Chapter 18

The Theorems of Green, Stokes, and Gauss

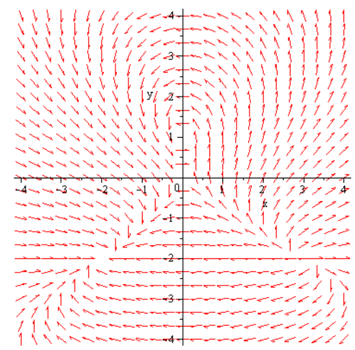


Figure 18.0.1:

Imagine a fluid or gas moving through space. Its density may vary from point to point. Also its velocity vector may vary from point to point. Figure 18.0.1 shows a typical situation. The diagram shows a flow in the plane because it's easier to sketch and show the vectors there than in space.

The figure resembles the slope fields you saw in Section 3.6 but now, instead of short segments, we have vectors, which may be long, and point in a definite direction. This particular flow has a whirlpool. The speed tends to increase from left to right. On the left, fluid is entering the region bounded by the dashed loop. On the right, it tends to leave that region. Several questions come to mind:

- Is the amount of fluid in the region bounded by the dashed curve increasing or decreasing (or unchanged)?
- In (three-dimensional) space, the similar question asks:
  Is the amount of fluid in a region bounded by a given surface tending to increase, decrease, or remain constant?
- At a given point how strong is the tendency of the fluid to rotate?
  In other words, if we put a little propeller in the fluid would it turn? If so, in which direction, and how fast or slow?

Chapter 18 provides techniques for answering these questions. These techniques will apply more generally, to a general "vector field." Applications come from magnetics as well as fluid flow.

There is another way to look at this chapter.

The Fundamental Theorem of Calculus relates the integral of a function f on an interval, $[a, b]$, to the behavior of another function, an antiderivative of

f, F, at the boundary of the interval:

$$\int_a^b f(x)\ dx = F|_b^a = F(b) - F(a).$$

We will discover three different relations between integrals of certain functions over regions to integrals of other functions over their boundaries. We will see how these ideas are important in the study of gravity, magnetism, and fluid flow. Throughout we assume that all partial derivatives of the first and second orders exist and are continuous.

18.1 Conservative Vector Fields

SHERMAN: We did not use "differential form" in Section 15.3. I had to modify this introduction.

In Section 15.3 we defined integrals of the form

$$\int_C (P\ dx + Q\ dy + R\ dz). \tag{18.1.1}$$

where P, Q, and R are scalar functions of x, y, and z and C is a curve in space. Similarly, in the xy plane, for scalar functions of x and y, P and Q, we have

$$\int_C (P\ dx + Q\ dy).$$

Instead of three scalar fields, P, Q, and R, we could think of a single vector function $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$. Such a function is called a **vector field**, in contrast to a scalar field. It's hard to draw a vector field defined in space. However, it's easy to sketch one defined only on a plane. Figure 18.1.1 shows part of a vector field that has two whirlpools.

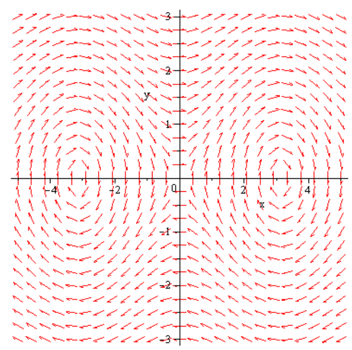


Figure 18.1.1:

SHERMAN: Better: a weather map with winds. Source?

Introducing the formal vector $\overrightarrow{dr} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, we may rewrite (18.1.1) as

$$\int_C \mathbf{F} \cdot\ dr.$$

The vector notation is compact and emphasizes the idea of a vector field. However, the clumsy notations

$$\int_C (P\ dx + Q\ dy + R\ dz) \qquad \text{and} \qquad \int_C (P(x, y, z)\ dx + Q(x, y, z)\ dy + R(x, y, z)\ dz):$$

do have two useful purposes: to prove theorems and to carry out calculations. We use one of these forms on those occasions we have a need to refer to the individual scalar components of F.

Conservative Vector Fields

Recall the definition of a conservative vector field from Section 15.3.

> **DEFINITION** *(Conservative Field)* A vector field $\mathbf{F}$ defined in some planar or spatial region is called **conservative** if
>
> $$\int_{C_1} \mathbf{F} \cdot\ dr = \int_{C_2} \mathbf{F} \cdot\ dr$$
>
> whenever C_1 and C_2 are any two simple curves in the region with the same initial and terminal points.

Another way to view "conservative"

An equivalent definition of a conservative vector field $\mathbf{F}$ is that for any simple closed curve C in the region $\oint_C \mathbf{F}\cdot\ dr = 0$, as Theorem 18.1.1 implies. A closed curve is a curve that begins and ends at the same point, forming a loop. It is simple if it passes through no point — other than its start and finish points — more than once. A curve that starts at one point and ends at a different point is simple if it passes through each point just once. Figure 18.1.2 shows some curves that are simple and some that are not.

The symbol $\oint$ indicates "closed curve".

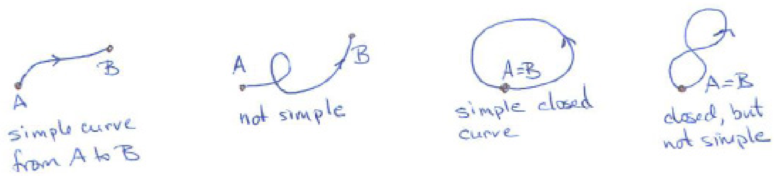


Figure 18.1.2:

Theorem 18.1.1. *A vector field* $\mathbf{F}$ *is conservative if and only if* $\oint_C \mathbf{F}\cdot\ dr = 0$ *for every simple closed curve in the region where* $\mathbf{F}$ *is defined.*

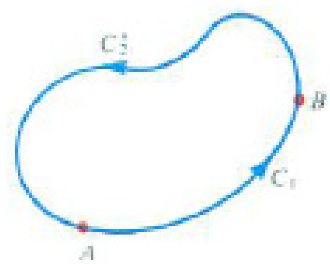


Figure 18.1.3:

Proof

Assume that $\mathbf{F}$ is a conservative and let C be simple closed curve that starts and ends at the point A. Pick a point B on the curve and break C into two curves:C_1 from A to B and $C+2^*$ from B to A, as indicated in Figure 18.1.3.

Let C_2 be the curve C_2^* traversed in the opposite direction, from A to B. Then, since $\mathbf{F}$ is conservative,

$$\begin{aligned} \oint\limits_C \mathbf{F}\cdot\ dr &= \int\limits_{C_1} \mathbf{F}\cdot\ dr + \int\limits_{C_2^*} \mathbf{F}\ dr \\ &= \int\limits_{C_1} \mathbf{F}\cdot\ dr - \int\limits_{C_2} \mathbf{F}\cdot\ dr = 0. \end{aligned}$$

Note the sign change.

On the other hand, assume that $\mathbf{F}$ has the property that $\oint_C \mathbf{F}\cdot\ dr = 0$ for any simple closed curve C in the region. Let C_1 and C_2 be two simple curves in the region, starting at A and ending at B. Let C_2 be C_2 taken in the reverse direction. (See Figure 18.1.4.) Then C_1 followed by C_2 is a closed curve C from A back to A. Thus

$$0 = \oint\limits_C \mathbf{F}\cdot\ dr = \int\limits_{C_1} \mathbf{F}\cdot\ dr + \int\limits_{C_2} \mathbf{F}\cdot\ dr = \int\limits_{C_1} \mathbf{F}\cdot\ dr - \int\limits_{C_2} \mathbf{F}\cdot\ dr.$$

(a) (b)

Figure 18.1.4:

Consequently,

$$\int_{C_1} \mathbf{F}\cdot\ dr = \int_{C_2} \mathbf{F}\cdot\ dr.$$

This conclude both directions of the argument. •

In this proof we tacitly assumed that C_1 and C_2 overlap only at their endpoints, A and B. Exercise 25treats the case when the curves intersect elsewhere also.

Every Gradient Field is Conservative

To Doug: We define "-C" back in earlier chapter? SHERMAN: Yes, in Section 15.3.

Whether a particular vector field is conservative is important in the study of gravity, electro-magnetism, and thermodynamics. In the rest of this section we describe says to determine whether a vector field $\mathbf{F}$ is conservative.

The first method that may come to mind is to evaluate $\oint \mathbf{F}\cdot d\mathbf{r}$ for every simple closed curve and see if it is always 0. If you find a case where it is not 0, then $\mathbf{F}$ is not conservative. Otherwise you face the task of evaluating a never-ending list of integrals checking to see if you always get 0. That is a most impractical test. The use of partial derivatives will help us obtain much simpler tests. The first test involves gradients.

Gradient Fields Are Conservative

The fundamental theorem of calculus asserts that $\int_a^b f'(x)\ dx = f(b) - f(a)$. The next theorem asserts that $\int_C \nabla f\cdot d\mathbf{r} = f(B)-f(A)$, where f is a function of two or three variables and C is a curve from A to B. Because of its resemblance to the fundamental theorem of calculus, Theorem 18.1.2 is sometimes called the **fundamental theorem of vector fields**.

Any vector field that is the gradient of a scalar field turns out to be conservative. That is the substance of Theorem 18.1.2, which says, "The circulation of a gradient field of a scalar function f along a curve is the difference in values of f at the end points."

Theorem 18.1.2. *Let F be a scalar field defined in some region in the plane or in space. Then the gradient field $\mathbf{F} = \nabla f$ is conservative. In fact, for any points A and B in the region,*

$$\int_C \nabla f \cdot \ dr = f(B) - f(A).$$

Proof

For simplicity take the planar case. Let C be given by the parameterization $\mathbf{r} = \mathbf{G}(t)$ for t in $[a, b]$. Let $\mathbf{G}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then,

$$\int_C \nabla f \cdot d\mathbf{r} \ = \ \int_C \left(\frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy \right) = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right)\ dt.$$

The integrand $(\partial f/\partial x)(dx/dt) + (\partial f/\partial y)(dy/dt)$ is reminiscent of the chain rule in Section 16.3. To be specific, if we introduce the function H defined by the formula

$$H(t) = f(x(t), y(t)),$$

then the chain rule gives

$$\frac{dH}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Thus

$$\int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right)\ dt = \int_a^b \frac{dH}{dt}\ dt = H(b) - H(a)$$

by the fundamental theorem of calculus. But

$$H(b) = f(x(b), y(b)) = f(B)$$

and

$$H(a) = f(x(a), y(a)) = f(A).$$

Consequently,

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A), \tag{18.1.2}$$

and the theorem is proved. •

In differential form Theorem 18.1.2 reads

> If f is defined as the xy-plane, and C starts at A and ends at B,
>
> $$\int_C \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\right) = f(B) - f(A) \tag{18.1.3}$$
>
> If f is defined in space, then,
>
> $$\int_C \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz\right) = f(B) - f(A). \tag{18.1.4}$$

Note that the one vector equation (18.1.2) covers both cases (18.1.3) and (18.1.4). This illustrates an advantage of vector notation.

It is a much more pleasant task to evaluate $f(B) - f(A)$ than to compute a line integral.

EXAMPLE 1 Let $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, which is defined everywhere except at the origin. (a) Find the gradient field $\mathbf{F} = \nabla f$, (b) Compute $\int_C \mathbf{F} \cdot\, d\mathbf{r}$ where C is any curve from $(1, 2, 2)$ to $(3, 4, 0)$.
SOLUTION (a) Straightforward computations show that

$$\frac{\partial f}{\partial x} = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{\partial f}{\partial y} = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{\partial f}{\partial z} = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}.$$

So

$$\nabla f = \frac{-z\mathbf{i} - y\mathbf{j} - z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}. \tag{18.1.5}$$

If we let $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = \|\mathbf{r}\|$, and $\widehat{\mathbf{r}} = \mathbf{r}/r$, then (18.1.5) can be written more simply as

$$\mathbf{F} = \nabla f = \frac{-\mathbf{r}}{r^3} = \frac{\widehat{\mathbf{r}}}{r^2}.$$

This vector field, $F = \widehat{\mathbf{r}}/r^2$, is called the **inverse square central field**. It will play an important role, in both two and three dimensions, in Chapter 18.

(b) For any curve C from $(1, 2, 2)$ to $(3, 4, 0)$,

$$\begin{aligned}\int_C \nabla f \cdot\, dr &= f(3, 4, 0) - f(1, 2, 2) = \frac{1}{\sqrt{3^2 + 4^2 + 0^2}} - \frac{1}{\sqrt{1^2 + 2^2 + 2^2}} \\ &= \frac{1}{5} - \frac{1}{3} = -\frac{2}{15}.\end{aligned}$$

◇

In this example $\|\nabla f\| = \frac{\|-vr\|}{r^3} = \frac{r}{r^3} = \frac{1}{r^2}$ and $\|f(x,y,z)\| = \frac{1}{r}$. In the study of gravity, ∇f measures gravitational attraction, and f measures "potential."

EXAMPLE 2 Evaluate $\oint_C (y\ dx + x\ dy)$ around a closed curve C taken counterclockwise.

SOLUTION In Section 15.3 it was shown that $\oint_C x\ dy = A$ and $\oint_C y\ dx = -A$, where A is the area of the region enclosed by C. Thus,

$$\oint_C (y\ dx + x\ dy) = -A + A = 0.$$

A second solution uses Green's theorem. Note that

$$\nabla(xy) = \frac{\partial(xy)}{\partial x}\mathbf{i} + \frac{\partial(xy)}{\partial y}\mathbf{j} = y\mathbf{i} + x\mathbf{j},$$

that is, the gradient of xy is $y\mathbf{i} + x\mathbf{j}$.

Hence, by Green's theorem,

$$\oint_C (y\ dx + x\ dy) = \oint_C \nabla(xy) \cdot d\mathbf{r} = xy|_A^B$$

where A and B are the endpoints of C. Because C is a closed curve, $A = B$ and so the integral is 0. ◇

SHERMAN: In V, you have many more stated theorems and corollaries. I think I prefer the approach we are using in VI, it has less to memorize and emphasizes a small number of fundamentals from which everything else can be derived.

A differential form $P(x,y,z)\ dx + Q(x,y,z)\ dy + R(x,y,z)\ dz$ is called **exact** if there is a scalar function f such that $P(x,y,z) = \partial f/\partial x$, $Q(x,y,z) = \partial f/\partial y$, and $R(x,y,z) = \partial f/\partial z$. In that case, the expression takes the form

$$\frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z}\ dz.$$

That is the same thing as saying that the vector field $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a gradient field: $\mathbf{F} = \nabla f$.

If F is Conservative Must It Be a Gradient Field?

The proof of the next theorem is similar to the proof that showed every continuous function has an antiderivative. We suggest you review that proof (page 1300) before reading the following proof.

The question may come to mind, *"If* $\mathbf{F}$ *is conservative, it is necessarily the gradient of some scalar function?"* The answer is "yes." This is the substance of the next theorem.

SHERMAN: Is "arcwise-connected" defined?

Theorem 18.1.3. *Let* **F** *be a conservative vector field defined in some arcwise-connected region in the plane (or in space). Then there is a scalar function* f *define in that region such that* $\mathbf{F} = \nabla f$.

Proof

Consider the case when **F** is planar, $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$. (The case where **F** is defined in space is similar.) Define a scalar function f as follows. Let (x,y) be a point in the region. Select a curve C in the region that starts at (a,b) and ends at (x,y).

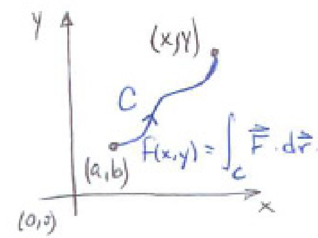


Figure 18.1.5:

Define $f(x,y)$ to be $\int_C \mathbf{F}\cdot\, d\mathbf{r}$. Since **F** is conservative, the number $f(x,y)$ depends only on the point (x,y) and not on the choice of C. (See Figure 18.1.5.)

All that remains is to show that $\nabla f = \mathbf{F}$; that is, $\partial f/\partial x = P$ and $\partial f/\partial y = Q$. We will go through the details for the first case, $\partial f/\partial x = P$. The reasoning for the other partial derivative is similar.

Let (x_0, y_0) be an arbitrary point in the region and consider the difference quotient whose limit is $\partial f/\partial x(x_0, y_0)$, namely,

$$\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h},$$

for h small enough so that $(x_0 + h, y_0)$ is also in the region.

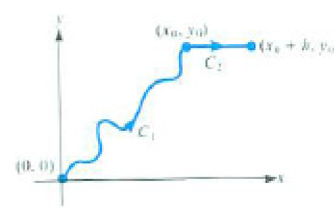


Figure 18.1.6:

Let C_1 be a curve from (a,b) to (x_0, y_0) and let C_2 be the straight path from (x_0, y_0) to $(x_0 + h, y_0)$. (See Figure 18.1.6.) Let C by the curve from $(0,0)$ to the point$(x_0 + h, y_0)$ formed by taking C_1 first and then continuing on C_2. Then

$$f(x_0, y_0) = \int_{C_1} \mathbf{F}\cdot\mathbf{r},$$

and

$$f(x_0+h, y_0) = \int_C \mathbf{F}\cdot\mathbf{r} = \int_{C_1} \mathbf{F}\cdot\mathbf{r} + \int_{C_2} \mathbf{F}\cdot\mathbf{r}.$$

Thus

$$\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \frac{\int_{C_2} \mathbf{F}\cdot\mathbf{r}}{h} = \frac{\int_{C_2} (P(x,y)\ dx + Q(x,y)\ dy)}{h}.$$

On C_2, y is constant, $y = y_0$; hence $dy = 0$. Thus $\int_{C_2} P(x,y)\ dy = 0$. Also,

$$\int_{C_2} P(x,y)\ dx = \int_{x_0}^{x_0+h} P(x, y_0)\ dx.$$

By the Mean-Value Theorem for definite integrals, there is a number x^* be-

See Section 6.3 for the MVT for Definite Integrals tween x_0 and $x_0 + h$ such that

$$\int_{x_0}^{x_0+h} P(x, y_0)\, dx = P(x^*, y_0)h.$$

Hence

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = \lim_{h\to 0} \frac{1}{h} \int_{x_0}^{x_0+h} P(x, y_0)\, dx = P(x_0, y_0).$$

Consequently,

$$\frac{\partial f}{\partial x}(x_0, y_0) = P(x_0, y_0),$$

as was to be shown.

In a similar manner, we can show that

$$\frac{\partial f}{\partial y}(x_0, y_0) = Q(x_0, y_0).$$

•

For a vector field **F** defined throughout some region in the plane (or space) the following three properties are therefore equivalent: In Figure 18.1.7, an

Three views of a conservative field

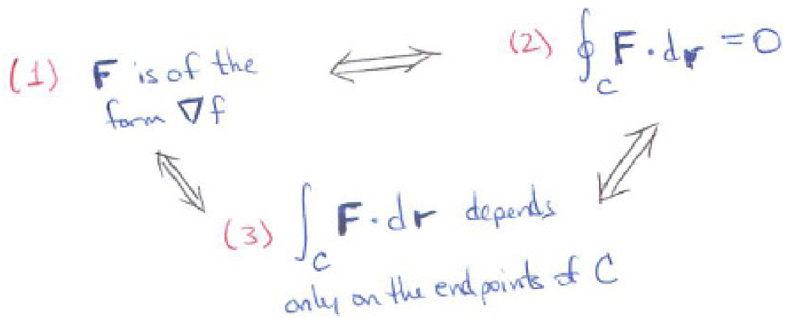


Figure 18.1.7:

arrow (⇒) means "implies" and a double arrow (⇔) means "if and only if" or "is equivalent to." This figure tells us that any one of the three properties, (1),(2), or (3), completely describes a conservative field. We used property (3) as the definition.

Almost A Test For Being Conservative

Figure 18.1.7 describes three ways of deciding whether a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + r\mathbf{k}$ is conservative. Now we give a simple way to tell that it is not conservative. The method is simpler than finding a particular line integral $\int_C \mathbf{F} \cdot\ d\mathbf{r}$ that is not 0.

Remember that we have assumed that all of the functions we encounter in this chapter have continuous first and second partial derivatives.

The test depends on the fact that the two order in which are may compute a second-order mixed partial derivative give the same result. (We used this fact in Section 16.8 in a thermodynamics context.)

Consider an expression of the form $P\ dx + Q\ dy + R\ dz$ (or equivalently a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$). If $\mathbf{F}$ is exact, then $\mathbf{F}$ is a gradient and there is a scalar function f such that

$$\frac{\partial f}{\partial x} = P, \qquad \frac{\partial f}{\partial y} = Q, \qquad \frac{\partial f}{\partial z} = R.$$

Since

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Similarly we find

$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \qquad \text{and} \qquad \frac{\partial P}{\partial z} = \frac{\partial P}{\partial x}.$$

These three equations can be rewritten as

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0, \qquad \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0, \qquad \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = 0. \tag{18.1.6}$$

If at least one of these three equations (18.1.6) doesn't hold, then $P\ dx + Q\ dy + R\ dz$ is not exact (and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is not conservative).

EXAMPLE 3 Show that $\cos(y)\ dx + \sin(xy)\ dy + \ln(1+x)\ dz$ is not exact.
SOLUTION Checking whether the first equation in (18.1.6) holds we compute

$$\frac{\partial(\sin(xy))}{\partial x} - \frac{\partial(\cos(y))}{\partial y},$$

which equals

$$y\cos(xy) + \sin(y),$$

which is not 0. There's no need to check the remaining two equations in (18.1.6). The expression $\sin(xy)\ dx + \cos(y)\ dy + \ln(1 + x)\ dz$ is not exact.

(Equivalently, the vector field $\sin(xy)\mathbf{i} + \cos(y)\mathbf{j} + \ln(1+x)\mathbf{k}$ is not a gradient field, hence not conservative.) ◇

We can restate the three equations (18.1.6) as a single vector equation, by introducing a 3 by 3 formal determinant

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix} \tag{18.1.7}$$

Expanding this as though the nine entries were numbers, we get

$$\mathbf{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \mathbf{j}\left(\frac{\partial R}{\partial x} - \frac{\partial R}{\partial z}\right) + \mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right). \tag{18.1.8}$$

If the three scalar equations in (18.1.8) hold, then (18.1.8) is the **0**-vector. In view of the importance of the vector (18.1.8), it is given a name.

> **DEFINITION** *(Curl of a Vector Field)* The **curl** of the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the vector field given by the formula (18.1.7) or (18.1.8). It is denoted $\nabla \times \mathbf{F}$.

The formal determinant (18.1.7) is like the one for the cross product of two vectors. For this reason, it is also denoted **curl F** (read as "del cross F"). That's a lot easier to write than (18.1.8), which refers to the components. Once again we see the advantage of vector notation.

The definition also applies to a vector field $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ in the plane. Writing $\mathbf{F}$ as $P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} + 0\mathbf{k}$, we find that

$$\nabla \times \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k},$$

since $\partial Q/\partial z = 0$ and $\partial P/\partial z = 0$. The physical meaning of $\nabla \times \mathbf{F}$ will be explored later in this chapter.

EXAMPLE 4 Compute the curl of $\mathbf{F} = xyz\mathbf{i} + x^2\mathbf{j} - xy\mathbf{k}$.
SOLUTION The curl of $\mathbf{F}$ is given by

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & x^2 & -xy, \end{pmatrix}$$

which is short for

$$\begin{aligned} &\left(\frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(x^2)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial z}(xyz)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial y}(xyz)\right)\mathbf{k} \\ &= (-x-0)\mathbf{i} - (-y-xy)\mathbf{j} + (2x-xz)\mathbf{k} \\ &= -x\mathbf{i} + (y+xy)\mathbf{j} + (2x-xz)\mathbf{k}. \end{aligned}$$

◇

If any case, for vector fields in space or in the xy-plane we have this theorem.

Theorem 18.1.4. *If* $\mathbf{F}$ *is a conservative vector field, then* $\nabla \times \mathbf{F} = \mathbf{0}$.

You may wonder why the vector field $\mathbf{curl}\,\mathbf{F}$ obtained from the vector field $\mathbf{F}$ is called the "curl of $\mathbf{F}$." Here we came upon the concept purely mathematically, but, as you will see in Section 18.5 it has a physical significance: If $\mathbf{F}$ describes a fluid flow, $\mathbf{curl}\,\mathbf{F}$, the curl of $\mathbf{F}$, describes the tendency of the fluid to form whirlpools.

The Converse of Theorem ?? Isn't True

Warning: The converse of Theorem 18.1.4 is false.

All would be delightful if the converse of Theorem th17-1-5 were true. Unfortunately, it is not. There are vector fields $\mathbf{F}$ whose curls are $\mathbf{0}$ that are not conservative. Example 5 provides one such $\mathbf{F}$. Its curl is $\mathbf{0}$ but it is not conservative.

EXAMPLE 5 Let $\mathbf{F} = \frac{-y\mathbf{i}}{x^2+y^2} + \frac{x\mathbf{j}}{x^2+y^2}$. Show that (a) $\nabla \times \mathbf{F} = \mathbf{0}$, but (b) $\mathbf{F}$ is not conservative.
SOLUTION (a) We must compute

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0, \end{pmatrix}$$

which equals

$$\left(\frac{\partial(0)}{\partial y} - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right))\right)\mathbf{i} - \left(\frac{\partial(0)}{\partial x} - \frac{\partial}{\partial z}\left((\frac{-y}{x^2+y^2}\right))\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)\mathbf{k}.$$

The $\mathbf{i}$ and $\mathbf{j}$ components are clearly 0, and a direct computation shows that the $\mathbf{k}$ component is

$$\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2} = 0.$$

Thus the curl of $\mathbf{F}$ is $\mathbf{0}$.

(b) To show that $\mathbf{F}$ is *not* conservative, it suffices to exhibit a closed curve C such that $\oint_C \mathbf{F} \cdot \, d\mathbf{r}$ is not 0. One such choice for C is the unit circle parameterized counterclockwise by

$$x = \cos(\theta), \qquad y = \sin(\theta), \qquad 0 \leq \theta \leq 2\pi.$$

On this curve $x^2 + y^2 = 1$; so $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$. Figure 18.1.8 shows a few values of f at points on C. Clearly $\int_C \mathbf{F} \cdot \, d\mathbf{r}$, which measures circulation, is

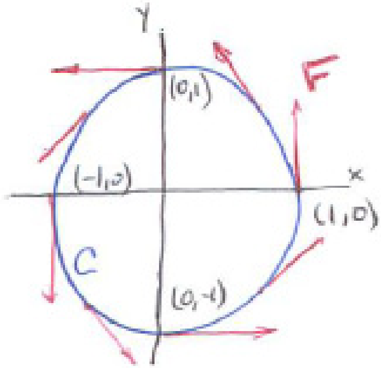


Figure 18.1.8:

positive, not 0. However, if you have any doubt, here is the computation of $\int_C \mathbf{F} \cdot\ d\mathbf{r}$:

$$\begin{aligned}
\oint\limits_C \mathbf{F} \cdot\ d\mathbf{r} &= \oint\limits_C \left(\frac{-y\ dx}{x^2+y^2} + \frac{x\ dy}{x^2+y^2} \right) \\
&= \int\limits_0^{2\pi} \left(\frac{-\sin\theta d(\cos\theta)}{\cos^2\theta+\sin^2\theta} + \frac{\cos\theta d(\sin\theta)}{\cos^2\theta+\sin^2\theta} \right) \\
&= \int\limits_0^{2\pi} \frac{(\sin^2\theta+\cos^2\theta)\ d\theta}{\sin^2\theta+\cos^2\theta} = \int\limits_0^{2\pi} d\theta = 2\pi.
\end{aligned}$$

This establishes (b). ◇

The curl of **F** being **0** is not enough to assure us that the vector field **F** is conservative. An extra condition must be satisfied by **F**. This condition concerns the domain of **F**. This extra assumption will be developed for planar fields in Section 18.2 and for spatial fields $\mathbf{F}(x, y, z)$ in Section 18.5. Then we will have a simple test for determining whether a vector field is conservative.

Summary

We showed that a vector field being conservative is equivalent to its being the gradient of a scalar field. Then we defined the curl of a vector field. If a field is denoted **F**, the curl of **F** is a new vector field denoted **curl F** or $\nabla \times \mathbf{F}$. If **F** is conservative, then $\nabla \times \mathbf{F}$, is **0**. However, if the curl of **F** is **0**, it does not follow that **F** is conservative. An extra assumption (on the domain of **F**) must be added. That assumption will be described later in this chapter.

EXERCISES for 18.1 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4 answer "True" or "False" and explain.

1.[R] "If **F** is conservative, the $\nabla \times \mathbf{F} = \mathbf{0}$."

2.[R] "If $\nabla \times \mathbf{F} = \mathbf{0}$, the **F** is conservative."

3.[R] "If **F** is a gradient field, then $\nabla \times \mathbf{F} = \mathbf{0}$."

4.[R] "If $\nabla \times \mathbf{F} = \mathbf{0}$, then **F** is a gradient field."

5.[R] Using information in this section, describe various ways of showing a vector field **F** is *not* conservative.

6.[R] Using information in this section, describe various ways of showing a vector field **F** *is* conservative.

7.[R] In Example 1 we computed a certain line integral by using the fact that the vector field $(-x\mathbf{i} - y\mathbf{j} - z\mathbf{k})/(x^2+y^2+z^2)^{3/2}$ is a gradient field. Compute that integral directly, without using the information that the field is a gradient.

8.[R] Let $f(x,y,z) = e^{3x}\ln(z+y^2)$. Compute $\int_C \nabla f \cdot\, d\mathbf{r}$, where C is the straight path from $(1,1,1)$ to $(4,3,1)$.

9.[R] We obtained the first of the three equations in (18.1.6). Derive the other two.

10.[R] Find the curl of $\mathbf{F}(x,y,z) = e^{x^2}yz\mathbf{i} + x^3\cos^2 3y\mathbf{j} + (1+x^6)\mathbf{k}$.

11.[R] Find the curl of $\mathbf{F}(x,y) = \tan^2(3x)\mathbf{i} + e^{3x}\ln(1+x^2)\mathbf{j}$.

12.[R] Using theorems of this section, explain why the curl of a gradient is **0**, that is, $\mathbf{curl}\,\nabla f = \mathbf{0}$ for a scalar function $f(x,y,z)$.

13.[R] By a computation, show that for the scalar function $f(x,y,z)$, $\mathbf{curl}\,\nabla f = \mathbf{0}$.

14.[R] Let $f(x,y) = \cos(x+y)$. Evaluate $\int_C \nabla f \cdot\, d\mathbf{r}$, where C is the curve that lies on the parabola $y = x^2$ and goes from $(0,0)$ to $(2,4)$.

15.[M] If $\mathbf{F}$ and $\mathbf{G}$ are conservative, is $\mathbf{F}+\mathbf{G}$?

16.[M] If $\mathbf{F}$ and $\mathbf{G}$ are conservative, is $\mathbf{F}\times\mathbf{G}$?

17.[M] Assume that $\mathbf{F}(x,y)$ is conservative. Let C_1 be the straight path from $(0,0,0)$ to $(1,0,0)$, C_2 the straight path from $(1,0,0)$ to $(1,1,1)$. If $\int_C \mathbf{F}\ d\mathbf{r} = 3$ and $\int_C \mathbf{F}\ d\mathbf{r} = 4$, what can be said about $\int_C \mathbf{F}\ d\mathbf{r}$, where C is the straight path from $(0,0,0)$ to $(1,1,1)$?

18.[M] Let $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$. (this is the vector field in Example 2.)

(a) Compute $\int_{C_1} \mathbf{F}\cdot\ d\mathbf{r}$ where C_1 goes from $(2,0)$ to $(-2,0)$ on the top half of the circle $x^2 + y^2 = 4$.

(b) Compute $\int_{C_2} \mathbf{F}\cdot\ d\mathbf{r}$ when C_2 goes from $(2,0)$ to $(-2,0)$ on the bottom half of the circle $x^2 + y^2 = 4$.

(c) The curl of $\mathbf{F}$ is $\mathbf{0}$, yet the integrals in (a) and (b) are different. Is this a contradiction?

19.[M] Let $\mathbf{F}(x,y)$ be a field that can be written in the form $\mathbf{F}(x,y) = g(\sqrt{x^2+y^2})\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ where g is a scalar function. If we denote $x\mathbf{i}+y\mathbf{j}$ as $\mathbf{r}$, then $\mathbf{F}(x,y) = g(r)\widehat{\mathbf{r}}$, where $r = \|\mathbf{r}\|$ and $\widehat{\mathbf{r}} = \|\mathbf{r}\|/r$. Show that $\oint_C \mathbf{F}\cdot\ d\mathbf{r} = 0$, for any path $ABCDA$ of the form shown in Figure 18.1.9. (The path consists of two circular arcs and parts of two rays from the origin.)

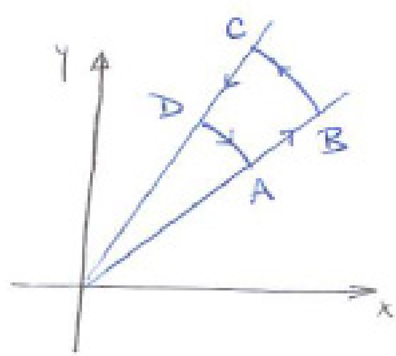


Figure 18.1.9:

20.[C] In view of the previous exercise, we may expect $\mathbf{F}(x,y) = g(\sqrt{x^2+y^2})\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ to be conservative. Show that it is by showing that $\mathbf{F}$ is the gradient of $\mathbf{G}(x,y) = H(\sqrt{x^2+y^2})$, where H is an antiderivative of g, that is, $H' = g$.

21.[C] The domain of a vector field $\mathbf{F}$ is all of the xy-plane. Assume that there are two points A and B such that $\int_C \mathbf{F}\, d\mathbf{r}$ is the same for all curves C from A to B. Deduce that $\mathbf{F}$ is conservative.

22.[C] A gas at temperature T_0 and pressure P_0 is brought to the temperature $T_1 > T_0$ and pressure $P_1 > P_0$. The work done in this process is given by the line integral in the TP plane

$$\int\limits_C \left(\frac{RT\ dP}{P} - R\ dT\right),$$

where R is a constant and C is the curve that records the various combinations of T and P during the process. Evaluate this integral over the following paths, shown in Figure 18.1.10.

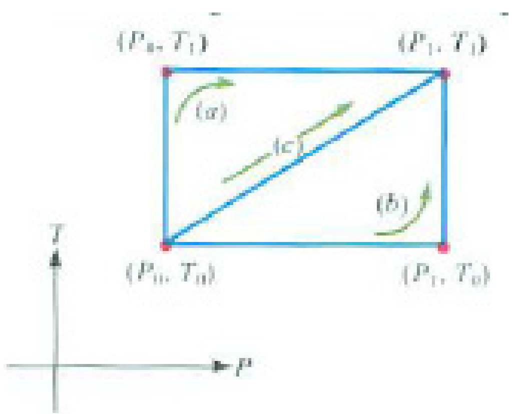


Figure 18.1.10:

(a) The pressure is kept constant at P_0 while the temperature is raised from T_0 to T_1; then the temperature is kept constant at T_1 while the pressure is raised from P_0 to P_1.

(b) The temperature is kept constant at T_0 while the pressure is raised from P_0 to P_1; then the temperature is raised from T_0 to T_1 while the pressure is kept constant at P_1.

(c) Both pressure and temperature are raised simultaneously in such a way that the path from (P_0, T_0) to (P_1, T_1) is straight.

Because the integrals are path dependent, the differential expression $RT\ dP/P - R\ dT$ defines a thermodynamic quantity that depends on the process, not just on the state. Vectorially speaking, the vector field $(RT/P)\mathbf{i} - R\mathbf{j}$ is not conservative.

23.[C] Assume that $\mathbf{F}(x, y)$ is defined throughout the xy-plane and that $\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r} = 0$ for every closed curve that can fit inside a disk of diameter 0.01. Show that $\mathbf{F}$ is conservative.

24.[C] This exercise completes the proof of Theorem 18.1.1 in the case when C_1 and C_2 overlap outside of their endpoints A and B. In that case; introduce a third simple curve from A to B that overlap C_1 and C_2 only at A and B. Then an argument similar to that in the proof of Theorem 18.1.1 can dispose of this case.

25.[M] In Theorem 18.1.1 we proved that $\partial f/\partial x = P$. Prove that $\partial f/\partial y = Q$.

26.[C] We proved that $\lim \frac{\int_{x_0}^{x_1+h} P(x,y_0)\ dx}{h}$ equals $P(x_0, y_0)$, by using the Mean Value Theorem for definite integrals. Find a new proof of this result that uses a Fundamental Theorem of Calculus.

18.2 Green's Theorem and Circulation

In this section we discuss a theorem that relates an integral of a vector field over a closed curve, C, to an integral of a related scalar function over the region, $\mathcal{R}$, whose boundary is C. We will also see what this means in terms of the circulation of a vector field.

Statement of Green's Theorem

Green's theorem in space is found in Section 18.3.

We begin by giving the statement of Green's theorem in the plane. We will explain each term used in this result, then see several applications of Green's theorem. The proof of Green's theorem can be found at the end of this section.

Theorem. *Green's Theorem in the Plane Let C be a simple, closed counterclockwise curve in the xy-plane, bounding a region R. Let P and Q be scalar functions defined at least on an open set containing R. Assume P and Q have continuous first partial derivatives. Then*

$$\oint_C (P\ dx + Q\ dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA.$$

SHERMAN: I've rewritten this, but does it really need to be here? Could it be deferred until the proof? Do you have a nice example to show what's possible when C is not simple? Maybe to be done as an exercise? Sam?

Recall, from Section 18.1 that a curve is closed when the curve starts and ends at the same point. It's simple when it does not intersect itself (except at the common endpoints). These restrictions on C ensure that this curve is the boundary of a region $\mathcal{R}$ in the xy-plane.

A set S in the xy-plane is "open" if for each point (x, y) in S there is a disk with center at (x, y) that is also in S. The disk that consists of all point (x, y) such that $x^2 + y^2$ is less than 1 is open. But the set that consists of that disk and its boundary is not open. (Why not?)

Now let $P(x, y)$ and $Q(x, y)$ be defined at least on an open set that contains the set R shown in Figure 18.1.1. That assures us that we can form the limits used in the definition of the partial derivatives of $P(x, y)$ and $Q(x, y)$ even for points on the boundary, namely, on C.

These details will be important when we give the proof of Green's Theorem at the end of the next section.

Since P and Q are independent of each other, Green's Theorem really consists of two theorems:

$$\int_C P\ dx = -\int_R \frac{\partial P}{\partial y}\ dA \qquad \text{and} \qquad \oint_C Q\ dy = \int_R \frac{\partial Q}{\partial x}\ dA. \tag{18.2.1}$$

EXAMPLE 1 In Section 15.3 we showed that if the counterclockwise curve C bounds a region $\mathcal{R}$, then $\oint_C y\ dx$ is the negative of the area of $\mathcal{R}$. Obtain this result with the aid of Green's Theorem.
SOLUTION Let $P(x, y) = y$, and $Q(x, y) = 0$. Then Green's Theorem says that

$$\oint_C y\ dx = -\int_R \frac{\partial y}{\partial y}\ dA.$$

Since $\partial y/\partial y = 1\partial$, it follows that $\oint y\ dx$ is $-\int_{\mathcal{R}} 1\ dA$, the negative of the area of $\mathcal{R}$. ◇

Green's Theorem and Circulation

What does Green's Theorem say about a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$? First of all, $\oint_C (P\ dx + Q\ dy)$ now becomes simply $\oint_C \mathbf{F} \cdot\ d\mathbf{r}$.

The right hand side of Green's Theorem looks a bit like the curl of a vector field in the plane. To be specific, we compute the curl of $\mathbf{F}$:

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P(x, y) & Q(x, y) &) \end{pmatrix} = 0\mathbf{i} - 0\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Thus the curl of $\mathbf{F}$, $\mathbf{curl}\,\mathbf{F}$, equals the vector function

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}. \qquad (18.2.2)$$

To obtain the (scalar) integrand on the right-hand side of (18.2.2), we "dot (18.2.2) with $\mathbf{k}$,"

$$\left(\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\right)] \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

So with the same assumptions as in the component form of Green's Theorem, we have this new formula.

Green's Theorem Expressed Terms of Circulation

Circulation expressed as a double integral:

> If the counterclockwise curve C bounds the region R, then
>
> $$\oint_C \mathbf{F} \cdot\ d\mathbf{r} = \int_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\ dA.$$

This is the planar version of Stokes' theorem, to be discussed in Section 18.5.

Recall that if **F** describes the flow of a fluid in the xy-plane, then $\oint_C \mathbf{F} \cdot d\mathbf{r}$ represents its circulation, or tendency to form whirlpools. This theorem tells us that the magnitude of the curl of **F** represents the tendency of the fluid to rotate. If the curl of **F** is **0** everywhere, then **F** is called **irrotational**.

This form of Green's theorem provides an easy way to show that a vector field **F** is conservative. It uses the idea of a simply-connected region. Informally "a simply-connected region in the xy-plane comes in one piece and has no holes." More precisely, any two points in the region can be joined by a curve that lies wholly within the region and any closed curve in the region can be shrunk gradually to a point while staying within the region.

For instance, the xy-plane is simply connected. So is the xy-plane without its positive x-axis. However, the xy-plane, without the origin is not simply connected, because a circular path around the origin cannot be shrunk to a point while staying within the region.

Now we can state the easy way to tell whether a vector filed is conservative.

Theorem. *If a vector field* **F** *is defined in a simply-connected region in the xy-plane and has $\nabla \times \mathbf{F} = \mathbf{0}$ throughout that region, then* **F** *is conservative.*

Proof

Let C be any simple closed curve in the region. We wish to prove that the circulation of **F** over C is **0**. Denoting the region that C bounds by $\mathcal{R}$ we have

$$\oint_C \mathbf{F} \cdot \, d\mathbf{r} = \int_{\mathcal{R}} (\mathbf{curl}\,\mathbf{F}) \; dA.$$

Since $\mathbf{curl}\,\mathbf{F}$ is **0** throughout R, $\oint_C \mathbf{F} \cdot \, d\mathbf{r} = 0$. •

In Example 5 in Section 18.1, there is a vector field whose curl is **0** but is not conservative. In view of the theorem just proved, its domain must not be simply connected. Indeed, the field is not defined at the origin.

EXAMPLE 2 Let $\mathbf{F}(x, y, z) = e^x y\mathbf{i} + (e^x + 2y)\mathbf{j}$.

1. Show that **F** is conservative.
2. Exhibit a scalar function f whose gradient is **F**.

SOLUTION

1. A straightforward calculation shows that $\nabla \times \mathbf{F} = \mathbf{0}$. Since $\mathbf{F}$ is defined throughout the xy-plane, a simply-connected region, Theorem 18.2 tells us that $\mathbf{F}$ is conservative.

2. By Section 18.1, we know that there is a scalar function f such that $\nabla f = \mathbf{F}$. There are several ways to find f. We show one of these methods here. Additional approaches are pursued in Exercises 7 and 8.

   The approach chosen here follows the construction in the proof of Theorem 18.1.3. Define $f(a, b)$ to equal $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any curve from $(0, 0)$ to (a, b). Any curve with the prescribed endpoints will do. For simplicity, choose C to be the curve that goes from $(0, 0)$ to (a, b) in a straight line. (See Figure 18.2.1.) When a is not zero, we can use x as a parameter are write this segment as: $x = t$, $y = (b/a)t$ for $0 \le t \le a$. (If $a = 0$, we would use y as a parameter.) Then

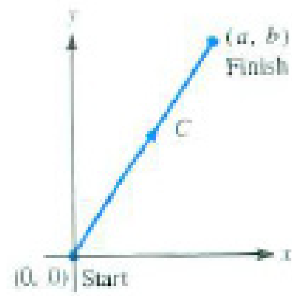


Figure 18.2.1:

$$
\begin{aligned}
f(a,b) &= \int_C \left(e^x y\, dx + (e^x + 2y)\, dy\right) = \int_0^a \left(e^t \frac{b}{a} t\, dt + \left(e^t + 2\frac{b}{a}t\right)\frac{b}{a}\, dt\right) \\
&= \frac{b}{a}\int_0^a \left(te^t + e^t + 2\frac{b}{a}t\right)\, dt = \frac{b}{a}\left((t-1)e^t + e^t + \frac{b}{a}t^2\right)\Big|_0^a = \frac{b}{a}\left(te^t + \frac{b}{a}t^2\right)\Big|_0^a = be^a + b^2.
\end{aligned}
$$

Note that in this constructive we already have $f(0, 0) = 0$.

   Since $f(a, b) = be^a + b^2$, we see that $f(x, y) = ye^x + y^2$ is the desired function.

$ye^x + y^2 + k$ for any constant k, also would be a potential.

◇

SHERMAN: In Section 18.4 we refer to the Two-Curve Case of Green's Theorem. Some adjustments are needed there, and here.

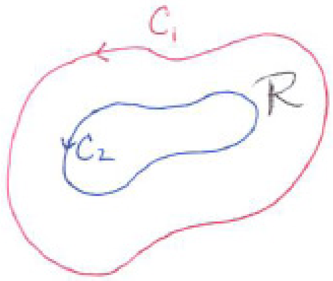
Green's Theorem — The Two-Curve Case

EXAMPLE 3 Figure 18.2.2(a) shows two curves C_1, and C_2 that enclose a ring-shaped region $\mathcal{R}$ in which $\nabla \times \mathbf{F}$ is $\mathbf{0}$. Show that the circulation of $\mathbf{F}$ over C_1 equals the circulation of $\mathbf{F}$ over C_2.
SOLUTION Cut $\mathcal{R}$ into two regions, each bounded by a simple curve, to which we can apply Theorem 18.2. Let C_3 bound one of the regions and C_4 bound the other, with the usual counterclockwise orientation. On the cuts, C_3 and C_4 go in opposite directions. On the outer curve C_3 and C_4 have the same orientation as C_1. On the inner curve they are the opposite orientation of C_2. (See Figure 18.1.2(b).) Thus

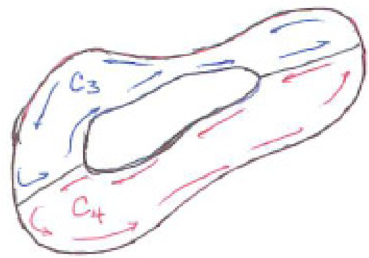
$$\int_{C_3} \mathbf{F}\cdot\ dr + \int_{C_4} \mathbf{F}\cdot\ dr = \int_{C_1} \mathbf{F}\cdot\ dr - \int_{C_2} \mathbf{F}\cdot\ dr. \qquad (18.2.3)$$

By Theorem 18.2 each integral on the left side of (18.2.3) is 0. Thus

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r} \qquad (18.2.4)$$



(a)



(b)

Figure 18.2.2:

◇

Example 3 tells us "as you move a closed curve within a region of zero-curl, you don't change the circulation."

How to Draw $\nabla \times \mathbf{F}$

For the planar vector field $\mathbf{F}$, its curl, $\nabla \times \mathbf{F}$ is of the form $a\mathbf{F}$. If a is positive, the curl points directly up from the page. Indicate this by the symbol $\odot$, which reminds you of the point of an arrow or the nose of a rocket. If a is negative, curl points down from the page. To show this, use the symbol $\oplus$, which suggests the feathers of an arrow or the fins of a rocket. Figure 18.2.3 illustrates their use.

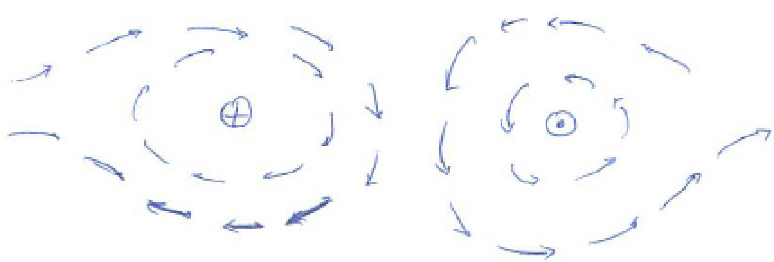


Figure 18.2.3:

DOUG: Some xrcss on this? or also in text? (it's standard in physics.)

Summary

We first expressed Green's theorem in terms of scalar functions

$$\oint_C (P\ dx + Q\ dy) = \int_R \left(\frac{\partial Q}{\partial \partial x} - \frac{\partial P}{\partial y} \right)\ dA.$$

We then translated it into a statement about the circulation of a vector field;

$$\oint_C \mathbf{F} \cdot\ d\mathbf{r} = \int_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\ dA.$$

With the aid of this theorem we were able to show the following important result:

> If the curl of **F** is **0** and if the domain of **F** is simply connected, then **F** is conservative.

Also, in a region in which $\nabla \times \mathbf{F} = \mathbf{0}$, the value of $\oint_C \mathbf{F} \cdot\ d\mathbf{r}$ does not change as you gradually change C to other curves in the region.

EXERCISES for 18.2 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 through 4 verify Green's Theorem for the given P and Q and curve C.

1.[R] $P = xy$, $Q = y^2$ C is the border of the square whose vertices are $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

2.[R] $P = x^2$, $Q = 0$ C is the boundary of the unit circle with center $(0,0)$.

3.[R] $P = e^y$, $Q = e^x$, C is the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

4.[R] $P = \sin(y)$, $Q = 0$, C is the boundary of the portion of the unit circle center $(0,0)$ in the first quadrant.

5.[R] Figure 18.2.4 shows a vector field for a fluid flow $\mathbf{F}$. At the indicated points A, B, C, and D tell when the curl of $\mathbf{F}$ is pointed up, down or is $\mathbf{0}$.

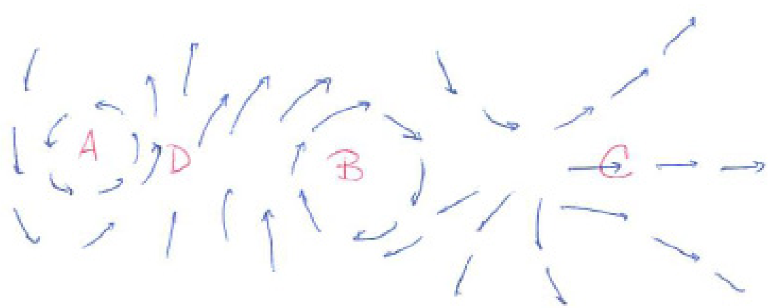


Figure 18.2.4:

6.[R] Assume that $\mathbf{F}$ describes a fluid flow. Let P be a point in the domain of $\mathbf{F}$ and C a small circular path around P swept out counterclockwise.

(a) The curl of $\mathbf{F}$ points upward, in what direction is the fluid tending to turn near P, clockwise or counterclockwise?

(b) Would you expect $\oint_C \mathbf{F} \cdot d\mathbf{r}$ to be positive or negative?

In Example 2 we constructed a potential function f for a vector field $\mathbf{F} = e^x y\mathbf{i} + (e^x + 2y)\mathbf{j}$ by using a straight path from $(0,0)$ to (a,b). Exercises 7 through 13 provide practice finding a potential function, f, with $\nabla f = \mathbf{F}$.

7.[R] In Example 2 we constructed a function f by using a straight path from $(0,0)$ to (a,b). Instead, construct f by using a path that consists of two line segments, the first from $(0,0)$ to $(a,0)$, and the second, from $(a,0)$ to (a,b).

8.[R] In Example 2 we constructed a function f by using a straight path from $(0,0)$ to (a,b). Instead, construct f by using a path that consists of two line segments, the first from $(0,0)$ to $(0,b)$, and the second from $(0,b)$ to (a,b).

9.[R] Another way to construct a potential function f for a vector field $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}$ is to work directly with the requirement that $\nabla f = \mathbf{F}$. That is, with

$$\frac{\partial f}{\partial x} = P(x,y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = Q(x,y).$$

(a) Integrate $\frac{\partial f}{\partial x} = e^x y$ with respect to x to conclude that $f(x,y) = e^x y + C(y)$. Note that the "constant of integration" can be any function of y. (Why?)

(b) Next, differentiate the result found in (a) with respect to y. This gives two formulas for $\frac{\partial f}{\partial y}$: $e^x + C'(y)$ and $e^x + 2y$. Use this fact to explain why $C'(y) = 2y$.

(c) Solve the equation for C found in (b).

(d) Combine the results of (a) and (c) to obtain the general form for a potential function for this vector field.

In Exercises 10 through 13

(a) check that $\mathbf{F}$ is conservative in the given domain.

(b) construct f such that $\nabla f = \mathbf{F}$, using integrals on curves.

(c) construct f such that $\nabla f = \mathbf{F}$, using antiderivatives.

10.[R] $\mathbf{F} = 3x^2y\ vi + x^3\mathbf{j}$

11.[R] $\mathbf{F} = y\cos(xy)\ vi + (x\cos(xy) + 2y)\mathbf{j}$

12.[R] $\mathbf{F} = (ye^{xy} + 1/x)\mathbf{i} + xe^{xy}\mathbf{j}$, $x > 0$

13.[R] $\mathbf{F} = \frac{2y\ln(x)}{x}\mathbf{i} + (\ln(x))^2\mathbf{j}$, $x > 0$

14.[R] Verify Green's Theorem when $\mathbf{F}(xy) = x\mathbf{i} + y\mathbf{j}$ and R is the disk of radius a and center at the origin.

15.[R] In Example 1 we used Green's Theorem to show that $\oint_C y\ dx$ is the negative of the area that C encloses. Use Green's Theorem to show that $\oint_C x\ dy$ equals that area. (We obtained this result previously, in Section 15.3.)

16.[R] Let A be a plane region with boundary C a simple closed curve swept out counterclockwise. Use Green's theorem to show that the area of A equals

$$\frac{1}{2}\oint(-y\ dx + x\ dy).$$

17.[R] See Example 16 to find the area of the region bounded by the line $y = x$ and the curve

$$\begin{cases} x &= t^6 + t^4 \\ y &= t^3 + t \end{cases} \qquad \text{for } t \text{ in } [0, 1].$$

18.[R] Say $\mathbf{curl}\,\mathbf{F}$ at $(0, 0)$ is -3. Let C sweep out the boundary of a circle of radius a, center at $(0, 0)$. When a is small, estimate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

19.[R] Which of these fields are conservative:

(a) $x\mathbf{i} - y\mathbf{j}$

(b) $\frac{x\mathbf{i}-y\mathbf{j}}{x^2+y^2}$

(c) $3\mathbf{i} + 4\mathbf{j}$

(d) $(6xy - y^3)\mathbf{i} + (4y + 3x^2 - 3xy^2)\mathbf{j}$

(e) $\frac{y\mathbf{i}-x\mathbf{j}}{1+x^2y^2}$

(f) $\frac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2}$

20.[R] Figure 18.2.5 shows a fluid flow $\mathbf{F}$. All the vectors are parallel, but their magnitudes increase from bottom to top. A small simple curve C is placed in the flow.

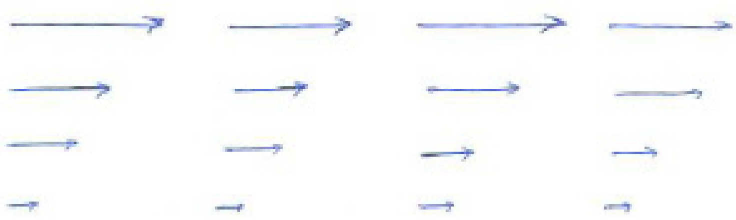


Figure 18.2.5:

(a) Is the circulation around C positive, negative, or 0? Justify your opinion.

(b) Assume that a wheel with small blades is free to rotate around it axis, which is perpendicular to the page, is inserted into this flow. Which way would it turn, or would it not turn at all?

SHERMAN: Isn't this the same field as in the previous Exercise? I know it's not given by a formula, but is this what you intend? It should not be difficult to make them different, if so desired.

21.[R] Let $\mathbf{F}(x, y) = y\mathbf{i}$.

(a) Sketch the field.

(b) Predict whether $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ is positive, negative or 0.

(c) Compute $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$.

(d) What would happen if you dipped a wheel with small blades free to rotate around its axis, which is perpendicular to the page, is inserted into this flow. (Don't just say, "It would get wet.")

22.[M] A curve is given parametrically by $x = t(1 - t^2)$, $y = t^2(1 - t^3)$, for t in $[0, 1]$.

(a) Sketch the points corresponding to $t = 0$, 0.2, 0.4, 0.6, 0.8, and 1.0, and use them to sketch the curve.

(b) Let $\mathcal{A}$ be the region enclosed by the curve. What difficulty arises when you try to compute the area of A by a definite integral involving vertical or horizontal cross sections?

(c) Use Exercise 16 to find the area of $\mathcal{A}$.

23.[M] Repeat Exercise 22 for $x = \sin(\pi t)$ and $y = t - t^2$, for t in $[0, 1]$. In (a), let $t = 0$, $1/4$, $1/2$, $3/4$, and 1.

24.[R] Use Exercise 16 to obtain the formula for area in polar coordinates:

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta.$$

Hint: Assume C is given parametrically as $x = r(\theta)\cos(\theta)$, $y = r(\theta)\sin(\theta)$, for $\alpha \leq \theta \leq \beta$.

25.[C] Assume that you know that Green's theorem is true when $\mathcal{A}$ is a triangle and C its boundary.

(a) Deduce that it therefore holds for quadrilaterals.

(b) Deduce that it holds for polygons.

26.[C] Assume that $\nabla \times \mathbf{F} = \mathbf{0}$ in the region $\mathcal{R}$ bounded by an exterior curve C_1 and two interior curves C_2 and C_3, as in Figure 18.2.6. Show that $\int_{C_1} \mathbf{F} \cdot\ d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot\ d\mathbf{r} + \int C_3 \mathbf{F} \cdot\ d\mathbf{r}$.

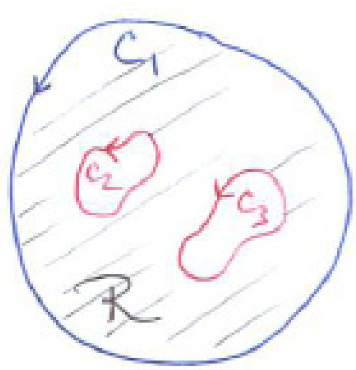


Figure 18.2.6:

27.[R] Check that the curl of the vector field in Example 2 is $\mathbf{0}$, as asserted.

28.[R] Explain in words, without explicit calculations, why the circulation of the

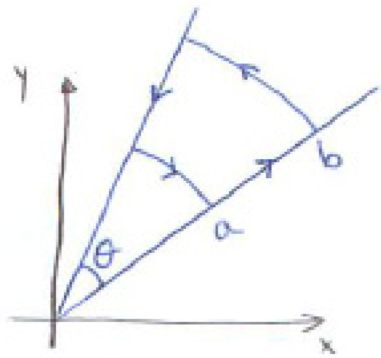


Figure 18.2.7:

field $f(r)\widehat{\mathbf{r}}$ around the curve in Figure 18.2.7 is zero. As usual, f is a scalar function, $r = ||\mathbf{r}||$, and $\widehat{r} = \mathbf{r}/r$.

In Exercises 29 to 32 let $\mathbf{F}$ be a vector field defined everywhere in the plane except a the point P shown in Figure 18.2.8. Assume that $\nabla \times \mathbf{F} = \mathbf{0}$ and that $\int_{C_1} \mathbf{F} \cdot \, d\mathbf{r} = 5$.

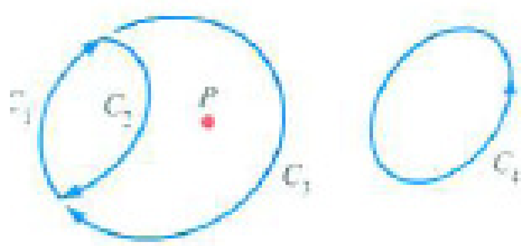


Figure 18.2.8:

29.[R] What, if anything, can be said about $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$?

30.[R] What, if anything, can be said about $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$?

31.[R] What, if anything, can be said about $\int_{C_4} \mathbf{F} \cdot d\mathbf{r}$?

32.[R] What, if anything, can be said about $\int_{C} \mathbf{F} \cdot d\mathbf{r}$, where C is the curve formed by C_1 followed by C_3?

In Exercises 33 to 36 show that the vector field is conservative and then construct a scalar function of which it is the gradient. Use the method in

Example 2.

33.[R] $2xy\mathbf{i} + x^2\mathbf{j}$

34.[R] $\sin(y)\mathbf{i} + (x\cos(y) + 3)\mathbf{j}$

35.[R] $(y + 1)\mathbf{i} + (x + 1)\mathbf{j}$

36.[R] $3y\sin^2(xy)\cos(xy)\mathbf{i} + (1 + 3x\sin^2(xy)\cos(xy))\mathbf{j}$

37.[R] Show that

(a) $3x^2y\ dx + x^3\ dy$ is exact.

(b) $3xy\ dx + x^2\ dy$ is not exact.

38.[R] Show that $(x\ dx + y\ dy)/(x^2 + y^2)$ is exact and exhibit a function f such that df equals the given expression. (That is, find f such that $\nabla f \cdot d\mathbf{r}$ agrees with the given differential form.)

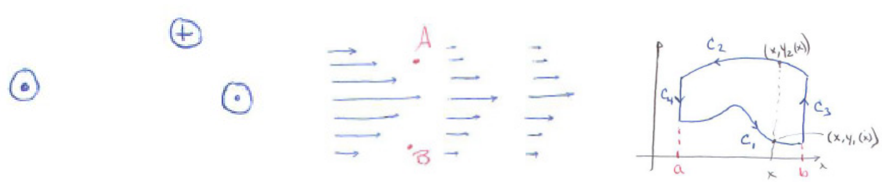
39.[R] Let $\mathbf{F} = \widehat{\mathbf{r}}/\|\mathbf{r}\|$ in the xy plane and let C be the circle of radius a and center $(0, 0)$.

(a) What does Green's theorem say about $\oint_C \mathbf{F} \cdot \mathbf{n}\ ds$?

(b) Evaluate $\oint_C \mathbf{F} \cdot \mathbf{n}\ ds$ without using Green's theorem.

(c) Let C now be the circle of radius 3 and center $(4, 0)$. Evaluate $\oint_C \mathbf{F} \cdot \mathbf{n}\ ds$, doing as little work as possible.

40.[R] Figure 18.2.9(a) shows the direction of a vector field at three points. Draw a vector field comparable with these values. (No zero-vectors, please.)



(a) (b) (c)

Figure 18.2.9:

41.[R] Consider the vector field in Figure 18.2.9(b). Will a paddle wheel turn at A? At B? If so, in which direction?

42.[C] We proved that $\int_{\mathcal{R}} \frac{\partial Q}{\partial y}\, dA = \int_C Q\, dy$ in a special case. Prove it with this more general case, in which we assume less about the region $\mathcal{R}$. Assume that $\mathcal{R}$ has the description $a \le x \le b$, $y_1(x) \le y \le y_2(x)$. Figure 18.2.9(c) shows such a region, which need not be convex. The curved path C breaks up into four paths, two of

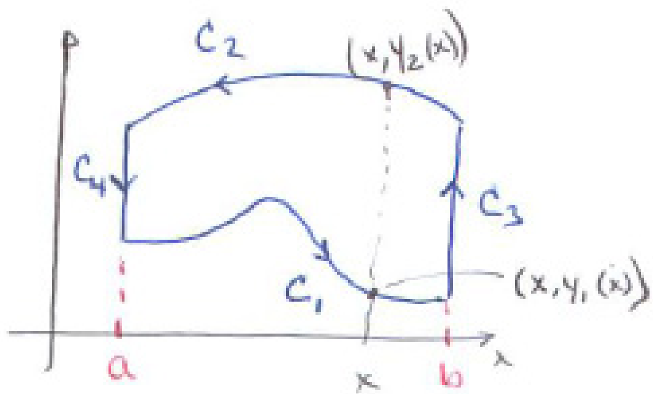


Figure 18.2.10:

which are straight (or may be empty), as would be in Figure 18.2.9(c).

43.[C] We proved the second part of (18.2.1), namely that $\oint_C Q\, dy = \int_R \partial Q/\partial x\, dA$. Prove the first part, $\oint_C P\, dx = -\int_R \partial p/\partial y\, dA$.

18.3 Green's Theorem, Flux, and Divergence

In the previous section we translated Green's theorem into a theorem about circulation and curl. That concerned the line integral of $\mathbf{F}\cdot\mathbf{T}$, the tangential component of $\mathbf{F}$, since $\mathbf{F}\cdot\ d\mathbf{r}$ is short for $(\mathbf{F}\cdot\mathbf{T})\ ds$. Now we will translate Green's theorem into a theorem about the line integral of $\mathbf{F}\cdot\mathbf{n}$, the normal component of $\mathbf{F}$.

Green's Theorem Expressed in Terms of Flux

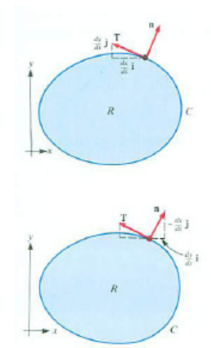


Figure 18.3.1:

Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ and C a counterclockwise closed curve. (We use M and N now, to avoid confusion with P and Q needed later.) To compute $\mathbf{F}\cdot\mathbf{n}$ in terms of M and N, we first express $\mathbf{n}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

The vector

$$\mathbf{T} = \frac{dx}{dx}\mathbf{i} + \frac{dy}{ds}\mathbf{j}$$

is tangent to the curve, has length 1, and points in the direction in which the curve is swept out. A typical $\mathbf{T}$ and $\mathbf{n}$ are shown in Figure 18.3.1. As Figure 18.3.1 shows, the exterior unit normal $\mathbf{n}$ has its x component equal to the y component of $\mathbf{T}$ and its y component equal to the negative of the component of $\mathbf{T}$. Thus

$$\mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

Consequently, if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, then

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{n} &= \oint_C (M\mathbf{i} + N\mathbf{j})\cdot\left(\frac{dy}{ds}\mathbf{i} - \frac{dx}{dx}\mathbf{j}\right)\ ds = \oint_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)\ ds \\ &= \oint_C (M\ dy - N\ dx) = \oint_C (-N\ dx + M\ dy). \qquad (18.3.1)\end{aligned}$$

In (18.3.1), $-N$ plays the role of P and M plays the role of Q in Green's Theorem. Since Green's Theorem states that

$$\oint_C (P\ dx + Q\ dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

we have

$$\oint_C (-N\ dx + M\ dy) = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial(-N)}{\partial y}\right)\ dA$$

or simply, if $\mathbf{F} = M\mathbf{i} + n\mathbf{j}$, then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\ ds = \int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\ dA.$$

In our customary "P and Q" notations, we have

Theorem. *Flux Expressed as a Double Integral If* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, *then*

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{\mathcal{R}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dA.$$

Note that $\nabla \cdot \mathbf{F}$ is a scalar function while $\nabla \times \mathbf{F}$ is a vector function.

The expression

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},$$

the sum of two partial derivatives, is call the **divergence** of $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. It is written $\nabla \cdot \mathbf{F}$ or $\operatorname{div}\mathbf{F}$. The latter notation is suggested by the "symbolic" dot product

$$\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right)\cdot(P\mathbf{i} + Q\mathbf{j}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

It is pronounce "del dot eff". Theorem 18.3 is called "the divergence theorem in the plane."

EXAMPLE 1 Compute the divergence of

(a) $\mathbf{F} = e^{xy}\mathbf{i} + \arctan(3x)\mathbf{j}$ and

(b) $\mathbf{F} = -x^2\mathbf{i} + 2xy\mathbf{j}$.

SOLUTION

(a) $\frac{\partial}{\partial x}e^{xy} + \frac{\partial}{\partial y}\arctan(3x) = ye^{xy} + 0 = ye^{xy}$

(b) $\frac{\partial}{\partial x}(-x^2) + \frac{\partial}{\partial y}(2xy) = -2x + 2x = 0.$

◇

The double integral of $\nabla\cdot\mathbf{F}$ over a region describes the amount of flow across the border of that region. It tells how rapidly the fluid is leaving (diverging) or entering (converging) the region. Hence the name "divergence".

In the next section we will be using the divergence of a vector field defined in space, $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where P, Q and R are functions of x, y, and z. As you may anticipate, it is defined as the sum of three partial derivatives

$$\nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

It will play a role in measuring flux across a surface.

EXAMPLE 2 Verify that $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$ equals $\int_R \nabla\cdot\mathbf{F}\,dA$, when $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$, R is the disk of radius a and center at the origin. C is the boundary curve of R.

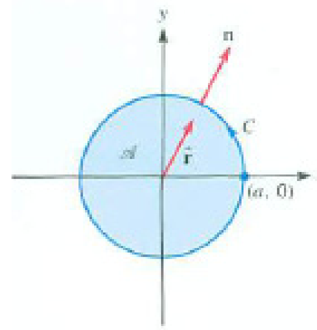


Figure 18.3.2:

SOLUTION First we compute $\oint_C \mathbf{F}\cdot\mathbf{n}\ ds$, where C is the circle boundary $\mathbf{R}$, taken counterclockwise. (See Figure 18.3.2.)

Since C is a circle centered at $(0,0)$, the unit exterior normal $\mathbf{n}$ is $\widehat{\mathbf{r}}$:

$$\mathbf{n} = \widehat{\mathbf{r}} = \frac{x\mathbf{i}+y\mathbf{j}}{\|x\mathbf{i}+y\mathbf{j}\|} = \frac{x\mathbf{i}+y\mathbf{j}}{a}.$$

Thus

$\oint_C ds$ = arclength of C

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{n}\ ds &= \oint_C (x\mathbf{i}+y\mathbf{j})\cdot\left(\frac{x\mathbf{i}+y\mathbf{j}}{a}\right)\ ds = \oint_C \frac{x^2+y^2}{a}\ ds \\ &= \oint_C \frac{a^2}{a}\ ds = a\oint_C\ ds = a(2\pi a) = 2\pi a^2. \qquad (18.3.2)\end{aligned}$$

Next we compute $\int_R \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\ dA$. Since $P = x$ and $Q = y$, $\partial P/\partial x + \partial Q/\partial y = 1+1 = 2$. Then

$$\int_R \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\ dA = \int_R 2\ dA,$$

which is twice the area of the disk R, hence $2\pi a^2$. This agrees with (18.3.2). ◇

As the next example shows, the double integral for flux provides an indirect way of computing $\oint \mathbf{F}\cdot\mathbf{n}\ ds$.

EXAMPLE 3 Let $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$. Evaluate $\oint \mathbf{F}\cdot\mathbf{n}\ ds$ over the curve that bounds the quadrilateral with vertices $(1,1)$, $(3,1)$, $(3,4)$, and $(1,2)$ shown in Figure 18.3.3.

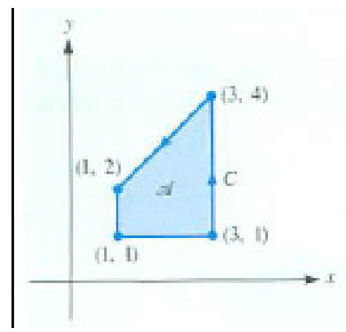


Figure 18.3.3:
See Exercise 23.

SOLUTION The line integral could be evaluated directly, but would require parameterizing each of the four edges of C. With Green's theorem we can instead evaluate an integral over a single plane region.

Let $\mathcal{A}$ be the region that C bounds. By Green's theorem

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{n}\ ds &= \int_{\mathcal{A}} \nabla\cdot\mathbf{F}\ dA = \int_{\mathcal{A}} \left(\frac{\partial(x^2)}{\partial x}+\frac{\partial(xy)}{\partial y}\right)\ dA \\ &= \int_{\mathcal{A}} (2x+x)\ dA = \int_{\mathcal{A}} 3x\ dA.\end{aligned}$$

Then

$$\int_{\mathcal{A}} 3x\ dA = \int_1^3 \int_1^{y(x)} 3x\ dy\ dx,$$

where $y(x)$ is determined by the equation of the line that provides the top edge of $\mathcal{A}$. We easily find that the line through $(1, 2)$ and $(3, 4)$ has the equation $y = x + 1$. Therefore,

$$\int_A 3x\ dA = \int_1^3 \int_1^{x+1} 3x\ dy\ dx.$$

The inner integration gives

$$\int_1^{x+1} 3x\ dy = 3xy|_{y=1}^{y=x+1} = 3x(x+1) - 3x = 3x^2.$$

The second integration gives

$$\int_1^3 3x^2\ dx = x^3|_1^3 = 27 - 1 = 26$$

◇

A Local View of $\nabla \cdot \mathbf{F}$

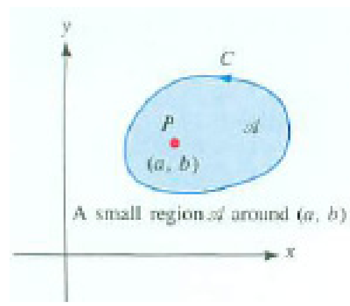


Figure 18.3.4:

We have presented a "global" view of $\nabla \cdot \mathbf{F}$, integrating it over a region R to get the total divergence across the boundary of R. But there is another way of viewing $\nabla \cdot \mathbf{F}$, "locally."

Let $P = (a, b)$ be a point in the plane and $\mathbf{F}$ a vector field describing fluid flow. Choose a very small region R around P of area A, and let C be its boundary, taken counterclockwise. (See Figure 18.3.4.) Then the net flow out of R is

$$\oint_C \mathbf{F} \cdot \mathbf{n}\ ds.$$

By Green's theorem, the net flow is also

$$\int_{\mathcal{R}} \nabla \cdot \mathbf{F}\ dA.$$

Now, since $\nabla \cdot \mathbf{F}$ is continuous and A is small, $\nabla \cdot \mathbf{F}$ is almost constant throughout R, staying close to the value at (a, b), namely, $(\nabla \cdot \mathbf{F})(a, b)$, the divergence of $\mathbf{F}$ at (a, b). Thus

$$\int_A \nabla \cdot \mathbf{F}\ dA \approx (\nabla \cdot \mathbf{F})(a, b)A.$$

or, equivalently,

$$\frac{\text{Net flow out of } R}{A} \approx (\nabla \cdot \mathbf{F})(a, b). \qquad (18.3.3)$$

This means that

$$\nabla \cdot \mathbf{F} \text{ at } P$$

is a measure of the rate at which fluid tends to leave a small region around P. Hence the name "divergence". If $\nabla \cdot \mathbf{F}$ is positive, fluid near P tends to get less dense (diverge). If $\nabla \cdot \mathbf{F}$ is negative, fluid near P tends to accumulate (converge).

Moreover,, (18.3.3) suggests a new definition of the divergence $\nabla \cdot \mathbf{F}$ at (a, b), namely

$$(\nabla \cdot \mathbf{F})(a, b) = \lim_{\text{Area of } \mathcal{R} \to 0} \frac{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}{\text{Area of } \mathcal{R}}$$

where R is any region enclosing (a, b) whose boundary C is a simple closed curve.

This definition appeals to our physical intuition. We began by defining $\nabla \cdot \mathbf{F}$ mathematically, as $\partial P/\partial x + \partial Q/\partial y$. We now see its physical meaning, which is independent of any coordinate system. ((Exercise 24 in Section 18.2 uses this fact to find divergence when $\mathbf{F}$ is given in polar coordinates.)

EXAMPLE 4 Estimate the flux of $\mathbf{F}$ across a small circle C of radius a if div $\mathbf{F}$ at the center of the circle is 3.

SOLUTION The flux of $\mathbf{F}$ across C is $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$, which equals $\int_R \nabla \cdot \mathbf{F}\, dA$, where R is the disk that C bounds. Since $\nabla \cdot \mathbf{F}$ is continuous, it changes little in a small enough disk, and we treat it as almost constant. Then $\int_R \nabla \cdot \mathbf{F}\, dA$ is approximately $(3)(\text{Area of} R) = 3(\pi a^2) = 3\pi a^2$. ◇

Proof of Green's Theorem

As Steve Whitaker of the chemical engineering department at the University of California at Davis says, "The concepts that one must understand to *prove* a theorem are frequently the concepts one must understand to *apply* the theorem." So read the proof slowly at least twice. It is not here just to show that Green's theorem is true. After all, it has been around for over 150 years, and no one has said it is false. Studying a proof strengthens one's understanding of the fundamentals.

In this proof we use the concepts of a double integral, an iterated integral, a line integral, and the fundamental theorem of calculus. The proof provides a quick review of four basic ideas.

We prove that $\oint_R Q\, dy = \int_R \frac{\partial Q}{\partial x}\, dA$. The proof that $\oint_C P\, dx = -\int \frac{\partial P}{\partial y}\, dA$ is similar.

To avoid getting involved in distracting details we assume that R is **strictly convex**: It has no dents and its border has no straight line segments. The basic ideas of the proof show up clearly in this special case. Thus R has the description $a \le x \le b$, $y(_1x) \le y \le y_2(x)$, as shown in Figure 18.3.5. We will express both

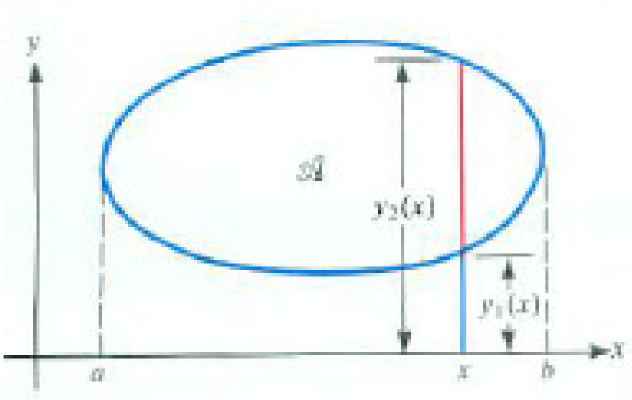


Figure 18.3.5:

$\int_{\mathcal{R}} \frac{\partial Q}{\partial y}\, dA$ and $\int_C Q\ dy$ as definite integrals over the interval $[a, b]$. First, we have

$$\int_{\mathcal{R}} \frac{\partial Q}{\partial y}\ dA = \int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial Q}{\partial y}\ dy\ dx.$$

By the Fundamental Theorem of Calculus,

$$\int_{y_1(x)}^{y_2(x)} \frac{\partial Q}{\partial y}\ dy = Q(x, y_2(x)) - Q(x, y_1(x)).$$

Hence

$$\int_A \frac{\partial Q}{\partial y}\ dA = \int_a^b \left(Q(x, y_2(x)) - Q(x, y_1(x))\right)\ dx. \tag{18.3.4}$$

Next, to express $\int_C -Q\ dx$ as an integral over $[a.b]$, break the closed path C into two successive paths, one along the bottom part of A, described by $y = y_1(x)$, the other along the top part of A, described by $y = y_2(x)$. Denote the bottom path C_1 and the top path C_2. (See Figure 18.3.6.)

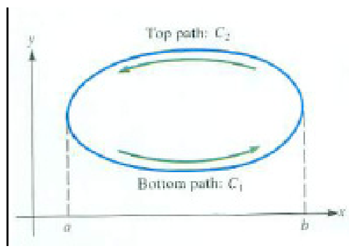


Figure 18.3.6:

Then

$$\oint_C (-Q)\ dx = \int_{C_1} (-Q)\ dx + \int_{C_2} (-Q)\ dx. \tag{18.3.5}$$

But

$$\int_{C_1} (-Q)\ dx = \int_{C_1} \left(-Q(x, y_1(x))\right)\ dx = \int_a^b \left(-Q(x, y_1(x))\right)\ dx,$$

and

$$\int_{C_2} (-Q)\ dx = \int_{C_2} \left(-Q(x, y_2(x))\right)\ dx = \int_a^b \left(-Q(x, y_2(x))\right)\ dx, = \int_a^b Q(x, y_2(x))\ dx.$$

thus by (18.3.5),

$$\begin{aligned}\oint_C (-Q)\ dx &= \int_a^b -Q(x, y_1(x))\ dx + \int_a^b Q(x, y_2(x))\ dx \\ &= \int_a^b [Q(x, y_2(x)) - Q(x, y_1(x))]\ dx.\end{aligned}$$

This is also the right side of (18.3.4) and concludes the proof.

Summary

We translated Green's theorem into a theorem about the flux of a vector field in the xy-plane. In symbols, the divergence theorem in the plane says that

$$\oint_C \mathbf{F} \cdot \mathbf{n}\ ds = \int_A \nabla \cdot \mathbf{F}\ dA.$$

"The integral of the normal component of **F** around a simple closed curve equals the integral of the divergence of **F** over the region that the curve bounds."

From this it follows that

$$\nabla \cdot \mathbf{F}(P) = \lim_{\text{Area of } \mathcal{R} \to 0} \frac{\oint_C \mathbf{F} \cdot \mathbf{n}\ ds}{\text{Area of } \mathcal{R}},$$

where C is the boundary of the region $\mathcal{R}$, which contains P.

This introduced the notion of the "divergence" of $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, namely the scalar field $\partial P/\partial x + \partial Q/\partial y$. (In the next section we have the divergence of $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, namely the sum $\partial P/\partial x + \partial Q/\partial y + \partial R/\partial z$.) We then showed the divergence of **F** at a point P is

$$\lim_{\text{Area of } \mathcal{R} \to 0} \frac{\text{Flux across } C}{\text{Area of } \mathcal{R}},$$

where C, the boundary of R, is chosen smaller and smaller. We concluded with a proof of Green's theorem, which amounts to a review of several basic concepts.

EXERCISES for 18.3 *Key:* R–routine, M–moderate, C–challenging

1.[R] State the divergence form of Green's theorem in symbols.

2.[R] State the divergence form of Green's theorem in words, using no symbols to denote the vector fields, etc.

3.[C] Let $\mathbf{F}(x, y)$ describe a fluid flow. Assume $\nabla \cdot \mathbf{F}$ is never 0 in a certain region R. Show that none of the stream lines in the region closes up to form a loop within $\mathcal{R}$. HINT: At each point P on a stream line, $\mathbf{F}(P)$ is tangent to that streamline.

4.[M] Find the area of the region bounded by the line $y = x$ and the curve

$$\begin{cases} x &= t^6 + t^4 \\ y &= t^3 + t \end{cases}$$

for t in $[0, 1]$. HINT: Use Green's Theorem.

5.[M] Let f be a scalar function. Let $\mathcal{R}$ be a convex region and C its boundary taken counterclockwise. Show that

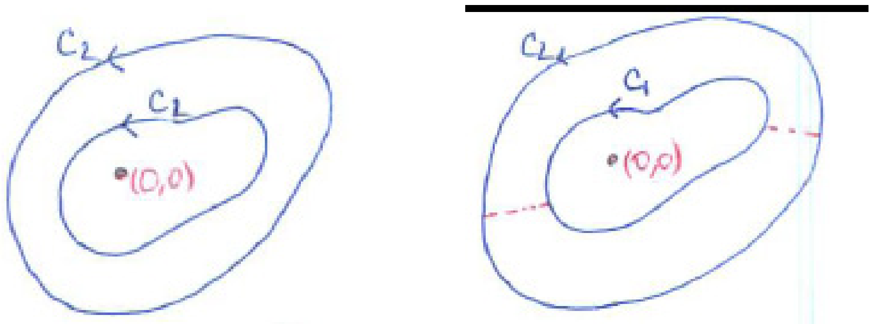
$$\int_{\mathcal{R}} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dA = \oint_C \left(\frac{\partial f}{\partial x}\, dy - \frac{\partial f}{\partial x}\, dx \right).$$

6.[C] Let $\mathcal{R}$ be a region in the xy plane bounded by the closed curve C. Let $f(x, y)$ be defined on the plane. Show that

$$\int_{\mathcal{R}} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x} \right)^2 dA = \oint_C D_{\mathbf{n}}(f)\, ds.$$

7.[C] Assume that $\mathbf{F}$ is defined everywhere in the xy-plane except at the origin and that the divergence of $\mathbf{F}$ is identically 0. Let C_1 and C_2 be two counterclockwise simple curves circling the origin C_1 lies within the region within C_2. Show that $\oint_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds = \int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds$. (See Figure 18.3.7(a).) HINT: Draw the dashed lines in Figure 18.3.7(b) to cut the region between C_1 and C_2 into two regions.

8.[C] (This continues Exercise 7.) Assume that $\mathbf{F}$ is defined everywhere in the xy-plane except at the origin and that the divergence of $\mathbf{F}$ is identically 0. Let C_1 and C_2 be two counterclockwise simple curves circling the origin. They may intersect. Show that $\oint_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds = \oint_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds$. The message to be learned from this Exercise is this: if the divergence of $\mathbf{F}$ is 0, you are permitted to replace a line integral over



(a) (b)

Figure 18.3.7:

a complicated curve by a line integral over a simpler curve.

9.[M] Let $\mathbf{F}$ be the vector field whose formula in polar coordinates is $\mathbf{F}(r,\theta) = r^n\widehat{\mathbf{r}}$, where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}$, $r = \|\mathbf{r}\|$, and $\widehat{\mathbf{r}} = \mathbf{r}/r$. Show that the divergence of $\mathbf{F}$ is $(n+1)r^{n-1}$. HINT: First express $\mathbf{F}$ in rectangular coordinates.

10.[M] A region with a hole is bounded by two oriented curves C_1 and C_2, as in Figure 18.3.8. which shows typical exterior-pointing unit normal vectors. Find an

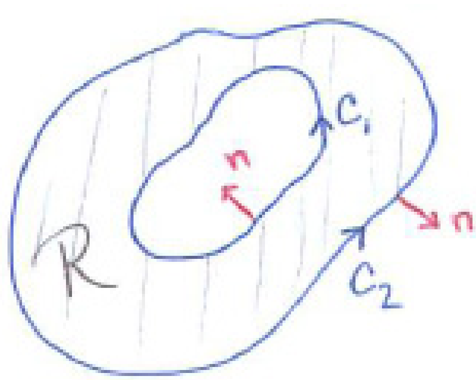


Figure 18.3.8:

equation expressing $\int_{\mathcal{R}} \nabla\cdot\mathbf{F}\, dA$ in terms of $\oint_{C_1} \mathbf{F}\cdot\mathbf{n}\, ds$ and $\oint_{C_2} \mathbf{F}\cdot\mathbf{n}\, ds$. HINT: Break R into two regions that have no holes, as in Exercises 7 and 8.

In Exercises 11 to 14 compute the divergence of the given vector fields.

11.[R] $\mathbf{F} = x^3y\mathbf{i} + x^2y^3\mathbf{j}$

12.[R] $\mathbf{F} = \arctan(3xy)\mathbf{i} + (e^{y/x})\mathbf{j}$

13.[R] $\mathbf{F} = \ln(x+y)\mathbf{i} + xy(\arcsin y)^2\mathbf{j}$

14.[R] $\mathbf{F} = y\sqrt{1+x^2}\mathbf{i} + \ln((x+1)^3(\sin(y))^{3/5}e^{x+y})\mathbf{j}$

In Exercises 15 to 18 compute $\int_{\mathcal{R}} \operatorname{div}\mathbf{F}\, dA$ and $\oint_C \mathbf{F}\cdot\mathbf{n}$ and verify the planar divergence theorem. The differential form of Green's theorem DOUG will be helpful.

SHERMAN: Do we use this name?

15.[R] $\mathbf{F} = 3x\mathbf{i} + 2y\mathbf{j}$, and $\mathcal{R}$ is the disk of radius 1 with center $(0,0)$.

16.[R] $\mathbf{F} = 5y^3\mathbf{i} - 6x^2\mathbf{j}$, and $\mathcal{R}$ is the disk of radius 2 with center $(0,0)$.

17.[R] $\mathbf{F} = xy\mathbf{i} + x^2y\mathbf{j}$, and $\mathcal{R}$ is the square with vertices $(0,0)$, $(a,0)$ (a,b) and $(0,b)$, where a, $b > 0$.

18.[R] $\mathbf{F} = \cos(x+y)\mathbf{i} + \sin(x+y)\mathbf{j}$, and $\mathcal{R}$ is the triangle with vertices $(0,0)$, $(a,0)$ and (a,b), (a,b), where a, $b > 0$.

In Exercises 19 to 22 use the planar divergence theorem to evaluate $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$ for the given $\mathbf{F}$, where C is the boundary of the given region R.

19.[R] $\mathbf{F} = e^x \sin y\mathbf{i} + e^{2x}\cos(y)\mathbf{j}$, and R is the rectangle with vertices $(0,0)$, $(1,0)$, $(0,\pi/2)$, and $(1,\pi/2)$.

20.[R] $\mathbf{F} = y\tan(x)\mathbf{i} + y^2\mathbf{j}$, and R is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

21.[R] $\mathbf{F} = 2x^3y\mathbf{i} - 3x^2y^2\mathbf{j}$, and R is the triangle with vertices $(0,1)$, $(3,4)$, and $(2,7)$.

22.[R] $\mathbf{F} = \frac{-\mathbf{i}}{xy^2} + \frac{\mathbf{j}}{x^2y}$, and R is the triangle with vertices $(1,1)$, $(2,2)$, and $(1,2)$. Hint: Write $\mathbf{F}$ with a common denominator.

23.[R] In Example 3 we found $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$ by computing a double integral. Instead, evaluate the line integral $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$ directly.

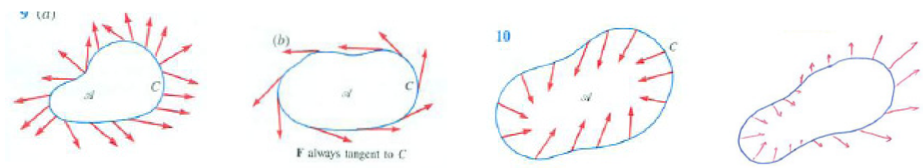
24.[R] A small curve C bounds a small region of area A.

(a) If $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = -2$, estimate $\nabla\times\mathbf{F}$ at points in R.

(b) Would you use $\odot$ or $\oplus$ to indicate the curl?

In Exercises 25 to 28, $\mathbf{F}$ is defined on the whole plane but indicated only at points on a curve C bounding a region A. What can be said about $\int_A \nabla\cdot\mathbf{F}\, dA$ in each case?

25.[R] See Figure 18.3.9(a).



(a) (b) (c) (d)

Figure 18.3.9:

26.[R] See Figure 18.3.9(b).

27.[R] See Figure 18.3.9(c).

28.[R] See Figure 18.3.9(d).

29.[R] Let $\mathbf{F}(x, y) = \mathbf{i}$, a constant field.

(a) Evaluate directly the flux of $\mathbf{F}$ around the triangular path, $(0, 0)$ to $(1, 0)$, to $(0, 1)$ back to $(0, 0)$.

(b) Use the divergence of $\mathbf{F}$ to evaluate the flux in (a).

30.[R] Let a be a "small number" and $\mathcal{R}$ be the square with vertices (a, a), $(-a, a)$, $(-a, -a)$, and $(a, -a)$, and C its boundary. If the divergence of $\mathbf{F}$ at the origin is 3, estimate $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$.

31.[R] Assume $\|\mathbf{F}(P)\| \leq 4$ for all points P on a curve of length L that bounds a region R of area A. What can be said about integral $\int_{\mathcal{R}} \nabla \cdot \mathbf{F}\, dA$.

32.[R] Let $\mathbf{F}(x, y) = \widehat{\mathbf{r}}/r$. Evaluate as simply as possible the flux of $\mathbf{F}$ across the rectangular curve whose vertices are $(-1, -2)$, $(3, -2)$, $(3, 4)$, and $(-1, 4)$.

33.[R] Let $\mathbf{F}(x, y) = \widehat{\mathbf{r}}/r$. Evaluate as simply as possible the flux of $\mathbf{F}$ across

(a) the triangle whose vertices are $(1, 0)$, $(2, 0)$, and $(2, 3)$,

(b) the triangle whose vertices are $(1, -1)$, $(1, 4)$, and $(-2, -1)$.

34.[R] Verify the divergence form of Green's theorem for $\mathbf{F} = 3x\mathbf{i} + 4y\mathbf{j}$ and C the square whose vertices are $(2, 0)$, $(5, 0)$, $(5, 3)$, and $(2, 3)$.

35.[M] (See Example 3 in Section 18.2.) The region R is bounded by the curves

SHERMAN: Is this the correct cross-reference?

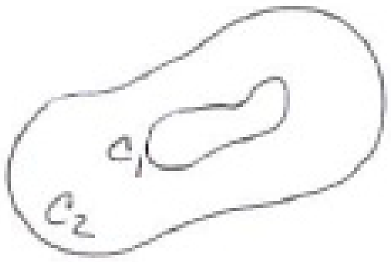


Figure 18.3.10:

C_1 and C_2, as in Figure 18.3.10.

(a) Show that $\oint_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds - \int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds = \int_R (\nabla \cdot \mathbf{F})\, dA$.

(b) If $\nabla \cdot \mathbf{F} = 0$, show that $\int_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds = \int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds$.

A vector field $\mathbf{F}$ is said to be **divergence free** when $\nabla \cdot \mathbf{F} = 0$ at every point in the field. **36.**[R] Figure 18.3.11 shows four vector fields. Two are divergence-free and two are not. Decide which two are not, copy them onto a sheet of drawing paper, and sketch a closed curve C for which $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ is not 0.

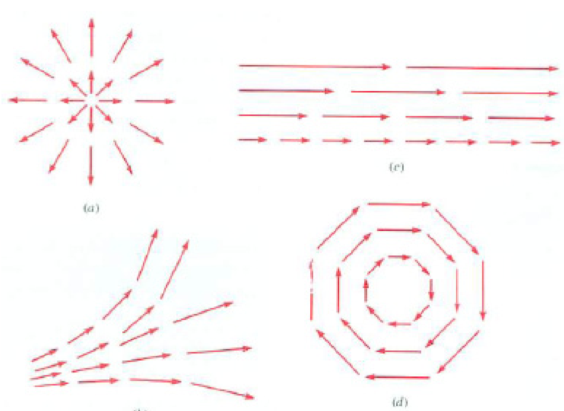


Figure 18.3.11:

37.[R] For a vector field $\mathbf{F}$ or scalar field f,

(a) Is the curl of the gradient of **F** always **0**?

(b) Is the divergence of the gradient of **F** always 0?

(c) Is the divergence of the curl of **F** always 0?

(d) Is the gradient of the divergence of **F** always **0**?

38.[R] Figure 18.3.12 describes the flow **F** of a fluid. Decide whether $\nabla \cdot \mathbf{F}$ is positive, negative, or zero at each of the points A, B, and C.

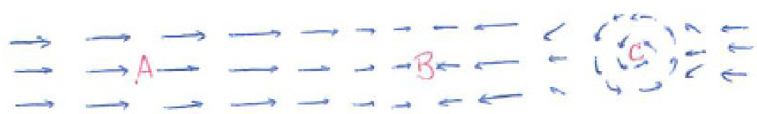


Figure 18.3.12:

SHERMAN: Do you care that $(0, 0)$ is not an interior point for this $\mathcal{R}$?

39.[R] If $\nabla \cdot \mathbf{F}$ at $(0, 0)$ is 3 estimate $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$, where C is the curve around the square whose vertices are $(0, 0)$, $(0.01, 0)$, $(0.01, 0.01)$, $(0, 0.01)$.

40.[C]

(a) Compute $\nabla \cdot \mathbf{F}$.

(b) Draw enough vectors for the field $\mathbf{F}(x, y) = (x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$ to show what it looks like.

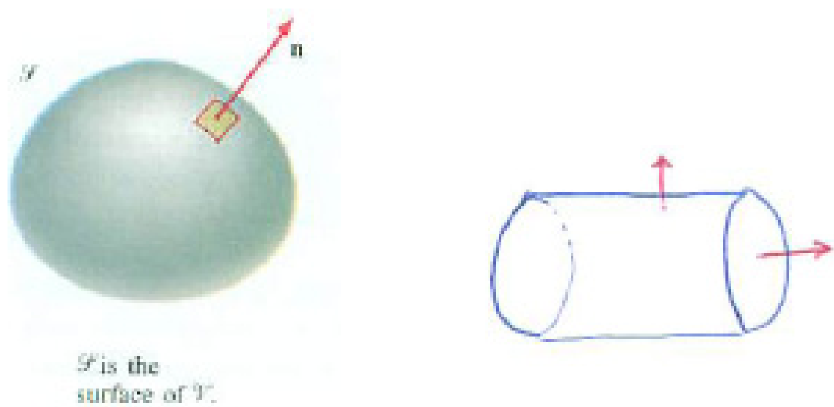
(c) Does your sketch in (b) agree with what you found for $\nabla \cdot \mathbf{F}$. in (a)? (If not, go back to (a) and redraw the vector field.)

18.4 The Divergence Theorem in Space (Gauss's Theorem)

In Sections 18.2 and 18.3 we developed Green's theorem and applied it in two forms for a vector field $\mathbf{F}$ in the plane. One form concerned the line integral of the normal component of $\mathbf{F}$, $\oint_C \mathbf{F} \cdot \mathbf{n}\ ds$. The other concerned the integral of the tangential component of $\mathbf{F}$, $\oint_C \mathbf{F} \cdot \mathbf{T} ds$, also written as $\oint_C \mathbf{F} \cdot d\mathbf{r}$. In this section we generalize the first form to space, in the **divergence theorem** in space. In Section 18.5 we will generalize the second form to space in Stokes' theorem.

The Divergence (or Gauss's) Theorem

Consider a region $\mathcal{R}$ in space bounded by a single connected surface $\mathcal{S}$. For instance, $\mathcal{R}$ may be a ball and $\mathcal{S}$ its surface. This is a case encountered in the elementary theory of electro-magnetism. In another case, $\mathcal{R}$ is a right circular cylinder and $\mathcal{S}$ is its surface, which consists of two disks and its curved side. See Figure 18.4.1(a). Both figures show typical unit exterior normals, perpendicular to the surface. The



(a) (b)

Figure 18.4.1:

divergence theorem relates an integral over the surface to our integral over the region it bounds.

Theorem. *Divergence Theorem —One-Surface Case Let $\mathcal{V}$ be the region in space bounded by the single connected surface $\mathcal{S}$. Let $\mathbf{n}$ denote the exterior unit normal of $\mathcal{V}$ along the boundary $\mathcal{S}$. Then*

$$\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\ ds = \int_{\mathcal{V}} \nabla \cdot \mathbf{F}\ dV$$

for any vector field $\mathbf{F}$ defined on $\mathcal{V}$.

Say the Theorem also.

In words: "The integral of the normal component of **F** over a surface equals the integral of the divergence of **F** over the solid region the surface bounds."

The integral $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dS$ is called the **flux** of the field **F** across the surface $\mathcal{S}$.

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and $\cos(\alpha)$, $\cos(\beta)$, and $\cos(\gamma)$ are the direction cosines of the exterior normal, then the divergence theorem reads

$$\int_{\mathcal{S}} (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k})\cdot(\cos(\alpha)\mathbf{i} + \cos(\beta)\mathbf{j} + \cos(\gamma)\mathbf{k})\ dS = \int_{\mathcal{V}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\ dV,$$

where $\cos(\alpha)$, $\cos(\beta)$, and $\cos(\gamma)$ are the direction cosines of the exterior normal. Evaluating the dot product puts the divergence theorem in the form

Direction cosines are defined in Section 14.4.

$$\int_{\mathcal{S}} (P\cos(\alpha) + Q\cos(\beta) + R\cos(\gamma))\ dS = \int_{\mathcal{V}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\ dV.$$

When the divergence theorem is expressed in this form, we see that it amounts to three scalar theorems:

$$\int_{\mathcal{S}} P\cos(\alpha)\, dS = \int_{\mathcal{V}} \frac{\partial P}{\partial x}\, dV, \quad \int_{\mathcal{S}} Q\cos(\beta)\, dS = \int_{\mathcal{V}} \frac{\partial Q}{\partial y}\, dV, \quad \text{and} \quad \int_{\mathcal{S}} R\cos(\gamma)\, dS = \int_{\mathcal{V}} \frac{\partial R}{\partial z}\, dV. \tag{18.4.1}$$

As is to be expected, establishing these three equations proves the divergence theorem. We delay the proof to the end of this section, after we have shown how the divergence theorem is applied.

EXAMPLE 1 Let $\mathcal{S}$ be a surface that bounds a solid region $\mathcal{R}$. Assume that the region lies outside $\mathcal{R}$. Evaluate

$$\int_{\mathcal{S}} \frac{\widehat{\mathbf{r}}}{r^2}\cdot\mathbf{n}\ dS.$$

SOLUTION The divergence of $\widehat{\mathbf{r}}/r^2$, the inverse-square central field in space, is 0. By the divergence theorem,

$$\int_{\mathcal{V}} \frac{\widehat{\mathbf{r}}}{r^2}\cdot\mathbf{n}\ dS = \int_{\mathcal{S}} (\operatorname{div}\frac{\widehat{\mathbf{r}}}{r^2})\ dS = \int_{\mathcal{S}} 0\ ds = 0.$$

So the flux of $\widehat{\mathbf{r}}/r^2$ in this case is 0. ◇

You could have guessed the result in this Example by thinking in terms of the solid angle and steradians. Why?

SHERMAN: Not introduced previously? Is this something that can be omitted, or moved to the Exercises? This does appear again later in this chapter, Section 18.6 and its Exercises.

Two-Surface Version of the Divergence Theorem

The divergence theorem also holds if the solid region has several holes like a piece of Swiss cheese. In this case, the boundary consists of several separate connected surfaces. The most important case is when there is just one hole and hence an inner surface $\mathcal{S}_1$ and an outer surface $\mathcal{S}_2$ as shown in Figure 18.4.2.

Theorem. *Divergence Theorem — Two-Surface Case. Let $\mathcal{V}$ be a region in space bounded by the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Let $\mathbf{n}^*$ denote the exterior normal along the boundary. Then*

$$\int_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}^*\, dS + \int_{\mathcal{S}_2} \mathbf{F}\cdot\mathbf{n}^*\, dS = \int_{\mathcal{V}} \operatorname{div}\mathbf{F}\, dV$$

for any vector field defined on $\mathcal{V}$.

The importance of this form of the divergence theorem is that it allows us to conclude that the fluxes(?) across two surfaces are the same provide these surfaces form the boundary of a solid where $\mathbf{F}$ is divergence-free.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two connected surfaces that form the boundary of the region $\mathcal{V}$. Let $\mathbf{F}$ be a vector field defined on $\mathcal{V}$ such that the divergence of $\mathbf{F}$, $\nabla\cdot\mathbf{F}$, is 0 throughout $\mathcal{V}$. Then

$$\int_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}\, dS = \int_{\mathcal{S}_2} \mathbf{F}\cdot\mathbf{n}\, dS \qquad (18.4.2)$$

The proof of this result closely parallels the derivation of (18.2.4) in Section 18.2.

The next example is a major application of (18.4.2), which enables us, if the divergence of $\mathbf{F}$ is 0, to replace the integral of $\mathbf{F}\cdot\mathbf{n}$ over a surface by an integral of $\mathbf{F}\cdot\mathbf{n}$ over a more convenient surface. This appears in physics books that do not mention that the flux of $\mathbf{F}(\mathbf{r}) = \widehat{r}/r^2$ is the solid angle subtended by the surface. For the moment, forget you ever heard of solid angle and steradians.

EXAMPLE 2 Let $\mathbf{F}(\mathbf{r}) = \widehat{r}/r^2$, the inverse square vector field with center at the origin. Let $\mathcal{S}$ be a surface that encloses the origin. Find the flux of $\mathbf{F}$ over the surface $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dS$.

SOLUTION Select a sphere with center at the origin that does not intersect $\mathcal{S}$. This sphere should be very small in order to miss $\mathcal{S}$. Call this spherical surface $\mathcal{S}_1$ and its radius α. Then, by (18.4.2),

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dS = \int_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}\, dS$$

Figure 18.4.2.

SHERMAN: We do not have an explicit two-curve result for Green's Theorem. Compare with (18.2.4) in Exercise 3 in Section 18.2.

SHERMAN: Should the formal statement be deleted, leaving a parallel presentation as for Green's Theorem in Section 18.2?

DOUG: Corollary a bad word?

SHERMAN: Corollary is used is Section 4.1, and again here. There are a few references to these corollaries in Section 6.3 and Section 18.5. Maybe we can find better choice of word for this. Maybe as Examples?

SHERMAN: I do not see that we made such a formal statement in Section 18.2.

SHERMAN: Where would they have heard about solid angle and steradians?

But $\int_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}\, dS$ is easy because the integrand $(\widehat{\mathbf{r}}/r^2)\cdot\mathbf{n}$ is constant: $\mathbf{n} = \widehat{\mathbf{r}}$ so $\widehat{\mathbf{r}}\cdot\mathbf{n} = 1$. Since the radius of the sphere $\mathcal{S}$ is a:

$$\int_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}\, dS = \int_{\mathcal{S}_1} \frac{1}{a^2}\, dS = \frac{1}{a^2}\int_{\mathcal{S}_1} dS = \frac{1}{a^2}4\pi a^2 = 4\pi.$$

◇

A **uniform** or **constant** vector field is a vector field where vectors at every point are all identical. Such fields are used in the next example.

EXAMPLE 3 Verify the divergence theorem for the constant field $\mathbf{F}(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and the surface $\mathcal{S}$ of a cube whose sides have length 5 and is situated as shown in Figure 18.4.3.

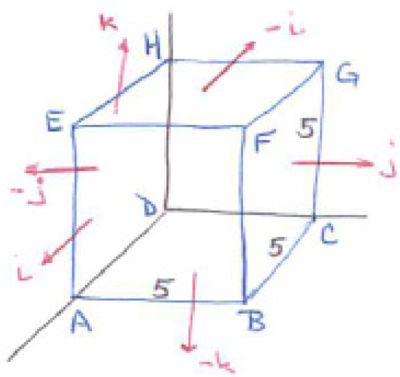


Figure 18.4.3:

SOLUTION To find $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dS$ we consider the integral of $\mathbf{F}\cdot\mathbf{n}$ over each of the six faces.

On the bottom face, $ABCD$ the unit exterior normal is $-\mathbf{k}$. Thus

$$\mathbf{F}\cdot\mathbf{n} = (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})\cdot(-\mathbf{k}) = -4.$$

So

$$\int_{ABCD} \mathbf{F}\cdot\mathbf{n}\, dS = \int_{ABCD} (-4)\, dS = -4\int_{ABCD} dS = (-4)(25) = -100.$$

The integral over the top face involves the exterior unit normal $\mathbf{k}$ instead of $-\mathbf{k}$. Then $\int_{EFGH} \mathbf{F}\cdot\mathbf{n}\, dS = 100$. The sum of these two integrals is 0. Similar computations show that the flux of $\mathbf{F}$ over the entire surface is 0.

The divergence theorem says that this flux equals $\int_{\mathcal{R}} \nabla\cdot\mathbf{F}\, dV$, where $\mathcal{R}$ is the solid cube. Now, $\operatorname{div}\mathbf{F} = \partial(2)/\partial x + \partial(3)/\partial y + \partial(4)/\partial z = 0 + 0 + 0 = 0$. So the integral of $\operatorname{div}\mathbf{F}$ over $\mathcal{R}$ is 0, thus verifying the divergence theorem. ◇

Why div F is Called the Divergence

Let $\mathbf{F}(x, y, z)$ be the vector field describing the **flow field** for a gas. That is, $\mathbf{F}(x, y, z)$ is the product of the density of the gas at (x, y, z) (mass per unit volume) and the velocity vector of the gas there.

The integral $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dS$ over a closed surface $\mathcal{S}$ represents the tendency of the gas to leave the region $\mathcal{R}$ that $\mathcal{S}$ bounds. If that integral is positive the gas is tending to escape or "diverge". If negative, the net effect is for the amount of gas in $\mathcal{R}$ to increase and become denser.

Let $\rho(x, y, z, t)$ be the density of the gas at time t at the point (x, y, z). Then $\int_{\mathcal{R}} \rho\, dV$ is the total mass of gas in $\mathcal{R}$ at a given time. So the rate at which the mass in $\mathcal{R}$ changes as given by the derivative

$$\frac{d}{dt}\int_{\mathcal{R}} \rho\, dV.$$

If ρ is sufficiently well-behaved, mathematicians assure us that we may "differentiate past the integral sign." Then

SHERMAN: Can we direct students to a prior discussion of this?

$$\frac{d}{dt}\int_{\mathcal{R}} \rho\, dV = \int_{\mathcal{R}} \frac{\partial p}{\partial t}\, dV.$$

Therefore

$$\int_{\mathcal{R}} \frac{\partial p}{\partial t}\, dV = \int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dS$$

since both represent the rate at which gas accumulates in or escapes from $\mathcal{R}$. But, by the divergence theorem, $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dS = \int_{\mathcal{R}} \nabla\cdot\mathbf{F}\, dV$, and so

SHERMAN: Reference? The Permanence Principle in Section 3.9 is the only thing along these lines that I recall seeing in this book. Do we need to add it? Where? Should this be an Exercise?

$$\int_{\mathcal{R}} \nabla\cdot\mathbf{F}\, dV = \int_{\mathcal{R}} \frac{\partial p}{\partial t}\, dV$$

or,

$$\int_{\mathcal{R}} (\nabla\cdot\mathbf{F} - \frac{\partial p}{\partial t})\, dV = 0. \qquad (18.4.3)$$

Equation 18.4.3 holds not just for the solid $\mathcal{R}$ but for any solid region within $\mathcal{R}$. By the zero-integral principle, the integrand must be zero thought $\mathcal{R}$, and we conclude that

$$\nabla\cdot\mathbf{F} = \frac{\partial p}{\partial t}.$$

This equation tells us that div $\mathbf{F}$ at a point P represents the rate gas is getting denser or lighter near P. That is why div $\mathbf{F}$ is called the "divergence of $\mathbf{F}$". Where div $\mathbf{F}$ is positive, the gas is dissipating. Where div $\mathbf{F}$ is negative, the gas is collecting.

See Exercise 36 in Section 18.3.

For this reason a vector field for which the divergence is 0 is called "incompressible". It is also called "divergence free".

We conclude this section with a proof of the divergence theorem.

Proof of the Divergence Theorem

We will prove the theorem in the special case that each line parallel to an axis meets the surface $\mathcal{S}$ in at most two points and $\mathcal{V}$ is convex. We prove the third equation in (18.4.1). The other two are established the same way.

We wish to show that

$$\int_{\mathcal{V}} R\cos(\gamma)\ dS = \int_{\mathcal{V}} \frac{\partial R}{\partial z}\ dV. \tag{18.4.4}$$

Let $\mathcal{A}$ be the projection of $\mathcal{S}$ on the xy plane. Its description is

$$a \leq x \leq b, \qquad y_1(x) \leq y \leq y_2(x).$$

The description of $\mathcal{V}$ is then

$$a \leq x \leq b, \qquad y_1(x) \leq y \leq y_2(x), \qquad z_1(x,y) \leq z \leq z_2(x,y).$$

Then (see Figure 18.4.4)

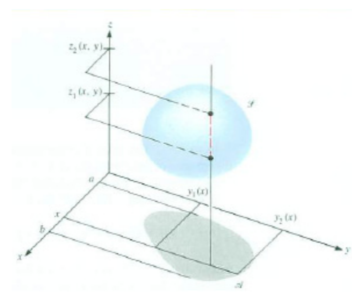


Figure 18.4.4:

$$\int_{\mathcal{V}} \frac{\partial R}{\partial z}\ dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z}\ dz\ dy\ dx. \tag{18.4.5}$$

The first integration gives

$$\int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z}\ dz = R(x,y,z_2) - R(x,y,z_1),$$

by the fundamental theorem of calculus. We have, therefore,

$$\int_{\mathcal{V}} \frac{\partial R}{\partial z}\ dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \left(R(x,y,z_2) - R(x,y,z_1)\right)\ dy\ dx,$$

hence

$$\int_{\mathcal{V}} \frac{\partial R}{\partial z}\ dV = \int_{\mathcal{A}} \left(R(x,y,z_2) - R(x,y,z_1)\right)\ dA.$$

This says that, essentially, on the "top half" of $\mathcal{V}$, where $0 < \gamma < \pi/2$, $dA = \cos(\gamma)\ dS$ is positive. And, on the bottom half of $\mathcal{V}$, where $\pi/2 < \gamma < \pi$, $dA = -\cos(\gamma)\ dS$. According to (17.5.1) in Section 17.5, the last integral equals

$$\int_{\mathcal{S}} R(x,y,z)\cos(\gamma)\ dS.$$

Thus

$$\int_{\mathcal{V}} \frac{\partial R}{\partial z}\ dV = \int_{\mathcal{S}} R\cos\gamma\ dS,$$

and (18.4.4) is established.

Similar arguments establish the other two equations in (18.4.1). •

Summary

We stated the divergence theorem for a single surface and for two surfaces. It enables one to calculate the flux of a vector field $\mathbf{F}$ in terms of an integral of its divergence $\nabla \cdot \mathbf{F}$ over the region. This is especially useful for fields that are divergence free. The most famous such field in space is the inverse-square central vector field. The flux of such a field depends on whether its center is inside or outside the surface. Specifically, if the center is at Q and the field is a constant multiple of $\frac{\overrightarrow{QP}}{\|\overrightarrow{QP}\|^3}$ its flux across a surface not enclosing Q of 0. If it encloses Q, its flux is $4\pi c$. This is a consequence of the divergence theorem. It also can be explained geometrically, in terms of solid angles.

SHERMAN: You appear to have $c/(4\pi)$. Shouldn't this be $4\pi c$?

EXERCISES for 18.4 *Key:* R–routine, M–moderate, C–challenging

1.[R] State Gauss's theorem in symbols.

2.[R] State Gauss's theorem only in words, not using symbols such as $\mathbf{F}$, $\nabla \cdot \mathbf{F}$, $\mathbf{n}$, $\mathcal{S}$, or $\mathcal{V}$.

3.[R] Explain why $\nabla \cdot \mathbf{F}$ at a point P can be expressed as a coordinate-free limit.

4.[R] What is the two-surface version of Gauss's theorem?

5.[R] Verify the divergence theorem for $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + 0\mathbf{k}$ and the surface $x^2 + y^2 + z^2 = 9$.

6.[R] Verify the divergence theorem for the field $\mathbf{F}(x,y,z) = x\mathbf{i}$ and the cube whose vertices are $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$, $(0,0,2)$, $(2,0,2)$, $(2,2,2)$, $(0,2,2)$.

7.[R] Verify the divergence theorem for $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and the tetrahedren whose four vertices are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

8.[R] Verify the two-surface version of Gauss's theorem for $\mathbf{F}(x,y,z) = (x^2+y^2+z^2)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ and the surfaces are the spheres of radii 2 and 3 centered at the origin.

9.[R] Let $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + (5z+6x)\mathbf{k}$, and let

$$\mathbf{G} = (3x + 4z^2)\mathbf{i} + (2y + 5x)\mathbf{j} + 5z\mathbf{k}.$$

Show that

$$\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{\mathcal{S}} \mathbf{G} \cdot \mathbf{n}\, dS,$$

where $\mathcal{S}$ is any surface bounding a region in space.

In Exercises 10 to 17 use the divergence theorem.

10.[R] Let $\mathcal{V}$ be the solid region bounded by the xy plane and the paraboloid $z = 9 - x^2 - y^2$. Evaluate $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = y^3\mathbf{i} + z^3\mathbf{j} + x^3\mathbf{k}$ and $\mathcal{S}$ is the boundary of $\mathcal{V}$.

11.[R] Evaluate $\int_{\mathcal{V}} \nabla \cdot \mathbf{F}\, dV$ for $\mathbf{F} = \sqrt{x^2+y^2+z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ and $\mathcal{V}$ the ball of radius 2 and center at $(0,0,0)$.

In Exercises 12 and 13 find $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$ for the given $\mathbf{F}$ and $\mathcal{S}$. **12.**[R] $\mathbf{F} = z\sqrt{x^2+z^2}\mathbf{i} + (y+3)\mathbf{j} - x\sqrt{x^2+z^2}\mathbf{k}$ and $\mathcal{S}$ is the boundary of the solid region between $z = x^2 + y^2$ and the plane $z = 4x$.

13.[R] $\mathbf{F} = x\mathbf{i} + (3y+z)\mathbf{j} + (4x+2z)\mathbf{k}$ and $\mathcal{S}$ is the surface of the cube bounded by the planes $x = 1$, $x = 3$, $y = 2$, $y = 4$, $z = 3$ and $z = 5$.

14.[R] Evaluate $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$ and $\mathcal{S}$ is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $z = 0$, and $z = 1$, with the face corresponding to $x = 1$ removed.

15.[R] Evaluate $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 2x\mathbf{k}$ and $\mathcal{S}$ is the boundary of the tetrahedron with vertices $(1,2,3)$, $(1,0,1)$ $(2,1,4)$, and $(1,3,5)$.

16.[R] Let $\mathcal{S}$ be a surface of area S that bounds a region $\mathcal{V}$ of volume V. Assume that $\|\mathbf{F}(P)\| \leq 5$ for all points P on the surface $\mathcal{S}$. What can be said about $\int_{\mathcal{V}} \nabla \cdot \mathbf{F}\, dV$?

17.[R] Evaluate $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ and $\mathcal{S}$ is the sphere of radius a and center $(0,0,0)$.

In Exercises 18 to 21 evaluate $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$ for $\mathbf{F} = \widehat{\mathbf{r}}/r^2$ and the given surfaces, doing as little calculation as possible.

18.[R] $\mathcal{S}$ is the sphere of radius 2 and center $(5,3,1)$.

19.[R] $\mathcal{S}$ is the sphere of radius 3 and center $(1,0,1)$.

20.[R] $\mathcal{S}$ is the surface of the box bounded by the planes $x = -1$, $x = 2$, $y = 2$, $y = 3$, $z = -1$, and $z = 6$.

21.[R] $\mathcal{S}$ is the surface of the box bounded by the planes $x = -1$, $x = 2$, $y = -1$, $y = 3$, $z = -1$, and $z = 4$.

22.[M] Assume that the flux of $\mathbf{F}$ across every sphere is 0. Must the flux of $\mathbf{F}$ across every cube be 0 also?

23.[R] If $\mathbf{F}$ is always tangent to a given surface $\mathcal{S}$ what can be said about the integral of $\nabla \cdot \mathbf{F}$ over the region that $\mathcal{S}$ bounds?

24.[M] Let $\mathbf{F}(\mathbf{r}) = f(r)\widehat{\mathbf{r}}$ be a central vector field in space that has zero divergence. Show that $f(r)$ must have the form $f(r) = a/r^2$ for some constant a. Hint: Consider the flux of $\mathbf{F}$ across the closed surface in Figure 18.4.5

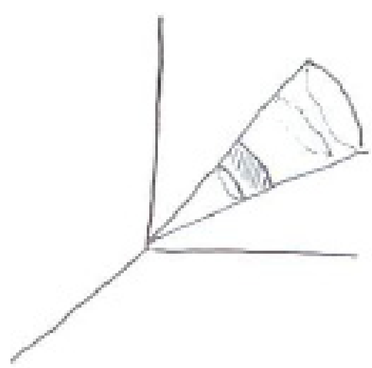


Figure 18.4.5:

25.[M] Let **F** be defined everywhere except at the origin and be divergence-free. Let
$c\mathcal{S}_1$ and $\mathcal{S}_2$ be two closed surfaces that enclose the origin. Explain why $\int_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}\,dS = \int_{\mathcal{S}_2} \mathbf{F}\cdot\mathbf{n}\,dS$. (The two surfaces may intersect.)

26.[C] Express $\nabla\cdot\mathbf{F}$ in cylindrical coordinates if **F** is given in cylindrical coordinates $\mathbf{F}(r,\theta,z) = P\widehat{\mathbf{r}} + Q\widehat{\widehat{\theta}} + R\widehat{\mathbf{z}}$. HINT: Review Exercise 24 in Section 18.2 and Exercise 9 in Section **??** for similar derivations in polar coordinates in the plane.

27.[C]

(a) Show that the proof in the text of the divergence theorem applies to tetrahedron. HINT: Choose your coordinate system carefully.

(b) Deduce that if the divergence theorem holds for tetrahedron then it holds for any polyhedron. HINT: Each polyhedron can be cut into tetrahedron.

28.[C] In Exercise 24 you were asked to show generally that the only central fields with zero divergence are the inverse square fields. Show this, instead, by computing the divergence of $\mathbf{F}(x,y,z) = f(r)\hat{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

29.[C] Let **F** be defined everywhere in space except at the origin. Assume that

$$\lim_{\|\mathbf{r}\|\to\infty} \frac{\mathbf{F}(\mathbf{r})}{\|\mathbf{r}\|^2} = \mathbf{0}$$

and that **F** is defined everywhere except at the origin, and is divergence free. What can be said about $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathcal{S}$ is the sphere of radius 2 centered at the origin?

30.[C] We proved one-third of the divergence theorem.

1. Prove that
$$\int_{\mathcal{S}} Q\cos(\beta)\,dS = \int_{\mathcal{V}} \frac{\partial Q}{\partial y}\,dV$$

2. Prove that
$$\int_{\mathcal{S}} P\cos(\alpha)\,dS = \int_{\mathcal{V}} \frac{\partial P}{\partial x}\,dV$$

18.5 Stokes' Theorem

Stokes' theorem in the xy plane asserts that

$$\oint_{C} \mathbf{F} \cdot \, d\mathbf{r} = \int_{\mathcal{A}} (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA,$$

where C is counterclockwise and C bounds the region $\mathcal{A}$. The general Stokes' theorem extends this result to closed curves in space. (It is this version that is usually called Stokes' theorem.) It asserts that if the closed curve C bounds a surface $\mathcal{S}$, then

$$\oint_{C} \mathbf{F} \cdot \, d\mathbf{r} = \int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.$$

As usual, the vector **n** is a unit normal to the surface. There are two such normals at each point on the surface. In a moment we describe which are to use.

In words, Stokes' theorem reads, "The circulation vector field around a closed curve is equal to the integral of the normal component of the curl of the field over the surface that the curve bounds.

Stokes' published his theorem in 1854 (without proof, for it appeared as a question on a Cambridge University examination.) By 1870 it was in common use. It is the most recent of the three major theorems discussed in this chapter, for green published his theorem in 1828 and Gauss published the divergence theorem in 1839.

Choosing the Normal n

In order to state Stokes' theorem precisely, we must describe what kind of surface $\mathcal{S}$ is permitted and which of the two possible normals **n** to choose.

In the case of a typical surface $\mathcal{S}$ that comes to mind, it is possible to assign at each point P a unit normal **n** in a continuous manner. On the surface shown in Figure 18.5.1, there are two ways to do this. They are shown in Figure 18.5.2. But, for the surface shown in Figure 18.4.3 (a Mobius band), it is impossible to make such a choice. If you start with choice (1) and move the normal continuously along

Figure 18.5.1:

Figure 18.5.2:

the surface, by the time you return to the initial point on the surface at stage (9), you have the opposite normal. A surface for which a continuous choice *can* be made is called **orientable** or **two-sided**. Stokes' theorem holds for orientable surfaces, which include, for instance, any part of the surface of a convex body, such as a ball, cube or cylinder.

Figure 18.5.3:

Follow the choices through nine stages— there's trouble.

Consider an orientable surface $\mathcal{S}$, bounded by a parameterized curve C so that the curve is swept out in a definite direction. If the surface is flat or almost flat, we can simply use the right-hand rule to choose **n**: The direction of **n** should match the thumb of the right hand if the fingers curl in the direction of C and the thumb and palm are perpendicular to the tangent plane to the surface. Figure 18.5.4 illustrates the choice of $\mathbf{n}$. For instance, if C is counterclockwise in the xy plane, this definition picks out the normal $\mathbf{k}$, not $-\mathbf{k}$.

Figure 18.5.4:

Theorem 18.5.1. *Stokes' theorem. Let $\mathcal{S}$ be an orientable surface bounded by the parameterized curve C. At each point of $\mathcal{S}$ let $\mathbf{n}$ be the unit normal chosen by the right-hand rule. Let $\mathbf{F}$ be a vector field defined on some region in space including $\mathcal{S}$. Then*

$$\oint_C \mathbf{F}\cdot\ d\mathbf{r} = \int_{\mathcal{S}} (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS.$$

Some Applications of Stokes' Theorem

Choosing a simpler surface

Stokes' theorem enables us to replace $\int_{\int}(\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS$ by a similar integral over a surface that might be simpler than $\mathcal{S}$. That is the substance of the following corollary of Stokes' theorem.

Corollary 18.5.2. *Let $\mathcal{S}_\infty$ and $\mathcal{S}_\in$ be two surfaces bounded by the same curve C and oriented so that they yield the same orientation on C. let $\mathbf{F}$ be a vector field defined on both $\mathcal{S}_\infty$ and $\mathcal{S}_\in$. Then*

$$\int_{\mathcal{S}_\infty} (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS = int_{\mathcal{S}_\in}(\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS$$

The two integrals are equal since both equal $\oint_C \mathbf{F}\cdot\ d\mathbf{r}$.

EXAMPLE 1 Let $\mathbf{F} = xe^z\mathbf{i} + (x+xz)\mathbf{j} + 3e^z\mathbf{k}$ and let $\mathcal{S}$ be the top half of the sphere $x^2+y^2+z^2=1$. Find $\int_{\mathcal{S}}(\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS$, where $\mathbf{n}$ is the outward normal. (See Figure 18.5.5.)
SOLUTION By Corollary DOUG,

Figure 18.5.5:

$$\int_{\mathcal{S}} (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS = \int_{\mathcal{S}^*} (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dS,$$

where $\mathcal{S}*$ is the flat base of the hemisphere. (On $\mathcal{S}*$ note that $\mathbf{k}$, not $-\mathbf{k}$, is the correct normal to use.)

A straightforward calculation shows that

$$\nabla\times\mathbf{F} = -x\mathbf{i} + xe^z\mathbf{j} + (z+1)\mathbf{k},$$

hence $(\nabla\times\mathbf{F})\cdot\mathbf{k} = z+1$. On $\mathcal{S}*$, $z=0$, so

$$\int_{\mathcal{S}^*} (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dS = \int_{\mathcal{S}*}\ dS = \pi.$$

thus the original integral over $\mathcal{S}$ is π. ◇

Just as there is a two-curve version of Green's theorem there is a two-curve version of Stokes' theorem.

Corollary 18.5.3. *OUG Stokes' theorem (two-curve version). Let $\mathcal{S}$ be an orientable surface whose boundary consists of the two closed curves C_1 and C_2. Give C_1 an orientation. Orient $\mathcal{S}$ consistent with the the right-hand rule, as applied to C_1. Give C_2 the same orientation as C_1. (If C_2 is moved on $\mathcal{S}$ to C_1, the orientation agree.) Then*

$$\oint_{C_1} \mathbf{F}\cdot\ d\mathbf{r} - \oint_{C_2} \mathbf{F}\cdot\ d\mathbf{r} = \int_{\mathcal{S}} (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS.$$

Proof F

igure 18.5.6 shows the typical situation.

Figure 18.5.6:

We will obtain the corollary from Stokes' theorem with the aid of the cancellation principle. Introduce lines AB and CD on $\mathcal{S}$, cutting $\mathcal{S}$ into two surfaces, $\mathcal{S}*$ and $\mathcal{S}**$. Now apply Stokes' theorem to $\mathcal{S}*$ and $\mathcal{S}**$. (See Figure 18.5.7.)

Figure 18.5.7:

Let $C*$ be the curve that bounds $\mathcal{S}*$, oriented so that where it overlaps C_1 it has the same orientation as C_1. Let $C**$ be the curve that bounds $\mathcal{S}**$, again oriented to match C_1. (See Figure 18.5.7.)

By Stokes' theorem,

$$\oint_{C*} \mathbf{F}\cdot\ d\mathbf{r} = \int_{\mathcal{S}*} (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS \qquad (18.5.1)$$

and

$$\oint_{C**} \mathbf{F}\cdot\ d\mathbf{r} = \int_{\mathcal{S}**} (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS. \qquad (18.5.2)$$

DOUG: where did we define "irrotational"

Adding 18.5.1 and 18.5.2 and using the cancellation principle gives

$$\oint_{C_1} \mathbf{F}\cdot\ d\mathbf{r} - \oint_{C_2} \mathbf{F}\cdot\ d\mathbf{r} = \int_{\mathcal{S}} (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS.$$

•

In practice, Corollary **??** is applied when $\nabla\times\mathbf{F} = \mathbf{0}$.

Corollary 18.5.4. *Let $\mathbf{F}$ be a field such that $\nabla\times\mathbf{F} = \mathbf{0}$. Let C_1 and C_2 be two closed curves that together bound an orientable surface $\mathcal{S}$ on which $\mathbf{F}$ is defined. If C_1 and C_2 are similarly oriented, then*

$$\oint_{C_1} \mathbf{F}\cdot\ d\mathbf{r} = \oint_{C_2} \mathbf{F}\cdot\ d\mathbf{r}.$$

Corollary **??** follow directly from Corollary **??** since $\int_{\mathcal{S}} (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS = 0$.

EXAMPLE 2 Assume that $\mathbf{F}$ is irrotational and defined everywhere except on the z axis. Given that $\oint_{C_1} \mathbf{F}\cdot\ d\mathbf{r} = 3$, find (a) $\oint_{C_2} \mathbf{F}\cdot\ d/vr$ and (b) $\oint_{C_3} \mathbf{F}\cdot\ d/vr$. (See Figure 18.5.8.)

SOLUTION (a) By Corollary **??**, $\oint_{C_2} \mathbf{F}\cdot\ d\mathbf{r} = \oint_{C_1} \mathbf{F}\cdot\ d\mathbf{r} = 3$. (b) By Stokes' theorem, $\oint_{C_3} \mathbf{F}\cdot\ d\mathbf{r} = 0$. ⋄

Figure 18.5.8:

Curl and Conservative Fields

In Sec. DOUG we learned that if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is defined on a simply connected region in the xy plane and if $\nabla \times \mathbf{F} = \mathbf{0}$, the $\mathbf{F}$ is conservative. Now that we have Stokes' theorem, this result can be extended to a field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ defined on a simply connected region in space.

A region in space is simply connected if each closed curve in the region can be gradually shrunk to a point while remaining in the region. The entire xyz-space is simply connected. So is the region that consists of all of space except a point or a bounded a line segment or a ball. However, if we delete the z-axis, what remains is not simply connected. (See Figure 18.5.9.)

Figure 18.5.9:

In the plane "simply connected" is the same thing as "no holes"; but in space this is not true.

Theorem 18.5.5. *Let* $\mathbf{F}$ *be defined on a simply connected region in space. If* $\nabla \times \mathbf{F} = \mathbf{0}$*, the* $\mathbf{F}$ *is conservative.*

Proof S

ketch of proof. Let C be a simple closed curve situated in the simply connected region. To avoid topological complexities, we assume that it bounds an orientable surface $\mathcal{S}$. To show that $\oint_C \mathbf{F} \cdot \, dr = 0$, we use the same short argument as in Sec. DOUG:

$$\oint_C \mathbf{F} \cdot \, d\mathbf{r} = \int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \int_{\mathcal{S}} 0 \, dS = 0.$$

•

It follows from Theorem 18.5.5 that every central field $\mathbf{F}$ is conservative: A straight-forward calculation shows that the curl of a central field is $\mathbf{0}$. Moreover, $\mathbf{F}$ is defined either throughout space or everywhere except at the center of the field.

Exercise DOUG of Sec. DOUG prevents a purely geometric argument for why a central field is conservative.

In Section DOUG we show how Stokes' theorem is applied in the theory of electromagnetism.

Why Curl is Called Curl

Let $\mathbf{F}$ be a vector field describing the flow of a fluid, as in Sec. DOUG. Stokes' theorem will give a physical interpretation of $\nabla \times \mathbf{F}$.

Consider a fixed point P_0 in space. Imagine a *small* circular disk $\mathcal{S}$ with center P_0. Let C be the boundary of $\mathcal{S}$ oriented in such a way that C and $\mathbf{n}$ fit the right-hand rule. (See Figure 18.5.10)

Figure 18.5.10:

Now examine the two sides of the equation

$$\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot \mathbf{T} \, ds. \tag{18.5.3}$$

The right side of Equation 18.5.3 measures the tendency of the fluid to move along C (rather than, say, perpendicular to it.) Thus $\oint_C \mathbf{F} \cdot \mathbf{T} \, ds$ might be thought of as the "circulation" or "whirling tendency" of the fluid along C. For each tilt of the small disk $\mathcal{S}$ at P_0–or, equivalently, each choice of unit normal vector $\mathbf{n}$ – $\oint_C \mathbf{F} \cdot \mathbf{T} \, ds$ measures a corresponding circulation. It records the tendency of a paddle wheel at P_0 with axis along $\mathbf{n}$ to rotate. (See Figure 18.5.11.)

Figure 18.5.11:

Consider the left side of (18.5.3). If $\mathcal{S}$ is small, the integrand is almost constant and the integral is approximately

$$(\nabla \times \mathbf{F})_{P_0} \cdot \mathbf{n} \cdot \text{Area of } \mathcal{S}, \tag{18.5.4}$$

where $(\nabla \times \mathbf{F})_{P_0}$ denotes the curl of P_0.

Keeping the center of $\mathcal{S}$ at P_0, vary the vector $\mathbf{n}$ by tilting the disk $\mathcal{S}$. For which choice of $\mathbf{n}$ will (18.5.4) be largest? Answer: For that $\mathbf{n}$ which has the same direction as the fixed vector $(\nabla \times \mathbf{F})_{P_0}$. With that choice of $\mathbf{n}$, (18.5.4) becomes

$$\|(\nabla \times \mathbf{F})_{P_0}\| \text{Area of } \mathcal{S}.$$

The physical interpretation of curl

Thus a paddle wheel placed in the fluid at P_0 *rotates most quickly when its axis is in the direction of* $\nabla \times \mathbf{F}$ at P_0. The *magnitude* of $\nabla \times \mathbf{F}$ is a measure of *how fast* the paddle wheel can rotate when placed at P_0. Thus $\nabla \times \mathbf{F}$ records the direction and magnitude of maximum circulation at a given point.

A vector Definition of Curl

In Sec. DOUG $\nabla \times \mathbf{F}$ was defined in terms of the partial derivatives of the components of $\mathbf{F}$. By Stokes' theorem, $\nabla \times \mathbf{F}$ is related to the circulation, $\oint_C vF \cdot \, d\mathbf{r}$. We exploit this relation to obtain a new view of $\nabla \times \mathbf{F}$, free of coordinates.

Let P_0 be a point in space and let $\mathbf{n}$ be a unit vector. Consider a small disk $\mathcal{S}_{\mathbf{n}}(\dashv)$, perpendicular to $\mathbf{n}$, whose center is P_0, and which has a radius a. Let $C_{\mathbf{n}}(a)$ be the boundary of $\mathcal{S}_{\mathbf{n}}(\dashv)$, oriented to be compatible with the right-hand rule. Then

$$\int_{\mathcal{S}_{\mathbf{n}}(\dashv)} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_{C_{\mathbf{n}}(a)} \mathbf{F} \cdot \, d\mathbf{r}.$$

As in our discussion of the physical meaning of curl, we see that

$$(\nabla \times \mathbf{F})_{P_0} \cdot \mathbf{n}(\text{Area of} \mathcal{S}_\backslash(\dashv)) \approx \oint_{\mathcal{C}_\backslash(\dashv)} \mathbf{F} \cdot \, \lceil\mathbf{r},$$

or

$$(\nabla \times \mathbf{F})_{P_0} \cdot \mathbf{n} \approx \frac{\oint_{C_n(a)} \mathbf{F} \cdot \, d\mathbf{r}}{\text{Area of} \mathcal{S}_\backslash(\dashv)}.$$

Thus

$$(\nabla \times \mathbf{F})_{P_0} \cdot \mathbf{n} = \lim_{a \to 0} \frac{\oint_{C_n(a)} \mathbf{F} \cdot \; d\mathbf{r}}{\text{Area of}\mathcal{S}_{\backslash}(\dashv)}. \tag{18.5.5}$$

Equation 18.5.5 gives meaning to the component of $(\nabla \times \mathbf{F})_{P_0}$ in any direction $\mathbf{n}$. So the magnitude and direction of $\nabla \times \mathbf{F}$ at P_0 can be described in terms of $\mathbf{F}$, without looking at the components of $\mathbf{F}$.

The magnitude of $(\nabla \times \mathbf{F})_{P_0}$ is the maximum value of

$$\lim_{a \to 0} \frac{\oint_{C_n(a)} \mathbf{F} \cdot \; d\mathbf{r}}{\text{Area of}\mathcal{S}_{\backslash}(\dashv)}, \tag{18.5.6}$$

for all unit vectors $\mathbf{n}$.
The direction of $(\nabla \times \mathbf{F})_{P_0}$ is given by the vector $\mathbf{n}$ that maximizes Equation 18.5.6.

EXAMPLE 3 Let $\mathbf{F}$ be a vector field such that at the origin $\nabla \times \mathbf{F} = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$. Estimate $\oint_C \mathbf{F} \cdot \; d\mathbf{r}$ if C encloses a disk of radius 0.01 in the xy plane with center $(0, 0, 0)$. C is swept out clockwise. (See Figure 18.5.12.)

SOLUTION Let $\mathcal{S}$ be the disk whose border is C. Choose the normal to $\mathcal{S}$ that is consistent with the orientation of C and the right-hand rule. That choice is $-\mathbf{k}$. Thus

$$(\nabla \times \mathbf{F}) \cdot (-\mathbf{k}) \approx \frac{\oint_C \mathbf{F} \cdot \; d\mathbf{r}}{\text{Area of}\mathcal{S}}.$$

Figure 18.5.12:

The area of $\mathcal{S}$ is $\pi(0.01)^2$ and $\nabla \times \mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. Thus

$$(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \cdot (-\mathbf{k}) \approx \frac{\oint_C \mathbf{F} \cdot \; d\mathbf{r}}{\pi(0.01)^2}.$$

¿From this it follows that

$$\oint_C \mathbf{F} \cdot \; d\mathbf{r} \approx -4\pi(0.01)^2.$$

◇

In a letter to the mathematician Tait written on November 7, 1870, Maxwell offered some names for $\nabla \times \mathbf{F}$:
Here are some rough-hewn names. Will you like a good Divinity shape their ends property so as to make them stick? ...
The vector part [$\nabla \times \mathbf{F}$] I would call the twist of the vector function. Here the work twist has nothing to do with a screw or helix. The word *turn* ... would be better than twist, for twist suggests a screw. Twirl is free from the screw motion and is sufficiently racy. Perhaps it is too dynamical for pure mathematicians, so for Cayley's sake I might say Curl (after the fashion of Scroll.)
His last suggestion, "curl," has stuck.

Proof of Stokes' Theorem

The proof uses Green's theorem, the normal to a surface $z = f(x, y)$, and expressing an integral over a surface to an integral over its shadow on a plane. The approach is straightforward. As usual, we begin by expressing the theorem in terms of components. We will assume that the surface $\mathcal{S}$ meets each line parallel to all axes in at most all points That permits us to project $\mathcal{S}$ onto each coordinate plane in an one-one fashion.

To begin we write $\mathbf{F}9x, y, z)$ as $P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, u, z)\mathbf{k}$, or, simply $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + r\mathbf{k}$. We will project $\mathcal{S}$ onto the xy-plane, so write the equation for $\mathcal{S}$ as $z - f(x, y) = 0$. A unit normal to $\mathcal{S}$ is

$$\mathbf{n} = \frac{-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2 + 1}}.$$

Let $C*$ be the projection of C on the xy plane, swept out counterclockwise. (Since the $\mathbf{k}$ component of $\mathbf{n}$ is positive, it is the correct normal, given by the right-hand rule.)

A straightforward computation shows that Stokes' theorem, expressed in components, reads

$$\int_C P\,dx + Q\,dy + R\,dz = \int_{\mathcal{S}} \frac{[\left(\frac{\partial R}{\partial x} - \frac{\partial Q}{\partial z}\right)\left(-\frac{\partial f}{\partial x}\right) - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\left(-\frac{\partial f}{\partial y}\right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)(1)}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}\,dS.$$

As expected, this equation reduces to these three equations, on P, on Q and on R.

We will establish the area on P, namely

$$\int_C P\,dx = \int_{\mathcal{S}} \frac{[\frac{\partial P}{\partial z}(-\frac{\partial f}{\partial y}) - frac\partial P\partial y(1)}{\sqrt{(\frac{\partial f}{\partial x}^2) + (\frac{\partial f}{\partial y})^2 + 1}}. \tag{18.5.7}$$

To change the integral over $\mathcal{S}$ to an integral over its projection on the xy-plane, we replace dS by $\sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1}$, changes the same time we project C out a counterclockwise curve $C*$. The square roots cancel leaving us with this equation in the xy-plane.

$$\int_{C*} P(x, y, f(x, y))\,ds = \int_R \left(-\frac{\partial P}{\partial z}\frac{\partial f}{\partial y} - \frac{\partial P}{\partial y}\right)\,dA. \tag{18.5.8}$$

Finally, we apply Green's theorem to the left side of (18.5.8). It asserts that

$$\int_{C*} P(x, y, f(x, y))\,ds = \int_R -\frac{\partial P(x, y, f(x, y))}{\partial y}\,dA.$$

But

$$\frac{\partial P(x,y,f(x,y))}{\partial y} = \frac{\partial p}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial f}{\partial y}. \qquad (18.5.9)$$

Combining (**??**) and (**??**) completes the proof.

In this proof we assumed that the surface $\mathcal{S}$ has a special form, meeting lines parallel to an axis just once. However, more general surfaces, such as the surface of a sphere or a polyhedron can be cut into pieces treated in the proof. Exercise DOUG shows why this observation then implies that Stokes theorem hold in these cases also.

NOTE TO DOUG: In V DOUG the "cancellation principle" I deleted it.

Summary

Stokes' theorem relates the circulation of a vector field over a closed curve C to the integral over a surface $\mathcal{S}$ that C bounds. The integrand over the surface is the component of the curl of the field perpendicular to the surface,

$$\int\limits_C \mathbf{F}\cdot\ d\mathbf{r} = \int\limits_{\mathcal{S}} 9\nabla \times \mathbf{F})\cdot \mathbf{n}\ dS.$$

The normal $\mathbf{n}$ is the area given by the right-hand rule.

EXERCISES for 18.5 *Key:* R–routine, M–moderate, C–challenging

DOUG: Too many xrcs's. Some repetition? Some go to Chapter Summary?

1.[C] Draw a picture showing $\mathcal{S}$, C, $\mathcal{S}*$, $C*$, and $\mathbf{n}$ with proof of Stokes' theorem.

2.[C] Carry out the calculations that yield the component form of Stokes' theorem.

3.[C] We dealt only with the component P. What is the analog of DOUG for Q? Prove it. (HINT: the steps would parallel the steps used for P.

4.[R] State Stokes' theorem (symbols permitted).

5.[R] State Stokes' theorem in words (symbols not permitted).

6.[M] Explain why DOUG holds if $\mathcal{S}_\infty$ and $\mathcal{S}_\in$ together form the boundary surface $\mathcal{S}$ of a solid region R. Use the divergence theorem, not Stokes' theorem.

7.[R] Let $F(r)$ be an antiderivative of $f(r)$. Show that $f(r)\frac{\mathbf{r}}{r}$ is the gradient of $F(r)$, hence is conservative.

8.[M] Show that a central field $f(r)\mathbf{r}/r$ is conservative by showing that it is irrotational and defined on a simply connected region. Suggestion: express $\mathbf{r}$ and r in terms of x, y and z.

9.[R]

(a) Using the fact that a gradient, ∇f, is conservative, show that the curl of a gradient is $\mathbf{0}$.

(b) Compute $\nabla \times \nabla f$ in terms of components to show that the curl of a gradient is $\mathbf{0}$.

10.[C]

Sam: The only conservative field in space that I know are the "inverse square central fields" with centers anywhere I please.

Jane: There are a lot more.

Sam: Oh?

Jane: Just start with any scalar function $f(x, y, z)$ with continuous partial derivation of the first and second orders. Then its gradient will be a conservative field.

Sam: O.K. But I bet there are still more.

Jane: No. I got them all.

Question: Who is right?

Exercises 11 to 13 concern the proof of Stokes' theorem.

11.[C] Carry out the calculations in the proof that translated Stokes' theorem into an equation involving the components P, Q, and R.

12.[C] In the proof of Stokes' theorem we used a normal $\mathbf{n}$. Show that it is the "correct" one, compatible with counterclockwise orientation of $C*$.

13.[C] Draw a picture of $\mathcal{S}$, $\mathcal{S}*$, C and $C*$ that appear in the proof.

Intro to Exercises

14.[M] Assume that $\mathbf{G}$ is the curl of another vector field $\mathbf{F}$, $\mathbf{G} = \nabla \times \mathbf{F}$. Let $\mathcal{S}$ be a surface that bounds a solid region V. Let C be a closed curve as the surface $\mathcal{S}$ breaking $\mathcal{S}$ into two pieces $\mathcal{S}_\infty$ and $\mathcal{S}_\in$.

15.[M] Using the divergence theorem, show that $\int_\mathcal{S} \mathbf{G} \cdot \mathbf{n}\ dS = 0$.

16.[M] Using Stokes' theorem, show that $\int_\mathcal{S} \mathbf{G} \cdot \mathbf{n}\ dS = 0$. Hint: Break the integral into integrals over $\mathcal{S}_\infty$ and $\mathcal{S}_\in$.

17.[R] Let $\mathbf{F} = e^{xy}\mathbf{i} + \tan 3yz\mathbf{j} + 5z\mathbf{k}$ and $\mathcal{S}$ be the tetrahedron whose vertices are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Let $\mathcal{S}_\infty$ be the base of $\mathcal{S}$ and $\mathcal{S}_\in$ consist of the other three faces. Find $\int_\mathcal{S}(\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS$. Hint: think about the preceding two exercises.

18.[R] Assume that $\mathbf{F}$ is defined everywhere except on the z axis and is irrotational. What, it anything, can be said about

$$\oint_{C_1} \mathbf{F} \cdot\ d\mathbf{r}, \qquad , \oint_{C_2} \mathbf{F} \cdot\ d\mathbf{r}, \qquad \oint_{C_3} \mathbf{F} \cdot\ d\mathbf{r}, \qquad \text{and} \qquad \oint_{C_4} \mathbf{F} \cdot\ d\mathbf{r},$$

where the curves are shown in Figure 18.5.13?

Figure 18.5.13:

In Exercises 19 to 22 verify Stokes' theorem for the given $\mathbf{F}$ and surface $\mathcal{S}$.

19.[R] $\mathbf{F} = xy^2\mathbf{i} + y^3\mathbf{j} + y^2z\mathbf{k}$; $\mathcal{S}$ is the top half of the sphere $x^2 + y^2 + z^2 = 1$.

20.[R] $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$; $\mathcal{S}$ is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

21.[R] $\mathbf{F} = y^5\mathbf{i} + x^3\mathbf{j} + z^4\mathbf{k}$; $\mathcal{S}$ is the portion of $z = x^2 + y^2$ below the plane $z = 1$.

22.[R] $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$, $\mathcal{S}$ is the portion of the cylinder $z = x^2$ inside the cylinder $x^2 + y^2 = 4$.

23.[R] Evaluate as simply as possible $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\ dS$, where $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j}$ and $\mathcal{S}$ is the surface of the cube bounded by the three coordinate planes and the planes $x = 1$, $y = 1$, $z = 1$, exclusive of the surface in the plane $x = 1$. (Let $\mathbf{n}$ be outward from the cube.)

24.[R] Using Stokes' theorem, evaluate $\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS$, where $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$, and $\mathcal{S}$ is the portion of the surface $z = 4 - (x^2 + y^2)$ above the xy plane. (Let $\mathbf{n}$ be the upward normal.)

In each of Exercises 25 to 28 use Stokes' theorem to evaluate $\oint_C \mathbf{F} \cdot\ d\mathbf{r}$ for the given $\mathbf{F}$ and C. In each case assume that C is oriented counterclockwise when viewed from above.

25.[R] $\mathbf{F} = \sin xy\mathbf{i}$; C is the intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

26.[R] $\mathbf{F} = e^x\mathbf{j}$; C is the triangle with vertices $(2, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 4)$.

27.[R] $\mathbf{F} = xy\mathbf{k}$; C is the intersection of the plane $z = y$ with the cylinder $x^2 - 2x + y^2 = 0$.

28.[R] $\mathbf{F} = \cos(x + z)\mathbf{j}$; C is the boundary of the rectangle with vertices $(1, 0, 0)$, $(1, 1, 1)$, $(0, 1, 1)$, and $(0, 0, 0)$.

29.[R] Let $\mathcal{S}_\infty$ be the top half and $\mathcal{S}_\in$ the bottom half of a sphere of radius a in space. Let $\mathbf{F}$ be a vector field defined on the sphere and let $\mathbf{n}$ denote an exterior normal to the sphere. What relation, if any, is there between $\int_{\mathcal{S}_\infty} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS$ and $\int_{\mathcal{S}_\in} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS$?

30.[R] Let $\mathbf{F}$ be a vector field throughout space such that $\mathbf{F}(P)$ is perpendicular to the curve C at each point P on C, the boundary of a surface $\mathcal{S}$. What can one conclude about

$$\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS?$$

31.[R] Let C_1 and C_2 be two closed curves in the xy plane that encircle the origin and are similarly oriented, as in Figure 18.5.14. Let $\mathbf{F}$ be a vector field defined

Figure 18.5.14:

throughout the plane except at the origin. Assume that $\nabla \times \mathbf{F} = \mathbf{0}$.

(a) Must $\oint_C = \mathbf{F}\ d\mathbf{r} = 0$?

(b) What, it any, relation exists between $\oint_{C_1} \mathbf{F} \cdot\ d\mathbf{r}$ and $\oint_{C_2} \mathbf{F} \cdot\ d\mathbf{r}$?

32.[R] Let $\mathbf{F}$ be defined everywhere in space except on the z axis. Assume also that $\mathbf{F}$ is irrotational, $\oint_{C_1} \mathbf{F} \cdot\ d\mathbf{r} = 3$, and $\oint_{C_2} \mathbf{F} \cdot\ d\mathbf{r} = 5$. (See Figure 18.5.15.) What if, anything, can be said about

(a) $\oint_{C_3} \mathbf{F} \cdot\ d\mathbf{r}$,

(b) $\oint_{C_4} \mathbf{F} \cdot\ d\mathbf{r}$?

Figure 18.5.15:

33.[R] Which of the following sets are connected? simply connected?

(a) A circle ($x^2 + y^2 = 1$)

(b) A disk ($x^2 + y^2 \leq 1$)

(c) The xy plane from which a circle is removed

(d) The xy plane from which a disk is removed

(e) The xy plane from which one point is removed

(f) xyz space from which one point is removed

(g) xyz space from which a sphere is removed

(h) xyz space from which a ball is removed

(i) A solid torus (doughnut)

(j) xyz space from which a solid torus is removed

(k) A coffee cup with one handle

34.[R] Which central fields have curl $\mathbf{0}$?

35.[R] Let $\mathcal{V}$ be the solid bounded by $z = x + 2$, $x^2 + y^2 = 1$, and $z = 0$. Let $\mathcal{S}_\infty$ be the portion of the plane $z = x+2$ that lies within the cylinder $x^2+y^2 = 1$. Let C be the boundary of $\mathcal{S}_\infty$, with a counterclockwise orientation (as viewed from above. Let $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + (x + 2y)\mathbf{k}$. Use Stokes' theorem for $\mathcal{S}_\infty$ to evaluate $\oint_C \mathbf{F} \cdot \, d\mathbf{r}$.

36.[R] (See Exercise 35.) Let $\mathcal{S}_\in$ be the curved surface of $\mathcal{V}$ together with the base of $\mathcal{V}$. Use Stokes' theorem for $\mathcal{S}_\in$ to evaluate $\oint_C \mathbf{F} \cdot \, d\mathbf{r}$.

37.[R] Verify Stokes' theorem for the special case when $\mathbf{F}$ has the form ∇f, that is, is a gradient field.

38.[R] Let $\mathbf{F}$ be a vector field defined on the surface $\mathcal{S}$ of a convex solid. Show that $\int_\mathcal{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 0$

(a) by the divergence theorem,

(b) by drawing a closed curve on C on $\mathcal{S}$ and using Stokes' theorem on the two parts into which C divides $\mathcal{S}$.

39.[R] Evaluate $\oint_C \mathbf{F} \, d\mathbf{r}$ as simply as possible if $\mathbf{F}(x, y, z) = -y\mathbf{i}/(x^2 + y^2) + x\mathbf{j}/(x^2 + y^2)$ and C is the intersection of the plane $z = 2x + 2y$ and the paraboloid $z = 2x^2 + 3y^2$ oriented counterclockwise as viewed from above.

40.[R] Let $\mathbf{F}9x, y)$ be a vector filed defined everywhere in the plane except at the origin. Assume that $\nabla \times \mathbf{F} = \mathbf{0}$. Let C_1 be the circle $x^2 + y^2 = 1$ counterclockwise; let C_2 be the circle $x^2 + y^2 = 4$ clockwise; let C_3 be the circle $(x - 2)^2 + y^2 = 1$ counterclockwise; let C_4 be the circle$(x - 1^2_) + y^2 = 9$ clockwise. Assuming that $\oint_{C_1} \mathbf{F} \cdot \, d\mathbf{r}$ is 5, evaluate

(a) $\oint_{C_2} \mathbf{F} \cdot \, d\mathbf{r}$

(b) $\oint_{C_3} \mathbf{F} \cdot \, d\mathbf{r}$

(c) $\oint_{C_4} \mathbf{F} \cdot \, d\mathbf{r}$.

41.[M] Let $\mathbf{F}(x, y, z) = \mathbf{r}/\|\mathbf{r}\|^a$, where $\mathbf{r} = \mathbf{i} + u\mathbf{j} + z\mathbf{k}$ and a is a fixed real number.

(a) Show that $\nabla \times \mathbf{F} = \mathbf{0}$.

(b) Show that $\mathbf{F}$ is conservative.

(c) Exhibit a scalar function f such that $\mathbf{F} = \nabla f$.

42.[M] Let $\mathbf{F}$ be defined throughout space and have continuous divergence and curl.

(a) For which $\mathbf{F}$ is $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS = 0$ for all spheres $\mathcal{S}$?

(b) For which $\mathbf{F}$ is $\int_C \mathbf{F} \cdot\, d\mathbf{r} = 0$ for all circles C?

(c) If $\int_C \mathbf{F} \cdot\, d\mathbf{r} = 0$ for all circles C, must $\int_C \mathbf{F} \cdot\, d\mathbf{r} = 0$ for all closed curves?

43.[M] Let C be the curve formed by the intersection of the plane $z = x$ and the paraboloid $z = x^2 + y^2$. Orient C to be counterclockwise when viewed from above. Evaluate $\oint_C (xyz\, dx + x^2\, dy + xz\, dz)$.

44.[M] Assume that Stokes' theorem is true for triangles. Deduce that it then holds for the surface $\mathcal{S}$ in Figure 18.5.16, consisting of the three triangles DAB, DBC, DCA, and the curve $ABCA$.

Figure 18.5.16:

45.[C] A Möbius band can be made by making a half-twist in a narrow rectangular strip, bringing the two ends together, and fastening them with glue or tape. See Figure 18.5.17.

Figure 18.5.17:

(a) Make a Möbius band.

(b) Letting a pencil represent a normal $\mathbf{n}$ to the band, check that the Möbius band is not orientable.

(c) If you form a band by first putting in a full twist (360°), is it orientable?

(d) What happens when you cut the bands in (b) and (c) down the middle? on third of the way from one edge to the other?

46.[C]

(a) Explain why the line integral of a central vector field around the path in Figure 18.5.18 is 0.

(b) Deduce from (a) and the coordinate-free view of curl that the curl of a central fields is **0**.

Figure 18.5.18:

47.[C]

(a) The proof of Stokes' theorem we gvae would not apply to surfaces that are more complicated, such as the 'top three fourths of a sphere," as shown in Figure 18.5.19. However, how called you cut $\mathcal{S}$ into pieces to each of which

Figure 18.5.19:

the proof applies? (Describe then in general terms, in words.)

(b) How could you use (a) to show that Stokes' theorem hold for C and $\mathcal{S}$ in Figure 18.5.19

18.6 Central Fields

DOUG: Should we switch the order of this and Stokes? (It refers to Stokes'!)

A special but important type of vector field appear in the study of gravity or the attraction or repulsion of electric charges. It is "central fields" that radiates from a point mass or point charge. Physicists invented these fields in order to avoid the mystery of "action at a distance." One particle acts on another directly, through the vector field it creates. This comforts students of gravitation of electromagnetism by glossing over the riddle of how an object can act upon and then without DOUG I CAN'T READ THE REST.

Central Fields

A **central field** is a continuous vector field defined everywhere in the plane (or in space) except, perhaps, at a point **0**, with these two properties:

1. Each vector points towards (or away from) **0**.
2. The magnitudes of all vectors at a given distance from **0** are equal.

0 is call the center of the field. A central field is also called "radially symmetric." There are various ways to think of a central vector field. For such a field in the plane, the vector at point on a circle with center **0** are perpendicular to the circle and have the same length, as shown in Figure 18.6.1.

Figure 18.6.1:

The same holds for central vector fields in space, with "circle" replace by "sphere."

The formula for a central vector field has a particularly simple form. Let the field be $\mathbf{F}$ and P any point other than **0**. Then there is a scalar function f, defined for all positive numbers, such that

$$\mathbf{F}(P) = f(|\overrightarrow{OP}|)\frac{\overrightarrow{OP}}{|\overrightarrow{OP}|}.$$

Here, $\overrightarrow{OP}/|\overrightarrow{OP}|$ is a unit vector parallel to the radius vector $\overrightarrow{OP}$. The magnitude of $\mathbf{F}(P)$ is $|f(|\overrightarrow{OP}|)|$. If $f(|\overrightarrow{OP}|)$ is positive, $\mathbf{F}(P)$ points away from **0**. If $f(|\overrightarrow{OP}|)$ is negative, $\mathbf{F}(P)$ points toward **0**.

Central Vector Fields in the Plane

Using polar coordinates with pole placed at the point **0**, we may express a central field in the form

$$\mathbf{F}(\mathbf{r}) = f(r)\widehat{\mathbf{r}},$$

where $r = |\mathbf{r}|$ and $\widehat{\mathbf{r}} = \mathbf{r}/r$. The magnitude of $\mathbf{F}(\mathbf{r})$ is $|f(r)|$.

We already met such a field in Section DOUG in the study of line integrals. That was the case, $f(r) = 1/r$, the "field varied as the inverse first power." The line integral for the normal component of this field along a curve gives the number of radians the curve subtends.

$\mathbf{F}(\mathbf{r}) = \frac{1}{r}\widehat{\mathbf{r}}$ can also be written

$$\mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}}{r^2}. \tag{18.6.1}$$

When glancing too quickly at (18.6.1), you might think its magnitude is inversely proportional to the square of r. However, the magnitude of the vector $\mathbf{r}$ in the DOUG is r; the magnitude of $\mathbf{r}/r^2$ is DOUG the reciprocal of the first power of r.

EXAMPLE 1 Evaluate the flux $\oint_C \mathbf{F} \cdot n \, ds$ for the central field $\mathbf{F}(x, y) = f(r)\widehat{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ over the closed curve shown in Figure 18.6.2.

Figure 18.6.2:

We have $a < b$ and the path goes from $A = (a, 0)$ to $B = (b, 0)$ to $C = (0, b)$, to $D = (0, a)$ and ends at $A = (a, 0)$.

SOLUTION On the parts from A to B and from C to D $\mathbf{F}$ is perpendicular to the exterior normal $\mathbf{n}$, so $\mathbf{F} \cdot \mathbf{n} = 0$, and these integrals DOUG nothing to the integral. On BC, $\mathbf{F}$ equals $f(b)\widehat{\mathbf{r}}$. there $\widehat{r} = \mathbf{n}$, so $\mathbf{F} \cdot \mathbf{n} = f(b)$ since $\mathbf{r} \cdot \mathbf{n} = 1$. Thus

$$\int_B^C \mathbf{F} \cdot \mathbf{n} \, ds = \int_B^C f(b) \, ds = f(b) \int_B^C \, ds = f(b)(\frac{1}{4})(2\pi b) = \frac{\pi b}{2} f(b)$$

On the arc DC, $\widehat{\mathbf{r}} = -\mathbf{n}$. A similar calculation shows that

$$\int_D^C \pi \cdot \mathbf{n} \, ds = -\frac{\pi}{2} a f(a).$$

Hence

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{n} \, ds &= 0 + \frac{\pi}{2} b f(b) + 0 - \frac{\pi}{2} a f(a) \\ &= \frac{\pi}{2}(b f(b) - a f(a)). \end{aligned}$$

◇

In order for a central field $f(r)\widehat{r}$ to have zero flux around all paths of the special type shown in DOUG, we must have

$$f(b)b - f(a)a = 0,$$

for all positive a and B. In particular,

$$f(b)b - f(1)1 = 0$$

or

$$f(b) = \frac{f(1)}{b}.$$

Thus $f(r)$ must be inversely DOUG to r, that is, there is a constant C such that

$$f(r) = \frac{c}{r}.$$

If $f(r)$ is not of the form c/r, the vector field $\mathbf{F}(x, y) = f(r)\widehat{r}$ does not have zero flux across these paths. In Exercise 2 you may compute $\nabla \cdot ((c/r)\widehat{\mathbf{r}})$ and show that it is zero.

> The only central vector field with center at the origin in the plane with zero divergence are these whose magnitude is inversely proportional to the distance from the origin.

We underline "in the plane," because in space only central fields with flux across closed surfaces a magnitude inversely proportional to the square of the distance, as we will see in a moment.

Knowing that the central field $\mathbf{F}\widehat{\mathbf{r}}/r$ has zero divergence enables us to evaluate easily line integrals of the form $\oint_C \frac{\widehat{\mathbf{r}}\ cdotd\mathbf{n}}{r}\,ds$, as the next example shows.

EXAMPLE 2 Let $\mathbf{F}(\mathbf{r}) = \widehat{\mathbf{r}}/r$. Evaluate $\oint_C \mathbf{F} \cdot \mathbf{n}\,ds$ where C is the counterclockwise circle of radius 1 and center $(2, 0)$, as shown in Figure 18.6.3.

Figure 18.6.3:

SOLUTION The field $\mathbf{F}$ has 0-diveregence throughout C and the region R that C bounds. By Green's theorem, the integral also equals the integral of the divergence over R:

$$\oint\limits_C \mathbf{F} \cdot \mathbf{n}\,ds = \int\limits_R \nabla \cdot \mathbf{F}\,dA. \qquad (18.6.2)$$

Since the divergence of $\mathbf{F}$ is 0, throughout R, the integral on the right side of (18.6.2) is 0. Therefore $\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = 0$. ◇

The next example involves a curve that surrounds a point where the vector field $\mathbf{F} = \widehat{r}/r$ is not defined.

EXAMPLE 3 Let C be a simple closed curve enclosing the origin. Evaluate $\oint \mathbf{F} \cdot \mathbf{n}\,ds$, where $\mathbf{F} = \widehat{\mathbf{r}}/r$.

SOLUTION Figure 18.6.4 shows C and a small circle D centered at the origin and

Figure 18.6.4:

situated in the region that C bounds. Without a formula describing C, we could not compute $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$ directly. However, since the divergence of $\mathbf{F}$ is 0 throughout the region bounded by C and D, we have, by DOUG in Section DOUG.

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_D \mathbf{F}\cdot\mathbf{n}\,ds. \tag{18.6.3}$$

The integral on the right-hand side of (18.6.3) is easy to compute directly. To do so, let the radius of D be a. Then for points P on D, $\mathbf{F}(P) = \widehat{\mathbf{r}}/a$. Now, $\widehat{\mathbf{r}}$ and $\mathbf{n}$ all the same unit vector. So $\widehat{\mathbf{r}}\cdot\mathbf{n} = 1$. Thus

$$\oint_D \mathbf{F}\cdot\mathbf{n}\,ds = \oint_D \frac{r\cdot\mathbf{n}}{a}\,ds = \int_D \frac{1}{a}\,ds = \frac{1}{a}2\pi a = 2\pi.$$

Hence $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 2\pi$. ◇

Central Fields in Space

A central field in space with center at the origin has the form $\mathbf{F}(x,y,z) = f(r)\overrightarrow{\mathbf{r}}$, where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, and $P = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$. We show that if the flux of $\mathbf{F}$ over any surface bounding certain special regions is zero then $f(r)$ must be inversely proportional to the square of r.

Compare to DOUG in the plane.

Consider the surface S shown in Figure 18.6.5. It consists of an octant of two DOUG spheres, are of radius 2, the other of radius b, a, b, together with the DOUG surfaces on the coordinate planes.

Figure 18.6.5:

Let R be the region that surface S bounds. On its three flat sides $\mathbf{F}$ is perpendicular to the exterior normal. On the outer sphere $\mathbf{F}(x,y,z)\cdot\mathbf{n} = f(f)$. On the inner sphere $\mathbf{F}(x,y,z)\cdot\mathbf{n} = -f(b)$. Thus

$$\begin{aligned}\oint_S \mathbf{F}\cdot\mathbf{n}\,dS &= f(b)(\frac{1}{8})(4\pi b^2) - f(a)(\frac{1}{8})4\pi a^2 \\ &= \frac{\pi}{2}(f(b)b^2 - f(a)a^2).\end{aligned}$$

Recall the surface area of a sphere of radius r is $4\pi r^2$.

Since this is to be 0 for all positive a and b, it follows that there is a constant C, such that

$$f(\mathbf{r}) = \frac{c}{r^2}.$$

The magnitude must be proportional to the "inverse square."

As we will see in Sec. DOUG the flux of an inverse-square central fields, $c\widehat{r}/r^2$, across any closed surface that bounds a region which does not contain the origin is zero.

A Geometric Application

As we will see later in this chapter this "inverse square" central field is at the heart of gravitational theory and electrostatics. Now we show how it is used in geometry, a result we will apply both in these fields.

In Sec. DOUG we showed how radian measure could be expressed in term of the line integral $\int_C (\widehat{\mathbf{r}}/r) \cdot \mathbf{n}\ ds$, that is, in terms of the line field whose magnitude is inversely proportional to the first power of the distance from the center. That was based on circular arcs in a plane. Now we move up one dimension and consider patches on surfaces of spheres, which will help us measure solid angles.

Let O be a point and $\mathcal{S}$ a surface such that each ray from O meets $\mathcal{S}$ in at most one point. Let $\mathcal{S}*$ be the unit sphere with center at O. The rays from O that meet $\mathcal{S}$ intersect $\mathcal{S}*$ in a set that we call $\mathcal{A}$, as shown in Figure 18.6.6. Let the area of $\mathcal{A}$ be A. The solid angle subtended by $\mathcal{S}$ at O is said to have a measure of A steradians (from *stereo*, the Greek word for space, and *radians*). For instance, a closed surface $\mathcal{S}$ that encloses O subtends a solid angle of 4π steradians, because the area of the unit sphere is 4π.

Figure 18.6.6:

EXAMPLE 4 Let $\mathcal{S}$ be part of the surface of a sphere of radius a, $\mathcal{S}_\dashv$, whose center is O. Find the angel subtended by $\mathcal{S}$ at O. (See Figure 18.6.7.)

Figure 18.6.7:

SOLUTION The entire sphere $\mathcal{S}$ subtends an angle of 4π steradians and has an area $4\pi a^2$. We therefore have the propotion

$$\frac{\text{Angle}\mathcal{S}\text{subtends}}{\text{Angle}\mathcal{S}_\dashv\text{subtends}} = \frac{\text{Area of}\mathcal{S}}{\text{Area of}\mathcal{S}_\dashv},$$

or

$$\frac{\text{Angle}\mathcal{S}\text{subtends}}{4\pi} = \frac{\text{Area of}\mathcal{S}}{4\pi a^2}.$$

Hence

$$\text{Angle}\mathcal{S}\text{subtends} = \frac{\text{Area of}\mathcal{S}}{\dashv^\in}\text{steradians.}$$

◇

EXAMPLE 5 Let $\mathcal{S}$ be a surface such that each ray from the point O meets $\mathcal{S}$ in at most one point. Find an integral that represents in steradians the solid angle that $\mathcal{S}$ subtends at O.

SOLUTION Consider a very small patch of $\mathcal{S}$. Call it $d\mathcal{S}$ and let its area be dS. If we can estimate the angle that this patch subtends at O, then we will have the local approximation that will tell us what integral represents the total solid angle subtended by $\mathcal{S}$.

Figure 18.6.8:

Let $\mathbf{n}$ be a unit normal at a point in the patch, which we regard as essentially flat, as in Figure 18.6.8. Let $d\mathcal{A}$ be the projection of the path $d\mathcal{S}$ on a plane perpendicular to $\mathbf{r}$, as shown in Figure 18.6.8. The area of $d\mathcal{A}$ is approximately dA, where

$$dA = \mathbf{n} \cdot \widehat{\mathbf{r}}\ dS.$$

Now, $d\mathcal{S}$ and $d\mathcal{A}$ subtend approximately the same sold angle, which according to Example 4 is about

$$\frac{dA}{\|\mathbf{r}\|^2} = \frac{\mathbf{n} \cdot \widehat{\mathbf{r}}}{\|\mathbf{r}\|^2}\ dS\text{steradians.}$$

Consequently $\mathcal{S}$ subtends a sold angle of

$$\int\limits_{\mathcal{S}} \frac{\mathbf{n} \cdot \widehat{\mathbf{r}}}{\|\mathbf{r}\|^2}\ dS\text{steradians.}$$

◇

We will make use of this important application of steradian measure of solid angles.

Let O be a point in the region bounded by the closed surface $\mathcal{S}$. Assume each ray from O meets $\mathcal{S}$ in exactly one point, and let $\mathbf{r}$ denote the position vector from O to that point. Then

$$\int\limits_{\mathcal{S}} \frac{\widehat{\mathbf{r}} \cdot \mathbf{n}}{r^2}\ dS = 4\pi. \qquad (18.6.4)$$

Incidentally, (18.6.4) is easy to establish when $\mathcal{S}$ is a sphere of radius a and center at 0. In that case $\widehat{\mathbf{r}} = \mathbf{n}$, so $\widehat{\mathbf{r}} \cdot \mathbf{n} = 1$. Also, $r = a$. Then (18.6.4) becomes $\int_{\mathcal{S}} (1/a^2)\ dS = (1/a^2)4\pi a^2 = 4\pi$. However, it is obvious that (18.6.4) hold for more generally, for instance $\mathcal{S}$ is a sphere and O is not its center, or when $\mathcal{S}$ is not a sphere.

EXAMPLE 6 Let $\mathcal{S}$ be the cube of side 2 bounded by the six planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$, shown in Figure 18.6.9. Find $\oint \frac{\widehat{\mathbf{r}} \cdot \mathbf{n}}{r^2}\ dS$, where $\mathcal{S}$ is one of the six faces of the cube.

SOLUTION Each of the six faces subtends the same solid angle at the origin. Since the entire surface subtends 4π steradians, each face subtends $4\pi/6 = 2\pi/3$ steradians. Then the flux over each face is

Figure 18.6.9:

$$\int\limits_{\mathcal{S}} \frac{\widehat{\mathbf{r}} \cdot \mathbf{n}}{r^2}\ dS = \frac{2\pi}{3}.$$

◇

DOUG: We should have $\cos(A, B)$ defined in Vec. algebra.

In physics books you will see the integral $\int_{\mathcal{S}} \frac{\widehat{\mathbf{r}}\cdot\mathbf{n}}{r^2}\, ds$ written as $\int_{\mathcal{S}} \frac{\widehat{\mathbf{r}}\cdot\mathbf{n}}{r^3}\, dS$, $\int_{\mathcal{S}} \frac{\widehat{\mathbf{r}}\cdot\overrightarrow{dS}}{r^2}$ or $\int_{\mathcal{S}} \frac{\widehat{\mathbf{r}}\cdot\overrightarrow{dS}}{r^3}$, $\int_{\mathcal{S}} \frac{\cos(\mathbf{r},\mathbf{n})}{r^2}\, dS$. The symbol $\overrightarrow{dS}$ is short for $\mathbf{n}\, dS$, and calls to mind Figure 18.6.10, which shows a small patch on the surface, together with an exterior normal unit vector.

Figure 18.6.10:

Summary

We investigated central vector fields. In the plane the only central field with divergence O are of the form $\mathbf{k}\widehat{\mathbf{r}}/r$ where $\mathbf{k}$ is a constant, "an inverse first power." In space the only central fields with divergence 0 are of the form $\mathbf{k}\widehat{r}/r^2$, "an inverse second power." This field can be used to express the size of a sold angle in steradians as an integral, $\int_{\mathcal{S}} \widehat{\mathbf{r}}\cdot\mathbf{n}\; ds/r^2$. In particular, it $\mathcal{S}$ encloses the center of the field, then $\int_{\mathcal{S}} \widehat{\mathbf{r}}\cdot\mathbf{n}\; ds/r^2 = 4\pi$.

EXERCISES for 18.6 *Key:* R–routine, M–moderate, C–challenging

1.[R] Define a central field in words, using no symbols.

2.[R] Define a central field with center at $\mathbf{0}$, in symbols.

3.[R] Give an example of a central field in the plane that

(a) does not have zero divergence,

(b) that does.

4.[R] Give an example of a central field in space that

(a) does not have zero divergence,

(b) that does

5.[C] Carry out the computation to show that the only central fields in space that have zero divergence have the form $\mathbf{F}(\mathbf{r}) = \mathbf{k}\widehat{r}/r^2$, if the origin of coordinates is at the center of the field.

6.[M] In Example DOUG the integral $\oint \widehat{r} \cdot \mathbf{n}/r \, ds$ turned out to be 0. How would you explain this geometrically in terms of subtended angles?

7.[R] Show that the curl of a central vector field in the plane is $\mathbf{0}$.

8.[R] Show that the curl of a central vector field in spce is $\mathbf{0}$.

9.[R] Let $\mathbf{F}(\mathbf{r}) = \widehat{r}/r$. Evaluate $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds$ as simply as you can for the 2 ellipses in Figure 18.6.11.

Figure 18.6.11:

10.[R] Figure 18.6.12 shows a cube of side 2 with one corner at the origin. Evaluate as easily as you can the integral of the function $\widehat{\mathbf{r}} \cdot \mathbf{n}/r^2$ over

(a) the square $EFGH$,

(b) the square $ABCD$,

(c) the entire surface of the cube.

Figure 18.6.12:

11.[C] If we DOUG in four-dimensional space instead of the plane or three-dimensional space, which central fields do you think would have zero divergence? Carry out the calculation to confirm your conjecture.

12.[M] Let **F** and **G** be central vector fields with different centers but whose magnitudes are the inverse property to the plane in the inverse first power.

(a) Show that the vector field **F** and **G** is not a central field.

(b) Show that the divergence of **F** adn **G** is 0.

13.[M] In Example DOUG, we evaluated a surface integral by integrating it in terms of the size of a subtended solid angle. Evaluate the integral directly, without that knowledge.

14.[M] Let $\mathcal{S}$ be the triangle whose vertices are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Evaluate $\int_{\mathcal{S}} \frac{\widehat{\mathbf{r}}\cdot\mathbf{n}}{r^2}\, ds$ by using steradians.

15.[M] Evaluate the integral in Exercise 17-6ex-1 directly (DOUG check that this is a reasonable integral.)

DOUG: Some review derivatives (use integral table to get messy f with single f') Throughout the final chapters. Some integral review. MAX MIN

16.[R] Let $\mathbf{F}(\mathbf{r}) = \widehat{r}/r^3$. Evaluate the flux of **F** over the sphere of radius 2 and center at the origin.

17.[R] Let $\mathbf{F} = \widehat{r}/r^2$ and S be the surface of the lopsided pyramid with square base, whose vertices are $(0,0,0)$, $(1,1,0)$, $(0,1,0)$, $(0,1,1)$, $(1,1,1)$.

(a) Sketch the pyramid.

(b) What is the integral of $\mathbf{F}\cdot\mathbf{n}$ over the square base?

(c) What is the integral of $\mathbf{F}\cdot\mathbf{n}$ over each of the remaining four faces?

(d) Evaluate $\oint_S \mathbf{F}\cdot\mathbf{n}\, ds$.

Examples of derivatives: NOTE TO DOUG: More and in other sections, too. lift from integral table

18.[R] Show that the derivative of $\frac{1}{3}\tan^3 x - \tan x + x$ is $\tan^4 x$.

19.[R] Use integration by parts to show that

$$\int \tan^n x\ dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\ dx.$$

20.[R] Show that $x\cos^{-1} - \sqrt{1-x^2}$ is an integral of $\cos^{-1} x$ (etc.)

21.[R] Find $\int xe^{ax}\ dx$.

22.[R]

(a) Use integration by parts to show that

$$\int x^m e^{ax}\ dx = \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1}e^{ax}\ dx.$$

(b) Establish the equation in (a) by differentiating the right hand side.

23.[R] Let $\mathbf{F}(x,y,z) = \frac{x\mathbf{i}+y\mathbf{j}+0\mathbf{k}}{x^2+y^2}$, a vector field in space.

(a) What is the domain of $\mathbf{F}$?

(b) Sketch $\mathbf{F}(1,1,0)$ and $\mathbf{F}(1,1,2)$ with tails at the given points.

(c) Show $\mathbf{F}$ is not a central field.

(d) Show its divergence is 0.

24.[C] Let C be the circle $x^2 + y^2 = 2$ in the xy plane. For each point Q in the disk bounded by C consider the central field with center Q, $\mathbf{F}(P) = \overrightarrow{PQ}/|PQ|^2$. Its magnitude is inversely proportional to the first power of the distance DOUG from Q. For each point Q consider the flux across C, by the associated central field, $\int |\overrightarrow{QP}|/|\overrightarrow{QP}|^2\ ds$.

(a) Evaluate directly the flux when Q is the origin $(0,0)$.

(b) If Q is not the origin, evaluate the flux of the associated central field.

(c) Evaluate the flux when Q lies in C.

25.[C] Let **F** the central field in the plane, with center at $(1,0)$ and with magnitude inversely proportional to the first power of the distance, $\mathbf{F}(x,y) = \frac{(x-1)\mathbf{i}+y\mathbf{j}}{|(x-1)\mathbf{i}+y\mathbf{j}|^2}$. Let C be the circle gradient with center at $(0,0)$.

(a) By thinking in terms of subtended angle, evaluate the flux $\oint \mathbf{F}\cdot\mathbf{n}\, ds$.

(b) Evaluate the flux by carrying out the integration.

26.[R] Let $\mathbf{F}(x,y)$ be an inverse-first-power central field in the plane $\mathbf{F}(x,y) = c\widehat{r}/r$, where $\mathbf{r} = s\mathbf{i} + y\mathbf{j}$. Compute the divergence of **F** which should turn out to be 0. Suggestion: First write $\mathbf{F}(x,y)$ as $\frac{cx\mathbf{i}+cy\mathbf{j}}{x^2+y^2}$.

27.[R] A pyramid is made of four congruent equilateral triangles. Find the number of steradians subtended by one face at the centroid of the pyramid. (No integration is necessary.)

28.[R] How many steradians does one face of a cube subtend at

(a) One of the four vertices not on the face?

(b) The center of the cube? (No integration is necessary.)

29.[C] This exercise gives a geometric way to see why a central force is conservative. Let $\mathbf{F}(x,y) = f(r)\widehat{r}$. Figure 18.6.13 show $\mathbf{F}(x,y)$ and a short vector $\overrightarrow{dr}$ and two circles.

Figure 18.6.13:

(a) Why is $\mathbf{F}(x,y)\cdot\ d\mathbf{r}$ approximately $f(r)\ dr$, where dr is the difference in the radii of the two circles.

(b) Let C be a curve from A to B, where $A = (a,\alpha)$ and $B = (b,\beta)$ in polar coordinates. Why is $\int_C \mathbf{F}\cdot\ d\mathbf{r} = \int_a^b f(r)\ dr$?

(c) Why is **F** conservative?

30.[M] Let **F** be a planar central field. Show that $\nabla\times\mathbf{F}$ is $\mathbf{0}$. (HINT: $\mathbf{F}(x,y) = \frac{g(\sqrt{x^2+y^2})(x\mathbf{i}+y\mathbf{j})}{\sqrt{x^2+y^2}}$ for some scalar function g.

31.[M] (This continues DOUG.) Show that $\mathbf{F}$ is a gradient field; to be specific $\mathbf{F} = \nabla g(\sqrt{x^2+y^2})$.

32.[M] By a direct computation, show that the divergence of $\mathbf{F}(x,y) = \widehat{\mathbf{r}}/r^n$, when $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, is $(1-n)/r^{n+1}$.

33.[M] By a direct computation, show that the divergence $\mathbf{F}(x,y,z) = \widehat{\mathbf{r}}/r^n$, when $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + 2\mathbf{k}$, is $(2-n)/r^{n+1}$.

18.7 Applying the Field $\frac{\mathbf{r}}{\|\mathbf{r}\|^2}$

Even if you are not an engineer or physicist, as someone living in the 21st century, you are surrounded by devises that depends on electricity. For that reason, we now introduce one of the four equations that explain all of the phenomena of electricity and magnetism. Later in the chapter we will turn to the other three equations, all of which are expressed in terms of vector fields. The chapter concludes with a detailed description of how James Clerk Maxwell, using just these four equations, predicted both the existence of radio waves and also that they travel with the speed of light. Everyone who uses a cell phone, watches TV, or microwaves dinner may be interested in how he did it. Our explanation does not assume any prior knowledge of physics.

The Electric Field Due To a Single Charge

The starting point is some assumptions about the fundamental electrical charges, electrons and protons. An electron has a negative charge and a proton has a positive charge of equal absolute value. Two like charges exert a force of repulse on each other. The opposite charges attract each other (just as the tow genders in daily life generally do.)

The force that charge q exert a charge q_0 is proportional to q and to q_0, hence proportional to the product qq_0. The force is also inversely proportional to the square of the distance separately q and q_0.

Let $\mathbf{r}$ be the vector from q to $q_)$, as in Figure 18.7.1.

Figure 18.7.1:

If q and q_0 are both protons or both electrons, the force pushes the charges further apart. If one is a proton and the other is an electron, the force draws them closer.

Assume that q is positive, that is, the charge of a proton. The magnitude force it exerts on charge q_0 is proportional to q and also proportional to q_0. It is also inversely proportional to the distance $r = |\hat{r}|$ that separates them. So the magnitude of the force is of the form

$$\mathbf{k}\frac{q\ q_0}{r^2}.$$

It is directed along the vector $\mathbf{r}$. If q_0 is also positive, it is in the same direction as $\mathbf{r}$. If q_0 is negative, it is in the direction of $-\mathbf{r}$. We can summarize these observations in one vector equation

$$\mathbf{F} = \mathbf{k}\frac{q\ q_0}{r^2}\mathbf{r}. \tag{18.7.1}$$

The constant $\mathbf{k}$ is positive.

For convenience in later calculations, $\mathbf{k}$ is replaced by $1/4\pi\epsilon_0$ (ϵ_0 = "epsilon zero" or "epsilon niel".) The value of ϵ_0 depends on the units in which charge, distance and force are measured. Then (18.7.1) is written

$$\mathbf{F} = \frac{q\ q_0}{4\pi\epsilon_0 r^2}\mathbf{r}.$$

To avoid thinking in terms of the mysterious "action at a distance," where a force is exerted without the aid of ropes or springs, physicists have invented the a vector field, takes on the role of ropes and springs. This is how they do it.

Consider a positive charge q at point C.

It "creates" a central inverse-square vector field $\mathbf{E}$ with center at C. It is defined everywhere except at C. Its value at a typical point P is

$$\mathbf{E}(P) = \frac{q\ \widehat{r}}{4\pi\epsilon_0 r^2}$$

where $\overrightarrow{r} = \overrightarrow{CP}$, as in Figure 18.7.2.

Figure 18.7.2:

The value of $\mathbf{E}$ depends only on q and the vector from C to P.

To find the force exerted by charge q on charge q_0 at P just DOUG $\mathbf{E}$ by q_0, obtaining

$$\mathbf{F} = q_0\mathbf{E} \tag{18.7.2}$$

The field $\mathbf{E}$, which is a sheer invention, can be calculated in principle by putting a charge q_0 at P, the DOUG force $\mathbf{F}$ and then divides $\mathbf{F}$ by q_0.

The Electric Field Due to a Distribution of Charge

Electrons and protons usually do not live in isolation. Instead, charge may be distributed on a line, a curve, a surface or in space.

Imagine a total charge Q occupying a region R in space. The density of the charge varies from point to point. Denote the density at P by $S(P)$. Like the density of mass it is defined as a limit as follows. Let $V(r)$ be a small ball of radius r and center at P. Let $Q(r)$ be the charge in that ball. Then we have the definition

$$S(P) = \lim_{r\to)} \frac{\text{charge in} V(r)}{\text{volume of} V(r)}.$$

The the charge in $V(r)$ is approximately the volume of $V(r)$ times $S(P)$. We will be interested only in uniform charges, where the density is constant, with the fixed value S. Thus the charge in a region of volume V is SV.

The field due to a uniform charge Q distributed in a region R is the sum of the fields due to the individual point charges in Q.

To estimate this field we partition R into small regions $R_1\ R_2, \dots \mathbb{R}_n$ and choose a point P_i in R_i , $i = 1, 2 \dots, n$. The volume of R_i is V_i. The charge in R_i us SV_i,

Figure 18.7.3:

where S is the density of the charge. Figure 18.7.3 shows this contribution to the values of the field at a point P.

Let $\mathbf{r}_i$ be the vector from P_i to P. Let $r_i = |\mathbf{r}_i|$. Then the field due to the charge in this small patch R_i is approximately

$$\frac{SV_i\ \widehat{r_i}}{4\pi\epsilon_0\ r_i}.$$

As an estimate of the field due to Q, we have the sum

$$\sum_{i=1}^{n} \frac{SV_c\widehat{r_i}}{4\pi\epsilon_0 r_i^2} = \sum_{i=1}^{n} \frac{S\widehat{r_i}}{4\pi\epsilon_0 r_i^2} = V_i.$$

Taking limits as all the regions R_i are chosen smaller, we have

$$\mathbf{E}(P) = \text{Field at} P = \int\limits_R \frac{s\widehat{r}}{4\pi\epsilon_0 r^2}\ dv$$

Factoring out the constant $S/4\pi\epsilon_0$, we have

$$\mathbf{E}(P) = \int\limits_R \frac{\widehat{r}}{r^2}\ dv$$

That is an integral over a solid region. If the charge is just on a surface S with uniform surface density σ, the field would be

$$\mathbf{E}(P) = \frac{\sigma}{4\pi\epsilon_0} \int\limits_S \frac{\widehat{r}}{r^2}\ dS.$$

If the charge lies on a line or a curve C, with uniform density λ, then

$$\mathbf{E}(P) = \frac{\lambda}{4\pi\epsilon_0} \int\limits_C \frac{\widehat{r}}{r^2}\ ds.$$

To illustrate the definition we compute one such field value directly. In Example 2 we solve the same problem much more simply.

EXAMPLE 1 A charge Q is uniformly distributed on a sphere of radius α, S. Find the electro static field $\mathbf{E}$ at a point B a distance b from the center of a sphere of radius α, $b > a$.

SOLUTION We evaluate

$$\frac{\sigma}{4\pi\epsilon_0} \int\limits_S \frac{\widehat{r}}{r^2}\ dS. \qquad (18.7.3)$$

Figure 18.7.4:

Note that $\sigma = Q/4\pi a^2$, since the charge is uniform on an area of $4\pi a^2$.

Place a rectangular coordinate system with its origin at the center of the sphere and the z-axis on B, so that $B = (0, 0, b)$, as in Figure 18.7.4.

Before we start to evaluate an integral, let us use the symmetry of the sphere to predict something about the vector $\mathbf{E}(B)$. Could it look like the vector $\mathbf{v}$, which is not parallel to the z axis, as in Figure 18.7.5?

Figure 18.7.5:

If you spin the sphere around the z-axis, the vector $\mathbf{v}$ would change. But the sphere is unchanged and so is the charge. So $\mathbf{E}(B)$ must be parallel to the z axis. That means we know its x and y components are both 0: $\mathbf{E}(B) = 0\mathbf{j} + 0\mathbf{k} + z\mathbf{k}$. So we must find just its z component, which is $\mathbf{E}(B) \cdot \mathbf{k}$.

Let (x, y, z) be a typical point in the sphere S. Then

$$\begin{aligned} r &= (0\mathbf{i} + 0\mathbf{j} + b\mathbf{k}) - (x\mathbf{i} + y\mathbf{j} - z\mathbf{k}) \\ &= -x\mathbf{i} - y\mathbf{j} + (b - z)\mathbf{k}. \end{aligned}$$

So

$$\begin{aligned} \frac{\widehat{r}}{r^2} = \frac{\mathbf{r}}{r^3} &= \frac{-x\mathbf{i} - y\mathbf{j} + (b - z)\mathbf{k}}{(\sqrt{x^2 + y^2 + b^2 - 2bz + z^2})^3} \\ &= \frac{-x\mathbf{i} - y\mathbf{j} + (b - z)\mathbf{k}}{(a^2 + b^2 - 2bz)^{3/2}}. \end{aligned}$$

We need only the z component of this,

$$\frac{-x\mathbf{i} - y\mathbf{j} + (b - z)\mathbf{k}}{(a^2 + b^2 - 2bz)^{3/2}} \cdot \mathbf{k} = \frac{b - z}{(a^2 + b^2 - 2bz)^{3/2}}.$$

The magnitude of $\mathbf{E}(B)$ is therefore

$$\frac{\sigma}{4\pi\epsilon_0} \int_S \frac{b - z}{(a^2 + b^2 - 2bz)^{3/2}}. \tag{18.7.4}$$

We evaluate the integral in

$$\int_S \frac{b - z}{\sqrt{z^2 + b^2 - 2bz}} \, ds. \tag{18.7.5}$$

To do this, introduce spherical coordinates in the standard position. We have $dS = a^2$ since $d\phi \, d\theta$ and $z = a\cos\phi$. So (18.7.5) becomes

$$\int_0^{2\pi} \int_0^{2\pi} \frac{(b - a\cos\phi)a^2 \sin\phi}{\sqrt{a^2 + b^2 - 2ab\cos\phi}^3} \, d\phi \, d\theta;$$

which reduces, after the first integrate within to θ

$$2\pi a^2 \int_0^{\pi} \frac{(b - z\cos\phi)\sin\phi \; d\phi}{(\sqrt{a^2 + b^2 - 2ab\cos\phi})^3} \tag{18.7.6}$$

Let $u = \cos\phi$, hence $du = -sin\phi \; d\phi$. This transforms (18.7.6) into

$$-2\pi z^2 \int_1^{-1} \frac{(b - au) \; du}{(\sqrt{a^2 + b^2 - 2abu})^3}. \tag{18.7.7}$$

Then to make a second substitution, $v = a^2 + b^2 - 2abu$.
As you may check, this changes (18.7.7) into

$$\frac{2\pi a^2}{4ab^2} \int_{(b-a)^2}^{(b+a)^2} \frac{v + b^2 a^2}{v^{3/2}} \; dv \tag{18.7.8}$$

Recent the integrand as the sum of $1/\sqrt{v}$ and $(b^2 - a^2)/v^{3/2}$, and use the fundamental DOUG of calculus, to show that DOUG equal $4\pi a^2/b^2$.
Combining this with (18.7.8) shows that

$$\mathbf{E}(B) = \frac{\sigma}{4\pi\epsilon_0} \frac{4\pi a^2}{b^2} \mathbf{k} = \frac{QK}{4\pi\epsilon_0 b^2}.$$

◇

The result in this example, $Q/(4\pi\epsilon_0 b^2)\mathbf{k}$ is the same as if all the change Q were at the center of the sphere. In other words, DOUG change as a sphere acts or external particles as though the whole charge was placed at its center. NOTE TO DOUG FROM SHERMAN: Rewrite sentence: This was DOUG for the gravitational field by Newton and proved geometrically in his DOUG of 1687.

Using Flux and Symmetry to Find E

We included Example 1 for two reasons. First, it's a nice review of some integration techniques. Second, it will help you appreciate a much simpler way to find the field $\mathbf{E}$ due to change distribution.

Picture a charge Q distributed outside of the region bound by a surface S, as in Figure 18.7.6.

Figure 18.7.6:

Because each point charge in Q produces zero flux across S, Q exerts zero-flux.

Consider a charge Q contained wholly within the region bounded by S. Recall that a point-charge q in Q exerts a flux of q/ϵ_0 across S.

Chop the solid R that the charge occupies into n small regions $R_1, R_2, \ldots, R_n$. In region R_i select a point P_i. Let the density of charge at P_i be $S(P_i)$. Thus the charge in R_i produces a flux of approximately $S(P_i)V_i/\epsilon_0$. Consequently

$$\sum_{u}^{i=1} \frac{S(P_i)V_i}{\epsilon_0}$$

estimate the flux produced by Q. Taking limits, we see that

$$\text{Flux across} S \text{produced by} Q = \int_R \frac{S(P_i)}{\epsilon_0}\, dV$$

But $\int_R S(P_i)\, dV$ is the total charge Q. Thus we have

$$\text{Flux} = \frac{Q}{\epsilon_0}.$$

Thus we have one of th four fundamental equations of electrostatics:

Gauss's Laws: The flux produced by a distribution of charge across a closed surface is the charge Q in the region bounded by S, divided by ϵ_0.

The charge outside of S produces no flux across S. More precisely, the negative flux across S cancels the positive flux.

Let's illustrate the power of Gauss's Law by applying it to the case in Example 1.

EXAMPLE 2 A charge Q is distributed uniformly on a sphere of radius a. Find the electrostatic field $\mathbf{E}$ at a point B at a distance b from the center of a sphere of radius a, $b > a$.

SOLUTION We don't need to introduce a coordinate system in Figure 18.7.7. By

Figure 18.7.7:

symmetry, the field at any point P outside the sphere is parallel to the vector $\overrightarrow{OP}$. Moreover, the magnitude of the filed is the same for all points at a given distance from 0. Call this magnitude, $f(r)$, where r is the distance from 0. We want to find $f(b)$.

To do this, imagine another sphere $S*$, with center 0 and radius b, as in Figure 18.7.8.

Figure 18.7.8:

The flux of $\mathbf{E}$ across $S*$ is $\int_{S*} \mathbf{E} \cdot \mathbf{n}\, dS$.

But $\mathbf{E} \cdot \mathbf{n}$ is just $f(b)$ since $\mathbf{E}$ and $\mathbf{n}$ are parallel and $\mathbf{E}$ and $\mathbf{E}(P)$ has magnitude $f(b)$ for all P or $S*$. Thus $\int_{S*} \mathbf{E} \cdot \mathbf{n}\, dS = \int_{S*} f(b)\, dS = f((b)) \int_{S*}\, dS = f(b)4\pi b^2$.

By Gauss's Law

$$\frac{Q}{\epsilon_0} = f(b)(4\pi b^2).$$

That tells us that

$$f(b) = \frac{Q}{4epsilon_0 b^2}.$$

This is the same result as in Example 1, but compare the work in each case. Symmetry and Gauss's Law provide an easy way to find the electrostatic field due to distribution of charge. ◇

The same approach shows that the field **E** produced by the spherical charge in Examples 1 and 2 inside the sphere is **0**. Let $f(r)$ be the magnitude of **E** at a distance r from the center of the sphere. For $r > a$, $f(r) = Q/(4\pi\epsilon_0 r^2)$; for $0 < r < a$, $f(r) = 0$. The graph of f is shown in Figure 18.7.9.

Figure 18.7.9:

If you are curious about $f(0)$ and $f(a)$, see Exercises DOUG and DOUG.

Summary

The field due to a point charge q at a point C is given by the formula $\mathbf{E}(P) = \frac{1}{4\pi\epsilon_0}\frac{q\widehat{r}}{r^2}$, where $\mathbf{R} = \overrightarrow{OP}$. This field "produces" a force $q_0\mathbf{E}(P)$ a charge q_0 located at P.

The field due to a distribution of charge is obtained by an integration, with

$$\mathbf{E}(P) = \frac{S}{4\pi\epsilon_0)} \int\limits_R \frac{\widehat{r}}{r^2}\, dV.$$

Here V is the solid occupied by the charge and S is the density, which we take to be constant.

We showed that a charge Q outside a surface produces a net flux of zero across the surface. However the flux produced by a charge within the surface is simply Q/ϵ_0. That is Gauss's Law.

We used Gauss's Law to find the field produced by a spherical distribution of charge.

EXERCISES for 18.7 *Key:* R–routine, M–moderate, C–challenging

1.[C] We showed that $\mathbf{E}(P) = \frac{S}{4\pi\epsilon_0}\int_R \frac{\widehat{r}}{r^2}\, dV$ if the charge density is constant. If the charge density varies find the integral for $\mathbf{E}(P)$.

2.[C] In Example 1, we used an integral to find the electrostatic field outside a uniformly charged sphere. Carry outs similar calculation to find the field inside the sphere. Warning: is the square root of $(b-a)^2$ still $b-a$?

3.[C] Use the approach in Example 2 to find the electrostatic field inside a uniformly charged sphere.

4.[R] Describe to a friend who knows no physics the field **E** "produced" by a point charge q.

5.[R] State Gauss's Law aloud several times.

6.[R] Why do you think that the constant k was replaced by $1/4\pi\epsilon_0$. (Later we will see why it is convenient to have ϵ_0 in the denominator.)

7.[R] A charge is distributed uniformly over a infinite plane. For any part of this surface of area A the charge is kA, where k is a constant. Find the field **E** due to the charge.

(a) Use symmetry to say as much as you can about its DOUG.

(b) Show that the magnitude is constant by applying Gauss's Theorem to a cylinder whose axes is perpendicular to the plane and which does not intersect the plane.

Figure 18.7.10:

(c) Find the magnitude by applying Gauss's Theorem to the cylinder in Figure 18.7.11. Let the area DOUG circular cross section be A and the area of

Figure 18.7.11:

its curved side be B.

8.[C] Find the field **E** in the Exercise DOUG by integrates over the whole plane.

(Don't use Gauss's Theorem.)

9.[R] A field **F** is called uniform if all its vectors are the same. Let $\mathbf{F}(x, y, z) = 3i$.

(a) Find the flux of **F** across each of the six faces of the cube in Figure 18.7.12 of side 3.

Figure 18.7.12:

(b) Find the total flux of **F** across the surface of the box.

(c) Verify the divergence theorem for this **F**.

10.[R] Let **F** be the uniform field $\mathbf{F}(x, y, z) = 2i+3j+0k$. Carry out the preceding exercise for this field.

11.[R] In Example DOUG, we computed the field **E** at a point outside a sphere of radius a, due to a uniform distribution of charge Q on the sphere. Fill in the missing calculations.

12.[R] Find the field **E** of the charge in Exercise DOUG at a point inside the sphere. Assume, as in Example DOUG, that this point has coordinates $(0, 0, b)$. Warning: The computations will be almost the same as in Example DOUG. However, be careful when computing the square root of $(b - a)^2$. Answer:

13.[R] Find the field **E** of the charge in Exercise DOUG at a point on the surface of the sphere. HINT: Let the point be $(0, 0, a)$. Answer: $Q/(8\pi\epsilon_0 a^2)$. Why is Gauss's Law not applicable here.

14.[R] See Exercise 13 If you placed the point at which **E** is evaluated at $(a, 0, 0)$ instead of at $(0, 0, a)$, what integral in spherical coordinates arises.

(a) would you like to evaluate it?

15.[R] Find the field **E** of the charge in DOUG at the center of the sphere. HINT: Use symmetry, don't integrate.

16.[R] Complete the graph in Figure DOUG.

17.[R] A charge is distributed uniformly along an infinite straight wire. The charge on a section of length l is kl. Find the field **E** due to this charge.

(a) Use symmetry to say as much as you can about its direction and magnitude.

(b) Find the magnitude by applying Gauss's Law to the cylinder of radius r adn height h shown in Figure 18.7.13

Figure 18.7.13:

(c) Find the force directly by an integral over the line, as in Example 1.

18.[R] Figure 18.7.14 shows four surfaces. Inside S_1 is a total charge Q_1, and in S_2 is a total charge Q_2. Find the total flux across each of the four surfaces.

Figure 18.7.14:

19.[R] Imagine that there is a uniform distribution of charge Q throughout a ball of radius a. Use Gauss's Law to find electrostatic field **E** produced by their charge

(a) at points outside the ball,

(b) at points inside the ball.

20.[R] (See Exercise DOUG) Let $f(r)$ be the magnitude of the field in Exercise DOUG at a distance r from the center of the ball. Graph $f(r)$ for $r \geq 0$.

21.[R] A charge Q lies partly under a closed surface S and partly outside. Let Q_1 be the amount inside and Q_2 the amount outside, as in Figure 18.7.15. What is

Figure 18.7.15:

the flux across S of the charge Q?

22.[R] Write up Example 1 in full, filling in all details.

23.[R] In Exercise DOUG you found the field **E** due to a charge uniformly spread on an infinite line. If the charge density is λ, **E** at a point at a distance a from the line is $[\lambda/(2\pi a\epsilon_0)]j$.

Now assume that the line occupies only the right half of the x-axis, $(0, \infty)$.

(a) Using the result in Exercise, show that the j-component of $\mathbf{E}(0, a)$ is $(\lambda/4\pi a\epsilon_0)j$.

(b) By integrating over $[0, \infty]$, show that the x-component of **E** at $(0, a)$ is $\lambda/(4\pi a\epsilon_0)i$.

(c) What angle does $\mathbf{E}(0, a)$ make with the y-axis?

(d) Why is Gauss's Law of no use in determining the x-component of **E** in this case.

18.8 Vector Functions in Other Coordinate Systems

We have expressed the gradient, divergence, and curl in terms of rectangular coordinates. However, students who apply vector analysis in engineering and physics courses will see these functions express in polar, cylindrical and spherical coordinates. This section introduces these expressions.

Polar Coordinates

Let $g(r,\theta)$ be a scalar function expressed in polar coordinates. Its gradient has the form $A(r,\theta)\widehat{r} + B(r,\theta)\widehat{\theta}$, where $\widehat{r}$ and $\widehat{\theta}$ are the unit vectors shown in Figure 18.8.1. The "radial vector" $\widehat{r}$ points in the direction of increasing r. The "tangential vector"

Figure 18.8.1:

$\widehat{\theta}$ points in the direction determined by increasing θ. Note that $\widehat{0}$ is tangent to the circle through (r,θ) with center at the pole.

Our goal is to find $A(r,\theta)$ and $B(r,\theta)$, which we may also denote simply as A and B.

One might guess, in analogy with rectangular coordinates, that $A(r,\theta)$ would be $\partial g/\partial r$ and $V(r,\theta)$ would be $\partial g/\partial\theta$. That guess is part right and part wrong, for we will show that

$$\nabla g = \frac{\partial g}{\partial r}\widehat{r} + \frac{1}{r}\frac{\partial g}{\partial \theta}\widehat{\theta} \tag{18.8.1}$$

One way to obtain (**??**) is labor-intensive and not illuminating: express $g, \widehat{r}, \widehat{\theta}$ in terms of $x, y, \widehat{\imath}, \widehat{\jmath}$ and, use the formula for gradient in terms of rectangular coordinates, the translate back to polar coordinates. This approach, whose only virtue is that it offers good practice applying the chain rule for partial derivatives, is outlined in Exercise DOUG.

We will use a simpler way which easily generalizes the cylindrical and spherical coordinates. It exploits the connection between a gradient and directional derivatives. In particular it shows why the coefficient $1/r$ appears in (18.8.1).

Recall that if $\widehat{u}$ is a unit vector, the directional derivation of g in the direction $\widehat{u}$ is just the dot product of ∇g with $\widehat{u}$:

$$D_u g = \nabla g \cdot \widehat{u}.$$

In particular,

$$D_{\widehat{r}} g = (A\widehat{r} + B\widehat{\theta}) \cdot \widehat{r} = A$$

and

$$D_{\theta} g = (A\widehat{r} + B\widehat{\theta}) \cdot \widehat{\theta} = B.$$

So all we do is find $D_{\widehat{r}}g$ and $D_{\widehat{\theta}}g$.
First,

$$\begin{aligned} D_{\widehat{r}}(g) &= \lim_{\Delta r \to} \frac{g(r+\Delta r, \theta - g(r,\theta)}{\Delta r} \\ &= \frac{\partial g}{\partial r}. \end{aligned}$$

So $A(r,\theta) = \partial g/\partial r(r,\theta)$. That explains part of (18.8.1).

Now we will see why B is <u>not</u> simply the partial derivation of g with respect to θ.

If we want to estimate a directional derivative at P of q in the direction $\widehat{u}$ we pick a nearby point Q a distance Δs away in the direction of $\widehat{u}$ and form the quotient

$$\frac{g(Q) - g(P)}{\Delta s} \tag{18.8.2}$$

Then we take the limit of (18.8.2) as $\Delta s \to 0$.

Now let $\widehat{u}$ be $\widehat{\theta}$, and let's examine (18.8.2) in case $P = (r,\theta)$ and $Q = (r, \theta + \Delta\theta)$. The numerator in (18.8.2) is

$$g(r, \theta + \Delta\theta) - g(r,\theta).$$

We draw a picture to find Δs, as in Figure 18.8.2.

Figure 18.8.2:

The distance between P and Q is <u>not</u> $\Delta\theta$. Rather it is approximately $r\Delta\theta$ (when $\Delta\theta$ is small). That tells us that Δs in (18.8.2) is not $\Delta\theta$ but $r\Delta\theta$. Therefore

$$\begin{aligned} D_{\theta}g &= \lim_{\Delta\theta \to 0} \frac{g(r, \theta + \Delta\theta) - g(r,\theta)}{r\Delta\theta} \\ &= \frac{1}{r} \lim_{\Delta\theta \to 0} \frac{g(r, \theta + \Delta\theta) - g(r,\theta)}{\Delta\theta} \\ &= \frac{1}{r}\frac{\partial g}{\partial \theta}. \end{aligned}$$

That is why there is a $1/r$ in the formula (18.8.1) for the gradient of g. It occurs because a change $\Delta\theta$ in the parameter θ causes a point to move about the distance $r\Delta\theta$.

Divergence in the Plane

The divergence of $\mathbf{F}(x,y) = P(x,y)\widehat{i} + Q)x,y)j$ is simply $\partial P/\partial x + \partial Q/\partial y$. But what is the divergence of a vector field described in polar coordinates, $\mathbf{G}(r,\theta) = A(r,\theta)\widehat{r} + B(r,\theta)\widehat{\theta}$. (By now you are on guard, $\nabla \cdot \mathbf{G}$ is <u>not</u> the sum of $\partial A/\partial r$ and $\partial B/\partial \theta$).

To find $\nabla \cdot \mathbf{G}$, DOUG the relation between $\nabla \cdot \mathbf{G}$ at $P = (r, \theta)$ and the flux across a small curve C that surrounds P.

$$\nabla \cdot \mathbf{G} = \lim \frac{\oint_C \mathbf{G} \cdot \mathbf{n}\, ds}{\text{Area within} C} \tag{18.8.3}$$

as the length of C approaches 0. Note that (18.8.3) provides a coordinate-free description of divergence in the plane.

We are free to choose the small curve C to make it easy to estimate the flux across it. The curve C that corresponds to small changes Δr and $\Delta\theta$ is convenient. See Figure 18.8.3.

Figure 18.8.3:

We will use (18.8.3) to find the divergence at $Q = (r, \theta)$. Now, Q is not inside C; rather it is on C. However, since $\mathbf{G}$ is continuous, $\mathbf{G}(Q)$ is the limit of values of $\mathbf{G}$ at points inside, so we may use (18.8.3).

To estimate the flux across C, we estimate the flux across each of the four parts of the curve. Because these sections are short when Δr and $\Delta\theta$ are small, we may estimate an integral over each part by multiplying the value of the integrand at any point of the section (even at an end point) by the length of the section. As usual, $\widehat{n}$ denotes an exterior unit vector perpendicular to C.

On QR and ST, $B\widehat{\theta}$ contributes to the flux (on RS and TQ it does not since $\mathbf{n} \cdot \theta$ is 0). On QR, $\widehat{\theta}$ is parallel to $\mathbf{n}$, as shown in Figure 18.8.4.

Figure 18.8.4:

However, at points in the opposite direction. So $\widehat{\theta} \cdot \widehat{\mathbf{n}}$ is -1. So across QR, the flux contributed by $B\widehat{\theta}$ is approximately

$$(B\widehat{\theta} \cdot \widehat{n})\Delta r = -B(r, \theta)\Delta r.$$

(We would get a better estimate by using $B(r + \frac{\Delta r}{2}, \theta)$ but $B(r, \theta)$ is good enough since B is continuous.)

On ST, $\widehat{\theta}$ on $\widehat{n}$ point in almost the same direction, hence $\widehat{\theta} \cdot \widehat{n}$ is close to 1 when $\Delta\theta$ is small. So on ST $B\widehat{\theta}$ contributes approximately $B(r, \theta + \Delta\theta)\Delta r$ to the flux.

All told, the total contribution of $B\widehat{\theta}$ to the flux across C is

$$B(r, \theta + \Delta\theta)\Delta r - B(r, \theta)\Delta r \tag{18.8.4}$$

The contribution of $A\widehat{r}$ to the flux is negligible on QR ans ST because there $\widehat{r}$ and *widehatn* are perpendicular. On TQ, $\widehat{r}$ and $\widehat{n}$ point in almost directly opposite directions, hence $\widehat{r} \cdot \widehat{n}$ is near -1. The flux of $A\widehat{r}$ there is approximately

$$A(r, \theta)(\widehat{r}, \widehat{n})r\Delta\theta = -A(r, \theta)r\Delta\theta. \tag{18.8.5}$$

On RS, which has radius $r+\Delta r$, $\widehat{r}$ and $\widehat{n}$ are almost identical, hence $\widehat{r}\cdot\widehat{n}$ is near 1. The contribution on RS, which has radius $r+\Delta r$ is approximately

$$A(r+\Delta r,\theta)(r+\Delta r)\Delta\theta. \tag{18.8.6}$$

Combining (18.8.4), (18.8.5) and (18.8.6), we see that the limit in (18.8.3) is the sum of two limits:

$$\lim_{\Delta r,\Delta\theta\to 0}\frac{A(r+\Delta r,\theta)(r+\Delta r)\Delta\theta - A(r,\theta)r\Delta\theta}{r\Delta r\Delta\theta} \tag{18.8.7}$$

The are within C is approximately, $r\Delta r\Delta\theta$.

and

$$\lim_{\Delta r,\Delta\theta\to 0}\frac{B(r,\theta+\Delta\theta,\theta)\Delta r) - B(r,\theta)\Delta r}{r\Delta r\Delta\theta} \tag{18.8.8}$$

The first limit (18.8.7) equals

$$\lim_{\Delta r,\Delta\theta\to 0}\frac{A(r+\Delta r,\Delta\theta)(r+\Delta r) - A(r,\theta)r}{\Delta r},$$

which is

$$\frac{1}{r}\frac{\partial(Ar)}{\partial r}.$$

Note that r appears in the coefficient, $1/r$, and also in the function, Ar, being differentiated.

The second limit (18.8.8) equals

$$\lim_{\Delta r,\Delta\theta\to 0}\frac{1}{r}\frac{B(r,\theta+\Delta\theta) - B(r,\theta)}{\Delta\theta},$$

hence is

$$\frac{1}{r}\frac{\partial B}{\partial\theta}.$$

Here r appears only once, in the coefficient.

All told, we have the desired formula:

$$\nabla\cdot(A\widehat{r}+B\widehat{\theta}) = \frac{1}{r}\frac{\partial(Ar)}{\partial r}+\frac{1}{r}\frac{\partial B}{\partial\theta}. \tag{18.8.9}$$

Curl in the Plane

The curl of $\mathbf{F}(x,y)=P(x,y)\mathbf{i}+Q(x,y)\mathbf{j}+0\mathbf{k}$, a vector field in the plane, is given by the formula

$$\nabla\times\mathbf{F} = \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}.$$

What is the formula for the curl of a vector field $\mathbf{G}(r,\theta)=A(r,\theta)\widehat{r}+B(r,\theta)\widehat{\theta}$? To find out we will reason as we did with divergence. This time we use

$$(\nabla\times\mathbf{G})\cdot\mathbf{k} = \lim\frac{\oint_C \mathbf{G}\cdot\pi\ ds}{\text{Area Bound by}C}.$$

See page DOUG.

Here C is a closed curve in the (r, θ) plane, and the limit is taken as the length of C approaches 0. The curl is evaluated at a fixed point, which is on or within C.

As is to be expected, we compute the circulation of $\mathbf{G} = A\widehat{r} + B\widehat{\theta}$ around the same curve used before.

On TQ and RS, $A\widehat{r}$ being perpendicular to the curve, contributes nothing to the circulation of $\mathbf{G}$ around C. On QR is contributes approximately

$$(A(r,\theta)(\widehat{r} \cdot \pi)\Delta r = A(r,\theta)\Delta r = A(r,\theta)\Delta r.$$

On ST, since there $\widehat{r} \cdot \pi = -1$, it contributes approximately

$$A(r,\theta+\Delta\theta)(r \cdot T)\Delta r = -A(r,\theta+\Delta\theta)\Delta r.$$

A similar computation shows that $B\widehat{\theta}$ contributes to the total circulation approximately

$$B(r+\Delta r,\theta)(r+\Delta r)\Delta\theta - B(r,\theta)r\Delta\theta.$$

therefore $\nabla \times \mathbf{G}$ in the sum of two limits:

$$\lim_{\Delta r, \Delta\theta \to 0} \frac{A(r,\theta)\Delta r - A(r,\theta+\Delta\theta)\Delta r}{r\Delta r\Delta\theta} = -\frac{1}{r}\frac{\partial A}{\partial \theta}$$

and

$$\lim_{\Delta r. \Delta\theta \to 0} \frac{B(r+\Delta r,\theta)(r+\Delta r)\Delta\theta - B(r,\theta)r\Delta\theta}{r\Delta r\Delta\theta} = \frac{1}{r}\frac{\partial(r,B)}{\partial r}.$$

All told, we have

$$\nabla \times (A\widehat{r} + B\widehat{\theta}) = \left(-\frac{1}{r}\frac{\partial A}{\partial \theta} + \frac{1}{r}\frac{\partial(rB)}{\partial r}\right)\mathbf{k}.$$

EXAMPLE 1 Find the curl of $r\theta^2\widehat{r} + r^3\tan\theta\widehat{\theta}$.
SOLUTION DOUG ⋄

Cylindrical Coordinates

In cylindrical coordinates the gradient of $g(r,\theta,z)$ is

$$\nabla g = \frac{\partial g}{\partial r}\widehat{r} + \frac{1}{r}\frac{\partial g}{\partial \theta}\widehat{\theta} + \frac{\partial g}{\partial z}\widehat{z}/ \qquad (18.8.10)$$

Here $\widehat{z}$ is the unit vector in the positive z direction, denoted $\mathbf{k}$ in Chapter DOUG. Note that (18.8.10) differs from (18.8.1) only by the extra term $(\partial y/\partial z)\widehat{z}$. You can obtain (18.8.10) by computing directional derivatives of g along $\widehat{r}$, $\widehat{\theta}$, and $\widehat{z}$. The calculation are similar to those that gave us the formulas for the gradient of $g(r,\theta)$.

The divergence of $\mathbf{G}(r,\theta,z) = A\widehat{r} + B\widehat{\theta} + C\widehat{z}$ is given by the formula

$$\nabla \cdot \mathbf{G} = \frac{1}{r}\left(\frac{\partial\theta(rA)}{\partial r} + \frac{\partial\partial B}{\partial\theta} + \frac{\partial(rC)}{\partial z}\right). \tag{18.8.11}$$

(The notation $\nabla \cdot \mathbf{G}$ might be misleading.) Note that the partial derivatives with respect to r and z are similar in that the factor r is present in both $\partial(rA)/\partial r$ and $\partial(rC)/\partial r$. You can obtain (18.8.11) by using the relation between $\nabla \cdot \mathbf{G}$ and the flux across the small surface determined by small change Δr, $\Delta\theta$, and Δz.

The curl of $\mathbf{G} = A\widehat{r} + B\widehat{\theta} + C\widehat{z}$ is given by a formal determinant:

$$\nabla \times \mathbf{G} = \frac{1}{r}\begin{vmatrix} \widehat{r} & r\widehat{\theta} & \widehat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A & rB & C \end{vmatrix} \tag{18.8.12}$$

To obtain this formula consider the circulation around three small closed curves lying in planes perpendicular to $\widehat{r}$, $\widehat{\theta}$ and $\widehat{z}$.

Spherical Coordinates

In mathematics texts, spherical coordinates are denoted ρ, ϕ, θ. In physics and engineering a different notation is standard. There ρ is replaced by r, θ is the angle with z-axis, and ϕ plays the role of the mathematicians θ, switching the roles of ϕ and θ. The formulas we state are in the mathematicians' notation.

The three basic unit vector for spherical coordinates are denoted $\widehat{\rho}$, $\widehat{\phi}$, $\widehat{\theta}$. For instance, $\widehat{\rho}$ points in the direction of increasing ρ. See Figure 18.8.5.

Figure 18.8.5:

Note that $\widehat{\phi}$ and $\widehat{\theta}$ are tangent to the sphere through P and center at the origin, while $\widehat{\rho}$ is perpendicular to that sphere. Also, any two of $\widehat{\rho}, \widehat{\phi}, \widehat{\theta}$ are perpendicular.

In obtaining the formulas for $\nabla \cdot \mathbf{G}$ and $\nabla \times \mathbf{G}$, we would use the region correspondence to small changes $\Delta\rho, \Delta\phi, \Delta\theta$, shown in Figure 18.8.6. The computation

Figure 18.8.6:

would yield these formulas: If $g(\rho, \phi, \theta)$ is a scalar function,

$$\nabla g = \frac{\partial g}{\partial\rho}\widehat{\rho} + \frac{1}{\rho}\frac{\partial g}{\partial\phi}\widehat{\phi} + \frac{1}{\rho\sin\phi}\frac{\partial g}{\partial\theta}\widehat{\theta}. \tag{18.8.13}$$

If $\mathbf{G}(\rho, \phi, \theta) = A\widehat{\rho} + B\widehat{\phi} + C\widehat{\theta}$

$$\nabla \cdot \mathbf{G} = \frac{1}{\rho^2}\frac{\partial(\rho^2 A}{\partial r} + \frac{1}{\rho \sin\phi}\frac{\partial(\sin\phi B)}{\partial\phi} + \frac{1}{\rho\sin\phi}\frac{\partial C}{\partial\theta} \qquad (18.8.14)$$

and

$$\nabla\times\mathbf{G} = \frac{1}{\rho}\left(\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}(\sin C) - \frac{1}{\rho\sin\phi}\frac{\partial B}{\partial\theta}\right)\widehat{\rho} + \left(\frac{1}{\sin\phi}\frac{\partial A}{\partial\theta} - \frac{\partial}{\partial\rho}(\rho C)\right)\widehat{\phi} + \left(\frac{\partial}{\partial\rho}(\rho B) - \frac{\partial A}{\partial\phi}\right)\widehat{\theta}\right) \qquad (18.8.15)$$

These can all be obtained by the method we used in the case of polar coordinates. In each case keep in mind that the change in ϕ or θ is not the same as the distance the corresponding point moves. However, a change in $\rho$is the same as the distance the corresponding point moves. Specifically, the distance between (ρ, ϕ, θ) and $(\rho, (\phi + \Delta\phi), \Delta\theta)$ is approximately $\rho\Delta\phi$ and the distance between (ρ, ϕ, θ) and $(\rho, \phi, \theta + \Delta\theta)$ is approximately $\rho\sin\phi\Delta\theta$.

An Application of Rotating Fluids

Consider a fluid rotating in a cylinder, for instance, in a centrifuge. If it rotates as a rigid body, then DOUG velocity at a distance r from the axis of rotation has the form

$$\mathbf{G}(r, \theta) = cr\widehat{\theta},$$

when c is a positive constant.

Then

$$\nabla\times\mathbf{G} = \frac{1}{r}\frac{\partial(c^2 r)}{\partial r}\mathbf{k} = 2c\mathbf{k}.$$

The curl is independent of r. That means that an imaginary paddle held in a fixed position would rotate at the same rate no matter where it is placed.

Now consider more general case, where

$$\mathbf{G}(r, \theta) = cr^n\widehat{\theta},$$

and n is an integer. Now

$$\nabla\times\mathbf{G} = \frac{1}{r}\frac{(cr^{n+1}}{\partial r}\mathbf{k} = c(n+1)r^{n-1}\mathbf{k}.$$

We just considered the case $n = 1$. If $n > 1$, the curl increases as r increase. The paddle wheel rotates faster if placed farther from the axis of rotation. The direction of rotation is the same as that of the fluid, counterclockwise.

Next consider the case $n = -2$. The speed of the fluid decreases as r increases. Now

$$\nabla\times\mathbf{G} = c(-2+1)r^{-2-1}\mathbf{k} = -cr^{-3}\mathbf{k}.$$

The minus sign before the coefficient c tells us that the paddle wheel spins clockwise even though the fluid rotates counterclockwise. The farther the paddle wheel is from the axis, the slower it rotates.

Summary

We expressed gradient, divergence and curl in several coordinate systems. Even though the basic unit vectors in each system may change direction from point to point, they remain perpendicular to each other. That simplified the computation of flux and circulation. The formulas are more complicated than there in rectangular coordinates because “distance a parameter moves”is not the same as “distance the corresponding point moves.”

EXERCISES for 18.8 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 through 4 find and draw the gradient of the given functions of (r, θ) at $(2, \pi/4)$

1.[R] r

2.[R] $r^2\theta$

3.[R] $e^{-r}\theta$

4.[R] $r^3\theta^2$

In Exercises 5 through 8 find the divergence of the given function

5.[R] $5\widehat{r} + r^2\theta\widehat{\theta}$

6.[R] $r^3\theta\widehat{r} + 3r$
$theta\widehat{\theta}$

7.[R] $r\widehat{r} + r^3\widehat{\theta}$

8.[R] $r\sin\theta\widehat{r} + r^2\cos\theta\widehat{\theta}$

In Exercise 9 through 12 compute the curl of the given function.

9.[R] $r\widehat{\theta}$

10.[R] $r^3\theta\widehat{r} + e^r\widehat{\theta}$

11.[R] $r\cos\theta\widehat{r} + r\theta\widehat{\theta}$

12.[R] $1/r^3\widehat{\theta}$

13.[R] What is the directional derivative of $r^2\theta^3$ in the direction

(a) $\widehat{r}$

(b) $\widehat{\theta}$

(c) $\widehat{i}$

(d) $\mathbf{j}$?

14.[R] What property of rectangular coordinates makes the formulas for gradient, divergence, and curl in those coordinate relatively simple?

15.[R] Estimate the flux of $r\theta\widehat{r}+r^2\theta^3\widehat{\theta}$ around the circle of radius 0.01 with center at $(r,\theta)=(2,\pi/6)$.

16.[R] Estimate the circulation of the function in the preceding exercise around the same circle.

When translating between rectangular and polar coordinates, it may be necessary to express $\widehat{r}$ and $\widehat{\theta}$ in terms of $\widehat{i}$ and $\widehat{j}$ and also $\widehat{i}$ and $\widehat{j}$ in terms of $\widehat{r}$ and $\widehat{\theta}$. Exercise 17 and 18 concern this matter.

17.[R] Drawing a picture, show that at (r,θ), which has rectangular coordinates (x,y)

(a) $\widehat{r}=\cos\theta\mathbf{i}+\sin\theta\mathbf{j}$ which equals $x/\sqrt{x^2+y^2}\mathbf{i}+y/\sqrt{x^2+y^2}\mathbf{j}$

(b) Show that $\widehat{\theta}=-\sin\theta\mathbf{i}+\cos\theta\mathbf{j}$, which equals $-y/\sqrt{x^2+y^2}\mathbf{i}+x/\sqrt{x^2+y^2}\mathbf{j}$.

So we have $\widehat{r}$ and $\widehat{\theta}$ in terms of $\widehat{i}$ and $\widehat{j}$:

$$DOUG\widehat{r}=\frac{x}{\sqrt{x^2+y^2}}\mathbf{i}+\frac{y}{\sqrt{x^2+y^2}}\mathbf{j}\widehat{\theta}=-\frac{y}{\sqrt{x^2+y^2}}\mathbf{i}+\frac{x}{\sqrt{x^2+y^2}}\mathbf{j}. \qquad (18.8.16)$$

18.[R] Show that if (x,y) has polar coordinates (r,θ), then DOUG $\widehat{i}=\cos\theta\theta\widehat{r}-\sin\theta\widehat{\theta}\ \widehat{j}=\sin\theta\theta\widehat{r}+\cos\theta\widehat{\theta}$ by solving the simultaneous equations 18.8.16 in the preceding exercise for $\widehat{i}$ and $\widehat{j}$.

In exercises 19 through 22

- find the gradient of the given function, using the formula for gradient in rectangular coordinates,
- find it by first expressing the function in polar coordinates and using DOUG,

show that the two results agree.

19.[R] $x^2 + y^2$

20.[R] $\sqrt{x^2 + y^2}$

21.[R] $3x + 2y$

22.[R] $x/\sqrt{x^2 + y^2}$

In Exercises 23 through 26

- find the gradient of the given function, using DOUG,
- find it by first expressing the function n rectangular coordinates,
- show that the two results agree.

23.[R] r^2

24.[R] $r^2 \cos\theta$

25.[R] $r \sin\theta$

26.[R] e^r

In Exercise 27 and 28

- find the divergence of the given vector filed in rectangular coordinates,
- find it by first expressing the function in polar coordinates,
- show that the result agree.

27.[R] $x^2\mathbf{i} + y^2\mathbf{j}$

28.[R] $xy\mathbf{i}$

In Exercises 29 and 30

- find the curl of the given vector field in rectangular coordinates,
- find it by first expressing the function in polar coordinates,
- show that the two results agree.

29.[R] $xy\mathbf{i} + x^2y^2\mathbf{j}$

30.[R] $(x/\sqrt{x^2+y^2}\mathbf{i}$

The next two exercises would be useful in developing the formula for the gradient in cylindrical and spherical coordinates.

31.[R] Approximately how far is it from the points (r, θ, z) to

(a) $(r + \Delta r, \theta, z)$,

(b) $(r, \theta + \Delta\theta, z)$,

(c) $r, \theta, z + \Delta z)$?

32.[R] Approximately how far is it from the point (ρ, ϕ, θ) to

(a) $(\rho + \Delta\rho, \phi, \theta)$,

(b) $(\rho, \phi + \Delta\phi, \theta)$,

(c) $\rho, \phi, \theta + \Delta\theta)$?

33.[M] Using the formulas for the gradient of $g(r, \phi, \theta)$, find the directional derivative of g in the direction

(a) $\widehat{\rho}$,

(b) $\widehat{\phi}$,

(c) $\widehat{\theta}$.

34.[M] Using the formulas for the gradient of $g(r, \theta, z)$, find the directional derivative of g in the direction

(a) $\widehat{r}$,

(b) $\widehat{\theta}$,

(c) $\widehat{z}$.

35.[M] Without using the formula for the gradient, do Exercise 33.

36.[M] Without using the formula for the gradient, do Exercise 34.

37.[M] Using as few mathematical symbols as you can, state the formula for the divergence of a vector field given relative to $\widehat{r}$ and $\widehat{\theta}$.

38.[M] Using as few mathematical symbols as you can, state the formula for the curl of a vector field given relative to $\widehat{r}$ and $\widehat{\theta}$.

39.[M] In the formula for the divergence of $A\widehat{r} + B\widehat{\theta}$, why does the term $(1/r)(\partial(rA)/\partial r$ and rA appear? Explain in detail why $1/r$ appears.

40.[M] Obtain the formula for the gradient in cylindrical coordinates.

41.[M] Obtain the formula for curl in cylindrical coordinates.

42.[M] Obtain the formula for divergence in cylindrical coordinates.

43.[M] Obtain the formula for the gradient in spherical coordinates.

44.[M] This exercise shows how to obtain the formula for the gradient of $g(r, \theta)$ in polar coordinates by starting with the formula for the gradient of $f(x, y)$ in rectangular coordinates. During the calculations you will have some happy moments as complicated expressions cancel and the identity $\cos^2\theta + \sin^2\theta = 1$ simplifies expressions.

Assume $g(r,\theta) = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$. To express $\nabla f = \partial f/\partial x\mathbf{i} + \partial f/\partial y\mathbf{j}$ in terms of polar coordinates, it is necessary to express $\partial f/\partial x$, $\partial f/\partial y$, $\mathbf{i}$, and $\mathbf{j}$ in terms of partial derivative of $g(r,\theta)$ and $\widehat{r}$ and $\widehat{\theta}$.

(a) Show that $\partial r/\partial x = \cos\theta, \partial r/\partial y = \sin\theta, \partial\theta/\partial x = -(\sin\theta)/r, \partial\theta/\partial y = (\cos\theta)/r$.

(b) Use the chain rule to express $\partial f/\partial x$ and $\partial f/\partial y$ in terms of partial derivatives of $g(r,\theta)$.

(c) Recalling the expression of $\widehat{i}$ and $\widehat{j}$ in terms of $\widehat{r}$ and $\widehat{\theta}$ in Exercise 18 obtain the gradient of $g(r,\theta)$ in polar coordinates.

18.9 Maxwell's Equations

Any point in space there is an electric field **E** and a magnetic field **B**. The electric field is due to charges (electrons, protons) whether stationary or moving. The magnetic field is due to moving charges.

To assure yourself that the magnetic field **B** is everywhere, hold up a pocket compass. The magnetic field, produced within the Earth, makes the needle point north.

All of electrical phenomenon and their applications can be explained by four equations. These equations allow **B** and **E** to vary in time. We state them for the simpler case; when B and E are constant, $\partial B/\partial t = \mathbf{0}$ and $\partial \mathbf{E}/\partial t = \mathbf{0}$. We met the first one in the previous section.

DOUG, this needs to be Roman numerals

1. $\int_S \mathbf{E} \cdot n \, dS = Q/E_0$, where S is a surface bounding a spatial region and Q is the change in that region. (Gauss's Law)
2. $\oint_C \mathbf{E} \cdot \, dr = 0$ for any closed curve C.
3. $\int_S \mathbf{B} \cdot n \, dS = 0$ for any surface S that bounds a spatial region.
4. $\frac{1}{\mu\epsilon_0} \oint_C \mathbf{B} \cdot \, dr = \frac{1}{\epsilon_0} \int_S \mathbf{j} \cdot \mathbf{n} \, dS$, where C bounds the surface S and j is the electric current flowing through S.

The constant ϵ_0 and μ_0 ("myu zero") depends on the unit used. They will be important in the next section.

Each of the four statements about integrals can be translated with information about the behavior of **E** or **B** at each point.

In derivative or "local" form the four principles read:

Doug: these need to be Roman numerals

1. $\nabla \cdot \mathbf{E} = q/\epsilon_0$, where q is the charge density (Couloumb's Law)
2. $\nabla \times \mathbf{E} = \mathbf{0}$
3. $\nabla \cdot \mathbf{B} = \mathbf{0}$
4. $\frac{1}{\mu_0\epsilon_0} \nabla \times \mathbf{B} = \frac{j}{\epsilon_0}$

In may books $1/(\mu_0\epsilon_0)$ is replaced by C^2, where C is the speed of light. Why that is justified is an astonishing story told in detail at the end of this chapter.

Going Back and Forth, "local" to "global."

Examples 1 and 2 show that Gauss's Law is equivalent to Coulomb's.

EXAMPLE 1 Obtain Gauss's law from Coulomb's law.

SOLUTION Let $\mathcal{V}$ be the solid region whose boundary is $\mathcal{S}$. Then

$$\begin{aligned}\int_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}\,dS &= \int_{\mathcal{V}} \nabla\cdot\mathbf{E}\,dV && \text{divergence theorem}\\ &= \int_{\mathcal{V}} \frac{q}{\epsilon_0}\,dV && \text{Coulomb's law}\\ &= \frac{1}{\epsilon_0}\int_{\mathcal{V}} q\,dV\\ &= \frac{Q}{\epsilon_0}.\end{aligned}$$

◇

"Does Gauss's law imply Coulomb's law?" Example 2 shows that the answer is yes.

EXAMPLE 2 Deduce Coulomb's law from Gauss's law.
SOLUTION Let $\mathcal{V}$ be any spatial region and let $\mathcal{S}$ be its surface. Let Q be the total charge in $\mathcal{V}$. Then

$$\begin{aligned}\frac{Q}{\epsilon_0} &= \int_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}\,dS && \text{Gauss's law}\\ &= \int_{\mathcal{V}} \nabla\cdot\mathbf{E}\,dV && \text{divergence theorem.}\end{aligned}$$

On the other hand,

$$Q = \int_{\mathcal{V}} q\,dV,$$

where a is the charge density. Thus

$$\int_{\mathcal{V}} \frac{q}{\epsilon_0}\,dV = \int_{\mathcal{V}} \nabla\cdot\mathbf{E}\,dV,$$

or

$$\int_{\mathcal{V}} \left(\frac{q}{\epsilon_0} - \nabla\cdot\mathbf{E}\right)\,dV = 0,$$

for all spatial regions. Since the integrand is assumed to be continuous, the "vanishing-integral principle" tells us that it must be identically 0. That is,

$$\frac{q}{\epsilon_0} - \nabla\cdot\mathbf{E} = 0,$$

which give us Coulomb's law. ◇

EXAMPLE 3 Show that DOUG, $\oint_C \mathbf{E} \cdot\ d\mathbf{r} = 0$ for closed curves, implies that DOUG, $\nabla \times \mathbf{E} = \mathbf{0}$.
SOLUTION By Stokes' theorem,

$$\int_{calS} (\nabla \times \mathbf{E}) \cdot \mathbf{n}\ dS = 0$$

for any DOUG surface bounded by a closed curve. The zero-integral principle implies that $(\nabla \times \mathbf{E}) \cdot \mathbf{n} = 0$ at each point on the surface. Choosing $\mathcal{S}$ such that $\mathbf{n}$ is parallel to $\nabla \times \mathbf{E}$ (if $\nabla \times \mathbf{E}$ was not $\mathbf{0}$), implies that the magnitude of $\nabla \times \mathbf{E}$ is 0, hence $\nabla \times \mathbf{E}$ is $\mathbf{0}$. ◇

Maxwell, by studying the four equation, DOUG,deduced that there are radio waves, that they travel with the speed of light, and therefore light is an electromagnetic phenomenon. In DOUG we show how he accomplished this, in one of the greatest creative insights in the history of science.

The exercises present the analogy of the four equations in integral form for the general case whose **B** and **E** vary with time. It is here that **B** and **E** became tangled with each other; both appearing in the same equation. In this generality they are known as Maxwell's Equations, in honor of James Clerk Maxwell (1831-1879), who put them in their final form in 1865.

Mathematical Origin of Electricity

Benjamin Franklin, in his book *Experiments and Observations Made in Philadelphia*, published in 1751, made electricity into a science. For his accomplishments, Franklin was elected a Foreign Associate of the French Academy of Sciences, an honor not bestowed on another American for over a century. In 1873, Maxwell completed the theory that Franklin had begun. At the time that Newton Published his *Principia* on the gravitational field (1687), electricity and magnetism were the subjects of little scientific study. But the experiments of Franklin, Oersted, Henry, AmpDOUGre, Faraday, and others in the eighteenth and early nineteenth centuries gradually built up a mass of information subject to mathematical analysis. All the phenomena cold be summarized in four equations, which in their final form appeared in Maxwell's *Treatise on Electricity and Magnetism*, published in 1873. For a fuller treatment, see *The Feynman Lectures on Physics*, vol. 2, Addison-Wesley, Reading, Mass., 1964. NOTE TO DOUG: Or a McGraw text??

Summary

We stated the four equations that describe electrostatic and magnetic fields that do not vary with time. Then we showed how to use the divergence theorem or Stokes' theorem to translate between their global and local forms. The exercises include the four equations in their general form, where **E** and **B** vary with time.

EXERCISES for 18.9 *Key:* R–routine, M–moderate, C–challenging

1.[R] Obtain DOUG from DOUG.

2.[R] Obtain DOUG from DOUG.

3.[R] Obtain DOUG from DOUG.

4.[R] Obtain DOUG from DOUG.

5.[R] Obtain DOUG from DOUG.

Using terms such as "circulation," "flux," "current," "change density," in Exercise **??** through **??** express the given equation in words.

6.[R] DOUG

7.[R] DOUG

8.[R] DOUG

9.[R] DOUG

10.[R] Which of the four laws tell us that an electric current produces a magnetic field?

11.[R] Which of the four laws tells us that a magnetic field produces an electric current?

In this section we assumed that the fields **E** and **B** do not vary in time, that is, $\partial\mathbf{E}/\partial t = \mathbf{0}$ and $\partial\mathbf{B}/\partial t = \mathbf{0}$. The general case, in empty space, where **E** and **B** depend on time, is also described by four equations, which we call $1, 2, 3, 4$. Numbers 1 and 3, do not involve time. They are DOUG and DOUG.

1. $\nabla \cdot \mathbf{E} = q/\epsilon_0$
2. $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$
3. $\nabla \cdot \mathbf{B} = 0$
4. $\nabla \times \mathbf{B} = \mu_0 g + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

(Here j is the current.)

12.[R] Which equation implies that a changing magnetic field creates an electric field?

13.[R] Which equation implies that a changing electrostatic field creates a magnetic field?

14.[R] Show that (**??**) is equivalent to

$$\oint_C \mathbf{E}\cdot\ dt = -\frac{\partial}{\partial t}\int_{\mathcal{S}} \mathbf{B}\cdot\mathbf{n}\ dS$$

Here, C bounds S. You may assume that $\partial/\partial t \int_{\mathcal{S}} \mathbf{B}\cdot\mathbf{n}\, dS$ equals $\int_{\mathcal{S}}(\partial B/\partial t)\cdot\mathbf{n}\, dS$.

15.[R] Show that (**??**) is equivalent to

$$\oint_C \mathbf{B}\cdot\ d\mathbf{r} = \mu_0\int_{\mathcal{S}} j\cdot n\ dS + \mu_0\epsilon_0\frac{\partial}{\partial t}\int_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}\ dS$$

(The circulation of **B** is related to the total current through the surface $\mathcal{S}$ that C bounds and to the rate at which the flux of **E** through $\mathcal{S}$ changes.)

16.[R] Use Green's theorem to evaluate $\oint_C (xy\ dx + e^x\ dy)$, where C is the curve that goes from $(0,0)$ to $(2,0)$ on the x axis and returns from $(2,0)$ to $(0,0)$ on the parabola $y = 2x - x^2$.

17.[R] Let C be the circle of radius 1 with center $(0,0)$.

(a) What does Green's theorem say about the line integral

$$\oint_C [(x^2 - y^3)\ dx + (y^2 + x^3)\ dy]?$$

(b) Use Green's theorem to evaluate the integral in (a).

(c) Evaluate the integral in (a) directly.

18.10 Chapter Summary

The two key theorems in this chapter are Gauss' Theorem and Stokes' Theorem. Gauss' Theorem says that

Gauss' Theorem

Gauss' Theorem (framed boxed?)

$$\int_R \nabla \cdot \mathbf{F}\, dV = \int_{\mathcal{S}} \mathbf{F}\dot{\mathbf{n}}\, dS \qquad \text{where } \mathcal{S} \text{ encloses the solid region } R.$$

If $\mathbf{F}$ represents gas- or fluid-flow, the surface integral measures the rate it escapes (or enters) R across $\mathcal{S}$ — the *flux*. The *divergence*, $\nabla \cdot \mathbf{F}$ measures this change locally. So both integrals measure the rate of total change of the mass of R. If the divergence is 0 throughout R, the field $\mathbf{F}$ is called *incompressible*.

Stokes' Theorem concerns a closed curve C in space (or in a plane) and any surface $\mathcal{S}$ whose boundary is C. It says that

Stokes' Theorem

Stokes' Theorem (framed boxed?)

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

The physical interpretation of Stokes' Theorem is that the circulation around the curve is equal to the integral of the normal component of the curl over the surface.

There are two unit normals to a surface at each point. The "correct" one is determined by the orientation of the curve. If the curl of $\mathbf{F}$ is identically $\mathbf{0}$, $\mathbf{F}$ is called "irrotational." In general, the curl describes the tendency of $\mathbf{F}$ to rotate.

We offer a sample showing how Gauss' Theorem is applied. Section 18.9 illustrates the role of curl in establishing the foundations of electromagnetic theory. Section 18.8 explained how to obtain formulas for the gradient, divergence, and curl in other coordinate systems. This introduction should lessen the shock when you encounter, and apply, these ideas in later courses.

If the surface in Stokes' Theorem lies in a plane, we obtain a version of Green's Theorem. Green's Theorem can also be expressed like Gauss' Theorem. It says that, for a curve C bounding a region R in the plane,

$$\oint_C (P\ dx + Q\ dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA.$$

The integrand, in differential form, can be expressed as either the tangential component ($\mathbf{F} \cdot \mathbf{T}$) or normal component ($\mathbf{F} \cdot \mathbf{n}$, with $\mathbf{n} = \mathbf{k}$) of suitable fields $\mathbf{F}$. These give the planar versions of Gauss' and Stokes' Theorems.

Of central interest in the study of gravity or electromagnetism are the central fields, which have the form $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{r}$. In space all such fields have curl $\mathbf{0}$, but only the case when $f(r)$ is inversely proportional to the square of r (or is 0) is the divergence 0.

A field $\mathbf{F}$ whose integral $\int_C \mathbf{F} \cdot\ d\mathbf{r}$ depends only on the endpoints of the curve C is called *conservative*. Equivalently, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for closed curves. There is a simple way to check whether a field is conservative: see if its curl is $\mathbf{0}$ and the region is simply connected (closed curves can be shrunk to a point while staying in the region). A field is conservative if and only if it is the gradient of some scalar field f: $\mathbf{F} = \nabla f$. This is good news since it's generally easier to work with a scalar field than with a vector field. For instance, in physics, the scalar field is called the potential, and its gradient is related to the force.

Along the way, there were several applications including the use of steradians to evaluate an important integral. The flux of $\mathbf{r}/r^2$ over any closed surface is always 4π:

$$\int_S \frac{\mathbf{r}}{r^2} \cdot \mathbf{n}\, dS = 4\pi \qquad \text{where } S \text{ is a closed surface.}$$

EXERCISES for 18.S *Key:* R–routine, M–moderate, C–challenging