Maple: A Comprehensive Tool for Introductory ODEs

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Maple: Pro and Con

- Advantages
  - one tool, many uses
    - analytic computations
    - numeric approximations
    - graphical/visual representations
  - skills applicable beyond ODEs
  - wide availability at a reasonable cost
    - variety of site licenses
    - Student Version (Brooks/Cole, <$100)

Maple: Pro and Con

- Drawbacks
  - steep learning curve
  - changes emphasis of course
  - deterioration of manipulative skills

Minimal Hardware Requirements

Platform	Operating System	Disk	RAM
Windows	Windows 3.1.1+	14-21MB	4MB
Mac/PPC	System 6.0.7+	14-21MB	4MB
DOS	DOS 5.0+	14-21MB	4MB
Linux	Linux 1.0	30MB	8MB
UNIX	*various*	30MB	8MB
⋮	⋮	⋮	⋮

Requirements for Maple V, Release 3. A full listing of platforms and requirements is available on the WWW:

`http://www.maplesoft.com/Products/MapleV/MapleAvail.html`

Maple & Introduction to ODEs

- Analytical Methods
  - exact solutions: explicit and implicit
  - solution techniques
  - theory/structure of solutions
- Numerical Methods
  - examine convergence/approximation
  - implement numerical methods (including systems)
  - compute with complicated/realistic problems
- Graphical/Visual Representations
  - direction fields & phase portraits
  - qualitative analysis

Picard Iterates and Maple

- Picard's Method for $\quad \dfrac{dy}{dt} = f(t, y(t)), \quad y(t_0) = y0$
  - Given ϕ_0
  - Define, for all $k = 0, 1, 2, \ldots,$

$$\phi_{k+1}(t) = y_0 + \int_{t_0}^{t} f(x, \phi_k(x))\, dx.$$

- Maple implementation

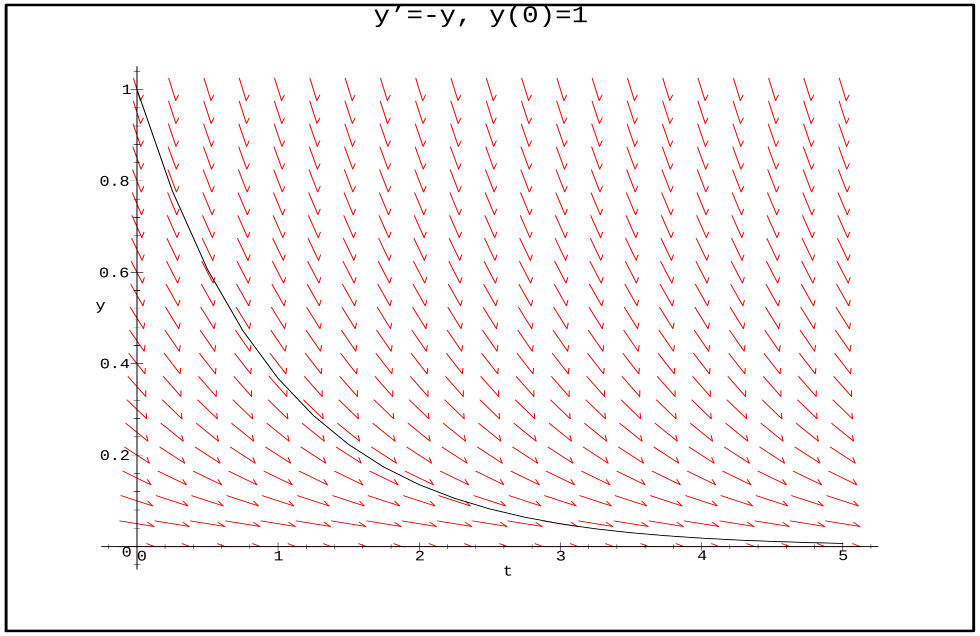
```
> restart;
> with( DEtools ):
> with( plots ):

> picard := proc(f, init, phi)
>   local t0, y0;
>   t0 := init[1];
>   y0 := init[2];
>   unapply( y0 + int( f(x,phi(x)), x=t0..t ), t );
> end:
```

Picard Iterates: Example 1

Consider the initial value problem: $$\frac{dy}{dy} = -y, \quad y(0) = 1$$

A first step is to visualize the solution to this problem

```
> DF := DEplot( diff(y(t),t)=-y(t), [t,y], t=0..5, {[0,1]},
>                 y=0..1, arrows=THIN, title=‘y’=-y, y(0)=1‘ ):
> display( DF );
```



Picard Iterates: Example 1

To use `picard` it is necessary to define:

- the RHS of the differential equation,

```
> RHS := (t,y) -> -y;
```

$$RHS := (t, y) \to -y$$

- the initial condition,

```
> IC := [ 0, 1 ];
```

$$IC := [0, 1]$$

- the initial guess,

```
> phi[0] := t -> 0;
```

$$\phi_0 := 0$$

- and the number of iterations

```
> N := 8;
```

$$N := 8$$

Picard Iterates: Example 1

Computation of the Picard iterates is easy to write in a loop

```
> for k from 0 to N-1 do
>   phi[k+1] := picard( RHS, IC, phi[k] );
> od;
```

$$\phi_1 := 1$$

$$\phi_2 := t \to 1 - t$$

$$\phi_3 := t \to 1 - t + \frac{1}{2}\,t^2$$

$$\vdots$$

$$\phi_8 := t \to 1 + \frac{1}{24}\,t^4 + \frac{1}{2}\,t^2 - \frac{1}{5040}\,t^7 - \frac{1}{120}\,t^5 + \frac{1}{720}\,t^6 - \frac{1}{6}\,t^3 - t$$

Picard Iterates: Example 1

Analysis is easier when terms are ordered

```
> for k from 0 to N-1 do
>   phi[k+1] := picard( RHS, IC, phi[k] ):
>   print( sort( simplify( phi[k+1](t) ) ) );
> od:
```

$$1$$

$$-t+1$$

$$\frac{1}{2}t^2-t+1$$

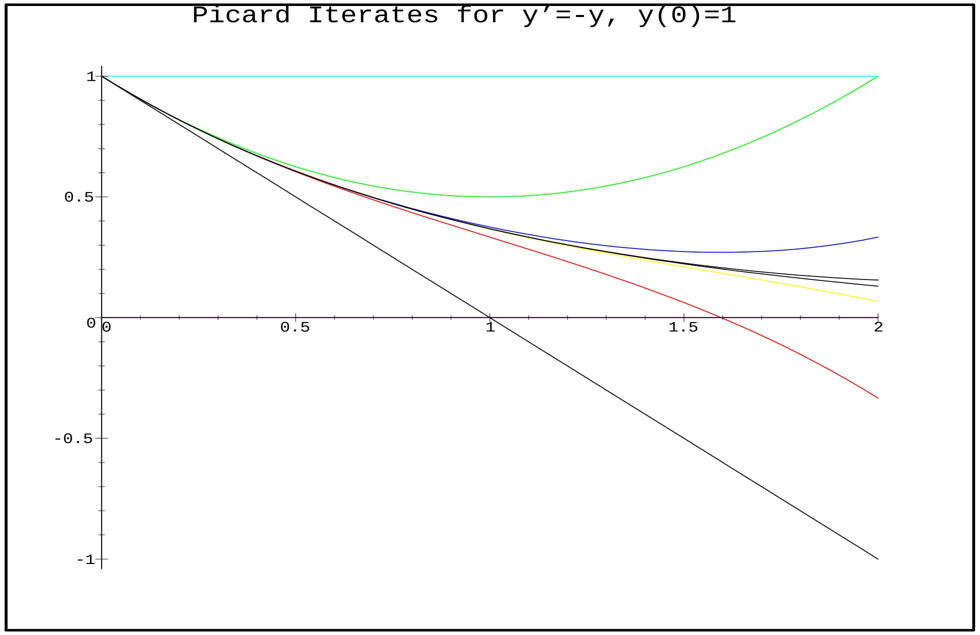
$$\vdots$$

$$-\frac{1}{5040}t^7+\frac{1}{720}t^6-\frac{1}{120}t^5+\frac{1}{24}t^4-\frac{1}{6}t^3+\frac{1}{2}t^2-t+1$$

Picard Iterates: Example 1

Convergence can be demonstrated graphically

```
> plot( { phi[i] $ i=0..N }, 0..2,
>        title=‘Picard Iterates for y’=-y, y(0)=1‘ );
```



Picard Iterates: Example 1

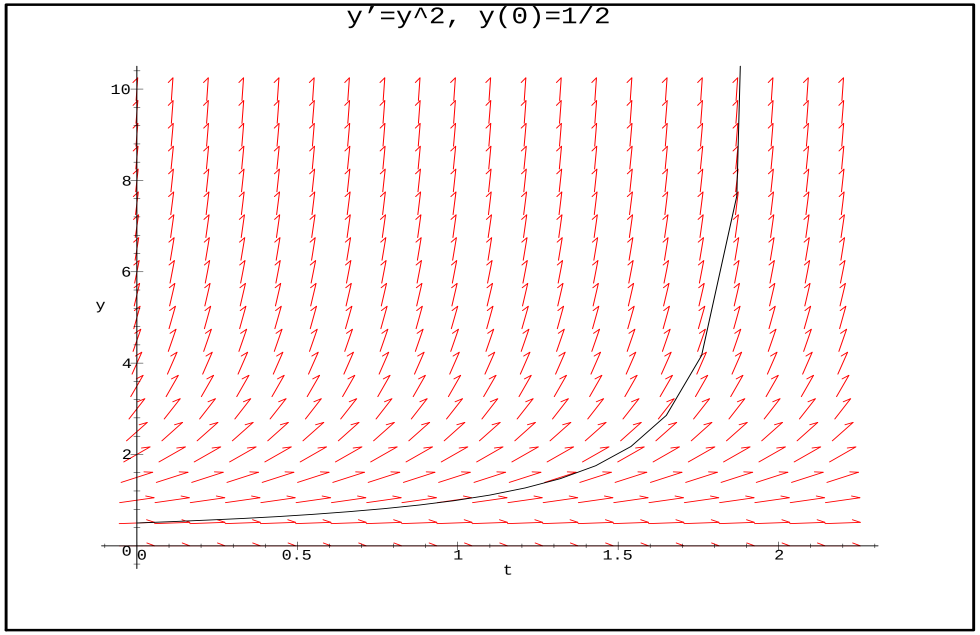
- Discussion
  - this sequence should look familiar: $\displaystyle\sum_{n=0}^{\infty} \frac{(-t)^n}{n!}$
  - provides additional justification that the solution is $y = e^{-t}$
  - animating the iterates can also be effective

```
> for k from 0 to N do
>  P[k]:=display([ DF, plot( phi[k],0..5,color=GREEN) ]);
> od:
> display( [ P[i] $ i=0..N ], insequence=true );
```

Picard Iterates: Example 2

Consider, now, the initial value problem: $\dfrac{dy}{dt} = y^2, \qquad y(0) = \dfrac{1}{2}$

```
> DF:=DEplot(diff(y(t),t)=y(t)^2, [t,y], t=0..2.2, {[0,1/2]},
>              y=0..10, arrows=THIN, title=`y'=y^2, y(0)=1/2` ):
> display( DF );
```



Picard Iterates: Example 2

Define the initial value problem and initial guess.

```
> RHS := (t,y) -> y^2;
```

$$RHS := (t, y) \to y^2$$

```
> IC := [ 0, 1/2 ];
```

$$IC := \left[0, \frac{1}{2}\right]$$

```
> phi[0] := t -> 0;
```

$$\phi_0 := 0$$

```
> N := 7;
```

$$N := 7$$

Picard Iterates: Example 2

Compute the first 7 iterates (last two omitted – 8 pages of output)

```
> for k from 0 to N-1 do
>   phi[k+1] := picard( RHS, IC, phi[k] );
>   print( sort( simplify( "(t) ) ) );
> od:
```

$$\frac{1}{2}$$

$$\frac{1}{4}\,t+\frac{1}{2}$$

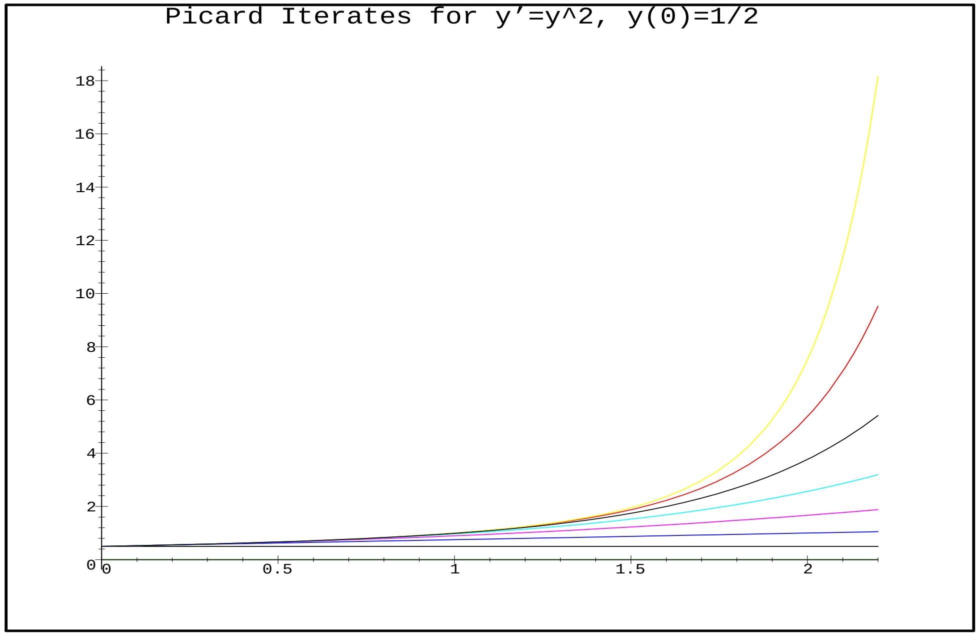
$$\frac{1}{48}\,t^3+\frac{1}{8}\,t^2+\frac{1}{4}\,t+\frac{1}{2}$$

$$\frac{1}{16128}\,t^7+\frac{1}{1152}\,t^6+\frac{1}{192}\,t^5+\frac{1}{48}\,t^4+\frac{1}{16}\,t^3+\frac{1}{8}\,t^2+\frac{1}{4}\,t+\frac{1}{2}$$

$$\begin{aligned}\frac{1}{3901685760}\,t^{15}&+\frac{1}{130056192}\,t^{14}+\frac{1}{9289728}\,t^{13}+\frac{1}{1032192}\,t^{12}\\&+\frac{5}{774144}\,t^{11}+\frac{11}{322560}\,t^{10}+\frac{43}{290304}\,t^9+\frac{71}{129024}\,t^8+\frac{29}{16128}\,t^7\\&+\frac{1}{192}\,t^6+\frac{13}{960}\,t^5+\frac{1}{32}\,t^4+\frac{1}{16}\,t^3+\frac{1}{8}\,t^2+\frac{1}{4}\,t+\frac{1}{2}\end{aligned}$$

Picard Iterates: Example 2

```
> plot( { phi[i] $ i=0..N }, 0..2.2,
>         title=`Picard Iterates for y'=y^2, y(0)=1/2` );
```



Picard Iterates: Example 2

- Discussion
  - Iterates converge to $\displaystyle\sum_{n=0}^{\infty} \frac{t^n}{2^{n+1}} = \frac{1}{2-t}$ (for $|t| < 2$)
  - Finite time "blow-up" is more clearly seen when the iterates are animated, and the exact solution is included in the plots

```
> for k from 0 to N do
>   P[k]:=display([DF, plot(phi[k],0..2.2,color=GREEN)]);
> od:
> display( [ P[i] $ i=0..N ], insequence=true );
```

Picard Iterates: Concluding Comments

- symbolic, numerical, and graphical tools in Maple have been used to provide an interactive introduction to Picard iterates and their role in the existence theory for IVPs

- additional benefits that can be obtained from this discussion:
  - an early introduction to power series solutions
  - rigorous discussion of existence/uniqueness theorem
  - discussion of analogous ideas for systems of IVPs