# Overview of Calculus III

The first two parts of this book have focused on calculus of a single variable. The final third of this book extends the basic calculus ideas — limit, derivative, and integral — to two- and three-dimensions.

# Chapter 14

# Vectors

Vectors are sometimes represented as arrows.

Section 14.1 introduces vectors and their arithmetic. Section 14.2 examines the dot product, which is a number. This includes the geometry of the dot product and its role in projections. (A projection is related to the shadow cast by parallel rays of light.)

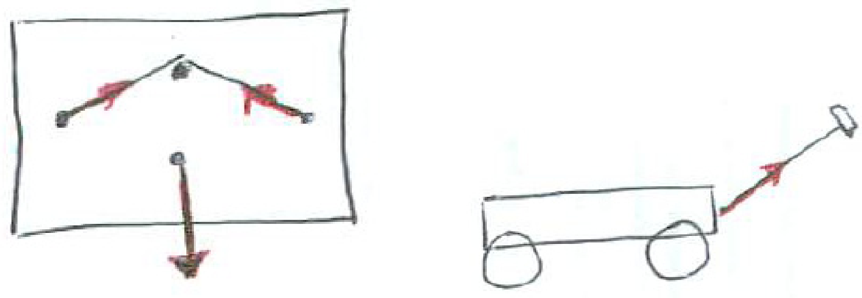
Section 14.3 examines the cross product, which is a vector. Determinants are reviewed, and the scalar triple product (a number) is introduced and used to find the volume of a parallelepiped.

Section 14.4 develops a number of fundamental properties of lines and planes, in terms of vectors. The distance from a point to a line or plane is developed, a parametric description of a line is given, using the dot and cross product. These ideas are used to talk about flows.

This algebra was developed primarily in response to James Clerk Maxwell's *Treatise on Electricity and Magnetism*, published in 1873. Josiah Gibbs, who in 1863 earned the first doctorate in engineering awarded in the United States and became a mathematical physicist, put vector analysis in its present form. His *Elements of Vector Analysis*, published in 1881, introduced the notation used in this chapter. Maxwell's contributions will be studied in greater detail in Chapter 18.

## 14.1 The Algebra of Vectors

You have lived with vectors all your life. When you hanged a picture on wire you dealt with three vectors: one describes the downward force of gravity and two describe the force of the wires pulling up to oppose gravity, as in Figure 14.1.1(a)



(a) (b)

Figure 14.1.1:

When you pull a wagon the force you use is represented by a vector, as in Figure 14.1.1(b). The harder you pull, the larger the vector.

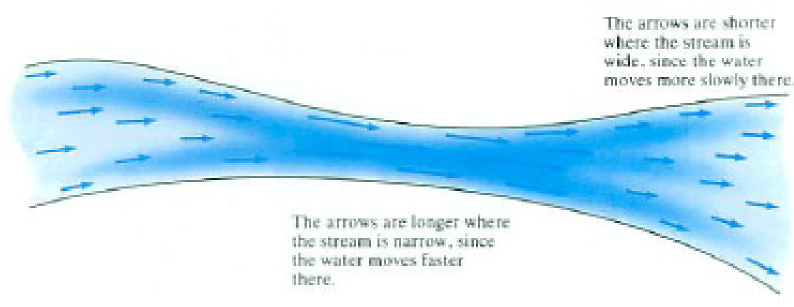



Figure 14.1.2:

A vector has a direction and a magnitude. You may think of it as an arrow, whose length and direction carry information. Vectors are of use in describing the flow of a fluid, as in Figure 14.1.2, or the wind, or the strength and direction of a magnetic field.

### Vectors in the Plane

A vector in the $xy$ plane is an ordered pair of numbers $x$ and $y$, denoted $\langle x, y\rangle$. Its magnitude, or length, is $\sqrt{x^2+y^2}$. Though the notation resembles that for a point, $(x, y)$, we treat vectors quite differently. We can add them, subtract them and multiply them by a number. Two additional products of vectors are introduced in Sections 14.2 and 14.3.

We represent a vector by an arrow whose tail is at $(0, 0)$ and whose head (or "tip") is at $(x, y)$, as in Figure 14.1.3.

More generally, we represent $\langle x, y\rangle$ by any pair of points $P = (a_1, a_2)$ and $Q = (b_1, b_2)$ if $b_1 - a_1 = x$ and $b_2 - a_2 = y$, as in Figure 14.1.4.

We speak then of "the vector from $P$ to $Q$" and denote it $\overrightarrow{PQ}$. A vector $\langle x, y\rangle$ will be denoted by bold face letters, such as **A**, **B**, **r**, **v**, and **a**. In handwriting or on the blackboard they are decorated with a bar or arrow on top, for instance $\overrightarrow{A}$ or $\overline{A}$. A vector of length 1 is called a unit vector and is topped with a little hat, as in $\widehat{r}$, which is read aloud as "r hat".

Here is how we operate on vectors. Let $\mathbf{A} = \langle a_1, a_2\rangle$ and $\mathbf{B} = \langle b_1, b_2\rangle$ be vectors and let $c$ be a number.


Figure 14.1.3: The arrow represents the vector $\langle x, y\rangle$.

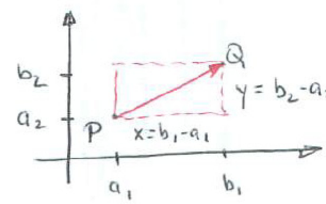
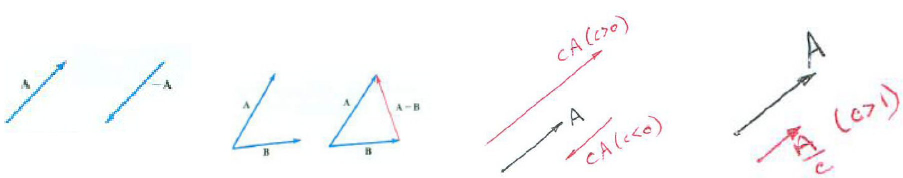



Figure 14.1.4:


(a) (b) (c) (d)

Figure 14.1.5:




(a) (b) (c) (d)

Figure 14.1.6:

| Operation | Definition | Geometry | Comment |
|---|---|---|---|
| $\mathbf{A}+\mathbf{B}$ | $\langle a_1+b_1, a_2+b_2\rangle$ | Figure 14.1.5 | The tail of **B** is placed at the head of **A** |
| $-\mathbf{A}$ | $\langle -a_1, -a_2\rangle$ | Figure 14.1.6(a) | $-\mathbf{A}$ points in opposite direction of **A** |
| $\mathbf{A}-\mathbf{B}$ | $\langle a_1-b_1, a_2-b_2\rangle$ | Figure 14.1.6(b) | What you add to **B** to get **A** |
| $c\mathbf{A}$ | $\langle ca_1, ca_2\rangle$ | Figure 14.1.6(c) | Parallel to **A** and $\lvert c\rvert$ times as long as **A** |
| $\frac{\mathbf{A}}{c}$ | $\left\langle \frac{a_1}{c}, \frac{a_2}{c}\right\rangle$ | Figure 14.1.6(d) | Parallel to **A** and $\frac{1}{c}$ times as long as **A** $(c \neq 0)$ |

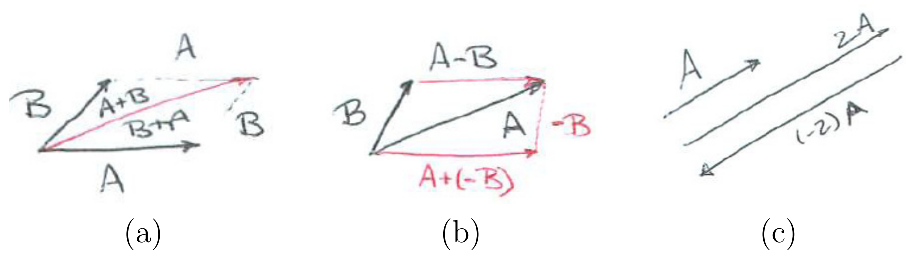



Figure 14.1.7:

The operation of addition obeys the usual rules of addition of numbers. For instance, $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$ and $\mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{B})+\mathbf{C}$. Also $\mathbf{A}-\mathbf{B} = \mathbf{A}+-\mathbf{B}$. This is easy to establish using the definitions. In terms of arrows it makes sense; see Figure 14.1.7(a).

$\mathbf{A}-\mathbf{B}$ and $\mathbf{A}+(-\mathbf{B})$ appears as opposite sides of a parallelogram. Figure 14.1.7(a) shows both $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}+\mathbf{A}$; they are equal.

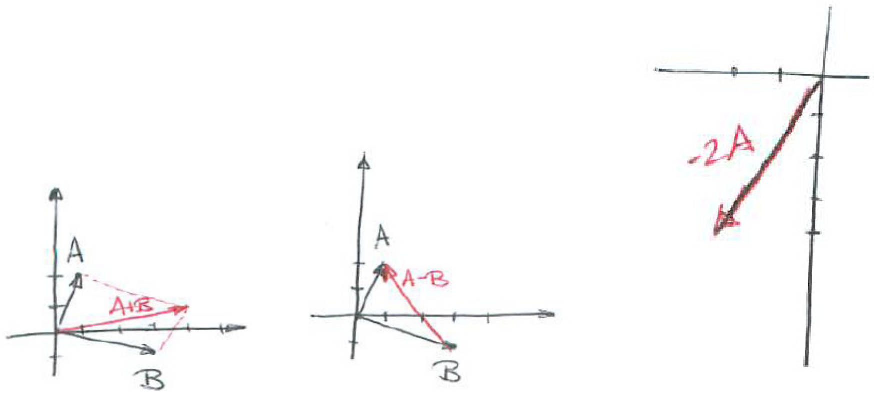
The magnitude of $\langle x, y\rangle$ is $\sqrt{(cx)^2+(cy)^2} = \sqrt{c^2}\sqrt{x^2+y^2}$, that is, $|c|$ times the magnitude of $\langle x, y\rangle$. If $c$ is positive $\langle cx, cy\rangle$ and $\langle x, y\rangle$ point in the same direction. If $c$ is negative they point in opposite direction, as the arrows in Figure 14.1.7(c) illustrate for $c = 2$ or $-2$.

When talking about numbers, such as $c$, $x$, and $y$, in the context of vectors, we call them **scalars**. Thus in $c\overrightarrow{A}$ the scalar $c$ is multiplying the vector **A**.

The vector $\langle 0, 0\rangle$ is denoted **0** and is called the **zero vector**.

**EXAMPLE 1** Let $\mathbf{A} = \langle 1, 2\rangle$, $\mathbf{B} = \langle 3, -1\rangle$ and $c = -2$. Complete $\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$ and $c\mathbf{A}$. Then draw the corresponding arrows.
*SOLUTION*

$$\begin{aligned} A+B &= \langle 1,2\rangle + \langle 3,-1\rangle = \langle 1+3, 2+(-1)\rangle = \langle 4,1\rangle \\ A-B &= \langle 1,2\rangle - \langle 3,-1\rangle = \langle 1-3, 2-(-1)\rangle = \langle -2,3\rangle \\ cA &= -2\langle 1,2\rangle = \langle -2,(1), -2(2)\rangle = \langle -2,-4\rangle \end{aligned}$$




(a) (b) (c)

Figure 14.1.8:

Note that $\mathbf{A} - \mathbf{B}$ and $\mathbf{A} + \mathbf{B}$ lie on the two diagonals of a parallelogram. (See Figure 14.1.8.) ◇

Before we can make the similar definition for vectors in space, we must introduce an appropriate coordinate system.

## Coordinates in Space

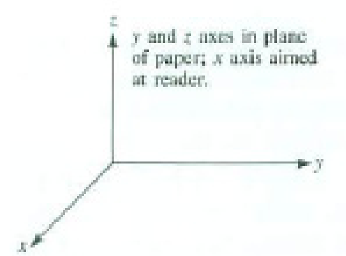



Figure 14.1.9: ARTIST: A "right hand" should be added to this figure.

First, pick a pair of perpendicular intersecting lines to serve as the $x$ and $y$ axes. The positive parts of these axes are indicated by arrows. These two lines determine the $xy$ **plane**. The line perpendicular to the $xy$ plane and meeting the $x$ and $y$ axes will be called the $z$-axis. The point where the three axes meet is called the **origin**. The 0 of the $z$-axis will be put at the origin. But which half of the $z$-axis will have positive numbers and which half will have the negative numbers? It is customary to determine this by the **right-hand rule**. Moving in the $xy$ plane through a right angle from the positive $x$-axis to the positive $y$-axis determines a sense of rotation around the $z$-axis. If the fingers of the right hand curl in that sense, the thumb points in the direction of the *positive* $z$-axis, as shown in Figure 14.1.9.

Any point $Q$ in space is now described by three numbers: First, two numbers specify the $x$ and $y$ coordinates of the point $P$ in the $xy$ plane directly below (or above) $Q$; then the height of $Q$ above (or below) the $xy$ plane is recorded by the $z$ coordinate of the point $R$ where the plane through $Q$ and parallel to the $xy$ plane meets the $z$-axis. The point $Q$ is then denoted $(x, y, z)$. See Figure 14.1.10.

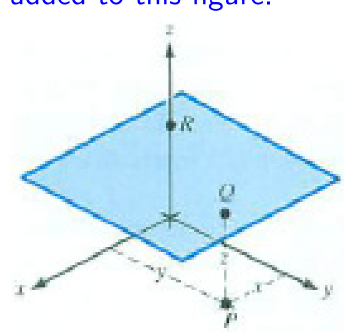
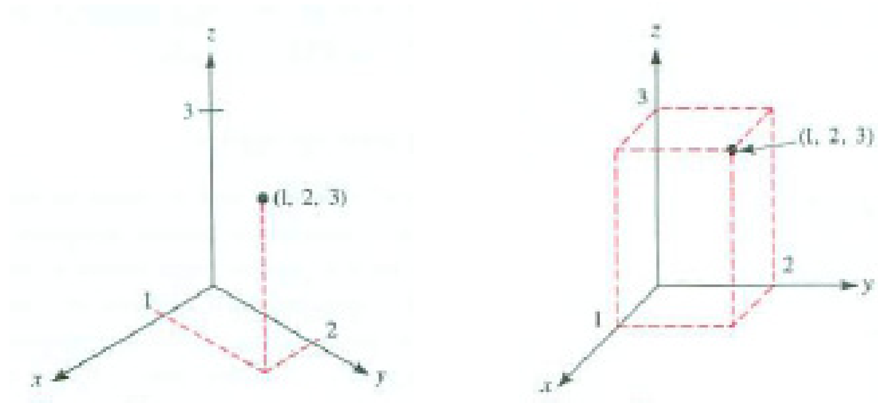



Figure 14.1.10:

The points $(x, y, z)$ for which $z = 0$ lie in the $xy$ plane. There are an infinite number of these points. The points $(x, y, z)$ for which $x = 0$ lie entirely in the plane determined by the $y$ and $x$ axes, which is called the $yz$ **plane**. Similarly, the equation $y - 0$ describe the $xz$ **plane**. The $xy$, $xz$ and $yz$ planes are called the **coordinate planes**.

**EXAMPLE 2** Plot the point $(1, 2, 3)$.




(a) (b)

Figure 14.1.11:

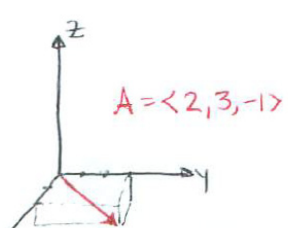
*SOLUTION* One way is to first plot the point $(1, 2)$ in the $xy$ plane. Then, on a line perpendicular to the $xy$ plane at that point, show the point $(1, 2, 3)$ as done in Figure 14.1.11(a).

Another way is to draw a box whose edges are parallel to the axes and which has the origin $(0, 0, 0)$ and $(1, 2, 3)$ as done in Figure 14.1.11(b). (This time, the $y$ and $z$ axes make a right angle.) ◇

Just as the axes in the $xy$ plane divide the plane with four quadrants, the three coordinate planes divide space with eight octants.

## Vectors in Space

The only difference between a vector in space and a vector in the $xy$ plane is that it has three components, $x$, $y$, and $z$, and is written $\langle x, y, z\rangle$. Its length or magnitude is defined as $\sqrt{x^2 + y^2 + z^2}$. The definition of the sum and difference of such vectors is so similar to the definition for planar vectors that we will not list them. For instance, $\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle$ is $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$. The biggest difference is that they are harder to draw, even though each can be suggested by our arrow. It may help visualize such a vector by drawing a box in which it is a main diagonal. For instance, to draw the vector $\langle 2, 3, -1\rangle$ you may draw the box shown in Figure 14.1.12

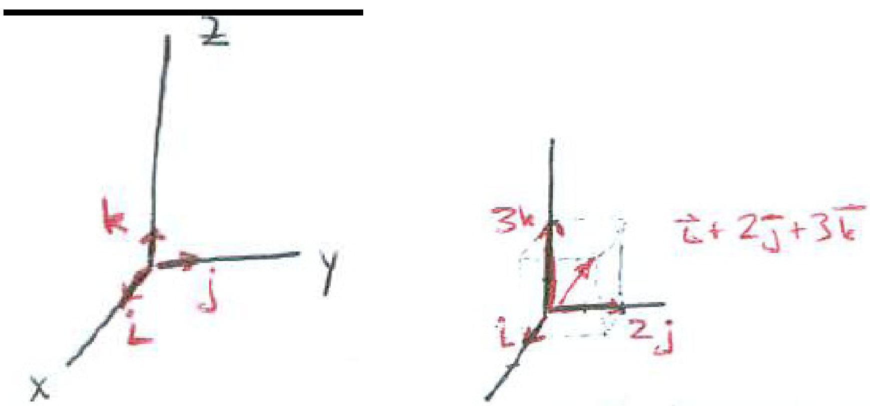


This representation of **A** has its tail at the arrow. Of course the arrow and box could be drawn with the tail of the arrow anywhere else.

## The Standard Unit Vectors

The three most important unit vectors indicate the positive directions of the positive $x$, $y$, and $z$ axes. They will be denoted **i**, **j** and **k**, respectively. For instance, $\mathbf{i} = \langle 1, 0, 0 \rangle$. The vectors $\langle x, y, z \rangle$ can also be written $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

**EXAMPLE 3** Draw **i**, **j**, **k** and $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
*SOLUTION* Figure 14.1.13(a) shows **i**, **j**, **k** and Figure 14.1.13(b) shows




(a) (b)

Figure 14.1.13:

$\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. ◇

The magnitude of **A** is indicated by $\|\mathbf{A}\|$. $\|\mathbf{A}\|$ is a scalar and $\mathbf{A}/\|\mathbf{A}\|$ is a vector.

The vector $\frac{\mathbf{A}}{\|\mathbf{A}\|}$ is a unit vector for any non-zero vector **A**. To see this, we let $\mathbf{A} = \langle x, y, z \rangle$ and compute $\mathbf{A}/\|\mathbf{A}\|$:

$$\frac{\mathbf{A}}{\|\mathbf{A}\|} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} = \langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \rangle.$$

The square of the length of $\mathbf{A}/\|\mathbf{A}\|$ is

$$\left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right)^2 + \left( \frac{y}{\sqrt{x^2 + y^2 + z^2}} \right)^2 + \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^2 = \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1.$$

Thus $\mathbf{A}/\|\mathbf{A}\|$ is a unit vector.

Example 4 shows how vectors can be used to establish geometric properties.

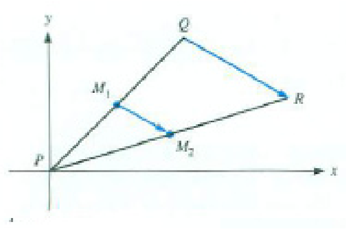



Figure 14.1.15:
Figure 14.1.14:

**EXAMPLE 4** Prove that the line which joins the midpoints of two sides of a triangle is parallel to the third side and half as long.

*SOLUTION* Let the triangle have vertices $P$, $Q$, and $R$. Let the midpoint of side $PQ$ be $M$ and the midpoint of side $PR$ be $N$ as in Figure 14.1.14.

Introduce an $xy$ coordinate system in the plane of the triangle. Through its origin could be anywhere in the plane, we should put it at $P$ in order to simplify the calculations. (See Figure 14.1.15.)

We wish to show that the vector $\overrightarrow{MN}$ is $\frac{1}{2}\overrightarrow{QR}$. To do so, we compute $\overrightarrow{MN}$ and $\overrightarrow{QR}$ in terms of vectors involving $P$, $Q$, and $R$.

First of all, $\overrightarrow{PM} = \frac{1}{2}\overrightarrow{PQ}$ and $\overrightarrow{PN} = \frac{1}{2}\overrightarrow{PR}$. Thus

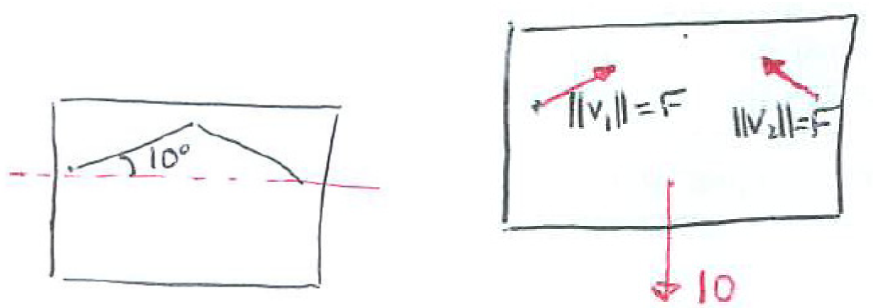
$$\overrightarrow{MN} = \frac{1}{2}\overrightarrow{PR} - \frac{1}{2}\overrightarrow{PQ} = \frac{1}{2}(\overrightarrow{PR} - \overrightarrow{PQ}) = \frac{1}{2}(\overrightarrow{QR}).$$

◇

The next example shows the importance of thinking vectorally. Not thinking that way, one of the other had a picture fall and break a vase.

**EXAMPLE 5** A picture weighing 10 pounds has a wire on the back, which rests on a picture hook, as shown in Figure 14.1.16(a). Find the force (tension) on the wire.

*SOLUTION* There are three vectors involved. One is straight down, with magnitude 10 lbs. and two are along the wire, with unknown magnitude $F$: $\|\mathbf{v}_1\| = F = \|\mathbf{v}_2\|$.




(a) (b)

Figure 14.1.16:

To balance the downward force of gravity, each end of the wire must have a vertical component of 5 lbs. Since the angle with the horizontal is 10° we must have $F\sin(10°) = 5$ or $F = 5/\sin(10°) \approx 29$ pounds. That is much greater than the weight of the painting and creates quite a pull on the screws at the bases of the wire. This force can (sadly, we learned) eventually pull a screw out of the wall. ◇

## Summary

We introduced the notion of a vector $\langle x, y\rangle$ in the $xy$ plane or $\langle x, y, z\rangle$ in space and defined their addition and subtraction. Furthermore we defined the operation of a scalar $c$ as a vector $\langle x, y, z\rangle$, as $\langle cx, cy, cz\rangle$.

We visualized vectors with the aid of arrows, which could be drawn anywhere in the $xy$ plane or in space.

Each vector in the $xy$-plane can be written as $x\mathbf{i} + y\mathbf{j}$. Vector in space can be written as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

**EXERCISES for Section 14.1** *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 and 2 use the plane of your paper as the $xy$ plane.

**1.**[R] Draw the vector $2\mathbf{i} + 3\mathbf{j}$, placing its tail at (a) $(0, 0)$, (b) $(-1, 2)$, (c) $(1, 1)$.

**2.**[R] Draw the vector $-\mathbf{i} + 2\mathbf{j}$, placing its tail at (a) $(0, 0)$, (b) $(3, 0)$, (c) $(-2, 2)$.

In Exercises 3 to 6 draw the vector $\mathbf{A}$ and enough extra lines to show how it is situated in space.

**3.**[R] $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$,

(a) tail at $(0, 0, 0)$,

(b) tail at $(1, 1, 1)$.

**4.**[R] $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$,

(a) tail at $(0, 0, 0)$,

(b) tail at $(2, 3, 4)$.

**5.**[R] $\mathbf{A} = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$,

(a) tail at $(0, 0, 0)$,

(b) tail at $(1, 1, -1)$.

**6.**[R] $\mathbf{A} = \mathbf{j} + \mathbf{k}$,

(a) tail at $(0, 0, 0)$,

(b) tail at $(-1, -1, -1)$.

In Exercises 7 to 10 plot the points $P$ and $Q$, draw the vector $\overrightarrow{PQ}$, express it in the form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and find its length.

**7.**[R] $P = (0, 0, 0), Q = (1, 3, 4)$

**8.**[R] $P = (1, 2, 3), Q = (2, 5, 4)$

**9.**[R] $P = (2, 5, 4), Q = (1, 2, 2)$

**10.**[R] $P = (1, 1, 1), Q = (-1, 3, -2)$

In Exercises 11 and 12 express the vector $\mathbf{A}$ in the form $x\mathbf{i} + y\mathbf{j}$. North is along the positive $y$-axis and east is along the positive $x$-axis.

**11.**[R]

(a) $\|\mathbf{A}\| = 10$ and $\mathbf{A}$ points northwest;

(b) $\|\mathbf{A}\| = 6$ and $\mathbf{A}$ points south;

(c) $\|\mathbf{A}\| = 9$ and $\mathbf{A}$ points southeast;

(d) $\|\mathbf{A}\| = 5$ and $\mathbf{A}$ points east.

**12.**[R]

(a) $\|\mathbf{A}\| = 1$ and $\mathbf{A}$ points southwest;

(b) $\|\mathbf{A}\| = 2$ and $\mathbf{A}$ points west;

(c) $\|\mathbf{A}\| = \sqrt{8}$ and $\mathbf{A}$ points northeast;

(d) $\|\mathbf{A}\| = 1/2$ and $\mathbf{A}$ points south.

**13.**[M] The wind is 30 miles per hour to the northeast. An airplane is traveling 100 miles per hour relative to the air, and the vector from the tail of the plane to its front tip points to the southeast. (See Figure 14.1.17.)

(a) What is the speed of the plane relative to the ground?

(b) What is the direction of the flight relative to the ground?

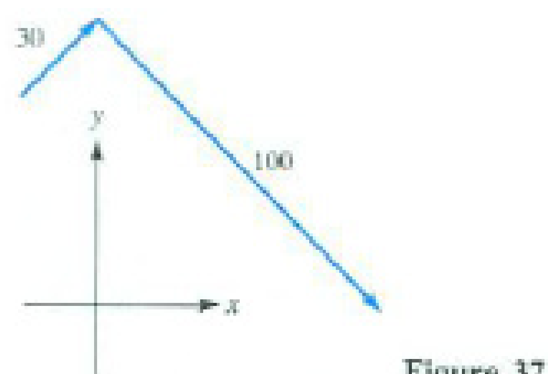



Figure 14.1.17:

**14.**[M] (See Exercise 13.) The jet stream is moving 200 miles per hour to the southeast. A plane with a speed of 550 miles per hour relative to the air is aimed to the northwest.

(a) Draw the vectors representing the wind and the plane relative to the air. (Choose a scale and make an accurate drawing.)

(b) Using your drawing, estimate the speed of the plane relative to the ground.

(c) Compute the speed in (b) exactly.

**15.**[R] Compute $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$ if

(a) $\mathbf{A}=\langle -1,2,3\rangle$ and $\mathbf{B}=\langle 7,0,2\rangle$.

(b) $\mathbf{A}=3\mathbf{j}+4\mathbf{k}$ and $\mathbf{B}=6\mathbf{i}+7\mathbf{j}$.

**16.**[R] Compute $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$ if

(a) $\mathbf{A}=\langle 1/2,1/3,1/6\rangle$ and $\mathbf{B}=\langle 2,3,-1/3\rangle$.

(b) $\mathbf{A}=2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $\mathbf{B}=-\mathbf{i}+5\mathbf{j}+6\mathbf{k}$.

**17.**[R] Compute and sketch $c\mathbf{A}$ if $\mathbf{A}-2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and $c$ is

(a) 2,

(b) $-2$,

(c) $\frac{1}{2}$,

(d) $-\frac{1}{2}$.

**18.**[R] Express each of the following vectors in the form $c(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$ for suitable $c$:

(a) $\langle 4,6,8\rangle$

(b) $-2\mathbf{i}-3\mathbf{j}-4\mathbf{k}$

(c) $\mathbf{0}$

(d) $\frac{2}{11}\mathbf{i}+\frac{3}{11}\mathbf{j}+\frac{4}{11}\mathbf{k}$

**19.**[R] If $\|\mathbf{A}\|=6$, find the length of the following vectors

(a) $-2\mathbf{A}$

(b) $\mathbf{A}/3$

(c) $\mathbf{A}/\|\mathbf{A}\|$

(d) $-\mathbf{A}$

(e) $\mathbf{A}+2\mathbf{A}$.

**20.**[R] If $\|\mathbf{A}\|=3$, find the length of the following vecrps

(a) $-4\mathbf{A}$

(b) $13\mathbf{A}-7\mathbf{A}$

(c) $\mathbf{A}/\|\mathbf{A}\|$

(d) $\mathbf{A}/0.05$

(e) $\mathbf{A}-\mathbf{A}$.

**21.**[R]

(a) Find a unit vector $\mathbf{u}$ that has the same direction as $\mathbf{A}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

(b) Draw $\mathbf{A}$ and $\mathbf{u}$, with their tails at the origin.

**22.**[R]

(a) Find a unit vector $\mathbf{u}$ that has the same direction as $\mathbf{A}=2\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.

(b) Draw $\mathbf{A}$ and $\mathbf{u}$, with their tails at the origin.

**23.**[R] Using the definition of addition of vectors $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$, show the $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ and $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$.

**24.**[R] Using the definition of addition of vectors show that $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

**25.**[R] Which unit vector points in the same direction as $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$?

**26.**[R] Sketch a unit vector pointing in the same direction as $3\mathbf{i} + 4\mathbf{j}$.

**27.**[M] *(Midpoint formula)* Let $A$ and $B$ be two points in space. Let $M$ be their midpoint. Let $\mathbf{A} = \overrightarrow{OA}$, $\mathbf{B} = \overrightarrow{OB}$, and $\mathbf{M} = \overrightarrow{OM}$.

(a) Show that $\mathbf{M} = \mathbf{A} + \frac{1}{2}(\mathbf{B} - \mathbf{A})$.

(b) Deduce that $\mathbf{M} = (\mathbf{A} + \mathbf{B})/2$. *Hint:* Draw a picture.

**28.**[M] Let $A$ and $B$ be two distinct points in space. Let $C$ be the point on the line segment $AB$ that is twice as far from $A$ as it is from $B$. Let $\mathbf{A} = \overrightarrow{OA}$, $\mathbf{B} = \overrightarrow{OB}$, and $\mathbf{C} = \overrightarrow{OC}$. Show that $\mathbf{C} = \frac{1}{3}\mathbf{A} + \frac{2}{3}\mathbf{B}$. *Hint:* Draw a picture.

**29.**[M] Show that $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $6\mathbf{i} + 9\mathbf{j} + 12\mathbf{k}$ are parallel.

**30.**[M] Show that $\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $-2\mathbf{i} + 6\mathbf{j} - 12\mathbf{k}$ are parallel.

**31.**[M] This exercise outlines a proof of the distributive rule: $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$. Write $\mathbf{A}$ and $\mathbf{B}$ in components, and obtain the rule by expressing both $c(\mathbf{A} + \mathbf{B})$ and $c\mathbf{A} + c\mathbf{B}$ in components.

**32.**[M]

(a) Show that the vectors $\mathbf{u}_1 = \frac{1}{2}\mathbf{i} + (\sqrt{3}/2\mathbf{j}$ and $\mathbf{u}_2 = (\sqrt{3}/2\mathbf{i} - \frac{1}{2}\mathbf{j}$ are perpendicular unit vectors. *Hint:* What angles do they make with the $x$-axis?

(b) Find scalars $x$ and $y$ such that $\mathbf{i} = x\mathbf{u}_1 + y\mathbf{u}_2$.

**33.**[M]

(a) Show that the vectors $\mathbf{u}_1 = (\sqrt{2}/2\mathbf{i}) + (\sqrt{2}/2\mathbf{j})$ and $\mathbf{u}_2 = (-\sqrt{2}/2\mathbf{i}) + (\sqrt{2}/2\mathbf{j})$ are perpendicular unit vectors. *Hint:* Draw them.

(b) Express $\mathbf{i}$ in the form of $x\mathbf{u}_1 + y\mathbf{u}_2$. *Hint:* Draw $\mathbf{i}$,$\mathbf{u}_1$, and $\mathbf{u}_2$.

(c) Express $\mathbf{j}$ in the form $x\mathbf{u}_1 + y\mathbf{u}_2$.

(d) Express $-2\mathbf{i} + 3\mathbf{j}$ in the form $x\mathbf{u}_1 + y\mathbf{u}_2$.

**34.**[M]

(a) Draw a unit vector $\mathbf{u}$ tangent to the curve $y = \sin x$ at $(0, 0)$.

(b) Express $\mathbf{u}$ in the form $x\mathbf{i} + y\mathbf{j}$.

**35.**[M]

(a) Draw a unit vector $\mathbf{u}$ tangent to the curve $y = x^3$ at $(1, 1)$.

(b) Express $\mathbf{u}$ in the form $x\mathbf{i} + y\mathbf{j}$.

**36.**[M]

(a) What is the sum of the five vectors shown in Figure 14.1.18?

(b) Sketch the figure corresponding to the sum $\mathbf{A} + \mathbf{C} + \mathbf{D} + \mathbf{E} + \mathbf{B}$.

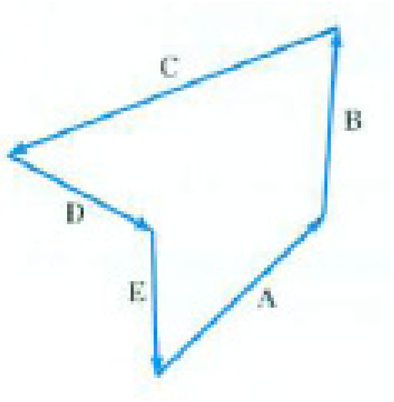



Figure 14.1.18:

**37.**[M] A rectangular box has sides of length $x$, $y$, and $z$. Show that the length of a longest diagonal (arc joining opposite corner) is $\sqrt{x^2+y^2+z^2}$. HINT: Use the Pythogorean Theorem, twice.

**38.**[M] See Example 5 concerning hanging a picture. What would be the tension in the wire if it were at an angle of

(a) $60°$ instead of $10°$ to the horizontal,

(b) $5°$ instead of $10°$ to the horizontal?

**39.**[C]

(a) Draw the vectors $\mathbf{A} = 2\mathbf{i}+\mathbf{j}$, $\mathbf{B} = 4\mathbf{i}-\mathbf{j}$, and $\mathbf{C} = 5\mathbf{i}+2\mathbf{j}$.

(b) With the aid of the drawing show that there are scalars $x$ and $y$ such that $\mathbf{C} = x\mathbf{A}+y\mathbf{B}$.

(c) Using the drawing in (a), estimate $x$ and $y$.

(d) Find $x$ and $y$ exactly.

**40.**[C] (See Exercise 13.) Let **A** and **B** be two nonzero and nonparallel vectors in the $xy$ plane. Let **C** be any vector in the $xy$ plane. Show with the aid of a sketch that there are scalars $x$ and $y$ such that $\mathbf{C} = x\mathbf{A}+y\mathbf{B}$.

**41.**[C] Let **A**, **B** and **C** be three vectors that do not all lie in one plane. Let **D** be any vector in space. Show with the aid of a sketch that there are scalars $x$, $y$, and $z$ such that $\mathbf{D} = x\mathbf{A}+y\mathbf{B}+z\mathbf{C}$.

**42.**[C] Let $A$, $B$ and $C$ be the vertices of a triangle. Let $\mathbf{A} = \overrightarrow{OA}$, $\mathbf{B} = \overrightarrow{OB}$, and $\mathbf{C} = \overrightarrow{OC}$.

(a) Let $P$ be the point that is on the line segment joining $A$ to the midpoint of the edge $BC$ and twice as far from $A$ as from the midpoint. Show that $\overrightarrow{OP} = (\mathbf{A}+\mathbf{B}+\mathbf{C})/3$.

(b) Use (a) to show that the three medians of a triangle are concurrent.

**43.**[C] The midpoints of a quadrilateral in space are joined to form another quadrilateral. Prove that this second quadrilateral is a parallelogram.

**44.**[C]

(a) Using an appropriate diagram, explain why $\|\mathbf{A}+\mathbf{B}\| \le \|\mathbf{A}\| + \|\mathbf{B}\|$. (This is called the **triangle inequality**.

(b) For which pairs of vectors **A** and **B** is $\|\mathbf{A}+\mathbf{B}\| = \|\mathbf{A}\| + \|\mathbf{B}\|$?

**45.**[C] From Exercise 44 deduce that for any four real numbers $x_1$, $y_1$, $x_2$, and $y_2$,

$$x_1x_2 + y_1y_2 \le \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}.$$

When does equality hold?

## 14.2 The Dot Product of Two Vectors

The dot product is a number, or scalar.

The "dot product" or "scalar product" is a number that is defined for every pair of vectors. Consider a rock being pulled along level ground by a


(a) (b)

Figure 14.2.1:

rope inclined at at fixed angle to the ground. Let the force applied to the rock be represented by the vector $\mathbf{F}$. The force $\mathbf{F}$ can be expressed as the sum of a vertical force $\mathbf{F}_2$ and a horizontal force $\mathbf{F}_1$, as shown in Figure 14.2.1(b).

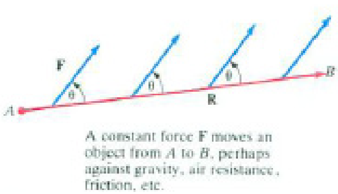



Figure 14.2.2:

How much work is done by the force $\mathbf{F}$ in moving the rock along the ground? The physicist defines the work accomplished by a constant force $\mathbf{F}$ (whatever direction it may have) as the product of the component of $\mathbf{F}$ in the direction of motion and the distance traveled. Say that the force $\mathbf{F}$, as shown in Figure 14.2.2, moves an object along a straight line from the tail to the head of $\mathbf{R}$.

By definition

$$\text{Work} = \underbrace{\|\mathbf{F}\| \cos(\theta)}_{\text{Force in Direction of } \mathbf{R}} \cdot \underbrace{\|\mathbf{R}\|}_{\text{Distance traveled}}$$

where $\theta$ is the angle between $\mathbf{R}$ and $\mathbf{F}$.

The force $\mathbf{F}_2$ in Figure 14.2.1 accomplishes no work. The work accomplished by $\mathbf{F}$ in pulling the rock is the same as that accomplished by $\mathbf{F}_1$.

## The Dot Product

This important physical concept illustrates the dot product of two vectors, which will be introduced after the following definition.

> **DEFINITION** *(Angle between two nonzero vectors.)* Let **A** and **B** be two nonparallel and nonzero vectors. They determine a triangle and an angle $\theta$, shown in Figure 14.2.3. The **angle between A and B** is $\theta$. Note that
>
> $$0 < \theta < \pi$$
>
> If **A** and **B** are parallel, the angle between them is 0 (if they have the same direction) or $\pi$ (if they have opposite directions). The angle between 0 and any other vector is not defined.

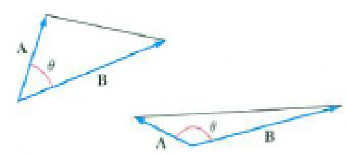



Figure 14.2.3:

The angle between **i** and **j** is $\pi/2$. The angle between $\mathbf{A} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{B} = 3\mathbf{i}$ is $3\pi/4$, as Figure 14.2.4 shows. The angle between **k** and $-\mathbf{k}$ is $\pi$; the angle between $2\mathbf{i}$ and $5\mathbf{i}$ is 0.

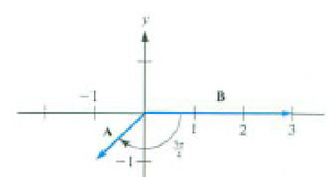



Figure 14.2.4:

In the next section we define another product of **A** and **B**, which will be a vector.

Sometimes we write $\cos(\mathbf{A}, \mathbf{B})$ instead of $\cos(\theta)$.

> **DEFINITION** *(Dot product)* Let **A** and **B** be two nonzero vectors. Their **dot product** is the number
>
> $$\|\mathbf{A}\|\|\mathbf{B}\|\cos(\theta),$$
>
> where $\theta$ is the angle between **A** and **B**. If **A** or **B** is 0, their dot product is 0. The dot product is denoted $\mathbf{A}\cdot\mathbf{B}$. It is a scalar and is also called the **scalar product** of **A** and **B**.

The dot product satisfies several useful identities, which follow from the definition:

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= \mathbf{B}\cdot\mathbf{A} && \text{(the dot product is commutative)} \\ \mathbf{A}\cdot\mathbf{A} &= \|\mathbf{A}\|^2 && \\ (c\mathbf{A})\cdot\mathbf{B} &= c(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot(c\mathbf{B}) && (c \text{ is a scalar}) \\ \text{and} \quad \mathbf{0}\cdot\mathbf{A} &= 0. && \end{aligned}$$

For instance, to establish that $\mathbf{A}\cdot\mathbf{A} = \|\mathbf{A}\|^2$, we calculate $\mathbf{A}\cdot\mathbf{A}$:

$$\mathbf{A}\cdot\mathbf{A} = \|\mathbf{A}\|\|\mathbf{A}\|\cos(\theta) = \|\mathbf{A}\|^2,$$

since the angle $\theta$ between **A** and **A** is 0, and $\cos(0) = 1$.

**EXAMPLE 1** Find the dot product $\mathbf{A}\cdot\mathbf{B}$ if $\mathbf{A} = 3\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = -5\mathbf{i}$.
*SOLUTION* Inspection of Figure 14.2.5 shows that $\theta$, the angle between **A** and **B**, is $3\pi/4$. Also,

$$\|\mathbf{A}\| = \sqrt{3^2 + 3^2} = \sqrt{18} \text{ and } \|\mathbf{B}\| = \sqrt{5^2 + 0^2} = 5.$$

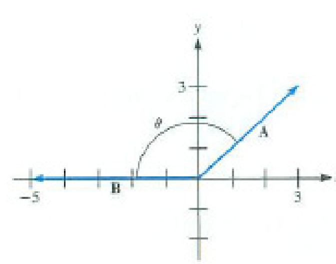



Figure 14.2.5:

Thus

$$\mathbf{A}\cdot\mathbf{B} = \|\mathbf{A}\|\|\mathbf{B}\|\cos\theta = \sqrt{18}\cdot\left(\frac{-\sqrt{2}}{2}\right) = -15.$$

$\diamond$

**EXAMPLE 2** Find

1. $\mathbf{i}\cdot\mathbf{j}$,
2. $\mathbf{i}\cdot\mathbf{i}$,
3. $2\mathbf{k}\cdot(-3\mathbf{k})$.

Recall that $\mathbf{i}$ and $\mathbf{j}$ are perpendicular, be definition.

*SOLUTION*

1. The angle between $\mathbf{i}$ and $\mathbf{j}$ is $\pi/2$. Thus

$$\mathbf{i}\cdot\mathbf{j} \;=\; \|\mathbf{i}\|\|\mathbf{j}\|\cos\left(\frac{\pi}{2}\right) = 1\cdot 1\cdot 0 = 0.$$

This is a special case of the fact that $\mathbf{A}\cdot\mathbf{A} = \|\mathbf{A}\|^2$.

2. The angle between $\mathbf{i}$ and $\mathbf{i}$ is 0. Thus

$$\mathbf{i}\cdot\mathbf{i} \;=\; \|\mathbf{i}\|\|\mathbf{i}\|\cos(0) = 1\cdot 1\cdot 1 = 1.$$

3. The angle between $2\mathbf{k}$ and $-3\mathbf{k}$ is $\pi$. Thus

$$2\mathbf{k}\cdot(-3\mathbf{k}) \;=\; \|2\mathbf{k}\|\|-3\mathbf{k}\|\cos(\pi) = 2\cdot 3\cdot(-1) = -6.$$

$\diamond$ Computations like those in Example 2 show that $a\mathbf{i}\cdot b\mathbf{i} = ab$, $a\mathbf{j}\cdot b\mathbf{j} = ab$, and $a\mathbf{k}\cdot b\mathbf{k} = ab$, while $a\mathbf{i}\cdot b\mathbf{j} = 0$, $a\mathbf{i}\cdot b\mathbf{k} = 0$, and $a\mathbf{j}\cdot b\mathbf{k} = 0$.

In particular, $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$, while $\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0$.

## The Geometry of the Dot Product

Let $\mathbf{A}$ and $\mathbf{B}$ be nonzero vectors and $\theta$ the angle between them. Their dot product is

$$\mathbf{A}\cdot\mathbf{B} = \|\mathbf{A}\|\|\mathbf{B}\|\cos(\theta).$$

Observe that, by definition, the zero vector, $\mathbf{0}$, is perpendicular to every vector in the $xy$ plane.

The quantities $\|\mathbf{A}\|$ and $\|\mathbf{B}\|$, being the lengths of vectors, are positive. However, $\cos(\theta)$ can be positive, zero, or negative. Note that $\cos(\theta) = 0$ only when $\theta = \pi/2$, that is when $\mathbf{A}$ and $\mathbf{B}$ are perpendicular. So the dot product provides a way of telling whether $\mathbf{A}$ and $\mathbf{B}$ are perpendicular:

A test for perpendicularity

Let **A** and **B** be nonzero vectors. If $\mathbf{A}\cdot\mathbf{B} = 0$, then **A** and **B** are perpendicular. Conversely, if **A** and **B** are perpendicular, then $\mathbf{A}\cdot\mathbf{B} = 0$.

As Figure 14.2.6 shows, **A** can be expressed as the sum of a vector parallel to **B** and a vector perpendicular to **B**.

The vector parallel to **B** we call the **projection** of **A** on **B**, denoted $\text{proj}_{\mathbf{B}}\,\mathbf{A}$. The vector perpendicular to **B** is then $\mathbf{A} - \text{proj}_{\mathbf{B}}\,\mathbf{A}$.

The length of $\text{proj}_{\mathbf{B}}\,\mathbf{A}$ is $\|\mathbf{A}\||\cos\theta|$, which equals $\frac{|\mathbf{A}\cdot\mathbf{B}|}{\|\mathbf{B}\|}$. If $\theta$ is less than $\pi/2$, $\text{proj}_{\mathbf{B}}\,\mathbf{A}$ points in the same direction as **B**.

If $\pi/2 < \theta \leq \pi$, then $\text{proj}_{\mathbf{B}}\,\mathbf{A}$ points in the direction opposite to that of **B**. In either case, since $\mathbf{B}/\|\mathbf{B}\|$ is the unit vector in the direction of **B**, we have

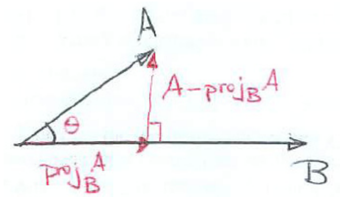



Figure 14.2.6:

Let **A** and **B** be vectors. $\text{proj}_{\mathbf{B}}\,\mathbf{A} = \frac{\mathbf{A}\cdot\mathbf{B}}{\|B\|}\frac{\mathbf{B}}{\|\mathbf{B}\|}$

- If $\mathbf{A}\cdot\mathbf{B}$ is positive, then the angle between the vectors is less than $\pi/2$. In this case $\text{proj}_{\mathbf{B}}\,\mathbf{A}$ points in the same direction as **B**.
- If $\mathbf{A}\cdot\mathbf{B}$ is negative, then the angle between the vectors is greater than $\pi/2$. In this case $\text{proj}_{\mathbf{B}}\,\mathbf{A}$ points in the opposite direction as **B**.

If $\mathbf{A}\cdot\mathbf{B}$ is negative, then the angle between **A** and **B** is obtuse (greater than $\pi/2$). Figure 14.2.7 shows this situation. As Figure 14.2.7 illustrates, $\text{proj}_{\mathbf{B}}\,\mathbf{A}$ points in the direction opposite that of **B**.

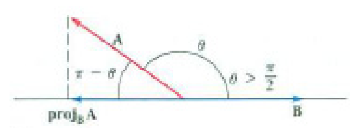


Figure 14.2.7:

## Computing $\mathbf{A}\cdot\mathbf{B}$ in Terms of Their Components

We defined $\mathbf{A}\cdot\mathbf{B}$, using the geometric interpretation of **A** and **B**. But what if **A** and **B** are given in terms of their components, $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle$? How would we find $\mathbf{A}\cdot\mathbf{B}$ in that case?

The answer turns out to be quite simple:

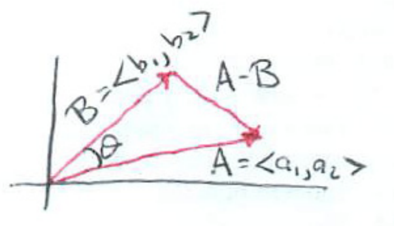
If $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle$, then $\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$.

The dot product is the sum of three numbers. Each number is a product of corresponding components.

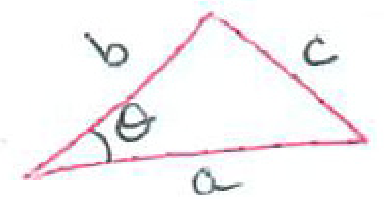
For vectors in the $xy$-plane, the result is a bit shorter:

| |
|---|
| If $\mathbf{A} = \langle a_1, a_2 \rangle$ and $\mathbf{B} = \langle b_1, b_2 \rangle$, then $\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2$. |

A proof of the Law of Cosines is defined in Exercise 45




(a)




(b)

Figure 14.2.8:

For convenience we establish the second result. Our reasoning rests on the Law of Cosines. It says that in a triangle where sides have lengths $a$, $b$, and $c$, and angle $\theta$ opposite the side with length $c$, as in Figure 14.2.8(b), $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.

Then

$$\|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\|\mathbf{A}\|\|\mathbf{B}\|\cos(\theta),$$

which tells us that

$$\|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A} \cdot \mathbf{B}, \qquad (14.2.1)$$

All that's left is to complete the three squares and solve for $\mathbf{A} \cdot \mathbf{B}$.

Translating (14.2.1) into components, we have

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2\mathbf{A} \cdot \mathbf{B}$$

or

$$a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2\mathbf{A} \cdot \mathbf{B}.$$

Thus

$$-2(a_1b_1 + a_2b_2) = -2\mathbf{A} \cdot \mathbf{B},$$

from which it follows, as the night follows the day, that

$$\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2.$$

The argument in the case of space vectors is practically the same, as doing Exercise 38 will show.

**EXAMPLE 3** Find $\cos(\mathbf{A},\mathbf{B})$ when $\mathbf{A} = \langle 6,3\rangle$ and $\mathbf{B} = \langle -1,1\rangle$.
*SOLUTION* We know that $\mathbf{A}\cdot\mathbf{B} = \|\mathbf{A}\|\|\mathbf{B}\|\cos(\mathbf{A},\mathbf{B})$. Thus

$$\begin{aligned} 6\cdot(-1)+3\cdot(1) &= \sqrt{2^2+3^2}\sqrt{(-1)+1^2}\cos(\mathbf{A},\mathbf{B}) \\ \text{or} \qquad -3 &= \sqrt{26}\cos(\mathbf{A},\mathbf{B}), \\ \text{e conclude that} \qquad \cos(\mathbf{A},\mathbf{B}) &= -3/\sqrt{26}. \end{aligned}$$

◇

Clearly $\theta$ is an obtuse angle. A calculator would estimate $\theta$, if we were curious. Figure 14.2.9 shows that the answer is reasonable.

As Example 3 illustrates

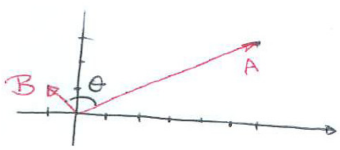



Figure 14.2.9:

$$\cos(\theta) = \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}$$

**EXAMPLE 4**

1. Find the projection of $\mathbf{A} = 2\mathbf{i}+\mathbf{j}$ on $\mathbf{B} = -3\mathbf{i}+2\mathbf{j}$.
2. Express $\mathbf{A}$ as the sum of a vector parallel to $\mathbf{B}$ and a vector perpendicular to $\mathbf{B}$.

*SOLUTION*

1. In this case

$$\begin{aligned} \text{proj}_{\mathbf{B}}\,\mathbf{A} &= \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|}\frac{\mathbf{B}}{\|\mathbf{B}\|} \\ &= \frac{(2\mathbf{i}+\mathbf{j})\cdot(-3\mathbf{i}+2\mathbf{j})}{|-3\mathbf{i}+2\mathbf{j}|}\frac{-3\mathbf{i}+2\mathbf{j}}{|-2\mathbf{i}+2\mathbf{j}|} \\ &= \frac{(-6+2)}{\sqrt{13}}\frac{(-3\mathbf{i}+2\mathbf{j})}{\sqrt{13}} \\ &= \frac{-4}{13}(-3\mathbf{i}+2\mathbf{j}) = \frac{12}{13}\mathbf{i}-\frac{8}{13}\mathbf{j}. \end{aligned}$$

Figure 14.2.10 shows the vector $\mathbf{A}$, $\mathbf{B}$, and $\text{proj}_{\mathbf{B}}\,\mathbf{A}$.

In this case $\mathbf{A}\cdot\mathbf{B}$ is negative, the angle between $\mathbf{A}$ and $\mathbf{B}$ is obtuse, and $\text{proj}_{\mathbf{B}}\,\mathbf{A}$ points in the direction opposite to the direction of $\mathbf{B}$.

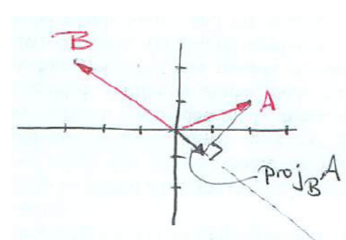



Figure 14.2.10:

2. The vector $\mathbf{A} - \text{proj}_{\mathbf{B}}\,\mathbf{A}$ is perpendicular to $\mathbf{B}$ and we have

$$\begin{aligned}\mathbf{A} &= (\text{proj}_{\mathbf{B}}\,\mathbf{A}) + (\mathbf{A} - \text{proj}_{\mathbf{B}}\,\mathbf{A}) \\ &= \left(\frac{12}{13}\mathbf{i} - \frac{8}{13}\mathbf{j}\right) + \left(2\mathbf{i} + \mathbf{j} - \left(\frac{12}{13}\mathbf{i} - \frac{8}{13}\mathbf{j}\right)\right) \\ &= \underbrace{\left(\frac{12}{13}\mathbf{i} - \frac{8}{13}\mathbf{j}\right)}_{\text{parallel to }\mathbf{B}} + \underbrace{\left(\frac{14}{13}\mathbf{i} + \frac{21}{13}\mathbf{j}\right)}_{\text{perpendicular to }\mathbf{B}}.\end{aligned}$$

◇

The scalar $\mathbf{A} \cdot (\mathbf{B}/\|\mathbf{B}\|)$ is the component of $\mathbf{A}$ in the direction of $\mathbf{B}$, denoted $\text{comp}_{\mathbf{B}}(\mathbf{A})$. It can be positive, negative, or zero. Its absolute value is the length of $\text{proj}_{\mathbf{B}}(\mathbf{A})$.

**EXAMPLE 5** Find $\text{proj}_{\mathbf{B}}(\mathbf{A})$ and $\text{comp}_{\mathbf{B}}(\mathbf{A})$ when $\mathbf{A} = \mathbf{i}+3\mathbf{j}$ and $\mathbf{B} = \mathbf{i}-\mathbf{j}$.

*SOLUTION* Since $\|\mathbf{B}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\mathbf{A} \cdot \mathbf{B} = 1 - 3 = -2$,

$$\text{proj}_{\mathbf{B}}(\mathbf{A}) = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|}\frac{\mathbf{B}}{\|\mathbf{B}\|} = \frac{-2}{\sqrt{2}}\frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}} = -\mathbf{i} + \mathbf{j}$$

and $\text{comp}_{\mathbf{B}}(\mathbf{A}) = (\mathbf{A} \cdot \mathbf{B})/\|\mathbf{B}\| = -2/\sqrt{2} = -\sqrt{2}$. This agrees with Figure 14.2.11. ◇

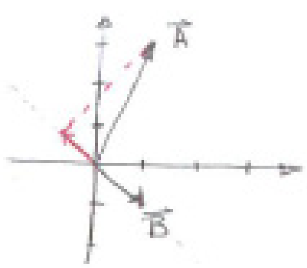



Figure 14.2.11:

## Properties of the Dot Product

With the aid of the formula for the dot product in terms of components, it is easy to establish the following properties:

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A} && \text{commutative} \\ \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} && \text{distributive} \\ c\mathbf{A} \cdot \mathbf{B} &= c(\mathbf{A} \cdot \mathbf{B}) && c \text{ a scalar.}\end{aligned}$$

$$3 - 2 - 2\cos(\theta) = \cos(\mathbf{A}, \mathbf{B}) = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}. \qquad (14.2.2)$$

Equation (**??**) tells us how to find the cosine of the angle between two vectors. With the aid of a calculator, we then can find the angle itself. Note that if $\cos(\theta) > 0$, then $0 < \theta < \pi/2$, and when $\cos(\theta) < 0$, then $\pi/2 < \theta \leq \pi$.

**EXAMPLE 6** Show that the vectors $\langle 2, -3, 4\rangle$ and $\langle 1, 2, 1\rangle$ are perpendicular.

*SOLUTION* We want to show that the angle $\theta$ between the vector in $\pi/2$. To do this we show $\cos(\theta) = 0$. Now,

$$\cos(\theta) = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{(1 \cdot 2) + 2(-3) + 1 \cdot 4}{|\mathbf{A}||\mathbf{B}|} = \frac{2 - 6 + 4}{|\mathbf{A}||\mathbf{B}|} = 0.$$

Therefore the vectors are perpendicular. ◇

Example 6 illustrates this test for two vectors being perpendicular to each other.

Two nonzero vectors are perpendicular if their dot product is 0.

**The Dot Product in Business and Statistics**

Imagine that a fast food restaurant sells 30 hamburgers, 20 salads, 15 soft drinks, and 13 orders of french fries. This is recorded by the four-dimensional "vector" $\langle 30, 20, 15, 13\rangle$. A hamburger sells for \$1.99, a salad for \$1.50, a soft drink for \$1.00, and an order of french fries for \$1.10. The "price vector" is $\langle 1.99, 1.50, 1.00, 1.10\rangle$. The dot product of these two vectors, $30(1.99) + 20(1.50) + 15(1.00) + 13(1.10)$, would be the total amount paid for all items. Descriptions of the economy use "production vectors," "cost vectors," "price vectors," and "profit vectors" with many more than the four componenets of our restaurant example.

In statistics the coefficient of correlation is defined in terms of a dot product. For instance, you may determine the height and weight of $n$ persons. Let the height of the $i$th person be $h_i$ and the weight be $w_i$. Let $h$ be the average of the $n$ heights and $w$ be the average of the $n$ weights. Let $\mathbf{H} = \langle h_1 - h, h_2 - h, \cdots, h_n - h\rangle$ and $\mathbf{W} = \langle w_1 - w, w_2 - w, \cdots, w_n - w\rangle$. Then coefficient of correlation between the heights and weights is defined to be

$$\frac{\mathbf{H}\cdot\mathbf{W}}{\|\mathbf{H}\|\|\mathbf{W}\|}.$$

In analogy with vectors in the plane or space,

$$\mathbf{HW} = \sum_{i=1}^{n}(h_i - h)(w_i - w), \|\mathbf{H}\| = \sqrt{\sum_{i=1}^{n}(h_i - h)^2}, \|\mathbf{W}\| = \sqrt{\sum_{i=1}^{n}(w_i - w)^2}.$$

It turns out that the coefficient of correlation is simply the cosine of the angle between the points $\mathbf{H} = \langle h_1 - h, h_2 - h, \cdots, h_n - h\rangle$ and $\mathbf{W}\langle w_1 - w, w_2 - w, \cdots, w_n - w\rangle$ in $n$-dimensional space.

## Summary

We defined the dot (scalar) product of two vectors $\mathbf{A}$ and $\mathbf{B}$ geometrically as $\|\mathbf{A}\|\|\mathbf{B}\|\cos(\theta)$, where $\theta$ is the angle between them. We then obtained a formula for $\mathbf{A}\cdot\mathbf{B}$ in terms of their components, as $\langle a_1, a_2\rangle\cdot\langle b_1, b_2\rangle = a_1b_1 + a_2b_2$ and a similar formula for the dot product of two space vectors.

The dot product enabled us to express a vector $\mathbf{A}$ as the sum of a vector parallel to $\mathbf{B}$ ($\operatorname{proj}_{\mathbf{B}}\mathbf{A}$) and a vector perpendicular to $\mathbf{B}$ ($\mathbf{A} - \operatorname{proj}_{\mathbf{B}}\mathbf{A}$).

When their dot product is 0, two non-zero vectors are perpendicular.

The zero-vector, $\mathbf{0}$, is considered to be perpendicular to every vector.

More generally, we can use the dot product to find the angle $\theta$ between two vectors:

$$\cos(\theta) = \cos(\mathbf{A}, \mathbf{B}) = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}.$$

**EXERCISES for Section 14.2** *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4 compute $\mathbf{A} \cdot \mathbf{B}$.

**1.**[R] $\mathbf{A}$ has length 3, $\mathbf{B}$ has length 4, and the angle between $\mathbf{A}$ and $\mathbf{B}$ is $\pi/4$.

**2.**[R] $\mathbf{A}$ has length 2, $\mathbf{B}$ has length 3, and the angle between $\mathbf{A}$ and $\mathbf{B}$ is $3\pi/4$.

**3.**[R] $\mathbf{A}$ has length 5, $\mathbf{B}$ has length $\frac{1}{2}$, and the angle between $\mathbf{A}$ and $\mathbf{B}$ is $\pi/2$.

**4.**[R] $\mathbf{A}$ is the zero vector $\mathbf{0}$, and $\mathbf{B}$ has length 5.

In Exercises 5 to 8 compute $\mathbf{A} \cdot \mathbf{B}$ using the formula in terms of components.

**5.**[R] $\mathbf{A} = -2\mathbf{i} + 3\mathbf{j}$, $\mathbf{B} = 4\mathbf{i} + 4\mathbf{j}$

**6.**[R] $\mathbf{A} = 0.3\mathbf{i} + 0.5\mathbf{j}$, $\mathbf{B} = 2\mathbf{i} - 1.5\mathbf{j}$

**7.**[R] $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$, $\mathbf{B} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

**8.**[R] $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{B} = 2\mathbf{i} + +3\mathbf{j} - 5\mathbf{k}$

**9.**[R]

(a) Draw the vectors $7\mathbf{i} + 12\mathbf{j}$ and $9\mathbf{i} - 5\mathbf{j}$.

(b) Do they seem to be perpendicular?

(c) Determine whether they are perpendicular by examining their dot product.

**10.**[R]

(a) Draw the vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + \mathbf{j} - \mathbf{k}$.

(b) Do they seem to be perpendicular?

(c) Determine whether they are perpendicular by examining their dot product.

**11.**[R]

(a) Estimate the angle between $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = 5\mathbf{i} + 12\mathbf{j}$ by drawing them.

(b) Find the angle between $\mathbf{A}$ and $\mathbf{B}$.

**12.**[R] Let $P = (6, 1)$, $Q = (3, 2)$, $R = (1, 3)$, and $S = (4, 5)$.

(a) Draw the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$.

(b) Using the diagram in (a) estimate the angle between $\overrightarrow{PQ}$ and $\overrightarrow{RS}$.

(c) Using the dot product, find the $\cos(\overrightarrow{PQ}, \overrightarrow{RS})$, that is, the cosine of the angle between $\overrightarrow{PQ}$ and $\overrightarrow{RS}$.

(d) Using (c) and a calculator, find the angle in (b).

**13.**[R] Find the angle between $2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

**14.**[R] Find the angle betwen $\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $3\mathbf{i}+6\mathbf{j}-3\mathbf{k}$.

**15.**[R] Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ if $A = (1, 3)$, $B = (7, 4)$, $C = (2, 8)$, and $D = (1, -5)$.

**16.**[R] Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ if $A = (1, 2, -5)$, $B = (1, 0, 1)$, $C = (0, -1, 3)$, and $D = (2, 1, 4)$.

**17.**[R] Find the length of the projection of $-4\mathbf{i} + 5\mathbf{j}$ on the line through $(2, -1)$ and $(6, 1)$.

(a) By making a drawing and estimating the length by eye.

(b) By using the dot product.

**18.**[R]

(a) Find a vector $\mathbf{C}$ parallel to $\mathbf{i} + 2\mathbf{j}$ and a vector $\mathbf{D}$ perpendicular to $\mathbf{i} + 2\mathbf{j}$ such that $-3\mathbf{i} + 4\mathbf{j} = \mathbf{C} + \mathbf{D}$.

(b) Draw the vectors in (a) to check that your answer is reasonable.

**19.**[R]

(a) Find a vector $\mathbf{C}$ parallel to $2\mathbf{i} - \mathbf{j}$ and a vector $\mathbf{D}$ perpendicular to $2\mathbf{i} - \mathbf{j}$ such that $3\mathbf{i} + 4\mathbf{j} = \mathbf{C} + \mathbf{D}$.

(b) Draw the vectors in (a) to check that your answer is reasonable.

**20.**[M] Give an example of a vector in the $xy$ plane that is perpendicular to $3\mathbf{i} - 2\mathbf{j}$.

**21.**[M] Give an example of a vector that is perpendicular to $5\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

Exercises 22 to 26 refer to the cube in Figure 14.2.12.

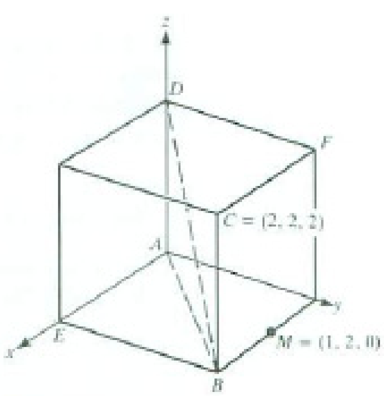



Figure 14.2.12:

**22.**[M] Find $\cos(\overrightarrow{AC}, \overrightarrow{BD})$, the cosine of the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

**23.**[M] Find $\cos(\overrightarrow{AF}, \overrightarrow{BD})$, the cosine of the angle between $\overrightarrow{AF}$ and $\overrightarrow{BD}$.

**24.**[M] Find $\cos(\overrightarrow{AC}, \overrightarrow{AM})$, the cosine of the angle between $\overrightarrow{AC}$ and $\overrightarrow{AM}$.

**25.**[M] Find $\cos(\overrightarrow{MD}, \overrightarrow{MF})$, the cosine of the angle between $\overrightarrow{MD}$ and $\overrightarrow{MF}$.

**26.**[M] Find $\cos(\overrightarrow{EF}, \overrightarrow{BD})$, the cosine of the angle between $\overrightarrow{EF}$ and $\overrightarrow{BD}$.

**27.**[R] How far is the point $(1, 2, 3)$ from the line through the points $(1, 4, 2)$ and $(2, 1, -4)$?

**28.**[M] If $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$ and $\mathbf{A}$ is not $\mathbf{0}$, must $\mathbf{B} = \mathbf{C}$?

**29.**[C] If $\|\mathbf{A}\| = 3$ and $\|\mathbf{B}\| = 5$,

(a) how large can $\|\mathbf{A} + B\|$ be?

(b) how small?

**30.**[C] By considering the dot product of the two unit vectors $\mathbf{u}_1 = \cos\theta_1 \mathbf{i} + \sin\theta_1 \mathbf{j}$ and $\mathbf{u}_2 = \cos\theta_2 \mathbf{i} + \sin\theta_2 \mathbf{j}$, prove that

$$\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2.$$

**31.**[C] Consider a tetrahedron (not necessarily regular). It has six edges. Show that the line segment joining the midpoints of two opposite edges is perpendicular to the line segment joining another pair of opposite edges if anly only if the remaining two edges are of the same length.

**32.**[C] The output of a firm that manufactures $x_1$ washing machines, $x_2$ refrigerators, $x_3$ dishwashers, $x_4$ stoves, and $x_5$ clothes dryers is recorded by the five-dimensional production vector $\mathbf{P} = \langle x_1, x_2, x_3, x_4, x_5 \rangle$. Similarly, the cost vector $\mathbf{C} = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ records the cost of producing each item; for instance, each refrigerator costs the firm $y_2$ dollars.

(a) What is the economic significance of $\mathbf{P} \cdot \mathbf{C} = \langle 20, 0, 7, 9, 15\rangle \cdot \langle 50, 70, 30, 20, 10\rangle$?

(b) If the firm doubles the production of all items in (a), what is its new production vector?

**33.**[C] Let $P_1$ be the profit from selling a washing machine and $P_2$, $P_3$, $P_4$, and $P_5$ be defined analogously for the firm of Exercise 32. (Some of the $P$'s may be negative.) What does it mean to the firm to have $\langle P_1, P_2, P_3, P_4, P_5\rangle$ "perpendicular" to $\langle x_1, x_2, x_3, x_4, x_5\rangle$?

ercise is
eep only
mmary?
oughts?

**34.**[C] If $a_1$, $a_2$, $b_1$, $b_2$ are four numbers, explain why

$$|a_1b_1 + a_2b_2| \le \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}.$$

**35.**[R] Prove that $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

(a) using the geometric definition of the dot product,

(b) using the formula for the dot product in terms of components.

**36.**[R] Prove that $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$

(a) using the geometric definition of the dot product,

(b) using the formula for the dot product in terms of components.

**37.**[C] Don't try to obtain the equation $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$ geometrically. If you use the geometric definition of the dot product, what does that distributive law say? Picture $\mathbf{B}$ and $\mathbf{C}$ in a horizontal plane and $\mathbf{A}$ not in that plane, as in Figure 14.2.13.

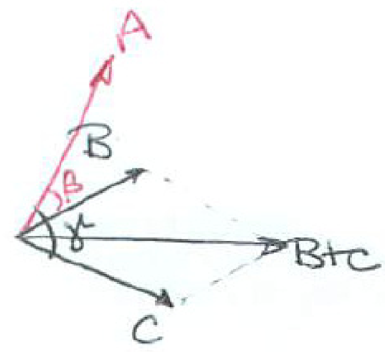



Figure 14.2.13:
It's not so obvious is it?

**38.**[R] Prove that $\langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3$ HINT: Read the proof in the case of planar vectors on page 966.

**39.**[C] Let $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ be unit vectors such that each two are perpendicular. Let $\mathbf{A}$ be a vector.

(a) Draw a picture that shows that there are scalars $x$, $y$, and $z$ such that $\mathbf{A} = x\mathbf{u}_1 + y\mathbf{u}_2 + z\mathbf{u}_3$.

(b) Express $x$ as a dot product.

(c) Express $x - z$ as a dot product.

**40.**[M]

(a) Let $\mathbf{A}$ be a vector in the $xy$ plane and $\mathbf{u}_1$ and $\mathbf{u}_2$ perpendicular unit vectors in that plane. If $\mathbf{A} \cdot \mathbf{u}_1 = 0$ and $\mathbf{A} \cdot \mathbf{u}_2 = 0$, must $\mathbf{A} = \mathbf{0}$?

(b) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be nonparallel unit vectors in the $xy$ plane. If $\mathbf{A} \cdot \mathbf{v}_1$ and $\mathbf{A} \cdot \mathbf{v}_2 = 0$, must $\mathbf{A} = \mathbf{0}$?

**41.**[C] A firm sells $x$ chairs at $C$ dollars per chair and $y$ desks at $D$ dollars per week. It costs the firm $c$ dollars to make a chair and $d$ dollars to make a desk. What is the economic interpretation of

(a) $Cx$?

(b) $(x\mathbf{i} + y\mathbf{j}) \cdot (C\mathbf{i} + D\mathbf{j})$?

(c) $(x\mathbf{i} + y\mathbf{j}) \cdot (c\mathbf{i} + d\mathbf{j})$?

(d) $(x\mathbf{i} + y\mathbf{j}) \cdot (C\mathbf{i} + D\mathbf{j}) > (x\mathbf{i} + y\mathbf{j}) \cdot (c\mathbf{i} + d\mathbf{j})$?

**42.**[C] A force $\mathbf{F}$ of 10 newtons has the direction of the vector $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. This force pushes an object on a ramp in a straight line from the point $(3, 1, 5)$ to the point $(4, 3, 7)$, where coordinates are measured in meters. How much work does the force accomplish?

**43.**[C] Show that if the two diagonals of the parallelogram are perpendicular, then the four sides have the same length (forming a rhombus). HINT: Use the dot product.

**44.**[C] Some molecules consist of 4 atoms arranged as the vertices of a regular tetrahedron, for instance at the points labeled $A$, $B$, $C$, and $D$ in Figure 14.2.14.

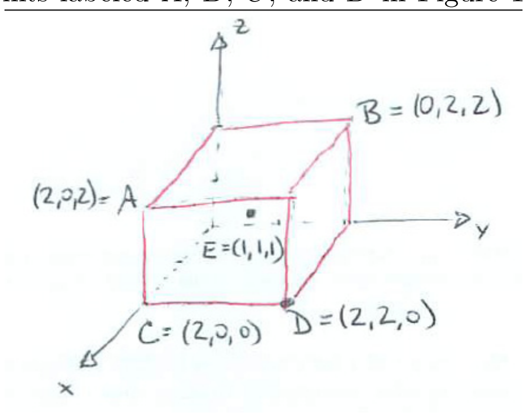



Figure 14.2.14:

(a) Show that $A$, $B$, $C$, and $D$ are vertices of a regular tetrahedron. HINT: Show that the four faces are equilateral triangles.

(b) Chemists are interested in the angle $\theta = AEB$. Show that $\cos(\theta) = -1/3$.

(c) Find $\theta$ (approximately).

**45.**[M] The key to obtaining the expression for the dot product in terms of components is from trigonometry: the Law of Cosines. In view of this, it makes sense to see why the Law of Cosines is true. The proof is quite easy, since it consists just of two applications of the Pythagorean Theorem. Figure 14.2.15 shows a triangle with sides $a$, $b$, $c$, with angle $\theta$ opposite side $c$. (We are concerned, for the moment, in the case when $\theta$ is less than $\frac{\pi}{2}$.

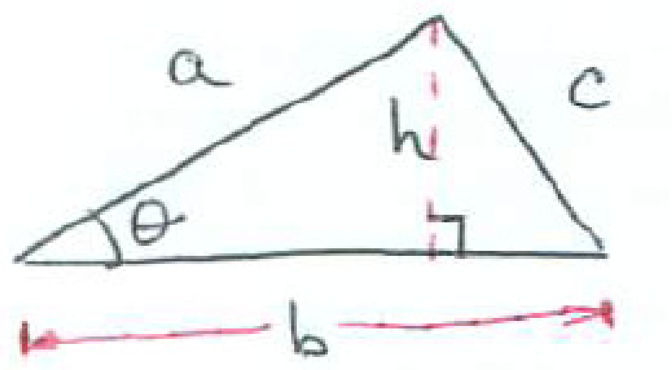



Figure 14.2.15:

(a) Show that $h^2 = a^2 - a^2\cos^2(\theta)$.

(b) Show that $h^2 = c^2 - (b - a\cos(theta))^2$.

(c) By equating the two expressions for $h^2$ found in (a) and (b), obtain the Law of Cosines.

**46.**[C] In the Exercise 45 the altitude of length $h$ meets the side of length $b$. If $\theta > \pi/2$, that altitude has its base outside of side $b$. Prove the Law of Cosines in this case.

**47.**[R] What is $\text{proj}_{\mathbf{B}}\,\mathbf{A}$ if $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$?

**48.**[C] How far is the point $(2, 3, 5)$ from the line through the origin and $(1, -1, 2)$?

**49.**[R] Express the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ as the sum of a vector parallel to $\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and a vector perpendicular to $\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

**50.**[M]

**Jane:** I don't like the way the author found how to express **A** as the sum of a vector parallel to **B** and a vector perpendicular to **B**.

**Sam:** It was O.K. for me. But I had to memorize a formula.

**Jane:** My goal is to memorize nothing. I simply write $\mathbf{A} = x\mathbf{B} + \mathbf{C}$, when **C** is perpendicular to **A**. Then *I* dot with **B**, getting

$$\mathbf{A}\cdot\mathbf{B} = x\mathbf{B}\cdot\mathbf{B} + \mathbf{C}\cdot\mathbf{B}.$$

Since **C** is perpendicular to I and behold, I have

$$x = \frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{B}\cdot\mathbf{B}}$$

So the vector parallel to **B** i

**Sam:** Cool. So why did the author stuff?

**Jane:** Maybe they wanted to rei of the dot product and the r

**Sam:** O.K. But how do I get the ular to **B**?

**Jane:** Simple...

Complete Jane's reply.

# 14.3 The Cross Product of Two Vectors

The dot product of two vectors is a scalar. The product of two vectors we define in this section is a vector. This vector has the property that it is perpendicular to each of the given vector.

## Definition of the Cross Product

Let $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ be two non-zero vectors that are not parallel. We will construct a vector **C** that is perpendicular to both **A** and **B**. Of course **C** is not unique since any vector parallel to **C** is also perpendicular to **A** and **B**.

Let $\mathbf{C} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. We want $\mathbf{C} \cdot \mathbf{A}$ and $\mathbf{C} \cdot \mathbf{B}$ to be 0. This gives us the equations

$$a_1x + a_2y + a_3z = 0 \tag{14.3.1}$$

$$b_1x + b_2y + b_3z = 0 \tag{14.3.2}$$

We eliminate $x$ by subtracting $b_1$ times (14.3.1) from $a_1$ times (14.3.2), as follows.

$a_1$ times (14.3.2)
$$a_1b_1x + a_1b_2y + a_1b_3z = 0 \tag{14.3.3}$$

$b_1$ times (14.3.1)
$$b_1a_1x + b_1a_2y + b_1a_3z = 0 \tag{14.3.4}$$

Subtracting the bottom equation (14.3.4) from the top equation (14.3.3) gives us

$$(a_1b_2 - a_2b_1)y + (a_1b_3 - a_3b_1)z = 0 \tag{14.3.5}$$

A simple non-zero solution of (14.3.5) is

$$y = -(a_1b_3 - a_3b_1), \quad z = a_1b_2 - a_2b_1$$

This is like solving $2y + 3z = 0$ by letting $y = -3$ and $z = 2$.

To find the corresponding $x$, substitute the value found for $y$ and $z$ into (14.3.1). As Exercise 39 shows, the straightforward algebra yields

$$x = a_2b_3 - a_3b_2.$$

So the vector

$$(a_2b_2 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \tag{14.3.6}$$

is perpendicular to **A** and **B**. It is denoted $\mathbf{A} \times \mathbf{B}$ and is called the **vector product** of **A** and **B** or the cross product of **A** and **B**. This vector is defined even if **A** and **B** are parallel or if one of them is 0.

## Determinants and the Cross Product

The expression (14.3.6) for the cross product is not easy to memorize. Fortunately, determinants provide a convenient memory aid.

Four numbers arranged in a square from a matrix of order 2, for instance

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

The determinant of this matrix is the number $a_1b_2 - a_2b_1$, denoted

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \qquad \text{or} \qquad \det \begin{pmatrix} cca_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

Each term in the cross product, (14.3.6), is itself the determinant of a matrix of order 2, namely

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Nine numbers arranged in a square for a matrix of order 3, for instance

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Its determinant is defined with the aid of determinants of order 2:

$$c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2. \end{vmatrix}$$

The coefficient of each $c_i$ is plus or minus the determinant of the matrix of order 2 that remains when the row and column in which $c_i$ appears are deleted, as shown in Figure 14.3.1 for the coefficient of $c_i$.

Therefore we can write (14.3.6) as a determinant of a matrix, and we have

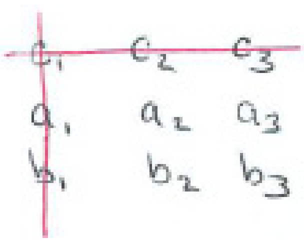



Figure 14.3.1:

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \tag{14.3.7}$$

**DEFINITION** *(Cross product (vector product).)* Let

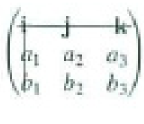
$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{k} + a_3\mathbf{k} \text{ and } \mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.$$

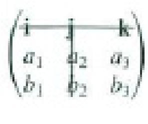
The vector

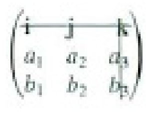
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

is called the **cross product** (or **vector product**) of **A** and **B**. It is denoted $\mathbf{A} \times \mathbf{B}$.

The determinant for $\mathbf{A} \times \mathbf{B}$ is expanded along its first row:







Delete the two lines through **i**. The determinant of the remaining square is the coefficient of **i** in $\mathbf{A} \times \mathbf{B}$.

Delete the two lines through **j**. The determinant of the remaining square is the coefficient of **j** in $\mathbf{A} \times \mathbf{B}$.

Delete the two lines through **k**. The determinant of the remaining square is the coefficient of **k** in $\mathbf{A} \times \mathbf{B}$.

**EXAMPLE 1** Compute $\mathbf{A} \times \mathbf{B}$ if $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{B} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.
*SOLUTION* By definition,

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 3 & 4 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} \\ &= -13\mathbf{i} + 7\mathbf{j} + 11\mathbf{k} \end{aligned}$$

◇

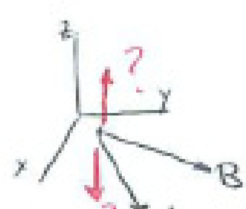
The cross cross product has these properties:

Recall: The zero vector is, by definition, perpendicular to every vector.

1. $\mathbf{A} \times \mathbf{B}$ is perpendicular to both **A** and **B**.
2. $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$.
3. $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ if **A** and **B** are parallel or at least one of them is 0.
4. $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.

See Exercises 27 and 28.

The first property holds because that is how we constructed the cross product. The second and third are established by straightforward computations, using (14.3.7). Exercises 16 and 17 take care of property 4.



## The Direction of $\mathbf{A} \times \mathbf{B}$?

We know that $\mathbf{A} \times \mathbf{B}$ is perpendicular to $\mathbf{A}$ and $\mathbf{B}$, but there are two possible directions, as Figure 14.3.2 shows,

To find out, take a specific case and we compute $\mathbf{i} \times \mathbf{j}$:

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + \mathbf{k} = \mathbf{k}.$$

Left-handed people must use their right hand here.

This suggests the general situation. The direction of $\mathbf{A} \times \mathbf{B}$ is given by the right hand rule:

Curl the fingers of the right hand to go from $\mathbf{A}$ and $\mathbf{B}$. The thumb points in the direction of $\mathbf{A} \times \mathbf{B}$.

**EXAMPLE 2** Check that the right hand rule is correct in the case for $\mathbf{j} \times \mathbf{i}$.

*SOLUTION*

$$\mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} - \mathbf{k} = -\mathbf{k}.$$

In this case, $\mathbf{j} \times \mathbf{i}$, points downward, the opposite of $\mathbf{i} \times \mathbf{j}$.

The right hand rule is illustrated in Figure 14.3.4.

The thumb indeed points downward. ◇

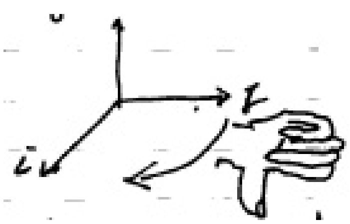


Figure 14.3.4:

## How Long is $\mathbf{A} \times \mathbf{B}$

To find a geometric meaning for $\|\mathbf{A} \times \mathbf{B}\|$ we will find $|\mathbf{A} \times \mathbf{B}|^2$ with the aid of (4). That is, we will compute $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{A} \times \mathbf{B})$ and interpret the results. By (4)

Check these steps by multiplying everything out.

$$\begin{aligned} \|\mathbf{A} \times \mathbf{B}\|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 + a_3^2b_3^2 + a_3^2b_1^2 + a_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2 - 2(a_2a_3b_2b_3 + a_1a_3b_1b_3 + a_1a \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= \|\mathbf{A}\|^2\|\mathbf{B}\|^2 - (\mathbf{A} \cdot \mathbf{B})^2 \\ &= \|\mathbf{A}\|^2\|\mathbf{B}\|^2 - (\|\mathbf{A}\|\|\mathbf{B}\|\cos(\theta))^2 \\ &= \|\mathbf{A}\|^2\|\mathbf{B}\|^2(1 - \cos^2(\theta)) \\ &= \|\mathbf{A}\|^2\|\mathbf{B}\|^2\sin^2(\theta). \end{aligned}$$

Then

$$\|\mathbf{A} \times \mathbf{B}\| = \|\mathbf{A}\|\|\mathbf{B}\|\sin(\theta) \qquad \sin(\theta) \text{ is not negative since } 0 \le \theta \le \pi. \tag{14.3.8}$$

We then have

Let $\mathbf{A}$ and $\mathbf{B}$ be nonzero vectors and $\theta$ the angle between them. Then $\|\mathbf{A}\times\mathbf{B}\| = \|\mathbf{A}\|\|\mathbf{B}\|\sin(\theta)$.

With the aid of this fact we now give a simple geometric meaning for the length of $\mathbf{A}\times\mathbf{B}$. A glance at the parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$ shows that it area is

$$\underbrace{\|\mathbf{A}\|}_{base}\underbrace{\|\mathbf{B}\|\sin(\theta)}_{height} = \text{area of parallelogram}$$

So now we have a simple geometric description of the length of $\mathbf{A}\times\mathbf{B}$.

Figure 14.3.5: This figure

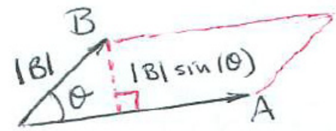



Figure 14.3.6:

In some texts the cross product is defined geometrically: It is the vector where length is the area of the parallelogram mentioned above and where direction is given by the right and rule. Then the author must obtain its formula in terms of components.

The length of $\mathbf{A}\times\mathbf{B}$ is the area of the parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$.

**EXAMPLE 3** Find the area of the parallelogram spanned by $\mathbf{A} = a_1\mathbf{i}+a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i}+b_2\mathbf{j}$.
*SOLUTION* First write $\mathbf{A}$ as $a_1\mathbf{i}+a_2\mathbf{j}+0\mathbf{k}$ and $b_\mathrm{i}+b_2\mathbf{j}+0\mathbf{k}$. Then the area of this parallelogram is the length of $\mathbf{A}\times\mathbf{B}$. So we compute $\mathbf{A}\times\mathbf{B}$.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = (a_1b_2 - a_2b_1)\mathbf{k}.$$

The area is therefore $|a_1b_2 - a_2b_1|$. In other words, it is the absolute value of the determinant

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

◇ The next example is typical of the geometric applications of the cross product.

**EXAMPLE 4** Find a vector perpendicular to the plane determined by the three points $P = (1,3,2)$, $Q = (4,-1,1)$, and $R = (3,0,2)$.
*SOLUTION* The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ lie in a plane (see Figure 14.3.7). The vector $\mathbf{N} = \overrightarrow{PQ}\times\overrightarrow{PR}$ being perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, is perpendicular to the plane. Now, $\overrightarrow{PQ} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k}$ and $\overrightarrow{PR} = 2\mathbf{i} - 3\mathbf{j} + 0\mathbf{k}$.
Thus

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -1 \\ 2 & -3 & 0 \end{vmatrix} = -3\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

◇

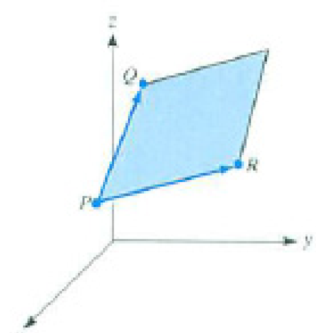



Figure 14.3.7:

## The Scalar Triple Product

The scalar $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is called the **scalar triple product**. It has an important geometric meaning. (The vector $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is also called the **vector triple product**.)

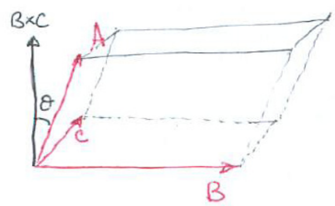



Figure 14.3.8:

The vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ span a parallelepiped, as shown in Figure 14.3.8. The angle between $\mathbf{B}\times\mathbf{C}$ and $\mathbf{A}$ is $\theta$ (which could be greater than $\pi/2$). The area of the base of the parallelogram is $\|\mathbf{B}\times\mathbf{C}\|$. The height of the parallelepiped is $\|\mathbf{A}\|\,|\cos(\theta)|$. Thus its volume is the absolute value of

$$\underbrace{|\mathbf{A}|\cos\theta}_{\text{height}}\ \underbrace{|\mathbf{B}\times\mathbf{C}|}_{\text{area of base}}\,.$$

This is the definition of the dot of product of $\mathbf{A}$ and $(\mathbf{B}\times\mathbf{C})$.

> $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is plus or minus the volume of the parallelepiped spanned by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

The scalar triple product can also be expressed as a determinant. To see why, note that the dot product of $\mathbf{A}$ and $\mathbf{B}\times\mathbf{C}$ is

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = a_1\begin{vmatrix} b_2 & b_3\\ c_2 & c_3\end{vmatrix} + a_2(-\begin{vmatrix} b_1 & b_3\\ c_1 & c_3\end{vmatrix}) + a_3\begin{vmatrix} b_1 & b_2\\ c_1 & c_2\end{vmatrix}. \qquad (14.3.9)$$

Comparison of Dot Product and Vector Product

| $\mathbf{A}\cdot\mathbf{B}$ | $\mathbf{A}\times\mathbf{B}$ |
|---|---|
| $\mathbf{B}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}$ | $\mathbf{A}\times\mathbf{B} = -$ |
| $\lvert\mathbf{A}\cdot\mathbf{B}\rvert = \|\mathbf{A}\|\|\mathbf{B}\|\lvert\cos(\theta)\rvert$ | $\|\mathbf{A}\times\mathbf{B}\| = \|\mathbf{A}\|$ |
| $\mathbf{A}\cdot\mathbf{B}=0$ is a test for perpendicularity | $\mathbf{A}\times\mathbf{B}=\mathbf{0}$ is a test fo |
| formula in components involves $a_ib_i$ (same indices) | formula in components involv |

What ... en you cross ... with a mos... : You can't ... with a vec... hapter ... nary?]

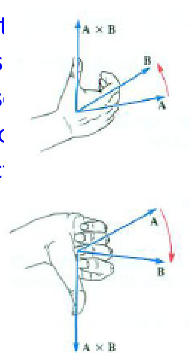



Figure 14.3.9:

Equation (14.3.9) can now be recognized the determinant of a matrix of order 3:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

So this determinant is plus or minus the volume of the parallelepiped spanned by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

This should not be a surprise. As Example 3 showed, the determinant $\begin{vmatrix} a_1 & a_2\\ v_1 & b_2\end{vmatrix}$ is plus or minus the area of the parallelogram spanned by the vectors $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$.

## Summary

We constructed a vector **C** perpendicular to vectors **A** and **B** by demanding that $\mathbf{C}\cdot\mathbf{A} = 0$ and $\mathbf{C}\cdot\mathbf{B} = 0$. A convenient formula for such a vector

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

It is denoted $\mathbf{A}\times\mathbf{B}$ and called the vector product or cross product of **A** and **B**. It also may be described as the vector whose length is the area of the parallelogram spanned by **A** and **B** and whose direction is given by the right-hand rule (the finger curling from **A** and **B**). These are some of its properties:

1. $\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A})$ (anticommunitive)
2. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ *is not* usually equal to $(\mathbf{A}\times\mathbf{B})\times\mathbf{C})$ (not associative)
3. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{B}\cdot\mathbf{A})\mathbf{C}$ (See Exercise 17.)
4. $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) = (\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{B})$
5. $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \pm$ volume of parallelepiped spanned by **A**, **B**, and **C**.

Item 5 appeared in finding the length of $\mathbf{A}\times\mathbf{B}$. It will be used in the next chapters.

ve some
ut lines,
ion 14.4
mary for
pter 14.

**EXERCISES for Section 14.3** *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4 compute and sketch $\mathbf{A} \cdot \mathbf{B}$.

**1.**[R] $\mathbf{A} = \mathbf{k}$, $\mathbf{B} = \mathbf{j}$

**2.**[R] $\mathbf{A} = \mathbf{i}+\mathbf{j}$, $\mathbf{B} = \mathbf{i}-\mathbf{j}$

**3.**[R] $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$,

$\mathbf{B} = \mathbf{i} + \mathbf{j}$

**4.**[R] $\mathbf{A} = \mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j}$

In Exercises 5 and 6, find $\mathbf{A} \times \mathbf{B}$ and check that it is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

**5.**[R] $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{j} + 4\mathbf{k}$
$\mathbf{B} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$

**6.**[R] $\mathbf{A} = \mathbf{i} - \mathbf{j}$, $\mathbf{B} =$

In Exercises 7 to 10 use the cross product to find the area of each region.

**7.**[R] The parallelogram three of whose vertices are $(0, 0, 0)$, $(1, 5, 4)$, and $(2, -1, 3)$.

**8.**[R] The parallelogram three of whose vertices are $(1, 2, -1)$, $(2, 1, 4)$, and $(3, 5, 2)$.

**9.**[R] The triangle two of whose sides are $\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} - \mathbf{j}$.

**10.**[R] The triangle two of whose sides are $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

In Exercises 11 to 14 find the volumes of the parallelepipeds spanned by the given vectors.

**11.**[R] $\langle 2, 1, 3\rangle$, $\langle 3, -1, 2\rangle$, $\langle 4, 0, 3\rangle$

**12.**[R] $3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{i} - \mathbf{j} - \mathbf{k}$.

**13.**[R] $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$, where $P = (1, 1, 1)$, $Q =$

$(2, 1, -2)$, $R = (3, 5, 2)$, and $S = (1, -1, 2)$.

**14.**[R] $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$, where $P = (0, 0, 0)$, $Q = (3, 3, 2)$, $R = (1, 4, -1)$, and $S = (1, 2, 3)$.

**15.**[R] Evaluate $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B})$.

**16.**[R] Prove that $\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$ in two ways:

(a) using the algebraic definition of the cross product;

(b) using the geometric description of the cross product.

**17.**[R] Show that if $\mathbf{B} = c\mathbf{A}$, then $\mathbf{A} \times \mathbf{B} = \mathbf{0}$:

(a) using the algebraic definition of the cross product;

(b) using the geometric description of the cross product.

**18.**[M] Show that the points $(0, 0, 0)$, $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ and $(x_3, y_3, z_3)$ lie on a plane if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

**19.**[M]

(a) If $\mathbf{B}$ is parallel to $\mathbf{C}$, is $\mathbf{A} \times \mathbf{B}$ parallel to $\mathbf{A} \times \mathbf{C}$?

(b) If $\mathbf{B}$ is perpendicular to $\mathbf{C}$, is $\mathbf{A} \times \mathbf{B}$ perpendicular to $\mathbf{A} \times \mathbf{C}$?

**20.**[M] Let $\mathbf{A}$ be a nonzero vector. If $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ and $\mathbf{A} \cdot \mathbf{B} = 0$, must $\mathbf{B} = \mathbf{0}$?

**21.**[R] Show that $\mathbf{A}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B})\mathbf{A}-(\mathbf{A}\cdot\mathbf{A})\mathbf{B}$.

**22.**[R] Show that $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = ((\mathbf{A} \times \mathbf{B}) \cdot \mathbf{D})\mathbf{C} - ((\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C})\mathbf{D}$. HINT: Think of $\mathbf{A} \times \mathbf{B}$ as a single vector, $\mathbf{E}$.

**23.**[M]

(a) Give an example of a vector perpendicular to the vector $3\mathbf{i} - \mathbf{j} + \mathbf{k}$.

(b) Give an example of a unit vector perpendicular to the vector $3\mathbf{i} - \mathbf{j} + \mathbf{k}$.

**24.**[M] Let $\mathbf{u}$ be a unit vector and $\mathbf{B}$ be a vector. What happens as you keep "crossing by $\mathbf{u}$," that is, as you form the sequence $\mathbf{B}$, $\mathbf{u} \times \mathbf{B}$, $\mathbf{u} \times (\mathbf{u} \times \mathbf{B})$ and so on? (See Exercise 21)

**25.**[C] *(Crystallography)* A crystal is described by three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. They span a "fundamental" parallelepiped, whose copies fill out the crystal lattice. (See Figure 14.3.10.) The atoms are at the corners. In order to study the diffraction of x-rays and light through a crystal, crystallographers work with the "reciprocal lattice," as follows. Its fundamental parallelepiped is spanned by three vectors, $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$. The vector $\mathbf{k}_1$ is perpendicular to the parallelogram spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$ and has a length equal to the reciprocal of the distance between that parallelogram and the opposite parallelogram of the fundamental parallelepiped. The vectors $\mathbf{k}_2$ and $\mathbf{k}_3$ are defined similarly in terms of the other four faces of the fundamental parallelepiped.

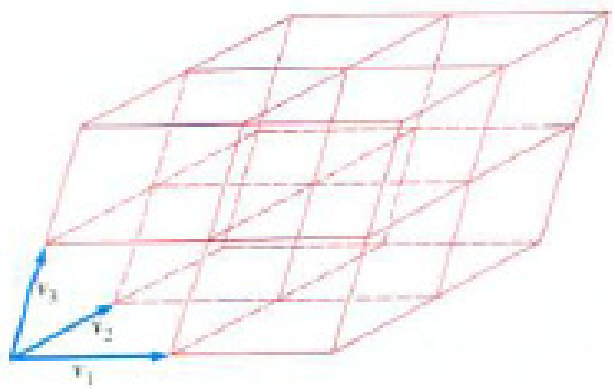



Figure 14.3.10:

(a) Show that $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ may be chosen to be

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}, \qquad \mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}, \qquad \mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

(b) Show that the volume of the fundamental parallelopiped determined by $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ is the reciprocal of the volume of the one determined by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

(c) Is the reciprocal of the reciprocal lattice the original lattice? For instance, is

$$\mathbf{v}_1 = \frac{\mathbf{k}_2 \times \mathbf{k}_3}{\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3)}?$$

**26.**[M] Let $\mathbf{B}$ and $\mathbf{C}$ be nonzero, nonparallel vectors and $\mathbf{A}$ a vector that is perpendicular neither to $\mathbf{B}$ nor $\mathbf{C}$.

(a) Why are their scalars $x$ and $y$ such that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = x\mathbf{B} + y\mathbf{C}?$$

(b) Why is $0 = x(\mathbf{A} \cdot \mathbf{B}) + y(\mathbf{A} \cdot \mathbf{C})$?

(c) Using (b), show that there is a scalar $z$ such that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = z[(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}].$$

(d) It would be nice if there were a simple geometric way to show that $z$ is a constant and equals 1. Of course we could show that $z = 1$ by writing $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in components and grinding out a tedious calculation. But that would hardly be instructive. Can you figure out why $z = 1$ in a simpler way?

(This identity, known as Jacobi's Identity, will come in handy in Chapter 18 when dealing with electric currents and magnetic fields.)

OMIT? In this section $\mathbf{A} \times \mathbf{B}$ was defined in terms of components, and then its geometric description was obtained. (This is the opposite of the way we dealt with the dot product. Exercises 27 to 29 outline a different approach to the cross product. We define $\mathbf{A} \times \mathbf{B}$ as follows. If $\mathbf{A}$ or $\mathbf{B}$ is $\mathbf{0}$ or if $\mathbf{A}$ is parallel to $\mathbf{B}$, we define $\mathbf{A} \times \mathbf{B}$ to be $\mathbf{0}$. Otherwise, $\mathbf{A} \times \mathbf{B}$ is the vector whose direction is given by the right-hand rule.

**27.**[R] Let $\mathbf{A}$ be a nonzero vector and $\mathbf{B}$ be a vector. Let $\mathbf{B}_1$ be the projection of $\mathbf{B}$ on a plane perpendicular to $\mathbf{A}$. Let $\mathbf{B}_2$ be obtained by rotating $\mathbf{B}_1$ $90^\circ$ in the direction given by the right-hand rule with thumb pointing in the same direction as $\mathbf{A}$

(a) Show that $\mathbf{A}\times\mathbf{B} = \mathbf{A}\times\mathbf{B}_1$. (Draw a clear diagram.)

(b) Show that $\mathbf{A}\times\mathbf{B} = \|\mathbf{A}\|\mathbf{B}_2$.

**28.**[R] Using Exercise 27(b), show that for $\mathbf{A}$ not $\mathbf{0}$, $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$. HINT: Draw a large, clear picture.

**29.**[R]

(a) From the distributive law $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$, and the fact that $\mathbf{D}\times\mathbf{E} = -\mathbf{E}\times\mathbf{D}$, deduce the distributive law $(\mathbf{B}+\mathbf{C})\times\mathbf{A} = \mathbf{B}\times\mathbf{A}+\mathbf{C}\times\mathbf{A}$.

(b) From the distributive law $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$, deduce that $\mathbf{A}\times(\mathbf{B}+\mathbf{C}+\mathbf{D}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}+\mathbf{A}\times\mathbf{D}$. HINT: Think of $\mathbf{B}+\mathbf{C}$ as a single vector $\mathbf{E}$.

**30.**[R] Check that $-13\mathbf{i}+7\mathbf{j}+11\mathbf{k}$ in Example 1 is perpendicular to $\mathbf{A}$ and to $\mathbf{B}$.

**31.**[R] Show, using (14.3.7), that $\mathbf{0}\times\mathbf{B} = \mathbf{0}$.

**32.**[R] Show, using (14.3.7), that $\mathbf{B}\times\mathbf{A} = -\mathbf{A}\times\mathbf{B}$.

**33.**[M] Using (14.3.7), show that if $\mathbf{B}$ is parallel to $\mathbf{A}$, then $\mathbf{A}\times\mathbf{B} = \mathbf{0}$. Suggestion: If $\mathbf{B}$ is parallel to $\mathbf{A}$, there is a scalar $t$ such that $\mathbf{B} = t\mathbf{A}$.

**34.**[M] In finding $|\mathbf{A}\times\mathbf{B}|^2$ we stated that

$$a_2^2b_3^2+a_3^2b_2^2+a_3^2b_1^2+a_1^2b_3^2+a_1^2b_2^2+a_2^2b_1^2-2(a_2a_3b_2b_3+a_1a_3b_1b_3+a_1a_2b_1b_2)$$

equals

$$a_1^2+a_2^2+a_3^2b_1^2+b_2^2+b_3^2-(a_1b_1+a_2b_2+a_3b_3).$$

Take nothing as faith. Check that the claim is correct.

**35.**[C] We showed that the direction of $\mathbf{i}\times\mathbf{j}$ is given by the right hand rule. Then we said that the right hand rule hold for any non-zero vector $\mathbf{A}$ and $\mathbf{B}$. Why is such a leap justified? HINT: Imagine moving a gradually changing pair of vectors through space, starting with $\mathbf{i}$ and $\mathbf{j}$ and ending with the pair $\mathbf{A}$ and $\mathbf{B}$.

**36.**[C]

(a) Thinking in terms of parallelograms, explain why $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is $+$ or $-$ $\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$.

(b) Using properties of 3 by 3 determents, decide which it's $+$ or $-$.

**37.**[C] In some expositions of the cross product, $\mathbf{a}\times\mathbf{b}$ is simply defined as the determinant of a matrix of order 3. If we start with this definition, use a property of determents to show that $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. (This approach bypasses the need to consider simultaneous equations. On the other hand, it may appear unmotivated.)

**38.**[M]

(a) How could you use cross products to produce a vector perpendicular to $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$? Give an example.

(b) How could you use cross product to produce two vectors perpendicular to $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and to each other? Give an example.

**39.**[R] Use the exhibited values for $y$ and $z$ when solving equations (14.3.3) and (14.3.4). Substitute these values into (14.3.1) and solve for $x$.

**40.**[R] By carrying out the necessary calculations, show that $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$. If you wish, you may use properties of determinants.

**41.**[M] Let $\mathbf{A}$ and $\mathbf{B}$ be non zero, nonparallel vectors. Show that $\mathbf{A} \times (\mathbf{A} \times \mathbf{B})$ is never equal to $(\mathbf{A} \times \mathbf{A} \times \mathbf{B})$. This shows that the cross product is not associative. You **cannot omit** the parentheses in $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

## 14.4 Lines, Planes and Components

This section uses the dot product and cross product to deal with lines, planes and projections ("shadows") of a vector or a line or on a plane.

### Equation of a Plane

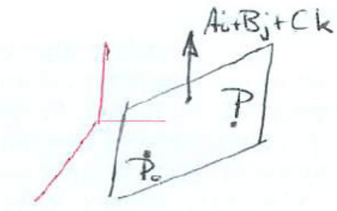



Figure 14.4.1:

We find an equation of the plane through the point $P_0 = (x_0, y_0, z_0)$ and perpendicular to the vector $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, shown in Figure 14.4.1.

Let $P = (x, y, z)$ be any point on the plane. The vector $\overrightarrow{P_0P}$ is perpendicular to $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. (Imagine sliding it so that $P_0$ coincides with the tail of $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.) Thus

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot ((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}) = 0.$$

So

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{14.4.1}$$

In (14.4.1) we have an equation for the plane. The vector $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is called a **normal** to the plane.

**EXAMPLE 1** Find an equation of the plane through $(2, -3, 4)$ and perpendicular to $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

*SOLUTION* An equation for the plane is

$$1(x - 2) + 2(x - (-3)) + 3(z - 4) = 0$$

which simplifies to

$$x + 2y + 3z - 8 = 0$$

◇

The graph of an equation of the form $Ax + By + Cz + D = 0$, where not all of $A$, $B$, and $C$ are 0 is a plane perpendicular to the vector $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. To show this, first pick any point $(x_0, y_0, z_0)$ that satisfies the equation: $Ax_0 + Bv_0 + C_0 + D = 0$. Subtracting this from the original equation gives

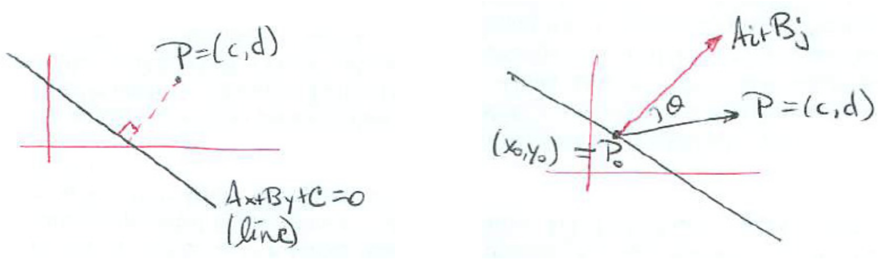
$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

which is an equation of the plane through $(x_0, y_0, z_0)$ perpendicular to $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

Similarly, we have

An equation for the line through $(x_0, y_0)$ and perpendicular to the vector $A\mathbf{i} + B\mathbf{j}$ is $A(x - x_0) + B(y - y_0) = 0$.

## Distance From a Point to the Line $Ax + By + C = 0$ or Plane $Ax + By + Cz + D = 0$




(a) (b)

Figure 14.4.2:

Let us find the distance from $P = (c, d)$ to the line whose equation is $Ax + By + C = 0$, shown in Figure 14.4.2(a).

Pick any point $P_0 = (x_0, y_0)$ on the line and place $A\mathbf{i} + B\mathbf{j}$ with its tail at $P_0$, as in Figure 14.4.2(b).

Let $\theta$ be the angle between $\overrightarrow{P_0P}$ and $A\mathbf{i} + B\mathbf{j}$. Then the distance from $P$ to the line is

cos($\theta$) could be negative

$$\begin{aligned} \|\overrightarrow{P_0P}\| |\cos(\theta)| &= \|\overrightarrow{P_0P}\| \frac{(A\mathbf{i} + B\mathbf{j}) \cdot ((c - x_0)\mathbf{i} + (d - y_0)\mathbf{j})}{\|P_0P\| \|A\mathbf{i} + B\mathbf{j}\|} \\ &= \frac{A(c - x_0) + B(d - y_0)}{\sqrt{A^2 + B^2}} \\ &= \frac{Ac + Bd - (Ax_0 + By_0)}{\sqrt{A^2 + B^2}}. \end{aligned}$$

Since $Ax_0 + By_0 + C = 0$, we have

Distance from $(c, d)$ to the line $Ax + By + C = 0$ is

$$\frac{|Ac + Bd + C|}{\sqrt{A^2 + B^2}}$$

In short, to find that distance simply substitute the coordinates of the point $(c, d)$ into the expression $Ax + By + C$ and divide by $\sqrt{A^2 + B^2}$ and take its absolute value.

**EXAMPLE 2** How far is the point $(1, 3)$ from the line $2x - 4y = 5$?
*SOLUTION* First, write the equation in the form $2x - 4y - 5 = 0$. Then the

distance is

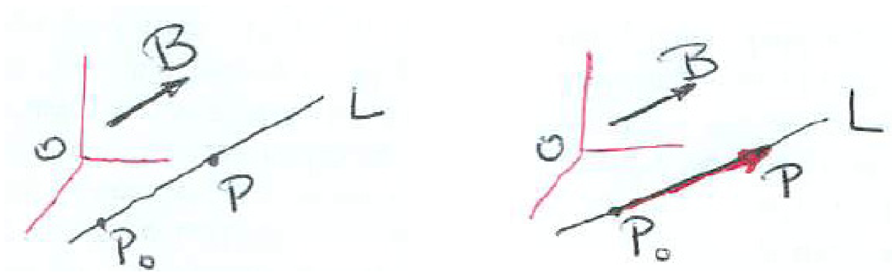
$$\frac{|2(1) - 4(3) - 5|}{\sqrt{2^2 + 4^2}} = \frac{|-15|}{\sqrt{20}} = \frac{3\sqrt{5}}{2}.$$

⋄ A similar result holds for the distance from a point $P = (x_0, y_0, z_0)$ to a plane:

> The distance from $(x_0, y_0, z_0)$ to the plane $Ax + By + C = 0$ is
>
> $$\frac{|Ax_0 + By_0 + C_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

## Using Vectors to Parameterize a Line

Let $L$ be the line through the point $P_0 = (x_0, y_0, z_0)$ parallel to the vector $\mathbf{B}$, shown in Figure 14.4.3(a).




(a) (b)

Figure 14.4.3:

Let $P$ be any point on $L$. Then the vector $\overrightarrow{P_0P}$ which is parallel to $\mathbf{B}$, is of the form $t\mathbf{B}$ for some scalar $t$. See Figure 14.4.3(b).

The $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \overrightarrow{OP_0} + t\mathbf{B}$. As $t$ varies the vector from 0 to $P$ varies, thus parameterizing the line $L$.

**EXAMPLE 3** The line $L$ passes through the point $(1, 1, 2)$ and is parallel to the vector $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$. Use this information to parameterize the line.
*SOLUTION* In this case $\overrightarrow{OP_0} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $B = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$. Thus

$$\begin{aligned} \overrightarrow{OP} &= \mathbf{i} + \mathbf{j} + 2\mathbf{k} + f(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) \\ &= (3t + 1)\mathbf{i} + (4t + 1)\mathbf{j} + (5t + 2)\mathbf{k}. \end{aligned}$$

If $P$ is the point $(x, y, z)$, then $\overrightarrow{OP}$ is the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Thus

$$\begin{cases} x = 3t + 1 \\ y = 4t + 1 \\ z = 5t + 2. \end{cases}$$

◇

One vector equation does the work of three scalar equation.

## Describing the Direction of Vectors and Lines

The direction of a vector in the plane is described by a single angle, the angle it makes with the positive $x$-axis. The direction of a vector in space involves three angles, two of which almost determine the third.

**DEFINITION** *(Direction of a vector.)* Let $\mathbf{A}$ be a nonzero vector in space. The angle between

$\mathbf{A}$ and $\mathbf{i}$ is denoted $\alpha$,

$\mathbf{A}$ and $\mathbf{j}$ is denoted $\beta$,

$\mathbf{A}$ and $\mathbf{k}$ is denoted $\gamma$.

The angles $\alpha$, $\beta$ and $\gamma$ are called the **direction angles of A**. (See Figure 14.4.4.)

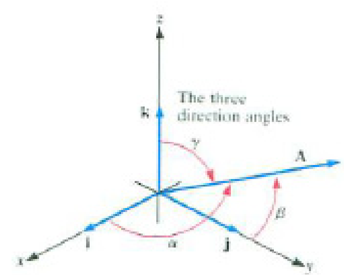



Figure 14.4.4:

**DEFINITION** *(Direction cosines of a vector)* The **direction cosines** of a vector are the cosines of its direction angles, $\cos(\alpha)$, $\cos(\beta)$, and $\cos(\gamma)$.

**EXAMPLE 4** The angle between a vector $\mathbf{A}$ and $\mathbf{k}$ is $\pi/6$. Find $\gamma$ and $\cos(\gamma)$ for

1. $\mathbf{A}$,
2. $-\mathbf{A}$.

*SOLUTION*

1. By definition, the direction angle $\gamma$ for $\mathbf{A}$ is $\pi/6$. It follows that $\cos(\gamma) = \cos(\pi/6) = \sqrt{3/2}$.
2. To find $\gamma$ and $\cos(\gamma)$ for $-\mathbf{A}$, we draw Figure 14.4.5. For $-\mathbf{A}$, $\gamma = 5\pi/6$ and $\cos(\gamma) = \cos(5\pi/6) = -\sqrt{3/2}$.

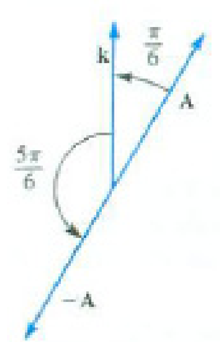



Figure 14.4.5:

◇

As Example 4 illustrates, if the direction angles of **A** are $\alpha$, $\beta$, $\gamma$, then the direction angles of $-\mathbf{A}$ are $\pi-\alpha$, $\pi-\beta$, and $\pi-\gamma$. The direction cosines of $-\mathbf{A}$ are the negatives of the direction cosines of **A**.

The three direction angles are not independent of each other, as is shown by the next theorem. Two of them determine the third up to sign.

**Theorem 14.4.1.** *If $\alpha$, $\beta$, $\gamma$ are the direction angles of the vector* **A**, *then* $\cos^2(\alpha)+\cos^2(\beta)+\cos^2(\gamma)=1$.

Proof

*It is no loss of generality to assume that* **A** *is a unit vector. Its component on the y-axis, for instance, is* $\cos(\beta)$, *as the right triangle OPQ in Figure 14.4.6 shows.* **A** *lies along the hypotenuse.*

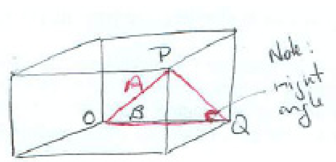



Figure 14.4.6:

*Since* **A** *is a unit vector,* $|\mathbf{A}|^2=1$, *and we have* $\cos^2(\alpha)+\cos^2(\beta)+\cos^2(\gamma)=1^2=1$. •

**EXAMPLE 5** The vector **A** makes an angle of $60^\circ$ with the $x$ and $y$ axes. What angle does it make with the $z$-axis?

*SOLUTION* Here $\alpha=60^\circ$ and $\beta=60^\circ$; hence

$$\cos(\alpha)=\frac{1}{2} \quad \text{and} \quad \cos(\beta)=\frac{1}{2}.$$

Since

$$\cos^2(\alpha)+\cos^2(\beta)+\cos^2(\gamma)=1,$$

it follows that

$$\begin{aligned}(\frac{1}{2})^2+(\frac{1}{2})^2+\cos^2(\gamma) &= 1,\\ \cos^2(\gamma) &= \frac{1}{2}.\end{aligned}$$

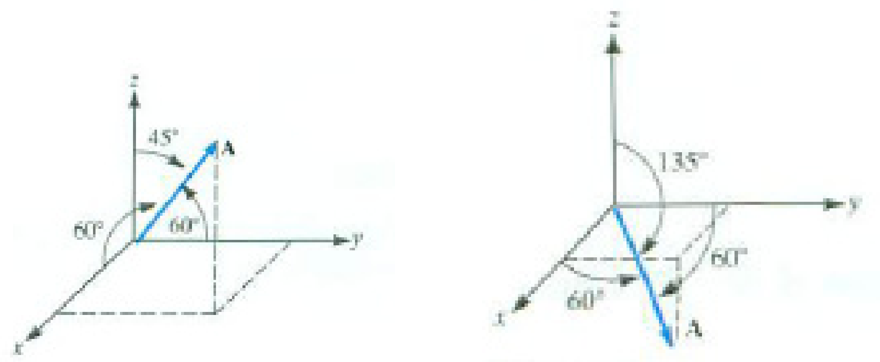
Thus

$$\cos(\gamma)=\frac{\sqrt{2}}{2} \quad \text{or} \quad \cos(\gamma)=-\frac{\sqrt{2}}{2}.$$

Hence

$$\gamma=45^\circ \quad \text{or} \quad \gamma=135^\circ.$$

Figures 14.4.7(a) and (b) show the two possibilities for **A**. ◇




(a) (b)

Figure 14.4.7:


(a) (b) (c)

Figure 14.4.8:

## Dot Products and Flow

SHERMAN: what word did you intend to go between imaginary and stick? My best guess is horizontal, but this makes no sense to me.

Let the vector $\mathbf{v}$ whose magnitude is $v$ describe the velocity of a river, as in Figure 14.4.8(a). Place an imaginary horizontal stick of length $L$ in the water. The amount of water crossing the stick depends on the position of the stick. If the stick is parallel to $\mathbf{v}$, no water crosses the stick. If the stick is perpendicular to $\mathbf{v}$ water crosses it. The question then arises, "How does the angle at which we place the stick affect the amount of water that crosses in a given time?

To answer this question, we begin by introducing a unit vector $\mathbf{n}$ perpendicular to the stick, and record its position, as in Figure 14.4.8(b). Let the angle between $\mathbf{n}$ and $\mathbf{v}$ be $\theta$.

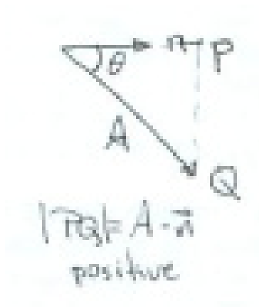
The amount of water that crosses the stick during time $\Delta t$ is proportional to the area of the parallelogram in Figure 14.4.8(c). The base of the parallelogram has length $v\Delta t$ (speed times time). The height is $L\cos(\theta)$. The area of the parallelogram is therefore

$$vL\cos(\theta)$$

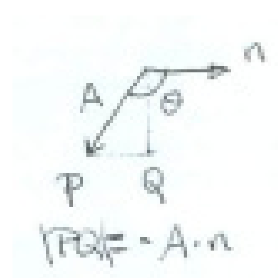
.

But $vL\cos(\theta)$ is equal to $\mathbf{v}\cdot\mathbf{n}$. So $\mathbf{v}\cdot\mathbf{n}$ measures the tendency of water to cross the stick.

As a check, when the stick is parallel to $\mathbf{v}$, $\theta = \pi/2$ and $\cos(\pi/2) = 0$. Then $\mathbf{v}\cdot\mathbf{n} = 0$ and no water crosses the stick. When the stick is perpendicular to $\mathbf{v}$, $\theta = 0$, and $\mathbf{v}\cdot\mathbf{n} = v$. For any angle $\theta < \pi/2$, $\mathbf{v}\cdot\mathbf{n} = v\cos(\theta)$ which is less than $v$. For any unit vector $\mathbf{n}$ and vector $\mathbf{A}$ the scalar $\mathbf{A}\cdot\mathbf{n}$ is called the




(a)




(b)

Figure 14.4.9:

**scalar component of A along n**. It equals $\|vA\|\cos(\theta)$, where $\theta$ is the angle between **A** and **n**. It can be positive or negative, as shown in Figure 14.4.9.

**EXAMPLE 6** When a stick is perpendicular to **v**, water crosses it at the rate of 100 cubic feet per second. When the stick is placed at an angle of $\pi/6$ to **v** at what rate does water cross it?

*SOLUTION* Figure 14.4.10 shows the position of the stick $PQ$.

The angle between the normal to the stick, **n**, and **v** is $\pi/2 - \pi/6 = \pi/3$. Let $x$ be the rate at which the water crosses the stick. Since the rate of flow across the stick is proportional to $v\cos(\theta)$, where $\theta$ is the angle between the normal **n** and **v**, we have

$$\frac{100}{v\cos(0)} = \frac{x}{v\cos(\pi/3)}.$$

this tells us that

$$\frac{100}{v} = \frac{x}{(v)(1/2)},$$

have $x = 50$. The flow is half the maximum possible. ⋄

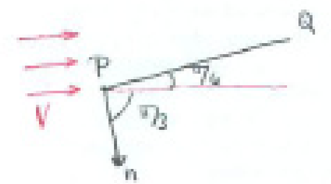



Figure 14.4.10:

## Summary

We used the dot product to obtain an equation of a plane (or line in the $xy$ plane) and to find the distance from a point to a line or plane. We also showed how to parameterize a line with the aid of a vector parallel to the line.

Direction angles and cosines of a vector were defined. Finally, we showed how the dot product describes the rate of flow across a line segment, a concept that will be needed in Chapters 17 and 18, where we deal with flows across curves and surfaces.

**EXERCISES for Section 14.4** *Key:* R–routine, M–moderate, C–challenging

exercises
d. Some
Chapter
ummary.

In each of Exercises 1 to 4 find an equation of the line through the given point and perpendicular to the given vector.

**1.**[R] $(2,3)$, $4\mathbf{i}+5\mathbf{j}$

**2.**[R] $(1,0)$, $2\mathbf{i}-\mathbf{j}$

**3.**[R] $(4,5)$, $1\mathbf{i}+3\mathbf{j}$

**4.**[R] $(2,-1)$, $\mathbf{i}+3\mathbf{j}$

In each of Exercises 5 to 8 find a vector in the $xy$ plane that is perpendicular to the given line.

**5.**[R] $2x-3y+8=0$

**6.**[R] $\pi x-\sqrt{2}y=7$

**7.**[R] $y=3x+7$

**8.**[R] $2(x-1)+5(y+2)=0$

**9.**[M] Find an equation of the plane through $(1,2,3)$ that contains the line given parametrically as $\overrightarrow{OP}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}+t(3\mathbf{i}+2\mathbf{j}+\mathbf{k})$.

**10.**[M] Is the point $(21,-3,28)$ on the line given parametrically as $\overrightarrow{OP}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}+t(4\mathbf{i}-\mathbf{j}+5\mathbf{k})$?

**11.**[M] A line segment has projections of lengths $a$, $b$, and $c$ on the coordinates axes. What, if anything, can be said about its length, $L$?

**12.**[C] A line segment has projections of lengths $d$, $e$, and $f$ on the coordinates planes. What, if anything, can be said about its length, $L$?

**13.**[C] Explain why the projection of a circle is an ellipse. HINT: Set up coordinate systems in the plane of the circle and in the plane of its shadow (which might as well be taken to be the $xy$ plane). Choose the axes for these coordinate systems to be as convenient as possible. Then express the equation of the shadow in terms of $x$ and $y$ by utilizing the equation of the circle.

**14.**[R] Find a vector perpendicular to the plane through $(2,1,3)$, $(4,5,1)$ and $(-2,2,3)$.

**15.**[R] How far is the point $(1,2,2)$ from the plane through $(0,0,0)$, $(3,5,-2)$, and $(2,-1,3)$?

**16.**[R] How far is the point $(1,2,3)$ from the line through $(-2,-1,3)$, and $(4,1,2)$?

**17.**[R] Find the parametric equations of the line through $(1,1,2)$ and perpendicular to the plane $3x-y+z=6$.

**18.**[R] How far apart are the lines whose vector equations are $2\mathbf{i}+4\mathbf{j}+\mathbf{k}+t(\mathbf{i}+\mathbf{j}+\mathbf{k})$ and $\mathbf{i}+3\mathbf{j}+2\mathbf{k}+s(2\mathbf{i}-\mathbf{j}-\mathbf{k})$?

**19.**[R] Find the point on the line through $(1,2,1)$ and $(2,-1,3)$ that is closest to the line through $(3,0,3)$ and parallel to the vector $\mathbf{i}+2\mathbf{j}+5\mathbf{k}$.

**20.**[R]

(a) Describe how you would find an equation for the plane through points $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$, and $P_3=(x_3,y_3,z_3)$?

(b) Find an equation for the plane through $(2,2,1)$, $(0,1,5)$ and $(2,-1,0)$.

**21.**[R]

(a) Describe how you would decide whether the line through $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$, is parallel to the line through $P_3=(x_3,y_3,z_3)$ and $P_4=(x_4,y_4,z_4)$?

(b) Is the line through $(1,2,-3)$ and $(5,9,4)$ parallel to the line through $(-1,-1,2)$ and $(1,3,5)$?

**22.**[R]

(a) Describe how you would decide whether the line through $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ is parallel to the plane $Ax+By+Cz+D=0$?

(b) Is the line through $(1,-2,3)$ and $(5,3,0)$ parallel to the plane $2x-y+z+3=0$?

**23.**[R]

(a) Describe how you would decide whether the line through $P_1$ and $P_2$ is parallel to the plane through $Q_1$, $Q_2$, and $Q_3$?

(b) Is the line through $(0,0,0)$ and $(1,1,-1)$ parallel to the plane through $(1,0,1)$, $(2,1,0)$, and $(1,3,4)$?

**24.**[R]

(a) How would you decide whether the plane through $P_1$, $P_2$ and $P_3$ is parallel to the plane through $Q_1$, $Q_2$, and $Q_3$?

(b) Is the plane through $(1,2,3)$, $(4,1,-1)$, and $(2,0,1)$ parallel to the plane through $(2,3,4)$, $(5,2,0)$, and $(3,1,2)$?

**25.**[M]

(a) How would you find the angle between the planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$?

(b) Find the angle between $x-y-z-1=0$ and $x+y+z+2=0$.

**26.**[C] Assume that the planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ met in a line $L$.

(a) How would you find a vector parallel to $L$?

(b) How would you find a point on $L$?

(c) Find parametric equations for the line that is the intersection of the planes $2x-y+3z+4=0$ and $3x+2y+5z+2=0$.

**27.**[C]

(a) How would you decide whether the four points $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$, $P_3=(x_3,y_3,z_3)$ and $P_4=(x_4,y_4,z_4)$ lie in a plane?

(b) Do the points $(1,2,3)$, $(4,1,-5)$, $(2,1,6)$, and $(3,5,3)$ lie in a plane?

**28.**[C] What is the angle between the line through $(1,2,1)$ and $(-1,3,0)$ and the plane $x+y-2z=0$?

**29.**[M]

(a) If you know the coordinates of point $P$ and parametric equations of line $L$, how would you find an equation of the plane that contains $P$ and $L$? (Assume $P$ is not on $L$.)

(b) Find an equation for the plane through $(1,1,1)$ that contains the line

$$\begin{cases} x = 2+t \\ y = 3-t \\ z = 4+2t. \end{cases}$$

**30.**[R]

(a) How many unit vectors are perpendicular to the plane $Ax+By+Cz+D=0$?

(b) How would you find one of them?

(c) Find a unit vector perpendicular to the plane $3x-2y+4z+6=0$.

**31.**[R]

(a) How would you go about producing a specific point on the plane $Ax+By+Cz+D=0$?

(b) Give the coordinates of a specific point that lies on the plane $3x-y+z+10=0$.

**32.**[R]

(a) How would you go about producing a specific point that lies on both planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$?

(b) Find a point that lies on both planes $3x+z+2=0$ and $x-y-z+5=0$.

**33.**[C] The planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ intersect in a line $L$. Find the direction cosines of a vector parallel to $L$.

**34.**[R]

(a) Let **A** and **B** be vectors in space. How would you find the area of the parallelogram they span?

(b) Find the area of the parallelogram spanned by $(2,3,1)$ and $(4,-1,5)$.

**35.**[C] How far is the point $(2,1,3)$ from the line through $(1,5,2)$ and $(2,3,4)$?

**36.**[C] How far is the point $P$ from the line through $Q$ and $R$.

**37.**[C] How far apart are the lines given parametrically as $2\mathbf{i}+\mathbf{j}-3\mathbf{k}+t(3\mathbf{i}-5\mathbf{j}+2\mathbf{k})$ and $3\mathbf{i}+\mathbf{j}+5\mathbf{k}+s(2\mathbf{i}+6\mathbf{j}+7\mathbf{k})$? (We use different letters, $s$ and $t$, for the parameters because they are independent of each other.)

**38.**[M]

(a) Sketch four points $P$, $Q$, $R$, and $S$, not all in one plane, such that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are not parallel. Explain way there is a unique pair of parallel planes one of which contains $P$ and $Q$ and are of which contains $R$ and $S$.

(b) Express a normal vector to these planes in terms of $P$, $Q$, $R$, and $S$.

**39.**[M] Find an equation for the plane through $P_1$ that is parallel to the non-parallel segments $P_2P_3$ and $P_4P_5$.

**40.**[C]

(a) Using properties of determinents, show that

$$\begin{vmatrix} x & y & 1 \\ a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \end{vmatrix} = 0$$

is the equation of a line through the points $(a_1,a_2)$ and $(b_1,b_2)$.

(b) What determinant of order 4 would give an angular equation for the plane through these given points?

**41.**[C]

(a) Review the Folium of Descartes in Section 9.3 on page 706.

(b) Show that the part in the fourth quadrant is asymptotic to the line $x+y+1=0$.

**42.**[M] Find where the line $L$ through $P_0=(2,1,3)$ and $P_1=(4,-2,5)$ meets the plane whose equation is $2x+y-4z+5=0$.

**43.**[M]

(a) Graph the line and the parabola. Identify, graphically, the point on the parabola closest to the line.

(b) Find, analytically, the point on the parabola $y=x^2$ closest to the line $y=x-3$.

(c) The tangent to the parabola at the point found in (b) looks as if it might be parallel to the line. Is it?

**44.**[C] Let $f$ be a differential function and $L$ a line that does not meet the graph of $F$. Assume that $P_0$ is the point as the graph that is nearest the line.

(a) Using calculus, show that the tangent there is parallel to $L$.

(b) Why is the result in (a) to be expected?

In Exercises 45 and 46, find the distance from the given point to the given line. **45.**[R] The point $(0, 0)$ to $3x + 4y - 10 = 0$

**46.**[R] The point $(3/2, 2/3)$ to $2x - y + 5 = 0$

In Exercises 47 and 48 find a normal and a unit normal to the given planes.

**47.**[R] $2x-3y+4z+11 = 0$

**48.**[R] $z = 2x - 3y + 4$

In Exercises 49 to 52 find the distance from the given point to the given plane.

**49.**[R] The point $(0, 0, 0)$ to the plane $2x-4y+3z+2 = 0$

**50.**[R] The point $(1, 2, 3)$ to the plane $x + 2y - 3z + 5 = 0$.

**51.**[R] The point $(2, 2, -1)$ to the plane that passes through $(1, 4, 3)$ and has a normal $2\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$.

**52.**[R] The point $(0, 0, 0)$ to the plane that passes through $(4, 1, 0)$ and is perpendicular to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

**53.**[R] Find the direction cosines of the vector $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

**54.**[R] Find the direction cosines of the vector from $(1, 3, 2)$ to $(4, -1, 5)$.

**55.**[R] Let $P_0 = (2, 1, 5)$ and $P_1 = (3, 0, 4)$. Find the direction cosines and direction angles of

(a) $\overrightarrow{P_0P_1}$ and

(b) $\overrightarrow{P_1P_0}$.

**56.**[R] Give parametric equations for the line through $(1/2, 1/3, 1/2)$ with direction numbers 2, $-5$ and 8 in

(a) scalar form,

(b) vector form.

**57.**[R] Give parametric equations for the line through $(1, 2, 3)$ and $(4, 5, 7)$ in

(a) scalar form,

(b) vector form.

**58.**[R] Give symmetric equations for the line through the points $(7, -1, 5)$ and $(4, 3, 2)$.

**59.**[R] A vector $\mathbf{A}$ has direction angles $\alpha = 70°$ and $\beta = 80°$. Find the third direction angle $\gamma$ and show the possibile angles for $\gamma$ on a diagram.

**60.**[M] Suppose that the three direction angles of a vector are equal. What can they be? Draw the cases.

**61.**[R] Find the angle between the line through $(3, 2, 2)$ and $(4, 3, 1)$ and the line through $(3, 2, 2)$ and $(5, 2, 7)$.

**62.**[R] Find the angle between the planes $2x + 3y + 4z = 11$ and $3x - y + 2z = 13$. The angle between two planes is the angle between their normals.

**63.**[R] Find where the line through $(1, 2)$ and $(3, 5)$ meets the line through $(1, -1)$ and $(2, 3)$.

**64.**[M] Find where the line through $(1, 2, 1)$ and $(2, 1, 3)$ meets the plane that is perpendicular to the vector $2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$ and passes through the point $(1, -2, -3)$.

**65.**[M] Are the three points $(1, 2, -3)$, $(1, 6, 2)$, and $(7, 14, 11)$ on a single line?

**66.**[R] Where does the line through $(1, 2, 4)$ and $(2, 1, -1)$ meet the plane $x + 2y + 5z = 0$?

**67.**[R] Give parametric equations for the line through $(1, 3, -5)$ that is perpendicular to the plane $2x - 3y + 4z = 11$.

**68.**[R] Give parametric equations for the line through $(1, 3, 4)$ that is parallel to the line through $(2, 4, 6)$ and $(5, 3, -2)$.

**69.**[C] A square of a side $a$ lies in the plane $2x + 3y + 2z = 8$. What is the area of its projection

(a) on the $xy$ plane?

(b) on the $yz$ plane?

(c) on the $xz$ plane?

**70.**[M] If $\alpha$, $\beta$, and $\gamma$ are direction angles of a vector, what is $\sin^2(\alpha) + \sin^2(\beta) + \sin^2(\gamma)$?

**71.**[M] Find the angle between the line through $(1, 3, 2)$ and $(4, 1, 5)$ and the plane $x - y - 2z + 15 = 0$.

**72.**[C] A disk of radius $a$ is situated in the plane $x + 3y + 4z = 5$. What is the area of its projection in the plane $2x + y - z = 6$?

**73.**[M] What point on the line through $(1, 2, 5)$ and $(3, 1, 1)$ is closest to the point $(2, -1, 5)$?

**74.**[C] Does the line through $(5, 7, 10)$ and $(3, 4, 5)$ meet the line through $(1, 4, 0)$ and $(3, 6, 4)$? If so, where?

**WARNING** *(Do Not Confuse Parameters from Different Curves)* Use parametric equations but give the parameters of the lines different names, such as $t$ and $s$.

**75.**[C] Develop a general formula for determining the distance from the point $P_1 = (x_1, y_1, z_1)$ to the line through the point $P_0 = (x_0, y_0, z_0)$ and parallel to the vector $\mathbf{A} = a_1\mathbf{i} + z_2\mathbf{j} + a_3\mathbf{k}$. The formula should be expressed in terms of the vectors $\overrightarrow{P_0P_1}$ and $\mathbf{A}$.

**76.**[C] How far is the point $(1, 2, -1)$ from the line through $(1, 3, 5)$ and $(2, 1, -3)$?

(a) Solve by calculus, minimizing a certain function.

(b) Solve by vectors.

**77.**[R] Find the direction cosines of the vector **A** shown in Figure 14.4.11. Hint: First draw a large diagram.

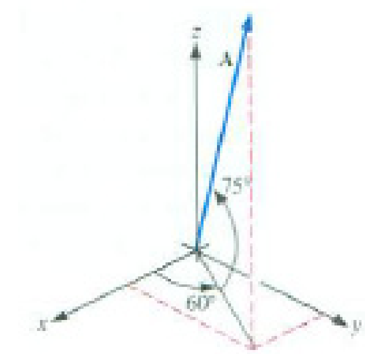



Figure 14.4.11:

**78.**[C] How small can the largest of three direction angles ever be?

**79.**[C] A plane $\pi$ is tilted at an angle $\theta$ to a horizontal plane. A convex region $R$ in $\pi$ has area $A$. Show that the area of its shadow ("projection") on the horizontal plane is $A\cos(\theta)$. Assume that the rays of light are perpendicular to the horizontal plane. (See Figure 14.4.12.)

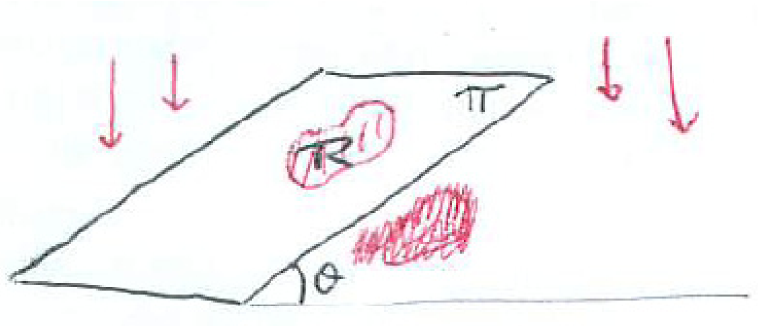


Figure 14.4.12:

**80.**[M]

(a) Find the point on the curve $y = \sin(x)$, $0 \leq x \leq \pi$, nearest the line $y = x/2 + 2$.

(b) Check your answer by sketching the curve to the line.

**81.**[M]

(a) Find the point on the curve $y = \sin x$, $0 \leq x \leq \pi$, nearest the line $y = 2x + 4$.

(b) Check your answer by drawing the curve and the line.

**82.**[R] Three points $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$, and $P_3 = (x_3, y_3, z_3)$ are the vertices of a triangle.

(a) What is the area of that triangle?

(b) What is the area of the projection of that triangle on the $xy$ plane?

**83.**[M] How can you decide whether the line through $P$ and $Q$ is parallel to the plane $Ax+By+Cz+D = 0$?

**84.**[M] Find where the line through $(1, 1)$ and $(2, 3)$ meets the line $x + 2y + 3 = 0$.

**85.**[R] Show that the line through $(1, 1, 1)$ and $(2, 3, 4)$ is perpendicular to the plane $x_1 + 2y + 3z + 4 = 0$.

**86.**[C] How would you decide whether the angle and a point $P = (x_0, y_0, z_0)$ are on the same side or opposite sides of the plane $Ax + Bx + Cz + D = 0$?

**87.**[M]

(a) Give an example of a vector perpendicular to the plane $2x + 3y - z + 4 = 0$.

(b) Give an example of a vector parallel to that plane.

**88.**[C] How would you decide whether the points $P$ and $Q$ are on the same side, or opposite sides, of the plane $Ax + By + Cz + D = 0$?

**89.**[R] A plane contains the points $P_0$, $P_1$, and $P_2$, which do not lie on a line. Find a vector perpendicular to the plane

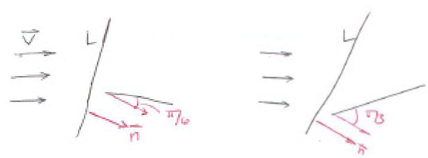
**90.**[C] Devise a procedure for determining whether the point $P = (x, y)$ is inside the triangle whose three vertices are $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$.

**91.**[C] Devise a procedure for determining whether the point $P = (x, y, x)$ is inside the four vertices are $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ $P_3 = (x_3, y_3, z_3)$ and $P_4 = (x_4, y_4, z_4)$.

**92.**[M] How far apart are the planes $Ax+By+Cz+D=0$ and $Ax+By+Cz+E=0$? Explain.

**93.**[R] We showed that the distance from $(c,d)$ to the line $Ax+By+C=0$ is $\frac{|Ac+Bd+C|}{\sqrt{A^2+B^2}}$. Show, following a similar argument, that the distance from $(c,d,e)$ to the plane $Ax+By+Cx+D=0$ is $\frac{|Ac+Bd+Ce+D|}{\sqrt{A^2+B^2+C^2}}$.

**94.**[M] What is the ratio of the flows across the two sticks in Figure 14.4.13(a) and (b)?



(a) (b)

Figure 14.4.13:

**95.**[R] Why is the angle $\theta$ shown in Figure 14.4.13 the same as the angle between $\overrightarrow{v}$ and $\widehat{\mathbf{n}}$.

**96.**[R] How far is the point $(1,5)$ from the line through $(4,2)$ and $(3,7)$? HINT: Draw a picture and think in terms of vectors.

**97.**[R] How far is the point $(1,2,-3)$ from the line through $(2,1,4)$ and $(1,5,-2)$?

**98.**[C] (Contributed by Melvyn Kopald Stein.) An industrial hopper is shaped as show
Its top and bottom are squares of
angle between the plane $ABD$ a
is 70°. The angle between the p
plane $ABC$ is 80°. What is the a
$ABC$ and plane $BCD$? NOTE: T
during the fabrication of the hopp
$ABC$ and $BCD$ are made from
heavy-gauge sheet metal bent al

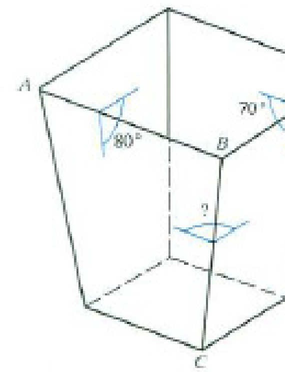



Figure 14.4.14:

**99.**[C]

(a) Let $L_1$ be the line through $P$
be the line through $P_2$ and
and $L_2$ are skew lines. How
point $R_1$ on $L_1$ and point
$\overrightarrow{R_1R_2}$ is perpendicular to bo

(b) Find $R_1$ and $R_2$ when $P_1$
$(1,1,1)$, $P_2=(0,2,0)$, $R_2=$

# 14.S Chapter Summary

DOUG/SHERMAN: Mention $\cos(\mathbf{A}, \mathbf{B})$ in text.

Because there are no limits in this chapter, it is, strictly speaking, not part of calculus. In the next chapter, which concern derivatives of functions whose inputs are scalars and whose outputs are vectors, we return to calculus.

The following table summarizes the basic concepts of vectors in space.

For plane vectors, disregard the third component.

Assume $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

| Symbol | Name | Geometric Descriptions | Algebraic Formula |
|---|---|---|---|
| $\mathbf{A}$ | Vector | Direction and magnitude (Figure) | $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ or $\langle a_1, a_2, a_3\rangle$ |
| $\|\mathbf{A}\|$ | Length (norm, magnitude) | Length of $\mathbf{A}$ | $\sqrt{a_1^2 + a_2^2 + a_3^2}$ |
| $-\mathbf{A}$ | Negative, or opposite, of $\mathbf{A}$ | Figure | $-a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$ or $\langle -a_1, -a_2, -a$ |
| $\mathbf{A} + \mathbf{B}$ | Sum of $\mathbf{A}$ and $\mathbf{B}$ | Figure | $(a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$ or $\langle a_1 + b_1,$ |
| $\mathbf{A} - \mathbf{B}$ | Difference of $\mathbf{A}$ and $\mathbf{B}$ | Figure | $(a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}$ or $\langle a_1 - b_1,$ |
| $c\mathbf{A}$ | Scalar multiple of $\mathbf{A}$ | Figure | $ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k}$ or $\langle ca_1, ca_2, ca$ |
| $\mathbf{A} \cdot \mathbf{B}$ | Dot, or scalar, product | $\|\mathbf{A}\|\|\mathbf{B}\|\cos(\theta)$ | $a_1b_1 + a_2b_2 + a_3b_3$ |
| $\mathbf{A} \times \mathbf{B}$ | Cross, or vector, product | Magnitude: area of parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$, $\|\mathbf{A}\|\|\mathbf{B}\|\sin(\theta)$ Direction: perpendicular to $\mathbf{A}$ and $\mathbf{B}$, direction by right-hand rule | $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b+3 \end{vmatrix}$ |
| $\mathrm{proj}_{\mathbf{B}}\,\mathbf{A}$ | (Vector) Projection of $\mathbf{A}$ on $\mathbf{B}$ | Figure | $(\mathbf{A} \cdot \mathbf{u})\mathbf{u}$, where $u = \mathbf{B}/\|\mathbf{B}\|$ |
| $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ | Scalar triple product | $\pm$ volume of parallelepiped spanned by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ | $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b+3 \\ c_1 & c_2 & c+3 \end{vmatrix}$ |
| $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ | Vector triple product | | |

Table 14.S.1:

Some Common Applications and Definitions

| | |
|---|---|
| $\mathbf{A} \cdot \mathbf{B} = 0$ | **A** is perpendicular to **B** (assuming neither **A** nor **B** is **0**) |
| $\mathbf{A} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$ | plane through $(x_0, y_0, z_0)$ perpendicular to **A** |
| $\frac{\lvert D \rvert}{\sqrt{A^2+B^2+C^2}}$ | distance from the plane $Ax + By + Cz + D = 0$ to the origin |
| $\frac{\lvert Ax_1+By_1+Cz_1+D \rvert}{\sqrt{A^2+B^2+C^2}}$ | distance from the plane $Ax + By + Cz + D = 0$ to the $(x_1, y_1, z_1)$ |
| $\frac{\mathbf{A} \cdot \mathbf{B}}{\lVert \mathbf{A} \rVert \lVert \mathbf{B} \rVert} = \cos(\theta)$ | $\theta$ is the angle between **A** and **B**, $0 < \theta < \pi$ |

When the angles between a vector **A** and **i**, **j**, $vk$ are respectively $\alpha$, $\beta$, and $\gamma$, the numbers $\cos(\alpha)$, $\cos(\beta)$, and $\cos(\gamma)$ are called the **direction cosines** of **A**. They are linked by the equation $\cos(\alpha)^2 + \cos(\beta)^2 + \cos(\gamma)^2 = 1$.

The line through $P_0 = (x_0, y_0, z_0)$ parallel to $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is given parametrically as

$$\begin{cases} x &= x_0 + a_1 t \\ y &= y_0 + a_2 t \\ z &= z_0 + a_3 t, \end{cases}$$

or vectorially as

$$\overrightarrow{OP} = \overrightarrow{OP_0} + t\mathbf{A}.$$

Also, the line has the description in the symmetric form

Assuming none of $a_1$, $a_2$, and $a_3$ are zero.

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}.$$

**EXERCISES for 14.S** *Key:* R–routine, M–moderate, C–challenging

**1.**[R] Find a vector perpendicular to the plane determined by the points $(1, 2, 1)$, $(2, 1, -3)$, and $(0, 1, 5)$.

**2.**[R] Find a vector perpendicular to the plane determined by the points $(1, 3, -1)$, $(2, 1, 1)$, and $(1, 3, 4)$.

**3.**[R] Find a vector that is perpendicular to the line through the points $(3, 6, 1)$ and $(2, 7, 2)$ and also to the line through the points $(2, 1, 4)$ and $(1, -2, 3)$.

**4.**[R] Find a vector perpendicular to the line through $(1, 2, 1)$ and $(4, 1, 0)$ and also to the line through $(3, 5, 2)$ and $(2, 6, -3)$.

**5.**[C] Figure 14.S.1 shows a tetrahedron $OABC$ with three edges of the indicated lengths.

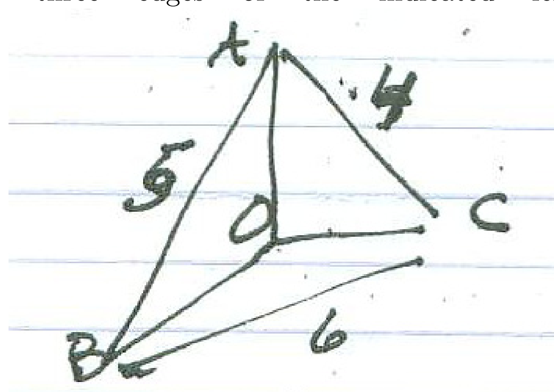



Figure 14.S.1:

(a) Find the coordinates of $A$, $B$, and $C$.

(b) Find the volume of the tetrahedron.

(c) Find the area of triangle $ABC$.

(d) Find the distance from $O$ to the plane in which triangle $ABC$ lies.

(e) Find the cosine of angle $ABC$.

# Calculus is Everywhere # 16

## Space Flight: The Gravitational Slingshot

For vector-algebra chapter

In a "slingshot" or "gravitational assist" a spacecraft picks up speed as it passes near a planet and exploits the planet's gravity. For instance, New Horizons, launched on January 19, 2006, enjoys a gravitational assist as it passed by Jupiter, February 27, 2007 on its long journey to Pluto. With the aid of that slingshot the speed of the spacecraft increased from 47,000 to 50,000 miles per hour (mph). As a result, it will arrive near Pluto in 2015, instead of 2018.

Before we see how this technique works, let's look at a simple situation on earth that illustrates the idea. Later we will replace the truck with a planet's gravitational field.

A playful lad throws a perfectly elastic tiny ball at 30 mph directly at a truck approaching him at 70 miles per hour, as shown in Figure C.16.1.

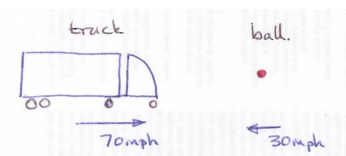



Figure C.16.1:

The truck driver sees the ball coming toward her at $70 + 30 = 100$ mph. The balls hits the windshield and, because the ball is perfectly elastic, the driver sees it bounce off at 100 mph in the opposite direction.

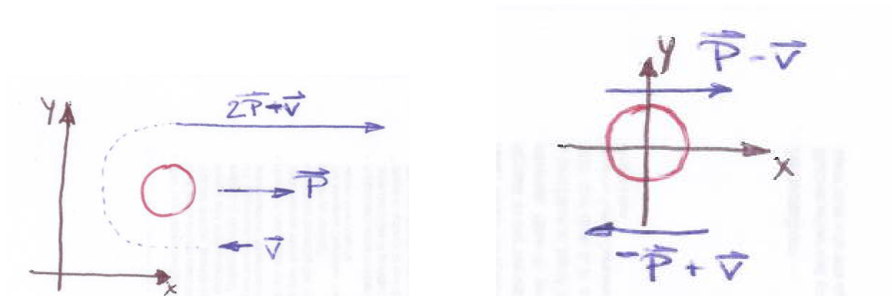
However, because the truck is moving in the same direction as the ball, the ball is moving through the air at $100 + 70 = 170$ mph as it returns to the boy. The ball has gained 140 mph, twice the speed of the truck.

Now, instead of picturing a truck, think of a planet whose velocity relative to the solar system is represented by the vector $\mathbf{P}$. A spacecraft, moving in the opposite direction with the velocity $\mathbf{v}$ relative to the solar system comes close to the planet.

An observer on the planet sees the spacecraft approaching with velocity $-vP + \mathbf{v}$. The spacecraft swings around the planet as gravity controls its orbit and sends it off in the opposite direction. Whatever speed it gained as it arrived, it loses as it exits. Its velocity vector when it exits is $-(-vP + \mathbf{v}) = \mathbf{P} - \mathbf{v}$, as viewed by the observer on the planet. Since the planet is moving through the solar system with velocity vector $\mathbf{P}$, the spacecraft is now moving through the solar system with velocity $\mathbf{P}+(\mathbf{P}-\mathbf{v}) = 2\mathbf{P}-\mathbf{v}$. See Figure C.16.2.

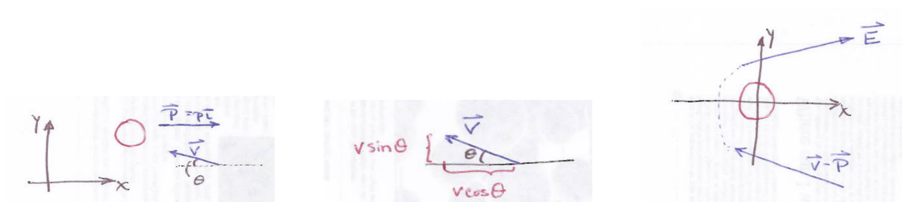
If $\mathbf{P} = 70\mathbf{i}$ and $\mathbf{v} = -30\mathbf{i}$, we have the vector $2(70\mathbf{i}) - (-30\mathbf{i}) = 170\mathbf{i}$, the case of the ball and truck.

But the direction of the spacecraft as it arrives may not be exactly opposite the direction of the planet. To treat the more general case, assume that $\mathbf{P} = p\mathbf{i}$, where $p$ is positive and $\mathbf{v}$ makes an angle $\theta$, $0 \leq \theta \leq \pi/2$, with $-\mathbf{i}$, as shown in Figure C.16.3(a). Let $v = |\mathbf{v}|$ be the speed of the spacecraft relative to the solar system. We will assume that the spacecraft's speed (relative to the planet) as it exits is the same as its speed relative to the planet on its arrival. (Figure C.16.3(b)) shows the arrival and exit vectors. Note that $\mathbf{E}$ and $\mathbf{v} - \mathbf{P}$




(a) (b)

Figure C.16.2: (a) The velocity vector relative to the solar system. (b) The velocity vector relative to the planet.




(a) (b) (c)

Figure C.16.3:

have the same $y$-components, but the $x$-component of $\mathbf{E}$ is the negative of the $x$-component of $\mathbf{v} - \mathbf{P}$.

Figure C.16.3(c) shows the arrival vector relative to the solar system. So, $\mathbf{v} = -w\cos(\theta)\mathbf{i} + v\sin(\theta)\mathbf{j}$.

Relative to the planet we have

$$\begin{array}{lrcl} \text{Arrival Vector:} & \mathbf{v} - \mathbf{P} & = & -p\mathbf{i} + (-v\cos(\theta)\mathbf{i} + v\sin(\theta)\mathbf{j}) \\ \text{Exit Vector:} & \mathbf{E} & = & p\mathbf{i} + v\cos(theta)\mathbf{i} + v\sin(\theta)\mathbf{j} \end{array}$$

The exit vector relative to the solar system, $\mathbf{E}$, is therefore

$$\mathbf{E} = (2p + v\cos(\theta))\mathbf{i} + v\sin(theta)\mathbf{j}.$$

The magnitude of $\mathbf{E}$ is

$$\sqrt{(2p + v\cos(\theta))^2 + (v\sin(theta))^2} = \sqrt{v^2 + 2pv\cos(\theta) + 4p^2}.$$

When $\theta = 0$, we have the case of the truck and ball or the planet and spacecraft in Figure C.16.2. Then $\cos(\theta) = 1$ and $|\mathbf{E}| = \sqrt{v^2 + 2pv + 4p^2} = v + 2p$, in agreement with our earlier observations.

The scientists controlling a slingshot carry out much more extensive calculations, which take into consideration the masses of the spacecraft and the planet, and involve an integration while the spacecraft is near the planet. Incidentally, the diameter of Jupiter is 86,000 miles.

"Near" in the case of the slingshot around Jupiter means 1.4 million miles. If the spacecraft gets too close, the atmosphere slows down or destroys the craft.

The gravity assist was proposed by Michael Minovitch in 1963 when he was still a graduate student at UCLA. Before then it was felt that to send a spacecraft to the outer solar system and beyond would require launch vehicles with nuclear reactors to achieve the necessary thrust.

# Calculus is Everywhere # 17
# How to Find Planets around Stars

Astronomers have discovered that other stars than the sun have planets circling them. How do they do this, given that the planets are too small to be seen? It turns out that they combine some vector calculus with observations of the star. Let us see what they do.

Imagine a star $S$ and a planet $P$ in orbit around $S$. To describe the situation, we are tempted to choose a coordinate system attached to the star. In that case the star would appear motionless, hence having no acceleration. However, the planet exerts a gravitational force $F$ on the star and the equation force = mass × acceleration would be violated. After introducing the appropriate mathematical tools, we will choose a proper coordinate system.

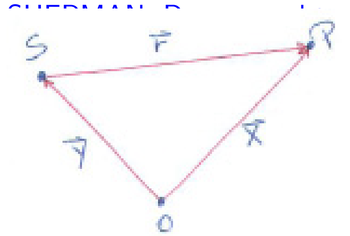



Figure C.17.1:

Let $\mathbf{X}$ be the position vector of the planet $P$ and $\mathbf{Y}$ be the position vector of the star $S$, relative to our inertial system. Let $M$ be the mass of the sun and $m$ the mass of planet $P$. Let $\mathbf{r} = \mathbf{X} - \mathbf{Y}$ be the vector from the star to the planet, as shown in Figure C.17.1.

The gravitational pull of the star on the planet is proportional to the product between them:

$$\mathbf{F} = \frac{-GmM\mathbf{r}}{r^3}.$$

Here $G$ is a universal constant, that depends on the units used to measure mass, length, time, and force. Equating the force with mass times acceleration, we have

$$M\mathbf{X}'' = \frac{-GmM\mathbf{r}}{r^3}.$$

Thus

$$\mathbf{X}'' = \frac{-Gm\mathbf{r}}{r^3}.$$

Similarly, by calculating the force that the planet exerts on the star, we have

$$\mathbf{Y}'' = \frac{Gm\mathbf{r}}{r^3}.$$

The center of gravity of the system consisting of the planet and the star, which we will denote $C$ (see Figure C.17.2), is given by

$$\mathbf{C} = \frac{M\mathbf{Y} + m\mathbf{X}}{M + m}.$$

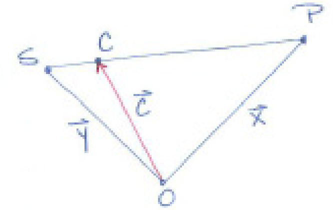



Figure C.17.2:

The center of gravity is much closer to the start than to the planet. In the case of our sun and Earth, the center of gravity is a mere 300 miles from the center of the sun.

The acceleration of the center of gravity is

$$\mathbf{C}'' = \frac{M\mathbf{Y}'' + m\mathbf{X}''}{M+m} = \frac{1}{M+m}\left(M\left(\frac{Gm\mathbf{r}}{r^3}\right) + m\left(\frac{-Gm\mathbf{r}}{r^3}\right)\right) = \mathbf{0}.$$

Because the center of gravity has $\mathbf{0}$-acceleration, it is moving at a constant velocity relative to the coordinate system we started with. Therefore a coordinate system rigidly attached to the center of gravity may also serve as an inertial system in which the laws of physics still hold.

We now describe the position of the star and planet to this new coordinate system. Star $S$ has the vector $\mathbf{x}$ from $\mathbf{C}$ to it and planet $P$ has the vector $\mathbf{y}$ from $\mathbf{C}$ to it, as shown in Figure C.17.3. Note that $\mathbf{r} = \mathbf{x} - \mathbf{y}$.

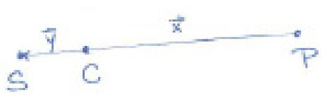



Figure C.17.3:

To obtain a relation between $\mathbf{x}$ and $\mathbf{y}$, we first express each in terms of $\mathbf{r}$. We have

$$\mathbf{y} = \mathbf{Y} - \vec{OC} = \mathbf{Y} - \frac{M\mathbf{Y} - m\mathbf{X}}{M+m} = \frac{m}{M+m}\mathbf{Y} + \frac{m}{M+m}\mathbf{X}.$$

Letting $k = m/M$, a very small quantity, we have

$$\mathbf{y} = \frac{k}{1+k}(\mathbf{Y} - \mathbf{X}) = \frac{-k}{1+k}\mathbf{r}. \tag{C.17.1}$$

Since $\mathbf{r} = \mathbf{x} - \mathbf{y}$, it follows that $\mathbf{x} = \mathbf{r} + \mathbf{y}$, hence

$$\mathbf{x} = \mathbf{r} + \left(\frac{-k}{1+k}\right)\mathbf{r} = \frac{1}{1+k}\mathbf{r}. \tag{C.17.2}$$

Combining (C.17.1) and (C.17.2) shows that

$$\boxed{\mathbf{y} = -k\mathbf{x}.} \tag{C.17.3}$$

SHERMAN: First use of "second inertial system;" what is the first?

Equation (C.17.3) tells us a good deal about the relation between the orbits of the star and planet in terms of the second inertial system:

1. The star and planet remain on opposite sides of $C$ on a straight line through $C$.

2. The star is always much closer to $C$ than the planet is.

3. The orbit of the star is similar in shape to the orbit of the planet, but smaller and reflected through $C$.

4. If the orbit of the star is periodic so is the orbit of the planet, and both have the same period.

Equation (C.17.3) is the key to the discover of planets around stars. The astronomers look for a star that "wobbles" a bit. That wobble is the sign that the star is in orbit around the center of gravity of it and some planet. Moveover, the time it takes for the planet to orbit the star is simply the time it takes for the star to oscillate back and forth once.

The reference cited below shows that the star and the planet sweep out elliptical orbits in the second coordinate system (the one relative to $C$).

Astronomers have found over two hundred stars with planets, some with several planets. A registry of these **exoplanets** is maintained at `http://exoplanets.org/`.

Reference: Robert Osserman, *Kepler's Laws, Newton's Laws, and the Search for New Planets*, Am. Math. Monthly **108** (2001), pp. 813–820.

## EXERCISES

**1.**[R] The mass of the sun is about 330,000 times that of Earth. The closest Earth gets to the sun is about 91,341,000 miles, and the farthest from it is about 94,448,000 miles. What is the closest the center of the sun gets to the center of gravity of the sun-Earth system? What is the farthest it gets from it? HINT: It lies within the sun itself.

**2.**[M] Find the condition that must be satisfied if the center of gravity of a sun-planet system will lie outside the sun.

SHERMAN: See `http://en.wikipedia.org/wiki/Center_of_mass`, particularly the animations at the end of the section on "Barycenter in astrophysics and astronomy".