Author’s Commentary on Chapters 1-18

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General Principles

As we have written this book, we sought to retain the strengths from the Fifth Edition of Stein and Barcellos with improvements designed to address weaknesses of other current calculus texts. The changes are so substantial that our current work can be viewed as a first edition, with a new title. Throughout this project we have adhered to a few general principles:

- One section = one theme = one lecture. Exceptions to this guideline are some sections that summarize several themes and serve as references.
- We encourage the reader to pay close attention to the concepts and definitions. The proofs we present are selected to be student-friendly and instructive.
- Overall, the level of exposition increases steadily in recognition of the student’s growing mathematical maturity.
- As we wrote, we kept in mind that the reader should be well prepared to deal with the challenges of the final chapters, especially Chapter 18 (the Theorems of Green, Stokes, and Gauss).
- Proofs should not only show that something is true, but reveal in a student-friendly way the underlying reason why it is true.
- The chapter summary sections provide an overview that a single section cannot. (See, for example, Section 11.6.)

Calculus is Everywhere (CIE)

Each chapter concludes with one or more sections entitled "Calculus is Everywhere," usually related to the contents of the chapter.

As the Table of Contents shows, they cover a wide variety of applications. Most students take calculus for its applications, and these sections give them an opportunity to browse or study some in depth. Applications are too important to be treated slightly. Hence the CIEs. They illustrate again our principle of doing one thing at a time and giving it proper attention.

Chapter 1: Pre-Calculus Review

This chapter is a brief review and a reference; it gets right to the point. The examples provide background for later work. The transcendental functions are introduced early. (See Exercises 35 to 39 in Section 1.1.) Section 1.2 reinforces the exponential and logarithmic functions, with a more detailed discussion of logarithms in Section 1.5 for the many who need this. The discussion of geometric series lays the foundation for their use in finding the derivative (Section 3.2) and the definite integral of integer powers (Section 6.2).

The discussion of “modeling” is deferred until the Calculus is Everywhere (CIE) at the end of Chapter 1. This organization supports our philosophy of one topic per section.

Chapter 2: Introduction to Calculus

The one theme in Section 2.1 is slope, used to provide estimates of tangent lines. There are no limits, and no limit notation. This motivates Section 2.2, which, again, focuses on one theme, limits. Note that Section 2.2 determines the four limits needed in developing formulas for derivatives in Chapter 3.

Exercises throughout the book encourage the reader to reflect on the concepts. Examples include Exercises 41, 42, and 43 in Section 2.3. The focus on concepts is a habit that should become ingrained long before encountering vector analysis, especially the big theorems in Chapter 18.

The development of continuous functions, in Section 2.5, is built around the Extreme Value, Intermediate, and Permanence Properties. The proofs are not given until Sections 3.8 and 3.9, when students have experience with the uses of these Properties. This may be the first appearance of the Permanence Principle in a calculus text. It is used several times and, incidentally, provides a nice example in Section 3.9 of the utility of the ε–δ treatment of limits.

Each chapter concludes with a summary that offers an overall perspective and emphasis not possible in the individual sections.

Chapter 3: The Derivative

The derivative is introduced in Section 3.1 in the traditional way, by velocity and the tangent line. The development of the key limits, in Section 2.2, allows this section to be short – and focused. Because certain limits were determined earlier and the idea of estimating a tangent line by the "nearby point" method already appeared, these two sections truly focus on the concept of the derivative.

In Section 3.3, the sum, product, and quotient rules for derivatives are developed without using any “student unfriendly” tricks, such as adding and subtracting $f(x)g(x+\Delta x)$. When the chain rule appears in Section 3.4 the focus is on having students become comfortable with its use; the rigorous proof is left to an exercise (Exercise 75). The derivatives of inverse functions are developed, in Section 3.5, using only the chain rule. This approach provides students with the opportunity to work with antiderivatives, long before they meet the definite integral (Chapter 6). This presentation, which does not depend upon implicit differentiation, is consistent with this chapter’s sole theme of developing the differentiation formulas. (Applications of derivatives are discussed in Chapters 4 and 5.)

Exercises 76 and 86 in Section 3.5 are two of the “Sam and Jane” exercises that appear throughout the book. These exercises add a light touch and invite students to think on their own.

Antiderivatives are introduced, in Section 3.6, well before the definite integral appears in Chapter 6. The separation of these two ideas is intentional, to give students a chance to more fully understand antiderivatives as the inverse operation to differentiation. Slope fields are essential in our discovery of some of the fundamental properties of antiderivatives.

Students who have developed a solid understanding of derivatives and their interpretation in terms of velocity, will have no trouble with the discussion of the second derivative and acceleration in Section 3.7.

The rigorous, yet student-friendly, definition of limits is at the end of Chapter 3 (Sections 3.8 and 3.9), but could be covered at the end of Chapter 2. We prefer this organization to not delay the definition of derivatives. Also, it gives the reader more time to mature. The geometry and reasoning for limits at infinity are easier than for limits at a number. That is why we put them first. Note that the proof of the Permanence Principle provides a good example of why a precise definition of a limit is sometimes needed.

This chapter summary re-emphasizes the essentials. Note that the derivatives of all the elementary functions are obtained early, giving the student more time to practice and also providing more varied exercises. In fact, many sections throughout the book include exercises for practicing just the differentiation algorithms. (Another illustration of doing one thing at a time.)

<u>Chapter 4: Derivatives and Curve Sketching</u>

The theme of this chapter is interpreting the first and second derivatives, particularly as used to graph a function. Note that throughout Chapter 4 all of the elementary functions and their derivatives are available.

While the presentation is built around the idea of graphing a function, it goes beyond the development of an algorithm for graphing functions and emphasizes the qualitative properties that can be obtained from the first derivative (Section 4.2) and second derivative (Section 4.3) of a function. The first few exercises in Section 4.1 start this process by guiding the reader to pay attention to the statement and substance of the Theorem of the Interior Extremum, Rolle's Theorem, and Mean Value Theorem.

The chapter concludes with proofs of these theorems that are accessible to students in the middle of their first semester of calculus. Student understanding of the proofs is reinforced by the exercises in Section 4.4, several of which present alternate arguments that sometimes have logical errors that students might make on their own.

<u>Chapter 5: More Applications of Derivatives</u>

The qualitative information about functions developed in Chapter 4 forms a strong foundation for the applications of differentiation, beginning with optimization (Section 5.1).

The introduction to implicit differentiation, in Section 5.2, comes well after we used just the chain rule to differentiate inverse functions. (Section 3.5). This separation allows related rates to be included in this section, as an application of implicit differentiation.

We use cars on several occasions to bring ideas down to earth. For example, the introduction to Section 5.3 uses cars to introduce the central idea of The Growth Theorem: "If a car starts from rest, the larger its acceleration, the further it will travel in a given time." That shows why the higher derivatives control the growth of a function.

We introduce Taylor polynomials and their errors as early as possible, in Section 5.4. We do this for several reasons:

1. To shake up students who think they already know calculus and are coasting.
2. To show the importance of the higher derivatives in estimating errors.
3. To make the treatment of Taylor series, later, much easier and natural and suitable for one section = one lecture.
4. To introduce the concept of error analysis early.

The Growth Theorem is used not only to analyze the error in Taylor polynomials, but also the error in the various methods for estimating a definite integral (Section 6.5). These discussions of error plant the seeds needed to understand Taylor series (Chapter 12).

Chapter 6: The Definite Integral

In the opening section we use three motivations for the definite integral, both to anticipate later applications and to discourage the student from identifying it only with area. We do not use the formula for the sum of squares for two reasons: (1) The student probably hasn't seen it, and (2) using a geometric "trick" will deepen appreciation of the Fundamental Theorem (FTC) later.

Section 6.2 covers more than one concept, but because the antiderivative was introduced way back in Section 3.6, much of it is primarily reference material. Section 6.3 begins the development of the Fundamental Theorem of Calculus (FTC). In view of the importance of the FTC, we give two different motivations for it, in Section 6.4. To reinforce the idea that a definite integral is a limit of a sum, the chapter concludes with an early introduction to numerical estimates for definite integrals.

Chapter 7: Applications of the Definite Integral

The emphasis in this chapter is on the process of obtaining the definite integral: slice, approximate, add, limit. In particular, Section 7.3 shows how to set up a definite integral by first making a local estimate.

Chapter 8: Computing Antiderivatives

We start, in Section 8.1, by acknowledging that there are shortcuts, tables and technologies that can be used to find many antiderivatives.

There are the traditional sections on substitution (Section 8.2) and integration by parts (Section 8.3), and a separate section devoted completely to the algebra of partial fractions. (The antiderivatives of the terms appearing in a partial fraction decomposition of a function are treated in Section 8.2.) This organization is consistent with our principle of one major idea per section.

Many of the exercises in this chapter involve the applications introduced in Chapter 7. This ordering allows students to practice the setup skills learned in Chapter 7 while developing experience applying the integration techniques introduced in Chapter 8. Also, to help students develop the ability to determine the appropriate techniques to use on an integral, not all exercises in a section involve the technique learned in that section.

Chapter 9: Polar Coordinates and Plane Curves

In Section 9.2 we consider the problem of finding area in polar coordinates. We did not include arc length in polar coordinates here, to avoid having two big ideas in one section.

In Section 9.4 we focus on one idea, arc length. Note that we obtain the formula in polar coordinates, intuitively rather than deriving it from the formula in rectangular coordinates. (The reader is asked to derive the formula by the latter method in an exercise.) This approach explains why the formula in polar coordinates is so simple and makes it easier to remember and use. The brief summary emphasizes the main ideas in the section.

In Section 9.5 we use an informal approach in order to focus quickly and clearly on the essentials. We return to one of the underlying themes: the curves of interest locally look like straight lines. A more rigorous development is relegated to the exercises. However, we do not pretend to be rigorous here. After all, surface area is a slippery concept, best handled in more advanced courses.

In Section 9.6 we treat curvature in the plane. We do this for three reasons. First, it provides background for the vector treatment in Section 15.2. Second, it is important in its own right. Third, if obtained as a corollary of the vector treatment, it would make Section 15.2 too long, violating our goal of having one section equaling one lecture.

Chapter 10: Sequences and Their Applications

Chapters 10, 11, and 12 concern sequences and series. We devote Chapter 10 exclusively to sequences and Chapter 11 to series in order to provide a clear-cut separation of the notions of sequence and series.

Chapter 11: Series

Because of their importance, Chapter 10 includes a thorough discussion of the limit of geometric sequences $\{ r^n \}$ (even if $r = 0.9999$). The discussion of infinite geometric series in Section 11.2 is based on a well-developed background: finite geometric series in Chapter 1 and geometric sequences $\{ r^n \}$in Section 10.1.

The proof of the Integral Test, in Section 11.2 reinforces the definition of the definite integral and the fact (from Section 10.1) that bounded and monotone sequences must converge – without necessarily finding the sum of the series. Sections 11.4 (Comparison Tests) and 11.5 (Ratio Tests) are concise and to-the-point.

Section 11.6 develops the absolute convergence test in a student-friendly manner. No tricks, such as writing a_n as $(a_n + | a_n |) - | a_n |$.

Chapter 12: Applications of Series

The applications of series introduced in this chapter are Taylor series, power series, and Fourier series. Having seen Taylor polynomials and their errors in Chapter 5[1] and the standard series tests in Chapter 11, students are well prepared for Taylor series – and their applications – in Chapter 12. In addition to approximating a function by the first few terms of its Taylor series, Taylor series are used in Section 12.2 to evaluate limits and to estimate definite integrals. This experience provides a solid basis for introducing power series in Section 12.3 and developing rules for their manipulation in Section 12.4.

While this is the end of the story in many calculus courses, we continue the student's mathematical development with a review of complex numbers (Section 12.5) and the relationship between exponential and trigonometric functions (Section 12.6). The final application is a brief introduction to Fourier series.

Chapter 13: Differential Equations

Chapter 13, on differential equations, will be written when it is decided exactly how much to cover. The decision depends in part on the state of technology, which is rapidly changing. This material could appear here as a separate chapter, or at the end of the book, or as a supplement on the web.

Chapter 14: Vectors

Chapter 14 focuses only on the algebra of vectors. The major innovation is a motivation for the algebraic definition of the cross product (Section 14.3). The last step in the preparation for traditional vector calculus is the discussion of lines, planes, and components in Section 14.4.

Chapter 15: Derivatives and Integrals of Vector Functions

Chapter 15 treats the calculus of vector functions. We begin with a discussion of the geometry of vector functions: position, velocity, and acceleration in Section 15.1 and curvature and components of acceleration in Section 15.2.

The last two sections in Chapter 15 introduce line integrals. This early exposure to line integrals and conservative fields should help prepare the student to use them in Chapter 18.

Chapter 16: Partial Derivatives

A highlight of Chapter 16, on partial derivatives, is a new section on their use in thermodynamics (Section 16.8). Using only the chain rule, it develops two key theorems that students will see in an elementary thermodynamics course, where the usual explanation is more mysterious than student-

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[1] If there Is not enough time in the first semester to cover Taylor polynomials, Section 5.4 can be delayed until the start of Chapter 12.

friendly. This new section also serves to reinforce understanding of the chain rule, introduced in Section 16.3.

Chapter 17: Plane and Solid Integrals

Chapter 17 deals with integrals over a region in the plane (Section 17.1), a region in space (Section 17.4), and over a surface (Section 17.7). Each type of integral is defined with practically the same notation used in defining integrals over an interval.

All the machinery needed in the next chapter is in place.

Chapter 18: The Theorems of Green, Stokes, and Gauss

Chapter 18 may well be the part of calculus that students will use the most. It has been significantly revised and expanded, based on one of the author's experience sitting in on a sophomore-level electro-magnetism course. Ideally, all sections should be covered.   The final four sections apply and reinforce the first five. If time restraints prevent the syllabus from including all those four sections, students may still read them and later use them as references.

The first six sections except Section 18.4 are standard.   The final three sections can be covered in class or left to the students to use as a way to test and reinforce their understanding, as well as a reference. contain a thorough discussion of conservative vector fields (Section 18.1) Green's Theorem (Sections 18.2 and 18.3) and central fields (Section 18.4). Gauss' Theorem (Section 18.5) and Stokes' Theorem (Section 18.6) are extensions of Green's Theorem to vector fields in space.

In Section 18.7, Gauss' Law is applied to electrostatics and the inverse square radial vector field in space. In Section 18.8, the vector differential operators grad, div, and curl are expressed in polar, cylindrical, and spherical coordinates. We take great care to show why the formulas are not what one would guess. The reasoning here reinforces the earlier sections.

This chapter, and the book, concludes with the derivation of the four Maxwell's equations (Section 18.9), in both their local and global forms.