DERIVATIVES

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
3. $\frac{d}{dx}(\sin(x)) = \cos(x)$
4. $\frac{d}{dx}(\cos(x)) = -sin(x)$
5. $\frac{d}{dx}(\tan(x)) = \sec^2(x)$
6. $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
7. $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
8. $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
9. $\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$
10. $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$
11. $\frac{d}{dx}(\text{arcsec}(x)) = \frac{1}{|x|\sqrt{x^2-1}}$
12. $\frac{d}{dx}(e^x) = e^x$
13. $\frac{d}{dx}(a^x) = a^x(\ln(a))$
14. $\frac{d}{dx}(\sinh(x)) = \cosh(x)$
15. $\frac{d}{dx}(\cosh(x)) = \sinh(x)$

ANTIDERIVATIVES

1. $\int x^n \, dx = \frac{1}{n+1}x^{n+1} \quad n \neq -1$
   $\int \frac{dx}{x} = \ln(x),\ x > 0 \quad \text{or} \quad \ln|x|,\ x \neq 0$
2. $\int e^x \, dx = e^x$
3. $\int \sin(x) \, dx = -\cos(x)$
4. $\int \cos(x) \, dx = \sin(x)$
5. $\int \tan(x) \, dx = \ln|\sec(x)| = -\ln|\cos(x)|$
6. $\int \cot(x) \, dx = \ln|\sin(x)| = -\ln|\csc(x)|$
7. $\int \sec(x) \, dx = \ln|\sec(x) + \tan(x)| = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|$
8. $\int \csc(x) \, dx = \ln|\csc(x) - \cot(x)| = \ln\left|\tan\left(\frac{x}{2}\right)\right|$
9. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$

10. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a}\arcsin\left(\frac{x}{a}\right),\ a > 0$

11. $\displaystyle\int \frac{dx}{|x|\sqrt{x^2 - a^2}} = \frac{1}{a}\,\text{arcsec}\left(\frac{x}{a}\right)$

Expressions Containing $ax + b$

12. $\displaystyle\int (ax + b)^n\ dx = \frac{1}{a(n+1)}(ax+b)^{n+1}$

13. $\displaystyle\int \frac{dx}{ax + b} = \frac{1}{a}\ln|ax + b|$

14. $\displaystyle\int \frac{dx}{(ax+b)^2} = \frac{-1}{a(ax+b)}$

15. $\displaystyle\int \frac{x\ dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\ln|ax+b|$

16. $\displaystyle\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right|$

17. $\displaystyle\int \frac{dx}{x^2(ax+b)} = \frac{-1}{bx} + \frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right|$

18. $\displaystyle\int \sqrt{ax+b}\ dx = \frac{2}{3a}\sqrt{(ax+b)^3}$

19. $\displaystyle\int x\sqrt{ax+b}\ dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$

20. $\displaystyle\int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a}\sqrt{ax+b}$

21. $\displaystyle\int \frac{\sqrt{ax+b}}{x}\ dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$

22. $\displaystyle\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right|,\ b > 0$

23. $\displaystyle\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}},\ b < 0$

24. $\displaystyle\int \frac{dx}{x^2\sqrt{ax+b}} = \frac{-\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$

25. $\displaystyle\int \sqrt{\frac{cx+d}{ax+b}}\ dx = \frac{\sqrt{ax+b}\sqrt{cx+d}}{a} + \frac{ad-bc}{2a}\int \frac{dx}{\sqrt{ax+b}\sqrt{cx+d}}$

Expressions Containing $ax^2 + c$, $x^2 \pm p^2$, and $p^2 - x^2$, $p > 0$

26. $\displaystyle\int \frac{dx}{p^2 - x^2} = \frac{1}{2p} \ln\left|\frac{p+x}{p-x}\right|$

27. $\displaystyle\int \frac{dx}{ax^2 + c} = \begin{cases} \frac{1}{\sqrt{ac}} \arctan\left(x\sqrt{\frac{a}{c}}\right) & a > 0,\ c > 0 \\ \frac{1}{2\sqrt{-ac}} \ln\left|\frac{x\sqrt{a}-\sqrt{-c}}{x\sqrt{a}+\sqrt{-c}}\right| & a > 0,\ c < 0 \\ \frac{1}{2\sqrt{-ac}} \ln\left|\frac{\sqrt{c}+x\sqrt{-a}}{\sqrt{c}-x\sqrt{-a}}\right| & a < 0,\ c > 0 \end{cases}$

28. $\displaystyle\int \frac{dx}{(ax^2+c)^n} = \frac{1}{2(n-1)c}\frac{x}{(ax^2+c)^{n-1}} + \frac{2n-3}{2(n-1)c}\int \frac{dx}{(ax^2+c)^{n-1}} \qquad n > 1$

29. $\displaystyle\int x(ax^2+c)^n\ dx = \frac{1}{2a}\frac{(ax^2+c)^{n+1}}{n+1} \qquad n \neq 1$

30. $\displaystyle\int \frac{x}{ax^2+c}\ dx = \frac{1}{2a}\ln\left|ax^2+c\right|$

31. $\displaystyle\int \sqrt{x^2 \pm p^2}\ dx = \frac{1}{2}\left(x\sqrt{x^2 \pm p^2} \pm p^2 \ln\left|x + \sqrt{x^2 \pm p^2}\right|\right)$

32. $\displaystyle\int \sqrt{p^2 - x^2}\ dx = \frac{1}{2}\left(x\sqrt{p^2 - x^2} + p^2 \arcsin\left(\frac{x}{p}\right)\right)$

33. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm p^2}} = \ln\left|x + \sqrt{x^2 \pm p^2}\right|$

34. $\displaystyle\int (p^2 - x^2)^{3/2}\ dx = \frac{x}{4}\left(p^2 - x^2\right)^{3/2}\Big) + \frac{3p^2x}{8}\sqrt{p^2 - x^2} + \frac{3p^4}{8}\arcsin\left(\frac{x}{p}\right)$

Expressions Containing $ax^2 + bx + c$

35. $\displaystyle\int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{1}{\sqrt{b^2-4ac}} \ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & b^2 > 4ac \\ \frac{2}{\sqrt{4ac-b^2}} \arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) & b^2 < 4ac \\ \frac{-2}{2ax+b} & b^2 = 4ac \end{cases}$

36. $\displaystyle\int \frac{dx}{(ax^2+bx+c)^{n+1}} = \frac{2ax+b}{n(4ac-b^2)(ax^2+bx+c)^n} + \frac{2(2n-1)a}{n(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^n}$

37. $\displaystyle\int \frac{x\ dx}{ax^2+bx+c} = \frac{1}{2a}\ln\left|ax^2+bx+c\right| - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c}$

38. $\displaystyle\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right| & a > 0 \\ \frac{1}{\sqrt{-a}} \arcsin\left(\frac{-2ax-b}{\sqrt{b^2-4ac}}\right) & a < 0 \end{cases}$

39. $\displaystyle\int \frac{x\ dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$

40. $\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Expressions Containing Powers of Trigonometric Functions

41. $\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$

42. $\int \sin^3(ax)\,dx = \frac{-1}{a}\cos(ax) + \frac{1}{3a}\cos^3(ax)$

43. $\int \sin^n(ax)\,dx = -\frac{\sin^{(n-1)}(ax)\cos(ax)}{na} + \frac{n-1}{n}\int \sin^{(n-2)}(ax)\,dx$, $n \geq 2$ positive integer

44. $\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}$

45. $\int \cos^3(ax)\,dx = \frac{1}{a}\sin(ax) - \frac{1}{3a}\sin^3(ax)$

46. $\int \cos^n(ax)\,dx = \frac{\cos^{(n-1)}(ax)\sin(ax)}{na} + \frac{n-1}{n}\int \cos^{(n-2)}(ax)\,dx$, $n \geq$ positive integer

47. $\int \tan^2(ax)\,dx = \frac{1}{a}\tan(ax) - x$

48. $\int \tan^3(ax)\,dx = \frac{1}{2a}\tan^2(ax) + \frac{1}{a}\ln|\cos(ax)|$

49. $\int \tan^n(ax)\,dx = \frac{\tan^{(n-1)}(ax)}{a(n-1)} - \int \tan^{(n-2)}(ax)\,dx$, $n \neq 1$

50. $\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax)$

51. $\int \sec^3(ax)\,dx = \frac{1}{2a}\sec(ax)\tan(ax) + \frac{1}{2a}\ln|\sec(ax)+\tan(ax)|$

52. $\int \sec^n(ax)\,dx = \frac{\sec^{(n-2)}(ax)\tan(ax)}{a(n-1)} - \frac{n-2}{n-1}\int \sec^{(n-2)}(ax)\,dx$, $n \neq 1$

53. $\int \frac{dx}{1 \pm \sin(ax)} = \mp\frac{1}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$

Expressions Containing Algebraic and Trigonometric Functions

54. $\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$

55. $\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$

56. $\int x^n\sin(ax)\,dx = \frac{-1}{a}x^n\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)\,dx \quad n$ positive

57. $\int x^n \cos(ax)\, dx = \frac{1}{a}x^n \sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\, dx \qquad n \text{ positive}$

58. $\int \sin(ax)\cos(bx)\, dx = \frac{-\cos((a-b)x)}{2(a-b)} - \frac{\cos((a+b)x)}{2(a+b))} \qquad a^2 \neq b^2$

Expressions Containing Exponential and Logarithmic Functions

59. $\int xe^{ax}\, dx = \frac{1}{a^2}e^{ax}(ax-1)$

60. $\int xb^{ax}\, dx = \frac{1}{a^2}\frac{b^{ax}}{(\ln(b))^2}(a\ln(b)x - 1)$

61. $\int x^n e^{ax}\, dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx$

62. $\int e^{ax}\sin(bx)\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx))$

63. $\int e^{ax}\cos(bx)\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos(bx) + b\sin(bx))$

64. $\int \ln(ax)\, dx = x\,(\ln(ax) - 1)$

65. $\int x^n \ln(ax)\, dx = x^{n+1}\left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2}\right) \qquad n = 0, 1, 2, \ldots$

66. $\int (\ln(ax))^2\, dx = x^2\left((\ln(ax))^2 - 2\ln(ax) + 2\right)$

Expressions Containing Inverse Trigonometric Functions

67. $\int \arcsin(ax)\, dx = x\arcsin(ax) + \frac{1}{a}\sqrt{1-a^2x^2}$

68. $\int \arccos(ax)\, dx = x\arccos(ax) - \frac{1}{a}\sqrt{1-a^2x^2}$

69. $\int \text{arcsec}(ax)\, dx = x\,\text{arcsec}(ax) - \frac{1}{a}\ln\left|ax + \sqrt{a^2x^2-1}\right|$

70. $\int \text{arccsc}(ax)\, dx = x\,\text{arccsc}(ax) + \frac{1}{a}\ln\left|ax + \sqrt{a^2x^2-1}\right|$

71. $\int \arctan(ax)\, dx = x\arctan(ax) - \frac{1}{2a}\ln\left(1 + a^2x^2\right)$

72. $\int \text{arccot}(ax)\, dx = x\,\text{arccot}(ax) + \frac{1}{2a}\ln\left(1 + a^2x^2\right)$

Some Special Integrals

73. $\int_0^{\pi/2} \sin^n(x)\ dx = \int_0^{\pi/2} \cos^n(x)\ dx = \begin{cases} \frac{1\cdot3\cdot5\cdot7\cdots(n-1)}{2\cdot4\cdot6\cdot8\cdots(n)}\frac{\pi}{2} & n \text{ even} \\ \frac{2\cdot4\cdot6\cdot8\cdots(n-1)}{1\cdot3\cdot5\cdot7\cdots(n)} & n \text{ odd} \end{cases}$

74. $\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$

Calculus

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Contents

Preface

As we wrote each section of this book, we kept in our minds an image of the student who will be using it. The student will be busy, taking other demanding classes besides calculus. Also that student may well need to understand the vector analysis chapter, which represents the culmination of the theory and applications within the covers of this book.

That image shaped both the exposition and the exercises in each section.

A section begins with a brief introduction. Then it quickly moves to an informal presentation of the central idea of the section, followed by examples. After the student has a feel for the core of the section, a formal proof is given.

Those proofs are what hold the course together and serve also as a constant review. For this reason we chose student-friendly proofs, adequately motivated. For instance, instead of the elegant, short proof that absolute convergence of a series implies convergence, we employed a longer, but more revealing proof. We avoid pulling tricks out of thin air; hence our new motivation of the cross product. Where one proof will do, we do not use two. Also, rather than proving the theorem in complete generality, we may treat only a special case, if that case conveys the flow of the general proof.

As we assembled the exercises we labeled them R (routine), M (medium), and C (challenging), to make sure we had enough of each type. The R- exercises focus on definitions and algorithmic calculations. The M-type require more thought. The C-type either demand a deeper understanding or offer an alternative view of the material.

In order to keep the sections as short as feasible, we concentrated on the mathematics. We avoided bringing in too many applications in the text, which would not only make the sections too long to be read by a busy student, but would not do justice to the applications. However, because the applications are the reason most students study the subject, each chapter concludes with a thorough treatment of an application in a section called "Calculus is Everywhere." Because each stands alone, students and instructors are free to deal with it as they please, depending on time available and interest: skip it, glance at it, browse through it, or read it carefully. The presence of the Calculus is Everywhere sections allowed us to replace exercises that start with a long description of an application and end with a trivial bit of calculus. Our guiding theme is do one thing at a time, whether it's exposition, an example, or an exercise.

As we worked on each section we asked ourselves several questions: Is it the right length? Does it get to the point quickly? Does it focus on just one idea and correspond to one lecture? Are there enough examples? Are there enough exercises, from routine to challenging?

Curvature is treated twice, first in the plane, without vectors, and later, in space, with vectors. We do this for two reasons. First, it provides the student background for appreciating the vector approach. Second, it reduces the vector treatment section to a reasonable length.

Many students will use vector analysis in engineering and physics courses. One of us sat in on a sophomore level electromagnetic course in order to find out how the concepts were applied and what was expected of the students. That inspired a major revision of that chapter.

In addition, the new edition reaches limits and derivatives as early as possible, and as simply as possible. Also, we introduce the Permanence Property, which implies that a continuous function that is positive at a number remains positive nearby. This is referred to several times; hence we gave it a name.

The controversy about what to do about epsilon-delta proofs will never end. Therefore in our text the instructor is free to choose what to do about such proofs. To make our treatment student-friendly, we broke it into two sections. The first section treats limits at infinity because the diagrams are easier and the concept is more accessible. The second deals with limits at a number. A rigorous proof is given there of the Permanence Property, illustrating the power of the epsilon-delta approach to demonstrate something that is not intuitively obvious. Later in the book the rigorous approach appears only in some C-level exercises, giving the instructor and student an opportunity to reinforce that approach if they so choose.

Throughout the book we include exercises that ask only for computing a derivative or an integral. These exercises are intended to keep those skills sharp. We do not want to assign to exercises that explore a new concept the additional responsibility of offering extensive practice in calculations. This illustrates our general principle: do only one thing at a time, and do it clearly.

One of our objectives was to develop throughout the chapters the mathematical maturity a student needs to understand the vector analysis in the final chapter. For instance, we often include an exercise which asks the student to state a theorem in their own words without mathematical symbols. We had found while doing some pro-bono tutoring that students do not read a theorem carefully. No wonder they didn't know what to do when a supposedly routine exercise asked them to verify a theorem in a particular case.

Notes to the Instructor

§1.1 A review and a reference. It gets right to the point. The examples provide background for later work. Exercises 35 to 39 bring in the transcendental functions early.

§1.2 Reinforces the exponential and logarithmic functions early and its summary emphasizes the most difficult functions, logarithms. We save "modeling" for later, abiding by our principle, "one section, one main idea." Exercise 52 asks students to think on their own, to be ready for the last third of the book.

§1.3 Quickly builds all the functions needed. We do this for two reasons: to give the students more time to deal with them and to have them available for examples and exercises.

Following our policy of doing just one thing at a time, we develop limits in Chapter 2, separating them from their application in Chapter 3, which introduces the derivative.

§2.2 Focuses on the basic limits needed in Chapter 3. The binomial theorem is not used because many students are not familiar or comfortable with it.

§2.5 Introduces the Permanence Property, which is used several times in later chapters. Hence, we give it a name.

§2.6 Chapter summaries offer an overall perspective and emphasis not possible in an individual section.

§3.1 Introduces the derivative in the traditional way, by velocity and the tangent line. Because of the earlier development of the key limits, this section can be kept short.

§3.3 By using the Δ-notation, we obtain the derivatives of $f + g$, fg, and f/g without using any "student unfriendly" tricks, such as adding and subtracting $f(x)g(x)$.

§3.4 The rigorous proof of the chain rule is left as an exercise with detailed sketch. That enables the student reading the text to concentrate on learning how to apply the chain rule.

§3.5 Obtains the derivatives of the inverse functions, using the chain rule. There is no need to wait until implicit differentiation is discussed. That way the chapter can focus on obtaining the differentiation formulas. Exercises 76 and 86 are two of the "Sam and Jane" exercises that add a light touch and invite the students to think on their own.

§3.6 Introduces antiderivatives well before the definite integral appears in Chapter 6, so that the two concepts are adequately separated in time. Slope fields will be used later.

§3.7 Note that the higher derivatives will be put to work as early as Section 5.4, which concerns Taylor polynomials.

§3.8 and 3.9 We delayed the precise definitions of limits in order to give the students more time to work with limits before facing these definitions. These sections are optional. Section 3.8 is easier. One may separate the two sections by several days to let the first one sink in. Note that Example 2 in Section 3.9 shows how useful a precise definition is, as it justifies the Permanence Principle.

§3.9 Emphasizes the essentials and invites more practice in differentiation. Throughout the remaining chapters we include exercises on straightforward differentiation.

Chapter 4 Concentrates on just one theme: using f' and f'' to graph a function. This provides a strong foundation for Chapter 5, which includes optimization.

§5.3 Shows how a higher derivative influences the growth of a function and sets the stage for Section 5.4, Taylor polynomials and their errors. The growth theorem of Section 5.3 is used in exercises in Chapter 6 to obtain the error in approximating a definite integral by the trapezoidal or Simpson's methods.

§5.6 Exercise 39 raises interesting questions about exponential growth.

§6.1 This section keeps to a readable length by avoiding involvement with a formula for the sum $1^2 + 2^2 + \cdots + n^2$.

§6.2 Anticipates the formula $F(b) - F(a)$ for evaluating a definite integral.

§6.5 Exercises such as 44 and 45 are not as hard as one would expect, because the steps are outlined. Such exercises review several important concepts.

Overview of Calculus I

There are two main concepts in calculus: the derivative and the integral. Two scenarios that could occur in your car introduce both concepts.

Scenario A
Your speedometer is broken, but your odometer works. Your passenger writes down the odometer reading every second. How could you estimate the speed, which may vary from second to second?

The speedometer measures your current speed. The odometer measures the total distance covered.

This scenario is related to the "derivative," the key concept of differential calculus. The derivative tells how rapidly a quantity changes if we know how much of it there is at any instant. (If the change is at a constant rate, the rate of change is just the total change divided by the total time, and no derivative is needed.)

The second scenario is the opposite.

Scenario B
Your odometer is broken, but your speedometer works. Your passenger writes down the speed every second. How could you estimate the total distance covered?

This scenario is related to the "definite integral," the key concept of integral calculus. This integral represents the total change in a varying quantity, if you know how rapidly it changes — even if the rate of change is not constant. (If the speed stays constant, you just multiply the speed times the total time, and no integral is needed.)

Both the derivative and the integral are based on limits, treated in Chapter 2. Chapter 3 defines the derivative, while Chapters 4 and 5 present some of its applications. Chapter 6 defines the integral.

As you would expect by comparing the two scenarios, the derivative and the integral are closely related. This connection is the basis of the Fundamental Theorem of Calculus (Section 6.4), which shows how the derivative provides a shortcut for computing many integrals.

Summary of Calculus I

The limit is the fundamental concept that forms the foundation for all of calculus. Limits are introduced in Chapter 2.

Chapters 3 through 5 were devoted to one of the two basic concepts in calculus, the derivative, defined as the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

It tells how rapidly a function changes for inputs near x. That is local information.

Chapter 6 introduced the other major concept in calculus, the definite integral, also defined as a limit

$$\int_a^b f(x)\ dx = \lim_{\max \Delta x_i} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

For a continuous function this limit exists. $\int_a^b f(x)\ dx$ can be viewed as the (net) area under the graph of $y = f(x)$ above the interval $[a, b]$. Both the definite integral and an antiderivative of a function are called "integrals." Context tells which is meant. An antiderivative is also called an "indefinite integral."

The definite integral, in contrast to the derivative, gives global or overall information.

Integrand: $f(x)$	Integral: $\int_a^b f(x)\ dx$
velocity	change in position
speed (= |velocity|)	distance traveled
length of cross-section of plane region	area of region
area of cross-section of solid	volume of solid
rate bacterial colony grows	total growth

As the first and last of these applications show, if you compute the definite integral of the rate at which some quantity is changing, you get the total change. To put this in mathematical symbols, let $F(x)$ be the quantity present at time x. Then $F'(x)$ is the rate at which it changes.

That equation gives a shortcut for evaluating many common definite integrals. However, finding an antiderivative can be tedious or impossible.

For instance, e^{x^2} does not have an elementary antiderivative. However, continuous functions do have antiderivatives, as slope fields suggest. Indeed $G(x) = \int_a^x f(t)\ dt$ is an antiderivative of the integrand.

One way to estimate a definite integral is to employ one of the sums $\sum_{i=1}^{n} f(c_i)\Delta x_i$ that appear in its definition. A more accurate method, which uses the same amount of arithmetic, uses trapezoids. The trapezoidal estimate takes the form

$$\int_a^b f(x)\ dx \approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right),$$

where consecutive x_is are a fixed distance $hx = (b-a)/n$ apart.

In the even more accurate Simpson's estimate the graph is approximated by parts of parabolas, n is even, and the estimate is

$$\int_a^b f(x)\ dx \approx \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{x-1}) + 4f(x_{n-1}) + f(x_n)\right).$$

Long Road to Calculus

It is often stated that Newton and Leibniz invented calculus in order to solve problems in the physical world. There is no evidence for this claim. Rather, as with their predecessors, Newton and Leibniz were driven by curiosity to solve the "tangent" and "area" problems, that is, to construct a general procedure for finding tangents and areas. Once calculus was available, it was then applied to a variety of fields, notably physics, with spectacular success.

The first five chapters have presented the foundations of calculus in this order: functions, limits and continuity, the derivative, the definite integral, and the fundamental theorem that joins the last two. This bears little relation to the order in which these concepts were actually developed. Nor can we sense in this approach, which follows the standard calculations syllabus, the long struggle that culminated in the creation of calculus.

The origins of calculus go back over 2000 years to the work of the Greeks on areas and tangents. Archimedes (287–212 B.C.) found the area of a section of a parabola, an accomplishment that amounts in our terms to evaluating $\int_0^b x^2 dx$. He also found the area of an ellipse and both the surface area and the volume of a sphere. Apollonius (around 260–200 B.C.) wrote about tangents to ellipses, parabolas, and hyperbolas, and Archimedes discussed the tangents to a certain spiral-shaped curve. Little did they suspect that the "area" and "tangent" problems were to converge many centuries later.

With the collapse of the Greek world, symbolized by the Emperor Justinian's closing in A.D. 529 of Plato's Academy, which had survived for a thousand years, it was the Arab world that preserved the works of Greek mathematicians. In its liberal atmosphere, Arab, Christian, and Jewish scholars worked together, translating and commenting on the old writings, occasionally adding their own embellishments. For instance, Alhazen (A.D. 965–1039) computed volumes of certain solids, in essence evaluating $\int_0^b x^3 dx$ and $\int_0^b x^4 dx$.

It was not until the seventeenth century that several ideas came together to form calculus. In 1637, both Descartes (1596–1650) and Fermat (1601–165) introduced analytic geometry. Descartes examined a given curve with the aid of algebra, while Fermat took the opposite tack, exploring the geometry hidden in a given equation. For instance, Fermat showed that the graph of $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is always an ellipse, hyperbola, parabola, or one of their degenerate forms.

In this same period, Cavalieri (1598–1647) found the area under the curve $y = x^n$ for $n = 1, 2, 3, \ldots, 9$ by a method the length of whose computations grew rapidly as the exponent increased. Stopping at $n = 0$, he conjectured that the pattern would continue for larger exponents. In the next 20 years, several mathematicians justified his guess. So, even the calculation of the area under $y = x^n$ for a positive integer n, which we take for granted, represented a hard-won triumph.

"What about the other exponents?" we may wonder. Before 1665 there were no other exponents. Nevertheless, it was possible to work with the function which we denote $y = x^{p/q}$ for positive integers p and q by describing it as the function y such that $y^q = x^p$. (For instance, $y = x^{2/3}$ would be the function y that satisfies $y^3 = x^2$.) Wallis (1616–1703) found the area by a method that smacks more of magic than of mathematics. However, Fermat obtained the same result with the aid of an infinite geometric series.

The problem of determining tangents to curves was also in vogue in the first half of the seventeenth century. Descartes showed how to find a line perpendicular to a curve at a point P (by constructing a circle that meets the curve only at P); the tangent was then the line through P perpendicular to that line. Fermat found tangents in a way similar to ours and applied it to maximum-minimum problems.

Newton (1642-1727) arrived in Cambridge in 1661, and during the two years 1665–1666, which he spent at his family's farm to avoid the plague, he developed the essentials of calculus — recognizing that finding tangents and calculating areas are inverse problems. The first integral table ever compiled is to be found in one of his manuscripts of this period. But Newton did not publish his results at that time, perhaps because of the depression in the book trade after the Great Fire of London in 1665. During those two remarkable years he also introduced negative and fractional exponents, thus demonstrating that such diverse operations as multiplying a number by itself several times, taking its reciprocal, and finding a root of some power of that number are just special cases of a single general exponential function a^x, where x is a positive integer, -1, or a fraction, respectively.

Independently, however, Leibniz (1646–1716) also invented calculus. A lawyer, diplomat, and philosopher, for whom mathematics was a serious avocation, Leibniz established his version in the years 1673–1676, publishing his researches in 1684 and 1686, well before Newton's first publication in 1711. To Leibniz we owe the notations dx and dy, the terms "differential calculus" and "integral calculus," the integral sign, and the work "function." Newton's notation survives only in the symbol $\dot{x}$ for differentiation with respect to time, which is still used in physics.

It was to take two more centuries before calculus reached its present state of precision and rigor. The notion of a function gradually evolved from "curve" to "formula" to any rule that assigns one quantity to another. The great calculus text of Euler, published in 1748, emphasized the function concept by including not even one graph.

In several texts of the 1820s, Cauchy (1789–1857) defined "limit" and "continuous function" much as we do today. He also gave a definition of the definite integral, which with a slight change by Riemann (1826–1866) in 1854 became the definition standard today. So by the mid-nineteenth century the discoveries of Newton and Leibniz were put on a solid foundation.

In 1833, Liouville (1808–1882) demonstrated that the fundamental theorem could not be used to evaluate integrals of all elementary functions. In fact, he showed that the only values of the constant k for which $\int \sqrt{1-x^2}\sqrt{1-kx^2}dx$ is elementary are 0 and 1.

Still some basic questions remained, such as "What do we mean by area?" (For instance, does the set of points situated within some square and having both coordinates rational have an area? If so, what is this area?) It was as recently as 1887 that Peano (1858–1932) gave a precise definition of area — that quantity which earlier mathematicians had treated as intuitively given.

The history of calculus therefore consists of three periods. First, there was the long stretch when there was no hint that the tangent and area problems were related. Then came the discovery of their intimate connection and the exploitation of this relation from the end of the seventeenth century through the eighteenth century. This was followed by a century in which the loose ends were tied up.

The twentieth century saw calculus applied in many new areas, for it is the natural language for dealing with continuous processes, such as change with time. In that century mathematicians also obtained some of the deepest theoretical results about its foundations.

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Pronunciation

Descartes	"Day-CART"
Fermat	"Fair-MA"
Leibniz	"LIBE-nits"
Euler	"OIL-er"
Cauchy	"KOH-shee"
Riemann	"REE-mahn"
Liouville	"LYU-veel"
Peano	"Pay-AHN-oh"

Overview of Calculus II

The first part of this book was mainly about the derivative and the definite integral. The derivative measures a rate of change. The integral measures total change of a quantity that has a varying rate of change. The derivative and definite integral are linked by the Fundamental Theorem of Calculus. Both concepts are defined with the aid of limits, the basis of calculus.

The next six chapters apply the derivative and integral in a variety of contexts. Chapters 7 and 8 apply the definite integral and describe a few ways to find antiderivatives. Chapter 9, which stands by itself, concerns the geometry of curves and the physics of objects moving in a curved path. The next three chapters emphasize power series, which you may think of as "polynomials of infinite degree." That functions such as e^x and $\sin(x)$ can be represented by power series gives a way to compute them. With the aid of power series and complex numbers we show that the trigonometric functions can be expressed in terms of exponential functions (a relation applied, for instance, in the theory of alternating currents). Chapter 13, which discusses equations involving derivatives, could be studied any time after Chapter 8.

Summary of Calculus II

Overview of Calculus III

The first two parts of this book have focused on calculus of a single variable. The final third of this book extends the basic calculus ideas — limit, derivative, and integral — to two- and three-dimensions.

Summary of Calculus III