Chapter 17

Plane and Solid Integrals

In Chapter 2 we introduced the derivative, one of the two main concepts in calculus. Then in Chapter 15 we extended the idea to higher dimensions. In the present chapter, we generalize the concept of the definite integral, introduced in Chapter 6, to higher dimensions.

Take a moment to review the definite integral. Instead of using the notation of Chapter 6, we will restate the definition in a notation that easily generalizes to higher dimension.

We started with an interval $[a, b]$, which we will call I, and a continuous function f defined at each point P of I. Then we cut I into n short intervals $I_1, I_2, \ldots, I_u$, chose a point P_1 in I_1, P_2 in I_2, $\ldots$, P_n in I_n. See Figure 17.0.1. Denoting the length of I_i by L_i, we formed the sum

$$\sum_{i=1}^{n} f(P_i)L_i.$$

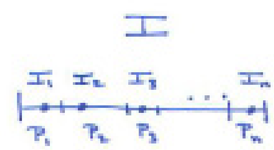


Figure 17.0.1:

The limit of these sums as all the subintervals are chosen shorter and shorter is the definite integral of f over interval I. We denoted it $\int_b^a f(x)\,dx$. We now denote it $\int_I f(P)dL$. This notation tells us that we are integrating a function, f, over an interval I. The dL reminds us that the integral is the limit of approximations formed as the sum of products of the function value and the length of an interval.

We will define integrals of functions over plane regions, such as square and disks, over solid regions, such as tubes and balls, and over surfaces such as the surface of a ball, in the same way. You can probably conjecture already what the definition will be. These integrals are needed to compute total mass if we know the density at each point, or total gravitational attraction, or center of gravity, and so on.

It is one thing to define these higher-dimensional integrals. It is another to calculate them. Most of our attention will be devoted to seeing how to compute them with the aid of so-called "iterated integrals," which involve integrals over intervals, the type defined in Chapter 6.

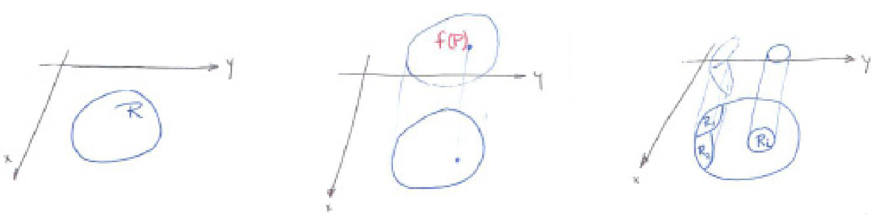
17.1 The Double Integral: Integrals Over Plane Areas

We suggest you re-read the introduction to this chapter and the definition of the definite integral $\int_a^b f(x)\,dx$ before going on.

The goal of this section is to define the integral of a function defined in a region of a plane. With only a slight tweaking of this definition, we will define later in the chapter integrals over surfaces and solids.

Volume Approximated by Sums

Let R be a region in the xy plane, bounded by curves. For convenience, assume R is convex (no dents), for example, an ellipse, a disk, a parallelogram, a rectangle, or a square. We draw R in perspective in Figure 17.1.1(a). Imagine



(a) (b) (c)

Figure 17.1.1:

that there is a surface above R (perhaps an umbrella). The height of the surface above point P on R is $f(P)$, as shown in Figure 17.1.1(b)

If you know $f(P)$ for every point P how would you estimate the volume, V, of the solid under the surface and above R?

Just as we used rectangles to estimate the area of regions back in Section 6.1, we will use cylinders to estimate the volume of a solid. Recall, from Section 7.4, that the volume of a cylinder is the product of its height and the area of its base.

Inspired by the approach in Section 6.1, we cut R into n small regions R_1, R_2, ..., R_n. Each R_i has area A_i. Choose points P_1 in R_1, P_2 in R_2, ..., P_n in R_n. Then we build a cylinder over each little region R_i. Its height will be $f(P_i)$. There will then be n cylinders. The total volume of these cylinders is

$$\sum_{i=1}^{n} f(P_i)A_i. \qquad (17.1.1)$$

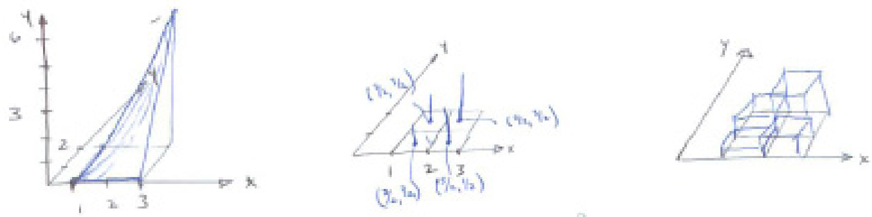
As we choose the regions R_1, R_2, ..., R_n, smaller and smaller, the sum (17.1.1) approaches the volume V, if f is a continuous function.

EXAMPLE 1 Estimate the volume of the solid under the saddle $z = xy$

SHERMAN: Changed left edge from 0 to 1 so that base and height are not the same.

and above the rectangle R whose vertices are $(1, 0)$, $(2, 0)$, $(2, 3)$, and $(1, 3)$.

SOLUTION Figure 17.1.2(a) shows the solid region in question.



(a) (b) (c)

Figure 17.1.2:

The highest point is above $(2, 3)$, where $z = 6$. So the solid fits in a box whose height is 6 and whose base has area 4. So we know the volume is at most $4 \cdot 6 = 24$.

To estimate the volume we cut the rectangular box into four 1 by 1 squares and evaluate $z = xy$ at, say, the center of the squares, as shown in Figure 17.1.2(b).

Then we form a cylinder for each square. The base is the square and the height is determined by the value of xy at the center of the square. These are shown in Figure 17.1.2(c). (The cylinder over rectangle boxes.)

Then the total volume is

$$\underbrace{\frac{3}{4}}_{\text{height}} \cdot \underbrace{1}_{\text{area of base}} + \underbrace{\frac{5}{4}}_{\text{height}} \cdot \underbrace{1}_{\text{area of base}} + \underbrace{\frac{9}{4}}_{\text{height}} \cdot \underbrace{1}_{\text{area of base}} + \underbrace{\frac{15}{4}}_{\text{height}} \cdot \underbrace{1}_{\text{area of base}} = 8 \tag{17.1.2}$$

This estimate is then 8 cubic units. We know this is an overestimate (Why?) By cutting the base into smaller pieces and using more cylinders we could make a more accurate estimate of the volume of the solid. ◇

Density

Before we consider a "total mass" problem we must define the concept of "density." Consider a piece of sheet metal, which we view as part of a plane. It is homogeneous, "the same everywhere." Let R be any region in it, of area A and mass m. The quotient m/A is the same for all regions R, and is called the "density."

It may happen that the material, unlike sheet metal is not uniform. For instance, a towel that was just used to dry dishes. As R varies, the quotient

m/A, or "average density in R," also varies. Physicists define the **density at a point** as follows.

They consider a small disk R of radius r and center at P, as in Figure 17.1.4. Let $m(r)$ be the mass in that disk and $A(r)$ be the area of the disk (πr^2). The

$$\text{"Density at } P\text{"} = \lim_{r\to 0} \frac{m(r)}{A(r)}.$$

Figure 17.1.3:

σ is Greek for our letter "s", the initial letter of "surface." $\sigma(P)$ denotes the density of a surface or "lamina" at P.

Thus density is denoted $\sigma(P)$, "sigma of P,"

With the physicists, we will assume the density $\sigma(P)$ exists at each point and that it is a continuous function. In addition, we will assume that if R is a very small region of area A and P is a point in that region then the product $\sigma(P)A$ is an approximation of the mass in R.

Total Mass Approximated by Sums

Assume that a flat region R is occupied by a material of varying density. The density at point P in R is $\sigma(P)$. Estimate M, the total mass in R.

As expected, we cut R into n small regions R_1, R_2, …, R_i has area A_i. We next choose points P_1 in R_1, P_2 in $R_2, \dots, P_u$ in R_n. Then we estimate the mass in each little region R_i, as shown in Figure 17.1.4. The mass in R_i is approximately

$$\underbrace{\sigma(P_i)}_{\text{density}} \cdot \underbrace{A_i}_{\text{area}}$$

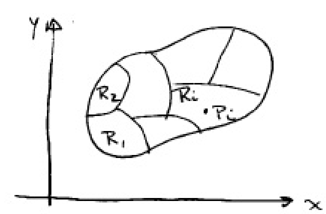


Figure 17.1.4: This example has $i = 7$ subregions.

Thus

$$\sum_{i=1}^{n} \sigma(P_1)A_i \tag{17.1.3}$$

is the total estimate. As we divide R into smaller and smaller regions, , the sums (17.1.2) approaches the total mass M, if σ is a continuous function.

EXAMPLE 2 A rectangular **lamina**, of varying density occupies the rectangle with corners at $(0,0)$, $(2,0)$, $(2,3)$, and $(0,3)$ in the xy plane. Its density at (x,y) is xy grams per square cm. Estimate its mass by cutting it into six 1 by 1 squares and evaluating the density at the center of each square.

SOLUTION One such square is shown in Figure 17.1.5. The density at its center is $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. Since its area is $1 \times 1 = 1$, an estimate of σ, its mass, is

$$\underbrace{\frac{1}{4}}_{\text{density}} \cdot \underbrace{1}_{\text{area}} = \frac{1}{4} \text{ grams.}$$

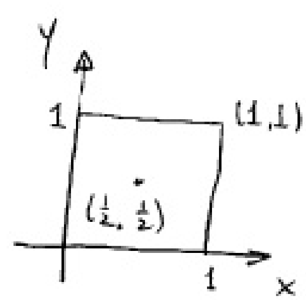


Figure 17.1.5:

Similar estimates for the remaining six small squares gives a total estimate of

$$\frac{1}{4}\cdot 1+\frac{3}{4}\cdot 1+\frac{3}{4}\cdot 1+\frac{9}{4}\cdot 1+\frac{5}{4}\cdot 1+\frac{15}{4}\cdot 1 = 9 \text{ grams}$$

Thus sum is identical to the sum (17.1.2), which estimates a volume. $\diamond$

The arithmetic in Examples 1 and 2 show that totally unrelated problems, one in volume, the other in mass, lead to the same estimates. Moreover, as the rectangle is cut into smaller pieces, the estimate would become closer and closer to the volume or the mass. These estimates, similar to the estimates $\sum_{i=1}^{n}(f(c_i)\Delta x_i$ that appears in the definition of the definite integral $\int_a^b f(x)\,dx$, brings us to the definition of "double integral". It is called the double integral because the domain of the function is in the two-dimensional plane.

———————-

The Double Integral

The definition of the double integral is almost the same as that of $\int_a^b f(x)\,dx$, the integral over an interval. The only differences are:

1. instead of dividing an interval into smaller intervals, we divide a planar region into smaller planar regions,

2. instead of a function defined on an interval, we have a function defined on a planar region, and

3. we need a quantitative way to say that a "little" region is "small."

To meet the need described in (3) we define the "diameter" of a planar region. The **diameter** of a region bounded by a curve is the maximum distance between two points in the region. For instance, the diameter of a square of side s is $s\sqrt{2}$ and the diameter of a disk is the same as its traditional diameter that we know from geometry.

With that aside taken care of, we are ready to define a double integral.

DEFINITION *(Double Integral)* Let R be a region in a plane bounded by curves and f a continuous numerical function defined at least on R. Partition R into smaller regions R_1, R_2, ..., R_n of respective areas A_1, A_2, ..., A_n. Choose a point P_1 in R_1, P_2 in R_2, ..., P_n in R_n and form the approximating (Riemann) sum

$$\sum_{i=1}^{n} f(P_i)A_i. \tag{17.1.4}$$

Form a sequence of such partitions such that as one goes out in the sequence of partitions, the sequence of diameters of the largest region in each partition approaches 0. Then the sums (17.1.4) approach a limit, which is called "the integral of f over R" or the "double integral" of f over R. It is denoted

$$\int_R f(P)\,dA.$$

Before looking at some examples, we make four brief remarks:

1. It is called a double integral because R lies in a plane, which has dimension 2.

2. We use the notion of a diameter of a region only to be able to define the double integral.

3. It is proved in advanced calculus that the sums do indeed approach a limit.

4. Other notations for a double integral are discussed near the end of this section.

Our discussion of integrals over a plane region started with two important illustrations . The rest of this section is devoted to these applications in the context of double integrals.

Volume Expressed as a Double Integral

Consider a solid S and its projections ("shadows") R on a plane, as in Figure 17.1.6. Assume that for each point P in R the line through P perpendicular to R intersects S in a line segment of length $C(P)$. Then

"The double integral of cross-section is the volume."

$$\text{Volume of } S = \int_R C(P)\,dA.$$

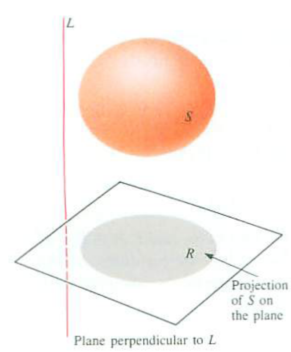


Figure 17.1.6: ARTIST: Delete the line L, and the current caption. Add a point P in R and draw the vertical line through P,

Mass Expressed as a Double Integral

Consider a plane distribution of mass through a region R, as shown in Figure 17.1.7. The density may vary throughout the region. Denote the density at P by $\sigma(P)$ (in grams per square centimeters). Then

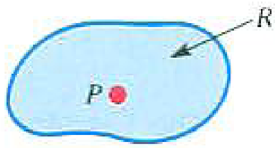


Figure 17.1.7:

"The double integral of density is the total mass."

$$\text{Mass in } R = \int_R \sigma(P)\, dA$$

Average Value as a Double Integral

The average value of $f(x)$ for x is the interval $[a, b]$ was defined in Section 6.3 as

$$\frac{\int_a^b f(x)\, dx}{\text{length of interval.}}$$

We make a similar definition for a function defined on a two-dimensional region.

DEFINITION *(Average value)* The **average value** of f over the region R is

$$\frac{\int_R f(P)\, dA}{\text{Area of } R}.$$

If $f(P)$ is positive for all P in R, there is a simple geometric interpretation of the average of f over R. Let S be the solid situated below the graph of f (a surface) and above the region R. The average value of f over R is the height of the cylinder whose base is R and whose volume is the same as the volume of S. (See Figure 17.1.8. The integral $\int_R f(P)\, dA$ is called "an integral over a plane region" to distinguish it from $\int_a^b f(x)\, dx$, which, for contrast, is called, "an integral over an interval."

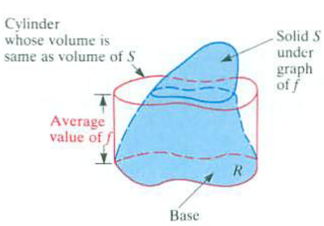


Figure 17.1.8:

/mnoteSHERMAN: Duplicitous? Or needed? Shorten to margin note? Recall that $\int_R f(P)\, dA$ is often denoted $\iint_R f(P)\, dA$, with the two integral signs emphasizing that the integral is over a plane set. However, the symbol dA, which calls to mind areas, is an adequate reminder.

The integral of the function $f(P) = 1$ over a region is of special interest. The typical approximating sum $\sum_{i=1}^n f(P_i)A_i$ then equals $\sum_{i=1}^n 1 \cdot A_i = A_1 + A_2 + \cdots + A_n$, which is the area of the region R that is being partitioned. Since *every* approximating sum has this same value, it follows that

$$\lim_{n\to\infty} \sum_{i=1}^n f(P_i)A_i = \text{Area of } R.$$

Integral	Interpretation
$\int_R 1\ dA$	Area of R
$\int_R \sigma(P)\ dA$, $\sigma(P) =$ density	Mass of R
$\int_R c(P)\ dA$, $c(P) =$ length of cross section of solid	Volume of R

Table 17.1.1:

Consequently

The integral of a constant function, 1, gives area.

$$\int\limits_R 1\ dA = \text{Area of } R.$$

This formula will come in handy on several occasions. The 1 is often omitted, in which case we write $\int_R\ dA =$ Area of R. This table summarizes some of the main applications of the double integral $\int_R\ dA$:

Properties of Double Integrals

Integrals over plane regions have properties similar to those of integrals over intervals:

1. $\int_R cf(P)\ dA = c\int_R f(P)\ dA$ for any constant c.
2. $\int_R [f(P) + g(P)]\ dA = \int_R f(P)\ dA + \int_R g(P)\ dA$.
3. If $f(P) \le g(P)$ for all points P in R, then $\int_R f(P)\ dA \le \int_R g(P)\ dA$.
4. If R is broken into two regions, R_1 and R_2, overlapping at most on their boundaries, then

$$\int\limits_R f(P)\ dA = \int\limits_{R_1} f(P)\ dA + \int\limits_{R_2} f(P)\ dA.$$

For instance, consider **3** when $f(P)$ and $g(P)$ are both positive. Then $\int_R f(P)\ dA$ is the volume under the surface $z = f(P)$ and above R in the xy plane. Similarly $\int_R g(P)\ dA$ is the volume under $z = f(P)$ and above R. Then **3** asserts that the volume of a solid is not larger than the volume of a solid that contains it. (See Figure 17.1.9.)

SHERMAN: This summary needs to be written.

Summary

A Word about 4-Dimensional Space

We can think of 2-dimensional space as the set of ordered pairs (x, y) of real numbers. The set of ordered triplets of real numbers (x, y, z) represents 3-dimensional space. The set of ordered quadruplets of real numbers (x, y, z, t) represents 4-dimensional space.

It is easy to show that 4-dimensional space is a very strange place.

In 2-dimensional space the set of points of the form $(x, 0)$, the x-axis, meets the set of points of the form $(0, y)$, the y-axis, in a point, namely the origin $(0, 0)$. Now watch what can happen in 4-space. The set of points of the form $(x, y, 0, 0)$ forms a plane congruent to our familiar xy-plane. The set of points of the form $(0, 0, z, t)$ forms another such plane. So far, no surprise. But notice what the intersection of those two planes is. Their intersection is just the point (0,0,0,0). Can you picture two endless planes meeting in a single point? If so, tell us how.

EXERCISES for Section 17.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] In the estimates for the volume in Example 1, the centers of the squares were used as the P_i's. Make an estimate for the volume in Example 1 by using the same partition but taking as P_i

(a) the lower left corner of each R_i,

(b) the upper right corder of each R_i.

(c) What do (a) and (b) tell about the volume of the solid?

2.[R] Estimate the mass in Example 2 using the partition of R into six squares and taking as the P_i's

(a) upper left corners,

(b) lower right corners.

3.[R] Let R be a set in the plane whose area is A. Let f be the function such that $f(P) = 5$ for every point P in R.

(a) What can be said about any approximating sum $\sum_{i=1}^{n} f(P_i)A_i$ formed for this R and this f?

(b) What is the value of $\int_R f(P)\, dA$?

4.[R] Let R be the square with vertices $(1,1)$, $(5,1)$, $(5,5)$, and $(1,5)$. Let $f(P)$ be the distance from P to the y-axis.

(a) Estimate $\int_R f(P)\, dA$ by partitioning R into four squares and using midpoints as sampling points.

(b) Show that $16 \leq \int_R f(P)\, dA \leq 80$.

5.[R] Let f and R be as in Example 1. Use the estimate of $\int_R f(P)\, dA$ obtained in the text to estimate the average of f over R.

6.[R] Assume that for all P in R, $m \leq f(P) \leq M$, where m and M are constants. Let A be the area of R. By examining approximating sums, show that

$$mA \leq \int\limits_R f(P)\, dA \leq MA.$$

7.[R]

(a) Let R be the rectangle with vertices $(0,0)$, $(2,0)$, $(2,3)$, and $(0,3)$. Let $f(x,y) = \sqrt{x+y}$. Estimate $\int_R \sqrt{x+y}\, dA$ by participating R into six squares and choosing the sampling points to be their centers.

(b) Use (a) to estimate the average value of f over R.

8.[R]

(a) Let R be the square with vertices $(0,0)$, $(0.8,0)$, $(0.8,0.8)$, and $(0,0.8)$. Let $f(P) = f(x,y) = e^{xy}$. Estimate $\int_R e^{xy}\, dA$ by partitioning R into 16 squares and choosing the sampling points to be their centers.

(b) Use (a) to estimate the average value of $f(P)$ over R.

(c) Show that $0.64 \leq \int_R f(P)\, dA \leq 0.64e^{0.64}$.

9.[R]

(a) Let R be the triangle with vertices $(0,0)$, $(4,0)$, and $(0,4)$ shown in Figure 17.1.10. Let $f(x,y) = x^2y$. Use the partition into four triangles and sampling points shown in the diagram to estimate $\int_R f(P)\, dA$.

(b) What is the maximum value of $f(x,y)$ in R?

(c) From (b) obtain an upper bound on $\int_R f(P)\, dA$.

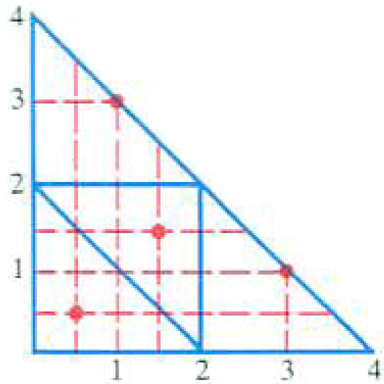


Figure 17.1.10:

10.[R]

(a) Sketch the surface $z = \sqrt{x^2 + y^2}$.

(b) Let $\mathcal{V}$ be the region in space below the surface in (a) and above the square R with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. Let V be the volume of $\mathcal{V}$. Show that $V \leq \sqrt{2}$.

(c) Using a partition of R with 16 squares, find an estimate for V that is too large.

(d) Using the partition in (c), find an estimate for V that is too small.

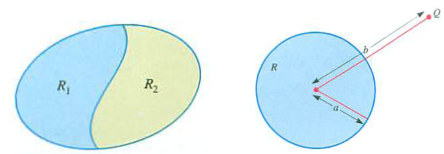
11.[R] The amount of rain that falls at point P during one year is $f(P)$ inches. Let R be some geographic region, and assume areas are measured in square inches.

(a) What is the meaning of $\int_R f(P)\, dA$?

(b) What is the meaning of

$$\frac{\int_R f(P)\, dA}{\text{Area of } R}?$$

12.[M] A region R in the plane is divided into two regions R_1 and R_2. The function $f(P)$ is defined throughout R. Assume that you know the areas of R_1 and R_2 (they are A_1 and A_2) and the average of f over R_1 and the average of f over R_2 (they are f_1 and f_2). Find the average of f over R. (See Figure 17.1.11(a).)



(a) (b)

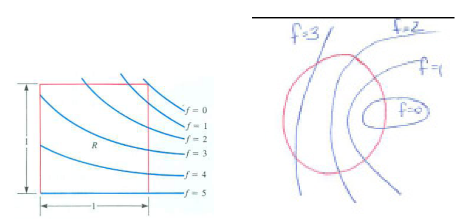
Figure 17.1.11:

13.[M] A point Q on the xy plane is at a distance b from the center of a disk R of radius $a (a < b)$ in the xy plane. For P in R let $f(P) = 1/\overrightarrow{PQ}$. Find positive numbers c and d such that:

$$c < \int_R f(P)\, dA < d.$$

(The numbers c and d depend on a and b.) See Figure 17.1.11(b).

14.[M] Figure 17.1.12(a) shows the parts of some level curves of a function $z = f(x, y)$ and a square R. Estimate $\int_R f(P)\, dA$, and describe your reasoning.



(a) (b)

Figure 17.1.12:

15.[M] Figure 17.1.12(b) shows the parts of some level curves of a function $z = f(x, y)$ and a unit circle

R. Estimate $\int_R f(P)\, dA$, and describe your reasoning.

16.[C]

(a) Let R be a disk of radius 1. Let $f(P)$, for P in R, be the distance from P to the center of the disk. By cutting R into narrow circular rings with center at the center of the disk, evaluate $\int_R f(P)\, dA$.

(b) Find the average of $f(P)$ over R.

Exercises 17 and 18 introduce an idea known as **Monte Carlo methods** for estimating a double integral using randomly chosen points. These methods tend to be rather inefficient because the error decreases on the order of $1/\sqrt{n}$, where n is the number of random points. That is a slow rate. These methods are used only when it's possible to choose n very large.

17.[C] This exercise involves estimating an integral by choosing points randomly. A computing machine can be used to generate random numbers and thus random points in the plane which can be used to estimate definite integrals, as we now show. Say that a complicated region R lies in the square whose vertices are $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$, and a complicated function f is defined in R. The machine generated 100 random points (x,y) in the square. Of these, 73 lie in R. The average value of f for these 73 points is 2.31.

(a) What is a reasonable estimate of the area of R?

(b) What is a reasonable estimate of $\int_R f(P)\, dA$?

18.[C] Let R be the disk bounded by the unit circle $x^2+y^2=1$ in the xy plane. Let $f(x,y)=e^{x^2y}$ be the temperature at (x,y).

(a) Estimate the average value of f over R by evaluating $f(x,y)$ at twenty random points in R. (Adjust your program to select each of x and y randomly in the interval $[-1,1]$. In this way you construct a random point (x,y) in the square whose vertices are $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$. Consider only those points that lie in R.)

(b) Use (a) to estimate $\int_R f(P)\, dA$.

(c) Show why $\pi/e \le \int_R f(P)\, dA \le \pi e$.

19.[C] Sam is heckling again. "As usual, the authors made this harder than necessary. They didn't need to introduce "diameters." Instead they could have used good old area. They could have taken the limit as all the areas of the little pieces approached 0. I'll send them a note."
Is Sam right?

In making finer and finer partitions as $n \to \infty$ we saw that each R_i is small in the sense it fits in a disk of radius r_n, where $r_n \to 0$ as $n \to \infty$. The Exercises 20 to 23 in this section explore another way to control the size of a region.

20.[C] Consider a region R in the plane. The diameter, d of R, is defined as the greater distance between two points in R. Find the diameter of

(a) a disk of radius r,

(b) and equilateral triangle of side length s,

(c) a square whose sides have length s.

21.[C]

(a) Show that a region of diameter d can always fit into a disk of diameter $2d$.

(b) Can it alway fit into a disk of diameter d?

22.[C] If a region has diameter d,

(a) how small can its area be?

(b) show that area is less than or equal to $\pi d^2/2$

SHERMAN: Is this in polar coordinate area? If so, move to Section 17.3 or Chapter Summary.

23.[C] The unit square can be partitioned with nine congruent squares.

(a) What is the diameter of each of these small squares?

(b) It is possible to partition that square into nine regions whose largest diameter is 5/11. Show that 5/11 is smaller than the

24.[R] Some practice differentiate

25.[R] Some practice integrals, e

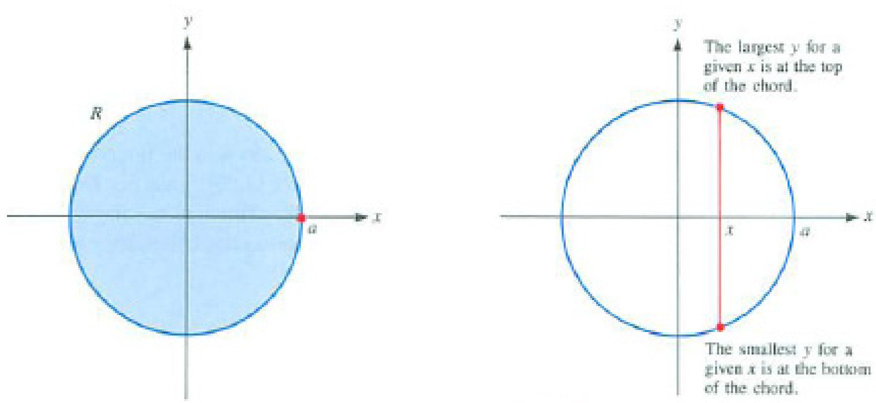
17.2 Computing $\int_R f(P)\, dA$ Using Rectangular Coordinates

In this section, we will show how to use rectangular coordinates to evaluate the integral of a function f over a plane region R, $\int_R f(P)\, dA$. This method requires that both R and f be described in rectangular coordinates. We first show how to describe plane regions R in rectangular coordinates.

Describing R in Rectangular Coordinates

Some examples illustrate how to describe planar regions by their cross sections in terms of rectangular coordinates.

EXAMPLE 1 Describe a disk R of radius a in a rectangular coordinates.



(a) (b)

Figure 17.2.1:

SOLUTION Introduce an xy coordinate system with its origin at the center of the disk, as in Figure 17.2.1(a). A glance at the figure shows that x ranges from $-a$ to a. All that remains is to tell how y varies for each x in $[-a, a]$.

Figure 17.2.1(b) shows a typical x in $[-a, a]$ and corresponding cross section. The circle has the equation $x^2 + y^2 = a^2$. The top half has the description $y = \sqrt{a^2 - x^2}$ and the bottom half, $y = -\sqrt{z^2 - y^2}$. So, for each x in $[-a, a]$, y varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$. (As a check, test $x = 0$. Does y vary from $-\sqrt{a^2 - 0^2} = -a$ to $\sqrt{a^2 - 0^2} = a$? It does, as an inspection of Figure 17.2.1(b) shows.)

All told, this is the description of R by vertical cross sections:

$$-a \le x \le a, \qquad -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}.$$

◇

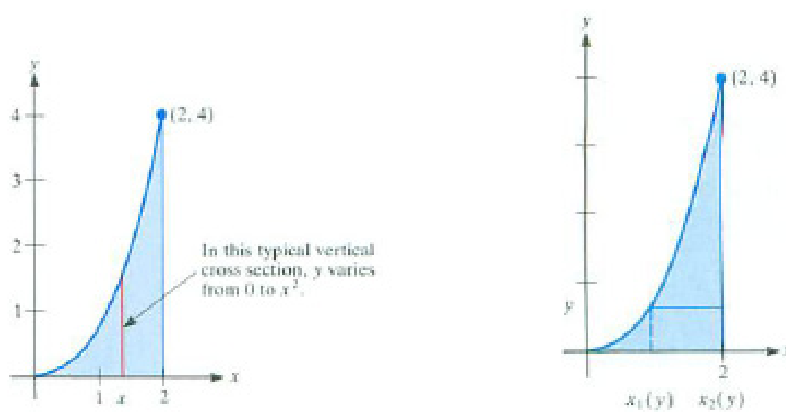
EXAMPLE 2 Let R be the region bounded by $y = x^2$, the x-axis, and the line $x = 2$. Describe R in terms of cross sections parallel to the y-axis.

SOLUTION A glance at R in Figure 17.2.2(a) shows that for points (x, y) in R, x ranges from 0 to 2. To describe R completely, we shall describe the behavior of y for any x in the interval $[0, 2]$.

Hold x fixed and consider only the cross section above the point $(x, 0)$. It extends from the x-axis to the curve $y = x^2$; for any x, the y coordinate varies from 0 to x^2. The compact description of R by vertical cross sections is:

$$0 \leq x \leq 2, \qquad 0 \leq y \leq x^2.$$

◇



(a) (b)

Figure 17.2.2:

EXAMPLE 3 Describe the region R of Example 2 by cross sections parallel to the x-axis, that is, horizontal cross sections.

SOLUTION A glance at R in Figure 17.2.2(b) shows that y varies from 0 to 4. For any y in the interval $[0, 4]$, x varies from a smallest value $x_1(y)$ to a largest value $x_2(y)$. Note that $x_2(y) = 2$ for each value of y in $[0, 4]$. To find $x_1(y)$, utilize the fact that the point $(x_1(y), y)$ is on the curve $y = x^2$, that is,

$$x_1(y) = \sqrt{y}.$$

The compact description of R in terms of horizontal cross sections is

$$0 \leq y \leq 4, \qquad \sqrt{y} \leq x \leq 2.$$

$$0 \le x \le 4, \quad 0 \le y \le 2$$

and

$$4 \le x \le 6, \quad 0 \le y \le 6 - x.$$

◇

EXAMPLE 4 Describe the region R whose vertices are $(0,0)$, $(0,6)$, $(4,2)$, and $(0,2)$ by vertical cross sections and then by horizontal cross sections. (See Figure 17.2.3.)

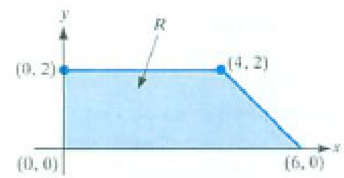


Figure 17.2.3:

SOLUTION Clearly, x varies between 0 and 6. For any x in the interval $[0,4]$, y ranges from 0 to 2 (independently of x). For x in $[4,6]$, y ranges from 0 to the value of y on the line through $(4,2)$ and $(6,0)$. This line has the equation $y = 6 - x$. The description of R by vertical cross sections therefore requires two separate statements:

Use of horizontal cross sections provides a simpler description. First, y goes from 0 to 2. For each y in $[0,2]$, x goes from 0 to the value of x on the line $y = 6 - x$. Solving this equation for x yields $x = 6 - y$.

The compact description in terms of horizontal cross-sections is much shorter:

$$0 \le y \le 2, \qquad 0 \le x \le 6 - y.$$

◇

These examples are typical. First, determine the range of one coordinate, and then see how the other coordinate varies for any fixed value of the first coordinate.

Evaluating $\int_R f(P)\, dA$ by Iterated Integrals

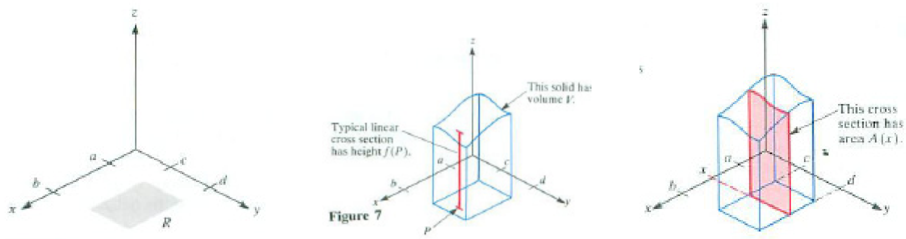
We will offer an intuitive development of a formula for computing double integrals over plane regions.

We first develop a way for computing a double integral over a rectangle. After applying this formula in Example 5, we make the slight modification needed to evaluate double integrals over more general regions.

Consider a rectangular region R whose description by cross sections is

$$a \le x \le b, \qquad c \le y \le d,$$

as shown in Figure 17.2.4(a). If $f(P) \le 0$ for all P in R, then $\int_R f(P)\, dA$ is the volume V of the solid whose base is R and which has, above P, height $f(P)$. (See Figure 17.2.4(b).) Let $A(x)$ be the area of the cross section made by a



(a) (b) (c)

Figure 17.2.4:

plane perpendicular to the x-axis and having abscissa x, as in Figure 17.2.4(c). As was shown in Section 5.1,

$$V = \int_b^a A(x)\ dx.$$

But the area $A(x)$ is itself expressible as a definite integral:

$$A(x) = \int_c^d f(x,y)\ dy.$$

Note that x is held fixed throughout the integration to find $A(x)$. This reasoning provides an iterated integral whose value is $V = \int_R f(P)\ dA$, namely,

$$\int_R f(P)\ dA = V = \int_a^b A(x)\ dx = \int_a^b \left(\int_c^d f(x,y)\ dy \right)\ dx.$$

In short

$$\boxed{\int_R f(P)\ dA = \int_a^b \left(\int_c^d f(x,y)\ dy \right)\ dx.}$$

An integral over a rectangle expressed an iterated integral

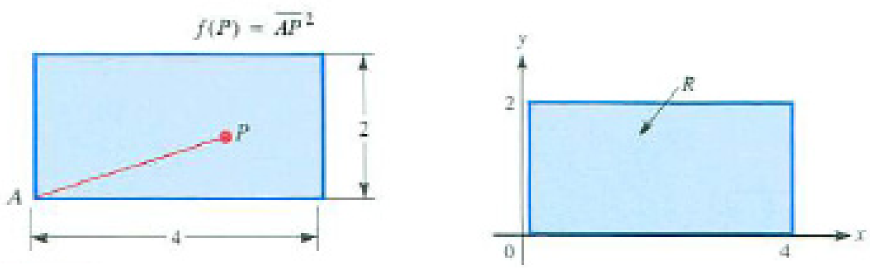
Of course, cross sections by planes perpendicular to the y-axis could be used. Then similar reasoning shows that

$$\int_R f(P)\ dA = \int_c^d \left(\int_a^b f(x,y)\ dx \right)\ dy.$$

The quantities $\int_a^b \left(\int_c^d f(x,y)\ dy\right)\ dx$ and $\int_c^d \left(\int_a^b f(x,y)\ dx\right)\ dy$ are called **iterated integrals**. Usually the brackets are omitted and are written $\int_a^b \int_c^d f(x,y)\,dy\,dx$ and $\int_c^d \int_a^b f(x,y)\ dx\ dy$.

The order of dx and dy matters; the differential that is on the left tells which integration is performed first.

EXAMPLE 5 Compute the double integral $\int_R f(P)\ dA$, where R is the rectangle shown in Figure 17.2.5(a) and the function f is defined by $f(P) = \overline{AP}^2$.



(a) (b)

Figure 17.2.5:

SOLUTION Introduce xy coordinates in the convenient manner depicted in Figure 17.2.5(b). Then f has this description in rectangular coordinates:

$$f(x,y) = \overline{AP}^2 = x^2 + y^2.$$

To describe R, observe that x takes all values from 0 to 4 and that for each x the number y takes all values between 0 and 2. Thus

$$\int_R f(P)\ dA = \int_0^4 \left(\int_0^2 (x^2+y^2)\ dy\right)\ dx.$$

We must first compute the inner integral

The cross-sectional area $A(x)$.

$$\int_0^2 (x^2+y^2)\ dy, \qquad \text{where } x \text{ is fixed in } [0,4].$$

To apply the Fundamental Theorem of Calculus, first find a function $F(x,y)$ such that

$$\frac{\partial F}{\partial y} = x^2 + y^2.$$

Keep in mind that x is constant during this first integration.

$$F(x,y) = x^2y + \frac{y^3}{3}$$

is such a function. The appearance of x in this formula should not disturb us, since x is fixed for the time being. By the Fundamental Theorem of Calculus,

$$\int_0^2 (x^2+y^2)\ dy \quad = \quad \left(x^2y + \frac{y^3}{3}\right)\Bigg|_{y=0}^{y=2} = \left(x^2\cdot 2 + \frac{2^3}{3}\right) - \left(x^2\cdot 0 + \frac{0^3}{3}\right) = 2x^2 + \frac{8}{3}.$$

The notation $|_{y=0}^{y=2}$ reminds us that y is replaced by 0 and 2.

The formula $2x^2 + \frac{8}{3}$ is the area $A(x)$ discussed earlier in this section. Now compute

$$\int_0^4 A(x)\ dx = \int_0^4 (2x^2 + \frac{8}{3})\ dx.$$

By the Fundamental Theorem of Calculus,

$$\int_0^4 \left(2x^2 + \frac{8}{3}\right)\ dx = \left(\frac{2x^3}{3} + \frac{8x}{3}\right)\Bigg|_0^4 = \frac{160}{3}.$$

How do these compare with the estimates in Section 17.1?

Hence the two-dimensional definite integral has the value $\frac{160}{3}$. The volume of the region in Problem 1 of Sec. 16.1 is $\frac{160}{3}$ cubic inches. The mass in Problem 2 is $\frac{160}{3}$ grams. $\diamond$

If R is not a rectangle, the repeated integral that equals $\int_R f(P)\ dA$ differs from that for the case where R is a rectangle only in the intervals of integration. If R has the description

$$a \le x \le b \qquad y_1(x) \le y \le y_2(x),$$

by cross sections parallel to the y-axis, then

$$\boxed{\int_R f(P)\ dA = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x,y)\ dy\right]\ dx.}$$

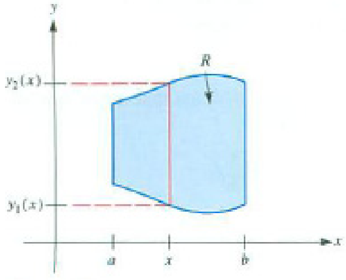


Figure 17.2.6:

Similarly, if R has the description

$$c \le y \le d \qquad x_1(y_{\le} x \le x_2(y),$$

by cross sections parallel to the x-axis, then

$$\int_R f(P)\, dA = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x,y)\, dx \right) dy.$$

The intervals of integration are determined by R; the function f influences only the integrand. (See Figure 17.2.7.)

In the next example R is the region bounded by $y = x^2$, $x = 2$, and $y = 0$; the function is $f(x,y) = 3xy$. The integral $\int_R 3xy\, dA$ has at least three interpretations:

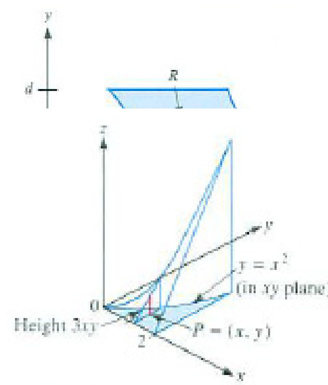


Figure 17.2.8:

1. If at each point $P = (x,y)$ in R we erect a line segment above P of length $3xy$, then the integral is the volume of the resulting solid. (See Figure 17.2.8.)
2. If the density of matter at (x,y) in R is $3xy$, then $\int_R 3xy\, dA$ is the total mass in R.
3. If the temperature at (x,y) in R is $3xy$ then $\int_R 3xy\, dA$ divided by the area of R is the average temperature in R.

EXAMPLE 6 Evaluate $\int_R 3xy\, dA$ over the region R shown in Figure 17.2.9.

This is the same R as in Examples 2 and 3.

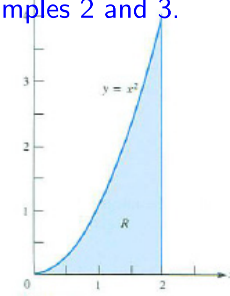


Figure 17.2.9:

SOLUTION If cross sections parallel to the y-axis are used, then R is described by

$$0 \le x \le 2 \qquad 0 \le y \le x^2.$$

Thus

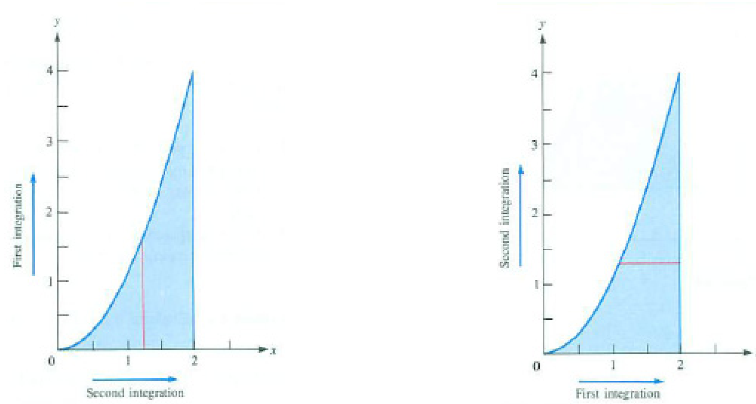
$$\int_R 3xy\, dA = \int_0^2 \left(\int_0^{x^2} 3xy\, dy \right) dx,$$

which is easy to compute. First, with x fixed,

$$\int_0^{x^2} 3xy\, dy = \left. \left(3x\frac{y^2}{2} \right) \right|_{y=0}^{y=x^2} = 3x\frac{(x^2)^2}{2} - 3x\frac{0^2}{2} = \frac{3x^5}{2}.$$

Then,

$$\int_0^2 \frac{3x^5}{2}\, dx = \left. \frac{3x^6}{12} \right|_0^2 = 16.$$



(a) (b)

Figure 17.2.10:

Figure 17.2.10(a) shows which integration is performed first.

The region R can also be described in terms of cross sections parallel to the x-axis:

$$0 \leq y \leq 4 \qquad \sqrt{y} \leq x \leq 2.$$

In this case, the double integral is evaluated as:

$$\int_R 3xy \; dA = \int_0^4 \left(\int_{\sqrt{y}}^2 3xy \; dx \right) \; dy,$$

which, as the reader may verify, equals 16. See Figure 17.2.10(b). $\diamond$

In Example 6 we could evaluate $\int_R f(P) \; dA$ by cross sections in either direction. In the next example we don't have that choice.

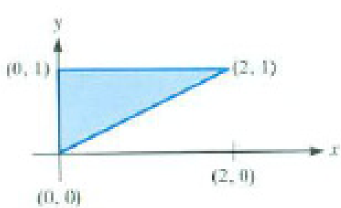


Figure 17.2.11:

EXAMPLE 7 A triangular lamina is located as in Figure 17.2.11. Its density at (x, y) is e^{y^2}. Find its mass, that is $\int_R f(P) \; dA$, where $f(x, y) = e^{y^2}$.

SOLUTION The description of R by vertical cross sections is

$$0 \leq x \leq 2, \qquad \frac{x}{2} \leq y \leq 1.$$

Hence

$$\int_R f(P) \; dA = \int_0^2 \left(\int_{x/2}^1 e^{y^2} \; dy \right) \; dx.$$

Since e^{y^2} does not have an elementary antiderivative, the Fundamental Theorem of Calculus is useless in computing

$$\int_{x/2}^{1} e^{y^2}\ dy.$$

So we try horizontal cross sections instead. The description of R is now

$$0 \le y \le 1, \qquad 0 \le x \le 2y.$$

This leads to a different iterated integral, namely:

$$\int_R f(P)\ dA = \int_0^1 \left(\int_0^{2y} e^{y^2}\ dx \right)\ dy.$$

The first integration, $\int_0^2 e^{y^2}\ dx$, is easy, since y is fixed; the integrand is constant. Thus

Note that the integrand does not depend on x.

$$\int_0^{2y} e^{y^2}\ dx \ = \ e^{y^2} \int_0^{2y} 1\ dx = e^{y^2} x \Big|_{x=0}^{x=2y} = e^{y^2} 2y.$$

The second definite integral in the repeated integral is thus $\int_0^1 e^{y^2} 2y\ dy$, which can be evaluated by the Fundamental Theorem of Calculus, since $d(e^{y^2})/dy = e^{y^2} 2y$:

$$\int_0^1 e^{y^2} 2y\ dy = e^{y^2}\Big|_0^1 = e^{1^2} - e^{0^2} = e - 1.$$

The total mass is $e - 1$. $\diamond$

Notice that computing a definite integral over a plane region R involves, first, a wise choice of an xy-coordinate system; second, a description of R and f relative to this coordinate system; and finally, the computation of two successive definite integrals over intervals. The order of these integrations should be considered carefully since computation may be much simpler in one than the other. This order is determined by the description of R by cross sections.

Summary

We showed that the integral of $f(P)$ over a plane region R can be evaluated by an iterated integral, where the limits of integration are determined by R

(not by f). If each line parallel to the y-axis meets R in at most two points then R has the description

$$a \le x \le b, \qquad y_1(x) \le y \le y_2(x)$$

and

$$\int_R f(P)\ dA = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)\ dy \right)\ dx.$$

If each line parallel to the x-axis meets R in at most two points, then, similarly, R can be described in the form

$$c \le y \le d \qquad x_1(y) \le x \le x_2(y)$$

and

$$\int_R f(P)\ dA = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x,y)\ dx \right)\ dy.$$

A Few Words on Notation

We use the notation $\int f(P)\ dA$ or $\int_R f(P)\ dA$ for a (double) integral over a plane region, $\int f(P)\ dS$ or $\int_{\mathcal{S}} f(P)\ dS$ for an integral over a surface, and $\int f(P)\ dV$ or $\int_R f(P)\ dV$ for a (triple) integral over a region in space. The symbols dA, dS, and dV indicate the type of set over which the integral is defined.

Many people traditionally use repeated integral signs to distinguish dimensions. For instance they would write $\int f(P)\ dA$ as $\iint f(P)dA$ or $\iint f(x,y)\ dxdy$. Similarly, they denote a triple integral by $\iiint f(P)\ dxdydz$. We use the single-integral-sign notation for all integrals for three reasons:

1. it is free of any coordinate system
2. it is compact (uses the fewest symbols): $\int$ for "integral", $f(P)$ or f for the integrand, and dA, dS, or dV for the set
3. it allows the symbols $\iint$ and $\iiint$ to be reserved for use exclusively for iterated integrals.

Iterated integrals are a completely different mathematical object. Each integral in an iterated integral is an integral over an interval. Note that this means we we write dx (or dy or dz) only when we are talking about an integral over an interval.

EXERCISES for Section 17.2 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 12 provide practice in describing plane regions by cross sections in recangular coordinates. In each exercise, describe the region by (a) vertical cross sections and (b) horizontal cross sections.

1.[R] The triangle whose vertices are $(0,0)$, $(2,1)$, $(0,1)$.

2.[R] The triangle whose vertices are $(0,0)$, $(2,0)$, $(1,1)$.

3.[R] The parallelogram with vertices $(0,0)$, $(1,0)$, $(2,1)$, $(1,1)$.

4.[R] The parallelogram with vertices $(2,1)$, $(5,1)$, $(3,2)$, $(6,2)$.

5.[R] The disk of radius 5 and center $(0,0)$.

6.[R] The trapezoid with vertices $(1,0)$, $(3,2)$, $(3,3)$,$(1,6)$.

7.[R] The triangle bounded by the lines $y = x$, $x+y = 2$, and $x+3y = 8$.

8.[R] The region bounded by the ellipse $4x^2+y^2 = 4$.

9.[R] The triangle bounded by the lines $x = 0$, $y = 0$, and $2x+3y = 6$.

10.[R] The region bounded by the curves $y = e^x$, $y = 1-x$, and $x = 1$.

11.[R] The quadrilateral bounded by the lines $y = 1$, $y = 2$, $y = x$, $y = x/3$.

12.[R] The quadrilateral bounded by the lines $x = 1$, $x = 2$, $y = x$, $y = 5-x$.

In Exercises 13 to 16 draw the regions and describe them by horizontal cross sections.

13.[R] $0 \le x \le 2$, $2x \le y \le 3x$

14.[R] $1 \le x \le 2$, $x^3 \le y \le 2x^2$

15.[R] $0 \le x \le \pi/4$, $0 \le y \le \sin x$ and $\pi/4 \le x \le \pi/2$, $0 \le y \le \cos x$

16.[R] $1 \le x \le e$, $(x-1)/(e-1? \le y \le \ln x$

In Exercises 17 to 22 evaluate the iterated integrals.

17.[R] $\int_0^1 \left(\int_0^x (x+2y)\ dy\right)\ dx$

18.[R] $\int_1^2 \left(\int_x^{2x}\ dy\right)\ dx$

19.[R] $\int_0^2 \left(\int_0^{x^2} xy^2\ dy\right)\ dx$

20.[R] $\int_1^2 \left(\int_0^y e^{x+y}\ dx\right)\ dy$

21.[R] $\int_1^2 \left(\int_0^{\sqrt{y}} yx^2\ dx\right)\ dy$

22.[R] $\int_0^1 \left(\int_0^x y\sin(\pi x)\ dy\right)\ dx$

23.[R] Complete the calculation of the second iterated integral in Example 6.

24.[R]

(a) Sketch the solid region S below the plane $z = 1+x+y$ and above the triangle R in the place with vertices $(0,0)$, $(1,0)$, $(0,2)$.

(b) Describe R in terms of coordinates.

(c) Set up an iterated integral for the volume of S.

(d) Evaluate the expression in (c), and show in the manner of Figure 17.2.10(a) and 17.2.10(b) which integration you performed first.

(e) Carry out (c) and (d) in the other order of integration.

25.[R] Let S be the solid region below the paraboloid $z = x^2+2y^2$ and above the rectangle in the xy plane with vertices $(0,0)$, $(1,0)$, $(1,2)$, $(0,2)$. Carry out the steps of Exercise 24 in this case.

26.[R] Let S be the solid region below the saddle $z = xy$ and above the triangle in the xy plane with vertices $(1,1)$, $(3,1)$, and $(1,4)$. Carry out the steps of Exercise 24 in this case.

27.[R] Let S be the solid region below the saddle $z = xy$ and above the region n the first quadrant of the xy plane bounded by the parabolas $y = x^2$ and $y = 2x^2$ and the line $y = 2$. Carry out the steps of Exercise 24 in this case.

28.[R] Find the mass of a thin lamina occupying the finite region bounded by $y = 2x^2$ and $y = 5x - 3$ and whose density at (x, y) is xy.

29.[R] Find the mass of a thin lamina occupying the triangle whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$ and whose density at (x, y) is $1/(1 + x^2)$.

30.[R] The temperature at (x, y) is $T(x, y) = \cos(x + 2y)$. Find the average temperature in the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 2)$.

31.[R] The temperature at (x, y) is $T(x, y) = e^{x-y}$. Find the average temperature in the region in the first quadrant bounded by the triangle with vertices $(0, 0)$, $(1, 1)$, and $(3, 1)$.

In each of Exercises 32 to 35 replace the given iterated integral by an equivalent one with the order of integration reversed. First sketch the region R of integration.

32.[R] $\int_0^2 \left(\int_0^{x^2} x^3y\ dy\right)\ dx$

33.[R] $\int_0^{\pi/2} \left(\int_0^{\cos x} x^2\ dy\right)\ dx$

34.[R] $\int_0^1 \left(\int_{x/2}^{x} xy\ dy\right)\ dx + \int_1^2 \left(\int_{x/2}^{1} xy\ dy\right)\ dx$

35.[R] $\int_{-1/\sqrt{2}}^{0} \left(\int_{-x}^{\sqrt{1-x^2}} x^3y\ dy\right)\ dx + \int_0^1 \left(\int_0^{\sqrt{1-x^2}} x^3y\ dy\right)\ dx$

In Exercises 36 to 39 evaluate th
First sketch the region of integrati

36.[R] $\int_0^1 \left(\int_x^1 \sin(y^2)\ dy\right)\ dx$

37.[R] $\int_0^1 \left(\int_{\sqrt{x}}^1 \frac{dy}{\sqrt{1+y^3}}\right)\ dx$

39.[R] $\int_2^4 \Big(\int_y$

38.[R] $\int_0^1 \left(\int_{\sqrt[3]{y}}^1 \sqrt{1+x^4}/dx\right)\ dy$

40.[R] Let $f(x, y) = y^2e^{y^2}$ and le
bounded by $y = a$, $y = x/2$, and y
a is positive.

(a) Set up two repeated integral

(b) Evaluate the easier one.

41.[R] Let R be the finite region b
$y = \sqrt{x}$ and the line $y = x$. Let $f($
$y \neq 0$ and $f(x, 0) = 1$. Compute $\int$

17.3 Computing $\int_R f(P)\ dA$ Using Polar Coordinates

This section shows how to evaluate $\int_R f(P)\ dA$ by using polar coordinates. This method is especially appropriate when the region R has a simple description in polar coordinates, for instance, if it is a disk or cardioid. As in Section 17.2, we first examine how to describe cross sections in polar coordinates. Then we describe the iterated integral in polar coordinates that equals $\int_R f(P)\ dA$.

Describing R in Polar Coordinates

In describing a region R in polar coordinates, we first determine the range of θ and then see how r varies for any fixed value of θ. (The reverse order is seldom useful.) Some examples show how to find how r varies for each θ.

EXAMPLE 1 Let R be the disk of radius a and center at the pole of a polar coordinate system. (See Figure 17.3.1.) Describe R in terms of cross sections by rays emanating from the pole.

SOLUTION To sweep out R, θ goes from 0 to 2π. Hold θ fixed and consider the behavior of r on the ray of angle θ. Clearly, r goes from 0 to a, independently of θ. (See Figure 17.3.1.) The complete description is

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le a.$$

◇

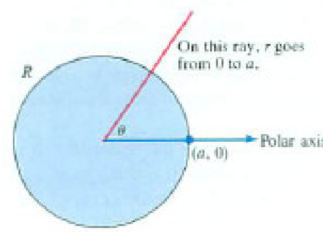


Figure 17.3.1:

EXAMPLE 2 R Let R be the region between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$. Describe R in terms of cross sections by rays from the pole. (See Figure 17.3.2.)

SOLUTION To sweep out this region, use the rays from $\theta = -\pi/2$ to $\theta = \pi/2$. for each such θ, r varies from $2\cos\theta$ to $4\cos\theta$. The complete description is

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \qquad 2\cos\theta \le r \le 4\cos\theta.$$

◇

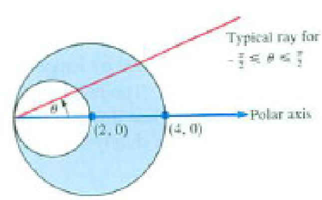
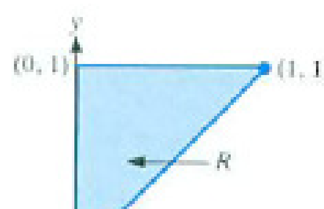


Figure 17.3.2:

As Examples 1 and 2 suggest, polar coordinates provide simple descriptions for regions bounded by circles. The next example shows that polar coordinates may also provide simple descriptions of regions bounded by straight lines, especially if some of the lines pass through the origin.

EXAMPLE 3 Let R be the triangular region whose vertices, in rectangular coordinates, are $(0,0)$, $(1,1)$, and $(0,1)$. Describe R in polar coordinates.



SOLUTION Inspection of R in Figure 17.3.3 shows that θ varies from $\pi/4$ to $\pi/2$. For each θ, r goes from 0 until the point (r, θ) is on the line $y = 1$, that is, on the line $r\sin(\theta) = 1$. Thus the upper limit of r for each θ is $1/\sin(\theta)$. The description of R is

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}, \qquad 0 \leq r \leq \frac{1}{\sin(\theta)}.$$

$\diamond$ In general, cross sections by rays lead to descriptions of plane regions of the form:

$$\alpha \leq \theta \leq \beta, \qquad r_1(\theta) \leq r \leq r_2(\theta).$$

A Basic Difference Between Rectangular and Polar Coordinates

Before we can set up an iterated integral in polar coordinates for $\int_R f(P)\,dA$ we must contrast certain properties of rectangular and polar coordinates.

Consider all points (x, y) in the plane that satisfy the inequalities

$$x_0 \leq x \leq x_0 + \Delta x \qquad \text{and} \qquad y_0 \leq y \leq y_0 + \Delta y,$$

where x_0, Δx, y_0 and Δy are fixed numbers with Δx and Δy positive. The set is a rectangle of sides Δx and Δy shown in Figure 17.3.4(a). The area of this rectangle is simply the product of Δx and Δy; that is,

$$\text{Area} = \Delta x \Delta y. \tag{17.3.1}$$

This will be contrasted with the case of polar coordinates.

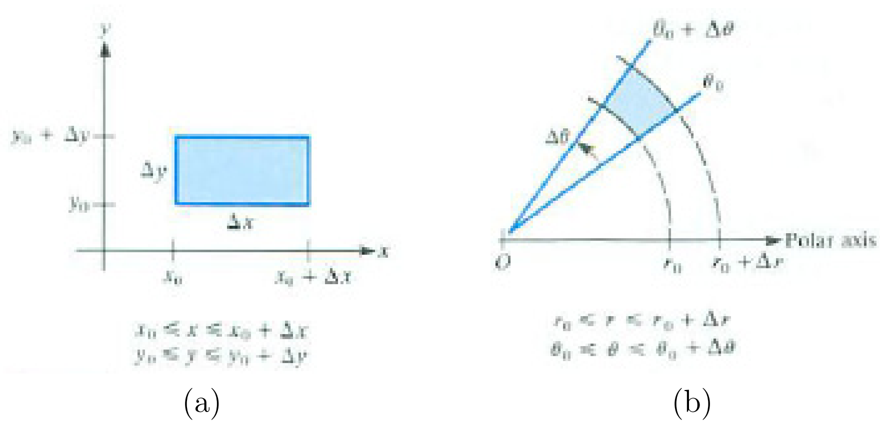


Figure 17.3.4:

Consider the set in the plane consisting of the points (r, θ) such that

$$r_0 \leq r \leq r_0 + \Delta r \qquad \text{and} \qquad \theta_0 \leq \theta \leq \theta_0 + \Delta\theta,$$

where r_0, Δr, θ_0 and $\Delta\theta$ are fixed numbers, with r_0, Δr, θ_0 and $\Delta\theta$ all positive, as shown in Figure 17.3.4(b).

The exact area is found in Exercise 32.

When Δr and $\Delta\theta$ are small, the set is approximately a rectangle, one side of which has length Δr and the other, $r_0\Delta\theta$. So its area is approximately $r_0\Delta r\Delta\theta$. In this case,

$$\text{Area} \approx r_0\Delta r\Delta\theta. \tag{17.3.2}$$

The area is *not* the product of Δr and $\Delta\theta$. (It couldn't be since $\Delta\theta$ is in radians, a dimensionless quantity – "arc length subtended on a circle divided by length of radius" – so $\Delta r\Delta\theta$ has the dimension of length, not of area.) The presence of this extra factor r_0 will be reflected in the integrand we use when integrating in polar coordinates.

It is necessary to replace dA by $r\ dr\ d\theta$, not simply by $dr\ d\theta$.

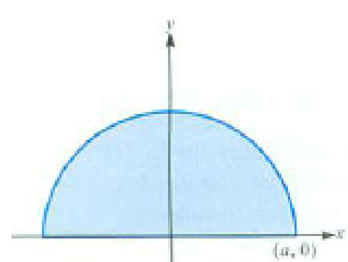
How to Evaluate $\int_R f(P)\ dA$ by an Iterated Integral in Polar Coordinates

The method for computing $\int_R f(P)\ dA$ with polar coordinates involves an iterated integral where the dA is replaced by $r\ dr\ d\theta$. A more detailed explanation of why the r must be added is given at the end of this section.

Notice the factor r in the integrand.

Evaluating $\int_R f(P)\ dA$ in Polar Coordinates

1. Express $f(P)$ in terms of r and θ: $f(r, \theta)$.
2. Describe the region R in polar coordinates:
$$\alpha \leq \theta \leq \beta, \qquad r_1(\theta) \leq r \leq r_2(\theta).$$
3. Evaluate the iterated integral:
$$\int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r\ dr\ d\theta.$$



EXAMPLE 4 Let R be the semicircle of radius a shown in Figure 17.3.5. Let $f(P)$ be the distance from a point P to the x-axis. Evaluate $\int_R f(P)\,dA$ by an iterated integral in polar coordinates.

SOLUTION In polar coordinates, R has the description

$$0 \leq \theta \leq \pi, \qquad 0 \leq r \leq a.$$

Notice the extra r in the integrand.

The distance from P to the x-axis is, in rectangular coordinates, y. Since $y = r\sin(\theta)$, $f(P) = r\sin(\theta)$. Thus,

$$\int\limits_R f(P)\,dA = \int\limits_0^\pi \left(\int\limits_0^a (r\sin(\theta))r\,dr \right)\,d\theta.$$

From here on the calculation are like those in the preceding section.

The calculation of the iterated integral is like that for an iterated integral in rectangular coordinates. First, evaluate the inside integral:

$$\int\limits_0^a r^2\sin(\theta)\,dr = \sin(\theta)\int\limits_0^a r^2\,dr = \sin(\theta)\left(\frac{r^3}{3}\right)\Big|_0^a = \frac{a^3\sin(\theta)}{3}.$$

The outer integral is therefore

$$\begin{aligned}\int\limits_0^\pi \frac{a^3\sin\theta}{3}\,d\theta &= \frac{a^3}{3}\int\limits_0^\pi \sin\theta\,d\theta = \frac{a^3}{3}(-\sin\theta)\Big|_0^\pi \\ &= \frac{a^3}{3}[(-\cos\pi)-(-\cos 0)] = \frac{a^3}{3}(1+1) = \frac{2a^3}{3}.\end{aligned}$$

Thus

$$\int\limits_R y\,dA = \frac{2a^3}{3}.$$

◇

Example 5 refers to a ball of radius a. Generally, we will distinguish between a **ball**, which is a solid region, and a **sphere**, which is only the surface of a ball.

EXAMPLE 5 A ball of radius a has its center at the pole of a polar coordinate system. Find the volume of the part of the ball that lies above the plane region R bounded by the curve $r = a\cos(\theta)$. (See Figure 17.3.6.)

SOLUTION It is necessary to describe R and f in polar coordinates, where $f(P)$ is the length of a cross section of the solid made by a vertical line through P. R is described as follows: r goes 0 to $a\cos(\theta)$ for each θ in $[-\pi/2, \pi/2]$, that is,

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \qquad 0 \leq r \leq a\cos\theta.$$

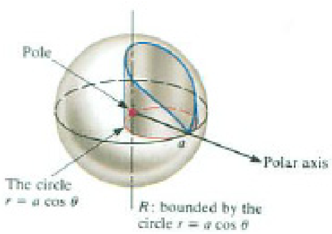


Figure 17.3.6:

To express $f(P)$ in polar coordinates, consider Figure 17.3.7, which shows the top half of a ball of radius a. By the Pythagorean Theorem,

$$r^2 + (f(r,\theta))^2 = a^2.$$

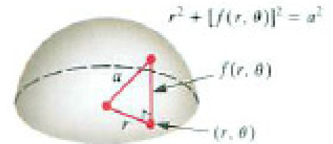


Figure 17.3.7:

Thus

$$f(r,\theta) = \sqrt{a^2 - r^2}.$$

Consequently,

$$\text{Volume} = \int_R f(P)\ dA = \int_{-\pi/2}^{\pi/2} \left(\int_0^{a\cos(\theta)} \sqrt{z^2 - r^2} r\ dr \right) d\theta.$$

Exploiting symmetry, compute half the volume, keeping θ in $[0, \pi/2]$, and then double the result:

Remember to double.

$$\begin{aligned}
\int_0^{a\cos(\theta)} \sqrt{a^2 - r^2} r\ dr &= \left. \frac{-(a^2 - r^2)^{3/2}}{3} \right|_0^{a\cos(\theta)} = -\left(\frac{(a^2 - a^2\cos^2(\theta))^{3/2}}{3} - \frac{(a^2)^{3/2}}{3} \right) \\
&= \frac{a^3}{3} - \frac{(a^2 - a^2\cos^2(\theta))^{3/2}}{3} = \frac{a^3}{3} - \frac{a^3(1 - \cos^2(\theta))^{3/2}}{3} \\
&= \frac{a^3}{3}(1 - \sin^3(\theta)).
\end{aligned}$$

(The trigonometric formula used above, $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$, is true when $0 \le \theta \le \pi/2$ but not when $-\pi/2 \le \theta \le 0$.)

Then comes the second integration:

$$\begin{aligned}
\int_0^{\pi/2} \frac{a^3}{3}(1 - \sin^3(\theta))\ d\theta &= \frac{a^3}{3} \int_0^{\pi/2} (1 - (1 - \cos^2(\theta))\sin(\theta))\ d\theta \\
&= \frac{a^3}{3} \int_0^{\pi/2} 1 - \sin(\theta) - \cos^2(\theta)\sin(theta)\ d\theta \\
&= \frac{a^3}{3} \left. \left(\theta + \cos(\theta) - \frac{\cos^3(\theta)}{3} \right) \right|_0^{\pi/2} \\
&= \frac{a^3}{3} \left[\frac{\pi}{2} - \left(1 - \frac{1}{3} \right) \right] = a^3 \left(\frac{3\pi - 4}{18} \right).
\end{aligned}$$

The total volume is twice is large:

We remembered.

$$a^3 \left(\frac{3\pi - 4}{9} \right).$$

◇

EXAMPLE 6 A circular disk of radius a is formed of a material which had a density at each point equal to the distance from the point to the center.

(a) Set up an iterated integral in rectangular coordinates for the total mass of the disk.

(b) Set up an iterated integral in polar coordinates for the total mass of the disk.

(c) Compute the easier one.

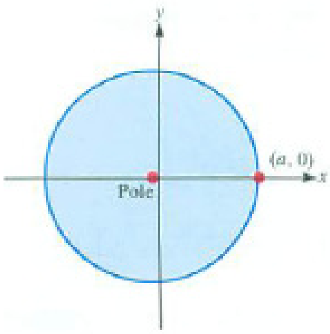


Figure 17.3.8:

SOLUTION The disk is shown in Figure 17.3.8.

(a) (Rectangular coordinates) The density $\sigma(P)$ at the point $(P) = (x, y)$ is $\sqrt{x^2 + y^2}$. The disk has the description

$$-a \le x \le a, \qquad -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}.$$

Thus

$$\text{Mass} = \int_R \sigma(P)\,dA = \int_{-a}^{a} \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2 + y^2}\,dy \right) dx.$$

(b) (Polar coordinates) The density $\sigma(P)$ at $P = (r, \theta)$ is r. The disk has the description

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le a.$$

Thus

$$\text{Mass} = \int_R \sigma(P)\,dA = \int_0^{2\pi} \left(\int_0^a r \cdot r\,dr \right) d\theta = \int_0^{2\pi} \left(\int_0^a r^2\,dr \right) d\theta.$$

(c) Even the first integration in the iterated integral in (a) would be tedious. However, the iterated integral in (b) is a delight: The first integration gives

$$\int_0^a r^2\,dr = \left. \frac{r^3}{3} \right|_0^a = \frac{a^3}{3}.$$

The second integration gives

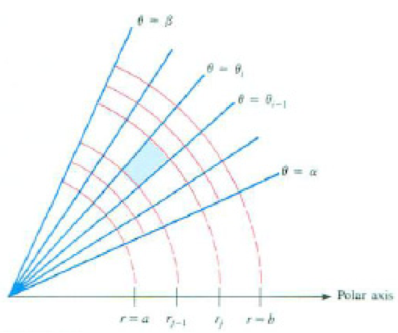
$$\int_0^{2\pi} \frac{a^3}{3}\, d\theta = \left.\frac{a^3\theta}{3}\right|_0^{2\pi} = \frac{2\pi a^3}{3}.$$

The total mass is $2\pi a^3/3$.

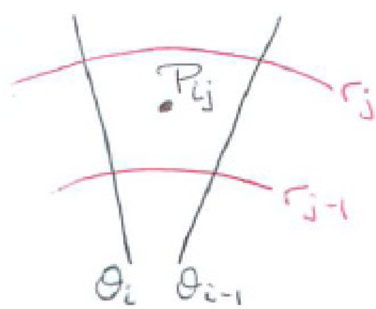
◇

A Fuller Explanation of the Extra r in the Integrand

Consider $\int_R f(P)\, dA$ as the region in the plane bound by the circle $r = a$ and $r = b$ and the range $\theta = \alpha$ and $\theta = \beta$. Break it into n^2 little pieces with the aid of the partitions $r_0 = a, r_1, r_i, r_n = b$ and $\theta_0 = \alpha, \theta_1, \theta_j, \theta_n = \beta$. For convenience, assume that all $r_i - r_{i-1}$ are equal to Δr and all $\theta_j - \theta_{j-1}$ are equal to $\Delta\theta$. (See Figure 17.3.9(a).) The typical patch, shown in Figure 17.3.9(b),



(a)



(b)

Figure 17.3.9: (b) P_{ij} is $\left(\frac{r_j + r_{j+1}}{2}, \frac{\theta_j + \theta_{i-1}}{2}\right)$

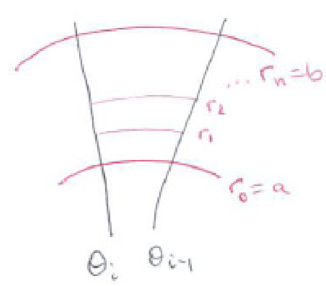
has area, exactly

$$A_i j = \frac{(r_j + r_{j-1})}{2}(r_j - r_{j-1})(\theta_i - \theta_{i-1}),$$

as shown in Exercise 6.

Then the sum of the n^2 terms of the form $f(P_{ij})A_{ij}$ is an estimate of $\int_R f(P)\, dA$.

Let us look closely at the summand for the n patches between the rays $\theta = \theta_{i-1}$ and $\theta = \theta_i$, as shown in Figure 17.3.10.



The sum is

$$\sum_{j=1}^{n} f\left(\frac{r_j + r_{j-1}}{2}, \frac{\theta_i + \theta_{i-1}}{2}\right) \frac{r_j + r_{j+1}}{2} \Delta r \Delta\theta. \qquad (17.3.3)$$

In (17.3.3), θ_i, θ_{i-1}, and $\Delta\theta$ are constants. If we define $g(r, \theta)$ to be $f(r, \theta)r$, then the sum is

$$\left(\sum_{i-1}^{n} g\left(\frac{r_j + r_{j+1}}{2}, \frac{\theta_i + \theta_{i-1}}{2}\right) \Delta r\right) \ \Delta\theta. \qquad (17.3.4)$$

The sum in brackets in (17.3.4) is an estimate of

$$\int_a^b g\left(r, \frac{\theta_j + \theta_{j-1}}{2}\right) \ dr.$$

Thus the sum, corresponding to the region between the rays $\theta = \theta_i$ and $\theta = \theta_{i-1}$, is

$$\sum_{i=1}^{n} \int_a^b g\left(r, \frac{\theta_i + \theta_{i-1}}{2}\right) \ dr \ \Delta\theta. \qquad (17.3.5)$$

Now let $h(\theta) = \int_a^b g(r, \theta) \ dr$. Then (17.3.5) equals

$$\sum_{i=1}^{n} h\left(\frac{\theta_i + \theta_{i-1}}{2}\right) \ \Delta\theta.$$

This is an estimate of $\int_a^b f(\theta) \ d\theta$. Hence the sum of all n^2 little terms of the form $f(P_{ij})A_{ij}$ is an approximation of

$$\int_\alpha^\beta h(\theta) \ d\theta \ = \ \int_\alpha^\beta \left(\int_a^b g(r, \theta) \ dr\right) \ d\theta = \int_\alpha^\beta \left(\int_a^b f(r, \theta)r \ dr\right) \ d\theta.$$

The extra factor r appears as we obtained the first integral, $\int_a^b f(r, \theta)r \ dr$. The sum of the n^2 terms A_{ij}, which we knew approximated the double integral $\int_R f(P) \ dA$, we now see approximate also the iterated integral (17.3.6). Taking limits as $n \to \infty$ show that the iterated integral equals the double integral.

Summary

We saw how to calculate an integral $\int_R f(P) \ dA$ by introducing polar coordinates. In this case, the plane region R can be described, in polar coordinates, as

$$\alpha \leq \theta \leq \beta, \qquad r_1(\theta) \leq r \leq r_2(\theta)$$

then

$$\int\limits_R f(P)\ dA = \int\limits_\alpha^\beta \int\limits_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r\ dr\ d\theta.$$

The extra r in the integrand is due to the fact that a small region corresponding to changes dr and $d\theta$ has area area approximately $r\ dr\ d\theta$ (not $dr\ d\theta$). Polar coordinates are convenient when either the function f or the region R has a simple description in terms of r and θ.

EXERCISES for Section 17.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 6 draw and describe the given regions in the form $\alpha \le \theta \le \beta$, $r_1(\theta) \le r \le r_2(\theta)$.

1.[R] The region inside the curve $r = 3 + \cos(\theta)$.

2.[R] The region between the curve $r = 3 + \cos(\theta)$ and the curve $r = 1 + \sin(\theta)$.

3.[R] The triangle whose vertices have the rectangular coordinates $(0,0)$,$(1,1)$, and $(1,\sqrt{3})$.

4.[R] The circle bounded by the curve $r = 3\sin(\theta)$.

5.[R] The region shown in Figure 17.3.11.

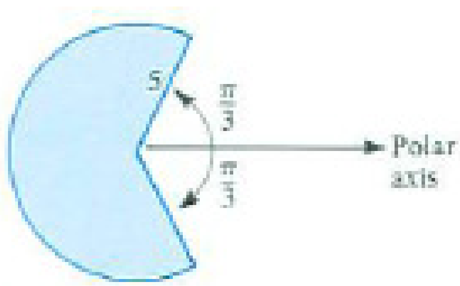


Figure 17.3.11:

6.[R] The region in the loop of the three-leaved rose, $r = \sin(3\theta)$, that lies in the first quadrant.

7.[R]

(a) Draw the region R bounded by the lines $y = 1$, $y = 2$, $y = x$, $y = x/\sqrt{3}$.

(b) Describe R in terms of horizontal cross sections,

(c) Describe R in terms of vertical cross sections,

(d) Describe R in terms of cross sections by polar rays.

8.[R]

(a) Draw the region R whose description is given by

$$-2 \le y \le 2, \qquad -\sqrt{4 - y^2} \le x \le \sqrt{4 - y^2}.$$

(b) Describe R by vertical cross sections.

(c) Describe R by cross sections obtained using polar rays.

9.[R] Describe in polar coordinates the square whose vertices have rectangular coordinates $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

10.[R] Describe the trapezoid whose vertices have rectangular coordinates $(0,1)$, $(1,1)$, $(2,2)$, $(0,2)$.

(a) in polar coordinates,

(b) by horizontal cross sections,

(c) by vertical cross sections.

In Exercises 5 to 14 draw the regions and evaluate $\int_R r^2\, dA$ for the given regions R.

11.[R] $-\pi/2 \le \theta \le \pi/2$, $0 \le r \le 1 + \cos(\theta)$

$0 \le r \le \cos(\theta)$

12.[R] $0 \le \theta \le \pi/2$, $0 \le r \le \sin^2(\theta)$

14.[R] $0 \le \theta \le 0.3$, $0 \le r \le \sin 2(\theta)$

13.[R] $0 \le \theta \le 2\pi$,

In Exercises 15 to 18 draw R and evaluate $\int_R y^2\, dA$ for the given regions R.

15.[R] The circle of radius a, center at the pole.

16.[R] The circle of radius a with center at $(a,0)$ in polar coordinates.

17.[R] The region within the cardioid $r = 1 + \sin\theta$.

18.[R] The region within one leaf of the four-leaved rose $r = \sin 2\theta$.

In Exercises 19 and 20, use iterated integrals in polar coordinates to find the given point.

19.[R] The center of mass of the region within the cardioid $r = 1 + \cos(\theta)$.

20.[R] The center of mass of the region within the leaf $r = \cos 3(\theta)$ that lies along the polar axis.

The average of a function $f(P)$ over a region R in the plane is defined as $\int_R f(P)\ dA$ divided by the area of R. In each of Exercises 21 to 24, find the average of the given function over the given region.

21.[R] $f(P)$ is the distance from P to the pole; R is one leaf of the three-leaved rose, $r = \sin(3\theta)$.

22.[R] $f(P)$ is the distance from P to the x-axis; R is the region between the rays $\theta = \pi/6$, $\theta = \pi/4$, and the circles $r = 2$, $r = 3$.

23.[R] $f(P)$ is the distance from P to a fixed point on the border of a disk R of radius a. (Hint: Choose the pole wisely.)

24.[R] $f(P)$ is the distance from P to the x-axis; R is the region within the cardioid $r = 1 + \cos(\theta)$.

In Exercises 25 to 28 evaluate the given iterated integrals using polar coordinates. Pay attention to the elements of each exercise that makes it appropriate for evaluation in polar coordinates.

25.[R] $\int_0^1 \left(\int_0^x \sqrt{x^2+y^2}\ dy \right)\ dx$

26.[R] $\int_0^1 \left(\int_0^{\sqrt{1-x^2}} x^3\ dy \right)\ dx$

28.[R] $\int_1^2 \left(\int_{x/\sqrt{3}}^{\sqrt{3}x} (x^2+y^2)^{3/2}\ dy \right)\ dx$

27.[R] $\int_0^1 \left(\int_x^{\sqrt{1-x^2}} xy\ dy \right)\ dx$

29.[R] Evaluate the integrals over the given regions.

(a) $\int_R \cos(x^2+y^2)\ dA$; R is the portion in the first quadrant of the disk of radius a centered at the origin.

(b) $\int_R \sqrt{x^2+y^2}\ dA$; R is the triangle bounded by the line $y = x$, the line $x = 2$, and the x-axis.

30.[R] Find the volume of the region above the paraboloid $z = x^2+y^2$ and below the plane $z = x+y$.

31.[R] The area of a region R is equal to $\int_R 1\ dA$. Use this to find the area of a disk of radius a. (Use an iterated integral in polar coordinates.)

32.[R] Find the area of the shaded region in Figure 17.3.4(b) as follows:

(a) Find the area of the ring between two circles, one of radius r_0, the other of radius $r_0 + \Delta r$.

(b) What fraction of the area in (a) is included between two rays whose angles differ by $\Delta\theta$?

(c) Show that the area of the shaded region in Figure 17.3.4(b) is precisely

$$\left(r_0 + \frac{\Delta r}{2} \right) \Delta r \Delta\theta.$$

33.[R] Evaluate the repeated integral

$$\int_{-\pi/2}^{\pi/2} \left(\int_0^{a\cos(\theta)} \sqrt{a^2-r^2}\ r\ dr \right)\ d\theta$$

directly. The result should still be $a^3(3\pi-4)/9$. (In Example 5 we computed half the volume and doubled the result.)

Caution: Use trigonometric formulas with care.

Prior to beginning Exercise 34, consider the following two quotes:

> Once when lecturing to a class he [the physicist Lord Kelvin] used the word "mathematician" and then interrupting himself asked the class: "Do you know what a mathematician is?" Stepping to his blackboard he wrote upon it: $\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$. Then putting his finger on what he had written, he turned to his class and said, "A mathematician is one to whom this is as obvious as that twice two makes four is to you."

S. P. Thompson, in *Life of Lord Kelvin* (Macmillan, London, 1910).

> Many things are not accessible to intuition at all, the value of $\int_0^{\infty} e^{-x^2}\ dx$ for instance.

J. E. Littlewood, "Newton and the Attraction of the Sphere", *Mathematical Gazette*, vol. 63, 1948.

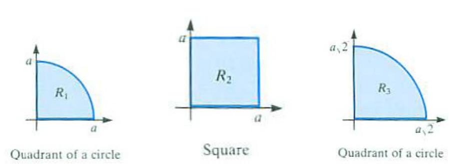
34.[M] This exercise shows that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$. Let R_1, R_2, and R_3 be the three regions indicated in Figure 17.3.12, and $f(P) = e^{-r^2}$ where r is the distance from P to the origin. Hence, $f(r,\theta) = e^{-r^2}$ in polar coordinates and in rectangular coordinates $f(x,y) = e^{-x^2-y^2}$. NOTE: Observe that R_1 is inside R_2 and R_2 is inside R_3.

(a) Show that $\int_{R_1} f(P)\,dA = \frac{\pi}{4}\left(1 - e^{-a^2}\right)$ and that $\int_{R_3} f(P)\,dA = \frac{\pi}{4}\left(1 - e^{-2a^2}\right)$.

(b) By considering $\int_{R_2} f(P)\,dA$ and the results in (a), show that

$$\frac{\pi}{4}\left(1 - e^{-a^2}\right) < \left(\int\limits_0^\infty e^{-x^2}\,dx\right)^2 < \frac{\pi}{4}\left(1 - e^{-2a^2}\right).$$

(c) Show that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.



(a) (b) (c)

Figure 17.3.12:

35.[R] Figure 17.3.13 shows the "bell curve" or "normal curve" often used to assign grades in large classes. Using the fact established in Exercise 34 that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi/2}$, show that the area under the curve in Figure 17.3.13 is 1.

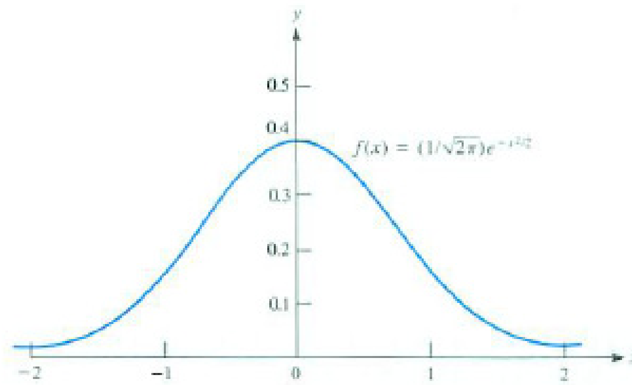


Figure 17.3.13:

36.[R] (The spread of epidemics.) In the theory of a spreading epidemic it is assumed that the probability that a contagious individual infects an individual D miles away depends only on D. Consider a population that is uniformly distributed in a circular city whose radius is 1 mile. Assume that the probability we mentioned is proportional to $2 - D$. For a fixed point Q let $f(P) = 2 - \overline{PQ}$. Let R be the region occupied by the city.

(a) Why is the exposure of a person residing at Q proportional to $\int_R f(P)\,dA$, assuming that contagious people are uniformly distributed throughout the city?

(b) Compute this definite integral when Q is the center of town and when Q is on the edge of town.

(c) In view of (b), which is the safer place?

Transportation problems lead to integrals over plane sets, as Exercises 37 to 40 illustrate.

37.[R] Show that the average travel distance from the center of a disk of area A to points in the disk is precisely $2\sqrt{A}/(3\sqrt{})\pi \approx 0.376\sqrt{A}$.

38.[R] Show that the average travel distance from the center of a regular hexagon of area A to points in the hexagon is

$$\frac{\sqrt{2A}}{3^{3/4}}\left(\frac{1}{3}+\frac{\ln 3}{4}\right) \approx 0.377\sqrt{A}.$$

39.[R] Show that the average travel distance from the center of a square of area A to points in the square is $(\sqrt{2}+\ln(\tan(3\pi/8)))\sqrt{A/6} \approx 0.383\sqrt{A}$.

40.[R] Show that the average travel distance from the centroid of an equilateral triangle of area A to points in the triangle is

$$\frac{\sqrt{A}}{3^{9/4}}\left(2\sqrt{3}+\ln(\tan(\frac{5\pi}{12}))\right) \approx 0.404\sqrt{A}$$

NOTE: The centroid of a triangle is its center of mass.

In Exercises 37 to 40 the distance is the ordinary straight-line distance. In cities the usual street pattern suggests that the "metropolitan" distance between the points (x_1, y_1) and (x_2, y_2) should be measured by $|x_1 - x_2| + |y_1 - y_2|$.

41.[M] Show that if in Exercise 37 metropolitan distance is used, then the average is $8\sqrt{A}/(3\pi^{3/2}) \approx 0.470\sqrt{A}$.

42.[M] Show that if in Exercise 40 metropolitan distance is used, then the average is $\sqrt{A/2}$. In most cities the metropolitan average tends to be about 25 percent larger than the direct-distance average.

43.[C]

Sam: The formula in this section for integrating in polar coordinates is wrong. I'll get the right formula. We don't need the factor r.

Jane: But the book's formula gives the correct answers.

Sam: I don't care. Let $f(r,\theta)$ be positive and I'll show how to integrate over the set R bounded by $r = b$ and $r = a$, $b > a$, and $\theta = \beta$ and $\theta = \alpha$. We have $\int_R f(P)\ dA$ is the volume under the graph of f and above R. Right?

Jane: Right.

Sam: The area of the cross-section corresponds to a fixed angle θ is $\int_a^b f(r,\theta)\ dr$. Right?

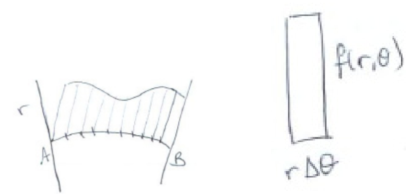
Jane: Right.

Sam: So I, just integrate cross-sectional areas as θ goes from α to β, and the volume is therefore $\int_\alpha^\beta(\int_a^b f(r,\theta)\ dr)\ d\theta$. Perfectly straightforward. I hate to overthrow a formula that's been around for three centuries.

What does Jane say next?

44.[C]

Jane: I won't use a partition. Instead, look at the area under the graph of f and above the circle of radius r. I'll draw this fence for you (see Figure 17.3.14(a).



(a) (b)

Figure 17.3.14:

To estimate its area I'll cut the arc AB into n sections of equal length by angle $\theta_0 = a \ldots$.

Then break AB into n short area, each of length $r\Delta\theta$. (Remember, Sam, how radians are defined.) The typical small approach to the shaded area looks like Figure 17.3.14(b). That's just an estimate of $\int_\alpha^\beta f(r,\theta)r\ d\theta$. Here r is fixed. Then I integrate the cross-sectional area as r goes from a to b. The total volume is then $\int_a^b\int_\alpha^\beta f(r,\theta)r\ d\theta\ dr$. But $\int_R f(r,\theta)\ dA$ is the volume.

Sam: All right.

Jane: At least it gives the r factor.

Sam: But you had to assume f is positive.

Jane: Well, if it isn't just add a big positive number k to f, then $g = f + k$ is positive. From then on its easy. If it's so far g it's so far f.

Check that Jane is right about g and f.

17.4 The Triple Integral: Integrals Over Solid Regions

In this section we define integrals over solid regions in space and show how to compute them by iterated integrals using rectangular coordinates. Throughout we assume the regions are bounded by smooth surfaces and the functions are continuous.

The Triple Integral

Let R be a region in space bounded by some surface. For instance, R could be a ball, a cube, or a tetrahedron. Let f be a function refined at least on R.

For each positive integer n break R into n small region $R_1, R_2, \dots R_n$. Choose a point P, in R_1, P_2 in R_2, ..., P_n in R_n. Let the volume of R_i be V_i. Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(P_i)V_i$$

exists. It is denoted

$$\int\limits_R f(P)\ dV \tag{17.4.1}$$

and is called the integral of f over R or the **triple integral** of f over R.

NOTE:

1. As in the preceding section, we define small. For each n let r_n be the smallest number such that each R_i in the partition fits inside a ball of radius r_n. We assume that $r_n \to 0$ as $n \to \infty$.
2. The notation $\int\int\int_R f(P)\ dV$ is commonly used, but, we stick to using one integral sign, $\int_R f(P)\ dV$ to emphasize that the triple integral is not a repeated integral.
3. The notation $\int\int\int f(x, y, z)\ dV$ is also used, but, again, we prefer not to refer to a particular coordinate system.

EXAMPLE 1 If $f(P) = 1$ for each point P in a solid region R, compute $\int_R f(P)\ dV$.

SOLUTION Each approximating sum $\sum_{i=1}^{n} f(P_i)V_i$ has the value

$$\sum_{i=1}^{n} 1 \cdot V_i = V_1 + V_2 + \cdots + V_n = \text{Volume of } R.$$

Hence

$$\int_R f(P)\ dV = \text{Volume of } R,$$

a fact that will be useful for computing volumes. ◇

Average of a function

The **average value** of a function f defined on a region R in space is defined as

$$\frac{\int_R 1\ dV}{\text{Volume of } R}.$$

This is the analog of the definition of the average of a function over an interval (Section 6.3) or the average of a function over a plane region (Section 17.1). If f describes the density of matter in R, then the average value of f is the density of a *homogeneous* solid occupying R and having the same total mass as the given solid.

Think about it. If the number

$$\frac{\int_R f(P)\ dV}{\text{Volume of } R}.$$

is multiplied by the volume of R, the result is

$$\int_R f(P)\ dV,$$

which is the total mass.

SHERMAN: I have a feeling I've read this before, but didn't find it in a quick search. Is this a repeat? If, should one be removed?

"Density" at a point is defined for lamina; with balls replacing disks. For a positive number r, let $m(r)$ be the mass in a ball with center P and radius r. Let $V(r)$ be the volume of the ball of radius r. Then the density at P is defined as

$$\lim_{r\to 0}\frac{m(r)}{V(r)}.$$

An Interpretation of $\int_R f(P)\ dV$.

Triple integrals appear in the study of gravitation, rotating bodies, centers of gravity, and electro-magnetic theory. The simplest way to think of them is to interpret $f(P)$ as the density at P of some disturbance of matter and, then, $\int_R f(P)\ dV$ is the total mass in a region R.

We can't picture $\int_R f(P)\ dV$ as measuring the volume of something. We could do this for $\int_R f(P)\ dA$, because we could use two dimensions for describing the region of integration and then the third dimension for the values of the function, obtaining a surface in three-dimensional space. However, with

$\int_R f(P)\, dV$, we use up three dimensions just describing the region of integration. We need four-dimensional space to show the values of the function. But it's hard to visualize such a space, no matter how hard we squint.

A Word about Four-Dimensional Space

We can think of 2-dimensional space as the set of ordered pairs (x, y) of real numbers. The set of ordered triplets of real numbers (x, y, z) represents 3-dimensional space. The set of ordered quadruplets of real numbers (x, y, z, t) represents 4-dimensional space.

It is easy to show 4-D space is a very strange place.

In 2-dimensional space the set of points of the form $(x, 0)$, the y-axis, meets the set of points of the form $(0, y)$, the y-axis, in a point, namely the origin $(0, 0)$. Now watch what can happen in 4-space. The set of points of the form $(x, y, 0, 0)$ forms a plane congruent to our familiar xy-plane. The set of points of the form $(0, 0, z, t)$ forms another such plane. So far, no surprise. But notice what the intersection of those two planes is. Their intersection is just the point $(0, 0, 0, 0)$. Can you picture two endless planes meeting in a single point? If so, please tell us how.

Describing a Solid Region

In order to evaluate triple integrals, it is necessary to describe solid regions in terms of coordinates.

A description of a typical solid region in rectangular coordinates has the form

This is the order x, y, then z. There are six possible orders, as you may check.

$$a \le x \le b, \qquad y_1(x) \le y \le y_2(x), \qquad z_1(x, y) \le z \le z_2(x, y).$$

The inequalities on x and y describe the "shadow" or projection of the region on the xy plane. The inequalities for z then tell how z varies on a line parallel to the z-axis and passing through the point (x, y) in the projection. (See Figure 17.4.1.)

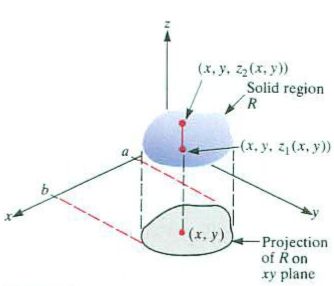
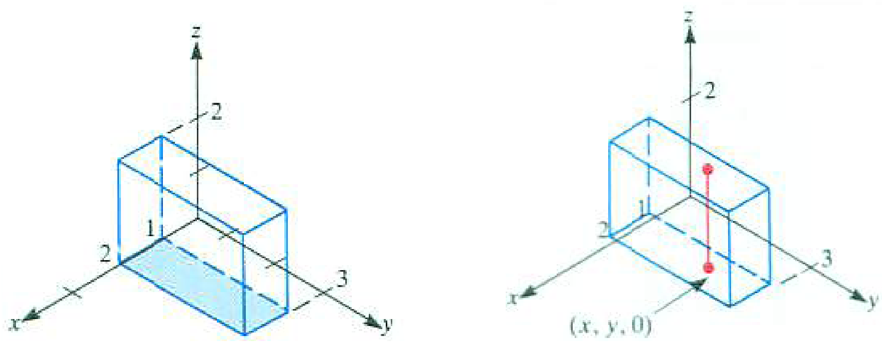


Figure 17.4.1:

EXAMPLE 2 Describe in terms of x, y, and z the rectangular box shown in Figure 17.4.2(a).

SOLUTION The shadow of the box on the xy plane has a description $1 \le x \le 2$, $0 \le y \le 3$. For each point in this shadow, z varies from 0 to 2, as shown in Figure 17.4.2(b). So the description of the box is

$$1 \le x \le 2, \qquad 0 \le y \le 3, \qquad 0 \le z \le 2,$$



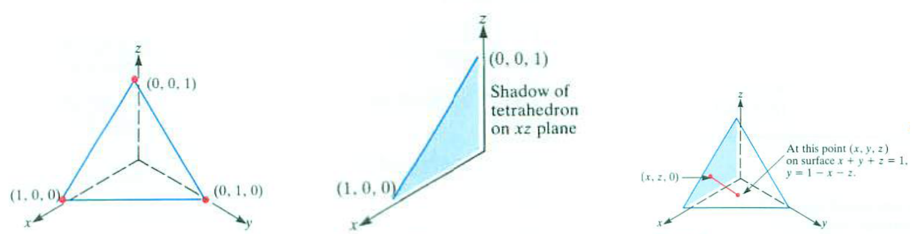
(a) (b)

Figure 17.4.2:

which is read from left to right as "x goes from 1 to 2; for each such x, the variable y goes from 0 to 3; for each such x and y, the variable z goes from 0 to 2."

Of course, we could have changed the order of x and y in the description of the shadow or projected the box on one of the other two coordinate planes. (All told, there are six possible descriptions.) ◇

EXAMPLE 3 Describe by cross sections the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$, as shown in Figure 17.4.3(a).



(a) (b) (c)

Figure 17.4.3:

SOLUTION For the sake of variety, project the tetrahedron onto the xz plane. The shadow is shown in Figure 17.4.3(b). A description of the shadow is

$$0 \leq x \leq 1, \qquad 0 \leq z \leq 1 - x,$$

since the slanted edge has the equation $x + z = 1$. For each point (x, z) in this shadow, y ranges from 0 up to the value of y that satisfies the equation

$x+y+z=1$, that is, up to $y=1-x-z$. (See Figure 17.4.3(c).) A description of the tetrahedron is

$$0 \le x \le 1, \qquad 0 \le z \le 1-x, \qquad 0 \le y \le 1-x-z.$$

That is, x goes from 0 to 1; for each x, z goes from 0 to $1-x$; for each x and z, y goes from 0 to $1-x-z$. $\diamond$

EXAMPLE 4 Describe in rectangular coordinates the ball of radius 4 whose center is at the origin.

SOLUTION The shadow of the ball on the xy plane is the disk of radius 4 and center $(0,0)$. Its description is

$$-4 \le x \le 4, \qquad -\sqrt{16-x^2} \le y \le \sqrt{16-x^2}.$$

Hold (x,y) fixed in the xy plane and consider the way z varies on the line parallel to the z-axis that passes through the point $(x,y,0)$. Since the sphere that bounds the ball has the equation

$$x^2+y^2+z^2=16,$$

for each appropriate (x,y), z varies from

$$-\sqrt{16-x^2-y^2} \qquad \text{to} \qquad \sqrt{16-x^2-y^2}.$$

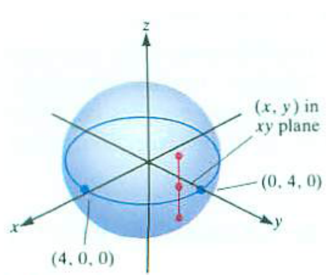


Figure 17.4.4:

This describes the line segment shown in Figure 17.4.4.

The ball, therefore, has a description

$$-4 \le x \le 4, \qquad -\sqrt{16-x^2} \le y \le \sqrt{16-x^2}, \qquad \sqrt{16-x^2-y^2} \le z \le \sqrt{16-x^2-$$

$\diamond$

Iterated Integrals for $\int_R f(P)\ dV$

The iterated integral in rectangular coordinates for $\int_R f(P)\ dV$ is similar to that for evaluating integrals over plane sets. It involves three integrations instead of two. The limits of integration are determined by the description of R in rectangular coordinates. If R has the description

$$a \le x \le b, \qquad y_1(x) \le y \le y_2(x), \qquad z_1(x,y) \le z \le z_2(x,y),$$

then

$$\int_R f(P)\ dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\ dz \right) dy\ dx.$$

An example illustrates how this formula is applied. In Exercise 31 an argument for its plausibility is presented.

EXAMPLE 5 Compute $\int_R z\,dV$, where R is the tetrahedron in Example 3.

SOLUTION A description of the tetrahedron is

$$0 \le y \le 1, \qquad 0 \le x \le 1-y, \qquad 0 \le z \le 1-x-y.$$

Hence

$$\int_R z\,dV = \int_0^1 \left(\int_0^{1-y} \left(\int_0^{1-x-y} z\,dz \right) dx \right) dy.$$

Compute the inner integral first, treating x and y as constants. By the Fundamental Theorem,

$$\int_0^{1-x-y} z\,dz = \left.\frac{z^2}{2}\right|_{z=0}^{z=1-x-y} = \frac{(1-x-y)^2}{2}.$$

The next integration, where y is fixed, is

$$\int_0^{1-y} \frac{(1-x-y)^2}{2}\,dx = -\left.\frac{(1-x-y)^3}{6}\right|_{x=0}^{x=1-y} = -\frac{0^3}{6} + \frac{(1-y)^3}{6} = \frac{(1-y)^3}{6}.$$

The third integration is

$$\int_0^1 \frac{(1-y)^3}{6}\,dy = -\left.\frac{(1-y)^4}{24}\right|_0^1 = -\frac{0^4}{24} + \frac{1^4}{24} = \frac{1}{24}.$$

This completes the calculation that

$$\int_R z\,dV = \frac{1}{24}.$$

◇

Summary

We defined $\int_R f(P)\,dV$, where R is a region in space. The volume of a solid region R is $\int_R dV$ and, if $f(P)$ is the density of matter near P, then $\int_R f(P)\,dV$ is the total mass. We also showed how to evaluate these integrals by introducing rectangular coordinates.

There are six possible orders.

The general approach is to, first, describe R, for instance, as

$$a \leq x \leq b, \qquad y_1(x) \leq y \leq y_2(x), \qquad z_1(x, y) \leq z \leq z_2(x, y).$$

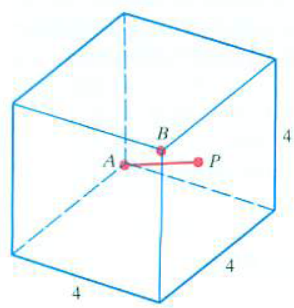
Then

$$\int_R f(P)\ dV = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\ dx \right)\ dy \right)\ dx.$$

EXERCISES for Section 17.4 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 4 concern the definition of $\int_R f(P)\,dV$.

1.[R] A cube of side 4 centimeters is made of a material of varying density. Near one corner A it is very light; at the opposite corner it is very dense. In fact, the density $f(P)$ (in grams per cubic centimeter) at any point P in the cube is the square of the distance from A to P (in centimeters). See Figure 17.4.5.



The density at P is the square of the distance $\overline{AP}$. P is a typical point in the cube.

Figure 17.4.5:

(a) Find upper and lower estimates for the mass of the cube by partitioning it into eight cubes.

(b) Using the same partition as in (a), estimate the mass of the cube, but select as the P_i's the centers of the four rectangular boxes.

(c) Estimate the mass of the cube described in the opening problem by cutting it into eight congruent cubes and using their centers as the P_i's.

(d) What does (c) say about the average density in the cube?

2.[R] How would you define the average distance from points of a certain set in space to a fixed point P_0?

3.[R] If R is a ball of radius r and $f(P) = 5$ for each point in R, compute $\int_R f(P)\,dV$ by examining approximating sums. Recall that the ball has volume $4/3\pi r^3$.

4.[R] If R is a three-dimensional set and $f(P)$ is never more than 8 for all P in R.

(a) what can we say about the maximum possible value of $\int_R f(P)\,dV$?

(b) what can we say about the average of f over R?

In Exercises 5 to 10 draw the solids described.

5.[R] $1 \le x \le 3$, $0 \le y \le 2$, $0 \le z \le x$

6.[R] $0 \le x \le 1$, $0 \le y \le 1$, $1 \le z \le 1 + x + y$

7.[R] $0 \le y \le 1$, $0 \le x \le y^2$, $y \le z \le 2y$

8.[R] $0 \le y \le 1$, $y^2 \le x \le y$, $0 \le z \le x + y$

9.[R] $-1 \le z \le 1$, $-\sqrt{1-z^2} \le x \le \sqrt{1-z^2}$, $-\frac{1}{2} \le y \le \sqrt{1-x^2-z^2}$

10.[R] $0 \le z \le 3$, $0 \le y \le \sqrt{9-z^2}$, $0 \le x \le \sqrt{9-y^2-z^2}$

In Exercises 11 to 14 evaluate the iterated integrals.

11.[R] $\int_0^1 \left(\int_0^2 \left(\int_0^x z\,dz\right)\,dy\right)\,dx$.

12.[R] $\int_0^1 \left(\int_{x^3}^{x^2} \left(\int_0^{x+y} z\,dz\right)\,dy\right)\,dx$.

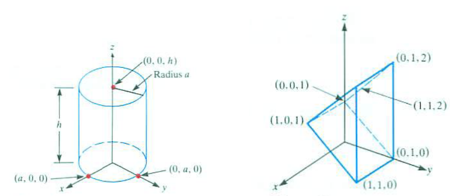
13.[R] $\int_2^3 \left(\int_x^{2x} \left(\int_0^1 (x+z)\,dz\right)\,dy\right)\,dx$.

14.[R] $\int_0^1 \left(\int_0^x \left(\int_0^3 (x^2+y^2)\,dz\right)\,dy\right.$

15.[R] Describe the solid cylinder of radius a and height h shown in Figure 17.4.6(a) in rectangular coordinates

(a) in the order first x, then y, then z,

(b) in the order first x, then z, then y.



(a) (b)

Figure 17.4.6:

16.[R] Describe the prism shown in Figure 17.4.6(b) in rectangular coordinates, in two ways:

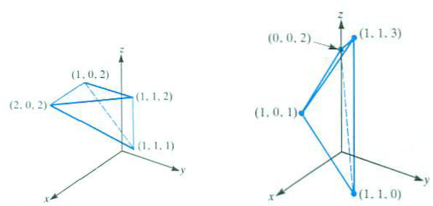
(a) First project it onto the xy plane.

(b) First project it onto the xz plane.

17.[R] Describe the tetrahedron shown in Figure 17.4.7(a) in rectangular coordinates in two ways:

(a) First project it onto the xy plane.

(b) First project it onto the xz plane.



(a) (b)

Figure 17.4.7:

18.[R] Describe the tetrahedron whose vertices are given in Figure 17.4.7(b) in rectangular coordinates as follows:

(a) Draw its shadow on the xy plane.

(b) Obtain equations of its top and bottom planes.

(c) Give a parametric description of the tetrahedron.

19.[R] Let R be the tetrahedron whose vertices are $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$, where a, b, and c are positive.

(a) Sketch the tetrahedron.

(b) Find the equation of its top surface.

(c) Compute $\int_R z\ dV$.

20.[R] Compute $\int_R z\ dV$, where R is the region above the rectangle whose vertices are $(0,0,0)$, $(2,0,0)$, $(2,3,0)$, and $(0,3,0)$ and below the plane $z = x+2y$.

21.[R] Find the mass of the cube in Exercise 1. (See Figure 17.4.1.)

22.[R] Find the average value of the square of the distance from a corner of a cube of side a to points in the cube.

23.[R] Find the average of the square of the distance from a point P in a cube of side a to the center of the cube.

24.[R] A solid consists of all points below the surface $z = xy$ that are above the triangle whose vertices are $(0,0,0)$, $(1,0,0)$, and $(0,2,0)$. If the density at (x,y,z) is $x+y$, find the total mass.

25.[R] Compute $\int_R xy\ dV$ for the tetrahedron of Example 3.

26.[R]

(a) Describe in rectangular coordinates the right circular cone of radius r and height h if its axis is on the positive z-axis and its vertex is at the origin. Draw the cross sections for fixed x and fixed x and y.

(b) Find the z coordinate of its centroid.

27.[R] The temperature at the point (x,y,z) is e^{-x-y-z}. Find the average temperature in the tetrahedron whose vertices are $(0,0,0)$, $(1,1,0)$, $(0,0,2)$, and $(1,0,0)$.

28.[R] The temperature at the point (x,y,z), $y>0$, is $e^{-x}/\sqrt{y}$. Find the average temperature in the region bounded by the cylinder $y = x^2$, the plane $y = 1$, and the plane $z = 2y$.

29.[R] Without using a repeated integral, evaluate $\int_R x \, dV$, where R is a spherical ball whose center is $(0, 0, 0)$ and whose radius is a.

30.[R] The work done in lifting a weight of w pounds a vertical distance of x feet is wx foot-pounds. Imagine that through geological activity a mountain is formed consisting of material originally at sea level. Let the density of the material near point P in the mountain be $g(P)$ pounds per cubic foot and the height of P be $h(P)$ feet. What definite integral represents the total work expended in forming the mountain? This type of problem is important in the geological theory of mountain formation.

31.[R] In Section 17.2 an intuitive argument was presented for the equality

$$\int_R f(P) \, dA = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x, y) \, dy \right) dx.$$

Here is an intuitive argument for the equality

$$\int_R f(P) \, dV = \int_{x_1}^{x_2} \left(\int_{y_1(x)}^{y_2(x)} \left(\int_{x_1(x,y)}^{x_2(x,y)} f(x, y, z) \, dz \right) dy \right) dx.$$

To start, interpret $f(P)$ as "density."

(a) Let $R(x)$ be the plane cross section consisting of all points in R with abscissa x. Show that the average density in $R(x)$ is

$$\frac{\int_{y_1(x)}^{y_2(x)} [\left(\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z) \, dz \right) dy}{\text{Area of } R(x)}$$

(b) Show that the mass of R between the plane sections $R(x)$ and $R(x + \Delta x)$ is approximately

$$\int_{y_1(x)}^{y_2(x)} \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z) \, dz \right) dy \, \Delta x.$$

(c) From (b) obtain a repeated integral in rectangular coordinates for $\int_R f(P) \, dV$.

17.5 Cylindrical and Spherical Coordinates

Rectangular coordinates provide convenient descriptions of solids bounded by planes. In this section we describe two other coordinate systems, cylindrical – ideal for describing circular cylinders — and spherical — ideal for describing spheres, balls, and cones. Both will be used in the next section to evaluate multiple integrals by iterated integrals.

CYLINDRICAL COORDINATES

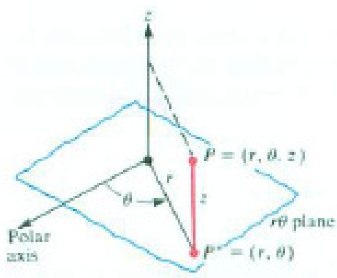
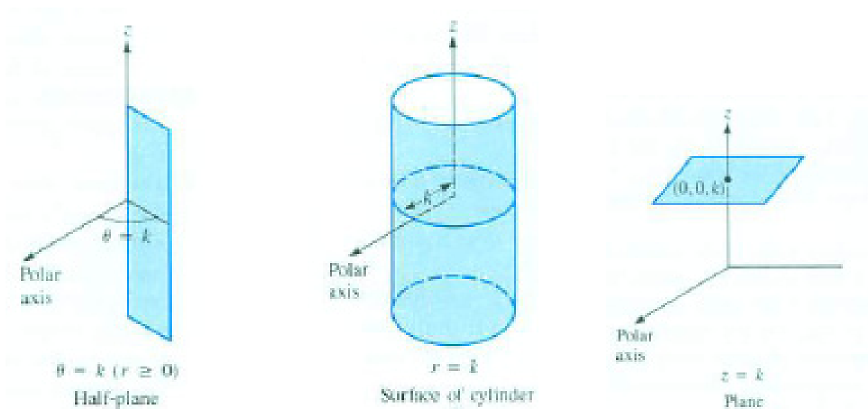


Figure 17.5.1:

Cylindrical coordinates combine polar coordinates in the plane with the z of rectangular coordinates in space. Each point P in space receives the name (r, θ, z) as in Figure 17.5.1. We are free to choose the direction of the polar axis; usually it will coincide with the x-axis of an (x, y, z) system. Note that (r, θ, z) is directly above (or below) $P^* = (r, \theta)$ in the $r\theta$ plane. Since the set of all points $P = (r, \theta, z)$ for which r is some constant is a circular cylinder, this coordinate system is convenient for describing such cylinders. Just as with polar coordinates, cylindrical coordinates of a point are not unique.



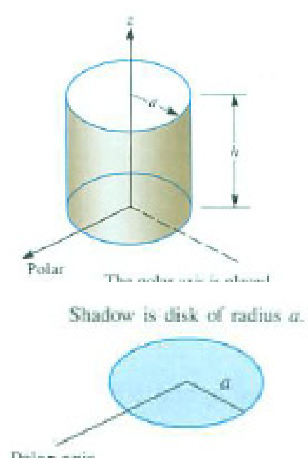
(a) (b) (c)

Figure 17.5.2:

Figure 17.5.2 shows the surfaces $\theta = k$, $r = k$, and $z = k$, where k is a positive number.

EXAMPLE 1 Describe a solid cylinder of radius a and height h in cylindrical coordinates. Assume that the axis of the cylinder is on the positive z-axis and the lower base has its center at the pole, as in Figure 17.5.3.

SOLUTION The shadow of the cylinder on the $r\theta$ plane is the disk of radius



a with center at the pole shown in Figure 17.5.4. Its description is

$$0 \leq \theta \leq 2\pi, \qquad 0 \leq r \leq a.$$

For each point (r, θ) in the shadow, the line through the point parallel to the z-axis intersects the cylinder in a line segment. On this segment z varies from 0 to h for every (r, θ). (See Figure 17.5.5.) Thus a description of the cylinder is

$$0 \leq \theta \leq 2\pi, \qquad 0 \leq r \leq a, \qquad 0 \leq z \leq h.$$

◇

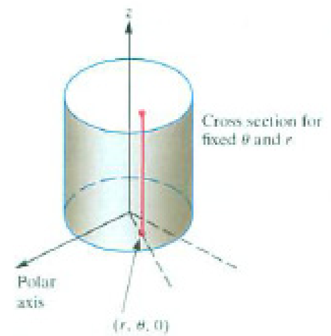


Figure 17.5.5:

EXAMPLE 2 Describe in cylindrical coordinates the region in space formed by the intersection of a solid cylinder of radius 3 with a ball of radius 5 whose center in on the axis of the cylinder. Place the cylindrical coordinate system as shown in Figure 17.5.6.

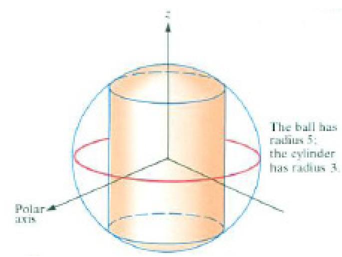


Figure 17.5.6:

SOLUTION Note that the point $P = (r, \theta, z)$ is a distance $\sqrt{r^2 + z^2}$ from the origin O, for, by the pythagorean theorem, $r^2 + z^2 = O\bar{P}^2$. (See Figure 17.5.7.) We will use this fact in a moment.

Now consider the description of the solid. First of all, θ varies from 0 to 2π and r from 0 to 3, bounds determined by the cylinder. For fixed θ and r, the cross section of the solid is a line segment determined by the sphere that bounds the ball, as shown in Figure 17.5.7(b). Now, since the sphere has radius 5, for any point (r, θ, z) on it,

$$r^2 + z^2 = 25 \qquad \text{or} \qquad z \pm \sqrt{25 - r^2}.$$

Thus, on the line segment determined by fixed r and θ, z varies from $-\sqrt{25 - r^2}$ to $\sqrt{25 - r^2}$.

The solid has this description:

$$0 \leq \theta \leq 2\pi, \qquad 0 \leq r \leq 3, \qquad -\sqrt{25 - r^2} \leq z \leq \sqrt{25 - r^2}.$$

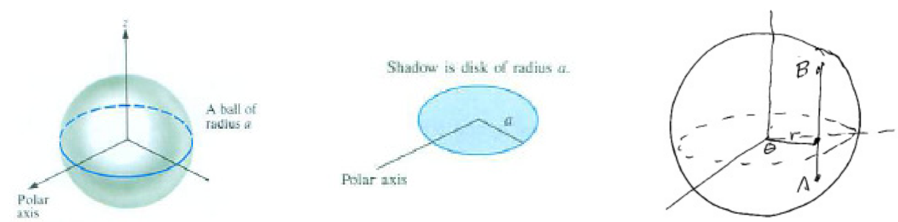
◇

EXAMPLE 3 Describe a ball of radius a in cylindrical coordinates.

SOLUTION Place the origin at the center of the ball, as in Figure 17.5.7(a). The shadow of the ball on the (r, θ) plane is a disk of radius a, shown in Figure 17.5.7(b) in perspective. This shadow is described by the equations

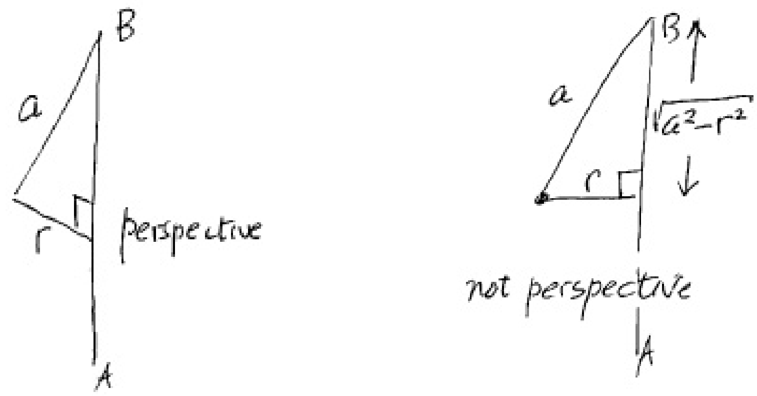
$$0 \leq \theta \leq 2\pi, \qquad 0 \leq r \leq a.$$

All that is left is to see how z varies for a given r and θ. In other words, how does z vary on the line AB in Figure 17.5.7(c)?



(a) (b) (c)

Figure 17.5.7:



(a) (b)

Figure 17.5.8:

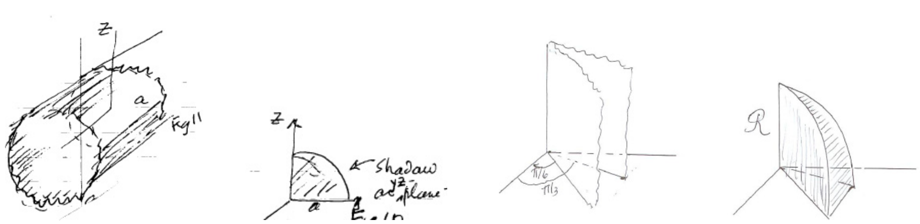
If r is a, then z "varies" from 0 to 0, as Figure 17.5.7(c) shows. If r is 0, then z varies from $-a$ to a. The bigger r is, the shorter AB is. Figure 17.5.8 presents the necessary geometry, first in perspective. With the aid of Figure 17.5.8, we see that z varies from $-\sqrt{a^2-r^2}$ to $\sqrt{a^2-r^2}$. You can check this by testing the easy cases, $r=0$ and $r=a$. All told,

$$\underbrace{0 \leq \theta \leq 2\pi, 0 \leq r \leq a}_{\text{The shadow}}, \underbrace{-\sqrt{a^2-r^2} \leq z \leq \sqrt{a^2-r^2}}_{\text{Range of } z \text{ for each } \theta \text{ and } r}$$

◇

EXAMPLE 4 Draw the region R bounded by the surfaces $r^2+z^2=a^2$, $\theta=\pi/6$, and $\theta=\pi/3$, situated in the first octant.

SOLUTION In the rz-plane, $r^2+z^2=a^2$ describes a circle of radius a, center at the origin. There is no restriction on θ. Thus it is a circular cylinder with its axis along the polar axis, as shown in Figure 17.5.9(a) in perspective. The shadow of R, which lies in the first octant, on the rz-plane is a quarter circle, shown in Figure 17.5.9(b).



(a) (b) (c) (d)

Figure 17.5.9:

Next we draw the half planes $\theta=\pi/6$ and $\theta=\pi/3$, as in Figure 17.5.9(c) showing at least the part in the first octant.

Finally we put Figure 17.5.9(a) and (c) together in (d), to show R.

R has three planar surfaces and one curved surface. The two curved edges are parts of ellipses, not parts of circles.

The description of R is

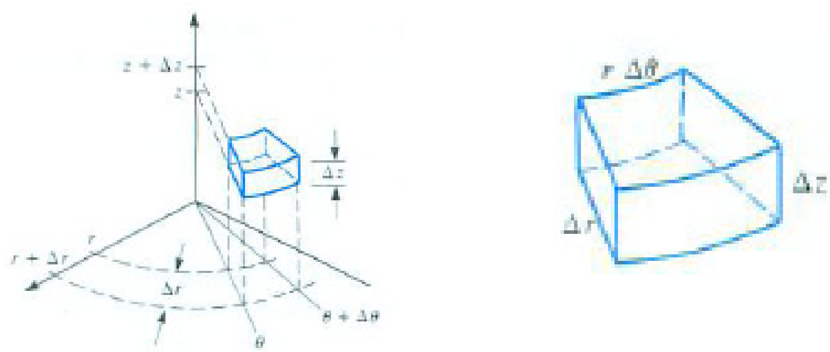
$$0 \leq r \leq a, \qquad 0 \leq z \leq \sqrt{a^2-r^2}, \qquad \pi/6 \leq \theta \leq \pi/3.$$

◇

Note that the shading and dashed hidden line help make the diagram clearer.

THE VOLUME SWEPT OUT BY Δr, Δ, and $\Delta\theta$

To use polar coordinates to evaluate an integral over a plane set we needed to know that the area of the little region corresponding to small changes Δr and $\Delta\theta$ is roughly $r\Delta r\Delta\theta$. In order to evaluate integrals over solids using an iterated integral in cylindrical coordinates, we will need to estimate the volume of the small region correspond to small changes Δr, $\Delta\theta$, Δz in the three coordinates.



(a) (b)

Figure 17.5.10:

The set of all points (r, θ, z) whose r coordinates are between r and $r + \Delta r$, whose θ coordinates are between θ and $\theta + \Delta\theta$, and whose z coordinates are between z and $z + \Delta z$ is shown in Figure 17.5.10(a). It is a solid with four flat surfaces and two curved surfaces.

When Δr is small, the area of the flat base of the solid is approximately $r\Delta r\Delta\theta$, as shown in Section 9.2 and as we saw when working with polar coordinates in the plane. Thus, when Δr, $\Delta\theta$, and Δz are small, the volume ΔV of the solid in Figure 17.5.10(b) is

$$\Delta V = (\text{Area of base})(\text{height}) \approx r\Delta r\Delta\theta\Delta z.$$

That is,

$$\boxed{\Delta V \approx r\Delta r\Delta\theta\Delta z.}$$

Just as the factor r appears in iterated integrals in polar coordinates, the same factor appears in iterated integrals in cylindrical coordinates.

SPHERICAL COORDINATES

The third standard coordinate system in space is **spherical coordinates**, which combines the θ of cylindrical coordinates with two other coordinates.

In spherical coordinates a point P is described by three numbers:

ρ is pronounced "row" or "roe"; it is the Greek letter for r. The letter ϕ is pronounced "fee" or "fie."

ρ the distance from P to the origin O, θ the same angle as in cylindrical coordinates, ϕ the angle between the positive z-axis and the ray from O to P.

In physics and engineering r is used instead of ρ.

The point P is denoted $P = (\rho, \theta, \phi)$. Note the order: first ρ, then θ, then ϕ. See Figure 17.5.11. Note that ϕ is the same as the direction angle of $\overline{OP}$ with k, $0 \le \phi \le \pi$. The surfaces $\rho = k$ (a sphere), $\phi = k$ (a cone), and $\theta = k$ (a half plane) are shown in Figure 17.5.12.

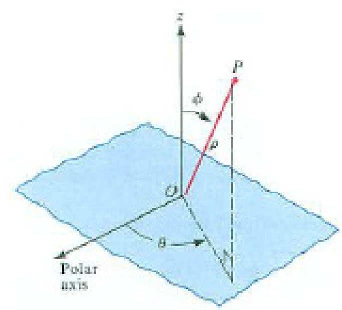
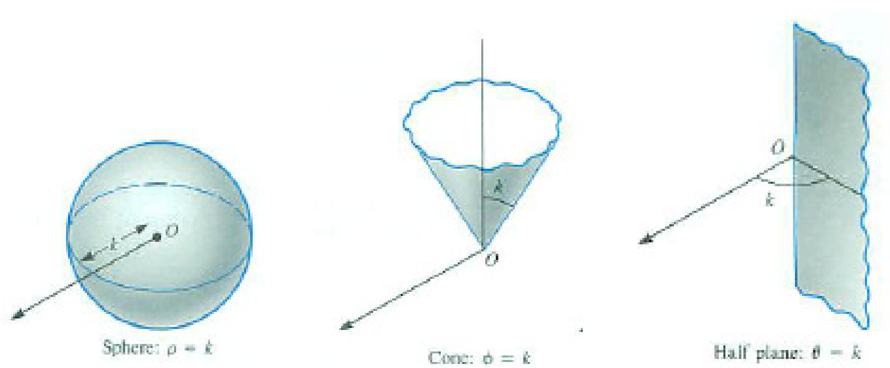


Figure 17.5.11:



(a) (b) (c)

Figure 17.5.12: (a) θ and ϕ vary, (b) ρ and θ vary, (c) ρ and ϕ vary.

When ϕ and θ are fixed and ρ varies, we describe a ray, as shown in Figure 17.5.13.

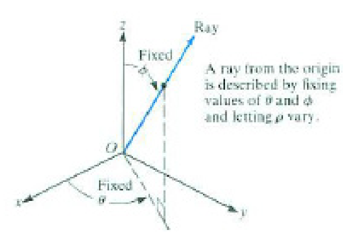
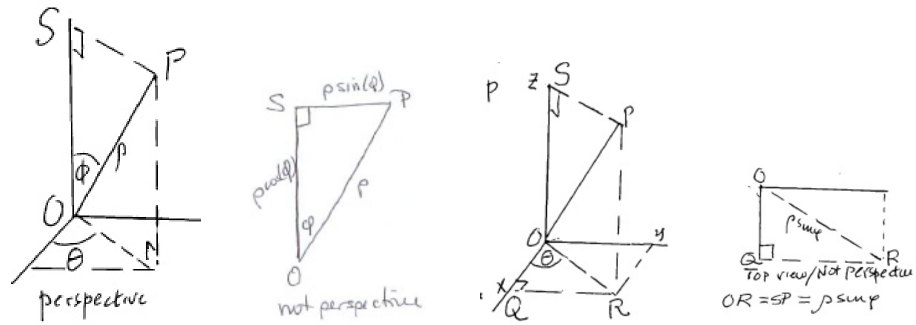


Figure 17.5.13:

RELATION TO RECTANGULAR COORDINATES

Figure 17.5.14 displays the relation between spherical and rectangular coordinates of a point $P = (\rho, \theta, \phi) = (x, y, z)$.

Note, in particular, right triangle OSP has hypotenuse OP and a right angle at S, and right triangle OQR has a right angle at Q.



(a) (b) (c) (d)

Figure 17.5.14:

First of all, $z = \rho\cos(\phi)$. Then $OR = \rho\sin(\phi)$. Finally $x = OR\cos(\theta) = \rho\sin(\phi)\cos(\theta)$ And $y = OR\sin(\theta) = \rho\sin(\phi)\sin(\theta)$.

EXAMPLE 5 Figure 17.5.15 shows a point given in spherical coordinates. Find its rectangular coordinates.

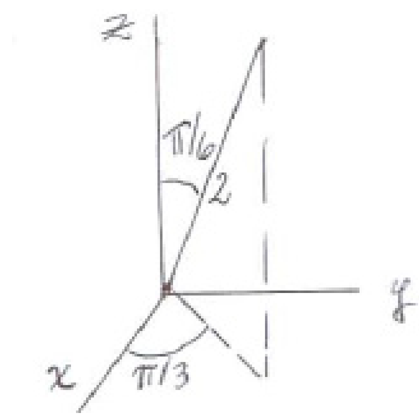


Figure 17.5.15:

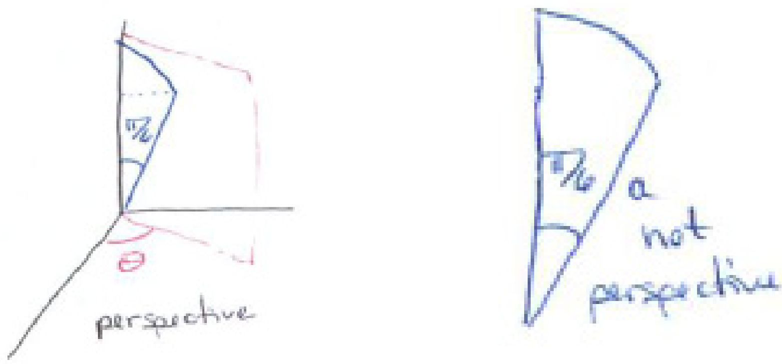
SOLUTION In this case, $\rho = 2$, $\theta = \pi/3$, $\phi = \pi/6$. Thus

$$\begin{aligned} x &= 2\sin(\pi/6)\cos\frac{\pi}{3} = 2\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2} \\ y &= 2\sin(\frac{\pi}{6})\sin(\frac{\pi}{3}) = 2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}2}{=}\frac{\sqrt{3}}{2} \\ z &= 2\cos(\frac{\pi}{6} = 2\frac{\sqrt{3}}{2} = \sqrt{3}. \end{aligned}$$

As a check, $x^2 + y^2 + z^2$ should equal ρ^2, and it does, for $(1/2)^2 + (\frac{\sqrt{3}}{2}) + (\sqrt{3})^2 = \frac{1}{4} + \frac{3}{4} + 3 = 4 = 2^2$. ◇

The next example exploits spherical coordinates to describe a cone and a ball.

EXAMPLE 6 The region R consists of the portion of a ball of radius a that lies within a cone of half angle $\pi/6$. The vertex of the cone is at the center of the ball.



(a) (b)

Figure 17.5.16:

SOLUTION R is shown in Figure 17.5.17. It resembles an ice cream cone, the dry cone topped with spherical ice cream.

Because R is a solid of revolution (around the z-axis), $0 \leq \theta \leq 2\pi$. The section of R corresponding to a fixed angle θ is the intersection of R with a half plane, shown in Figure 17.5.16.

In this sector of a disk, ϕ goes from 0 to $\pi/6$, independent of θ. Finally, a fixed θ and ϕ determine a ray on which ρ goes from 0 to a, as in Figure 17.5.18. ◇

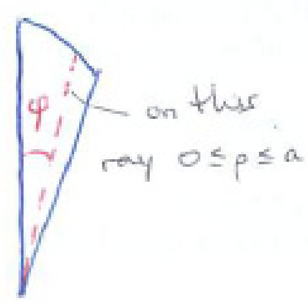


Figure 17.5.18:

The next example describes a ball in rectangular and spherical coordinates.

EXAMPLE 7 Describe a ball of radius a in rectangular and spherical coordinates.

SOLUTION In each case we put the origin of the coordinate system at the center of the ball.

Rectangular coordinates: The shadow of the ball on the xy-plane is a disk of radius a, described by

$$-a \leq x \leq a, \qquad -\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}.$$

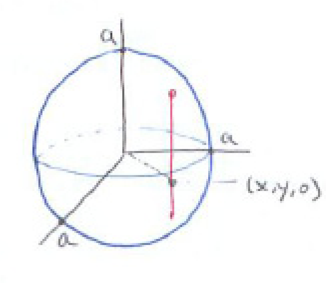


Figure 17.5.19:

For each point (x, y) in that projection, z varies along the line AB in Figure 17.5.19.

Since the equation of the sphere is $x^2+y^2+z^2 = a^2$ at A, z is $-\sqrt{z^2 - x^2 - y^2}$, and at B is $\sqrt{a^2 - x^2 - y^2}$. The entire description is

$$-a \le x \le a, \qquad -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}, \qquad -\sqrt{a^2 - x^2 - y^2} \le z \le \sqrt{z^2 - x^2}$$

Spherical coordinates: This time the shadow on the xy-plane plays no role. Instead, we begin with

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi,$$

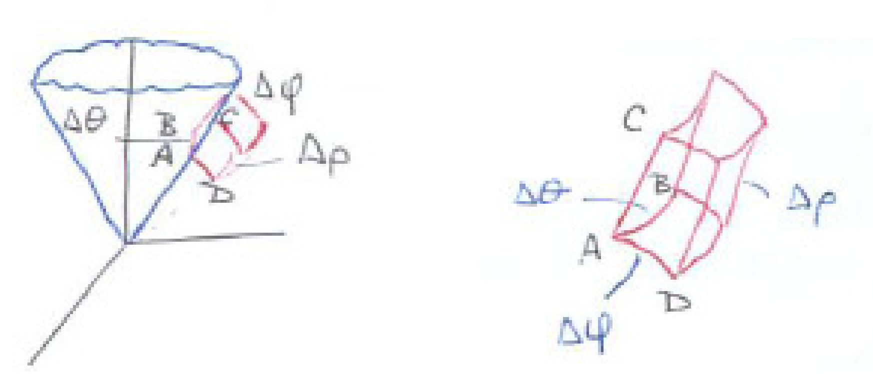
which sweeps out all the rays from the origin. On each such ray ρ goes from 0 to a. The complete description involves only constants as bounds:

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le \rho \le a.$$

Since the range of each variable is not influenced by other variables, the three restraints can be given in any order. ◇

THE VOLUME SWEPT OUT BY Δrho, Δphi, and $\Delta\theta$

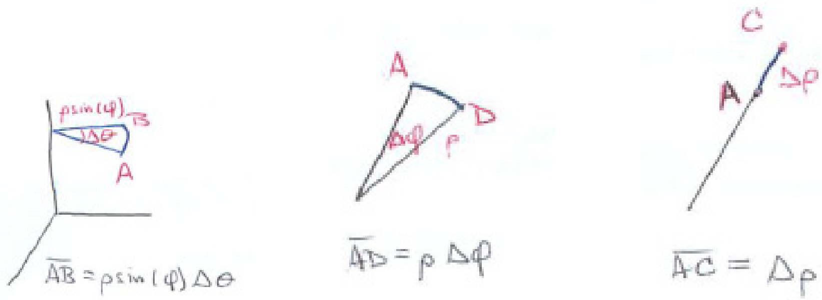
In the next section we will need an estimate of the volume of the little curvy "box-like" region bounded by spheres with radii ρ and $\rho + \Delta\rho$, the half-planes with angles θ and $\theta + \Delta\theta$, and the cones with half-angles ϕ and $\phi + \Delta\phi$. This region is shown in Figure 17.5.20. Two of its surfaces are flat, two are spherical, and two are patches on cones.



(a) (b)

Figure 17.5.20:

AB and AD are arcs of circles, while AC is straight



(a) (b) (c)

Figure 17.5.21:

The product of the length of AB, AC and AD is an estimate of the volume of the little box. Figure 17.5.21 shows how to find these lengths.

Therefore the volume of the small box is approximately $(\rho\sin(\phi)\Delta\theta)(\rho\Delta\phi)(\Delta\rho)$:

$$\boxed{\Delta V \approx \rho^2 \sin(\phi)\Delta\rho\Delta\phi\Delta\theta}$$

Just as we added an r to an integrand in polar coordinates, we must, in the next section, and the factor $\rho^2\sin(\phi)$ to an integrand when using an iterated integral in spherical coordinates.

Summary

This section described cylindrical and spherical coordinates. The volume of the small box corresponding to small changes in the three cylindrical coordinates is approximately $r\Delta r\Delta\theta\Delta z$. Because of the presence of the factor r, we must adjoin an r to the integrand when using an iterated integral in cylindrical coordinates.

Similarly, $\rho^2\sin(\phi)$ must be added to an integrand when using an iterated integral in spherical coordinates.

The next section illustrates the computations using these coordinates.

EXERCISES for Section 17.5 *Key:* R–routine, M–moderate, C–challenging

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Stewart)

1.[R] On the region in Example 2 draw the set of points described by (a) $z = 2$, (b) $z = 3$, (c) $z = 4.5$.

2.[R] For the cylinder in Example 1 draw the set of points described by (a) $r = a/2$, (b) $\theta = \pi/4$, (c) $z = h/3$.

3.[R]

(a) In the formula $\Delta V \approx r\Delta r\Delta\theta\Delta z$, which factors have the dimension of length?

(b) Why would you expect three such factors?

4.[R]

(a) In the formula $\Delta V \approx \rho^2\Delta\rho\Delta\theta\Delta\phi$, which factors have the dimension of length?

(b) Why would you expect three such factors?

5.[R] Drawing one clear, large diagram, show how to express rectangular coordinates in terms of cylindrical coordinates.

6.[R] Drawing one clear, large diagram, show how to express rectangular coordinates in terms of spherical coordinates.

7.[R] Find the cylindrical coordinates of $(x, y, z) = (3, 3, 1)$, including a clear diagram.

8.[R] Find the spherical coordinates of $(x, y, z) = (3, 3, 1)$,including a clear diagram.

In Exercises 9 to 11 (a) draw the set of points described, and (b) describe that set in words.

9.[R] ρ and ϕ fixed, θ varies.

10.[R] ρ and θ fixed, ϕ varies.

11.[R] θ and ϕ fixed, ρ varies.

12.[R] What is the equation of a sphere of radius a centered at the origin in

(a) spherical,

(b) cylindrical ,

(c) rectangular coordinates?

13.[R] Explain why if $P = (x, yz) = (\rho, \theta, \phi)$, in spherical coordinates, that $x^2 + y^2 + z^2 = \rho^2$. Hint: Draw a box.

14.[R] Describe the region in Example 6 in cylindrical coordinates in the order $\alpha \leq \theta \leq \beta$, $r_1(\theta) \leq r \leq r_2(\theta)$, $z_1(r, \theta) \leq z \leq z_2(r, \theta)$.

15.[R] Like Exercise 14, but in the order $a \leq z \leq b$, $\theta_1(z) \leq \theta \leq \theta_2(z)$, $r_1(\theta, z) \leq r \leq r_2(\theta, z)$.

16.[R] Sketch the region in the first octant bounded by the planes $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$ and the sphere $\rho = a$.

17.[R] Estimate the area of the bottom face of the curvy box shown in Figure 17.5.20. It lies on the sphere of radius ρ.

18.[A] cone of half-angle $\pi/6$ is cut by a plane perpendicular to its axis at a distance 4 from its vertex.

(a) Place it conveniently on a cylindrical coordinate system.

(b) Describe it in cylindrical coordinates.

19.[R] Like the preceding exercise, but use spherical coordinates.

20.[R] A cone has its vertex at the origin and its axis along the positive z-axis. It is made by revolving a line through the origin that has an angle A with the z-axis, about the z-axis. Describe it in

(a) spherical coordinates,

(b) cylindrical coordinates, and

(c) rectangular coordinates.

21.[R] Use spherical coordinates to describe the surface in Figure 17.5.22. It is part of a cone of half vertex angle B with the z-axis as its axis, situated within a sphere of radius a centered at the origin.

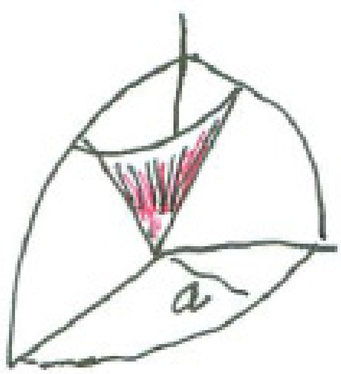


Figure 17.5.22:

22.[R] A triangle ABC is inscribed in a circle, with AB a diameter of the circle.

(a) Using elementary geometry, show that angle ACB is a right angle.

(b) Instead, using the equation of a circle in rectangular coordinates, show that AC and BC are perpendicular.

(c) Use (a) or (b) to show that the graph in the plane of $r = b\cos(\theta)$ is a circle of diameter b.

(d) In view of the preceding exercise, show that the equation of the circle in Figure 17.5.22 is $r = 2a\cos(\theta)$.

23.[R] (See Exercise 22.) A ball of radius a has a diameter coinciding with the interval $[0, 2a]$ on the x-axis. Describe the ball in spherical coordinates.

24.[R] The ray described in spherical coordinates by $\theta = \frac{\pi}{6}$ and $\phi = \frac{\pi}{4}$ makes an angle A with the x-axis.

(a) Draw a picture that shows the three angles.

(b) Find $\cos(A)$.

25.[R]

(a) If you describe the region in Example 2 in the order $0 \le \theta \le 2\pi$, $z_1(\theta) \le z \le z_2(\theta)$, $r_1(\theta, z) \le r \le r_2(\theta, z)$, what complication arises?

(b) Describe the region using the order given in (a).

By differentiating, verify the equations in Exercises 26 to 27.

26.[R] $\int |fracdx(x^3\sqrt{z^2+x^2}\frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right| - \frac{1}{2a}\arctan\frac{x}{a}$. $-\frac{\sqrt{a^2+x^2}}{2a^2+x^2} + \frac{1}{2a^3}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$.

27.[R] $\int \frac{x^2]dx}{a^4-x^4} =$

28.[R] What is the distance between $P_1 = (\rho_1, \theta_1, \phi_1)$ and $P_2 = (\rho_2, \theta_2, \phi_2)$?

29.[R] The points $P_1 = (\rho_1, \theta_1, \phi_1)$ and $P_2 = (\rho_1, \theta_2, \phi_2)$ both lie on a sphere of radius ρ_1. Assuming that both are in the first octant, find the great circle distance between them. NOTE: If the sphere is the earth's surface, ρ is approximately 3960 miles, ϕ is the complement of the latitude, and θ is related to longitude.

30.[R] At time t a particle moving along a curve is at the point $(\rho(t), \theta(t), \phi(t))$ What is its speed?

31.[R] How far apart are the points (r_1, θ_1, z_1) and (r_2, θ_2, z_2) in the first octant?

(a) Draw a large clear diagram.

(b) Find the distance.

32.[R] A bug is wandering on the
whose description is $0 \leq \theta \leq 2\pi, 0$
It is at the point $(3, 0, 2)$ and want
on the surface to $(3, \pi, 0)$. The bug
down, keeping $\theta = 0$, and then tal
on the base along a diameter. I
path? If not, what is?

17.6 Iterated integrals for $\int_R f(P)\ dV$ in Cylindrical or Spherical Coordinates

In Section 17.2 we evaluated an integral of the form $\int_R f(P)\ dA$ by an iterated integral in polar coordinates. In this method it is necessary to multiply the integrand by an "r." This is necessary because the small patch determined by small increments in r and θ is not $\Delta r \Delta\theta$ but $r\Delta r\Delta\theta$. Similarly, when developing iterated integrals using cylindrical coordinates, an extra r must be adjoined to the integrand. In the case of spherical coordinates one must adjoin $\rho^2\rho\sin(\phi)$. These adjustments are based on the estimates of the volumes of the small curvy boxes made in the previous section.

A few examples will illustrate the method, which is: Describe the solid R and the integrand in the most convenient coordinate system. Then use that description to set up an iterated integral, being sure to include the appropriate extra factor in the integrand.

ITERATED INTEGRALS IN CYLINDRICAL COORDINATES

To evaluate $\int_R f(P)\ dV$ in cylindrical coordinates we express the integrand in cylindrical coordinates and describe the region R in cylindrical coordinates. It must be kept in mind that dV is replaced by $r\ dz\ dr\ d\theta$. There are six possible orders of integration, but the most common one is: z varies first, then r, finally θ:

$$\boxed{\int_R f(P)\ dV = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} \left(\int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r,\theta,z) r\ dz\right)\ dr\ d\theta.}$$

Note, as with polar coordinates, the extra factor r.

EXAMPLE 1 Find the volume of a ball R of radius a using cylindrical coordinates.

SOLUTION Place the origin of a cylindrical coordinate system at the center of the ball, as in Figure 17.6.1.

The volume of the ball is $\int_R 1\ dV$. The description of R in cylindrical coordinates is

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le a, \qquad -\sqrt{a^2-r^2} \le z \le \sqrt{a^2-r^2}.$$

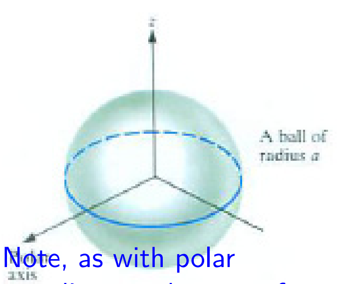


Figure 17.6.1:

The iterated integral for the volume is thus

$$\int_R 1\ dV = \int_0^{2\pi} \left(\int_0^a \left(\int_{-\sqrt{a^2-r^s}}^{\sqrt{a^2-r^2}} 1 \cdot r\ dz \right)\ dr \right)\ d\theta.$$

Evaluation of the first integral, where r and θ are fixed, yields

$$\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\ dz \quad = \quad rz|_{z=-\sqrt{z^2-r^2}}^{z=\sqrt{a^2-r^2}} = 2r\sqrt{a^2-r^2}.$$

Note that the order of integration is determined by the order of the variables in describing R.

Evaluation of the second integral, where θ is fixed, yields

$$\int_0^a 2r\sqrt{a^2-r^2}\ dr = \left.\frac{-2(a^2-r^2)^{3/2}}{3}\right|_{r=0}^{r=a} = \frac{2a^3}{3}.$$

Finally, evaluation of the third integral gives

$$\int_0^{2\pi} \frac{2a^3}{3}\ d\theta \quad = \quad \frac{2a^3}{3} \int_0^{2\pi}\ d\theta = \frac{2a^3}{3}\ cdot 2\pi = \frac{4}{3}\pi a^3.$$

◇

EXAMPLE 2 Find the volume of the region R inside the cylinder $x^2+y^2 = a$, above the xy-plane, and below the plane $z = x + 2y + 9$. Use cylindrical coordinates.

SOLUTION We wish to evaluate $\int_R 1\ dV$ over the region R described in cylindrical coordinates R by

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le 3, \qquad 0 \le z \le r\cos(\theta) + 2r\sin(\theta) + 9.$$

Note, as with polar coordinates, the extra factor r.

(Here we replace the equation $z = x + 2y + 9$ by $z = r\cos(\theta) + 2r\sin(\theta) + 9$.)

The iterated integral takes the form

$$\int_0^{2\pi} \left(\int_0^3 \left(\int_0^{r\cos(\theta)+2r\sin(\theta)+9} 1 \cdot r\ dz \right)\ dr \right)\ d\theta.$$

r and θ are constant

Integration with respect to z gives

$$\int_0^{r\cos(\theta)+2r\sin(\theta)+9} r\ dz \quad = \quad r \int_0^{r\cos(\theta)+2r\sin(\theta)+9} dz = r^2\ cos(\theta) + 2r^2\sin(\theta) + 9r.$$

Then comes integration with respect to r, with θ constant:

$$\int_0^3 \left(r^2\cos(\theta) + 2r^2\sin(\theta) + 9r\right)\ dr = \frac{r^3}{3}\cos(\theta) + \frac{2r^3}{3}\sin(\theta) + \left.\frac{9r^2}{2}\right|_0^3 = 9\cos(\theta) + 18\sin(\theta) + \frac{81}{2}.$$

Finally, integration with respect to θ gives

$$\int_0^{2\pi} \left(9\cos(\theta) + 18\sin(\theta) + \frac{81}{2}\right)\ d\theta. \tag{17.6.1}$$

Because $\int_0^{2\pi}\cos(\theta)\ d\theta = 0 = \int_0^{2\pi}\sin(\theta)\ d\theta$, (17.6.1) reduces to $\int_0^{2\pi}\frac{81}{2}\ d\theta = 81\pi$.

The volume is 81π. ◇

Computing $\int_R f(P)\ dV$ in Spherical Coordinates

To evaluate a triple integral $\int_R f(P)\ dV$ in spherical coordinates, first describe the region R in spherical coordinates. Usually this will be in the order:

$$\alpha \le \theta \le \beta, \qquad \phi_1(\theta) \le \phi \le \phi_2(\theta), \qquad \rho_1(\theta,\phi) \le \rho \le \rho_2(\rho,\theta).$$

Sometimes the order of ρ and ϕ is switched:

$$\alpha \le \theta \le \beta \qquad \rho_1(\theta) \le \rho \le \rho_2(\theta) \qquad \phi_1(\rho,\theta) \le \phi \le \phi_2(\rho,\theta).$$

Then set up an iterated integral, being sure to express dV as $\rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$ (or $\rho^2\sin(\phi)\ d\phi\ d\rho\ d\theta$).

EXAMPLE 3 Find the volume of a ball of radius a, using spherical coordinates.

SOLUTION Place the origin of spherical coordinates at the center of the ball, as in Figure 17.6.2. The ball is described by

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le \rho \le a.$$

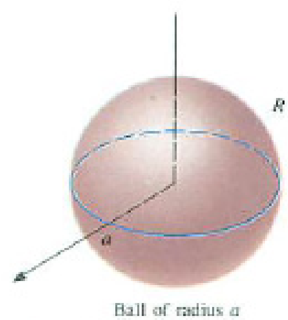


Figure 17.6.2:

Hence

$$\text{Volume of ball} = \int_R 1\ dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta.$$

The inner integral is

$$\int_0^a \rho^2\sin(\phi)\ d\rho = \sin\phi\int_0^a \rho^2\ d\rho = \frac{a^3\sin(\phi)}{3}.$$

The next integral is

$$\int_0^{\pi} \frac{a^3 \sin(\phi)}{3}\, d\phi \quad = \quad \left.\frac{-a^3 \sin(\phi)}{3}\right|_0^{\pi} = \frac{-a^3(-1)}{3} - \frac{-a^3(1)}{3} = \frac{2a^3}{3}.$$

The final integral is

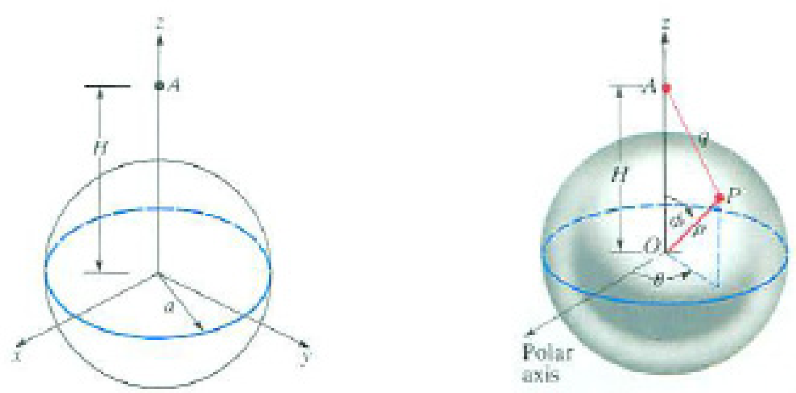
$$\int_0^{2\pi} \frac{2a^3}{3}\, d\theta \quad = \quad \frac{2a^3}{3}\int_0^{2\pi} d\theta = \frac{2a^3}{3}2\pi = \frac{4\pi a^3}{3}.$$

◇

An Integral in Gravity

The next example is of importance in the theory of gravitational attraction. It implies that a homogeneous ball attracts a particle (or satellite) as if all the mass of the ball were at its center.

EXAMPLE 4 Let A be a point at a distance H from the center of the ball, $H > a$. Compute $\int_R (\delta/q)\, dV$, where δ is density and q is the distance from a point P in R to A. (See Figure 17.6.3.)



(a) (b)

Figure 17.6.3:

SOLUTION First, express q in terms of spherical coordinates. To do so, choose a spherical coordinate system whose origin is at the center of the sphere and such that the ϕ coordinate of A is 0. (See Figure 17.6.3(b).)

Let $P = (\rho, \theta, \phi)$ be a typical point in the ball. Applying the law of cosines to triangle AOP, we find that

$$a^2 = H^2 + \rho^2 - 2\rho H \cos(\phi).$$

Hence

$$q = \sqrt{H^2 + \rho^2 - 2\rho H \cos(\phi)}.$$

Since the ball is homogeneous,

$$\delta = \frac{M}{\frac{4}{3}\pi a^3} = \frac{3M}{4\pi a^3}.$$

Hence

$$\int_R \frac{\delta}{q}\ dV = \int_R \frac{3M}{4\pi a^3 q}\ dV = \frac{3M}{4\pi a^3} \int_R \frac{1}{q}\ dV. \tag{17.6.2}$$

Now evaluate

$$\int_R \frac{1}{q}\ dV$$

by an iterated integral in spherical coordinates:

A case where integration with respect to ρ is not first

$$\int_R \frac{1}{q}\ dV = \int_0^{2\pi} \left(\int_0^a \left(\int_0^\pi \frac{\rho^2 \sin(\phi)}{\sqrt{H^2 + \rho^2 - 2\rho H \cos(\phi)}}\ d\phi \right)\ d\rho \right)\ d\theta.$$

We integrate with respect to ϕ first, rather than ρ, because it is easier in this case.

Evaluation of the first integral, where ρ and θ are constants, is accomplished with the aid of the fundamental theorem:

$$\begin{aligned} \int_0^\pi \frac{\rho^2 \sin\phi}{\sqrt{H^2 + \rho^2 - 2\rho H \cos(\phi)}}\ d\phi &= \left. \frac{\rho\sqrt{H^2 + \rho^2 - 2\rho H \cos(\phi)}}{H} \right|_{\phi=0}^{\phi=\pi} \\ &= \frac{\rho}{H}(\sqrt{H^2 + \rho^2 + 2\rho H} - \sqrt{H^2 + \rho^2 - 2\rho H}. \end{aligned}$$

Now, $\sqrt{H^2 + \rho^2 + 2\rho H} = H + \rho$. Since $\rho \leq a < H$, $H - \rho$ is positive and $\sqrt{H^2 + \rho^2 - 2\rho H} = H - \rho$.

Thus the first integral equals

$$\frac{\rho}{H}[H + \rho) - (H - \rho)] = \frac{2\rho^2}{H}.$$

Evaluation of the second integral yields

$$\int_0^a \frac{2\rho^2}{H}\ d\rho = \frac{2a^3}{3H}.$$

Evaluation of the third integral yields

$$\int_0^{2\pi} \frac{2a^3}{eH}\, d\theta = \frac{4\pi a^3}{3H}.$$

Hence

$$\int_R \frac{1}{q}\, dV = \frac{4\pi a^3}{3H}.$$

By (17.6.2)

$$\int_R \frac{\delta}{q}\, dV = \frac{3M}{4\pi a^3}\frac{4\pi a^3}{3H} = \frac{M}{H}.$$

Newton obtained this remarkable result in 1687.

This result, M/H, is exactly what we would get if all the mass were located at the center of the ball. ◇

SHERMAN: Compare this with your version. What is your M? I thought it was the object, i.e., the region together with its density, but you are using it also as the mass of the object. If you want the latter, you can't talk about "a mass M occupies a region R." I prefer to say we have an object that occupies a region R, with density δ, and mass M where $M = \int_R \delta(P)\, dV$. See what I have written. This may need to be changed elsewhere. That should not be difficult to do, but I want to see your comments first.

THE MOMENT OF INERTIA ABOUT A LINE

In the study of rotation of a object about an axis, one encounters the "moment of inertia", I of the object. It is defined as follows. The object occupies a region R. The density of the object at a typical point P is $\delta(P)$, so the mass of the object is $M = \int_R \delta(P)\, dV$. Usually the density is constant, in which case it is M divided by the volume of R (or M divided by the area of R if R is planar). Let $r(P)$ be the distance from P to a fixed line L. Then, by definition,

An object with constant density is called homogeneous.

$$I = \text{Moment of Inertia} = \int_R (r(P))^2 \delta(P)\, dV.$$

A similar definition holds for objects distributed on a planar region. The only difference is that dV is replaced by dA.

Exercise 31 shows that I plays the same role in a rotating body (such as a spinning skater) as mass does in an object moving

EXAMPLE 5 Compute the moment of inertia of a uniform mass M in the form of a ball of radius a around a diameter L.

SOLUTION The density $\delta(P)$, being constant, is $M/(\frac{4}{3}\pi a^3)$. We place the diameter L along the z-axis, as in Figure 17.6.4

As a check on our answer

Because the distance $r(P)$ is just r in cylindrical coordinates, we will first work in those coordinates. Then we will calculate the moment of inertia in spherical coordinates.

One description of the ball is

$$0 \le \theta \le 2\pi, \qquad -a \le z \le a, \qquad 0 \le r \le \sqrt{a^2 - z^2}.$$

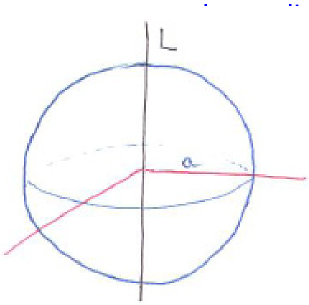


Figure 17.6.4:

Then

$$\begin{aligned} I &= \int_R \frac{M}{\frac{4}{3}\pi a^3} r^2\ dV = \frac{3M}{4\pi a^3}\int_R r^2\ dV \qquad \text{Note the introduction of the extra } r \\ &= \frac{3M}{4\pi a^3}\int_0^{2\pi}\int_{-a}^{a}\int_0^{\sqrt{a^2-z^2}} r^3\ dr\ dz\ d\theta \end{aligned}$$

The first integration is

$$\int\limits_0^{\sqrt{a^2-z^2}} r^3\ dr = \frac{r^4}{4}\bigg|_0^{\sqrt{a^2-z^2}} = \frac{(a^2-z^2)^2}{4}.$$

The second is

$$\begin{aligned} \int\limits_{-a}^{a} \frac{(a^2-z^2)^2}{4}\ dz &= \int\limits_{-a}^{a} \frac{a^4-2a^2z^2+z^4}{4}\ dz = \frac{1}{4}\left(a^4z - \frac{2a^2z^3}{3} + \frac{z^5}{5}\right)|_{-a}^{a} \\ &= \frac{1}{4}\left(a^5 - \frac{2a^5}{3} + \frac{a^5}{5}\right) - \frac{1}{4}\left(-a^5 + \frac{2a^5}{3} - \frac{a^5}{5}\right) = \frac{4}{15}a^5. \end{aligned}$$

The third is

$$\int\limits_0^{2\pi} \frac{4}{15}a^5\ d\theta = \frac{8\pi}{15}a^5.$$

Then remembering to include the factor $3M/4\pi a^3$, we have

$$I = \frac{3M}{4\pi a^3}\cdot\frac{8\pi}{15}a^5 = \frac{2}{5}Ma^2.$$

Because spherical coordinates provide a simple description of the ball, we will also use them to find I to see if the computations are easier. Now the distance $r(P)$ has a more complicated form, $\delta(P) = \delta(\rho,\theta,\phi) = \rho sin(\phi)$. The integral for the moment of inertia is

$$I = \frac{3M}{4\pi a^3}\int\limits_R (\rho\sin(\phi)^2\ dV.$$

The iterated integral for this multiple integral is

$$\int\limits_0^{2]pi}\left(\int\limits_0^{\pi}\left(\int\limits_0^{a} (\rho\sin(\phi))^2\rho^2\sin(\phi)\ d\rho\right)\ d\phi\right)\ d\theta.$$

The first integration is

$$\int\limits_0^{a} \rho^4\sin^3(\phi)\ d\rho = \frac{\rho^5}{5}\sin^3(\phi)\bigg|_{\rho=0}^{\rho=a} = \frac{a^5}{5\sin^3(\phi)}.$$

The second is

$$\int_0^\pi \frac{a^5}{5}\sin^3(\phi)\ d\phi = \frac{a^5}{5}\int_0^\pi \sin^3(\phi)\ d\phi.$$

Since the exponent, 3, is odd, we write $sin^3(\phi)$ as $(1-\cos^2(\phi))\sin(\phi)$ and have

$$\begin{aligned}\int_0^\pi \sin^3(\phi)\ d\phi &= \int_0^\pi (\sin(\phi)-\cos^2(\phi)\sin(\phi)) = (-\cos(\phi)+\frac{\cos^3(\phi)}{3}\bigg|_0^\pi \\ &= (-(-1)+\frac{(-1)^3}{3})-(-1+\frac{1}{3}) = \frac{4}{3}.\end{aligned}$$

The final integration is just

$$\int_0^{2\pi} \frac{a^5}{5}\cdot\frac{4}{3}\ d\theta = \frac{8}{15}\pi.$$

Note that this is 2/5 of our upper estimate, hence is plausible.

And, as expected, gives, again

$$I = (2/5)Ma^2.$$

◇

Summary

A multiple integral $\int_R f(P)\ dV$ may be evaluated by an iterated integral in cylindrical or spherical coordinates. In cylindrical coordinates the iterated integral takes the form

$$\int_{\theta_1}^{\theta_2}\left(\int_{r_1(\theta)}^{r_2(\theta)}\left(\int_{z_1(r,\theta)}^{z_2(r,\theta)} rf(r,\theta,z)\ dz\right)\ dr\right)\ d\theta.$$

The description of the region determines the range of integration on each of the three integrals over intervals. (Changing the order of the description of R changes the order of the integrations.) The factor r must be inserted into the integrand.

In spherical coordinates the iterated integral usually takes the form

$$\int_{\theta_1}^{\theta_2}\left(\int_{\phi_1(\theta)}^{\phi_2(\theta)}\left(\int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(r,\theta,\phi)\rho^2\sin(\phi)\ d\phi\right)\ d\phi\right)\ d\theta.$$

In this form, integration with respect to ρ is first, but as Example 4 illustrates, it may be convenient to integrate first with respect to ϕ. The factor $\rho^2\sin(\phi)$ must be inserted in the integrand.

EXERCISES for Section 17.6 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4: (a) draw the region, (b) set up an iterated integral in cylindrical coordinates for the given multiple integrals, and (c) evaluate the iterated integral.

1.[R] $\int_R r^2\ dV$, R is bounded by the cylinder $r = 3$ and the planes $z = 2x$ and $z = 3x$.

2.[R] $\int_R z\ dV$, R is bounded by the sphere $z^2 + r^2 = 25$, the $r\theta$ coordinate plane, and the plane $z = 2$.

3.[R] $\int_R rz\ dV$, R is the part of the ball bounded by $r^2 + z^2 = 16$ in the first octant.

4.[R] $\int_R \cos\theta/dV$, R is bounded by the cylinder $r = 2\cos(\theta)$ and the paraboloid $z = r^2$.

5.[R] Compute the volume of a right circular cone of height h and radius r using (a) spherical coordinates, (b) cylindrical coordinates, and (c) using rectangular coordinates.

6.[R] Find the volume of the region above the xy plane and below the paraboloid $z = 9 - r^2$ using cylindrical coordinates.

7.[R] A right circular cone of radius a and height h has a density at point P equal to the distance from P to the base of the cone. Find its mass, using spherical coordinates.

In Exercises 8 to 9 draw the region R and give a formula for the integrand $f(P)$ such that $\int_R\ dV$ is described by the given iterated integrals.

8.[R] $\int_0^{\pi/2}[\int_0^{\pi/4}(\int_0^{\cos\phi} \rho^3 \sin^2(\theta)\sin(\phi)\ d\rho)\ d\phi]\ d\theta.$

9.[R] $\int_0^{\pi/4}[\int_{\pi/6}^{\pi/2}(\int_0^{\sec\theta} \rho^3 \sin(\theta)\cos(\phi)\ d\rho)\ d\phi]\ d\theta.$

10.[R] Let R be the solid region inside both the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$. Let the density at (x, y, z) be $f(x, y, z) = z$. Set up iterated integrals for the mass in R using (a) rectangular coordinates, (b) cylindrical coordinates, (c) spherical coordinates. (d) Evaluate the iterated integral in (c).

11.[R] Find the average temperature in a ball of radius a if the temperature is the square of the distance from a fixed equatorial plane.

In each of Exercises 12 to 13 evaluate the iterated integral.

12.[R] $\int_0^{2\pi}\left(\int_0^1\left(\int_r^1 zr^3\cos^2\theta\ dz\right)\ dr\right)\ d\theta$

13.[R] $\int_0^{2\pi}\left(\int_0^1\left(\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} z^2r\ dz\right)\ d\right.$

14.[R] Let R be the solid region inside both the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$. Let the density at (x, y, z) be $f(x, y, z) = z$. Using cylindrical coordinates, find the mass of R.

15.[R] Using cylindrical coordinates, find the volume of the region below the plane $z = y + 1$ and above the circle in the xy plane whose center is $(0, 1, 0)$ and whose radius is 1. (Include a drawing of the region.) HINT: What is the equation of the circle in polar coordinates when the polar axis is along the positive x-axis?

16.[R] Find the average distance from the center of a ball of radius a to other points of the ball by setting up appropriate iterated integrals in the three types of coordinate systems and evaluating the easiest.

17.[R] A solid consists of that part of a ball of radius a that lies within a cone of half-vertex angle $\phi = \pi/6$, the vertex being at the center of the ball. Set up iterated integrals for $\int_R z\ dV$ in all three coordinate systems and evaluate the simplest.

In Exercises 18 to 23 evaluate the multiple integrals over a ball of radius a with center at the origin, without using an iterated integral (ϕ,θ, and z are cylindrical or spherical coordinates).

18.[R] $\int_R \cos(\theta)\ dV$

19.[R] $\int_R \cos^2\theta\ dV$

20.[R] $\int_R z\ dV$

21.[R] $\int_R (3+2\sin(\theta)\ dV$

22.[R] $\int_R \sin^2(\phi)\ dV$

23.[R] $\int_R \sin(\phi)\ dV$

24.[R] In polar, cylindrical, and spherical coordinates one must introduce an extra factor in the integrand when using an iterated integral. Why is that not necessary when using rectangular coordinates?

25.[R] Is $\sqrt{a^2}$ always equal to a?

26.[R] Using the method of Example 4 find the average value of q for all points P in the ball. Note that it is *not* the same as if the entire ball were placed at its center.

27.[C] Show that the result of Example 4 holds if the density $\delta(P)$ depends only on ρ, the distance to the center. (This is approximately the case with the planet Earth, which is not homogeneous.) Let $g(\rho)$ denote $\delta(\rho,\theta,\phi)$.

In Exercises 28 to 29check the equations by differentiation.

28.[R] $\tan\left(\frac{x}{2}\right) = 2\ln\left|\cos\left(\frac{x}{2}\right)\right| = \int \frac{x\ dx}{1+\cos(x)}$ $\int \frac{dx}{1+\cos(x)}$

29.[R] $x\tan\left(\frac{x}{2}\right) +$

30.[R]

(a) Find the exact volume of the little curvy box corresponding to the changes $\Delta\rho$, $\Delta\theta$, $\Delta\phi$.

(b) One hopes that the ratio between that exact volume and our estimate, $\rho^2\sin(\phi)\Delta\rho\Delta\theta\Delta\rho$ approaches 1 as $\Delta\rho$, $\Delta\theta$, $\Delta\phi$ approach 0. Show that it does. HINT: Recall the definition of a derivative.

(c) Show that the exact volume in (a) can be written in the form $(\rho^* t)^2\sin(\phi^*)\Delta\rho\Delta\phi\Delta\theta$, where ρ^* is between ρ and $\rho+\Delta\rho$ and ϕ^* is between ϕ and $\phi+\Delta\phi$.

31.[R] The kinetic energy of an object with mass m moving at the velocity v is $mv^2/2$. An object moving in a circle of radius r at the angular speed ω radians per unit time has velocity $r\omega$. (Why?) Thus its kinetic energy is $(mr^2/2)\omega^2$. Now consider a mass M that occupies the region R in space. Its density is $\Delta(P)$, which may vary from point to point. (If it is constant, it equals $M/(\text{Volume of R})$.) Let $f(P)$ be the distance from P to a fixed line L. If the mass is spinning around the axis L at the angular rate ω, show that its total kinetic energy is

$$\int_R \frac{1}{2}(f(P))^2\delta(P)w^2\ dv.$$

This can be written as

$$\text{Kinetic Energy} = (\frac{1}{2}I\omega^2.$$

Thus I plays the same role in rotational motion that mass m plays in linear motion in the formula $(\frac{1}{2}mv^2$.

Every spinning ice skater knows this. When spinning with her arms extended she has a certain amount of kinetic energy. If she suddenly puts her arms to her sides she decreases her moment of inertia but has not destroyed her kinetic energy. That forces her angular speed to increase. The larger the mass m is, the harder it is to start it moving and to stop it when it is moving. Similarly, the larger I is, the harder it is to stop the mass from spinning and to stop it when it is spinning.

In Exercises 32 to 36 the objects have a homogeneous (constant density) mass M. Find I.

32.[R] A rectangular box of dimensions, a, b, c around a line through its center and perpendicular to the face of dimensions a and b.

33.[R] A solid cylinder of radius a and height h around its axis.

34.[R] A solid cylinder of radius a and height h around a line on its surface.

35.[R] A hollow cylinder of height h, inner radius a, and outer radius b, about its axis.

36.[R] A solid cylinder of radius a and height h around a diameter in its base.

37.[R] In Example 2 what unpleasantness occurs when you describe the region in the order of the form $a \leq \theta \leq b$, $z_1(\theta) \leq z \leq z_2(\theta)$, $r_1(\theta, z) \leq r \leq r_2(\theta, z)$?

38.[R] Solve Example 2 using rectangular coordinates.

39.[R] Evaluate the moment of inertia in Example 5 using the description $0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$, $-\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2}$.

40.[R] Let R be a solid ball of radius a with center at the origin of the coordinate system

(a) Explain why $\int_R x^2\ dV = \frac{1}{3} \int_R (x^2 + y^2 + z^2)\ dV$.

(b) Evaluate the second integral by spherical coordinates.

(c) Use (b) to find $\int_R x^2\ dV$.

41.[M] Show that $\int_R (x^3 + y^3 + z^3)\ dV = 0$, where R is a ball whose center is the origin of a rectangular coordinate system. NOTE: Do not use an iterated integral. HINT: Use symmetry.

42.[R] A homogeneous object with mass M occupies the region R between concentric spheres of radii a and b, $a < b$. Let A be a point at a distance H from their center, $H < a$. Evaluate $\int_R \frac{\delta}{q}\ dV$, where δ is the density and $q = q(P)$ is the distance from H to any point P in R. (That the value of the integral does not involve H has an important consequence: A uniform hollow sphere exerts no gravitational force on objects in its interior.)

43.[R] In Example 4, H is greater than a. Solve the same problem for H less than a. NOTE: For some ρ, $\sqrt{H^2 + \rho^2 A - 2\rho H}$ equals $H - \rho$ and for some it equals $\rho - H$.

44.[C] (See Example 43.) Let A be a point in the plane of a disk but outside the disk. Is the average of the reciprocal of the distance from A to points in the disk equal to the reciprocal of the distance to the center of the disk?

45.[C] A certain ball of radius a is *not* homogeneous. However, its density at P depends only on the distance from P to the center of the ball. That is, there is a function $f(\rho)$ such that the density at $P = (\rho, \theta, \phi)$ is $f(\rho)$. Using an iterated integral, show that the mass of the ball is

$$4\pi \int_0^a f(\rho^2_\rho\ d\rho.$$

46.[C] Let R be the part of a ball of radius a removed by a cylindrical drill of diameter a whose edge passes through the center of the sphere.

(a) Sketch R.

(b) Notice that R consists of four congruent pieces. Find the volume of one of these pieces using cylindrical coordinates. Multiply by four to get the volume of R.

47.[C] Let R be the ball of radius a. For any point P in the ball other than the center of the ball, define $f(P)$ to be the reciprocal of the distance from P to the origin. The average value of r over R involves an improper integral, since the function blows up near the origin. Does this improper integral converge or diverge? What is the average value of f over R? *Suggestion:* Examine the integral over the region between concentric spheres of radii a and t, and let $t \to 0^+$.

17.7 Integrals Over Surfaces

In this section we define an integral over a surface and then show how to compute it by an iterated integral.

Definition of a Surface Integral

Consider a surface $\mathcal{S}$ such as the surface of a ball or part of the saddle $z = xy$. If f is a numerical function defined at least on $\mathcal{S}$, we will define the integral $\int_{\mathcal{S}} f(P)\ dS$. The definition is practically identical with the definition of the double integral, which is the special case when the surface is a plane.

We assume that the surfaces we deal with are smooth, or composed of a finite number of smooth pieces, and that the integrals we define exist.

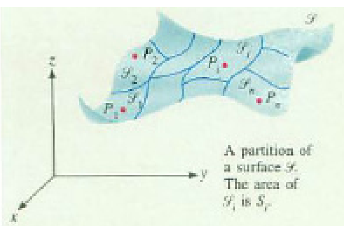


Figure 17.7.1:

DEFINITION *(Definite integral of a function f over a surface $\mathcal{S}$.)* Let f be a function that assigns to each point P in a surface $\mathcal{S}$ a number $f(P)$. Consider the typical sum

$$f(P_1)S_1 + f(P_2)S_2 + \cdots + f(P_n)S_n,$$

formed from a partition of $\mathcal{S}$, where S_i is the area of the ith region in the partition and P_i is a point in the ith region. (See Figure 17.7.1.) If these sums approach a certain number as the S_i are chosen smaller and smaller, the number is called the **integral of f over $\mathcal{S}$** and is written

$$\int_{\mathcal{S}} f(P)\ dS.$$

Surface integrals are also denoted $\iint_{\mathcal{S}} f(P)\ dS$.

If $f(P)$ is 1 for each point P in $\mathcal{S}$ then $\int_{\mathcal{S}} f(P)\ dS$ is the area of $\mathcal{S}$. If $\mathcal{S}$ is occupied by material of density $\sigma(P)$ at P then $\int_{\mathcal{S}} \sigma(P)\ dS$ is the total mass of $\mathcal{S}$.

First we show how to integrate over a sphere.

Integrating over a Sphere

If $\mathcal{S}$ is a sphere or part of a sphere, it is often convenient to evaluate an integral over it with the aid of spherical coordinates.

See Section 17.6 for a similar argument, where ρ was not constant.

If the center of a spherical coordinate system (ρ, θ, ϕ) is at the center of a sphere of radius a, then ρ is constant on the sphere $\rho = a$. As Figure 17.7.2 suggests, the area of the small region on the sphere corresponding to slight changes $d\theta$ and $d\phi$ is approximately

$$(a\ d\phi)\,(a \sin(\phi)\ d\theta) = a^2 \sin(\phi)\ d\theta\ d\phi.$$

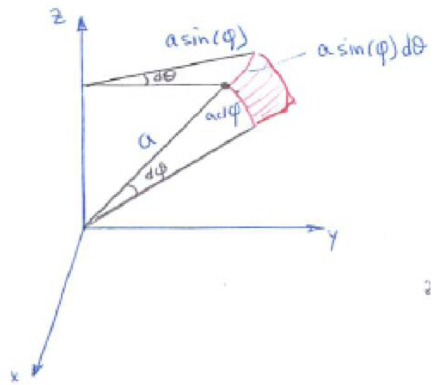


Figure 17.7.2:

Thus we may write

$$dS = a^2 \sin(\phi)\ d\theta\ d\phi$$

and evaluate

$$\int\limits_{\mathcal{S}} f(P)\ dS$$

in terms of a repeated integral in ϕ and θ. Example 1 illustrates this technique.

EXAMPLE 1 Let $\mathcal{S}$ be the top half of the sphere with radius a. Evaluate $\int_{\mathcal{S}} z\ dS$.

SOLUTION Since the sphere has radius a, $\rho = a$. The top half of the sphere is described by $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/2$. And, in spherical coordinates, $z = \rho\cos(\phi) = a\cos(\phi)$. Thus

$$\int\limits_{\mathcal{S}} z\ dS = \int\limits_{\mathcal{S}} (a\cos(\phi))\ dS = \int\limits_0^{2\pi} \left(\int\limits_0^{\pi/2} (a\cos(\phi))a^2\sin(\phi)\ d\phi \right)\ d\theta.$$

Now,

$$\begin{aligned}
\int\limits_0^{\pi/2} (a\cos(\phi))a^2\sin(\phi)\ d\phi &= a^3 \int\limits_0^{\pi/2} \cos(\phi)\sin(\phi)\ d\phi = a^3 \frac{(-\cos^2(\phi))}{2}\Bigg|_0^{\pi/2} \\
&= \frac{a^3}{2}[-0-(-1)] = \frac{a^3}{2}.
\end{aligned}$$

so that

$$\int_{\mathcal{S}} z \, dS = \int_0^{2\pi} \frac{a^3}{2} \, d\theta = \pi a^3.$$

◇

We can interpret the result in Example 1 in terms of average value. The **average value** of $f(P)$ over a surface $\mathcal{S}$ is defined as

$$\frac{\int_{\mathcal{S}} f(P) \, dS}{\text{Area of } \mathcal{S}}.$$

Geometric interpretation

Example 1 shows that the average value of z over the given hemisphere is

$$\frac{\int_{\mathcal{S}} z \, dS}{\text{Area of } \mathcal{S}} = \frac{\pi a^3}{2\pi a^2} = \frac{a}{2}.$$

"The average height above the equator is exactly half the radius."

A General Technique

When we faced an integral over a curve, $\int_C f \, ds$, we evaluated it by replacing it with $\int_a^b f \, \frac{ds}{dt} \, dt$, an integral over an interval $[a, b]$.

We will do something similar for an integral over a surface: We will replace an surface integral by a double integral over a set in a coordinate plane.

The basic idea is to replace a small patch on the surface $\mathcal{S}$ by its projection (shadow) or, say, the xy-coordinate plane. The area of the shadow is not the same as the area of the patch. With the aid of Figure 17.7.3 we will express the area of the shadow in terms of the tilt of the patch.

The unit normal vector to the patch is $\mathbf{n}$. The angle between $\mathbf{n}$ and $\mathbf{k}$ is γ. Call the area of the patch, dS, and the area of its projection, dA. Then

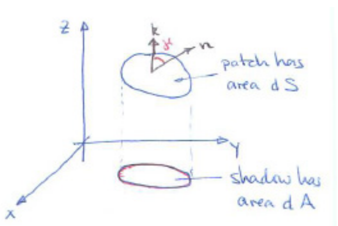


Figure 17.7.3:

$$\boxed{dA \approx |\cos(\gamma)| \, dS.}$$

Recall the discussion of direction angles and direction cosines in Section 14.4.

Notice that the angle γ is one of the direction angles of the unit normal vector, $\mathbf{k}$.

For instance, if $\gamma = 0$, then $dA = dS$. If $\gamma = \pi/2$, then $dA = 0$. We use the absolute value of $\cos(\gamma)$, since γ could be larger than $\pi/2$.

It follows, if $\cos(\gamma)$ is not 0, that

$$dS = \frac{dA}{|\cos(\gamma)|} \tag{17.7.1}$$

With the aid of (17.7.1), we replace an integral over $\mathcal{S}$ with an integral over its shadow in the xy plane.

The replacement is visible in the approximating sums involved in the integral over a surface.

Let $\mathcal{S}$ be a surface that meets each line parallel to the z-axis at most once. Let f be a function whose domain includes $\mathcal{S}$.

Consider an approximating sum for $\int_{\mathcal{S}} f(P)\ dS$, namely $\sum_{i=1}^{n} f(p_i)\Delta S_i$. The partition is shown in Figure 17.7.4.

Let R be the projection of $\mathcal{S}$ in the xy plane. The patch $\mathcal{S}_i$ with area S_i, projects down to R_i, of area A_i, and the point P_i on $\mathcal{S}_i$ points down to Q_i in R_i. Let γ_i be the angle between the normal at P_i and $\mathbf{k}$.

Then $f(P)S_i$ is approximately $\frac{f(P_i)}{|\cos(\gamma_i)|}A_i$. Thus an approximation of $\int_{\mathcal{S}} f(P)\ dS$ is

$$\sum_{i=1}^{n} \frac{f(P_i)}{|\cos\gamma_i|}A_i. \tag{17.7.2}$$

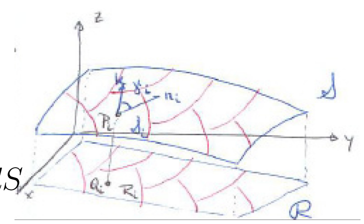


Figure 17.7.4:

Replacing an integral over a surface with an integral over a planar region.

Theorem 17.7.1. *Let $\mathcal{S}$ be a surface and let $\mathcal{A}$ be its projection on the xy plane. Assume that for each point Q on $\mathcal{A}$ the line through Q parallel to the z-axis meets $\mathcal{S}$ in exactly one point P. Let f be a function defined on $\mathcal{S}$. Define a function h on $\mathcal{A}$ by*

$$h(Q) = f(P).$$

Then

$$\int_{\mathcal{S}} f(P)\ dS = \int_{\mathcal{A}} \frac{h(Q)}{|\cos(\gamma)|}\ dA.$$

In this equation γ denotes the angle between $\mathbf{k}$ and a vector normal to the surface of $\mathcal{S}$ at P. (See Figure 17.7.5.)

In order to apply this result, we need to be able to compute $\cos(\gamma)$.

Computing $\cos(\gamma)$

We find a vector perpendicular to the surface in order to compute $\cos(\gamma)$. If $\mathcal{S}$ is the level surface of $g(x, y, z)$, that is $g(x, y, z) = c$, for some constant c, then the gradient ∇g is such a vector.

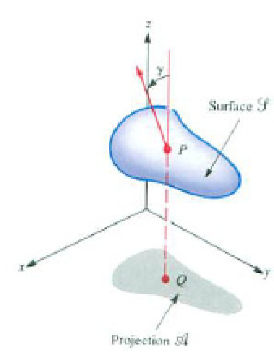


Figure 17.7.5:

If the surface $\mathcal{S}$ is given in the form $z = f(x,y)$, rewrite it as $z - f(x,y) = 0$. That means that $\mathcal{S}$ is a level surface of $g(x,y,z) = z - f(x,y)$, Theorem 17.7.2 shows what the formulas for $\cos(\gamma)$ look like. However, it is unnecessary, even distracting, to memorize them. Just remember that *a gradient provides a normal* to a level surface.

Theorem 17.7.2. *(a) If the surface $\mathcal{S}$ is part of the level surface $g(x,y,z) = c$, then*

$$|\cos(\gamma)| = \frac{|\frac{\partial g}{\partial z}|}{\sqrt{(\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2 + (\frac{\partial g}{\partial z})^2}}.$$

(b) If the surface $\mathcal{S}$ is given in the form $z = f(x,y)$, then

$$|\cos(\gamma)| = \frac{1}{\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1}}.$$

Proof

(a) A normal vector to $\mathcal{S}$ at a given point is provided by the gradient

$$\nabla g = \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}.$$

The cosine of the angle between $\mathbf{k}$ and ∇g is

$$\frac{\mathbf{k} \cdot \nabla g}{\|\mathbf{k}\|\|\nabla g\|} = \frac{\mathbf{k} \cdot (\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k})}{(1)\left(\cdot\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial g}{\partial y})^2 + (\frac{\partial g}{\partial z})^2}\right)};$$

hence

$$|\cos(\gamma)| = \frac{|\frac{\partial g}{\partial z}|}{\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial g}{\partial y})^2 + (\frac{\partial g}{\partial z})^2}}.$$

(b) Rewrite $z = f(x,y)$ as $z - f(x,y) = 0$. The surface $z = f(x,y)$ is thus the level surface $g(x,y,z) = 0$ of the function $g(x,y,z) = z - f(x,y)$. Note that

$$\frac{\partial g}{\partial x} = -\frac{\partial f}{\partial x}, \qquad \frac{\partial g}{\partial y} = -\frac{\partial f}{\partial y} \qquad \text{and} \qquad \frac{\partial g}{\partial z} = 1.$$

By the formula in (a),

$$|\cos(\gamma)| = \frac{1}{\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1}}$$

•

Theorem 17.7.2 is stated for projections on the xy plane. Similar theorems hold for projections on the xz or yz plane. The direction angle γ is then replaced by the corresponding direction angle, β or α, and the normal vector is dotted into $\mathbf{j}$ or $\mathbf{i}$. Just draw a picture in each case; there is no point in trying to memorize formulas for each situation.

EXAMPLE 2 Find the area of the part of the saddle $z = xy$ inside the cylinder $x^2 + y^2 = a^2$.

SOLUTION Let $\mathcal{S}$ be the part of the surface $z = xy$ inside $x^2 + y^2 = a^2$. Then

$$\text{Area of } \mathcal{S} = \int_{\mathcal{S}} 1 \, dS.$$

The projection of $\mathcal{S}$ on the xy plane is a disk of radius a and center $(0, 0)$. Call it $\mathcal{A}$, as in Figure 17.7.6. Then

$$\text{Area of } \mathcal{S} = \int_{\mathcal{S}} 1 \, dS = \int_{\mathcal{A}} \frac{1}{|\cos(\gamma)|} \, dA. \tag{17.7.3}$$

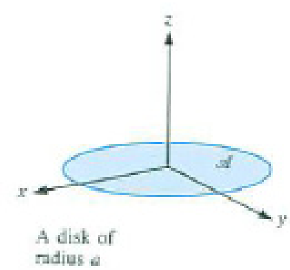


Figure 17.7.6:

To find the normal to $\mathcal{S}$ rewrite $z = xy$ as $z - xy = 0$. Thus $\mathcal{S}$ is a level surface of the function $g(x, y, z) = z - xy$. A normal to $\mathcal{S}$ is therefore

$$\begin{aligned} \nabla g &= \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k} \\ &= -y\mathbf{i} - x\mathbf{j} + \mathbf{k}. \end{aligned}$$

Then

$$\cos(\gamma) = \frac{\mathbf{k} \cdot \nabla g}{\|\mathbf{k}\| \|\nabla g\|} = \frac{\mathbf{k} \cdot (-y\mathbf{i} - x\mathbf{j} + \mathbf{k})}{\sqrt{y^2 + x^2 + 1}} = \frac{1}{\sqrt{y^2 + x^2 + 1}}.$$

The area of $\mathcal{S}$ is $\int_{\mathcal{A}} \sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1} \, dA$.

By (17.7.3),

$$\text{Area of } \mathcal{S} = \int_{\mathcal{A}} \sqrt{y^2 + x^2 + 1} \, dA. \tag{17.7.4}$$

Use polar coordinates to evaluate the integral in (17.7.4):

$$\int_{\mathcal{A}} \sqrt{y^2 + x^2 + 1} \, dA = \int_0^{2\pi} \int_0^a \sqrt{r^2 + 1} r \, dr \, d\theta.$$

The inner integration gives

$$\int_0^a \sqrt{r^2+1}f\ dr \quad = \quad \left.\frac{(r^2+1)^{3/2}}{3}\right|_0^a = \frac{(1+a^2)^{3/2}-1}{3}.$$

The second integration gives

$$\int_0^{2\pi} \frac{(1+a^2)^{3/2}-1}{3}\ d\theta = \frac{2\pi}{3}\left((1+a^2)^{3/2}-1\right).$$

◇

Summary

After defining $\int_{\mathcal{S}} f(P)\ dS$, an integral over a surface, we showed how to compute it when the surface is part of a sphere.

Replace dS by $a^2\sin(\phi)\ d\phi\ d\theta$, where a is the radius of the sphere.

If each line parallel to the z-axis meets the surface $\mathcal{S}$ in at most one point, an integral over $\mathcal{S}$ can be replaced by an integral over $\mathcal{A}$, the projection of $\mathcal{S}$ on the xy plane:

$$\int_{\mathcal{S}} f(P)\ dS = \int_{\mathcal{A}} \frac{h(Q)}{|\cos(\gamma)|}\ dA.$$

To find $\cos(\gamma)$, use a gradient. If the surface is a level surface of, $g(x,y,z)=c$, use ∇g. If it has the equation $z=f(x,y)$, rewrite the equation as $z-f(x,y)=0$. As a special case, if $\mathcal{S}$ is the graph of $z=f(x,y)$, then the area of $\mathcal{S}$

$$\text{Area of } \mathcal{S} = \int_{\mathcal{S}} dS = \int_{\mathcal{A}} \sqrt{(\partial f/\partial x)^2+(\partial f/\partial y)^2+1}\ dA.$$

EXERCISES for Section 17.7 *Key:* R–routine, M–moderate, C–challenging

1.[R] A small patch of a surface makes an angle of $\pi/4$ with the xy plane. Its projection on that plane has area 0.05. Estimate the area of the patch.

2.[R] A small patch of a surface makes an angle of 25° with the yz plane. Its projection on that plane has area 0.03. Estimate the area of the patch.

3.[R]

(a) Draw a diagram of the part of the plane $x+2y+3z = 12$ that lies inside the cylinder $x^2+y^2 = 9$.

(b) Find as simply as possible the area of the part of the plane $x+2y+3z = 12$ that lies inside the cylinder $x^2+y^2 = 9$.

4.[R]

(a) Draw a diagram of the part of the plane $z = x+3y$ that lies inside the cylinder $r = 1+\cos\theta$.

(b) Find as simply as possible the area of the part of the plane $z = x+3y$ that lies inside the cylinder $r = 1+\cos\theta$.

5.[R] Let $f(P)$ be the square of the distance from P to a fixed diameter of a sphere of radius a. Find the average value of $f(P)$ for points on the sphere.

6.[R] Find the area of that part of the sphere of radius a that lies within a cone of half-vertex angle $\pi/4$ and vertex at the center of the sphere, as in Figure 17.7.7.

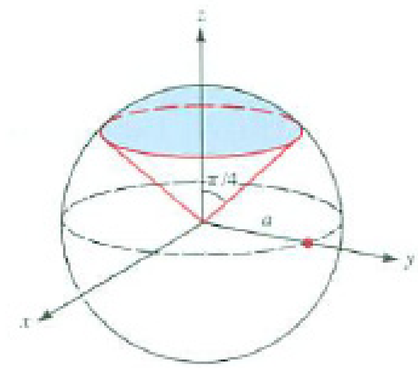


Figure 17.7.7:

In Exercises 7 and 8 evaluate $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dS$ for the given spheres and vectors fields ($\mathbf{n}$ is the outward unit normal.)

7.[R] The sphere $x^2+y^2+z^2 = 9$ and $\mathbf{F} = x^2\mathbf{i}+y^2vj+z^2\mathbf{k}$.

8.[R] The sphere $x^2+y^2+z^2 = 1$ and $\mathbf{F} = x^3\mathbf{i}+y^2\mathbf{j}$.

9.[R] Find the area of the part of the spherical surface $x^2+y^2+z^2 = 1$ that lies within the vertical cylinder erected on the circle $r = \cos\theta$ and above the xy plane.

10.[R] Find the area of that portion of the parabolic cylinder $z = \frac{1}{2}x^2$ between the three planes $y = 0$, $y = x$, and $x = 2$.

11.[R] Evaluate $\int_{\mathcal{S}} x^2y\, dS$, where $\mathcal{S}$ is the portion in the first octant of a sphere with radius a and center at the origin, in the following way:

(a) Set up an integral using x and y as parameters.

(b) Set up an integral using ϕ and θ as parameters.

(c) Evaluate the easier of (a) and (b).

12.[R] A triangle in the plane $z = x+y$ is directly above the triangle in the xy plane whose vertices are $(1,2)$, $(3,4)$, and $(2,5)$. Find the area of

(a) the triangle in the xy plane,

(b) the triangle in the plane $z = x + y$.

13.[R] Let $\mathcal{S}$ be the triangle with vertices $(1,1,1)$, $(2,3,4)$, and $(3,4,5)$.

(a) Using vectors, find the area of $\mathcal{S}$.

(b) Using the formula

$$\text{Area of } \mathcal{S} = \int_{\mathcal{S}} 1\ dS,$$

find the area of $\mathcal{S}$.

14.[R] Find the area of the portion of the cone $z^2 = x^2 + y^2$ that lies above one loop of the curve $r = \sqrt{\cos 2(\theta)}$.

15.[R] Let $\mathcal{S}$ be the triangle whose vertices are $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$. Let $f(x,y,z) = 3x + 2y + 2z$. Evaluate $\int_{\mathcal{S}} f(P)\ dS$.

In Exercises 16 and 17 let $\mathcal{S}$ be a sphere of radius a with center at the origin of a rectangular coordinate system.

16.[R] Evaluate each of these integrals with a minimum amount of labor.

(a) $\int_{\mathcal{S}} x\ dS$

(b) $\int_{\mathcal{S}} x^3\ dS$

(c) $\int_{\mathcal{S}} \frac{2x+4y^5}{\sqrt{2+x^2+3y^2}}\ dS$

17.[R]

(a) Why is $\int_{\mathcal{S}} x^2\ dS = \int_{\mathcal{S}} y^2\ dS$?

(b) Evaluate $\int_{\mathcal{S}}(x^2 + y^2 + z^2)\ dS$ with a minimum amount of labor.

(c) In view of (a) and (b), evaluate $\int_{\mathcal{S}} x^2\ dS$.

(d) Evaluate $\int_{\mathcal{S}}(2x^2 + 3y^3)\ dS$.

18.[R] An electric field radiates power at the rate of $k(\sin^2(\phi)/\rho^2$ units per square meter to the point $P = (\rho, \theta, \phi)$. Find the total power radiated to the sphere $\rho = a$.

19.[R] A sphere of radius $2a$ has its center at the origin of a rectangular coordinate system. A circular cylinder of radius a has its axis parallel to the z-axis and passes through the z-axis. Find the area of that part of the sphere that lies within the cylinder and is above the xy plane.

Consider a distribution of mass on the surface $\mathcal{S}$. Let its density at P be $\sigma(P)$. The **moment of inertia** of the mass around the z-axis is defined as $\int_{\mathcal{S}}(x^2 + y^2)\sigma(P)\ dS$. Exercises 20 and 21 concern this integral.

20.[R] Find the moment of inertia of a homogeneous distribution of mass on the surface of a ball of radius a around a diameter. Let the total mass be M.

21.[R] Find the moment of inertia about the z-axis of a homogeneous distribution of mass on the triangle whose vertices are $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$. Take a, b, and c to be positive. Let the total mass be M.

22.[R] Let $\mathcal{S}$ be a sphere of radius a. Let A be a point at distance $b > a$ from the center of $\mathcal{S}$. For P in $\mathcal{S}$ let $\delta(P)$ be $1/q$, where q is the distance from P to A. Show that the average of $\delta(P)$ over $\mathcal{S}$ is $1/b$.

23.[R] The data are the same as in Exercise 22 but $b < a$. Show that in this case the average of $1/q$ is $1/a$. (The average does *not* depend on b in this case.)

Exercises 24 to 26 concern integration over the curved surface of a cone. Spherical coordinates are also useful for integrating over a right circular cone. Place the origin at the vertex of the cone and the "$\phi = 0$" ray along the axis of the cone, as shown in Figure 17.7.8(a). Let α be the half-vertex angle of the cone.

On the surface of the cone ϕ is constant, $\phi = \alpha$, but ρ and θ vary. A small "rectangular" patch on the surface of the cone corresponding to slight changes $d\theta$ and $d\rho$ has area approximately

$$(\rho\sin(\alpha)\ d\theta)\ d\rho = \rho\sin(\alpha)\ d\rho\ d\theta.$$

(See Figure 17.7.8.) So we may write

$$dS = \rho\sin\alpha\ d\rho\ d\theta.$$

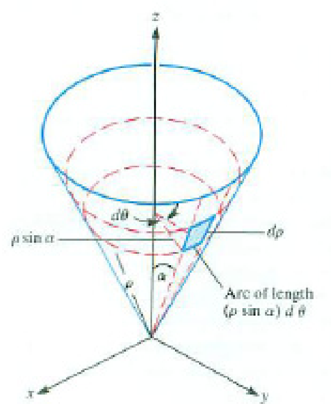


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Figure 17.7.8:

24.[R] Find the average distance from points on the curved surface of a cone of radius a and height h to its axis.

25.[R] Evaluate $\int_{\mathcal{S}} z^2\ dS$, where $\mathcal{S}$ is the *entire* surface of the cone shown in Figure 17.7.8(b), including its base.

26.[R] Evaluate $\int_{\mathcal{S}} x^2\ dS$, where $\mathcal{S}$ is the curved surface of the right circular cone of radius 1 and height 1 with axis along the z-axis.

Integration over the curved surface of a right circular cylinder is easiest in cylindrical coordinates. Consider such a cylinder of radius a and axis on the z-axis. A small patch on the cylinder corresponding to dz and $d\theta$ has area approximately $dS = a\ dz\ d\theta$. (Why?) Exercises 27 and 28 illustrate the use of these coordinates.

27.[R] Let $\mathcal{S}$ be the *entire* surface of a solid cylinder of radius a and height h. For P in $\mathcal{S}$ let $f(P)$ be the square of the distance from P to one base. Find $\int_{\mathcal{S}} f(P)\ dS$. Be sure to include the two bases in the integration.

28.[R] Let $\mathcal{S}$ be the curved part of the cylinder in Exercise 27. Let $f(P)$ be the square of the distance from P to a fixed diameter in a base. Find the average value of $f(P)$ for points in $\mathcal{S}$.

29.[R] The areas of the projections of a small flat surface patch on the three coordinate planes are 0.01, 0.02, and 0.03. Is that enough information to find the area of the patch? If so, find the area. If not, explain why not.

30.[R] Let $\mathbf{F}$ describe the flow of a fluid in space. (See Section 16.3 for fluid flow in a planar region.) $\mathbf{F}(P) = \delta(P)\mathbf{v}(P)$, where $\delta(P)$ is the density of the fluid at P and $\mathbf{v}(P)$ is the velocity of the fluid at P. Making clear, large diagrams, explain why the rate at which the fluid is leaving the solid region enclosed by a surface $\mathcal{S}$ is $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\ dS$, where $\mathbf{n}$ denotes the unit outward normal to $\mathcal{S}$.

31.[R] Let $\mathcal{S}$ be the smooth surface of a convex body. Show that $\int_{\mathcal{S}} z\cos(\gamma)\ dS$ is equal to the volume of the solid bounded by $\mathcal{S}$. HINT: Break $\mathcal{S}$ into two parts. In one part $\cos(\gamma)$ is positive; and the other it negative.

32.[M] Let $R(x,y,z)$ be a scalar function defined over a closed surface $\mathcal{S}$. (See Figure 17.7.9.)

(a) Show that

$$\int\limits_{\mathcal{S}} R(x,y,z)\cos(\gamma)\ dS = \int\limits_{\mathcal{A}} \left(P(x,y,z_2) - P(x,y,z_1)\right)\ dA,$$

where $\mathcal{A}$ is the projection of $\mathcal{S}$ on the xy plane and the line through $(x,y,0)$ parallel to the z-axis meets $\mathcal{S}$ at (x,y,z_1) and (x,y,z_2), with $z_1 \le z_2$.

(b) Let $\mathcal{S}$ be a surface of the type in (a). Evaluate $\int_{\mathcal{S}} x\cos\gamma\ dS$.

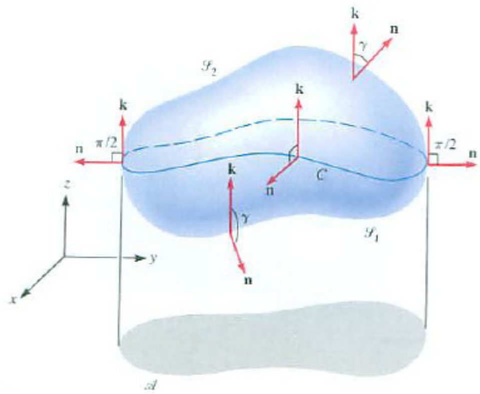


Figure 17.7.9:

33.[C]

(a) Let g be a differentiable function such that $g((x+y)/2) = ((g(x)+g(y))/2$ for all x and y. Show that $g(x) = kx + c$ for some constraints k and c. Hint: Differentiate.

(b) Let f be a differentiable
$(x+y)f(x+y) + (x-y)f($
all x and y. Deduce that the
and c such that $f(x) = k +$

34.[C] (Suggested by Exercises 22
tion $f(x) = 1/x$ has the remarkabl
average value of $f(d(P))$ over a sp
$f(H)$. Here $d(P)$ is the distance
point at a distance H for the cente
dius a, $a < H$. Show that the only
property have the form $k + c/x$ for
and c. Hint: Use part of the Funda
Calculus to remove integration. T
many come in handy.

17.8 Moments, Centers of Mass, and Centroids

Now that we can integrate over planar regions, surfaces, and solid regions, we can define and calculate the center of mass of a physical object. The center of mass is important in the eyes of a naval architect, who wants his ships not to tip over easily. A pole vaulter hopes that as she clears the bar her center of mass goes under it. Archimedes, the first person to study the center of mass, was interested in the stability of floating paraboloids.

The Center of Mass

A small boy on one side of a seesaw (which we regard as weightless) can balance a bigger boy on the other side. For example, the two boys in Figure 17.8.1 balance. (According to physical laws, each boy exerts a force on the seesaw, due to gravitational attraction, proportional to his mass.)

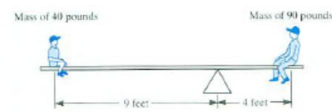


Figure 17.8.1:

The small mass with the long lever arm balances the large mass with the small lever arm. Each boy contributes the same tendency to turn–but in opposite directions.

This tendency is called the **moment**:

$$\text{Moment} = (\text{Mass}) \cdot (\text{Lever arm}),$$

where the lever arm can be positive or negative. To be more precise, introduce on the seesaw an x-axis with its origin 0 at the fulcrum, the point on which the seesaw rests. Define the moment about 0 of a mass m located at the point x on the x-axis to be the product mx. Then the bigger boy has a moment $(90)(4)$, which the smaller boy has a moment $(40)(-9)$. The total moment of the lever-mass system is 0, and the masses balance. (See Figure 17.8.2.)

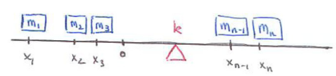


Figure 17.8.2:

If a mass m is located on a line with coordinate x, we define its moment about the point having coordinate k as the product $m(x-k)$.

Now consider several point masses $m_1, m_2, \dots, m_i$. If mass m_i is located at x_i, with $i = 1, 2, \dots, n$, then $\sum_{i=1}^{n} m_i(x_i - k)$ is the total moment of all the masses about the point k. If a fulcrum is placed at k, then the seesaw rotates clockwise if the total moment is greater that 0, rotates counterclockwise if it is less than 0, and is in equilibrium if the total moment is 0. See Figure 17.8.3.

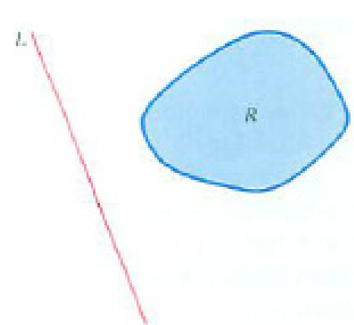


Figure 17.8.3:

To find where to place the fulcrum so that the total tendency to turn is 0, we find k such that

$$\sum_{i=1}^{n} m_i(x_i - k) = 0.$$

Writing this as

$$k \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i - x_i,$$

we see that

$$k = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}. \tag{17.8.1}$$

The number k given by (17.8.1) is called the **center of mass** or center of gravity of the system of masses. It is the point about which all the masses balance. *The center of mass is found by dividing the total moment about* 0 *by the total mass.* It is usually denoted $\bar{x}$.

(a) (b)

Figure 17.8.4:

Finding the center of mass of a finite number of "point masses" involves only arithmetic, no calculus. For example, suppose three masses are placed on a seesaw as in Figure 17.8.4(a). Introduce an x-axis with origin at mass $m_1 = 20$ pounds. Two additional masses are located at $x_2 = 4$ feet and $x_3 = 14$ feet with masses $m_2 = 10$ pounds and $m_3 = 50$ pounds, respectively. The total moment about $x = k$ is

$$M = 20(0-k) + 10(4-k) + 50(14-k) = 740 - 80k.$$

This moment vanishes when $M = 0$, that is, when $k = 740/80 = 9.25$.

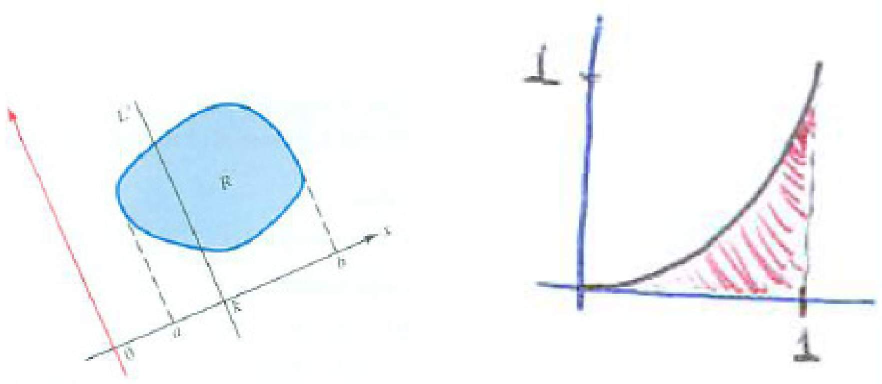
This is consistent with the formula for the center of mass:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{0 + 40 + 700}{10 + 20 + 50} = \frac{740}{80} = 9.25.$$

The seesaw balances when the fulcrum is placed 9.25 feet from the first mass. (See Figure 17.8.4(b).)

Now let us turn our attention to finding the center of mass of a continuous distribution of matter in a plane region. For this purpose, we consider double integrals.

Let R be a region in the plane occupied by a thin piece of metal whose density, $\sigma(P)$, varies. Let L be a line in the plane, as shown in Figure 17.8.5(a). We will find a formula for the unique line parallel to L, around which the mass in R balances.



(a) (b)

Figure 17.8.5:

To begin, let L' be *any* line parallel to L. We will compute the moment about L' and then see how to choose L' to make that moment equal to 0. To compute the moment of R about L', introduce an x-axis perpendicular to L with its origin at its intersection with L. Assume that L' passes through the x-axis at the point$x = k$, as in Figure 17.8.5(b). In addition, assume that each line parallel to L meets R either in a line segment or at a point on the boundary of R. The lever arm of the mass distributed throughout R varies from point to point.

We partition R into n small regions $R_1, R_2, \dots, R_n$. Call the area of R_i, A_i. In each of these regions the lever arm around L' varies only a little. So, if we pick a point P_1 in R_1, P_2 in R_2, $\dots$, P_n, in R_n, and the x-coordinate of P_i is x_i, then

$$\underbrace{(x_i - k)}_{\text{lever arm}}\underbrace{\sigma(P_i)A_i}_{\text{mass in } R_i}$$

is a local estimate of the turning tendency.

Thus

$$\sum_{i=1}^{n}(x_i - k)\sigma(P_i)A_i \tag{17.8.2}$$

would presumably be a good estimate of the total turning tendency around L'. Taking the limit of (17.8.2) as all R_i are chosen smaller and smaller, we expect

$$\int_R (x - k)\sigma(P)\, dA \tag{17.8.3}$$

to represent the turning tendency of the total mass around L'. The quantity

(17.8.3) is called the **moment of torque** of the mass distribution around L'.

EXAMPLE 1 Let R be the region under $y = x^2$ and above $[0,1]$ DOUG with the density $\sigma(x,y) = xy$. Find its moment around the line $x = 1/2$.
SOLUTION R is shown in Figure 17.8.6. The moment (17.8.3) equals

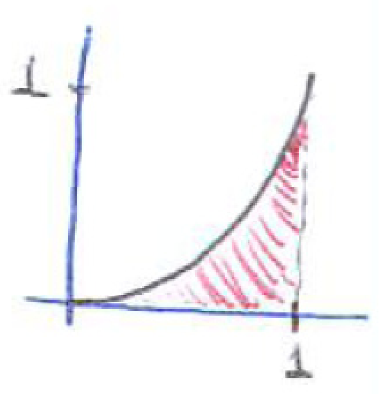


Figure 17.8.6:

$$\int_R \left(x - \frac{1}{2}\right) xy \; dA. \tag{17.8.4}$$

We evaluate this double integral by the iterated integral

$$\int_0^1 \left(\int_0^{x^2} \left(x - \frac{1}{2}\right) xy \; dy \right) \; dx.$$

See Exercise 2.

The first integration gives

$$\int_0^{x^2} \left(x - \frac{1}{2}\right) xy \; dy \quad = \quad (x - 1/2)x \int_0^{x^2} y \; dy = \frac{\left(x - \frac{1}{2}\right) x^5}{2}.$$

The second integration is

$$\int_0^1 \frac{\left(x - \frac{1}{2}\right) x^5}{2} = \int_0^1 \frac{2x^6 - x^5}{4} \; dx = \frac{5}{168}.$$

Since the total moment (17.8.4) is positive, the object would rotate clockwise around the line $x = \frac{1}{2}$. $\diamond$

Now that we have a way to find the moment around any line parallel to the y-axis we can find the line around which the moment is zero, the so-called "balancing line." We just solve for k in the equation

$$\int_R (x-k)\sigma(P)\ dA = 0.$$

Thus

$$\int_R x\sigma(P)\ dA = k\int_R \sigma(P)\ dA,$$

from which we find that

$$k = \frac{\int_R x\sigma(P)\ dA}{\int_R \sigma(P)\ dA}. \tag{17.8.5}$$

The denominator is the total mass. The numerator is the total torque. So we can think of k as "the average lever arm as integrated by the density."

That is therefore a unique balancing line parallel to the y axis. Call its x-coordinate $\overline{x}$ (read: "x bar"). Similarly, there is a unique balancing line parallel to the x axis. Call its y-coordinate $\overline{y}$. The point $(\overline{x}, \overline{y})$ is called the center of mass of the region R. We have:

The center of mass of a region R with density $\sigma(P)$ has coordinates $(\bar{x}, \bar{y})$ where

$$\overline{x} = \frac{\int_R x\sigma(P)\ dA}{\int_R \sigma(P)\ dA} \qquad \text{and} \qquad \overline{y} = \frac{\int_R y\sigma(P)\ dA}{\int_R \sigma(P)\ dA}.$$

The integral $\int_R x\sigma(P)\ dA$ is called the **moment of R around the y-axis**, and is denoted M_y. Similarly, $M_x = \int_R y\sigma(P)\ dA$.

If the density $\sigma(P)$ is constant, say, equal to 1 everywhere in R, then the two equations reduce to

$$\overline{x} = \frac{\int_R x\ dA}{\int_R\ dA} \qquad \text{and} \qquad \overline{y} = \frac{\int_R y\ dA}{\int_R\ dA}.$$

In this case the center of mass R is also called the **centroid** of the region, a purely geometric concept:

The centroid of the plane region R has the coordinates $(\bar{x}, \bar{y})$ where

$$\overline{x} = \frac{\int_R x\ dA}{\text{Area of } R} \qquad \text{and} \qquad \overline{y} = \frac{\int_R y\ dA}{\text{Area of } R)}. \tag{17.8.6}$$

EXAMPLE 2 Find the center of mass of the region in Example 1.

SOLUTION The density at (x, y) in R is given by $\sigma = xy$. We compute three double integrals: the mass $\int_R xy\ dA$ and the two moments $M_y = \int_R x(xy)\ dA$ and $M_x = \int_R y(xy)\ dA$.

We have

$$\int_R x^2y\ dA = \int_0^1 \left(\int_0^{x^2} x^2y\ dy \right)\ dx = \int_0^1 \frac{x^6}{2}\ dx = \frac{1}{14}.$$

Then

$$\int_R xy\ dA = \int_0^1 \left(\int_0^1 xy\ dy \right)\ dx = \int_0^1 \frac{x^5}{2}\ dx = \frac{1}{12}.$$

Finally,

$$\int_R xy^2\ dA = \int_1^0 \left(\int_1^0 xy^2\ dy \right)\ dx = \int_0^1 \frac{x^7}{3}\ dx = \frac{1}{24}.$$

Thus

$$\overline{x} = \frac{\frac{1}{14}}{\frac{1}{12}} = \frac{6}{7} \qquad \text{and} \qquad \overline{y} = \frac{\frac{1}{24}}{\frac{1}{12}} = \frac{1}{2}.$$

It is not surprising that $\overline{x}$ is greater than $1/2$, since in Example 17.8.1 we found that the object rotates clockwise around the line $x = 1/2$. ◇

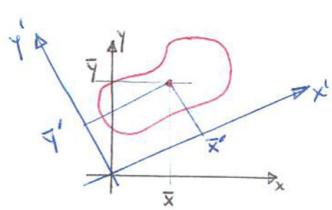


Figure 17.8.7:

An Important Point About an Important Point

We defined the center of mass $(\overline{x}, \overline{y})$ by first choosing an xy coordinate system. What if we choose an $x'y'$ coordinate system at an angle to the xy coordinate system? Would the center of mass computed in this system, $(\overline{x}', \overline{y}')$ be the same point as $(\overline{x}, \overline{y})$? See Figure 17.8.7. Fortunately, it is, as Exercise 59 shows.

Shortcuts for Computing Centroids

Assume that R is the region under $y = f(x)$ for x in $[a, b]$. Then the moment about the x-axis is

$$M_x = \int_R y\ dA.$$

Thus

$$M_y = \int_a^b \left(\int_0^{f(x)} y\ dy\right)\ dx = \int_a^b \frac{(f(x))^2}{2}\ dx = \frac{1}{2}\int_a^b (f(x))^2\ dx.$$

Thus, by (17.8.6)

$$\boxed{\overline{y} = \frac{\frac{1}{2}\int_a^b (f(x))^2\ dx}{\text{Area of } R}. \qquad (17.8.7)}$$

EXAMPLE 3 Find the centroid of the semicircular region of radius a shown in Figure 17.8.8.

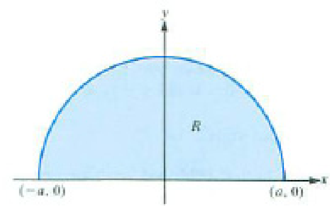


Figure 17.8.8:

SOLUTION By symmetry, $\overline{x} = 0$.

To find $\overline{y}$, use (17.8.7). The function f in this case is given by the formula $f(x) = \sqrt{a^2 - x^2}$, an even function. The moment of R about the x-axis is

$$\begin{aligned}
\int_{-a}^{a} \frac{(\sqrt{z^2 - x^2})^2}{2}\ dx &= \int_{-a}^{a} \frac{a^2 - x^2}{2}\ dx = 2\int_0^a \frac{a^2 - x^2}{2}\ dx \\
&= \int_0^a (a^2 - x^2)\ dx = \left(a^2 x - \frac{x^3}{3}\right)\Big|_0^a \\
&= (a^3 - \frac{a^3}{3}) - 0 = \frac{2}{3}a^3.
\end{aligned}$$

Thus

$$\overline{y} = \frac{\frac{2}{3}a^3}{\text{Area of } R} = \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi}.$$

Since $4/(3\pi) \approx 0.42$, the center of gravity of R is at a height of about $0.42a$.

◇

Centers of Other Masses

We developed the ideas of moments and centers of mass for masses situation in a plane. The definition generalizes easily to masses distributed on a curve (such as a wire) or in space (such as a potato).

In the case of a curve, the curve would have a linear density $\lambda(P)$. A short piece around P of length Δs would have mass approximately $\lambda(P)\Delta s$. Thus, the mass and moments of the curve would be

$$M = \int\limits_C \lambda(P)\ ds, \qquad M_y = \int\limits_C x\lambda(P)\ ds, \qquad \text{and} \qquad M_x = \int\limits_C y\lambda(P)\ ds.$$

We state the definition in the case of a solid object of density $\lambda(P)$ occupying the region R. We assume an xyz-coordinate system. The total mass is

$$M = \int\limits_R \delta(P)\ dV.$$

Now, there are three moments — one around each of the three coordinate planes:

$$M_{yz} = \int\limits_R x\delta(P)\ dV, M_{xz} = \int\limits_R y\delta(P)\ dV, M_{xy} \int\limits_R z\rho\ dV.$$

The center of mass is $(\overline{x}, \overline{y}, \overline{z})$, where

$$overlinex = \frac{\int_R x\delta(P)\ dV}{M}, \qquad \overline{y} = \frac{\int_R y\delta(P)\ dv}{M}, \qquad \overline{z} = \frac{\int_R z\delta(P)\ dV}{M}.$$

If $\delta(P) = 1$ for all P in R, the center of mass is called the centroid. In this case the mass is the same as the volume.

EXAMPLE 4 Find the centroid of a hemisphere of radius a.

SOLUTION We place the origin of an xyz-coordinate system at the center of the hemisphere, as in Figure 17.8.9.

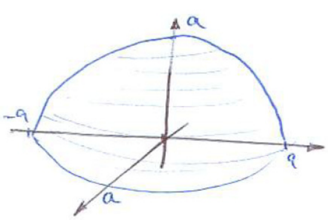


Figure 17.8.9:

First of all, by symmetry, the centroid must be at the z-axis. mnoteIf the centroid were not at the z-axis, you would get two centroids for the same object. (If you spin the hemisphere about the z-axis you get the same hemisphere back, which must have the same centroid.)

So $\overline{x} = \overline{y} = 0$. Calling the hemisphere R, we have

$$\overline{z} = \frac{\int_R z\ dV}{\text{Volume of } R}.$$

The volume of the hemisphere is half that of a ball, $(2/3)\pi a^3$. To evaluate the moment $\int_R z\ dV$, we bring in an iterated integral in spherical coordinates:

$$\int\limits_R z\ dV = \int\limits_0^{2\pi} \int\limits_0^{\pi/2} \int\limits_0^{a} (\rho\cos(\phi))\rho 2\sin(\phi)\ d\rho\ d\phi\ d\theta.$$

See Exercise 3.

Straightforward computations show that

$$\int_R z\ dV = \frac{\pi a^4}{4}.$$

Thus

$$\overline{z} = \frac{\frac{\pi a^4}{4}}{\frac{2}{3}\pi a^3} = \frac{3a}{8}.$$

The centroid is $(0, 0, \frac{3a}{8})$. ◇

EXAMPLE 5 Find the centroid of a homogeneous cone of height h and radius a.

SOLUTION As we just saw for the sphere in Example 4, symmetry tells us the centroid lies on the axis of the cone.

Introduce a spherical coordinate system with the origin at the vertex of the cone and with the axis of the cone lying on the ray $\phi = 0$, as in Figure 17.8.10. The half-vertex angle is $\arctan(a/h)$. The plane of the base of the cone is $z = h$ (in rectangular coordinates), hence

$$\rho\cos(\phi) = h.$$

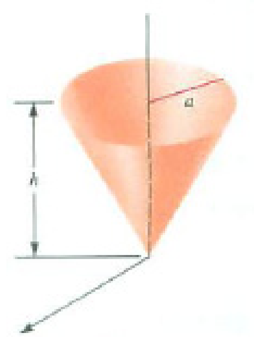


Figure 17.8.10:

In spherical coordinates, the cone's description is

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \arctan(a/h), \qquad 0 \le \rho \le h/\cos(\phi).$$

To find the centroid of the cone we compute $\int_R z\ dV$ and divide the results by the volume of the cone, which is $\frac{1}{3}\pi a^2 h$.

Now

$$\int_R z\ dV = \int_0^{2\pi} \int_0^{\arctan(a/h)} \int_0^{h/\cos(\phi)} \rho\cos(\phi)(\rho^2\sin(\phi))\ d\rho\ d\phi\ d\theta.$$

See Exercise 4

For the first integration, ϕ and θ are constant; hence

$$\int_0^{h/\cos(\phi)} \rho\cos(\phi)\rho^2\sin(\phi)\ d\rho \quad = \quad \cos(\phi)\sin(\phi) \int_0^{h/\cos(\phi)} \rho^3\ d\rho = \frac{h^4\sin(\phi)}{4\cos^3(\phi)}.$$

The second integration is

$$\int_0^{\arctan(a/h)} \frac{h^4\sin(\phi)}{4\cos^3(\phi)}\ d\phi \quad = \quad \frac{h^4}{4} \int_0^{\arctan(a/h)} \frac{\sin(\phi)}{\cos^3(\phi)}\ d\phi = \frac{a^2h^2}{8}.$$

	curve (C)	solid (R)
density	$\lambda(P)$	$\delta(P)$
M	$\int_C \lambda(P)\ ds$	$\int_S \delta(P)\ dV$
M_{yz}	$\int_C x\lambda(P)\ ds$	$\int_S x\delta(P)\ dV$
M_{xz}	$\int_C y\lambda(P)\ ds$	$\int_S y\delta(P)\ dV$
M_{xy}	$\int_C z\lambda(P)\ ds$	$\int_S z\delta(P)\ dV$

The final integral is simply:

$$\int_0^{2\pi} \frac{a^2h^2}{8}\ d\theta = \frac{a^2h^2}{8}2\pi = \frac{\pi a^2h^2}{4}.$$

Thus,

$$\overline{z} = \frac{\int_R z\ dV}{\text{Volume of } R} = \frac{\left(\frac{\pi a^2 h^2}{4}\right)}{\left(\frac{\pi a^2 h}{3}\right)} = \frac{3h}{4}.$$

The centroid of a cone is three-fourths of the way from the vertex to the base. ◇

Summary

We defined the moment about a line and used this concept to define the center of mass for a plane distribution of mass. The moment of a mass about a line L indicates the tendency of the mass to rotate about the line L. The center of mass for a region R is the point in the region where the region balances.

- The moment about the y-axis, M_y, is $\int_R x\delta(P)\ dA$.
- The moment about the x-axis, M_x, is $\int_R y\delta(P)\ dA$.

Then, the center of mass is $(\bar{x}, \bar{y})$ where

$$\overline{x} = \frac{M_y}{\text{Mass}}, \overline{y} = \frac{M_x}{\text{Mass}}.$$

If the density is constant, we have a purely geometric concept,

$$\overline{x} = \frac{\int_R x\ dA}{\text{Area of } R}, \overline{y} = \frac{\int_R y\ dA}{\text{Area of } R}.$$

These definitions generalize to curves and solids.

EXERCISES for Section 17.8 *Key:* R–routine, M–moderate, C–challenging

1.[R]

(a) How would you define the centroid of a curve? Call its (linear) density $\lambda(P)$.

(b) Find the centroid of a semicircle of radius a.

2.[R] Carryout the integrations in Example 1.

3.[R] Carryout the "straightforward calculations" in Example 4.

4.[R] Provide the details needed to complete the integrals in Example 5.

5.[R] Example 4 showed that the centroid of a hemisphere is less than halfway from the center to its surface. Why is that to be expected?

6.[M] If R is the region below $y = f(x)$ and above $[a, b]$, show that

$$\overline{x} = \frac{\int_a^b x f(x)\, dx}{\text{Area of } R}.$$

7.[M] The corners of a triangular piece of metal of constant density 1 are $(0, 0)$, $(1, 0)$, and $(0, 2)$.

(a) Is the line $y = 11x/5$ a balancing line?

(b) If not, if the metal rests on this line which way would it rotate?

DEFINITION *(Section of a region)* Let R be a convex set in the plane. A **section** of R is a part of R that is bounded by a chord and part of the the boundary, as shown as Figure 17.8.11

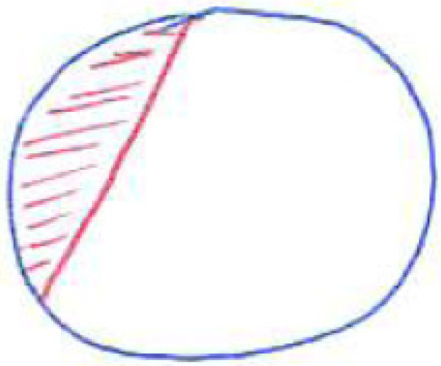


Figure 17.8.11:

8.[C] Consider a convex set R in the plane furnished with a density. Show that different sections have different centers of gravity.

9.[C] (See Exercise 8.) Is every point in R that is not on the boundary the center of mass of some section of R?

10.[C] Archimedes (287-212 B.C.) investigated the centroid of a section of a parabola. Consider the parabola $y = x^2$. The typical section is shown in Figure 17.8.12. M is the midpoint of the chord and N is the point on the parabola directly below M.

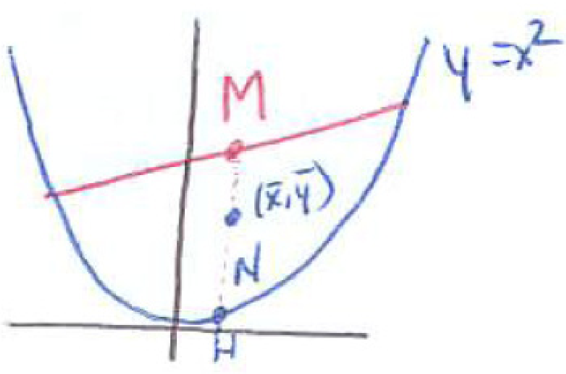


Figure 17.8.12:

He showed, without calculus, that the centroid is on the line MN, three-fifths of the way from N and M. Obtain his result with the aid of calculus.

11.[C] (See Exercise 10.) Is every point in the region bounded by the parabola the centroid of some section?

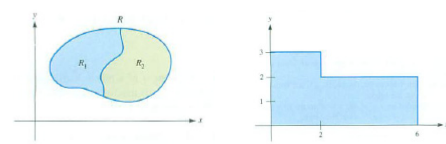
12.[R] Find the centroid of a solid paraboloid of revolution. This is the region above $z = x^2 + y^2$ and below the plane $z = c$. Archimedes solved this problem without calculus and used the result to analyze the equilibrium of a floating paraboloid. (If it is slightly tilted, will it come back to the vertical or topple over?) For details as how he did this 2200 years ago see S. Stein, *Archimedes: What Did He Do Besides Cry Eureka?*, Math. Assoc. America, 1999.

13.[C] (See Exercise 12.) The plane $z = c$ in Exercise 12 is perpendicular to the axis of the paraboloid. Archimedes was also interested in the case when the plane is not perpendicular to the axis. Find the centroid of the region below the tilted plane $z = cy$ and above the paraboloid $z = x^2 + y^2$.

14.[R] Using cylindrical coordinates, find $\overline{z}$ for the region below the paraboloid $z = x^2 + y^2$ and above the disk in the $r\theta$ plane bounded by the circle $r = 2$. (Include a drawing of the region.)

15.[R] Find the z coordinate, $\overline{z}$, of the centroid of the part of the saddle $z = xy$ that lies above the portion of the disk bounded by the circle $x^2 + y^2 = a^2$ in the first quadrant.

16.[M] A plane distribution of matter occupies the region R. It is cut into two pieces, occupying regions R_1 and R_2, as in Figure 17.8.13(a). The part in R_1 has mass M_1 and centroid $(\overline{x}_1, \overline{y}_1)$. The part in R_2 has mass M_2 and centroid $(\overline{x}_2, \overline{y}_2)$. Find the centroid $(\overline{x}, \overline{y})$ of the entire mass, which occupies R. [Express $(\overline{x}, \overline{y})$ in terms of M_1, M_2, $\overline{x}_1$, $\overline{x}_2$, $\overline{y}_1$ and $\overline{y}_2$.]



(a) (b)

Figure 17.8.13:

17.[M] Use the formula in Exercise 16 to find the center of mass of the homogeneous lamina shown in Figure 17.8.13(b).

In Exercises 18 to 25 find the centroid of the given regions R. (Exercises 22 to 25 require integral tables or techniques of Chapter 8.)

18.[R] R is bounded by $y = x^2$ and $y = 4$.

19.[R] R is bounded by $y = x^4$ and $y = 1$.

20.[R] R is bounded by $y = 4x - x^2$ and the x-axis.

21.[R] R is bounded by $y = x$, $x + y = 1$, and the x-axis.

22.[R] The region bounded by $y = e^x$ and the x-axis, between the lines $x = 1$ and $x = 2$.

23.[R] The region bounded by $y = \sin(2x)$ and the x-axis, between the lines $x = 0$ and $x = \pi/2$.

24.[R] The region bounded by $y = \sqrt{1+x}$ and the x-axis, between the lines $x = 0$ and $x = 3$.

25.[R] The region bounded by $y = \ln(x)$ and the x-axis between the lines $x = 1$ and $x = e$.

Exercises 26 to 28 concern Pappus's Theorem, which relates the volume of a solid of revolution to the centroid of the planar region R that is revolved to form the solid.

Theorem 17.8.1 (Pappus)**.** *Let R be a region in the plane and L a line in the plane that does not cross R (though it can touch R at its border). Then the volume of the solid formed by revolving R about L is equal to the product*

$$(\textit{Distance the centroid of } R \textit{ is rotated}) \cdot (\textit{Area of } R)\,.$$

26.[C]

(a) Prove Pappus's Theorem

(b) Use Pappus's Theorem to find the volume of the torus or "doughnut" formed by revolving a circle of radius 3 inches about a line 5 inches from its center.

27.[C] Use Pappus's Theorem to find the centroid of the half disk R of radius a.

28.[C] Use Pappus's Theorem to find the centroid of the right triangle in Figure 17.8.14.

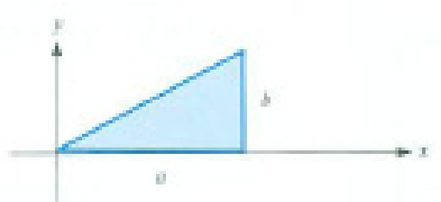


Figure 17.8.14:

29.[M] Consider a distribution of mass in a plane region R with density $\sigma(P)$ at P. Use the following steps to show that any line in the plane that passes through the center of the mass is a balancing line.

(a) For convenience, place the origin of the xy-coordinate system at the center of mass. That is, assume $(\bar{x}, \bar{y}) = (0,0)$. Show that $\int_R x\sigma(P)\ dA = 0$ and $\int_R y\sigma(P)\ dA = 0$.

(b) Let L be any line $ax+by=0$ through the origin. Show that the moment of the mass about L is

$$\int\limits_R \frac{ax+by}{\sqrt{a^2+b^2}}\sigma(P)\ dA.$$

HINT: What is the distance from a point (x, y) in R to the line $ax + by = 0$?

(c) From (a) and (b) deduce that the moment of the mass about L is 0. Thus all balancing lines for the mass pass through a single point. Any two of them therefore determine that point, which is called the center of mass. It is customary to use the two lines parallel to the x and y axes to determine that point.

30.[M] (See Exercise 29.) Show that the moment of a mass occupying a solid region R about any plane through its center of mass is 0.

31.[C] This exercise concerns hydrostatic pressure. (See Section 7.6.)

(a) Show that the pressure of water against a submerged vertical surface occupying the plane region R equals the pressure at the centroid of R times the area of R.

(b) Is the assertion in (a) correct if R is not vertical?

In each of Exercises 32 to 39 find the center of mass of the lamina occupying the given region and having the given density.

32.[R] The triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$; density at (x, y) is $x + y$.

33.[R] The triangle with vertices $(0, 0)$, $(2, 0)$, $(1, 1)$; density at (x, y) is y.

34.[R] The square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$; density at (x, y) equals to $y \arctan(x)$.

35.[R] The finite region bounded by $y = 1 + x$ and $y = 2^x$; density at (x, y) is $x + y$.

36.[R] The triangle with vertices $(0, 0)$, $(1, 2)$, $(1, 3)$; density at (x, y) is xy.

37.[R] The finite region bounded by $y = x^2$, the x-axis, and $x = 2$; density at (x, y) is e^x

38.[R] The finite region bounded by $y = x^2$ and $y = x + 6$, situated to the right of the y-axis; density at (x, y) is $2x$.

39.[R] The trapezoid with vertices $(0, 0)$, $(3, 0)$, $(2, 1)$, $(0, 1)$; density at (x, y) is $\sin(x)$.

40.[C] Let R be a region in a plane and P a point a distance $h > 0$ from the plane. P and R determine a cone with base R and vertex P, as shown in Figure 17.8.15. Let the area of R be A. What can be said about the distance of the centroid of the cone from the plane of R?

(a) What is that distance in the case of a right circular cone?

(b) Experiment with another cone with any convenient base of your choice.

(c) Make a conjecture.

(d) Explain why the conjecture is true.

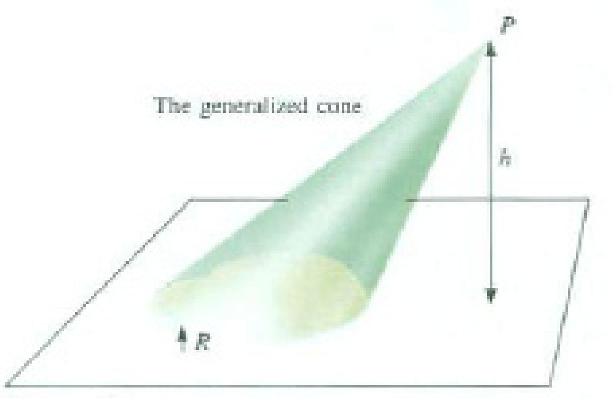


Figure 17.8.15:

In Exercises 41 and 42 find $\overline{z}$ for the given surfaces.

41.[M] The portion of the paraboloid $2z = x^2 + y^2$ below the plane $z = 9$.

42.[M] The portion of the plane $x + 2y + 3z = 6$ above the triangle in the xy plane whose vertices are $(0, 0)$, $(4, 0)$, and $(0, 1)$.

43.[R] In a letter of 1680 Leibniz wrote:

> Huygens, as soon as he had published his book on the pendulum, gave me a copy of it; and at that time I was quite ignorant of Cartesian algebra and also of the method of indivisibles, indeed I did not know the correct definition of the center of gravity. For, when by chance I spoke of it to Huygens, I let him know that I thought that a straight line drawn through the center of gravity always cut a figure into two equal parts; since that clearly happened in the case of a square, or a circle, an ellipse, and other figures that have a center of magnitude. I imagine that it was the same for all other figures. Huygens laughed when he heard this, and told me that nothing was further from the truth.

(Quoted in C.H. Edwards, *The Historical Development of the Calculus*, p. 239, Springer-Verlag, New York, 1979.)
Give an example showing that "nothing is further from the truth."

44.[R] Let a be a constant that is not less than 1. Let R be the region below $y = x^a$, above the x-axis, and between the lines $x = 0$ and $x = 1$.

(a) Sketch R for a large value of a.

(b) Compute the centroid $(\overline{x}, \overline{y})$ of R.

(c) Find $\lim_{a\to\infty} \overline{x}$ and $\lim_{a\to\infty} \overline{y}$.

(d) For large a, does the centroid of R lie in R?

45.[C] (Contributed by Jeff Lichtman) Let f and g be two continuous functions such that $f(x) \geq g(x) \geq 0$ for x in $[0,1]$. Let R be the region under $y = f(x)$ and above $[0,1]$; let R^* be the region under $y = g(x)$ and above $[0,1]$.

(a) Do you think the center of mass of R is at least as high as the center of mass of R^*? (An opinion only.)

(b) Let $g(x) = x$. Define $f(x)$ to be $\frac{1}{3}$ for $0 \leq x \leq \frac{1}{3}$ and $f(x)$ to be x if $\frac{1}{3} \leq x \leq 1$. (Note that f is continuous.) Find $\overline{y}$ for R and also for R^*. (Which is larger?)

(c) Let a be a constant, $0 \leq a \leq 1$. Let $f(x) = a$ for $0 \leq x \leq a$ and let $f(x) = x$ for $a \leq x \leq 1$. Find $\overline{y}$ for R.

(d) Show that the number a for which $\overline{y}$ defined in part (c) is a minimum is a root of the equation $x^3 + 3x - 1 = 0$.

(e) Show that the equation in (d) has only one real root q.

(f) Find q to four decimal places.

46.[M] This exercise shows that the three medians of a triangle meet at the centroid of the triangle. (A **median** of a triangle is a line that passes through a vertex and the midpoint of the opposite edge.)
Let R be a triangle with vertices A, B, and C. It suffices to show that the centroid of R lies on the median through C and the midpoint M of the edge AB. Introduce an xy coordinate system such that the origin is at A, and B lies on the x-axis, as in Figure 17.8.16.

(a) Compute $(\overline{x}, \overline{y})$.

(b) Find the equation of the median through C and M.

(c) Verify that the centroid lies on the median computed in (b).

(d) Why would you expect the centroid to lie on each median? (Just use physical intuition.)

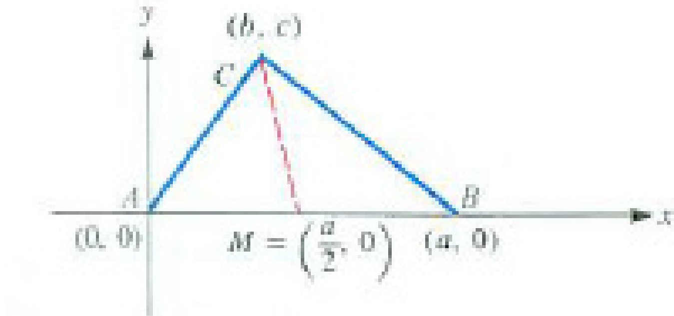


Figure 17.8.16:

47.[R] Cut an irregular shape out of cardboard and find three balancing lines for it experimentally. Are they concurrent; that is, do they pass through a common point?

48.[R] Let f and g be continuous functions such that $f(x) \geq g(x) \geq 0$ for x in $[a,b]$. Let R be the region above $[a,b]$ which is bounded by the curves $y = f(x)$ and $y = g(x)$.

(a) Set up a definite integral (in terms of f and g) for the moment of R about the y-axis.

(b) Set up a definite integral with respect to x (in terms of f and g) for the moment of R about the x-axis.

In Exercises 49 to 52 find (a) the moment of the given region R about the y-axis, (b) the moment of R about the x-axis, (c) the area of R, (d) $\overline{x}$, (e) $\overline{y}$. Assume the density is 1. (See Exercise 48.)

is set of
ed from
on 17.4.

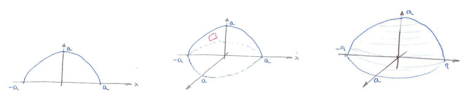
49.[R] R is bounded by the curves $y = x^2$, $y = x^3$.

50.[R] R is bounded by $y = x$, $y = 2x$, $x = 1$, and $x = 2$.

51.[R] R is bounded by the curves $y = 3^x$ and $y = 2^x$ between $x = 1$ and $x = e$.

52.[R] (Use a table of integrals or techniques from Chapter 8.) R is bounded by the curves $y = x - 1$ and $y = \ln(x)$, between $x = 1$ and $x = e$.

53.[M] Which do you think would have the highest centroid? The semicircular wire of radius a, shown in Figure 17.8.17(a); the top half of the surface of a ball of radius a, shown in Figure 17.8.17(b); the top half of a ball of radius a, shown in Figure 17.8.17(c).



(a) (b) (c)

Figure 17.8.17:

54.[C] Consider the parabolic surface $z = x^2 + y^2$ below the plane za^2.

(a) Set up a double integral in the xy-plane for the moment about the xy plane.

(b) Express this integral as an iterated integral in polar coordinates.

(c) Evaluate the integral.

(d) Find the centroid of the (cu

Exercises 55 to 58 concern the mom
that if the object is homogeneous, h
ume V, its density $\delta(P)$ is M/V.

55.[R] A homogeneous rectangular solid box has mass M and sides of lengths a, b, and c. Find its moment of inertia about an edge of length a.

56.[R] A rectangular homogeneous box of mass M has dimensions a, b and c. Show that the moment of inertia of the box about a line through its center and parallel to the side of length a is $M(b^2 + c^2)/12$.

57.[R] A right solid circular cone has altitude h, radius a, constant density, and mass M.

(a)

(b)

58.[R]
point i
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ular li
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59.[C] [An exercise showing that
does not depend on the choice of c

17.S Chapter Summary

This chapter generalizes the notion of a definite integral over an interval to integrals over plane sets, surfaces, and solids. These definitions are almost the same, the integral of $f(P)$ over a set being the limit of sums of the form $\sum f(P_i)\ \Delta A_i$, $\sum f(P_i)\ \Delta S_i$, or $\sum f(P_i)\ \Delta V_i$ for integrals over plane sets, surfaces, or solids, respectively.

If $f(P)$ denotes the density at P, then in each case, the integrals give the total mass.

The average value concept extends easily to functions of several variables. For instance, if $f(P)$ is defined on some plane region R, its average value over R is defined as

$$\frac{1}{area(R)} \int\limits_R f(P)\ dA.$$

Sometimes these "multiple integrals" (also known as "double" or "triple" integrals) can be calculated by repeated integrations over intervals, that is, as "iterated integrals." This requires a description of the region in an appropriate coordinate system and replaces dA or dV by an expression based on the area or volume of a small patch swept out by small changes in the coordinates, as recorded in Table 17.S.1.

Coordinate System	Substitution
Rectangular (2-d)	$dA = dx\ dy$
Rectangular (3-3)	$dV = dx\ dy\ dz$
Polar	$dA = r\ dr\ d\theta$
Cylindrical	$dV = r\ dr\ d\theta\ dz$
Cylindrical (surface)	$dS = r\ d\theta\ dz$
Spherical	$dV = \rho^2 \sin(\phi)\ d\phi\ d\rho\ d\theta$
Spherical (surface)	$dS = \rho^2 \sin(\phi)\ d\phi\ d\theta$

Table 17.S.1:

An integral over a surface S, $\int_S f(P)\ dS$, can often be replaced by an integral over the projection of S onto a plane R, replacing dS by $dA\cos(\gamma)$, where γ is the angle between a normal to S and a normal to R.

To remember these formulas, be able to draw Figures 17.S.1 and 17.S.2, including the labels.

EXERCISES for 17.S *Key:* R–routine, M–moderate, C–challenging

1.[R] The temperature at the point (x, y) at time t is $T(x, y, t) = e^{-tx}\sin(x + 3y)$. Let $f(t)$ be the average temperature in the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi/2$ at time t. Find df/dt.

2.[R] Let f be a function such that $f(-x, y) =$

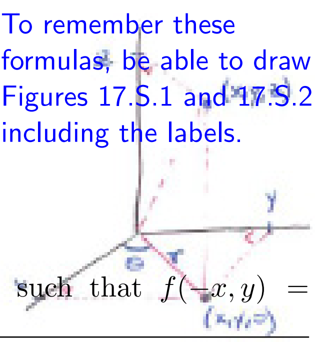


Figure 17.S.2:

Figure 17.S.1:

Key Facts

Formula	Significance
$\int_R 1\ dA$	Area of R
$\int_R 1\ dV$	Volume of R
$\frac{\int_R f(P)\ dA}{\text{Area of } R}$ or $\frac{\int_R f(P)\ dV}{\text{Volume of } R}$	Average value of f over R
$\int_R \sigma(P)\ dA$ or $\int_R \delta(P)\ dV$	Total mass of R, M (σ and δ denote density)
$\int_R y\sigma(P)\ dA$, $\int_R x\sigma(P)\ dA$	Moments, M_x and M_y about x and y axes, respectively. (A moment can be computed around any line in the plane.)
$\int_R f(P)\sigma(P)\ dA$, $\int_R f(P)\sigma(P)\ dV$ where $f(P)$ is the square of the distance from P to some fixed line L	Moment of inertia around L for planar and solid regions, respectively.
$\int_R x^2\sigma(P)\ dA$, $\int_R y^2\sigma(P)\ dA$	Second moments, M_{xx} and M_{yy} about x and y axes, respectively.
$\left(\frac{M_y}{M}, \frac{M_x}{M}\right)$	Center of mass, $(\bar{x}, \bar{y})$
$\int_R z\delta(P)\ dV$	Moment M_{xy}
$\int_R y\delta(P)\ dV$	Moment M_{xz}
$\int_R x\delta(P)\ dV$	Moment M_{yz}
$\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$	Center of mass of solid, $(\bar{x}, \bar{y}, \bar{z})$

Table 17.S.2:

Relations Between Rectangular Coordinates and Spherical or Cylindrical Coordinates

$$\begin{array}{ll} x = \rho\sin(\phi)\cos(\theta) & x = r\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) & y = r\sin(\theta) \\ z = \rho\cos(\phi) & z = z \end{array}$$

Table 17.S.3:

$-f(x, y)$.

(a) Give some examples of such functions.

(b) For what type regions R in the xy plane is $\int_R f(x, y)\, dA$ certainly equal to 0?

3.[R] Find $\int_R (2x^3y^2 + 7)\, dA$ where R is the square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$. Do this with as little work as possible.

4.[R] Let $f(x, y)$ be a continuous function. Define $g(x)$ to be $\int_R f(P)\, dA$, where R is the rectangle with vertices $(3, 0)$, $(3, 5)$, $(x, 0)$, and $(x, 5)$, $x > 3$. Express dg/dx as a suitable integral.

5.[R] Let R be a plane lamina in the shape of the region bounded by the graph of the equation $r = 2a\sin(\theta)$ $(a > 0)$. If the variable density of the lamina is given by $\sigma(r, \theta) = \sin(\theta)$, find the center of mass R.

In Exercises 6 to 9 find the moment of inertia of a homogeneous lamina of mass M of the given shape, around the given line.

6.[R] A disk of radius a, about the line perpendicular to it through its center.

7.[R] A disk of radius a, about a line perpendicular to it through a point on the circumference.

8.[R] A disk of radius a, about a diameter.

9.[R] A disk of radius a, about a tangent.

10.[C] Let $\mathcal{S}$ be the sphere of radius a and center at the origin. The integral $\int_{\mathcal{S}} (xz + y^2) dS$ can be done with little effort.

(a) Why is $\int_{\mathcal{S}} xz dS = 0$?

(b) Why is $\int_{\mathcal{S}} x^2 dS = \int_{\mathcal{S}} y^2 dS = \int_{\mathcal{S}} z^2 dS$?

(c) Why is $\int_{\mathcal{S}} y^2 dS = \int_{\mathcal{S}} (a^2/3) dS$?

(d) Show that $\int_{\mathcal{S}} (xz + y^2) dS = 4\pi a^2/3$.

11.[C] Let $f(P)$ and $g(P)$ be continuous functions defined on the plane region R.

(a) Show that

$$\left(\int_R f(P)g(P)\, dA\right)^2 \leq \left(\int_R f(P)^2\, dA\right)\left(\int_R g(P)^2\, dA\right).$$

HINT: Review the proof of the Cauchy-Schwarz inequality presented in the CIE on Average Speed and Class Size on page 600.

(b) Show that if equality occurs in the inequality in (a), then f is a constant times g.

12.[C] (Courtesy of G. D. Chakerian.) A solid region $\mathcal{S}$ is bounded below by the $x - y$ plane, above by the surface $z = f(P)$, and the sides by the surface of a cylinder, as shown in Figure 17.S.3.

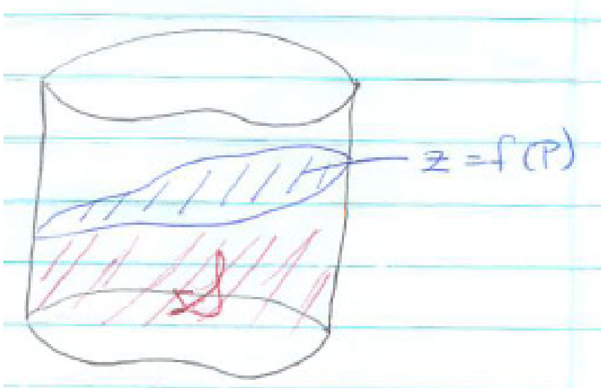


Figure 17.S.3:

The volume of $\mathcal{S}$ is V. If V is fixed, show that the top surface that minimizes the height of the centroid of $\mathcal{S}$ is a horizontal plane. NOTE: Water in a glass illustrates this, for nature minimizes the height of the centroid of the water. HINT: See Exercise 11.

Exercises 13 to 19 explore the average distance for all points on a curve or in a region. Recall that the distance from a point to a curve is the shortest distance from the point to the curve.

13.[M] Find the average distance from points in a disk of radius a to the center of the disk.

(a) Set up the pertinent definite integral in rectangular coordinates.

(b) Set it up in polar coordinates.

(c) Evaluate the easier integral in (a) and (b).

14.[M] Find the average distance from points in a square of side a to the center of the square.

(a) Set up the pertinent definite integral in rectangular coordinates.

(b) Set it up in polar coordinates.

(c) Evaluate the easier integral in (a) and (b).

15.[M] Find the average distance from points in a ball of radius a to the center of the ball.

(a) Set up the pertinent definite integral in rectangular coordinates.

(b) Set it up in spherical coordinates.

(c) Evaluate the easier integral in (a) and (b).

16.[M] Find the average distance from points in a cube of side a to the center of the cube.

(a) Set up the pertinent definite integral in rectangular coordinates.

(b) Set it up in polar coordinate

(c) Evaluate the easier integral i

17.[M] Find the average distanc square of side a to the border of th

(a) Set up the pertinent definite gular coordinates.

(b) Set it up in polar coordinate

(c) Evaluate the easier integral i

18.[M] Find the average distance a disk of radius a to the circular b

(a) Before doing any calculatio the average distance is great than $a/2$. Explain how you r

(b) Carry out the calculation usi ordinate system.

19.[C] Let A and B be two poi A curve (in the xy-plane) consists that the sum of the distances fron B is constant, say $2a$. Consider t to A as a function of arclength on average of that distance.

Calculus is Everywhere # 22

Solving the Wave Equation

In the *The Wave in a Rope* Calculus is Everywhere in the previous chapter we encountered the partial differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}. \tag{C.22.1}$$

Now we will solve this equation to find y as a function of x and t. First, we solve some simpler equations, which will help us solve (C.22.1).

EXAMPLE 6 Let $u(x, y)$ satisfy the equation $\partial u/\partial x = 0$. Find the form of $u(x, y)$.

SOLUTION Since $\partial u/\partial x$ is 0, $u(x, y)$, for a fixed value of y, is constant. Thus, $u(x, y)$ depends only on y, and can be written in the form $h(y)$ for some function h of a single variable.

On the other hand, any function $u(x, y)$ that can be written in the form $h(y)$ has the property that $\partial u/\partial x = 0$ is any function that can be written as a function of y alone. ◇

EXAMPLE 7 Let $u(x, y)$ satisfy

$$\frac{\partial^2 u}{\partial x \partial y} = 0. \tag{C.22.2}$$

Find the form of $u(x, y)$.

SOLUTION We know that

$$\frac{\partial\left(\frac{\partial u}{\partial y}\right)}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = 0.$$

By Example 6,

$$\frac{\partial u}{\partial y} = h(y) \qquad \text{for some function } h(y).$$

By the Fundamental Theorem of Calculus, for any number b,

$$u(x, b) - u(x, 0) = \int_0^b \frac{\partial u}{\partial y} dy = \int_0^b h(y) dy.$$

Let H be an antiderivative of h. Then

$$u(x,b) - u(x,0) = H(b) - H(0).$$

Replacing b by y shows that

$$u(x,y) = u(x,0) + H(y) - H(0).$$

That tells us that $u(x,y)$ can be expressed as the sum of a function of x and a function of y,

$$u(x,y) = f(x) + g(y). \tag{C.22.3}$$

◇

We will solve the wave equation (C.22.1) by using a suitable change of variables that transforms that equation into the one solved in Example 7.

The new variables are

$$p = x + ct \qquad \text{and} \qquad q = x - ct.$$

$x = \frac{1}{2}(p+q)$ and $t = \frac{1}{2c}(p-q)$.

One could solve these equations and express x and t as functions of p and q. We will apply the chain rule, where y is a function of p and q and p and q are functions of x and t, as indicated in Figure C.22.1. Thus $y(x,t) = u(p,q)$.

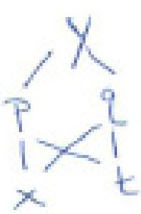


Figure C.22.1:

Keeping in mind that

$$\frac{\partial p}{\partial x} = 1, \qquad \frac{\partial p}{\partial t} = c, \qquad \frac{\partial q}{\partial x} = 1, \qquad \text{and } \frac{\partial q}{\partial t} = -c,$$

we have

$$\frac{\partial y}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} = \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}.$$

Then

$$\begin{aligned}
\frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}\right) \\
&= \frac{\partial}{\partial p}\left(\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}\right)\frac{\partial p}{\partial x} + \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}\right)\frac{\partial q}{\partial x} \\
&= \left(\frac{\partial^2 u}{\partial p^2} + \frac{\partial^2 u}{\partial p \partial q}\right)\cdot 1 + \left(\frac{\partial^2 u}{\partial q \partial p} + \frac{\partial^2 u}{\partial^2 q}\right)\cdot 1.
\end{aligned}$$

Thus

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 u}{\partial p^2} + 2\frac{\partial^2 u}{\partial p \partial q} + \frac{\partial^2 u}{\partial q^2}. \tag{C.22.4}$$

A similar calculation shows that

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial p^2} - 2\frac{\partial^2 u}{\partial p \partial q} + \frac{\partial^2 u}{\partial q^2}\right). \tag{C.22.5}$$

Substituting (C.22.4) and (C.22.5) in (C.22.1) leads to

$$\frac{\partial^2 u}{\partial p^2} + 2\frac{\partial^2 u}{\partial p \partial q} + \frac{\partial^2 u}{\partial q^2}. = \frac{1}{c^2}\left(c^2\right)\left(\frac{\partial^2 u}{\partial p^2} - 2\frac{\partial^2 u}{\partial p \partial q} + \frac{\partial^2 u}{\partial q^2}\right),$$

which reduces to

$$4\frac{\partial^2 u}{\partial p \partial q} = 0.$$

By Example 7, there are function $f(p)$ and $g(q)$ such that

$$y(x,t) = u(p,q) = f(p) + g(q).$$

or

$$y(x,t) = f(x+ct) + g(x-ct). \tag{C.22.6}$$

The expression (C.22.6) is the most general solution of the wave equation (C.22.1).

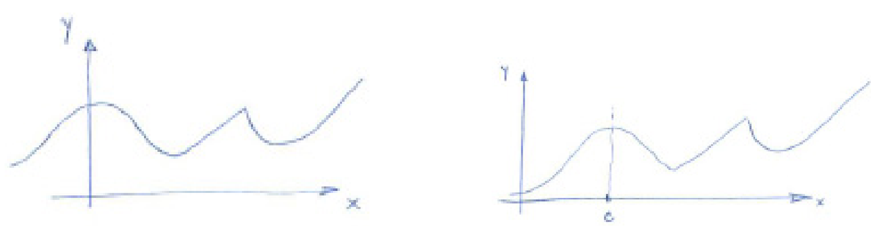
What does a solution (C.22.6) look like? What does the constant c tell us? To answer these questions, consider just

$$y(x,t) = g(x-ct). \tag{C.22.7}$$

Here t represents time. For each value of t, $y(x,t) = g(x-ct)$ is simply a function of x and we can graph it in the xy plane. For $t=0$, (C.22.7) becomes

$$y(x,0) = g(x).$$

That is just the graph of $y = g(x)$, whatever g is, as shown in Figure C.22.2(a).



(a) (b)

Figure C.22.2: (a) $t=0$, (b) $t=1$.

Now consider $y(x,t)$ when $t=1$, which we may think of as "one unit of time later." Then

$$y = y(x,1) = g(x - c\cdot 1) = g(x-c).$$

The value of $y(x, 1)$ is the same as the value of g at $x - c$, c units to the left of x. So the graph at $t = 1$ is the graph of f in Figure C.22.2(a) shifted to the *right* c units, as in Figure C.22.2(b).

As t increases, the initial "wave" shown in Figure C.22.2(a) moves further to the right at the constant speed, c. Thus c tells us the velocity of the moving wave. That fact will play a role in Maxwell's prediction that electro-magnetic waves travel at the speed of light, as we will see in the Calculus is Everywhere at the end of Chapter 18.

EXERCISES

1.[R] Which functions $u(x, y)$ have both $\partial u/\partial x$ and $\partial u/\partial y$ equal to 0 for all x and y?

2.[R] Let $u(x, y)$ satisfy the equation $\partial^2 u/\partial x^2 = 0$. Find the form of $u(x, y)$.

3.[R] Show that any function of the form (C.22.3) satisfies equation (C.22.2).

4.[R] Verify that any function of the form (C.22.6) satisfies the wave equation.

5.[M] We interpreted $y(x, t) = g(x - ct)$ as the description of a wave moving with s
Interpret the equation $y(x, t) = f($

6.[M] Let k be a positive constan

(a) What are the solutions to th

$$\frac{\partial^2 y}{\partial x^2} = k\frac{\partial^2}{\partial t}$$

(b) What is the speed of the "w