy Parachute Team Training Guide, 1990.
- Zim, *Parachutes*, 1942.
- Douglas B. Meade, *ODE models for the parachute problem*, SIAM Review **40**(2) June 1998, pp. 327–332.
- Douglas B. Meade, *Maple and the parachute problem: modelling with an impact*, MapleTech, **4**(1) 1997, pp. 68–76.