DERIVATIVES

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
3. $\frac{d}{dx}(\sin(x)) = \cos(x)$
4. $\frac{d}{dx}(\cos(x)) = -sin(x)$
5. $\frac{d}{dx}(\tan(x)) = \sec^2(x)$
6. $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
7. $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
8. $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
9. $\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$
10. $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$
11. $\frac{d}{dx}(\text{arcsec}(x)) = \frac{1}{|x|\sqrt{x^2-1}}$
12. $\frac{d}{dx}(e^x) = e^x$
13. $\frac{d}{dx}(a^x) = a^x(\ln(a))$
14. $\frac{d}{dx}(\sinh(x)) = \cosh(x)$
15. $\frac{d}{dx}(\cosh(x)) = \sinh(x)$

ANTIDERIVATIVES

1. $\int x^n\ dx = \frac{1}{n+1}x^{n+1} \qquad n \neq -1$

   $\int \frac{dx}{x} = \ln(x),\ x > 0 \quad \text{or} \quad \ln|x|,\ x \neq 0$
2. $\int e^x\ dx = e^x$
3. $\int \sin(x)\ dx = -\cos(x)$
4. $\int \cos(x)\ dx = \sin(x)$
5. $\int \tan(x)\ dx = \ln|\sec(x)| = -\ln|\cos(x)|$
6. $\int \cot(x)\ dx = \ln|\sin(x)| = -\ln|\csc(x)|$
7. $\int \sec(x)\ dx = \ln|\sec(x)+\tan(x)| = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$
8. $\int \csc(x)\ dx = \ln|\csc(x)-\cot(x)| = \ln\left|\tan\left(\frac{x}{2}\right)\right|$
9. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$

10. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a} \arcsin\left(\frac{x}{a}\right),\ a > 0$$

11. $$\int \frac{dx}{|x|\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\frac{x}{a}\right)$$

Expressions Containing $ax + b$

12. $$\int (ax+b)^n\ dx = \frac{1}{a(n+1)}(ax+b)^{n+1}$$

13. $$\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b|$$

14. $$\int \frac{dx}{(ax+b)^2} = \frac{-1}{a(ax+b)}$$

15. $$\int \frac{x\ dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2} \ln|ax+b|$$

16. $$\int \frac{dx}{x(ax+b)} = \frac{1}{b} \ln\left|\frac{x}{ax+b}\right|$$

17. $$\int \frac{dx}{x^2(ax+b)} = \frac{-1}{bx} + \frac{a}{b^2} \ln\left|\frac{ax+b}{x}\right|$$

18. $$\int \sqrt{ax+b}\ dx = \frac{2}{3a}\sqrt{(ax+b)^3}$$

19. $$\int x\sqrt{ax+b}\ dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$$

20. $$\int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a}\sqrt{ax+b}$$

21. $$\int \frac{\sqrt{ax+b}}{x}\ dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$$

22. $$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln\left|\frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}}\right|,\ b > 0$$

23. $$\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}} \arctan\sqrt{\frac{ax+b}{-b}},\ b < 0$$

24. $$\int \frac{dx}{x^2\sqrt{ax+b}} = \frac{-\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$$

25. $$\int \sqrt{\frac{cx+d}{ax+b}}\ dx = \frac{\sqrt{ax+b}\sqrt{cx+d}}{a} + \frac{ad-bc}{2a}\int \frac{dx}{\sqrt{ax+b}\sqrt{cx+d}}$$

Expressions Containing $ax^2 + c$, $x^2 \pm p^2$, *and* $p^2 - x^2$, $p > 0$

26. $$\int \frac{dx}{p^2 - x^2} = \frac{1}{2p} \ln\left|\frac{p+x}{p-x}\right|$$

27. $$\int \frac{dx}{ax^2 + c} = \begin{cases} \frac{1}{\sqrt{ac}} \arctan\left(x\sqrt{\frac{a}{c}}\right) & a > 0,\ c > 0 \\ \frac{1}{2\sqrt{-ac}} \ln\left|\frac{x\sqrt{a}-\sqrt{-c}}{x\sqrt{a}+\sqrt{-c}}\right| & a > 0,\ c < 0 \\ \frac{1}{2\sqrt{-ac}} \ln\left|\frac{\sqrt{c}+x\sqrt{-a}}{\sqrt{c}-x\sqrt{-a}}\right| & a < 0,\ c > 0 \end{cases}$$

28. $$\int \frac{dx}{(ax^2 + c)^n} = \frac{1}{2(n-1)c} \frac{x}{(ax^2+c)^{n-1}} + \frac{2n-3}{2(n-1)c} \int \frac{dx}{(ax^2+c)^{n-1}} \qquad n > 1$$

29. $$\int x(ax^2 + c)^n\ dx = \frac{1}{2a} \frac{(ax^2+c)^{n+1}}{n+1} \qquad n \neq 1$$

30. $$\int \frac{x}{ax^2 + c}\ dx = \frac{1}{2a} \ln\left|ax^2 + c\right|$$

31. $$\int \sqrt{x^2 \pm p^2}\ dx = \frac{1}{2}\left(x\sqrt{x^2 \pm p^2} \pm p^2 \ln\left|x + \sqrt{x^2 \pm p^2}\right|\right)$$

32. $$\int \sqrt{p^2 - x^2}\ dx = \frac{1}{2}\left(x\sqrt{p^2 - x^2} + p^2 \arcsin\left(\frac{x}{p}\right)\right)$$

33. $$\int \frac{dx}{\sqrt{x^2 \pm p^2}} = \ln\left|x + \sqrt{x^2 \pm p^2}\right|$$

34. $$\int (p^2 - x^2)^{3/2}\ dx = \frac{x}{4}\left(p^2 - x^2)^{3/2}\right) + \frac{3p^2x}{8}\sqrt{p^2 - x^2} + \frac{3p^4}{8} \arcsin\left(\frac{x}{p}\right)$$

Expressions Containing $ax^2 + bx + c$

35. $$\int \frac{dx}{ax^2 + bx + c} = \begin{cases} \frac{1}{\sqrt{b^2-4ac}} \ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & b^2 > 4ac \\ \frac{2}{\sqrt{4ac-b^2}} \arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) & b^2 < 4ac \\ \frac{-2}{2ax+b} & b^2 = 4ac \end{cases}$$

36. $$\int \frac{dx}{(ax^2 + bx + c)^{n+1}} = \frac{2ax+b}{n(4ac-b^2)(ax^2+bx+c)^n} + \frac{2(2n-1)a}{n(4ac-b^2)} \int \frac{dx}{(ax^2+bx+c)^n}$$

37. $$\int \frac{x\ dx}{ax^2 + bx + c} = \frac{1}{2a} \ln\left|ax^2 + bx + c\right| - \frac{b}{2a} \int \frac{dx}{ax^2 + bx + c}$$

38. $$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln\left|2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c}\right| & a > 0 \\ \frac{1}{\sqrt{-a}} \arcsin\left(\frac{-2ax-b}{\sqrt{b^2-4ac}}\right) & a < 0 \end{cases}$$

39. $$\int \frac{x\ dx}{\sqrt{ax^2 + bx + c}} = \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

40. $\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Expressions Containing Powers of Trigonometric Functions

41. $\int \sin^2(ax)\ dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$

42. $\int \sin^3(ax)\ dx = \frac{-1}{a}\cos(ax) + \frac{1}{3a}\cos^3(ax)$

43. $\int \sin^n(ax)\ dx = -\frac{\sin^{(n-1)}(ax)\cos(ax)}{na} + \frac{n-1}{n}\int \sin^{(n-2)}(ax)\ dx,\ n \geq 2$ positive integer

44. $\int \cos^2(ax)\ dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}$

45. $\int \cos^3(ax)\ dx = \frac{1}{a}\sin(ax) - \frac{1}{3a}\sin^3(ax)$

46. $\int \cos^n(ax)\ dx = \frac{\cos^{(n-1)}(ax)\sin(ax)}{na} + \frac{n-1}{n}\int \cos^{(n-2)}(ax)\ dx$, $n \geq$ positive integer

47. $\int \tan^2(ax)\ dx = \frac{1}{a}\tan(ax) - x$

48. $\int \tan^3(ax)\ dx = \frac{1}{2a}\tan^2(ax) + \frac{1}{a}\ln|\cos(ax)|$

49. $\int \tan^n(ax)\ dx = \frac{\tan^{(n-1)}(ax)}{a(n-1)} - \int \tan^{(n-2)}(ax)\ dx,\ n \neq 1$

50. $\int \sec^2(ax)\ dx = \frac{1}{a}\tan(ax)$

51. $\int \sec^3(ax)\ dx = \frac{1}{2a}\sec(ax)\tan(ax) + \frac{1}{2a}\ln|\sec(ax)+\tan(ax)|$

52. $\int \sec^n(ax)\ dx = \frac{\sec^{(n-2)}(ax)\tan(ax)}{a(n-1)} - \frac{n-2}{n-1}\int \sec^{(n-2)}(ax)\ dx,\ n \neq 1$

53. $\int \frac{dx}{1 \pm \sin(ax)} = \mp\frac{1}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$

Expressions Containing Algebraic and Trigonometric Functions

54. $\int x\sin(ax)\ dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$

55. $\int x\cos(ax)\ dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$

56. $\displaystyle\int x^n \sin(ax)\, dx = \frac{-1}{a} x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax)\, dx \qquad n \text{ positive}$

57. $\displaystyle\int x^n \cos(ax)\, dx = \frac{1}{a} x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax)\, dx \qquad n \text{ positive}$

58. $\displaystyle\int \sin(ax)\cos(bx)\, dx = \frac{-\cos((a-b)x)}{2(a-b)} - \frac{\cos((a+b)x)}{2(a+b))} \qquad a^2 \neq b^2$

Expressions Containing Exponential and Logarithmic Functions

59. $\displaystyle\int x e^{ax}\, dx = \frac{1}{a^2} e^{ax}(ax-1)$

60. $\displaystyle\int x b^{ax}\, dx = \frac{1}{a^2} \frac{b^{ax}}{(\ln(b))^2} \left(a \ln(b) x - 1\right)$

61. $\displaystyle\int x^n e^{ax}\, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx$

62. $\displaystyle\int e^{ax} \sin(bx)\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx))$

63. $\displaystyle\int e^{ax} \cos(bx)\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos(bx) + b\sin(bx))$

64. $\displaystyle\int \ln(ax)\, dx = x\left(\ln(ax) - 1\right)$

65. $\displaystyle\int x^n \ln(ax)\, dx = x^{n+1}\left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2}\right) \qquad n = 0, 1, 2, \ldots$

66. $\displaystyle\int (\ln(ax))^2\, dx = x^2\left((\ln(ax))^2 - 2\ln(ax) + 2\right)$

Expressions Containing Inverse Trigonometric Functions

67. $\displaystyle\int \arcsin(x)\, dx = x \arcsin(ax) + \frac{1}{a}\sqrt{1-a^2x^2}$

68. $\displaystyle\int \arccos(x)\, dx = x \arccos(ax) - \frac{1}{a}\sqrt{1-a^2x^2}$

69. $\displaystyle\int \operatorname{arcsec}(x)\, dx = x \operatorname{arcsec}(ax) - \frac{1}{a}\ln\left|ax + \sqrt{a^2x^2-1}\right|$

70. $\displaystyle\int \operatorname{arccsc}(x)\, dx = x \operatorname{arccsc}(ax) + \frac{1}{a}\ln\left|ax + \sqrt{a^2x^2-1}\right|$

71. $\displaystyle\int \arctan(x)\, dx = x \arctan(ax) - \frac{1}{2a}\ln\left(1+a^2x^2\right)$

72. $\displaystyle\int \operatorname{arccot}(x)\, dx = x \operatorname{arccot}(ax) + \frac{1}{2a}\ln\left(1+a^2x^2\right)$

Some Special Integrals

73. $$\int_0^{\pi/2} \sin^n(x)\ dx = \int_0^{\pi/2} \cos^n(x)\ dx = \begin{cases} \frac{1\cdot 3\cdot 5\cdot 7\cdots(n-1)}{2\cdot 4\cdot 6\cdot 8\cdots(n)}\frac{\pi}{2} & n \text{ even} \\ \frac{2\cdot 4\cdot 6\cdot 8\cdots(n-1)}{1\cdot 3\cdot 5\cdot 7\cdots(n)} & n \text{ odd} \end{cases}$$

74. $$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$$

Calculus

Sherman Stein   Douglas Meade

August 31, 2009

Contents

Preface

As we wrote each section of this book, we kept in our minds an image of the student who will be using it. The student will be busy, taking other demanding classes besides calculus. Also that student may well need to understand the vector analysis chapter, which represents the culmination of the theory and applications within the covers of this book.

That image shaped both the exposition and the exercises in each section.

A section begins with a brief introduction. Then it quickly moves to an informal presentation of the central idea of the section, followed by examples. After the student has a feel for the core of the section, a formal proof is given.

Those proofs are what hold the course together and serve also as a constant review. For this reason we chose student-friendly proofs, adequately motivated. For instance, instead of the elegant, short proof that absolute convergence of a series implies convergence, we employed a longer, but more revealing proof. We avoid pulling tricks out of thin air; hence our new motivation of the cross product. Where one proof will do, we do not use two. Also, rather than proving the theorem in complete generality, we may treat only a special case, if that case conveys the flow of the general proof.

As we assembled the exercises we labeled them R (routine), M (medium), and C (challenging), to make sure we had enough of each type. The R- exercises focus on definitions and algorithmic calculations. The M-type require more thought. The C-type either demand a deeper understanding or offer an alternative view of the material.

In order to keep the sections as short as feasible, we concentrated on the mathematics. We avoided bringing in too many applications in the text, which would not only make the sections too long to be read by a busy student, but would not do justice to the applications. However, because the applications are the reason most students study the subject, each chapter concludes with a thorough treatment of an application in a section called "Calculus is Everywhere." Because each stands alone, students and instructors are free to deal with it as they please, depending on time available and interest: skip it, glance at it, browse through it, or read it carefully. The presence of the Calculus is Everywhere sections allowed us to replace exercises that start with a long description of an application and end with a trivial bit of calculus. Our guiding theme is do one thing at a time, whether it's exposition, an example, or an exercise.

As we worked on each section we asked ourselves several questions: Is it the right length? Does it get to the point quickly? Does it focus on just one idea and correspond to one lecture? Are there enough examples? Are there enough exercises, from routine to challenging?

Curvature is treated twice, first in the plane, without vectors, and later, in space, with vectors. We do this for two reasons. First, it provides the student background for appreciating the vector approach. Second, it reduces the vector treatment section to a reasonable length.

Many students will use vector analysis in engineering and physics courses. One of us sat in on a sophomore level electromagnetic course in order to find out how the concepts were applied and what was expected of the students. That inspired a major revision of that chapter.

In addition, the new edition reaches limits and derivatives as early as possible, and as simply as possible. Also, we introduce the Permanence Property, which implies that a continuous function that is positive at a number remains positive nearby. This is referred to several times; hence we gave it a name.

The controversy about what to do about epsilon-delta proofs will never end. Therefore in our text the instructor is free to choose what to do about such proofs. To make our treatment student-friendly, we broke it into two sections. The first section treats limits at infinity because the diagrams are easier and the concept is more accessible. The second deals with limits at a number. A rigorous proof is given there of the Permanence Property, illustrating the power of the epsilon-delta approach to demonstrate something that is not intuitively obvious. Later in the book the rigorous approach appears only in some C-level exercises, giving the instructor and student an opportunity to reinforce that approach if they so choose.

Throughout the book we include exercises that ask only for computing a derivative or an integral. These exercises are intended to keep those skills sharp. We do not want to assign to exercises that explore a new concept the additional responsibility of offering extensive practice in calculations. This illustrates our general principle: do only one thing at a time, and do it clearly.

One of our objectives was to develop throughout the chapters the mathematical maturity a student needs to understand the vector analysis in the final chapter. For instance, we often include an exercise which asks the student to state a theorem in their own words without mathematical symbols. We had found while doing some pro-bono tutoring that students do not read a theorem carefully. No wonder they didn't know what to do when a supposedly routine exercise asked them to verify a theorem in a particular case.

Notes to the Instructor

§1.1 A review and a reference. It gets right to the point. The examples provide background for later work. Exercises 35 to 39 bring in the transcendental functions early.

§1.2 Reinforces the exponential and logarithmic functions early and its summary emphasizes the most difficult functions, logarithms. We save "modeling" for later, abiding by our principle, "one section, one main idea." Exercise 52 asks students to think on their own, to be ready for the last third of the book.

§1.3 Quickly builds all the functions needed. We do this for two reasons: to give the students more time to deal with them and to have them available for examples and exercises.

Following our policy of doing just one thing at a time, we develop limits in Chapter 2, separating them from their application in Chapter 3, which introduces the derivative.

§2.1 Focuses on the basic limits needed in Chapter 3. The binomial theorem is not used because many students are not familiar or comfortable with it.

§2.4 Introduces the Permanence Principle, which is used several times in later chapters. Hence, we give it a name.

§2.6 Chapter summaries offer an overall perspective and emphasis not possible in an individual section.

§3.1 Introduces the derivative in the traditional way, by velocity and the tangent line. Because of the earlier development of the key limits, this section can be kept short.

§3.3 By using the Δ-notation, we obtain the derivatives of $f + g$, fg, and f/g without using any "student unfriendly" tricks, such as adding and subtracting $f(x)g(x)$.

§3.4 The rigorous proof of the chain rule is left as an exercise with detailed sketch. That enables the student reading the text to concentrate on learning how to apply the chain rule.

§3.5 Obtains the derivatives of the inverse functions, using the chain rule. There is no need to wait until implicit differentiation is discussed. That way the chapter can focus on obtaining the differentiation formulas. Exercises **??** and **??** are two of the "Sam and Jane" exercises that add a light touch and invite the students to think on their own.

§3.6 Introduces antiderivatives well before the definite integral appears in Chapter 6, so that the two concepts are adequately separated in time. Slope fields will be used later.

§3.7 Note that the higher derivatives will be put to work as early as Section 5.4, which concerns Taylor polynomials.

§§3.8 and 3.9 We delayed the precise definitions of limits in order to give the students more time to work with limits before facing these definitions. These sections are optional. Section 3.8 is easier. One may separate the two sections by several days to let the first one sink in. Note that Example 2 in Section 3.9 shows how useful a precise definition is, as it justifies the Permanence Principle.

§3.10 Emphasizes the essentials and invites more practice in differentiation. Throughout the remaining chapters we include exercises on straightforward differentiation.

Chapter 4 Concentrates on just one theme: using f' and f'' to graph a function. This provides a strong foundation for Chapter 5, which includes optimization.

§5.3 Shows how a higher derivative influences the growth of a function and sets the stage for Section 5.4, Taylor polynomials and their errors. The growth theorem of Section 5.3 is used in exercises in Chapter 6 to obtain the error in approximating a definite integral by the trapezoidal or Simpson's methods.

§5.6 Exercise 39 raises interesting questions about exponential growth.

§6.1 This section keeps to a readable length by avoiding involvement with a formula for the sum $1^2 + 2^2 + \cdots + n^2$.

§6.2 Anticipates the formula $F(b) - F(a)$ for evaluating a definite integral.

§6.5 Exercises such as 43 and 44 are not as hard as one would expect, because the steps are outlined. Such exercises review several important concepts.

Overview of Calculus I

There are two main concepts in calculus: the derivative and the integral. Two scenarios that could occur in your car introduce both concepts.

> **Scenario A**
> Your speedometer is broken, but your odometer works. Your passenger writes down the odometer reading every second. How could you estimate the speed, which may vary from second to second?

The speedometer measures your current speed. The odometer measures the total distance covered.

This scenario is related to the "derivative," the key concept of differential calculus. The derivative tells how rapidly a quantity changes if we know how much of it there is at any instant. (If the change is at a constant rate, the rate of change is just the total change divided by the total time, and no derivative is needed.)

The second scenario is the opposite.

> **Scenario B**
> Your odometer is broken, but your speedometer works. Your passenger writes down the speed every second. How could you estimate the total distance covered?

This scenario is related to the "definite integral," the key concept of integral calculus. This integral represents the total change in a varying quantity, if you know how rapidly it changes — even if the rate of change is not constant. (If the speed stays constant, you just multiply the speed times the total time, and no integral is needed.)

Both the derivative and the integral are based on limits, treated in Chapter 2. Chapter 3 defines the derivative, while Chapters 4 and 5 present some of its applications. Chapter 6 defines the integral.

As you would expect by comparing the two scenarios, the derivative and the integral are closely related. This connection is the basis of the Fundamental Theorem of Calculus (Section 6.4), which shows how the derivative provides a shortcut for computing many integrals.

Chapter 1

Pre-Calculus Review of Functions

Calculus is the study of functions. To understand the concepts introduced in this text, it is important to have a solid understanding of functions.

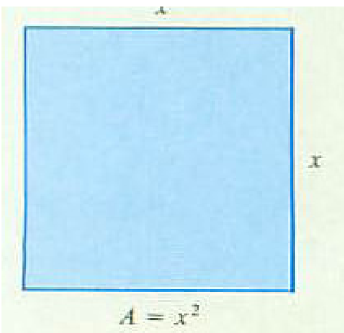
We begin this chapter with a review in Section 1.1 of the terminology for functions that will be used for the remainder of this book. In Section 1.2 fundamental types of functions are reviewed: power functions, exponentials, logarithms, and the trigonometric functions. Section 1.3 describes how functions can be combined to create new functions.

1.1 Functions

This section reviews several ideas related to functions: piecewise-defined function, one-to-one function, inverse function, and increasing or decreasing functions.

Definition of a Function

The area A of a square depends on the length of its side x and is given by the formula $A = x^2$. (See Figure 1.1.1.)



Similarly, the distance s (in feet) that a freely falling object drops in the first t seconds is described by the formula $s = 16t^2$. Each choice of t determines a specific value for s. For instance, when $t = 3$ seconds, $s = 16 \cdot 3^2 = 144$ feet.

Both of these formulas illustrate the notion of a function.

> **DEFINITION** *(Function.)* Let X and Y be sets. A **function** from X to Y is a rule (or method) for assigning one (and only one) element in Y to each element in X.

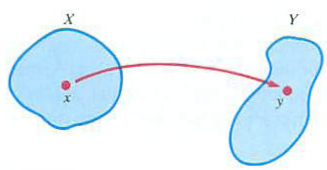


Figure 1.1.2:

The notion of a function is illustrated in Figure 1.1.2, where the element y in Y is assigned to the element x in X. Usually X and Y will be sets of numbers.

A function is often denoted by the symbol f. The element that the function assigns to the element x is denoted $f(x)$ (read "f of x"). In practice, though, almost everyone speaks interchangeably of the function f or the function $f(x)$.

If $f(x) = y$, x is called the **input** or **argument** and y is called the **output** or **value** of the function at x. Also, x is called the **independent variable** and y the **dependent variable**.

A function may be given by a formula, as in the function $A = x^2$. Because A depends on x, we say that "A is a function of x." Because A depends on only one number, x, it is called a function of a single variable. The first thirteen chapters concern functions of a single variable. The area A of a rectangle depends on its length l and width w; it is a function of *two* variables, $A = lw$. The last five chapters extend calculus to functions of more than one variable and to other functions.

Ways to write and talk about a function

The function that assigns to each argument x the value x^2 is usually described in a shorthand. For instance, we may write $x \mapsto x^2$ (and say "x goes to x^2" or "x is mapped to x^2"). Or we may say simply, "the formula x^2", "the function x^2", or, sometimes, just "x^2." Using this abbreviation, we might say, "How does x^2 behave when x is large?" Some people object to the shorthand

"x^2" because they fear that it might be misinterpreted as the number x^2, with no sense of a general assignment. In practice, the context will make it clear whether x^2 refers to a number or to a function.

EXAMPLE 1 Consider a circle of radius a, as shown in Figure 1.1.3. Let $f(x)$ be the length of chord AB of this circle at a distance x from the center of the circle. Find a formula for $f(x)$.

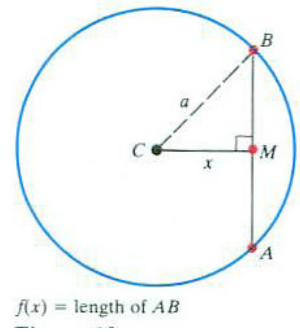


Figure 1.1.3:

SOLUTION We are trying to find how the length $\overline{AB}$ varies as x varies. That is, we are looking for a *formula* for $\overline{AB}$, the length of AB, in terms of x.

Before searching for the formula, it is a good idea to calculate $f(x)$ for some easy inputs. These values can serve as a check on the formula we work out.

In this case $f(0)$ and $f(a)$ can be read at a glance at Figure 1.1.3: $f(0) = 2a$ and $f(a) = 0$. (Why?) Now let us find $f(x)$ for all x in $[0, a]$.

Let M be the midpoint of the chord AB and let C be the center of the circle. Observe that $\overline{CM} = x$ and $\overline{CB} = a$. By the Pythagorean theorem, $\overline{BM} = \sqrt{a^2 - x^2}$. Hence $\overline{AB} = 2\sqrt{a^2 - x^2}$. Thus

$$f(x) = 2\sqrt{a^2 - x^2}.$$

Does the formula give the correct values at $x = 0$ and $x = a$? ◇

Domain and Range

The set of permissible inputs and the set of possible outputs of a function are an essential part of the definition of a function. These sets have special names, which we now introduce.

> **DEFINITION** *(Domain and range)* Let X and Y be sets and let f be a function from X to Y. The set X is called the **domain** of the function. The set of all outputs of the function is called the **range** of the function.

The range is not necessarily all of Y.

When the function is given by a formula, the domain is usually understood to consist of all the numbers for which the formula is defined.

In Example 1 the domain is the closed interval $[0, a]$ and the range is the closed interval $[0, 2a]$. (For interval notation see Appendix A.)

When using a calculator you must pay attention to the domain corresponding to a function key or command. If you enter a negative number as x and press the $\sqrt{x}$-key to calculate the square root of x your calculator will not be happy. It might display an E for "error" or start flashing, the calculator's standard signal for distress. Your error was entering a number not in the domain of the square root function.

Try it. What does your calculator do? Some advanced calculators go into "complex number" mode to handle square roots of negative numbers.

Try it. *No* calculator, however advanced, can permit division by zero.

You can also get into trouble if you enter 0 and press the $1/x$-key. The domain of $1/x$, the reciprocal function, consists of all numbers — except 0.

Graph of a Function

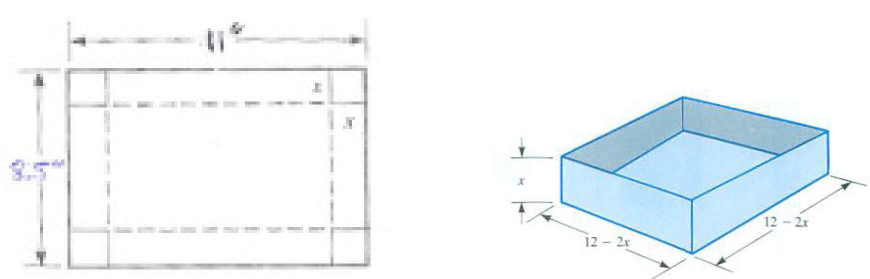
In case both the inputs and outputs of a function are numbers, we can draw a picture of the function, called its **graph**.

> **DEFINITION** *(Graph of a function)* Let f be a function whose inputs and output are numbers. The **graph** of f consists of those points (x, y) in the xy-plane such that $y = f(x)$.

The next example illustrates the usefulness of a graph. We will encounter this function again in Chapter 4.

EXAMPLE 2 A tray is to be made from a rectangular piece of paper by cutting congruent squares from each corner and folding up the flaps. The dimensions of the rectangle are $8\frac{1}{2}'' \times 11''$. Find how the volume of the tray depends on the size of the cutout squares.

SOLUTION Let the side of the cutout square be x inches, as shown in Figure 1.1.4(a). The resulting tray is shown in Figure 1.1.4(b).



(a) (b)

Figure 1.1.4: (a) A rectangular sheet with a square cutout from each corner. (b) The tray formed when the sides are folded up.

The volume $V(x)$ of the tray is the height, x, times the area of the base $(11 - 2x)(8.5 - 2x)$,

$$V(x) = x(11 - 2x)(8.5 - 2x). \qquad (1.1.1)$$

The domain of V contains all values of x that lead to an actual tray. This means that x cannot be negative, and x cannot be more than half of the shortest side.

Thus, the largest corners that can be cut out have sides of length 4.25". So, for this tray problem, the domain of interest is only the interval $[0, 4.25]$. Note the peculiar trays that are obtained when $x = 0$ or $x = 4.25$. What are their volumes?

Of course we are free to graph (1.1.1) viewed simply as a polynomial whose domain is $(-\infty, \infty)$.

A short table of inputs and corresponding outputs will help us to sketch the graph. Figure 1.1.5 displays the graph of $V(x)$.

x (in)	-1	0	1	2	3	4	4.25	5	6
$V(x)$ (in^3)	-136.5	0	5.85	63	37.5	6	0	-7.5	21

Table 1.1.1:

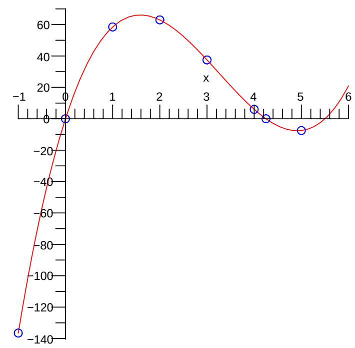


Figure 1.1.5:

When $11 - 2x = 0$, that is, when $x = \frac{11}{2} = 5.5$, $V(x) = 0$. When x is greater than $\frac{11}{2}$ all three factors in the formula for $V(x)$ are positive, and $V(x)$ becomes very large for large values of x.

For negative x, two factors in (1.1.1) are positive and one is negative. Thus $V(x)$ is negative and has large absolute value for negative inputs of large absolute value.

Which factor is negative?

Only the part of the graph above the interval $[0, 4.25]$ is meaningful in the tray problem. All other values of x have nothing to do with trays. ◇

If you want to test whether some curve drawn in the xy-plane is the graph of a function, check that each vertical line meets the curve no more than once. If the vertical line $x = a$ meets the curve twice, say at (a, b) and (a, c), there would be the two outputs b and c for the single input a.

Vertical Line Test

The input a is in the domain of f if and only if the vertical line $x = a$ intersects the graph of $y = f(x)$ exactly once. Otherwise, a is not in the domain of f.

For example, Figure 1.1.6 shows a graph that does not pass the vertical line test. The input-output table corresponding to this graph would have three entries for each input x between -2 and 2, two entries for $x = -2$ and $x = 2$ and exactly one entry for each input $x < -2$ or $x > 2$.

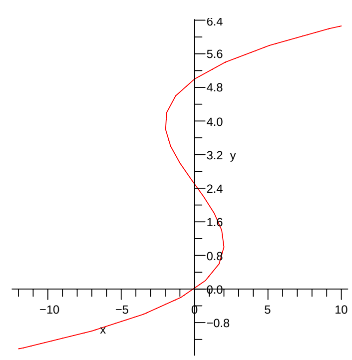


Figure 1.1.6:

In Example 2 the function is described by a single formula, $V(x) = x(11 - 2x)(8.5 - 2x)$. But a function may be described by different formulas for different intervals or individual points in its domain, as in the next example.

EXAMPLE 3 A hollow sphere of radius a has mass M, distributed uniformly throughout its surface. Describe the gravitational force it exerts on a particle of mass m at a distance r from the center of the sphere.

SOLUTION Let $f(r)$ be the force at a distance r from the center of the sphere. In an introductory physics course it is shown that the sphere exerts no force at all on objects in the interior of the sphere. Thus for $0 \le r < a$, $f(r) = 0$.

The sphere attracts an external particle as though all the mass of the sphere were at its center. Thus, for $r > a$, $f(r) = G\frac{Mm}{r^2}$, where G is the gravitational constant, which depends on the units used for measuring length, time, mass, and force.

It can be shown by calculus that for a particle on the surface, that is, for $r = a$ the force is $G\frac{Mm}{2a^2}$. The graph of f is shown in Figure 1.1.7. ⋄

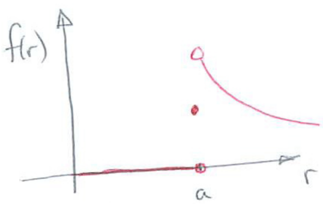


Figure 1.1.7:

The formula describing the function in Example 3 changes for different parts of its domain.

$$f(r) = \begin{cases} 0 & \text{if } 0 \le r < a \\ \frac{GMm}{2a^2} & \text{if } r = a \\ \frac{GMm}{r^2} & \text{if } r > a \end{cases}$$

Such a function is called a **piecewise-defined function**.

In a graph that consists of several different pieces, such as Figure 1.1.7, the presence of a point on the graph of a function is indicated by a solid dot (•) and the absence of a point by a hollow dot (◦).

Inverse Functions

If you know a particular output of the function $f(x) = x^3$ you can figure out what the input must have been. For instance, if $x^3 = 8$, then $x = 2$ – you can go backwards from output to input. However, you cannot do this with the function $f(x) = x^2$. If you are told that $x^2 = 25$, you do not know what x is. It can be 5 or -5. However, if you are told that $x^2 = 25$ and that x is positive, then you know that x is 5.

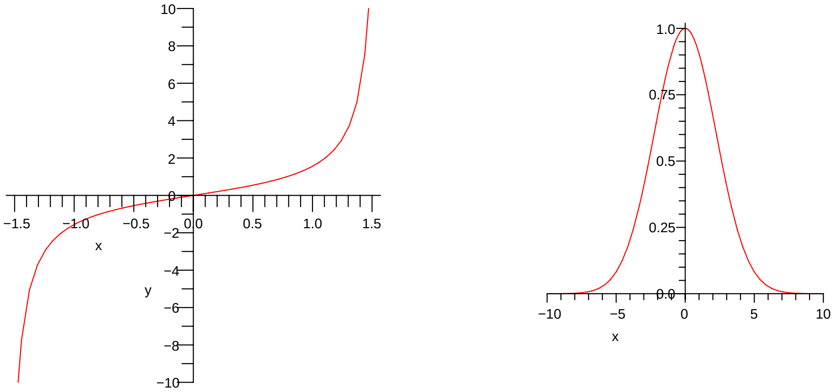
This brings us to the notion of a one-to-one function.

DEFINITION *(One-to-One Function)* A function f that does not assign the same output to two different inputs is **one-to-one**. That is, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

DEFINITION *(Inverse Function)* If f is a one-to-one function, the **inverse function** is the function g that assigns to each output of f the corresponding input. That is, if $f(x) = y$ then $g(y) = x$.

Horizontal Line Test

The graph of a one-to-one function never meets a horizontal line more than once. (See Figure 1.1.8.)



(a) (b)

Figure 1.1.8: The function in (a) is one-to-one as it passes the horizontal line test. The function in (b) does not pass the horizontal line test, so is not one-to-one.

The function $f(x) = x^3$ is one-to-one on the entire real line. A few entries in the tables for $f(x)$ and its inverse function are shown in Table 1.1.2(a) and (b), respectively.

input	1	2	$\frac{1}{2}$	3	-2
output	1	8	$\frac{1}{8}$	27	-8

(a)

input	1	8	$\frac{1}{8}$	27	-8
output	1	2	$\frac{1}{2}$	3	-2

(b)

Table 1.1.2: (a) Table of input and output value for $f(x) = x^3$. (b) Table of input and output values for the inverse of $f(x) = x^3$.

In this case an explicit formula for the inverse function can be found algebraically: if $y = x^3$ then $y^{1/3} = (x^3)^{1/3} = x$. Then $x = y^{1/3}$. Since it is customary to use the x-axis for the input and the y-axis for the output, it is convenient to rewrite $x = y^{1/3}$ as $y = x^{1/3}$. (Both say the same thing: "The output is the cube root of the input.".)

By the way, an inverse of a one-to-one function may not be given by a nice formula. For instance, $f(x) = 2x + \cos(x)$ is one-to-one, as will be easily shown in Chapter 4. However, the inverse function is not described by a convenient formula. Happily, we do not need to deal with an explicit formula for this particular inverse function.

The Graph of an Inverse Function

Suppose you know the graph of a one-to-one function. Then there is an easy way to draw the graph of the inverse function.

If (a, b) is a point on the graph of the function f, that is, $b = f(a)$, then (b, a) is a point on the graph of inv$f$, shown in Figure 1.1.9.

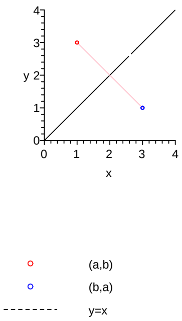
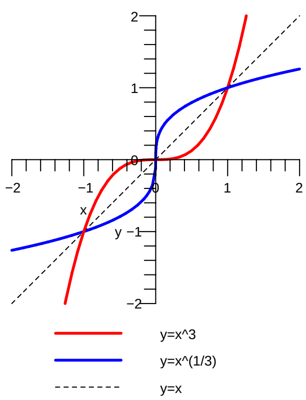


Figure 1.1.9: The point (b, a) is obtained by reflecting (a, b) around the line $y = x$.

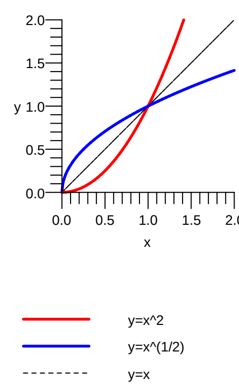
> Notation: The use of inv$f$ to denote the inverse function of f is based on the fact that many calculators have a button marked inv to indicate the inverse of a function. The mathematical notation for the inverse function of f is f^{-1} or inv$f$. *Note that the* -1 *is not an exponent, and in general the inverse and reciprocal functions are different:* f^{-1} *is not equal to* $\frac{1}{f}$.

EXAMPLE 4 Draw the graphs of (a) the inverse of the cubing function given by $f(x) = x^3$, and (b) the squaring function $g(x) = x^2$ restricted to $x \geq 0$.

SOLUTION See Figure 1.1.10. ◇



(a)



(b)

Figure 1.1.10: (a) Plots of $f(x) = x^3$ and $f^{-1}(x) = x^{1/3}$. (b) Plots of $g(x) = x^2$ $(x \geq 0)$ and $g^{-1}(x) = \sqrt{x}$.

EXAMPLE 5 Let $m \neq 0$ and b be constants and $f(x) = mx + b$. Show that f is one-to-one and describe its inverse function.

SOLUTION If $f(x_1) = f(x_2)$ we have

$$\begin{aligned} mx_1 + b &= mx_2 + b & \\ mx_1 &= mx_2 & \text{subtract } b \text{ from both sides} \\ x_1 &= x_2 & \text{divide both sides by } m \neq 0 \end{aligned}$$

Because $f(x_1) = f(x_2)$ only when $x_1 = x_2$, f is one-to-one.

This problem can also be analyzed graphically. The graph of $y = f(x)$ is the line with slope m and y-intercept b. (See Figure 1.1.11.) It passes the horizontal line test.

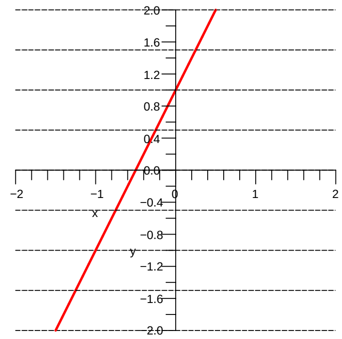


Figure 1.1.11:

To find the inverse function, solve the equation $y = f(x)$ to express x in terms of y:

$$\begin{aligned} y &= mx + b & \\ y - b &= mx & \text{subtract } b \text{ from both sides} \\ \tfrac{y-b}{m} &= x & \text{divide by } m \neq 0 \\ x &= \tfrac{y}{m} - \tfrac{b}{m} & \text{move } x \text{ to left-hand side} \\ y &= \tfrac{x}{m} - \tfrac{b}{m} & \text{interchange } x \text{ and } y. \end{aligned}$$

Reversing the roles of x and y in the final step is done only to present the inverse function in a form where the input is called x and the output is called y. Thus the inverse function has the formula

$$f^{-1}(x) = \frac{x}{m} - \frac{b}{m}.$$

The graph of the inverse function is also a line; its slope is $1/m$, the reciprocal of the slope of the original line, and its y-intercept is $-b/m$. (See Figure 1.1.12.) ◇

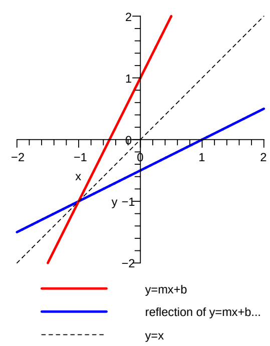


Figure 1.1.12:

Decreasing and Increasing Functions

There is another way to check whether a function is one-to-one on an interval. It uses the following concepts.

A function is **increasing** on an interval if whenever x_1 and x_2 are in the interval and x_2 is greater than x_1, then $f(x_2)$ is greater than $f(x_1)$. As you move along the graph of f from left to right, you go up. This is shown in Figure 1.1.14(a).

In the case of a **decreasing** function, you go down as you move from left to right: if $x_2 > x_1$ then $f(x_2) < f(x_1)$. (See Figure 1.1.14(b).)

For instance, consider $f(x) = \sin(x)$, whose graph is shown in Figure 1.1.13. On the interval $[-\pi/2, \pi/2]$ the values of $\sin(x)$ increase. On the interval

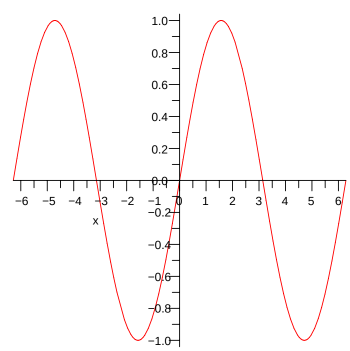


Figure 1.1.13:

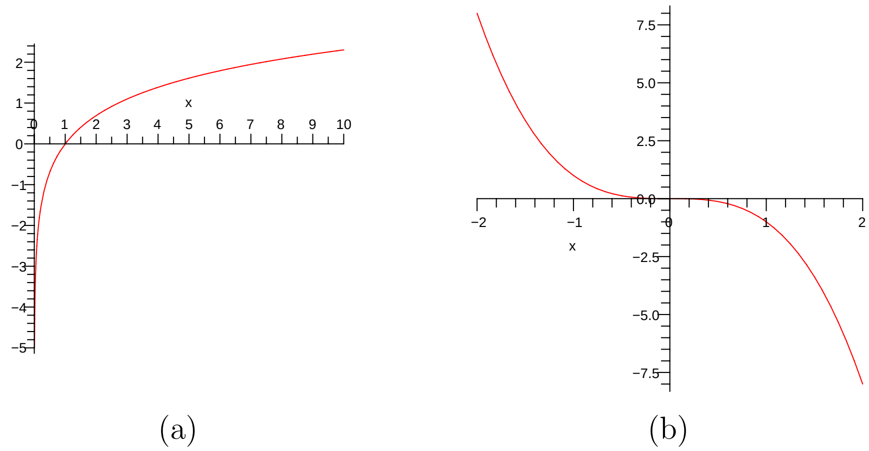


Figure 1.1.14: Graph of (a) an increasing function and (b) a decreasing function.

$[\pi/2, 3\pi/2]$ the values of $\sin(x)$ decrease. The function x^3 increases on its entire domain $(-\infty, \infty)$.

A **monotonic** function is a function that is either only increasing or only decreasing. A monotone function always passes the Horizontal Line Test, as the next example illustrates.

Monotone is from the Greek, *mono*=single, *tonos*=tone, which also gives us the word 'monotonous').

EXAMPLE 6 For $k \neq 0$ and $x > 0$, x^k is a monotonic function. The inverse of x^k is $x^{1/k}$. If $k = 0$, we have a constant function, $x^0 = 1$. The constant function does not pass the Horizontal Line Test; therefore it has no inverse. ◇

Because strict inequalities are used in the definitions of increasing and decreasing, we sometimes say these functions are *strictly increasing* or *strictly decreasing* on an interval. A function f is said to be **non-decreasing** on an interval if whenever x_1 and x_2 are in the interval and x_2 is greater than x_1, then $f(x_2) \geq f(x_1)$. The graph of a non-decreasing function is increasing except on intervals where it is constant. Likewise, f is **non-increasing** on an interval if whenever x_1 and x_2 are in the interval and x_2 is greater than x_1, then $f(x_2) \leq f(x_1)$.

The sign of a function's outputs provides another way to describe a function. A function that has only positive outputs is called a **positive function**; for instance, 2^x. A **negative function** has only negative outputs; for instance, $\frac{-1}{1+x^2}$. A **non-negative function** has outputs that are either positive or zero; for instance x^2. The outputs of a **non-positive function** are either negative or zero, for instance, $\sin(x) - 1$.

Summary

This section introduced concepts that will be used throughout the coming chapters: function, domain, range, graph, piecewise-defined function, one-to-one functions, inverse functions, increasing functions, decreasing functions, monotonic functions, non-decreasing functions, non-increasing functions, positive functions, negative functions, non-negative functions, and non-positive functions.

Two important observations are that every monotone function has an inverse function and the graph of the inverse function is the reflection across the line $y = x$ of the graph of the original function.

A function can be described in several ways: by a formula, such as $V(x) = x(11 - 2x)(8.5 - 2x)$, by a table of values, or by words, such as "the volume of a tray depends on the size of the cut-out square."

EXERCISES for 1.1

Key: R–routine, M–moderate, C–challenging

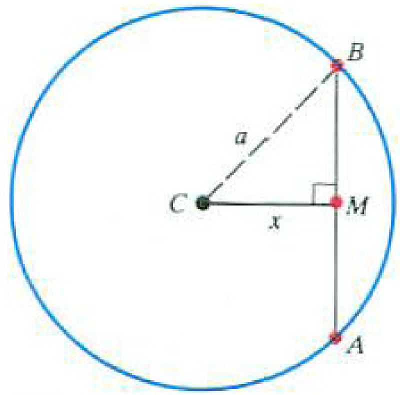


Figure 1.1.15: Exercises 1 to 4. ARTIST: Please add θ to denote the angle BCM.

Exercises 1 to 4 refer to Figure 1.1.15.

1.[R] Express the area of triangle ABC as a function of $x = \overline{CM}$

2.[R] Express the perimeter of triangle ABC as a function of x.

3.[R] Express the area of triangle ABC as a function of θ.

4.[R] Express the perimeter of triangle ABC as a function of θ.

In Example 2 a tray was formed from an $8\frac{1}{2}$" by 11" rectangle by removing squares from the corners. Find and graph the corresponding volume function for trays formed from sheets with the dimensions given in Exercises 5 to 8.

5.[R] 4" by 13"

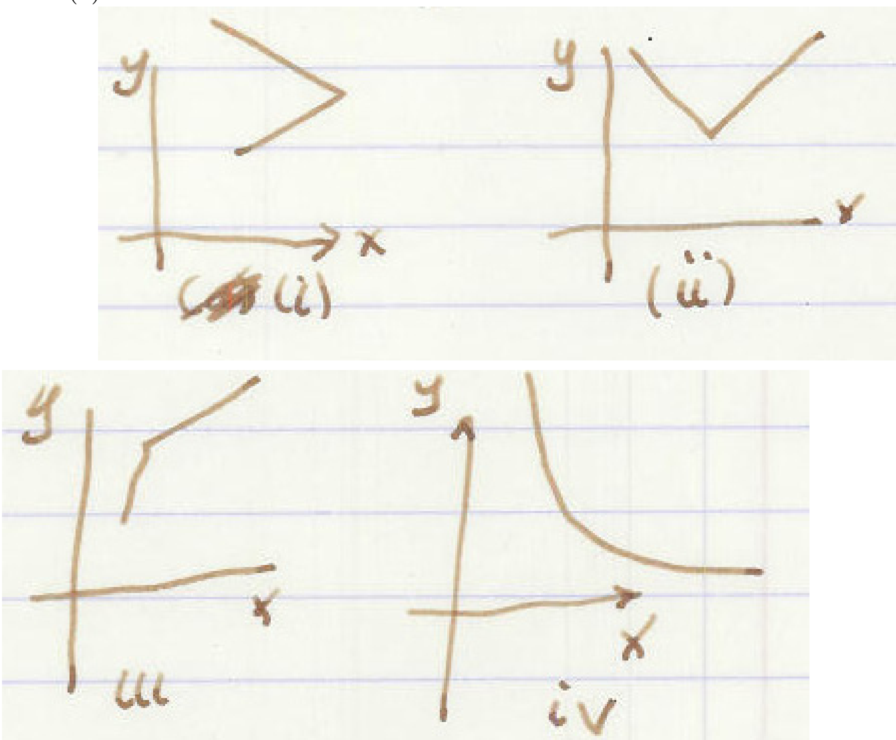
6.[R] 5" by 7"

7.[R] 6" by 6"

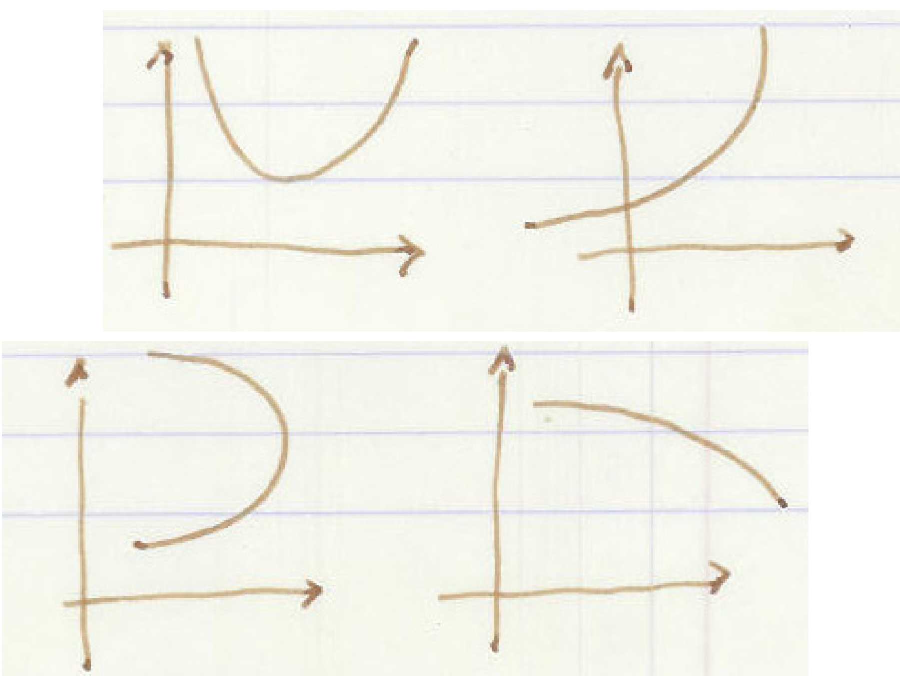
8.[R] 5" by 5"

In Exercises 9 and 10 decide which curves are graphs of (a) functions, (b) increasing functions, and (c) one-to-one functions.

9.[R]



10.[R]



11.[R] Let $f(x) = x^3$.

(a) Fill in this table

x	0	1/4	1/2	$-1/4$	$-1/2$	1	2
x^3							

(b) Graph f.

(c) Use the table in (a) to find seven points on the graph of f^{-1}.

(d) Graph f^{-1} (use the same axes as in (b)).

12.[R] Let $f(x) = \cos(x)$, $0 \leq x \leq \pi$ (angles in radians).

(a) Fill in this table

x	0	$\pi/6$	$\pi/4$	$2\pi/3$	$\pi/2$	$3\pi/4$	π
$\cos(x)$							

(b) Graph f.

(c) Use the table in (a) to find seven points on the graph of $\mathsf{inv}\cos$.

(d) Graph $\mathsf{inv}\cos$ (use the same axes as in (b)).

In Exercises 13 to 18 the functions are one-one. Find the formula for each inverse function, expressed in the form $y = g(x)$, so that the independent variable is labeled x. NOTE: If you have trouble with the use of logarithms in Exercise 17 or Exercise 18, read Appendix D.

13.[R] $y = 3x - 2$

14.[R] $y = x/2 + 7$

15.[R] $y = x^5$

16.[R] $y = 3\sqrt{x}$

17.[R] $y = 3^x$

18.[R] $y = 5(2^x)$

In Exercises 19 to 23 the slope of line L is given. Let L' be the reflection of L across the line $y = x$. What is the slope of the reflected line, L'? In each case sketch a possible L and its reflection, L'.

19.[R] L has slope 2.

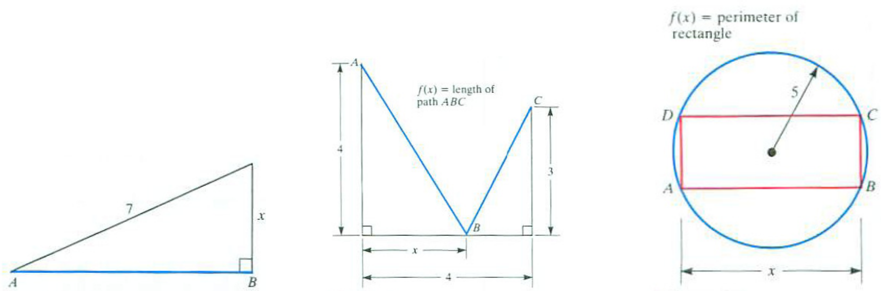
20.[R] L has slope 1.

21.[R] L has slope $1/10$.

22.[R] L has slope $-1/3$.

23.[R] L has slope -2.

In Exercises 24 to 33 state the formula for the function f and give the domain of the function.



(a) (b) (c)

Figure 1.1.16:

24.[R] $f(x)$ is the perimeter of a circle of radius x.

25.[R] $f(x)$ is the area of a disk of radius x.

26.[R] $f(x)$ is the perimeter of a square of side x.

27.[R] $f(x)$ is the volume of a cube of side x.

28.[R] $f(x)$ is the total surface area of a cube of side x.

29.[R] $f(x)$ is the length of the hypotenuse of the right triangle whose legs have lengths 3 and x.

30.[M] $f(x)$ is the length of the side AB in the triangle in Figure 1.1.16(a).

31.[M] For $0 \leq x \leq 4$, $f(x)$ is the length of the path from A to B to C in Figure 1.1.16(b).

32.[M] For $0 \leq x \leq 10$, $f(x)$ is the perimeter of the rectangle $ABCD$, one side of which has length x, inscribed in the circle of radius 5 shown in Figure 1.1.16(c).

33.[C] A person at point A, two miles from shore in a lake, is going to swim to the shore ST and then walk to point B, five miles from the shore. She swims at 1.5 miles per hour and walks at 4 miles per hour. If she reaches the shore at point P, x miles from S, let $f(x)$ denote the time for her combined swim and walk. Obtain a formula for $f(x)$. (See Figure 1.1.19(a).)

34.[M] A camper at A will walk to the river, put some water in a pail at P, and take it to the campsite at B.

(a) Express the distance $\overline{AP} + \overline{PB}$ as a function of x.

(b) Where should P be located to minimize the length of the walk, $\overline{AP} + \overline{PB}$? (See Figure 1.1.17.) HINT: Reflect B across the line L.

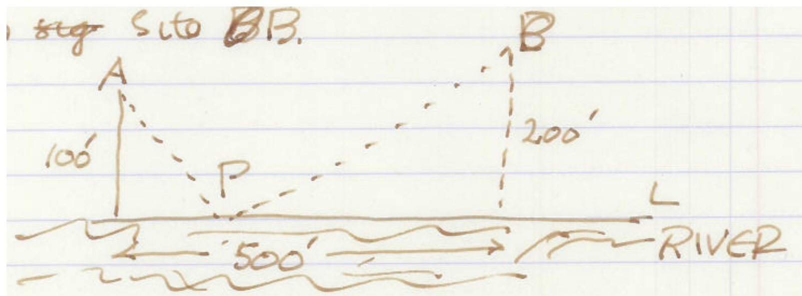


Figure 1.1.17: Sketch of situation in Exercise 34.

NOTE: A geometric trick solved (b). Chapter 4 develops a general procedure for finding the maximum or minimum of a function.

In Exercises 35 to 39 give (a) three functions that satisfy the equation for all positive x and y and (b) one function that does not.

35.[M] $f(x+y) = f(x) + f(y)$

36.[M] $f(x+y) = f(x)f(y)$

37.[M] $f(xy) = f(x) + f(y)$

38.[M] $f(xy) = f(x)f(y)$

39.[M] $f(x) = f(y)$

40.[M] The cost of life insurance depends on whether the person is a smoker or a non-smoker. The following chart lists the annual cost for a male for a million-dollar life insurance policy.

age (yrs)	20	30	40	50	60	70	80
cost for smoker ($)	1150	1164	1944	4344	9864	26500	104600
cost for non-smoker ($)	396	396	600	1490	3684	10900	41600

NOTE: A "smoker" is a person who has used tobacco during the previous three years.

(a) Plot the data and sketch the graphs on the same axes for both groups of males.

(b) A smoker at age 20 pays as much as a non-smoker of about what age?

(c) A smoker pays about how many times as much as a non-smoker of the same age?

41.[M] If f is an increasing function, what, if anything, can be said about f^{-1}?

42.[M] On a typical summer day in the Sacramento Valley the temperature is at a minimum of 60° at 7A.M. and a maximum of 95° at 4P.M..

(a) Sketch a graph that shows how the temperature may vary during the twenty-four hours from midnight to midnight.

(b) A closed shed with little insulation is in the middle of a treeless field. Sketch a graph that shows how the temperature inside the shed may vary during the same period.

(c) Sketch a graph that shows how the temperature in a well-insulated house may vary. Assume that in the evening all the windows and skylights are opened when the outdoor temperature equals the indoor temperature, and closed in the morning when the two temperatures are again equal.

NOTE: Use the same set of axes for all three graphs.

43.[M] The monthly average air and water temperatures in Myrtle Beach, SC, are shown in Table 1.1.3.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Air Temp (°)	56	60	68	76	83	88	91	89	85	77	69	60
Water Temp (°)	51	52	57	62	69	77	81	83	80	73	65	55

Table 1.1.3: Source: `http://www.myrtle-beach-resort.com/weather.htm`
NOTE: Assume, for convenience, that the temperatures in the table are the temperatures on the first day of each month.

(a) Sketch a graph that shows how the water temperature may vary during one calendar year, that is, from January 1 through December 31.

(b) Sketch a graph that shows how the difference between the air and water temperatures may vary during one calendar year. During what month is the temperature difference greatest? least?

(c) During February, the water temperature increases 5° in 28 days so the average daily change is $5/28 \approx 0.1786°/\text{day}$. For each month, estimate the average daily change in the water temperature from one day to the next. During which month is this daily change greatest? least?

(d) Repeat (b) and (c) for the air temperature data.

44.[M] This problem grew out of a question raised by the daughter of one of the authors, Rebecca Stein-Wexler, when cutting cloth for a dress. She wanted to cut out two congruent semicircles from a long strip of fabric 44 inches wide, as shown in Figure 1.1.18. The radius of the semicircles determines d, the length of fabric used, $d = f(r)$.

(a) Draw a picture to show that $f(22) = 44$.

(b) For $0 \leq r \leq 22$, determine d as a function of r, $d = f(r)$.

(c) For $22 \leq r \leq 44$, determine d as a function of r, $d = f(r)$.

(d) Obtain an equation expressing r as a function of d.

(e) She had 104 inches of fabric, and guesed that the largest semicircle she could cut set has a radius of about 30 inches. Use (c) to see how good her guess is.

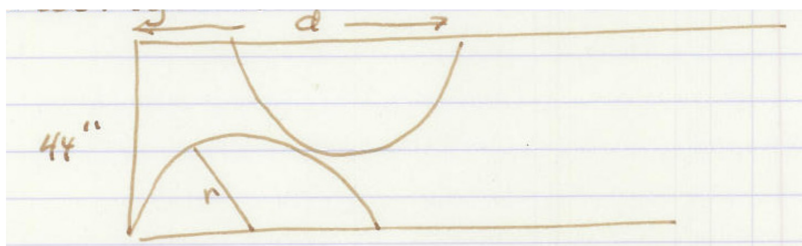


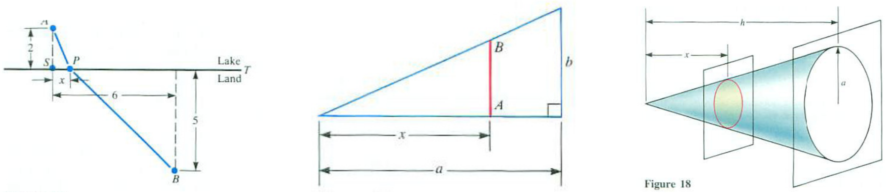
Figure 1.1.18: Exercise 44.

45.[C] Let $f(x)$ be the length of the segment AB in Figure 1.1.19(b).

(a) What are $f(0)$ and $f(a)$?

(b) What is $f(a/2)$?

(c) Find the formula for $f(x)$ and explain your solution.



(a) (b) (c)

Figure 1.1.19:

46.[C] Let $f(x)$ be the area of the cross-section of aright circular cone shown in Figure 1.1.19(c).

(a) What are $f(0)$ and $f(h)$?

(b) Find a formula for $f(x)$ and explain your solution.

47.[C] The cost of a ride in a New York city taxi is described by this formula: At the start the meter reads \$2.50. For every fifth of a mile, 40 cents is added. Graph the cost as a function of distance travelled. NOTE: The cost also depends on other factors. For every two minutes stopped in traffic, 40 cents is added. During the evening rush, 4–8 pm, there is a surcharge of one dollar. Between 8 pm and 6 am there is a surcharge of 50 cents. So the cost, which depends on distance travelled, time stopped, and time of day, is actually a function of three variables.)

48.[C] Let $g(d)$ be the radius of the largest pair of semicircles with diameters on the edge of the fabric, if the fabric is d inches long and 44 inches wide. The domain of g is $(0, \infty)$. Find g and graph it. NOTE: This is related to Exercise 44.

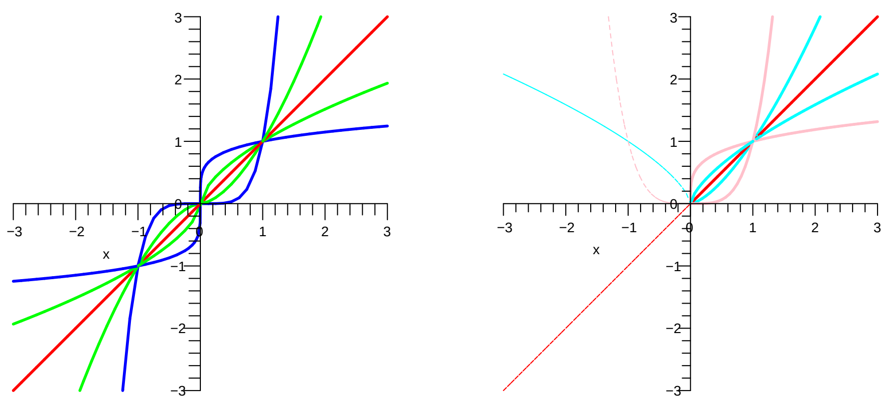
1.2 The basic functions of calculus

This section describes the basic functions in calculus. In the next section you will see how to use them as building blocks to build more complicated functions.

The Power Functions

The first group of functions consists of the **power functions** x^k where the exponent k is a fixed non-zero number and the base x is the input. If k is an odd integer, then x^k has an inverse, $x^{1/k}$, another power function. If k is an even integer and we restrict the domain of x^k to the positive numbers, then it is one-to-one, and has an inverse, again $x^{1/k}$, with, again, a domain consisting of all positive numbers.

In Section 1.1 it was shown that the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ for all x. Notice, however, $g(x) = x^4$ does not pass the horizontal line test unless the domain is restricted to, say, nonnegative inputs ($x \geq 0$). Thus, the inverse of $g(x) = x^4$ is $g^{-1}(x) = x^{1/4}$ only for $x \geq 0$.



(a) (b)

Figure 1.2.1: Graphs of power functions. (a) x^k for $k = 1$ (red), 5 (blue), 1/5 (blue), 5/3 (green), and 3/5 (green). (b) x^k for $k = 1$ (red), 4 (pink), 1/4 (pink), 3/2 (aqua), and 2/3 (aqua). Note that the pairs of blue and green graphs are inverses in (a), as are the pairs of (solid) pink and aqua graphs in (b). In (b) the graphs of x^4 and $x^{2/3}$ pass the horizontal line test only for $x \geq 0$, and the graphs of $x^{1/4}$ and $x^{3/2}$ are defined only for $x \geq 0$.

OBSERVATION *(Inverses of Power Functions)*

1. The inverse of a power function is another power function.

2. When $k = 0$, we obtain the function x^0, which is constant (with all outputs equal to 1), the very opposite of being one-to-one. Constant functions are discussed in more detail in Section 1.3.

3. When the exponent k is an even integer or a rational number (in lowest terms) whose numerator is even (2/3, 4/7, etc.) the graph of $y = x^k$ will not pass the horizontal line test unless the domain is reduced, typically by restricting it to $[0, \infty)$.

The Exponential and Logarithm Functions

Next we have the **exponential functions** b^x where the base b is fixed and the exponent x is the input. The inverses of exponential functions are not exponential functions. The inverses are called **logarithms** and are the next class of functions that we will consider. (A review of exponential and logarithmic functions is in Appendix D.)

Consider a function of the form b^x, where b is positive and fixed. In order to be concrete, let's take the case $b = 2$, that is, $f(x) = 2^x$.

As x increases, so does 2^x. So the function 2^x has an inverse function. In other words, if $y = 2^x$, then if we know the output y we can determine the input x, the exponent, uniquely. For instance, if $2^x = 8$ then $x = 3$. This is expressed as $3 = \log_2 8$ and it read as "the logarithm of 8, base 2, is 3." If $y = b^x$, then we write $x = \log_b y$.

Since we usually denote the independent variable (the input or argument) by x, and the dependent variable (the output, or value) by y, we will rewrite this as $y = \log_b(x)$.

The table of easy values of $\log_2(x)$ in Table 1.2.1 will help us graph $y = \log_2(x)$. Putting a smooth curve through the seven points in Table 1.2.1 yields the graph in Figure 1.2.2.

x	1	2	4	8	1/2	1/4	1/8
$\log_2(x)$	0	1	2	3	-1	-2	-3

Table 1.2.1: Table of easy values of $y = \log_2(x)$.

As x increases, $\log_2(x)$ grows very slowly. For instance $\log_2 1024 = 10$, as every computer scientist knows. For x between 0 and 1, $\log_2(x)$ is negative. As x moves from 1 towards 0, $|\log_2(x)|$ grows very large. For instance, $\log_2 \frac{1}{1024} = -10$.

Because $\log_2(x)$ is the inverse of the function 2^x, we could have sketched the graph of $y = \log_2(x)$ by first sketching the graph of $y = 2^x$ and reflecting it around the line $y = x$.

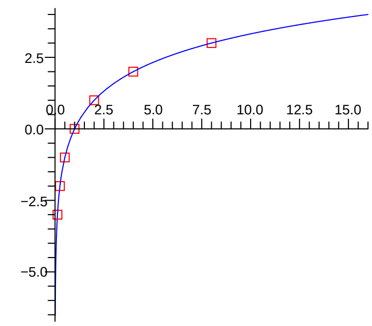


Figure 1.2.2: Plot of $y = \log_2(x)$ based on data in Table 1.2.1.

For any positive base b, $\log_b(x)$ is defined similarly. For x and b both positive numbers, the logarithm of x to the base b, denoted $\log_b(x)$, is the power to which we must raise b to obtain x. By the very definition of the logarithm

$$b^{\log_b(x)} = x.$$

(Whenever you see "$\log_b(x)$" you should think, "Ah, ha! The fancy name for an exponent.")

The Trigonometric Functions and Their Inverses

So far we have the power functions, x^k, the exponential functions, b^x, and the logarithm functions, $\log_b(x)$. The last major group of important functions consists of the trigonometric functions, $\sin(x)$, $\cos(x)$, $\tan(x)$, and their inverses (after we shrink their domains to make the functions one-to-one).

$\sin(x)$ and its inverse

In calculus we generally measure angles in radians. See also Appendix E.

The graph of the **sine function** $\sin(x)$ has period 2π and is shown in Figure 1.2.3. The range is $[-1, 1]$. On the domain $[-\pi/2, \pi/2]$, $\sin(x)$ is increasing and its values for these inputs already sweep out the full range, $[-1, 1]$.

When we restrict the domain of the function $\sin(x)$ to $[-\pi/2, \pi/2]$ it is a one-to-one function with range $[-1, 1]$. This means the sine function has an inverse with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$. The inverse sine function is denoted by $\arcsin(x)$, $\sin^{-1}(x)$, or $\mathsf{inv}\sin(x)$.

Let's stop for a moment to summarize our findings: For x in $[-1, 1]$, $\arcsin(x)$ is the angle in $[-\pi/2, \pi/2]$ whose sine is x. In equations:

$$y = \arcsin(x) \iff \sin(y) = x.$$

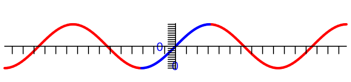


Figure 1.2.3:

For instance, $\arcsin(1) = \pi/2$ because the angle in $[-\pi/2, \pi/2]$ whose sine is 1 is $\pi/2$. Similarly, $\sin^{-1}(1/2) = \pi/6$, $\mathsf{inv}\sin(0) = 0$, $\arcsin(-1/2) = -\pi/6$, $\sin^{-1}(-1) = -\pi/2$. Drawing a unit circle will display these facts, as Figure 1.2.4 illustrates.

We could graph $y = \arcsin(x)$ with the aid of these five values. However, it's easier just to reflect the graph of $y = \sin(x)$ around the line $y = x$. (See Figure 1.2.5(a).)

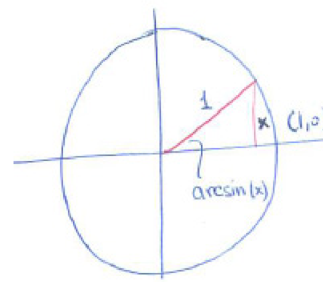
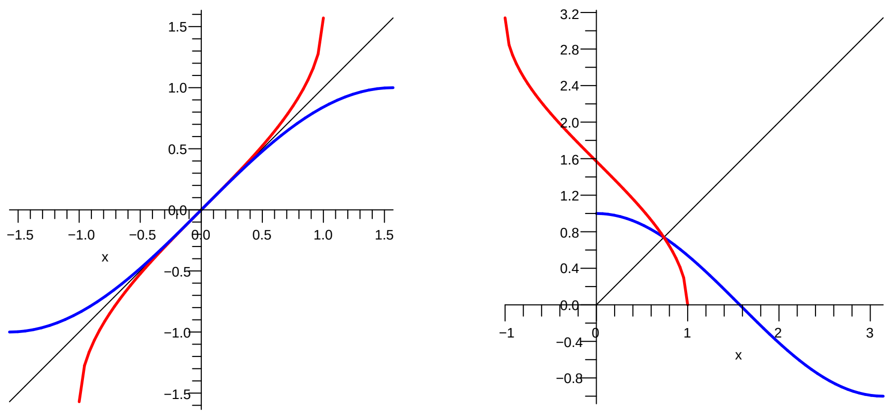


Figure 1.2.4:



(a) (b)

Figure 1.2.5: (a) The graph of $y = \arcsin(x)$ (red) is the graph of $y = \sin(x)$ (blue), with domain restricted to $[-\pi/2, \pi/2]$, reflected around the line $y = x$. (b) The graph of $y = \arccos(x)$ (red) is the graph of $y = \cos(x)$ (blue), with domain restricted to $[0, \pi]$, around the line $y = x$.

$\cos(x)$ and its inverse

The graph of the **cosine function** $\cos(x)$ is shown in Figure 1.2.6.

It is clearly not one-to-one, even if we restrict the domain to the domain used for $\sin(x)$, namely $[-\pi/2, \pi/2]$. In this case note that $\cos(x)$ is a decreasing function on $[0, \pi]$. So the cosine function is one-to-one on $[0, \pi]$. Moreover, the values of $\cos(x)$ for x in $[0, \pi]$ sweep out all possible values of the cosine function, namely $[-1, 1]$.

Because $\cos(x)$ is a one-to-one function on the domain $[0, \pi]$, it has an inverse function, called $\arccos(x)$, $\mathsf{inv}\cos(x)$, or simply $\cos^{-1}(x)$. Each of these is short for "the angle in $[0, \pi]$ whose cosine is x". For instance, $\arccos(0) = \pi/2$, $\cos^{-1}(1) = 0$, and $\mathsf{inv}\cos(-1) = \pi$. Moreover, because the range of the cosine function is the closed interval $[-1, 1]$, the domain of arccos is $[-1, 1]$. Figure 1.2.5(b) shows that the graph of $\arccos(x)$ is obtained by reflecting the graph of $\cos(x)$, with domain $[0, \pi]$, around the line $y = x$.

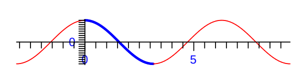


Figure 1.2.6: ARTIST: Please add a more visible vertical axis.

$\tan(x)$ and its inverse

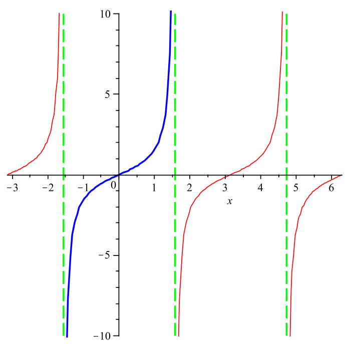


Figure 1.2.7:

The range of the function $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is $(-\infty, \infty)$, as Figure 1.2.7 shows.

When the inputs are restricted to $(-\pi/2, \pi/2)$, $\tan(x)$ is one-to-one, and therefore has an inverse function, denoted $\arctan(x)$, $\tan^{-1}(x)$, or $\mathsf{inv}\tan(x)$. The domain of the inverse tangent function is $(-\infty, \infty)$ and its range is $(-\pi/2, \pi/2)$.

For instance, $\tan^{-1}(0) = 0$, $\mathsf{inv}\tan(1) = \pi/4$, and as x increases, $\arctan(x)$ approaches $\pi/2$. Also, $\arctan(-1) = -\pi/4$, and when x is negative and becomes ever more negative (that is, $|x|$ becomes bigger and bigger) $\arctan(x)$ approaches $-\pi/2$. Figure 1.2.8 is the graph of $\arctan(x)$. It is the reflection of the blue part of the graph in Figure 1.2.7 across the line $y = x$. (See Figure 1.2.8.)

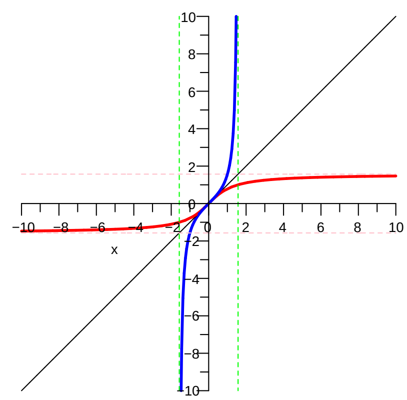


Figure 1.2.8: ARTIST: Please label the two curves as $y = \tan(x)$ and $y = \arctan(x)$.

EXAMPLE 1 Evaluate

$$\text{(a) } \sin(\sin^{-1}(0.3)), \quad \text{(b) } \sin(\tan^{-1}(3)), \quad \text{and (c) } \tan(\cos^{-1}(0.4)).$$

SOLUTION

(a) The expression $\sin^{-1}(0.3)$ is short for the angle in the interval $[-\pi/2, \pi/2]$ whose sine is 0.3. So, the sine of $\sin^{-1}(0.3)$ is 0.3.

(b) To find $\sin(\tan^{-1}(3))$, first draw the angle θ whose tangent is 3 (and lies in the interval $[-\pi/2, \pi/2]$. Figure 1.2.9 shows a simple way to draw this angle. To find the sine of θ, recall that sine equals "opposite/hypotenuse." By the Pythagorean Theorem, the hypotenuse is $\sqrt{3^2 + 1^2} = \sqrt{10}$. Thus, $\sin(\tan^{-1} 3) = 3/\sqrt{10}$.

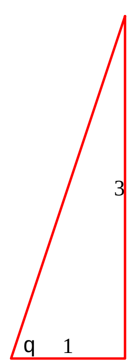


Figure 1.2.9: The traditional symbol for angles is the Greek letter θ (pronounced "theta"). ARTIST: Check that the angle is labeled as θ, not q.

(c) To evaluate $\tan(\cos^{-1}(0.4))$, first draw an angle whose cosine is $0.4 = \frac{2}{5}$, as in Figure 1.2.10, which is based on the fact that cosine equals equals "$\frac{\text{adjacent}}{\text{hypotenuse}}$." To find the tangent of this angle, we need the length of the other leg in Figure 1.2.10. By the Pythagorean Theorem this length is $\sqrt{5^2 - 2^2} = \sqrt{21}$.

From the relation $\tan(\theta) = \text{opposite/adjacent}$, we conclude that

$$\tan(\cos^{-1}(0.4)) = \sqrt{21}/2 \approx 2.291.$$

◇

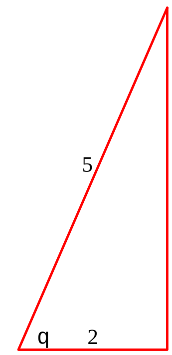


Figure 1.2.10: ARTIST: Check that the angle is labeled as θ, not q.

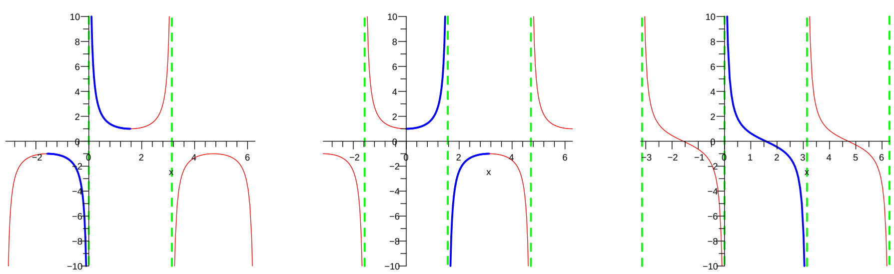
WARNING *(Notation for Inverse Functions)* The notation "$\sin^{-1}(x)$ can be confusing. It may be read as $(\sin(x))^{-1}$, the reciprocal of $\sin(x)$. After all, $\sin^2(x)$ means $(\sin(x))^2$. Though the notation "$\sin^{-1}(x)$" is shorter than "$\arcsin(x)$," we prefer the latter to avoid the risk of misinterpretation. Similar comments apply to $\tan^{-1}(x)$ and $\arctan(x)$ and to $\cos^{-1}(x)$ and $\arccos(x)$.

$\csc(x)$, $\sec(x)$, and $\cot(x)$ and their inverses

The **cosecant**, **secant**, and **cotangent** functions are defined in terms of the sine and cosine functions:

$$\csc(x) = \frac{1}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \text{and} \quad \cot(x) = \frac{\sin(x)}{\cos(x)}.$$

While we could write $\csc(x) = (\sin(x))^{-1}$, we do not because of the possible confusion with $\sin^{-1}(x) = \arcsin(x)$. Each of these functions is defined only when the denominator is not zero. Figure 1.2.11 shows their graphs.



(a) (b) (c)

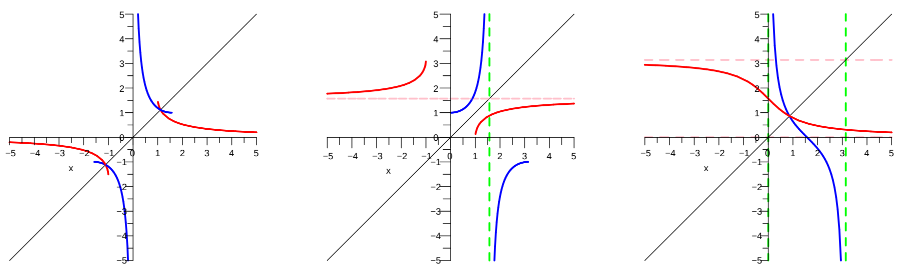
Figure 1.2.11: The graphs of (a) the cosecant, (b) the secant, and (c) the cotangent functions. ARTIST: Please add "cosecant,", "secant," and "cotangent" above each of graph, respectively.

Note that $|\sec(x)| \geq 1$ and $|\csc(x)| \geq 1$. In each case the range consists of two separate intervals: $[1, \infty)$ and $(-\infty, -1]$.

These three functions have inverses, when restricted to appropriate intervals. Table 1.2.2 contains a summary of the three inverse functions, $\csc^{-1} x$, $\sec^{-1} x$, and $\tan^{-1} x$. Figure 1.2.12 shows the graphs of csc, sec, and cot and their inverses.

function	domain (input)	range (output)
$\csc^{-1}(x)$	$(-\infty, -1]$ and $[1, \infty)$	all of $[-\pi/2, \pi/2]$ except 0
$\sec^{-1}(x)$	$(-\infty, -1]$ and $[1, \infty)$	all of $[0, \pi]$ except 0, that is $(0, \pi]$
$\cot^{-1} x$	$(-\infty, \infty)$	the open interval $(0, \pi)$

Table 1.2.2: Summary of the inverse cosecant, inverse secant, and inverse cotangent functions.



(a) (b) (c)

Figure 1.2.12: Graphs of (a) $y = \csc(x)$ and $y = \csc^{-1}(x)$, (b) $y = \sec(x)$ and $y = \sec^{-1}(x)$, and (c) $y = \cot(x)$ and $y = \cot^{-1}(x)$. Notice how the inverse function is the reflection of the original function across the line $y = x$.

Summary

This section reviewed the basic functions in calculus, x^k, b^x, $\sin(x)$, $\cos(x)$, $\tan(x)$, and their inverses. $\log_b(x)$, $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$. (The inverse of x^k, $k \neq 0$, is just another power function $x^{1/k}$).

The functions that may be hardest to have a feel for are the logarithms. Now, $\log_2(x)$ is typical of $\log_b(x)$, $b > 1$. These are its key features:

- its graph crosses the x-axis at $(1, 0)$ because $\log_2(1) = 0$ $(2^0 = 1)$,
- it is defined only for positive inputs, that is, the domain of $\log_2$ is $(0, \infty)$, because only positive numbers can be expressed in the form 2^x,

- it is an increasing function,
- it grows very slowly as the argument increases: $\log_2(8) = 3$, $\log_2(16) = 4$, $\log_2(32) = 5$, $\log_2(64) = 6$, and $\log_2(1024) = 10$,
- for values of x in $(0, 1)$, $\log_2(x)$ is negative ($x = 2^y < 1$ only when $y < 0$),
- for x near 0 (and positive), $|\log_2(x)|$ is large.

The case when the base b is less than 1 is treated in Exercise 54.

EXERCISES for 1.2 *Key:* R–routine, M–moderate, C–challenging

1.[R] Graph the power function $x^{3/2}$, $x \geq 0$, and its inverse.

2.[R] Graph the power function x^5 and its inverse.

3.[R] Explain your calculator's response when you try to calculate $\log_{10}(-3)$.

4.[R] Explain your calculator's response when you try to calculate $\arcsin(2)$.

5.[R]

(a) Graph 2^x and $(1/2)^x$ on the same axes.

(b) How could you obtain the second graph from the first?

6.[R]

(a) Graph 3^x and $(1/3)^x$ on the same axes.

(b) How could you obtain the second graph from the first?

7.[R] For any base b, $b^0 = 1$. What is the corresponding property of logarithms? Explain.

8.[R] For any base b, $b^{x+y} = b^x b^y$. What is the corresponding property of logarithms? Explain. NOTE: If you have trouble with this exercise, study Appendix D.

9.[R] Explain why $\log_b(1/x) = -\log_b(x)$. ("The log of the reciprocal of x is the negative of the log of x.")

10.[R] Explain why $\log_b(c^x) = x\log_b(c)$. ("The log of a number raised to a power x is x times the log of the number.")

11.[R]

(a) Evaluate $\log_2(x)$ and $\log_4(x)$ at $x = 1$, 2, 4, 8, 16, and 1/16.

(b) Graph $\log_2(x)$ and $\log_4(x)$ on the same axes (clearly label each point found in (a)).

(c) Compute $\frac{\log_4(x)}{\log_2(x)}$ for the six values of x in (a).

(d) Explain the phenomenon observed in (c).

(e) How would you obtain the graph of $\log_4(x)$ from that for $\log_2(x)$?

12.[R]

(a) Evaluate $\log_2(x)$ and $\log_8(x)$ at x=1, 2, 4, 8, 16, and 1/8.

(b) Graph $\log_2(x)$ and $\log_8(x)$ on the same axes (clearly label each point found in (a)).

(c) Compute $\frac{\log_8(x)}{\log_2(x)}$ for the six values of x in (a).

(d) Explain the phenomenon observed in (c).

(e) How would you obtain the graph of $\log_8(x)$ from that for $\log_2(x)$?

13.[R] Evaluate

(a) $\log_{10}(1000)$

(b) $\log_{100}(10)$

(c) $\log_{10}(0.01)$

(d) $\log_{10}(\sqrt{10})$

(e) $\log_{10}(10)$

14.[R] Evaluate

(a) $\log_3(3^{17})$

(b) $\log_3(1/9)$

(c) $\log_3(1)$

(d) $\log_3(\sqrt{3})$

(e) $\log_3(81)$

15.[R] Evaluate $5^{\log_5(17)}$.

16.[R] Evaluate $3^{-\log_3(21)}$.

17.[R] For positive x near 0, what happens to the functions 2^x, x^2 and $\log_2(x)$?

18.[R] For very large values of x what happens to the quotent $x^2/2^x$? Illustrate by using specific values for x.

19.[R] What happens to $(\log_2(x))/x$ for large values of x? Illustrate by citing specific x.

20.[R] Draw graphs of $\cos(x)$ for x in $[0, \pi]$, and $\arccos(x)$ on the same axes.

21.[R] Draw graphs of $\tan(x)$ for x in $(-\pi/2, \pi/2)$, and $\arctan(x)$ on the same axes.

In Exercises 22 to 38 evaluate the given expressions.

22.[R]

(a) $\sin^{-1}(1/2)$

(b) $\arcsin(1)$

(c) $\mathsf{inv}\sin(-\sqrt{3}/2)$

(d) $\arcsin(\sqrt{2}/2)$

23.[R]

(a) $\cos^{-1}(0)$

(b) $\mathsf{inv}\cos(-1)$

(c) $\arccos(1/2)$

(d) $\arccos(-1/\sqrt{2})$

24.[R]

(a) $\arctan(1)$

(b) $\mathsf{inv}\tan(-1)$

(c) $\arctan(\sqrt{3})$

(d) $\arctan(1000)$ (approximately)

25.[R]

(a) $\operatorname{arcsec}(2)$

(b) $\mathsf{inv}\sec(-2)$

(c) $\operatorname{arcsec}(\sqrt{2})$

(d) $\sec^{-1}(1000)$ (approximately)

26.[R]

(a) $\arcsin(0.3)$

(b) $\arccos(0.3)$

(c) $\arctan(0.3)$

(d) $\frac{\arcsin(0.3)}{\arccos(0.3)}$

NOTE: Observe that (c) and (d) are different.

27.[R] $\sin(\tan^{-1}(2))$.
28.[R] $\sin(\cos^{-1}(0.4))$.
29.[R] $\tan(\tan^{-1}(3))$.
30.[R] $\tan(\sin^{-1}(0.7))$.

31.[R] $\tan(\sec^{-1}(3))$.
32.[R] $\sec(\tan^{-1}(0.3))$.
33.[R] $\sin(\sec^{-1}(5))$.
34.[R] $\sec(\cos^{-1}(0.2))$.

35.[R] $\arctan(\tan(\frac{\pi}{3}))$.
36.[R] $\arcsin(\sin(\frac{-3\pi}{4}))$.
37.[R] $\arccos(\cos(\frac{5\pi}{2}))$.
38.[R] $\operatorname{arcsec}(\sec(\frac{-\pi}{3}))$.

In Exercises 39 to 42, use properties of logarithms to express $\log_{10} f(x)$ as simple as possible.
39.[M] $f(x) = \frac{(\cos(x))^7\sqrt{(x^2+5)^3}}{4+(\tan(x))^2}$
40.[M] $f(x) = \sqrt{(1+x^2)^5(3+x)^4\sqrt{1+2x}}$
41.[M] $f(x) = (x\sqrt{2+\cos(x)})^{x^2}$

42.[M] $f(x) = \sqrt{\frac{x(1+x)}{\sqrt{1+2x^3}}}$

43.[M] Imagine that your calculator fell on the floor and its multiplication and division keys stopped working. However, all the other keys, including the trigonometric, arithmetic, logarithmic, and exponential keys still functioned. Show how you would use your calculator to calculate the product and quotient of two positive numbers, a and b.

44.[M] (Richter Scale) In 1989, San Francisco and vicinity was struck by an earthquake that measured 7.1 on the **Richter scale**. The strongest earthquake in recent years had a Richter measure of 8.9 (Colombia-Equador in 1906 and Japan in 1933). A "major earthquake" typically has a measure of at least 7.5.
In his *Introduction to the Theory of Seismology*, Cambridge, 1965, pp. 271–272, K. E. Bullen explains the Richter scale as follows:
"Gutenburg and Richter sought to connect the magnitude M with the energy E of an earthquake by the formula

$$aM = \log_{10}\left(\frac{E}{E_0}\right)$$

and after several revisions arrived in 1956 at the result $a = 1.5$, $E_0 = 2.5\times10^{11}$ ergs."
NOTE: Energy E is measured in ergs. M is the number assigned to the earthquake on the Richter scale. E_0 is the energy of the smallest instrumentally recorded earthquake.

(a) Deduce that $\log_{10}(E) \approx 11.4 + 1.5M$.

(b) What is the ratio between the energy of the earthquake that struck Japan in 1933 ($M = 8.9$) and the San Francisco earthquake of 1989 ($M = 7.1$)?

(c) What is the ratio between the energy of the San Francisco earthquake of 1906 ($M = 8.3$) and that of the San Francisco earthquake of 1989 ($M = 7.1$)?

(d) Find a formula for E in terms of M.

(e) If one earthquake has a Richter measure 1 larger than that of another earthquake, what is the ratio of their energies?

(f) What is the Richter measure of a 10-megaton H-bomb, that is, of an H-bomb whose energy is equivalent to that of 10 millon tons of TNT?

NOTE: One ton of TNT releases an energy of 4.2×10^6 ergs.

45.[M] Translate the sentence, "She has a five-figure annual income" into logarithms. How small can the income be? How large?

46.[M] As of 2006 the largest known prime was $2^{30402457} - 1$.

(a) When written in decimal notation, how many digits will it have?

(b) How many pages of this book would be needed to print it? (One page can hold about 6,400 digits.)

NOTE: There is a prize of $250,000 for the discovery of the first billion-digit prime. Do a Google search for "largest prime".

47.[M]

(a) In many calculators the log key refers to base-ten logarithms. You can use it to find logarithms to any base $b > 0$. To see why, start with the equation $b^{\log_b(x)} = x$ and then take $\log_{10}$ of both sides. This gives the formula

$$\log_b(x) = \frac{\log_{10}(x)}{\log_{10}(b)}.$$

(b) Use (a) to find $\log_3(7)$. (Why should the result be between 1 and 2?)

(Semi-log graphs) In most graphs the scale on the y-axis is the same as the scale on the x-axis, or a constant multiple of it. However, to graph a rapidly increasing function, such as 10^x, it is convenient to "distort" the y-axis. Instead of plotting the point (x, y) at a height of, say, y inches, you plot it at a height of $\log_{10} y$ inches. So the datum $(x, 1)$ could be drawn with height zero, the datum $(x, 10)$, would have height 1, and the datum $(x, 100)$ would have height 2 inches. Instead of graphing $y = f(x)$, you graph $Y = \log_{10} f(x)$. In particular, if $f(x) = 10^x$, $y = \log_{10} 10^x = x$: the graph would be a straight line. To avoid having to calculate a bunch of logarithms, it is convenient to use semi-log graph paper, shown in Figure 1.2.13.

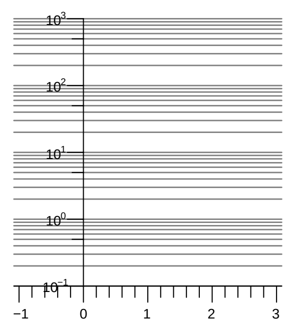


Figure 1.2.13:

48.[C] Using semi-log paper, graph $y = 2 \cdot 3^x$.

49.[C] Using semi-log paper, graph $y = \frac{2}{3^x}$.

50.[C] (Slide Rule) This exercise shows how to build a slide rule by exploiting the equation $\log_b(xy) = \log_b(x) + \log_b(y)$. We will use $\log_2$ for convenience.

Step 1. Mark on the bottom edge of a stick (or page) the numbers 2^0, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, and $2^4 = 16$, placing 2^n at a distance n cm from the left end. In other words, place each number x, at a distance $\log_2(x)$ cm from the left edge. Figure 1.2.14 shows only numbers with convenient integer logarithms, with base 2.

Step 2. Do the same thing as the top edge of another stick or sheet of paper.

Step 3. You now have a slide rule. To compute 4×8, say, with your slide rule, slide the bottom stick along the top stick until its left edge is next to the 4 of the top stick. The product 4×8 appears above the 8 on the lower stick. Why?

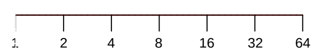


Figure 1.2.14:

51.[C] Newton computed the logarithms of 0.8, 0.9, 1.1, and 1.2 to 57 decimal places by hand using a method that you will learn about in Section 10.4.

(a) Show how to compute log(2), using log(1.2), log(0.8) and log(0.9).

(b) Show how to compute log(3), using log(2), log(1.2) and log(0.8).

(c) Show how to compute log(4), using log(2).

(d) Show how to compute log(5), using log(2) and log(0.8).

(e) How would you then compute log(6, log(8), log(9), and log(10).

(f) How would you then estimate log(11).

NOTE: You don't need to know the base. Why?

52.[M] The graph of $y = \log_2(x)$ consists of the part to the right of $(1, 0)$ and the part to the left of $(1, 0)$. Are the two parts congruent?

53.[C] Say that you have drawn the graph of $y = \log_2(x)$. Jane says that to get the graph of $y = \log_2(4x)$, you just raise that graph 2 units parallel to the y-axis. Sam says, "No, just shrink the x-coordinate of each point on the graph by a factor of 4." Who is right?

54.[C] Answer the following questions about $y = \log_b(x)$ where $0 < b < 1$.

(a) Sketch the graphs of $y = b^x$ and $y = \log_b(x)$ on the same set of axes.

(b) What is the domain of $\log_b$?

(c) What is the x-intercept? That is, solve $\log_b(x) = 0$.

(d) For what values of x is $\log_b(x)$ positive? negative?

(e) Is the graph of $y = \log_b(x)$ an increasing or decreasing function?

(f) What can you say about the values of $\log_b(x)$ when x is close to zero (and in the domain)?

(g) What can you say about the values of $\log_b(x)$ when x is a large positive number?

(h) What can you say about the values of $\log_b(x)$ when x is a large negative number?

55.[C] Let a, b, c, d be constants such that $ad - bc \neq 0$.

(a) Show that $y = (ax + b)/(cx + d)$ is one-to-one.

(b) For which a, b, c, d does the function in (a) equal its inverse function?

56.[C] Prove that $\log_3(2)$ is irrational, that is, not rational. HINT: Assume that it is rational, that is, equal to m/n for some integers m and n, and obtain a contradiction.

1.3 Building more functions from basic functions

In this section we complete the list of functions needed for calculus. Our starting point is the basic functions introduced in Section 1.1. We will use just two methods to build more complicated functions from x^k, b^x, $\sin(x)$, $\cos(x)$, $\tan(x)$, and their inverses. For instance we will see how to obtain

$$f(x) = \frac{\sin(2x) + 3 + 4x + 5x^2}{\log_2(x) + 3^{-5x} + \sqrt{1 + x^3}}. \tag{1.3.1}$$

Before we go into the details of how we construct new functions from old ones, we must introduce one more type of basic function. These functions are so simple, however, that they did not deserve to appear with the functions in the preceding section. They are the constant functions, whose graphs are horizontal lines. (See Figure 1.3.1.)

The Constant Functions

DEFINITION *(Constant Function)* A function $f(x)$ is **constant** if there is a number C such that $f(x) = C$ for all x in its domain. A special constant function is the **zero function**: $f(x) = 0$.

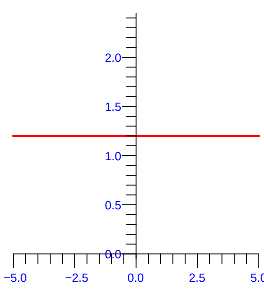


Figure 1.3.1:

Using the Four Arithmetic Operations: $+$, $-$, $\times$, $/$

Given two functions f and g, we can produce other functions from them by using the four operations of arithmetic:

$f + g$: for an input value x, the function assigns $f(x) + g(x)$ as the output

$f - g$: for an input value x, the function assigns $f(x) - g(x)$ as the output

fg: for an input value x, the function assigns $f(x)g(x)$ as the output

f/g: for an input value x with $g(x) \neq 0$, the function assigns $f(x)/g(x)$ as the output

The domains of $f+g$, $f-g$, and fg consist of the numbers that belong to both the domain of f and the domain of g. The domain of f/g is a little different because division by zero is not defined. The function f/g is defined for all numbers x that belong to the domain of f and the domain of g with the extra condition that $g(x) \neq 0$.

With the aid of these constructions we can build any polynomial or rational function from the simple function $f(x) = x$, called the **identity function**, and the constant functions.

A **polynomial** is a function of the form $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where the coefficients a_0, a_1, a_2, $\ldots$, a_n are numbers. If a_n is not zero, the **degree** of the polynomial is n. A **rational function** is the quotient of two polynomials. The domain of a polynomial is the set of all real numbers. The domain of a rational polynomial is all real numbers except those where the denominator is zero.

EXAMPLE 1 Use addition, subtraction, and multiplication to form the polynomial $F(x) = x^3 + 3x - 7$.

SOLUTION We first build each of the three terms: x^3, $3x$, and 7. The last of these is just a constant function. Multiplying the identity function x and the constant function 3 gives $3x$. The first term is obtained by first multiplying x and x to obtain x^2. Then multiplying x^2 and x yields x^3. Adding x^3 and $3x$ gives $x^3 + 3x$. Lastly, subtract the constant function 7 to obtain $x^3 + 3x - 7$.

Notice that each of the three functions involved in forming F is defined for all real numbers. As a result, the domain of F is also all real numbers, $(-\infty, \infty)$. ◇

Example 1 shows how to build any polynomial using $+$, $-$, and $\times$. Constructing rational functions also requires one use of the division operator.

But how would we build a function like $\sqrt{1 + 3x}$? This leads us to the most important technique for combining two or more functions to build more complicated functions.

Composite Functions

Given two functions f and g we can use the output of g as the input for f. That is, we can find $f(g(x))$. For instance, if $g(x) = 1 + 3x$ and f is the square root function, $f(x) = \sqrt{x}$, then $f(g(x)) = f(1 + 3x) = \sqrt{1 + 3x}$. This brings us to the definition of a composite function.

> **DEFINITION** *(Composition of functions)* Let X, Y, and Z be sets. Let g be a function from X to Y and let f be a function from Y to Z. Then the function that assigns to each element x in X the element $f(g(x))$ in Z is called the **composition** of f with g. It is denoted $f \circ g$, which is read as "f circle g" or as "f composed with g".

Some or all of the sets X, Y, and Z could be the same set.

Thinking of f and g as input-output machines we may consider $f \circ g$ as the machine built by hooking up the machine for f to process the outputs of the machine for g (see Figure 1.3.2).

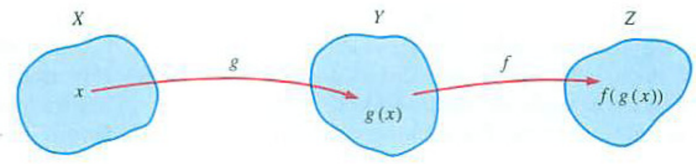


Figure 1.3.2: The output of the g machine, $g(x)$, becomes the input for the f machine. The result is the composition of f with g, $(f \circ g)(x) = f(g(x))$.

Most functions we encounter are composite functions. For instance, $\sin(2x)$ is the composition of $g(x) = 2x$ and $f(x) = \sin(x)$. Of course, we can hook up three or more functions to make even fancier functions. Consider $\sin^3(2x) = (\sin(2x))^3$. This function is built up as follows:

$$x \longrightarrow 2x \longrightarrow \sin(2x) \longrightarrow (\sin(2x))^3. \tag{1.3.2}$$

It is the composition of three functions: the first doubles the input, the second takes the sine of its input, and the third cubes its input.

The order does matter. If, instead, you cube first, then take the sine, and then double the input you obtain:

$$x \longrightarrow x^3 \longrightarrow \sin(x^3) \longrightarrow 2\sin(x^3). \tag{1.3.3}$$

When you enter a function on your calculator or on a computer, you have to be careful of the order in which the functions are applied as you evaluate a composite function. The specific way that you would evaluate $\sin(\log_{10}(240))$ depends on your calculator. On a traditional scientific calculator you enter 240 followed by the log10 key, and finally the sin key. On many of the newer graphing calculators you would press the sin key followed by the log10 key, then 240, followed by two right parentheses,)), and, finally, the Enter key. Note that these two approaches are different. If you press the sin key before log10, you will get $\log_{10}(\sin 240)$. For most computer software it is necessary to use parentheses to indicate inputs to functions. In this case you might enter sin(log10(240)).

Before pressing the sin key, be sure to check that your calculator is in radians mode.

To describe the build-up of a composite function it is convenient to use various letters, not just x, to denote the variables. This is illustrated in Examples 2 to 4.

If your calculator is in degree mode, you will find that $\sin(240°) < 0$ and so $\log_{10}(\sin(240°))$ is not defined.

EXAMPLE 2 Show how the function $\sqrt{4-x^2}$ is built up by the composition of functions. Find its domain.

SOLUTION The function $\sqrt{4-x^2}$ is obtained by applying the square-root function to the function $4 - x^2$. To be specific, let

$$g(x) = 4 - x^2 \qquad \text{and} \qquad f(u) = \sqrt{u}\ (u \geq 0). \tag{1.3.4}$$

Then

$$f(g(x)) = f(4-x^2) = \sqrt{4-x^2}. \tag{1.3.5}$$

The square-root function is defined for all $u \geq 0$ and the polynomial $g(x)$ is defined for all numbers. So $f(g(x))$ is defined only when $g(x) \geq 0$:

$$\begin{aligned} g(x) &\geq 0 \\ 4-x^2 &\geq 0 \\ 4 &\geq x^2 \\ 2 &\geq |x|. \end{aligned}$$

Thus, the domain of $\sqrt{4-x^2}$ is the closed interval $[-2, 2]$. ⋄

EXAMPLE 3 Express $1/\sqrt{1+x^2}$ as a composition of three functions. Find the domain of this function.

SOLUTION Call the input x. First, we compute $1+x^2$. Second, we take the square root of that output, getting $\sqrt{1+x^2}$. Third, we take the reciprocal of that result, getting $1/\sqrt{1+x^2}$. In summary, we form

$$u = 1+x^2, \qquad \text{then} \quad v = \sqrt{u} \qquad \text{then} \quad y = \frac{1}{v}. \tag{1.3.6}$$

Given x, we first evaluate the polynomial $1+x^2$, then apply the square-root function, then the reciprocal function.

The domain of a polynomial consists of all real numbers; the domain of the square-root function is $v \geq 0$; and the domain of the reciprocal function is all numbers except zero. Because $u = 1+x^2 \geq 1$, $v = \sqrt{u} = \sqrt{1+x^2}$ is defined for all x. Moreover, $v = \sqrt{1+x^2} \geq 1$, so that $y = \frac{1}{v} = 1/\sqrt{1+x^2}$ is defined for all real numbers x. ⋄ The function in Example 3 can also be written as the composition of two functions: $x \longrightarrow 1+x^2 \longrightarrow (1+x^2)^{-1/2}$.

EXAMPLE 4 Let f be the cubing function and g the cube-root function. Compute $(f \circ g)(x)$, $(f \circ f)(x)$ and $(g \circ f)(x)$.

SOLUTION In terms of formulas, $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$.

$$(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x}) = \left(\sqrt[3]{x}\right)^3 = x. \tag{1.3.7}$$

$$(f \circ f)(x) = f(f(x)) = f(x^3) = \left(x^3\right)^3 = x^9. \tag{1.3.8}$$

$$(g \circ f)(x) = g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x. \tag{1.3.9}$$

Observe that the domains of f and g are $(-\infty, \infty)$. Therefore, each of $f \circ g$, $f \circ f$, and $g \circ f$ is defined for all real numbers.

Notice that both $f \circ g$ and $g \circ f$ are the identity function. Whenever g is the inverse of f, $f \circ g$ and $g \circ f$ are the identity function. Each function undoes the action of the other. ◇

EXAMPLE 5 Give two different ways of obtaining the function $1/f(x)$ from the function f.

SOLUTION The first approach is to view $1/f(x)$ as the quotient of the constant function 1 and the function $f(x)$.

This function can also be viewed as a composition. The quotient $1/f(x)$ can be obtained in two steps: First, evaluate $f(x)$. Second, take the reciprocal of the result. So, if $g(x) = 1/x$, then

$$\frac{1}{f(x)} = g(f(x)). \tag{1.3.10}$$

Regardless of the way in which $1/f(x)$ is constructed, the domain is all real numbers for which $f(x) \neq 0$. ◇

EXAMPLE 6 Show that every power function x^k, $x > 0$, can be constructed as a composition using exponential or logarithmic functions.

SOLUTION The first step is to write $x = 2^{\log_2(x)}$. Then, $x^k = \left(2^{\log_2(x)}\right)^k$ or, by properties of exponentials, $x^k = 2^{k\log_2(x)}$. So x^k is the composition of three functions: First, find $\log_2(x)$, then multiply by the constant function k, and then raise 2 to the resulting power. ◇

That a power function can be expressed in terms of an exponential function will be used in Chapter 4.

OBSERVATION *(Consequences of Example 6)*

1. The construction in Example 6 provides a meaning to functions like $x^{\sqrt{2}}$ and $x^{-\pi}$ for $x > 0$.
2. As a result of Example 6 we could remove the power functions from our list of basic functions in Section 1.1. We choose not to do so because power functions with integer exponents are so common and in many instances we want to define a power function for all inputs (not just positive numbers).
3. It might seem surprising that the power functions can be expressed in terms of exponentials (and logarithms). An even more astonishing result is that trigonometric functions, such as $\sin(x)$, can also be expressed in terms of exponentials, as shown in Section 12.7.

Summary

This section showed how to build more complicated functions from power, exponential, and trigonometric functions and their inverses, and the constant functions. One method is to simply add, multiply, subtract, or divide outputs. The other method is the "composition of functions" in which one function operates on the output of a second function. Composite functions are extremely important, especially when we calculate derivatives beginning in Chapter 3.

> **WARNING** *(Traveler's Advisory about Notation)* Be careful when composing functions when one of them is a trigonometric function. For instance, what is meant by $\sin x^3$? Is it $\sin(x^3)$ or $(\sin(x))^3$? Do we first cube x, then take the sine, or the other way around? There is a general agreement that $\sin x^3$ stands for $\sin(x^3)$; you cube first, then take the sine.
>
> Spoken aloud, $\sin x^3$ is usually "the sine of x cubed," which is ambiguous. We can either insert a brief pause – "sine of (pause) x cubed" – to emphasize that x is cubed rather than $\sin(x)$, or rephrase it as "sine of the quantity x cubed."
>
> On the other hand $(\sin(x))^3$, which is by convention usually written as $\sin^3(x)$, is spoken aloud as "the cube of $\sin(x)$" or "sine cubed of x."
>
> Similar warnings apply to other trigonometric functions and logarithmic functions.

EXERCISES for 1.3

Key: R–routine, M–moderate, C–challenging

The function $y = \sqrt{1+x^2}$ is the composition of $s = 1 + x^2$ and $y = \sqrt{s}$. In Exercises 1 to 12 use a similar format to build the given functions as the composition of two or more functions.

1.[R] $\sin(2x)$

2.[R] $\sin^3(x)$

3.[R] $\sin(3x)$

4.[R] $\sin(x^3)$

5.[R] $\sin^2(x^3)$

6.[R] 2^{x^2}

7.[R] $(x^2+3)^{10}$

8.[R] $\log_{10}(1+x^2)$

9.[R] $1/(x^2+1)$

10.[R] $\cos^3(2x+3)$

11.[R] $\left(\frac{2}{3x+5}\right)^3$

12.[R] $\arcsin(\sqrt{x})$

13.[R] These tables show some of the values of functions f and g:

x	1	2	3	4	5
$f(x)$	6	8	9	7	10

x	6	7	8	9	10
$g(x)$	4	3	2	5	1

(a) Find $f(g(7))$.

(b) Find $g(f(3))$.

14.[R] Figure 1.3.3 shows the graphs of functions f and g.

(a) Estimate $f(g(0.6))$.

(b) Estimate $f(g(0.3))$.

(c) Estimate $f(f(0.5))$.

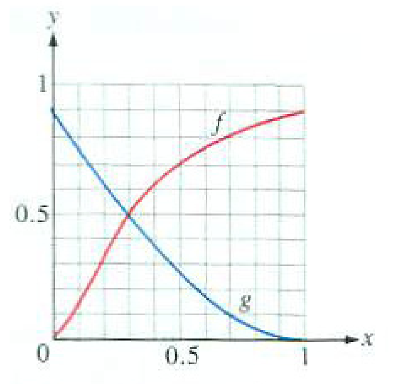


Figure 1.3.3:

In Exercises 15 and 24 write y as a function of x.

15.[R] $u = \sin(x)$, $y = u^2$

16.[R] $u = x^3$, $y = 1/u$

17.[R] $u = 2x^2 - 3$, $y = 1/u$

18.[R] $u = \sqrt{x}$, $y = u^2$

19.[R] $u = \sqrt{x}$, $y = \sin(u)$

20.[R] $u = x^2$, $y = 2^u$

21.[R] $v = 2x$, $u = v^2 - 1$, $y = u^2$

22.[R] $v = \sqrt{x}$, $u = 1 + v$, $y = u^2$

23.[R] $v = x + x^2$, $u = \sin(v)$, $y = u^3$

24.[R] $v = \tan(x)$, $u = 1 + v^2$, $y = \cos(u)$

25.[M] Let $f(x) = 2x^2 - 1$ and $g(x) = 4x^3 - 3x$. Show that $(f \circ g)(x) = (g \circ f)(x)$. [Rare indeed are pairs of polynomials that commute with each other under composition, as you may convince yourself by trying to find more examples.] NOTE: Of course, any two powers, such as x^3 and x^4, commute. (The composition of x^3 and x^4 in either order is x^{12}, as may be checked.)

26.[M] Let $f(x) = 1/(1 - x)$. What is the domain of f? of $f \circ f$? of $f \circ f \circ f$? Show that $(f \circ f \circ f)(x) = x$ for all x in the domain of $f \circ f \circ f$.

27.[M] Let $g(x) = x^2$. Find all first degree polynomials $f(x) = ax + b$, $a \neq 0$, such that $f \circ g = g \circ f$, that is, $f(g(x)) = g(f(x))$.

28.[M] Let $f(x) = x^3$. Is there a function $g(x)$ such that $(f \circ g)(x) = x$ for all numbers x? If so, how many such functions are there?

29.[M] Let $f(x) = x^4$. Is there a function $g(x)$ such that $(f \circ g)(x) = x$ for all negative numbers x? If so, how many such functions are there?

30.[M] Let $f(x) = x^4$. Is there a function $g(x)$ such that $(f \circ g)(x) = x$ for all positive numbers x? If so, how many such functions are there?

31.[M] Figure 1.3.4 shows the graph of a function f whose domain is $[0, 1]$. Let $g(x) = f(2x)$.

(a) What is the domain of g?

(b) Graph $y = g(x)$

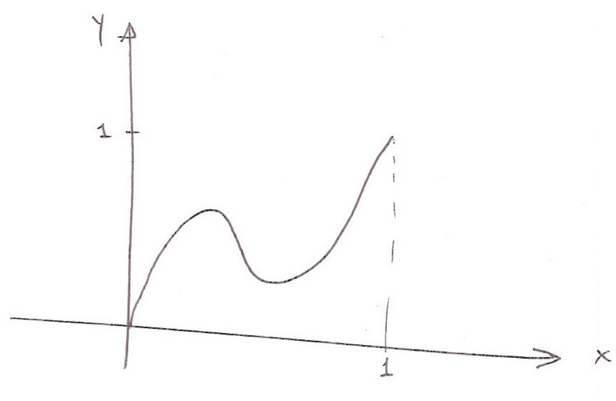


Figure 1.3.4:

32.[M] Show that there is a function $u(x)$ such that $\cos x = \sin u(x)$. NOTE: This shows that we didn't need to include $\cos x$ among our basic functions.

33.[M] Find a function $u(x)$ such that $3^x = 2^{u(x)}$.

34.[C] If f and g are one-to-one, must $f \circ g$ be one-to-one?

35.[C] If $f \circ g$ is one-to-one, must f be one-to-one? Must g be one-to-one?

36.[C] If f has an inverse, $\mathsf{inv}f$, compute $(f \circ \mathsf{inv}f)(x)$ and $((\mathsf{inv}f) \circ f)(x)$.

37.[C] Let $g(x) = x^2$. Find all second-degree polynomials $f(x) = ax^2 + bx + c$, $a \neq 0$, such that $f \circ g = g \circ f$, that is, $f(g(x)) = g(f(x))$.

38.[C] Let $f(x) = 2x + 3$. Find all functions of the form $g(x) = ax + b$, a and b constants, such that $f \circ g = g \circ f$.

39.[C] Let $f(x) = 2x + 3$. Find all functions of the form $g(x) = ax^2 + bx + c$, a, b, and c constants, such that $f \circ g = g \circ f$.

40.[C] Find all functions of the form $f(x) = 1/(ax + b)$, $a \neq 0$, such that $(f \circ f \circ f)(x) = x$ for all x in the domain of $f \circ f \circ f$.

41.[C] (Induction) This exercise rests on the identifies $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$, $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$, and $\cos^2 x + \sin^2 x = 1$.

(a) Show that $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = 2\cos^2(x) - 1$.

(b) Show that $\sin(3x) = 3\sin(x) - 4\sin^3(x)$ and $\cos(3x) = 4\cos 3(x) - 3\cos(x)$.

(c) Show that $\sin(4x) = \cos(x)(4\sin(x) - 8\sin^3(x))$ and $\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$.

(d) Use induction to show that for each positive integer n, $\cos(nx)$ is a polynomial in $\cos(x)$. That is, there is a polynomial P_n such that $\cos(nx) = P_n(\cos(x))$. NOTE: You will have to consider the form of $\sin(nx)$, n odd or even, in the induction.

(e) Explain why $P_n \circ P_m = P_m \circ P_n$. NOTE: This does not require the explicit formulas for P_n and P_m.

1.4 Chapter Summary

This chapter reviewed precalculus material concerning functions. Calculus begins in the next chapter when we answer questions such as "What happens to $(2^x - 1)/x$ as x gets very small?". The answers are used in Chapter 3 to settle questions such as "How rapidly does 2^x change for a slight change in x?" That is where we meet the derivative of a function.

Section 1.1 introduced the terminology of functions: input (argument), output (value), domain, range, independent variable, dependent variable, piecewise-defined function, inverse of a function, graph of a function, decreasing, increasing, non-increasing, non-decreasing, positive, and monotonic.

Section 1.2 reviewed the key function x^k and its inverse $x^{1/k}$ (constant exponent, variable base), b^x (constant base, variable exponent) and its inverse $\log_b(x)$ and the six trigonometric functions and their inverses (for instance $\sin(x)$ and $\arcsin(x)$). All angles are measured in radians, unless otherwise stated.

Section 1.3 described five ways of getting new functions from two function f and g, namely $f + g$, $f - g$, fg, f/g, and the composition $f \circ g$.

EXERCISES for 1.4 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 10 concern logarithms, important functions in calculus and its applications. Each property of a logarithm function is simply a translation of some property of an exponential function.

1.[R] Deduce from the equation $b^{x+y} = b^x b^y$ that $\log_b(uv) = \log_b(u) + \log_b(v)$.

2.[R] Show that $\log_b(b) = 1$, $\log_b(1) = 0$, and $\log_b(1/b) = -1$.

3.[R] Show why $\log_b(1/c) = -\log_b(c)$.

4.[R] Show why $\log_b(c/d) = \log_b(c) - \log_b(d)$.

5.[R] Show why $\log_b(c^d) = d \log_b(c)$.

6.[R] Why do only positive numbers have logarithms? (In Chapter 12 negative numbers have logarithms also, provided with the aid of complex numbers.)

7.[R] Evaluate (a) $\log_2(2^{43})$, (b) $\log_2(32)$, and (c) $\log_2(1/4)$.

Exercises 8 to 10 concern the relation between logarithms to different bases.

8.[R] Suppose that you want to obtain $\log_2(17)$ in terms of $\log_3(17)$.

(a) Which would be larger $\log_2(17)$ or $\log_3(17)$?

(b) Show that $\log_2(17) = (\log_2(3))\log_3(17)$. HINT: Take logarithms to the base 2 of both sides of the equation $3^{\log_3(17)} = 17$.

9.[R]

(a) Calculate (by hand) $\log_a(b)$, $\log_b(a)$, and $\log_a(b) \cdot \log_b(a)$ when $a = 2$ and $b = 8$.

(b) Starting with $a^{\log_a(b)} = b$ and taking logarithms to the base b, show that $\log_a(b) \cdot \log_b(a) = 1$.

10.[R] You can use your calculator with a key for base-ten logarithms to compute logarithms to any base.

(a) Show why $\log_b(x) = \frac{\log_{10}(x)}{\log_{10}(b)}$.

(b) Compute $\log_2(3)$.

NOTE: When using the formula in (a) it is easy to forget whether you multiply or divide by $\log_{10}(b)$. As a memory device keep in mind that when b is "large," $\log_b(x)$ is "small," so you want to divide by $\log_{10}(b)$.

11.[R] If your scientific calculator lacks a key to display a decimal approximation to π, how could you use other keys to display it?

12.[R] Drawing pictures, find (a) $\tan(\arcsin(1/2))$, (b) $\tan(\arctan(-1/2))$, and (c) $\sin(\arctan(3))$.

13.[R] If f and g are decreasing functions, what (if anything) can be said about (a) $f + g$, (b) $f - g$, (c) f/g, (d) f^2, and (e) $-f$?

14.[R] What type of function is $f \circ g$ if (a) f and g are increasing, (b) f and g are decreasing, (c) f is increasing and g is decreasing? Explain.

15.[R] If f is increasing, what (if anything), can be said about $g = \mathsf{inv}(f)$?

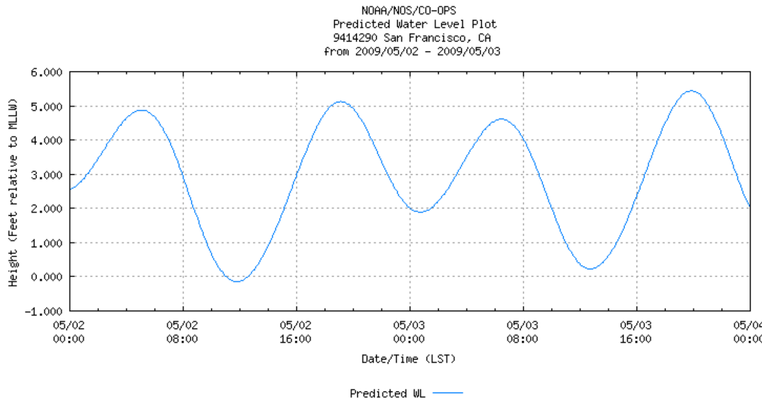


Figure 1.4.1: Source: `http://tidesandcurrents.noaa.gov/gmap3/`

16.[R] The predicted height of the tide at San Francisco for May 3, 2009 is shown in Figure 1.4.1.

(a) At what time was the tide falling the fastest?

(b) At what time was it rising the fastest?

(c) At what times was it changing most slowly?

(d) How high was the highest tide? The lowest?

(e) At what rate was the tide going down at 2 p.m.? NOTE: Express this answer in feet per hour.

17.[R] Evaluate as simply as possible.

(a) $\log_3\left(3^{17.21}\right)$,

(b) $\log_5\left(5^{\sqrt{2}}/25^{\sqrt{3}}\right)$,

(c) $\log_2\left(4^{123}\right)$,

(d) $\log_2\left((4^5)^6\right)$,

(e) $\tan(\arctan(3))$.

18.[M] Give an example of (a) an increasing function f defined for positive x such that $f(f(x)) = x^9$ and (b) a decreasing function g such that $g(g(x)) = x^9$.

19.[M] Graph each of the following functions

(a) $\sin(x)$, x in $[0, 2\pi]$,

(b) $\sin(3x)$, x in $[0, \pi/2]$,

(c) $\sin(x - \pi)$, x in $[0, 2\pi]$,

(d) $\sin(3x - \pi/6)$, x in $[0, \pi/2]$.

20.[M] Imagine that the exponential key, x^y, on your calculator is broken. How would you compute $(2.73)^{3.09}$?

21.[M] The equation $y = x - \mathrm{e}\sin(x)$, known as **Kepler's equation**, is important in the study of the motion of planets. Here e is the eccentricity of an elliptical orbit, y is related to time, and x is related to an angle. For more information, visit `http://en.wikipedia.org/wiki/Keplerian_problem` or do a Google search for `Kepler equation`. NOTE: Kepler's equation, with $\mathrm{e} = 1$, reappears in Example 2 in Section 7.5 (see page 630).
The function $f(x) = x - \sin(x)$ is increasing for all numbers x. (See Exercise 27.)

(a) Graph f.

(b) Explain why, even though it cannot be found explicitly, you know the equation $y = x - \sin(x)$ can be solved for x as a function of y ($x = g(y)$).

(c) How are the graphs of $y = x - \sin(x)$ and $y = g(x)$ related?

22.[C] Copy and label each of the following in Figure 2.6.1(b).

(a) $y = x^2$,

(b) $y = x^3$,

(c) $y = 2^x$,

(d) $y = \log_2(x)$,

(e) $y = \log_3(x)$, and

(f) $f(x) = \left(\frac{1}{2}\right)^x$.

23.[C] The equation $\log_a(b)\cdot\log_b(a) = 1$ makes one wonder, "Is $\log_a(b)\cdot\log_b(c)\log_c(a) = 1$?" What is the answer? Either exhibit positive a, b, and c for which the equation does not hold or else prove it always hold.

24.[C] Find all numbers a and b such that $\log_a(b)$ equals $\log_b(a)$.

25.[C] Some people, when careless, like to assume that the logarithm of a sum of two numbers is related to the logarithm of the individual numbers. This is a common error.

(a) Show that $\log_{100}(100 + 1/100)$ is not equal to $\log_{100}(100)$ plus $\log_{100}(1/100)$. HINT: Start by setting $\log(b + c)$ equal to $\log(b) + \log(c)$.

(b) Show that $\log_a(8 + 8/7)$ happens to equal $\log_a(8)$ plus $\log_a(8/7)$.

(c) Find all numbers b and c such that $\log_a(b + c)$ happens to equal the sum of $\log_a(b)$ and $\log_a(c)$.

26.[C] A solar cooker can be made in the shape of part of a sphere. The one in Figure 1.4.2 spans only $\pi/3$ (60°) at the center $\mathcal{O}$. For simplicity, we take the radius to be 1.

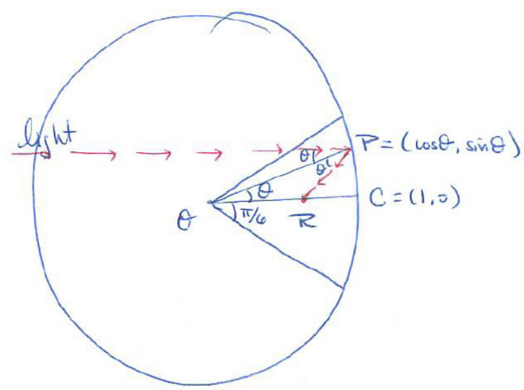


Figure 1.4.2:
Light parallel to $\mathcal{O}C$ strikes the cooker at $P = (\cos(\theta), \sin(\theta))$ and is reflected to a point R on the radius $\mathcal{O}C$.

(a) There are two angles of measure θ at P. Why is the top one equal to θ?

(b) Why is the bottom angle at P also θ?

(c) Show that $\overline{\mathcal{O}R} = 1/(2\cos(\theta))$.

(d) Show that the "heated part" of the x-axis has length $(1/\sqrt{3}) - (1/2) \approx 0.077$, or about 1/13th of the radius.

The Calculus is Everywhere section at the end of Chapter 3 describes a parabolic reflector, which reflects all of the light to a single point.

27.[C] Show that for $x < \pi/2$, $x - \sin(x)$ is an increasing function. HINT: Display x and $\sin(x)$ using a unit circle, for two values of x, a and b. NOTE: See also Exercise 21.

Calculus is Everywhere
Graphs Tell It All

The graph of a function conveys a great deal of information quickly. Here are four examples, all based on numerical data.

The Hybrid Car

A friend of ours bought a hybrid car that runs on a fuel cell at low speeds and on gasoline at higher speeds and a combination of the two power supplies in between. He also purchased the gadget that exhibits "miles-per-gallon" at any instant. With the driver glancing at the speedometer and the passenger watching the gadget, we collected data on fuel consumption (miles-per-gallon) as a function of speed. Figure 1.4.1 displays what we observed.

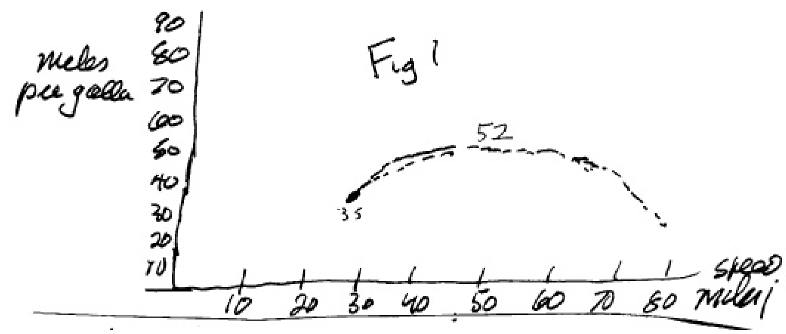


Figure 1.4.1:

The straight-line part is misleading, for at low speeds no gasoline is used. So 100 plays the role of infinity. The "sweet spot," the speed that maximizes fuel efficiency (as determined by miles-per-gallon), is about 55 mph, while speeds in the range from 40 mph to 70 mph are almost as efficient. However, at 80 mph the car gets only about 30 mpg.

To avoid having to use 100 to represent infinity, we also graph gallons-per-mile, the reciprocal of miles-per-gallon, as shown in Figure 1.4.2. In this graph the minimum occurs at 55 mph. And the straight line part of the graph on the speed axis (horizontal) records zero gallons per mile.

Life Insurance

The graphs in Figure 1.4.3 compare the cost of a million-dollar life insurance policy for a non-smoker and for a smoker, for men at various ages. A glance at

By definition, a "non-smoker" has not used any tobacco product in the previous three years.

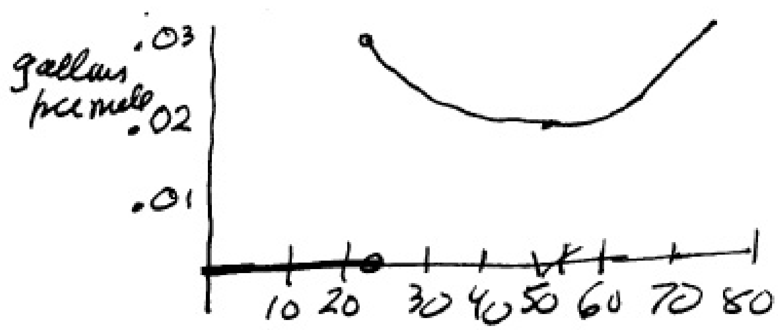


Figure 1.4.2:

the graphs shows that at a given age the smoker pays about three times what a non-smoker pays. One can also see, for instance, that a 20-year-old smoker pays more than a 40-year-old non-smoker.

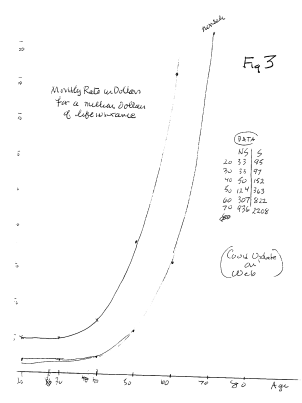


Figure 1.4.3: Source: American General Life Insurance Company advertisement

Traffic and Accidents

Figure 1.4.4 appears in S.K. Stein's, *Risk Factors of Sober and Drunk Drivers by Time of Day*, Alcohol, Drugs, and Driving 5 (1989), pp. 215–227. The vertical scale is described in the paper.

Glancing at the graph labeled "traffic" we see that there are peaks at the morning and afternoon rush hours, with minimum traffic around 3A.M.. However, the number of accidents is fairly high at that hour. "Risk" is measured by the quotient, "accidents divided by traffic." This reaches a peak at 1a.m.. The high risk cannot be explained by the darkness at that hour, for the risk rapidly decreases the rest of the night. It turns out that the risk has the same shape as the graph that records the number of drunk drivers.

It is a sobering thought that at any time of day a drunk's risk of being involved in an accident is on the order of one hundred times that of an alcohol-free driver at any time of day.

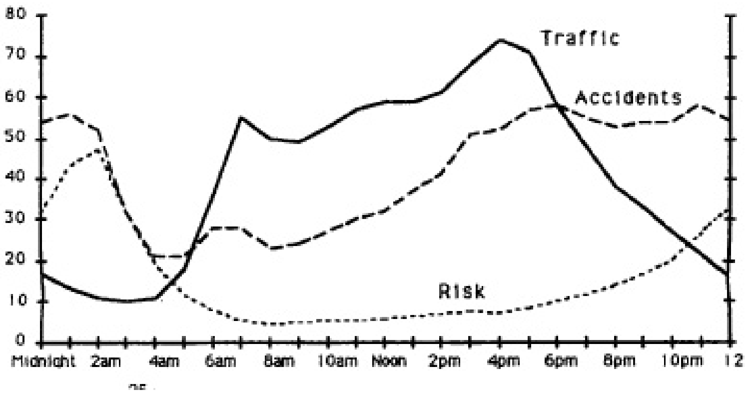


Figure 1.4.4:

Petroleum

The three graphs in Figure 1.4.5 show the rate of crude oil production in the United States, the rate at which it was imported, and their sum, the rate of consumption. They are expressed in millions of barrels per day, as a function of time. A barrel contains 42 gallons. (For a few years after the discovery of oil in Pennsylvania in 1859 oil was transported in barrels.)

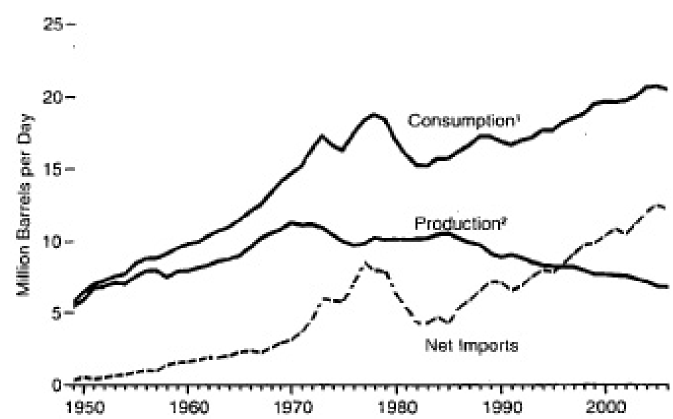


Figure 1.4.5: Source: Energy Information Administration (Annual Energy Review, 2006)

The graphs convey a good deal of history and a warning. In 1950 the United States produced almost enough petroleum to meet its needs, but by

2006 it had to import most of the petroleum consumed. Moreover, domestic production peaked in 1970.

The imbalance between production and consumption raises serious questions, especially as exporting countries need more oil to fuel their own growing economies, and developing nations, such as India and China, place rapidly increasing demands on world production. Also, since the total amount of petroleum in the earth is finite, it will run out, and the Age of Oil will end. Geologists, having gone over the globe with a "fine-tooth comb," believe they have already found all the major oil deposits. No wonder that the development of alternative sources of energy has become a high priority.

Chapter 2

Introduction to Calculus

There are two main concepts in calculus: the derivative and the integral. Underlying both is the concept of a limit. This chapter introduces limits, with an emphasis on developing both your understanding of limits and techniques for finding them.

The four limits introduced in Section 2.1 provide the foundation for computing many other limits, particularly the ones needed in Chapter 3.

2.1 Four special limits

This section develops the notion of a limit of a function, using four examples that play a key role in Chapter 3.

A Limit Involving x^n

Let a and n be fixed numbers, with n a positive integer.

What happens to the quotient $\dfrac{x^n - a^n}{x - a}$ as x is chosen nearer and nearer to a? (2.1.1)

To keep the reasoning down-to-earth, let's look at a typical concrete case:

$$\text{What happens to } \frac{x^3 - 2^3}{x - 2} \text{ as } x \text{ gets closer and closer to 2?} \qquad (2.1.2)$$

As x approaches 2, the numerator approaches $2^3 - 2^3 = 0$. Because 0 divided by anything (other than 0) is 0 we suspect that the quotient may approach 0. But the denominator approaches $2 - 2 = 0$. This is unfortunate because division by zero is not defined.

That $x^3 - 2^3$ approaches 0 as x approaches 2 may make the quotient small. That the denominator approaches 0 as x approaches 2 may make the quotient very large. How these two opposing forces balance determines what happens to the quotient (2.1.2) as x approaches 2.

We have already seen that it is pointless to replace x in (2.1.2) by 2 as this leads to $(2^3 - 2^3)/(2 - 2) = 0/0$, a meaningless expression.

Instead, let's do some experiments and see how the quotient behaves for specific values of x near 2; some less than 2, some more than 2. Table 2.1.1 shows the results as x increases from 1.9 to 2.1. You are invited to fill in the empty squares in the table below and add to the list with values of x even closer to 2.

Math is not a spectator sport. Check some of the calculations reported in Table 2.1.1.

As x increases the quotient increases.

The cases with $x = 1.99$ and 2.01, being closest to 2, should provide the best estimates of the quotient. This suggests that the quotient (2.1.2) approaches a number near 12 as x approaches 2, whether from below or from above.

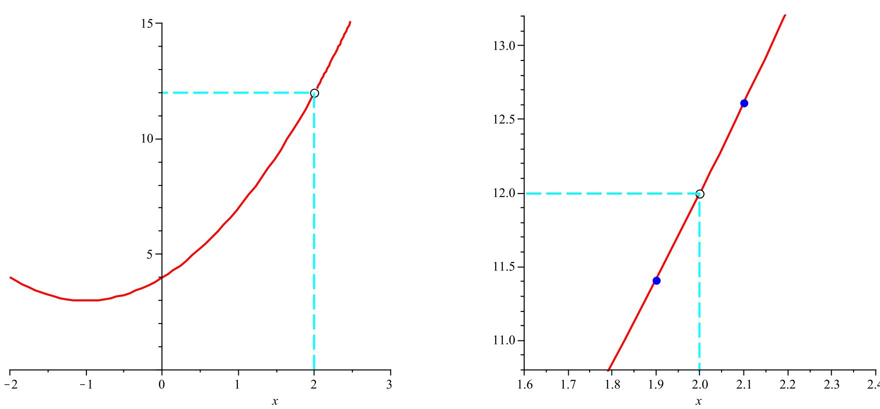
While the numerical and graphical evidence is suggestive, this question can be answered once-and-for-all with a little bit of algebra. You can check that $x^3 - 2^3 = (x - 2)(x^2 + 2x + 2^2)$. We have

$$\frac{x^3 - 2^3}{x - 2} = \frac{(x - 2)(x^2 + 2x + 2^2)}{x - 2} \qquad \text{for all } x \text{ other than 2.} \qquad (2.1.3)$$

x	x^3	x^3-2^3	$x-2$	$\frac{x^3-2^3}{x-2}$
1.90	6.859	-1.141	-0.1	11.41
1.99	7.8806	-0.1194	-0.01	11.94
1.999				
2.00	8.0000	0.0000	0.00	undefined
2.001				
2.01	8.1206	0.1206	0.01	12.06
2.10	9.261	1.261	0.1	12.61

Table 2.1.1: Table showing the steps in the evaluation of $\frac{x^3-2^3}{x-2}$ for four choices of x near 2.

Recall that a hollow dot on a graph indicates that that point is NOT on the graph.



(a) (b)

Figure 2.1.1: The graph of a $y=\frac{x^3-2^3}{x-2}$ suggests that the quotient approaches 12 as x approaches 2. In (b), zooming for x near 2 shows how the data in Table 2.1.1 also suggests the quotient approaches 12 as x approaches 2.

For $x = 2$, both sides of (2.1.3) become the meaningless expression 0/0. Canceling the $(x-2)$ in (2.1.3) shows that

$$\frac{x^3 - 2^3}{x - 2} = x^2 + 2x + 2^2, \qquad x \neq 2.$$

It is easy to see what happens to $x^2 + 2x + 2^2$ as x gets nearer and nearer to 2: $x^2 + 2x + x^2$ approaches $4 + 4 + 4 = 12$. This agrees with the calculations (see Table 2.1.1).

We say "the limit of $(x^3 - 2^3)/(x - 2)$ as x approaches 2 is 12" and use the shorthand

$$\lim_{x\to 2} \frac{x^3 - 2^3}{x - 2} = \lim_{x\to 2}(x^2 + 2x + 2^2) \tag{2.1.4}$$

$$= 3 \cdot 2^2 = 12. \tag{2.1.5}$$

Similar algebra yields the following limit, which will be used in the next chapter. (See Exercises 39 and 40.)

For any positive integer n and fixed number a,

$$\lim_{x\to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1} \tag{2.1.6}$$

A Limit Involving b^x

What happens to $\frac{2^x - 1}{x}$ and to $\frac{4^x - 1}{x}$ as x approaches 0?

Consider $(2^x - 1)/x$ first: As x approaches 0, $2^x - 1$ approaches $2^0 - 1 = 1 - 1 = 0$. Since the numerator and denominator in $(2^x - 1)/x$ both approach 0 as x approaches 0, we face the same challenge as with $(x^3 - 2^3)/(x - 2)$. There is a battle between two opposing forces.

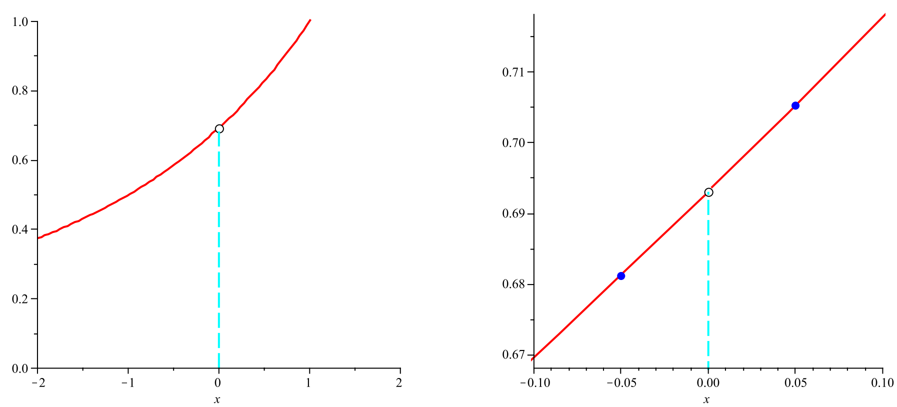
There are no algebraic tricks to help in this case. Instead, we will rely upon numerical data. While this motivation will be convincing, it is not mathematically rigorous. Later, in Appendix D, we will present a way to evaluate these limits that does not depend upon any numerical computations.

Table 2.1.2 records some results (rounded off) for four choices of x. You are invited to fill in the blanks and to add values of x even closer to 0.

WARNING *(Do not believe your eyes!)* The graphs in Figure 2.1.1(b) and Figure 2.1.2(b) are not the graphs of straight lines. They look straight only because the viewing windows are so small. Compare the labels on the axes in the two views in each of Figure 2.1.1 and Figure 2.1.2. That the graphs of many common functions look straight as you zoom in on a point will be important in Section 3.1.

x	2^x	$2^x - 1$	$\frac{2^x-1}{x}$
-0.01	0.993093	-0.006907	0.691
-0.001	0.999307	-0.000693	0.693
-0.0001			
0.0001			
0.001	1.000693	0.000693	0.693
0.01	1.006956	0.006956	0.696

Table 2.1.2: Numerical evaluation of $(2^x - 1)/x$ for four different choices of x. The numbers in the last column are rounded to three decimal places. See also Figure 2.1.2.



(a) (b)

Figure 2.1.2: (a) Graph of $y = (2^x - 1)/x$ for x near 0. (b) View for x nearer to 0, with the data points from Table 2.1.2. Note that there is no point for $x = 0$ since the quotient is not defined when x is 0.

It seems that, as x approaches 0, $(2^x - 1)/x$ approaches a number whose decimal value begins 0.693. We write

$$\lim_{x \to 0} \frac{2^x - 1}{x} \approx 0.693 \qquad \text{rounded to three decimal places.} \tag{2.1.7}$$

It is then a simple matter to find

$$\lim_{x \to 0} \frac{4^x - 1}{x}.$$

Recalling the algebraic identity for the difference of two squares, $a^2 - b^2 = (a - b)(a + b)$, we have $4^x - 1 = (2^x)^2 - 1^2 = (2^x - 1)(2^x + 1)$. Hence

$$\frac{4^x - 1}{x} = \frac{(2^x - 1)(2^x + 1)}{x} = (2^x + 1)\frac{2^x - 1}{x}.$$

As $x \to 0$, $2^x + 1$ approaches $2^0 + 1 = 1 + 1 = 2$ and $(2^x - 1)/x$ approaches (approximately) 0.693. Thus,

$$\lim_{x \to 0} \frac{4^x - 1}{x} \approx 2 \cdot 0.693 \approx 1.386 \qquad \text{rounded to three decimal places.}$$

We now have strong evidence about the values of $\lim\limits_{x \to 0} \dfrac{b^x - 1}{x}$ for $b = 2$ and $b = 4$. They suggest that the larger b is, the larger the limit is. Since $\lim_{x \to 0} \frac{2^x - 1}{x}$ is less than 1 and $\lim_{x \to 0} \frac{4^x - 1}{x}$ is more than 1, it seems reasonable that there should be a value of b such that $\lim_{x \to 0} \frac{b^x - 1}{x} = 1$. This special number is called e, **Euler's number**. We know that e is between 2 and 4 and that $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$. It turns out that e is an irrational number with an endless decimal representation that begins $2.718281828\ldots$. In Chapter 3 we will see that e is as important in calculus as π is in geometry and trigonometry.

Euler named this constant e, but no one knows why he chose this symbol.

In any case we have

Basic Property of e

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1, \qquad \text{and } e \approx 2.71828.$$

In Section 1.2 it was remarked that the logarithm with base b, $\log_b$, can be defined for any base $b > 0$. The logarithm with base $b = e$ deserves special attention. The $\log_e(x)$ is called the **natural logarithm**, and is typically written as $\ln(x)$ or $\log(x)$. Thus, in particular,

$$y = \ln(x) \quad \text{is equivalent to} \quad x = e^y.$$

Note that, as with any logarithm function, the domain of ln is the set of positive numbers $(0, \infty)$ and the range is the set of all real numbers $(-\infty, \infty)$.

In Exercise 38 it is shown that $\lim_{x\to 0} \frac{b^x-1}{x}$ is $\ln(b)$.

Often the exponential function with base e is written as exp. This notation is convenient when the input is complicated:

$$\exp\left(\frac{\sin^3(\sqrt{x})}{\cos(x)}\right) \qquad \text{is easier to read than} \qquad e^{\sin^3(\sqrt{x})/\cos(x)}.$$

Many calculators and computer languages use exp to name the exponential function with base e.

> **Three Important Bases for Logarithms**
> While logarithms can be defined for any positive base, three numbers have been used most often: 2, 10, and e. Logarithms to the base 2 are used in information theory, for they record the number of "yes – no" questions needed to pinpoint a particular piece of information. Base 10 was used for centuries to assist in computations. Since the decimal system is based on powers of 10, certain convenient numbers had obvious logarithms; for instance, $\log(1000) = \log(10^3) = 3$. Tables of logarithms to several decimal places facilitated the calculations of products, quotients, and roots. To multiply two numbers, you looked up their logarithms, and then searched the table for the number whose logarithm was the sum of the two logarithms. The calculator made the tables obsolete, just as it sent the slide rule into museums. However, a Google search for "slide rule" returns a list of more than 15 million websites full of history, instruction, and sentiment. The number e is the most convenient base for logarithms in calculus. Euler, as early as 1728, used e for the base of logarithms.

A Limit Involving $\sin(x)$

What happens to $\frac{\sin(x)}{x}$ as x gets nearer and nearer to 0?

Here x represents an angle, measure in radians. In Chapter 3 we will see that in calculus radians are much more convenient than degrees.

Appendix E includes a review of radians.

Consider first $x > 0$. Because we are interested in x near 0, we assume that $x < \pi/2$. Figure 2.1.3 identifies both x and $\sin(x)$ on a circle of radius 1, the **unit circle**.

To get an idea of the value of this limit, let's try $x = 0.1$. Setting our calculator in the "radian mode", we find

You should graph $y = \frac{\sin(x)}{x}$ on your calculator.

$$\frac{\sin(0.1)}{0.1} \approx \frac{0.099833}{0.1} = 0.99833. \tag{2.1.8}$$

Likewise, with $x = 0.01$,

$$\frac{\sin(0.1)}{0.01} \approx \frac{0.0099998}{0.01} = 0.99998. \tag{2.1.9}$$

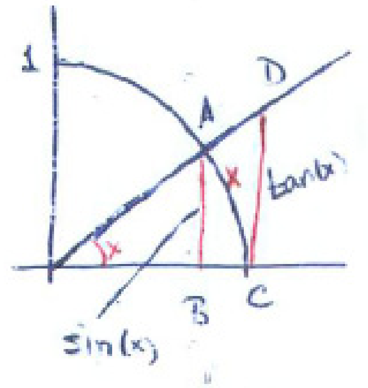


Figure 2.1.3: On the circle with radius 1, (a) x is the arclength subtended by an angle of x radians and $\sin(x) = \overline{AB}$.

These results lead us to suspect maybe this limit is 1.

Geometry and a bit of trigonometry can be used to show that $\lim_{x\to 0} \frac{\sin(x)}{x}$ is indeed 1. First, using Figure 2.1.3, we show that $\frac{\sin(x)}{x}$ is less than 1 for all positive x. Recall that $\sin(x) = \overline{AB}$. Now, $\overline{AB}$ is shorter than $\overline{AC}$, since a leg of a right triangle is shorter than the hypotenuse. Then $\overline{AC}$ is shorter than the circular arc joining A to C, since the shortest distance between two points is a straight line. Thus,

$$\sin(x) < \overline{AC} < x.$$

So $\sin(x) < x$. Since x is positive, dividing by x preserves the inequality. We have

$$\frac{\sin(x)}{x} < 1. \tag{2.1.10}$$

Next, we show that $\frac{\sin(x)}{x}$ is greater than something which gets near 1 as x approaches 0. Figure 2.1.3 again helps with this step.

The area of triangle OCD is greater than the area of the sector OCA. (The area of a sector of a disk of radius r subtended by an angle θ is $\theta r^2/2$.) Thus

$$\underbrace{\frac{1}{2}\cdot 1\cdot \tan(x)}_{\text{area of } \Delta OCD} > \underbrace{\frac{x\cdot 1^2}{2}}_{\text{area of sector } OCA}.$$

Multiplying this inequality by 2 simplifies it to

$$\tan(x) > x.$$

In other words,

$$\frac{\sin(x)}{\cos(x)} > x.$$

Now, multiplying $\cos(x)$ which is positive and dividing by x (also positive) gives

$$\frac{\sin(x)}{x} > \cos(x). \tag{2.1.11}$$

Putting (2.1.10) and (2.1.11) together, we have

$$\cos(x) < \frac{\sin(x)}{x} < 1. \tag{2.1.12}$$

Since $\cos(x)$ approaches 1 as x approaches 0 , $\frac{\sin(x)}{x}$ is squeezed between 1 and something that gets closer and closer to 1, $\frac{\sin(x)}{x}$ must itself approach 1.

We still must look at $\frac{\sin(x)}{x}$ for $x < 0$ as x gets nearer and nearer to 0. Define u to be $-x$. Then u is positive, and

$$\frac{\sin(x)}{x} = \frac{\sin(-u)}{-u} = \frac{-\sin u}{-u} = \frac{\sin u}{u}.$$

As x is negative and approaches zero, u is positive and approaches 0. Thus $\frac{\sin(x)}{x}$ approaches 1 as x approaches 0 through positive or negative values.

In short,

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \text{where the angle, } x\text{, is measured in radians.}$$

A Limit Involving $\cos(x)$

Knowing that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, we can show that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0. \tag{2.1.13}$$

All we will say about this limit now is that the numerator, $1 - \cos(x)$ is the length of BC in Figure 2.1.3. Exercises 26 and 27 outline how to establish this limit.

The Meaning of $\lim_{x\to\to 0} \frac{\sin(x)}{x} = 1$

When x is near 0, $\sin(x)$ and x are both small. That their quotient is near 1 tells us much more, namely, that x is a "very good approximation of $\sin(x)$."

That means that the difference $\sin(x) - x$ is small, even in comparison to $\sin(x)$. In other words, the "relative error"

$$\frac{\sin(x) - x}{\sin(x)} \tag{2.1.14}$$

approaches 0 as x approaches 0.
To show that this is the case, we compute

$$\lim_{x\to 0} \frac{\sin(x) - x}{\sin(x)}.$$

We have

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x}{\sin(x)} &= \lim_{x\to 0} \left(\frac{\sin(x)}{\sin(x)} - \frac{x}{\sin(x)} \right) \\
&= \lim_{x\to 0} \left(1 - \frac{x}{\sin(x)} \right) \\
&= \lim_{x\to 0} \left(1 - \frac{1}{\left(\frac{x}{\sin(x)} \right)} \right) \\
&= 1 - \frac{1}{1} = 0.
\end{aligned}$$

As you may check by graphing, the relative error in (2.1.14) stays less than 1 percent for x less than 0.24 radians, just under 14 degrees.

It is often useful to replace $\sin(x)$ by the much simpler quantity x. For instance, the force tending to return a swinging pendulum is proportional to $\sin(\theta)$, where θ is the angle that the pendulum makes with the vertical. As one physics book says, "If the angle is small, $\sin(\theta)$ is nearly equal to θ"; it then replaces $\sin(\theta)$ by θ.

Summary

This section discussed four important limits:

$$
\begin{aligned}
\lim_{x \to a} \frac{x^n - a^n}{x - a} &= na^{n-1} && (n \text{ a positive integer}) \\
\lim_{x \to 0} \frac{e^x - 1}{x} &= 1 && (e \approx 2.71828) \\
\lim_{x \to 0} \frac{\sin(x)}{x} &= 1 && (\text{angle in radians}) \\
\lim_{x \to 0} \frac{1 - \cos(x)}{x} &= 0 && (\text{angle in radians}).
\end{aligned}
$$

That is, $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$ says, informally, that $\frac{\text{exp(a small number} - 1)}{\text{same small number}}$ is near 1.

Each of these limits will be needed in Chapter 3, which introduces the derivative of a function.

The next section examines the general notion of a limit. This is the basis for all of calculus.

EXERCISES for 2.1

Key: R–routine, M–moderate, C–challenging

In each of Exercises 1 to 10 describe the two opposing forces involved in the limit. If you can figure out the limit on the basis of results in this section, give it. Otherwise, use a calculator to estimate the limit.

1.[R] $\lim_{x\to 2} \frac{x^4 - 16}{x - 2}$

2.[R] $\lim_{x\to 0} \frac{\sin(x)}{x\cos(x)}$

3.[R] $\lim_{x\to 0} (1 - x)^{1/x}$

4.[R] $\lim_{x\to 0} (\cos(x))^{1/x}$

5.[R] $\lim_{x\to 0} x^x$, $x > 0$

6.[R] $\lim_{x\to 0} \frac{\arcsin(x)}{x}$

7.[R] $\lim_{x\to 0} \frac{\tan(x)}{x}$ HINT: Write $\tan(x) = \sin(x)/\cos(x)$.

8.[R] $\lim_{x\to 0} \frac{\tan(2x)}{x}$

9.[R] $\lim_{x\to 0} \frac{8^x - 1}{2^x - 1}$ HINT: The numerator is the difference of two cubes; how does $b^3 - a^3$ factor?

10.[R] $\lim_{x\to 0} \frac{9^x - 1}{3^x - 1}$

Exercises 11 to 15 concern $\lim_{x\to a} \frac{x^n - a^n}{x - a}$.

11.[R] Using the factorization $(x - a)(x + a) = x^2 - a^2$ find $\lim_{x\to a} \frac{x^2 - a^2}{x - a}$.

12.[R] Using Exercise 11,

(a) find $\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$

(b) find $\lim_{x\to\sqrt{3}} \frac{x^2 - 3}{x - \sqrt{3}}$

13.[R]

(a) By multiplying it out, show that $(x - a)(x^2 + ax + a^2) = x^3 - a^3$.

(b) Use (a) to show that $\lim_{x\to a} \frac{x^3 - a^3}{x - a} = 3a^2$.

(c) By multiplying it out, show that $(x - a)(x^3 + ax^2 + a^2x + a^3) = x^4 - a^4$.

(d) Use (c) to show that $\lim_{x\to a} \frac{x^4 - a^4}{x - a} = 4a^3$.

14.[R]

(a) What is the domain of $(x^2-9)/(x-3)$?

(b) Graph $(x^2-9)/(x-3)$.

NOTE: Use a hollow dot to indicate an absent point in the graph.

15.[R]

(a) What is the domain of $(x^3-8)/(x-2)$?

(b) Graph $(x^3-8)/(x-2)$.

Exercises 16 to 19 concern $\lim_{x\to 0} \frac{e^x-1}{x}$.

16.[R] What is a definition of the number e?

17.[R] Use a calculator to compute $(2.7^x-1)/x$ and $(2.8^x-1)/x$ for $x = 0.001$. NOTE: This suggests that e is between 2.7 and 2.8.

18.[R] Use a calculator to estimate $(2.718^x-1)/x$ for $x = 0.1$, 0.01, and 0.001.

19.[R] Graph $y = (e^x-1)/x$ for $x \neq 0$.

Exercises 20 to 28 concern $\lim_{x\to 0} \frac{\sin(x)}{x}$ and $\lim_{x\to 0} \frac{1-\cos(x)}{x}$.

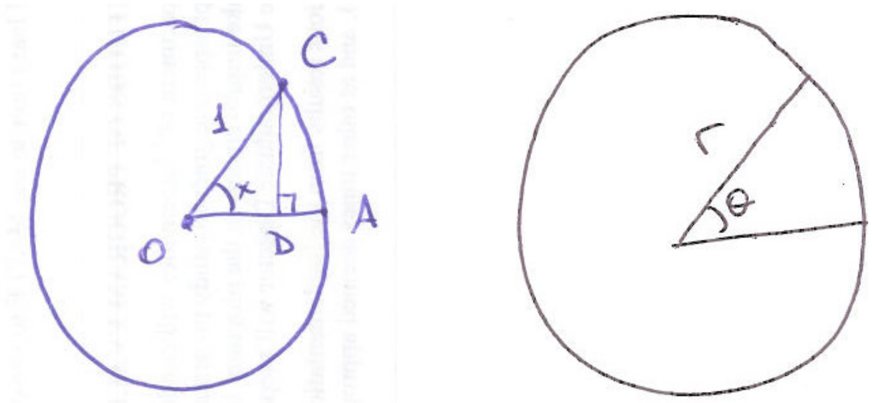
20.[R] Using the fact that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, find the limits of the following as x approaches 0.

(a) $\frac{\sin(3x)}{3x}$

(b) $\frac{\sin(3x)}{x}$

(c) $\frac{\sin(3x)}{\sin(x)}$

(d) $\frac{\sin^2(x)}{x}$



(a) (b)

Figure 2.1.4:

21.[R] Why is the arc length from A to C in Figure 2.1.4(a) equal to x?

22.[R] Why is the length of CD in Figure 2.1.4(a) equal to $\tan x$?

23.[R] Why is the area of triangle OCD in Figure 2.1.4(a) equal to $(\tan x)/2$?

24.[R] An angle of θ radians in a circle of radius r subtends a sector, as shown in Figure 2.1.4(b). What is the area of this sector? NOTE: For a review of trigonometry, see Appendix E.

25.[R]

(a) Graph $\sin(x)/x$ for x in $[-\pi, 0)$

(b) Graph $\sin(x)/x$ for x in $(0, \pi]$.

(c) How are the graphs in (a) and (b) related?

(d) Graph $\sin(x)/x$ for $x \neq 0$.

26.[R] When $x = 0$, $(1 - \cos(x))/x$ is not defined. Estimate $\lim_{x\to 0} \frac{1-\cos(x)}{x}$ by evaluating $(1 - \cos(x))/x$ at $x = 0.1$ (radians).

27.[R] To find $\lim_{x\to 0} \frac{1-\cos(x)}{x}$ first check this algebra and trigonometry:

$$\frac{1-\cos(x)}{x} = \frac{1-\cos(x)}{x}\frac{1+\cos(x)}{1+\cos(x)} = \frac{1-\cos^2(x)}{x(1+\cos(x))} = \frac{\sin^2(x)}{x(1+\cos(x))} = \frac{\sin(x)}{x}\frac{\sin(x)}{1+\cos(x)}.$$

Then show that

$$\lim_{x\to 0} \frac{\sin(x)}{x}\frac{\sin(x)}{1+\cos(x)} = 0.$$

28.[M] Show that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}.$$

Thus suggests that, for small values of x, $1-\cos(x)$ is close to $\frac{x^2}{2}$, so that $\cos(x)$ is approximately $1-\frac{x^2}{2}$.

(a) Use a calculator to compare $\cos(x)$ with $1-\frac{x^2}{2}$ for $x = 0.2$ and 0.1 radians. NOTE: 0.2 radians is about 11°.

(b) Use a graphing calculator to compare the graphs of $\cos(x)$ and $1-\frac{x^2}{2}$ for x in $[-\pi, \pi]$.

(c) What is the largest interval on which the values of $\cos(x)$ and $1-\frac{x^2}{2}$ differ by no more than 0.1? That is, for what values of x is it true that $\left|\cos(x) - (1-\frac{x^2}{2})\right| < 0.1$?

NOTE: See Exercise 27.

29.[M] In the design of the water sprinkler that appears in the "Calculus is Everywhere" in Chapter 5 $\lim_{\theta\to 0} \frac{\sin(4\theta)}{\sin(\theta)}$ appears. Find that limit.

30.[M]

(a) We examined $(2^x-1)/x$ only for x near 0. When x is large and positive 2^x-1 is large. So both the numerator and denominator of $(2^x-1)/x$ are large. The numerator influences the quotient to become large. The large denominator pushes the quotient toward 0. Use a calculator to see how the two forces balance for large values of x.

(b) Sketch the graph of $f(x) = (2^x-1)/x$ for $x > 0$. (Pay special attention to the behavior of the graph for large values of x.)

31.[M]

(a) When x is large and negative, what happens to $(2^x-1)/x$?

(b) Sketch the graph of $f(x) = (2^x-1)/x$ for $x < 0$. (Pay special attention to the behavior of the graph for large negative values of x.)

32.[M]

(a) Using a calculator, explore what happens to $\sqrt{x^2+x}-x$ for large positive values of x.

(b) Show that for $x>0$, $\sqrt{x^2+x}<x+(1/2)$.

(c) Using algebra, find what number $\sqrt{x^2+x}-x$ approaches as x increases. Hint: Multiply $\sqrt{x^2+x}-x$ by $\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$, an operation called **conjugating**.

33.[M] Using a calculator, examine the behavior of the quotient $(\theta-\sin(\theta))/\theta^3$ for θ near 0.

34.[M] Using a calculator, examine the behavior of the quotient $\left(\cos(\theta)-1+\frac{\theta^2}{2}\right)/\theta^4$ for θ near 0.

Exercises 35 to 38 concern $f(x)=(1+x)^{1/x}$, x in $(-1,0)$ and $(0,\infty)$.

35.[M]

(a) Why is $(1+x)^{1/x}$ not defined when $x=0$?

(b) For x near 0, $x>0$, $1+x$ is near 1. So we might expect $(1+x)^{1/x}$ to be near 1 then. However, the exponent $1/x$ is very large. So perhaps $(1+x)^{1/x}$ is also large. To see what happens, fill in this table.

x	1	0.5	0.1	0.01	0.001
$1+x$	2				
$1/x$	1				
$(1+x)^{1/x}$	2				

(c) For x near 0 but negative, investigate $(1+x)^{1/x}$ with the use of this table

x	-0.5	-0.1	-0.01	-0.001
$1+x$	0.5			
$1/x$	-2			
$(1+x)^{1/x}$	4			

36.[M] Graph $y=(1+x)^{1/x}$ for x in $(-1,0)$ and $(0,10)$.

Exercises 35 and 36 show that $\lim_{x\to 0}(1+x)^{1/x}$ is about 2.718. Thus suggests that the number e may equal $\lim_{x\to 0}(1+x)^{1/x}$. In Section 3.2 we show that this is the case. However, the next two exercises give persuasive arguments for this fact. Unfortunately, each argument has a big hole or "unjustified leap," which you are asked to find.

37.[C] Assume that all we know about the number e is that $\lim_{x\to 0}\frac{e^x-1}{x}=1$. We will write this as

$$\frac{e^x-1}{x}\sim 1,$$

and read this as "$(e^x-1)/x$ is close to 1 when x is near 0." Multiplying both sides by x gives

$$e^x-1\sim x.$$

Adding 1 to both sides of this gives

$$e^x\sim 1+x.$$

Finally, raising both sides to the power $1/x$ gives

$$(e^x)^{1/x}\sim(1+x)^{1/x}.$$

Hence

$$e\sim(1+x)^{1/x}.$$

This suggests that

$$e=\lim_{x\to 0}(1+x)^{1/x}.$$

The conclusion is correct. Most of the steps are justified. Which step is the "big leap"?

38.[C] Assume that $b=\lim_{x\to 0}(1+x)^{1/x}$. We will "show" that

$$\lim_{x\to 0}\frac{b^x-1}{x}=1.$$

First of all, for x near (but not equal to) 0

$$b\sim(1+x)^{1/x}.$$

Then

$$b^x\sim 1+x.$$

Hence

$$b^x-1\sim x.$$

Dividing by x gives

$$\frac{b^x-1}{x}\sim 1.$$

Hence

$$\lim_{x\to 0}\frac{b^x-1}{x}=1.$$

where is the "suspect step" this time?

39.[C] Let n be a positive integer and define $P_n(x) = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}$. This polynomial is equal to the quotient $\frac{x^n - a^n}{x-a}$. That is $(x-a)P_n(x) = x^n - a^n$. (This factorization is justified in Exercise 44 in Section 5.4.)

(a) Verify that $(x-a)P_2(x) = x^2 - a^2$. (Compare with Exercise 11)

(b) Verify that $(x-a)P_3(x) = x^3 - a^3$. (Compare with Exercise 13(a))

(c) Verify that $(x-a)P_4(x) = x^4 - a^4$. (Compare with Exercise 13(c))

(d) Explain why $(x-a)P_n(x) = x^n - a^n$ for all positive integers n.

40.[C] Use Exercise 39 to show that $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

41.[C] An intuitive argument suggested that $\lim_{\theta \to 0} (\sin\theta)/\theta = 1$, which turned out to be correct. Try your intuition on another limit associated with the unit circle shown in Figure 2.1.5.

(a) What do you think happens to the quotient

$$\frac{\text{Area of triangle } ABC}{\text{Area of shaded region}} \qquad \text{as } \theta \to 0?$$

More precisely, what does your intuition suggest is the limit of that quotient as $\theta \to 0$?

(b) Estimate the limit in (a) using $\theta = 0.01$.

NOTE: This problem is a test of your intuition. This limit, which arose during some research in geometry, is determined in Exercise 54 in Section 5.5. The authors guessed wrong, as has everyone they asked.

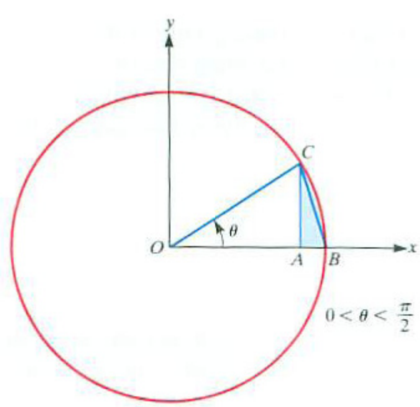


Figure 2.1.5:

2.2 The Limit of a Function: The General Case

Section 2.1 concerned four important limits:

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1},\quad \lim_{x\to 0}\frac{e^x-1}{x}=1,\quad \lim_{x\to 0}\frac{\sin(x)}{x}=1,\quad \lim_{x\to 0}\frac{1-\cos(x)}{x}=0.$$

Limits are fundamental to all of calculus. In this section, we pause to discuss the concept of a limit, beginning with the notion of a one-sided limit.

One-Sided Limits

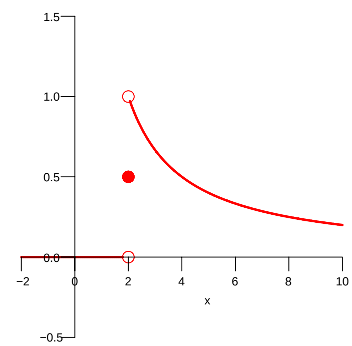


Figure 2.2.1:

The domain of the function shown in Figure 2.2.1 is $(-\infty,\infty)$. In particular, the function is defined when $x=2$ and $f(2)=1/2$. This fact is conveyed by the solid dot at $(2,1/2)$ in the figure. The hollow dots at $(2,0)$ and $(2,1)$ indicate that these points are not on the graph of this function (but some nearby points are on the graph).

Consider the part of the graph for inputs $x>2$, that is, for inputs to the right of 2. As x approaches 2 from the right, $f(x)$ approaches 1. This conclusion can be expressed as

$$\lim_{x\to 2^+} f(x)=1$$

and is read "the limit of f of x, as x approaches 2, from the right, is 1." Similarly, looking at the graph of f in Figure 2.2.1 for x to the left of 2, that is, for $x<2$, the values of $f(x)$ approach a different number, namely, 0. This is expressed with the shorthand

$$\lim_{x\to 2^-} f(x)=0.$$

It might sound strange to say the values of $f(x)$ "approach" 0 since the function values are exactly 0 for all inputs $x<2$. But, it is convenient, and customary, to use the word "approach" even for constant functions.

This illustrates the concept of the "right-hand" and "left-hand" limits, the two **one-sided limits**.

DEFINITION *(Right-hand limit of $f(x)$ at a)* Let f be a function and a some fixed number. Assume that the domain of f contains an open interval (a,c). If, as x approaches a from the right, $f(x)$ approaches a specific number L, then L is called the **right-hand limit** of $f(x)$ as x approaches a. This is written

$$\lim_{x\to a^+} f(x)=L$$

or

$$f(x) \to L \quad \text{as} \quad x \to a^+.$$

The assertion that

$$\lim_{x \to a^+} f(x) = L$$

is read "the limit of f of x as x approaches a from the right is L" or "as x approaches a from the right, $f(x)$ approaches L."

> **DEFINITION** *(Left-hand limit of $f(x)$ at a)* Let f be a function and a some fixed number. Assume that the domain of f contains an open interval (b, a). If, as x approaches a from the left, $f(x)$ approaches a specific number L, then L is called the **left-hand limit** of $f(x)$ as x approaches a. This is written
>
> $$\lim_{x \to a^-} f(x) = L$$
>
> or
>
> $$f(x) \to L \quad \text{as} \quad x \to a^-.$$

Notice that the definitions of the one-sided limits do not require that the number a be in the domain of the function f. If f is defined at a, we do not consider $f(a)$ when examining limits as x approaches a.

The Two-Sided Limit

If the two one-sided limits of $f(x)$ at $x = a$, $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$, exist and are equal to L then we say the limit of $f(x)$ as x approaches a is L.

$$\lim_{x \to a} f(x) = L \qquad \text{means} \qquad \lim_{x \to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to a^+} f(x) = L.$$

For the function graphed in Figure 2.2.1 we found that $\lim_{x \to 2^+} f(x) = 1$ and $\lim_{x \to 2^-} f(x) = 0$. Because they are different, the two-sided limit of $f(x)$ at 2, $\lim_{x \to 2} f(x)$, does not exist.

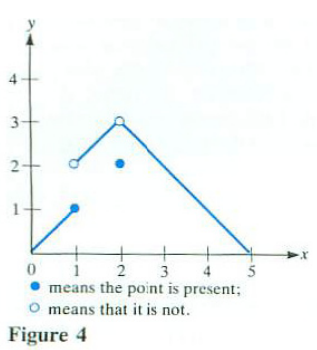


Figure 2.2.2:

EXAMPLE 1 Figure 2.2.2 shows the graph of a function f whose domain is the closed interval $[0, 5]$.

(a) Does $\lim_{x \to 1} f(x)$ exist?

(b) Does $\lim_{x \to 2} f(x)$ exist?

(c) Does $\lim_{x \to 3} f(x)$ exist?

SOLUTION

(a) Inspection of the graph shows that

$$\lim_{x\to 1^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 1^+} f(x) = 2.$$

Although the two one-sided limits exist, they are not equal. Thus, $\lim_{x\to 1} f(x)$ does not exist. In short, "f does not have a limit as x approaches 1."

(b) Inspection of the graph shows that

$$\lim_{x\to 2^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x\to 2^+} f(x) = 3.$$

Thus $\lim_{x\to 2} f(x)$ exists and is 3. That $f(2) = 2$, as indicated by the solid dot at $(2, 2)$, plays no role in our examination of the limit of $f(x)$ as $x \to 2$ (either one-sided or two-sided).

(c) Inspection, once again, shows that

$$\lim_{x\to 3^-} f(x) = 2 \qquad \text{and} \qquad \lim_{x\to 3^+} f(x) = 2.$$

Thus $\lim_{x\to 3} f(x)$ exists and is 2. Incidentally, the fact that $f(3) = 2$ is irrelevant in determining $\lim_{x\to 3} f(x)$.

◇

We now define the (two-sided) limit without referring to one-sided limits.

DEFINITION *(Limit of $f(x)$ at a.)* Let f be a function and a some fixed number. Assume that the domain of f contains open intervals (b, a) and (a, c), as shown in Figure 2.2.3. If there is a number L such that as x approaches a, from both the right and the left, $f(x)$ approaches L, then L is called the **limit** of $f(x)$ as x approaches a. This is expressed as either

$$\lim_{x\to a} f(x) = L \qquad \text{or} \qquad f(x) \to L \quad \text{as} \quad x \to a.$$

EXAMPLE 2 Let f be the function defined by by $f(x) = \dfrac{x^n - a^n}{x - a}$ where n is a positive integer. This function is defined for all x except a. How does it behave for x near a?

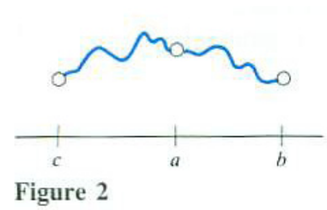


Figure 2.2.3: The function f is defined on open intervals on both sides of a.

SOLUTION In Section 2.1 and its Exercises we found that as x gets closer and closer to a, $f(x)$ gets closer and closer to na^{n-1}. This is summarized with the shorthand

$$\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1},$$

read as "the limit of $\frac{x^n-a^n}{x-a}$ as x approaches a is na^{n-1}." ◇

EXAMPLE 3 Investigate the one-sided and two-sided limits for the square root function at 0.

SOLUTION The function $\sqrt{x}$ is defined only for x in $[0, \infty)$. We can say that the right-hand limit at 0 exists since $\sqrt{x}$ approaches 0 as $x \to 0$ through positive values of x; that is, $\lim_{x\to 0^+} \sqrt{x} = 0$. Because $\sqrt{x}$ is not defined for any negative values of x, the left-hand limit of $\sqrt{x}$ at 0 does not exist. Consequently, the two-sided limit of $\sqrt{x}$ at 0, $\lim_{x\to 0} \sqrt{x}$, does not exist. ◇

EXAMPLE 4 Consider the function f defined so that $f(x) = 2$ if x is an integer and $f(x) = 1$ otherwise. For which a does $\lim_{x\to a} f(x)$ exist?

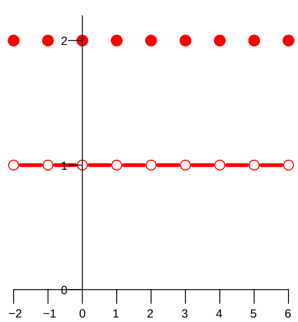


Figure 2.2.4:

SOLUTION The graph of f, shown in Figure 2.2.4, will help us decide. If a is not an integer, then for all x sufficiently near a, $f(x) = 1$. So $\lim_{x\to a} f(x) = 1$. Thus the limit exists for all a that are not integers.

Now consider the case when a is an integer. In deciding whether $\lim_{x\to a} f(x)$ exists we never consider the value of f at a, namely $f(a) = 2$. For all x sufficiently near an integer a, $f(x) = 1$. Thus, once again, $\lim_{x\to a} f(x) = 1$. The limit exists but is not $f(a)$.

Thus, $\lim_{x\to a} f(x)$ exists and equals 1 for every number a. ◇

EXAMPLE 5 Let $g(x) = \sin(1/x)$. For which a does $\lim_{x\to a} g(x)$ exist?

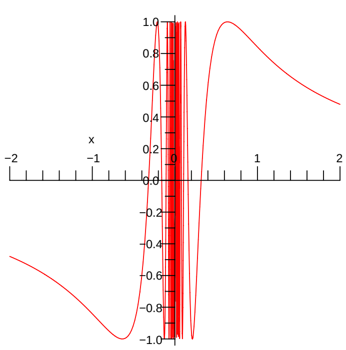


Figure 2.2.5: $y = g(x) = \sin(1/x)$.

SOLUTION To begin, graph the function. Notice that the domain of g consists of all x except 0. When x is very large, $1/x$ is very small, so $\sin(1/x)$ is small. As x approaches 0, $1/x$ becomes large. For instance, when $x = \frac{1}{2n\pi}$, for a non-zero integer n, $1/x = 2n\pi$ and therefore $\sin(1/x) = \sin(2n\pi) = 0$. Thus, the graph of $y = g(x)$ for x near 0 crosses the x-axis infinitely often. Similarly, $g(x)$ takes the values 1 and -1 infinitely often for x near 0. The graph is shown in Figure 2.2.5.

Does $\lim_{x\to 0} g(x)$ exist? Does $g(x)$ tend toward one specific number as $x \to 0$? No. The function oscillates, taking on all values from -1 to 1 (repeatedly) for x arbitrarily close to 0. Thus $\lim_{x\to 0} \sin(1/x)$ does not exist.

At all other values of a, $\lim_{x\to a} g(x)$ does exist and equals $g(a) = \sin(1/a)$. ◇

Infinite Limits at a

A function may assume arbitrarily large values as x approaches a fixed number. One important example is the tangent function. As x approaches $\pi/2$ from the left, $\tan(x)$ takes on arbitrarily large positive values. (See Figure 2.2.6.) We write

$$\lim_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty.$$

However, as $x \to \frac{\pi}{2}$ from inputs larger than $\pi/2$, $\tan(x)$ takes on negative values of arbitrarily large absolute value. We write

$$\lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty.$$

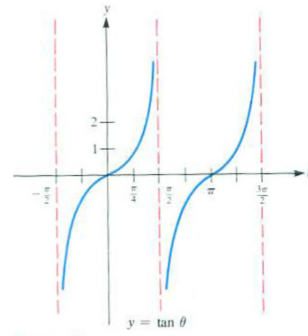


Figure 2.2.6:

DEFINITION *(Infinite limit of $f(x)$ at a)* Let f be a function and a some fixed number. Assume that the domain of f contains an open interval (a, c). If, as x approaches a from the right, $f(x)$ becomes and remains arbitrarily large and positive, then the limit of $f(x)$ as x approaches a is said to be positive infinity. This is written

$$\lim_{x \to a^+} f(x) = +\infty$$

or sometimes just

$$\lim_{x \to a^+} f(x) = \infty.$$

If, as x approaches a from the left, $f(x)$ becomes and remains arbitrarily large and positive, then we write

$$\lim_{x \to a^-} f(x) = +\infty.$$

Similarly, if $f(x)$ assumes values that are negative and these values remain arbitrarily large in absolute value, we write either

$$\lim_{x \to a^+} f(x) = -\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = -\infty,$$

depending upon whether x approaches a from the left or from the right.

Limits as $x \to \infty$

Sometimes it is useful to know how $f(x)$ behaves when x is a very large positive number (or a negative number of large absolute value).

EXAMPLE 6 Determine how $f(x) = 1/x$ behaves for

(a) large positive inputs

(b) negative inputs of large absolute value

(c) small positive inputs

(d) negative inputs of small absolute value

SOLUTION

x	$1/x$
10	0.1
100	0.01
1000	0.001

(a) To get started, make a table of values as shown in the margin. As x becomes arbitrarily large, $1/x$ approaches 0: $\lim_{x\to\infty} \frac{1}{x} = 0$. This conclusion would be read as "as x approaches ∞, $f(x)$ approaches 0."

(b) This is similar to (a), except that the reciprocal of a negative number with large absolute value is a negative number with a small absolute value. Thus, $\lim_{x\to-\infty} \frac{1}{x} = 0$.

(c) For inputs that are positive and approaching 0, the reciprocals are positive and large: $\lim_{x\to0^+} \frac{1}{x} = +\infty$.

(d) Lastly, the reciprocal of inputs that are negative and approaching 0 from the left are negative and arbitrarily large in absolute value: $\lim_{x\to0^-} \frac{1}{x} = -\infty$.

◇

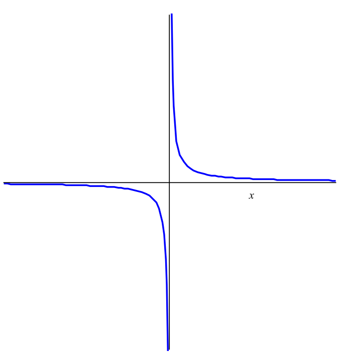


Figure 2.2.7:

More generally, for any fixed positive exponent p,

$$\lim_{x\to\infty} \frac{1}{x^p} = 0.$$

Limits of the form $\lim_{x\to\infty} P(x)$ and $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$, where P and Q are polynomials are easy to treat, as the following examples show.

Keep in mind that ∞ is not a number. It is just a symbol that tells us that something — either the inputs or the outputs of a function — become arbitrarily large.

EXAMPLE 7 Find $\lim_{x\to\infty} (2x^3 - 5x^2 + 6x + 5)$.

SOLUTION When x is large, x^3 is much larger than either x^2 or x. With this in mind, we use a little algebra to determine the limit:

$$2x^3 - 5x^2 + 6x + 5 = x^3\left(2 - \frac{5}{x} + \frac{6}{x^2} + \frac{5}{x^3}\right).$$

The expression in parentheses approaches 2, while x^3 gets arbitrarily large. Thus

$$\lim_{x\to\infty} \frac{2x^3 - 5x^2 + 6x + 5}{x^3} = \infty.$$

◇

EXAMPLE 8 Find $\lim_{x\to\infty} \frac{2x^3-5x^2+6x+5}{7x^4+3x+2}$.

SOLUTION We use the same technique as in Example 7.

$$\begin{array}{rrcl} & 2x^3 - 5x^2 + 6x + 5 & = & x^3\left(2 - \frac{5}{x} + \frac{6}{x^2} + \frac{5}{x^3}\right) \\ \text{and} & 7x^4 + 3x + 2 & = & x^4\left(7 + \frac{3}{x^3} + \frac{2}{x^4}\right) \\ \text{so that} & \frac{2x^3-5x^2+6x+5}{7x^4+3x+2} & = & \frac{x^3\left(2-\frac{5}{x}+\frac{6}{x^2}+\frac{5}{x^3}\right)}{x^4\left(7+\frac{3}{x^3}+\frac{2}{x^4}\right)} \\ & & = & \frac{1}{x}\frac{2-\frac{5}{x}+\frac{6}{x^2}+\frac{5}{x^3}}{7+\frac{3}{x^3}+\frac{2}{x^4}}. \end{array}$$

As x gets arbitrarily large, $\frac{1}{x}$ approaches 0, $2 - \frac{5}{x} + \frac{6}{x^2} + \frac{5}{x^3}$ approaches 2, and $7 + \frac{3}{x^3} + \frac{2}{x^4}$ approaches 7. Thus,

$$\lim_{x\to\infty} \frac{2x^3 - 5x^2 + 6x + 5}{7x^4 + 3x + 2} = 0.$$

◇

As these two examples suggest, the limit of a quotient of two polynomials, $\frac{P(x)}{Q(x)}$, is completely determined by the limit of the quotient of the highest degree term in $P(x)$ and in $Q(x)$.

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

where a_n and b_m are not 0. Then

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}.$$

In particular, if $m = n$, the limit is a_n/b_m. If $m > n$, the limit is 0. If $n > m$, the limit is infinite, either ∞ or $-\infty$, depending on the signs of a_n and b_n.

Summary

This section introduces the concept of a limit and introduce notations for the various types of limits. One-sided limits are the foundation for the two-sided limit as well as for infinite limits and limits at infinity.

It is important to keep in mind that when deciding whether $\lim_{x\to a} f(x)$ exists, you never consider $f(a)$. Perhaps a isn't even in the domain of the function. Even if a is in the domain, the value $f(a)$ plays no role in deciding whether $\lim_{x\to a} f(x)$ exists.

EXERCISES for 2.2 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 8 the limits exist. Find them.

1.[R] $\lim_{x\to 3} \frac{x^2-9}{x-3}$

2.[R] $\lim_{x\to 4} \frac{x^2-9}{x-3}$

3.[R] $\lim_{x\to 0} \frac{\sin(x)}{x}$

4.[R] $\lim_{x\to \frac{\pi}{2}} \frac{\sin(x)}{x}$

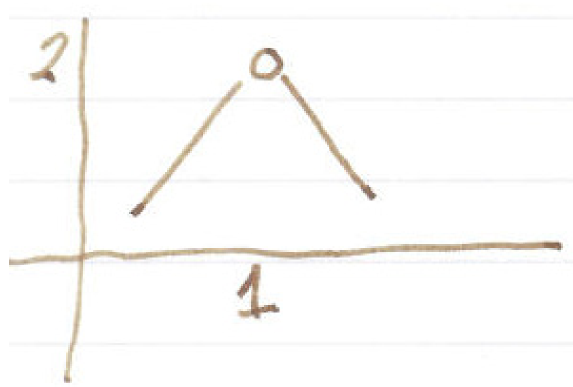
5.[R] $\lim_{x\to 0} \frac{e^x-1}{2x}$

6.[R] $\lim_{x\to 2} \frac{e^x-1}{2x}$

7.[R] $\lim_{x\to 0} \frac{1-\cos(x)}{3x}$

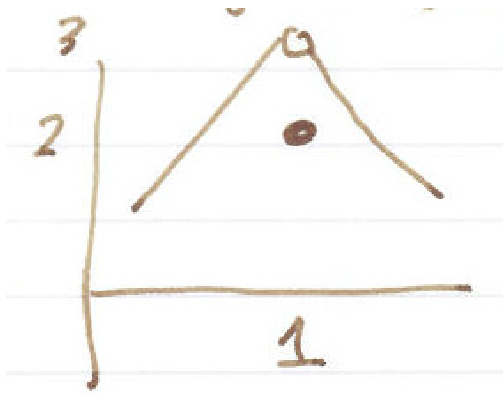
8.[R] $\lim_{x\to \pi} \frac{1-\cos(x)}{3x}$

In Exercises 9 to 12 the graph of a function $y = f(x)$ is given. Decide whether $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1^-} f(x)$, and $\lim_{x\to 1} f(x)$ exist. If they do exist, give their values.



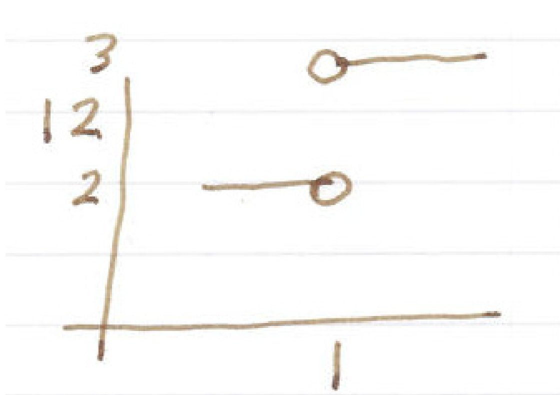
9.[R]

10.[R]



11.[R]

12.[R]



13.[R]

(a) Sketch the graph of $y = \log_2(x)$.

(b) What are $\lim_{x\to\infty} \log_2(x)$, $\lim_{x\to 4} \log_2(x)$, and $\lim_{x\to 0^+} \log_2(x)$?

14.[R]

(a) Sketch the graph of $y = 2^x$.

(b) What are $\lim_{x\to\infty} 2^x$, $\lim_{x\to 4} 2^x$, and $\lim_{x\to-\infty} 2^x$?

15.[R] Find $\lim_{x\to a} \dfrac{x^3-8}{x-2}$ for $a = 1$, 2, and 3.

16.[R] Find $\lim_{x\to a} \dfrac{x^4-16}{x-2}$ for $a = 1$, 2, and 3.

17.[R] Examine $\lim_{x\to a} \dfrac{e^x-1}{x-2}$ for $a = -1$, 0, 1, and 2.

18.[R] Find $\lim_{x\to a} \dfrac{\sin(x)}{x}$ for $a = \frac{\pi}{6}$, $\frac{\pi}{4}$, and 0.

In Exercises 19 to 24, find the given limit (if it exists).

19.[R] $\lim_{x\to\infty} 2^{-x}\sin(x)$

20.[R] $\lim_{x\to\infty} 3^{-x}\cos(2x)$

21.[R] $\lim_{x\to\infty} \dfrac{3x^5+2x^2-1}{6x^5+x^4+2}$

22.[R] $\lim_{x\to\infty} \dfrac{13x^5+2x^2+1}{2x^6+x+5}$

23.[R] $\lim_{x\to\infty} \dfrac{10x^6+x^5+x+1}{x^6}$

24.[R] $\lim_{x\to\infty} \dfrac{25x^5+x^2+1}{x^3+x+2}$

In Exercises 25 to 27, information is given about functions f and g. In each case decide whether the limit asked for can be determined on the basis of that information. If it can, give its value. If it cannot, show by specific choices of f and g that it cannot.

25.[M] Given that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 1$, discuss

(a) $\lim_{x\to\infty} (f(x)+g(x))$

(b) $\lim_{x\to\infty} (f(x)/g(x))$

(c) $\lim_{x\to\infty} (f(x)g(x))$

(d) $\lim_{x\to\infty} (g(x)/f(x))$

(e) $\lim_{x\to\infty} (g(x)/|f(x)|)$

26.[M] Given that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, discuss

(a) $\lim_{x\to\infty}(f(x) + g(x))$

(b) $\lim_{x\to\infty}(f(x) - g(x))$

(c) $\lim_{x\to\infty}(f(x)g(x))$

(d) $\lim_{x\to\infty}(g(x)/f(x))$

27.[M] Given that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} g(x) = \infty$, discuss

(a) $\lim_{x\to\infty}(f(x)/g(x))$

(b) $\lim_{x\to\infty}(f(x)g(x))$

(c) $\lim_{x\to\infty}(f(x) - 1)g(x)$

28.[M] Graph $f(x) = \cos(1/x)$, following these steps.

(a) What is the domain of f?

(b) Does $\lim_{x\to 0}\cos(1/x)$ exist?

(c) Graph $f(x) = \cos(1/x)$.

29.[M] Graph $f(x) = x\sin(1/x)$, following these steps.

(a) What is the domain of f?

(b) Graph the lines $y = x$ and $y = -x$.

(c) For which x does $f(x) = x$? When does $f(x) = -x$? (Notice that the graph of $y = f(x)$ goes back and forth between these lines.)

(d) Does $\lim_{x\to 0} f(x)$ exist? If so, what is it?

(e) Does $\lim_{x\to\infty} f(x)$ exist? If so, what is it?

(f) Graph $y = f(x)$.

30.[M] Let $f(x) = \frac{|x|}{x}$, which is defined except at $x = 0$.

(a) What is $f(3)$?

(b) What is $f(-2)$?

(c) Graph $y = f(x)$.

(d) Does $\lim_{x\to 0} f(x)$ exist? If so, what is it?

(e) Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it?

(f) Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it?

In Exercises 31 to 33, find $\lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$ for the following functions.

31.[M] $f(x) = 5x$

32.[M] $f(x) = x^2$

33.[M] $f(x) = e^x$

34.[M] Figure 2.2.8 shows a circle of radius a. Find

(a) $\lim_{\theta\to 0^+} \frac{\overline{AB}}{\widehat{CB}}$ NOTE: $\widehat{CB}$ is the length of the arc of the circle with radius a.

(b) $\lim_{\theta\to 0^+} \frac{\overline{AB}}{\overline{CD}}$

(c) $\lim_{\theta\to 0} \frac{\text{area of } ABC}{\text{area of } ABCD}$.

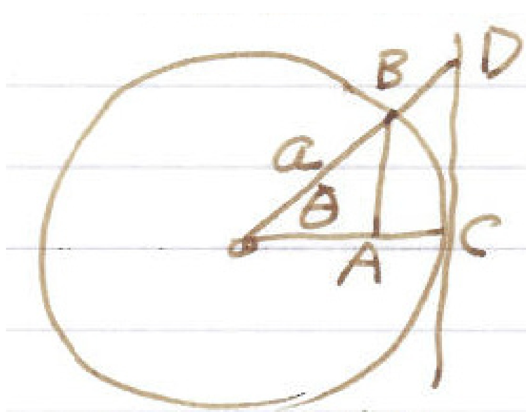


Figure 2.2.8: Exercise 34

35.[M] Let $f(x)$ be the diameter of the largest circle that fits in a $1 \times x$ rectangle.

(a) Find a formula for $f(x)$.

(b) Graph $y = f(x)$.

(c) Does $\lim_{x \to 1} f(x)$ exist?

36.[M] I am thinking of two numbers near 0. What, if anything, can you say about their

(a) product?

(b) quotient?

(c) difference?

(d) sum?

37.[M] I am thinking about two large positive numbers. What, if anything, can you say about their

(a) product?

(b) quotient?

(c) difference?

(d) sum?

38.[C] Find $\lim_{h \to 0} \frac{f(\theta + h) - f(\theta)}{h}$ for $f(x) = \sin(x)$. HINT: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.

39.[C] Find $\lim_{h \to 0} \frac{f(\theta + h) - f(\theta)}{h}$ for $f(x) = \cos(x)$. HINT: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

40.[C] Find $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$.

41.[C] Sam and Jane are discussing

$$f(x) = \frac{3x^2 + 2x}{x + 5} - 3x.$$

Sam: For large x, $2x$ is small in comparison to $3x^2$, and 5 is small in comparison to x. So the quotient $\frac{3x^2+2x}{x+5}$ behaves like $\frac{3x^2}{x} = 3x$. Hence, the graph of $y = f(x)$ is very close to the graph of the line $y = 3x$ when x is large.

Jane: "Nonsense. After all,

$$\frac{3x^2 + 2x}{x + 5} = \frac{3x + 2}{1 + (5/x)}$$

which clearly behaves like $3x+2$ for large x. Thus the graph of $y = f(x)$ stays very close to the line $y = 3x + 2$ when x is large.

Settle the argument.

42.[C] Sam, Jane, and Wilber are arguing about limits in a case where $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = \infty$.

Sam: $\lim_{x\to\infty} f(x)g(x) = 0$, since $f(x)$ is going toward 0.

Jane: Rubbish! Since $g(x)$ gets large, it will turn out that $\lim_{x\to\infty} f(x)g(x) = \infty$.

Wilber: You're both wrong. The two influences will balance out and you will see that $\lim_{x\to\infty} f(x)g(x)$ is near 1.

Settle the argument.

43.[C] Sam and Jane are arguing about limits in a case where $f(x) \geq 1$ for $x > 0$, $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0} g(x) = \infty$. What can be said about $\lim_{x\to 0^+} f(x)^{g(x)}$?

Sam: That's easy. Multiply a bunch of numbers near 1 and you get a number near 1. So the limit will be 1.

Jane: Rubbish! Since $f(x)$ may be bigger than 1 and you are multiplying it lots of times, you will get a really large number. There's no doubt in my mind: $\lim_{x\to 0} f(x)^{g(x)} = \infty$.

Settle the argument.

44.[C] An urn contains n marbles. One is green and the remaining $n-1$ are red. When picking one marble at random without looking, the probability is $1/n$ of getting the green marble, and $(n-1)/n$ of getting a red marble. If you do this experiment n times, each time putting the chosen marble back, the probability of not getting the green marble on any of the n experiments is $((n-1)/n)^n$.

(a) Let $p(n) = \left(\frac{n-1}{n}\right)^n$. Compute $p(2)$, $p(3)$, and $p(4)$ to at least three decimal digits (to the right of the decimal point).

(b) Show that as $n \to \infty$, $p(n)$ approaches the reciprocal of $\lim_{x\to 0} (1+x)^{1/x}$.

2.3 Continuous Functions

This section introduces the notion of a continuous function. While almost all functions met in practice are continuous, we must always remain alert that a function might not be continuous. We begin with an informal description and then give a more useful working definition.

An Informal Introduction to Continuous Functions

When we draw the graph of a function defined on some interval, we usually do not have to lift the pencil off the paper. Figure 2.3.1 shows this typical situation.

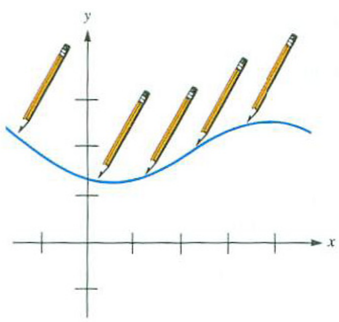


Figure 2.3.1:

A function is said to be **continuous** if, when considered on any interval in its domain, its graph can he traced without lifting the pencil off the paper. (The domain may consist of several intervals.) According to this definition any polynomial is continuous. So is each of the basic trigonometric functions, including $y = \tan(x)$, whose graph is shown in Figure 2.2.6 of Section 2.2.

You may be tempted to say "But $\tan(x)$ blows up at $x = \pi/2$ and I have to lift my pencil off the paper to draw the graph." However, $x = \pi/2$ is not in the domain of the tangent function. *On every interval in its domain,* $\tan(x)$ *behaves quite decently;* on such an interval we can sketch its graph without lifting the pencil from the paper. That is why $\tan(x)$ is continuous. The function $1/x$ is also continuous, since it "explodes" only at a number not in its domain, namely at $x = 0$. The function whose graph is shown in Figure 2.3.2 is not continuous. It is defined throughout the interval $[-2, 3]$, but to draw its graph you must lift the pencil from the paper near $x = 1$. However, when you consider the function *only for* x *in* $[1, 3]$*, then it is continuous.* By the way, a formula for the piecewise-defined function given graphically in Figure 2.3.2 is:

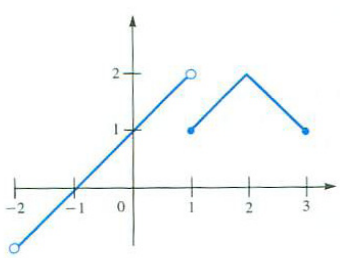


Figure 2.3.2:

$$f(x) = \begin{cases} x+1 & \text{for } x \text{ in } [-2,1) \\ x & \text{for } x \text{ in } [1,2) \\ -x+4 & \text{for } x \text{ in } [2,3]. \end{cases}$$

It is pieced together from three different continuous functions.

The Definition of Continuity

Our informal "moving pencil" notion of a continuous function requires drawing a graph of the function. Our working definition does not require such a graph. Moreover, it easily generalizes to functions of more than one variable in later chapters.

To get the feeling of this second definition, imagine that you had the information shown in the table in the margin about some function f. What would

x	$f(x)$
0.9	2.93
0.99	2.9954
0.999	2.9999997

you expect the output $f(1)$ to be?

It would be quite a shock to be told that $f(1)$ is, say, 625. A reasonable function should present no such surprise. The expectation is that $f(1)$ will be 3. More generally, we expect the output of a function at the input a to be closely connected with the outputs of the function at inputs near a. The functions of interest in calculus usually behave that way. In short, "What you expect is what you get." With this in mind, we define the notion of *continuity at a number* a. We first assume that the domain of f contains an open interval around a.

DEFINITION *(Continuity at a number a)* Assume that $f(x)$ is defined in some open interval that contains the number a. Then the function f is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$. This means that

1. $f(a)$ is defined (that is, a is in the domain of f).
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x)$ equals $f(a)$.

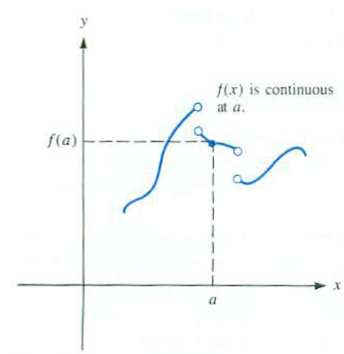


Figure 2.3.3:

As Figure 2.3.3 shows, whether a function is continuous at a depends on its behavior both at a and at inputs near a. Being continuous at a is a local matter, involving perhaps very tiny intervals about a.

To check whether a function f is continuous at a number a, we ask three questions:

Question 1: Is a in the domain of f?

Question 2: Does $\lim_{x\to a} f(x)$ exist?

Question 3: Does $f(a)$ equal $\lim_{x\to a} f(x)$?

If the answer is "yes" to each of these questions, we say that f is continuous at a.

If a is in the domain of f and the answer to Question 2 or to Question 3 is "no," then f is said to be **discontinuous at** a. If a is not in the domain of f, we do not speak of it being continuous or discontinuous there.

We are now ready to define a continuous function.

DEFINITION *(Continuous function)* Let f be a function whose domain is the x-axis or is made up of open intervals. Then f is a **continuous function** if it is continuous at each number a in its domain. A function that is not continuous is called a **discontinuous function**.

EXAMPLE 1 Use the definition of continuity to decide whether $f(x) = 1/x$ is continuous.

SOLUTION Let a be in the domain of f. In other words, a is not 0 so the answer to Question 1 is "yes." Since

$$\lim_{x \to a} \frac{1}{x} = \frac{1}{a},$$

the answer to Question 2 is "yes." Because

$$f(a) = \frac{1}{a},$$

the answer to Question 3 is also "yes." Thus $f(x) = 1/x$ is continuous at every number in its domain. Hence f is a continuous function.

Note that the conclusion agrees with the "moving pencil" picture of continuity. ◇

Not every important function is continuous. For instance, let $f(x)$ be the greatest integer that is less than or equal to x. We have $f(1.8) = 1$, $f(1.9) = 1$, $f(2) = 2$, and $f(2.3) = 2$. This function is often used in number theory and computer science, where it is denoted $[x]$ or $\lfloor x \rfloor$ and called the **floor** of x. People use the floor function every time they answer the question, "How old are you?" The next example examines where the floor function fails to be continuous.

EXAMPLE 2 Let f be the floor function, $f(x) = \lfloor x \rfloor$. Graph f and find where it is continuous. Is f a continuous function?

SOLUTION We begin with the following table to show the behavior of $f(x)$ for x near 1 or 2.

x	0	0.5	0.8	1	1.1	1.99	2	2.01
$\lfloor x \rfloor$	0	0	0	1	1	1	2	2

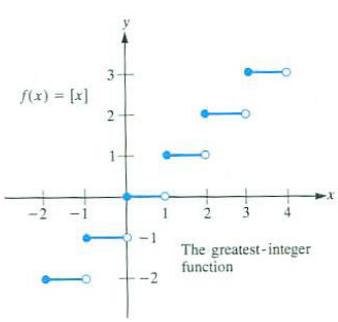


Figure 2.3.4:

For $0 \le x < 1$, $\lfloor x \rfloor = 0$. But at the input $x = 1$ the output jumps to 1 since $\lfloor 1 \rfloor = 1$. For $1 \le x < 2$, $\lfloor x \rfloor$ remains at 1. Then at 2 it jumps to 2. More generally, $\lfloor x \rfloor$ has a jump at every integer, as shown in Figure 2.3.4.

Let us show that f is not continuous at $a = 2$ by seeing which of the three conditions in the definition are not satisfied. First of all, Question 1 is answered "yes" since 2 lies in the domain of the function; indeed, $f(2) = 2$.

What is the answer to Question 2? Does $\lim_{x \to 2} f(x)$ exist? We see that

$$\lim_{x \to 2^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = 2.$$

Since the left-hand and right-hand limits are not equal, $\lim_{x\to 2} f(x)$ does not exist. Question 2 is answered "no."

Already we know that the function is not continuous at $a = 2$. Since the limit does not exist there is no point in considering Question 3. Because there is one point in the domain where $\lfloor x \rfloor$ is not continuous, this is a discontinuous function. More specifically, the floor function is discontinuous at $x = a$, whenever a is an integer.

Is f continuous at a if a is not an integer? Let us take the case $a = 1.5$, for instance.

Question 1 is answered "yes," because $f(1.5)$ is defined.
(In fact, $f(1.5) = 1$.)

Question 2 is answered "yes," since $\lim_{x\to 1.5} f(x) = 1$.

Question 3 is answered "yes," since $\lim_{x\to 1.5} f(x) = f(1.5)$.
(Both values are 1.)

The floor function is continuous at $a = 1.5$. Similarly, f is continuous at every number that is not an integer.

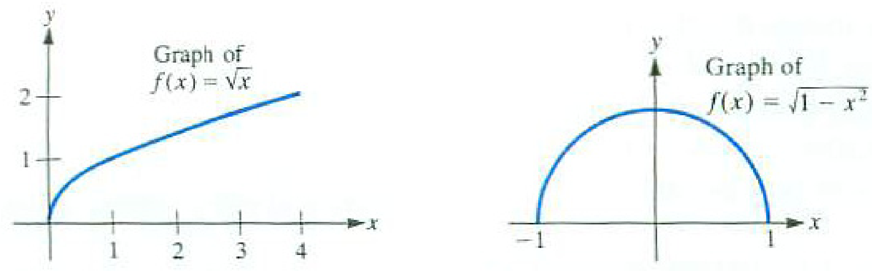
Note that $\lfloor x \rfloor$ is continuous on any interval that does not include an integer. For instance, if we consider the function only on the interval $(1.1, 1.9)$, it is continuous there. ◇

Continuity at an Endpoint

The functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1 - x^2}$ are graphed in Figures 2.3.5(a) and (b), respectively. We would like to call both of these functions continuous. However, there is a slight technical problem. The number 0 is in the domain of f, but there is no open interval around 0 that lies completely in the domain, as our definition of continuity requires. Since $f(x) = \sqrt{x}$ is not defined for x to the left of 0, we are not interested in numbers x to the left of 0. Similarly, $g(x) = \sqrt{1 - x^2}$ is defined only when $1 - x^2 \geq 0$, that is, for $-1 \leq x \leq 1$. To cover this type of situation we utilize one-sided limits to define **one-sided continuity**.

DEFINITION *(Continuity from the right at a number.)* Assume that $f(x)$ is defined in some closed interval $[a, c]$. Then the function f **is continuous from the right at** a if

1. $f(a)$ is defined
2. $\lim_{x\to a^+} f(x)$ exists
3. $\lim_{x\to a^+} f(x)$ equals $f(a)$



(a) (b)

Figure 2.3.5:

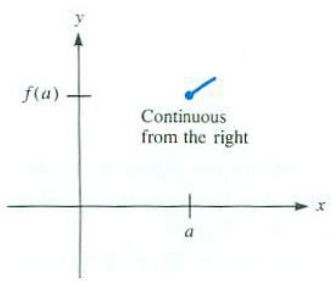


Figure 2.3.6:

Figure 2.3.6 illustrates this definition, which also takes care of the continuity of $g(x) = \sqrt{1 - x^2}$ at -1 in Figure 2.3.5(b). The next definition takes care of the right-hand endpoints.

DEFINITION *(Continuity from the left at a number a.)* Assume that $f(x)$ is defined in some closed interval $[b, a]$. Then the function f **is continuous from the left at** a if

1. $f(a)$ is defined
2. $\lim_{x \to a^-} f(x)$ exists
3. $\lim_{x \to a^-} f(x)$ equals $f(a)$

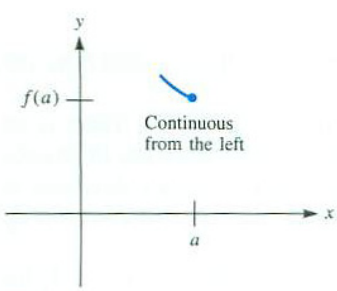


Figure 2.3.7:

Figure 2.3.7 illustrates this definition.

With these two extra definitions to cover some special cases in the domain, we can extend the definition of continuous function to include those functions whose domains may contain endpoints. We say, for instance, that $\sqrt{1 - x^2}$ is continuous because it is continuous at any number in $(-1, 1)$, is continuous from the right at -1, and continuous from the left at 1.

These special considerations are minor matters that will little concern us in the future. The key point is that $\sqrt{1 - x^2}$ and $\sqrt{x}$ are both continuous functions. So are practically all the functions studied in calculus.

The following example reviews the notion of continuity.

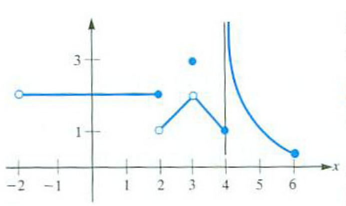


Figure 2.3.8:

EXAMPLE 3 Figure 2.3.8 is the graph of a certain (piecewise-defined) function $f(x)$ whose domain is the interval $(-2, 6]$. Discuss the continuity of $f(x)$ at (a) 6, (b) 4, (c) 3, (d) 2, (e) 1, and (f) -2.

SOLUTION

(a) Since $\lim_{x \to 6-} f(x)$ exists and equals $f(6)$, f is continuous from the left at 6.

(b) Since $\lim_{x\to 4} f(x)$ does not exist, f is not continuous at 4.

(c) Inspection of the graph shows that $\lim_{x\to 3} f(x) = 2$. However, Question 3 is answered "no" because $f(3) = 3$, which is *not* equal to $\lim_{x\to 3} f(x)$. Thus f is not continuous at 3.

(d) Though $\lim_{x\to 2-} f(x)$ and $\lim_{x\to 2+} f(x)$ both exist, they are not equal. (The left-hand limit is 2; the right-hand limit is 1.) Thus $\lim_{x\to 2} f(x)$ does not exist, the answer to Question 2 is "no," and f is discontinuous at $x = 2$.

(e) At 1, "yes" is the answer to all three questions: $f(1)$ is defined, $\lim_{x\to 1} f(x)$ exists (it equals 2) and, finally, it equals $f(1)$. f is continuous at $x = 1$.

(f) Since -2 is not even in the domain of this function, we do not speak of continuity or discontinuity of f at -2.

◇

As Example 3 shows, a function can fail to be continuous at a given number a in its domain for either of two reasons:

1. $\lim_{x\to a} f(x)$ might not exist
2. when, $\lim_{x\to a} f(x)$ does exist, $f(a)$ might not be equal to that limit.

Continuity and Limits

Some limits are so easy that you can find them without any work; for instance, $\lim_{x\to 2} 5^x = 5^2 = 25$. Others offer a challenge; for instance, $\lim_{x\to 2} \frac{x^3-2^3}{x-2}$.

If you want to find $\lim_{x\to a} f(x)$, and you know f is a continuous function with a in its domain, then you just calculate $f(a)$. In such a case there is no challenge and the limit is called *determinate*.

The interesting case for finding $\lim_{x\to a} f(x)$ occurs when f is not defined at a. That is when you must consider the influences operating on $f(x)$ when x is near a. You may have to do some algebra or computations. Such limits are called *indeterminate*.

The four limits encountered in Section 2.1, $\lim_{x\to a} \frac{x^n - a^n}{x - a}$, $\lim_{x\to 0} \frac{b^x - 1}{x}$, $\lim_{x\to 0} \frac{\sin(x)}{x}$, and $\lim_{x\to 0} \frac{1 - \cos(x)}{x}$ all required some work to find their value. These types of limits will be discussed in detail in Section 5.5.

We list the properties of limits which are helpful in computing limits.

Theorem 2.3.1 (Properties of Limits)**.** *Let g and h be two functions and assume that $\lim_{x\to a} g(x) = A$ and $\lim_{x\to a} h(x) = B$. Then*

Each of these properties remains valid when the two-sided limit is replaced with a one-sided limit.

Sum $\lim_{x\to a}(g(x)+h(x)) = \lim_{x\to a} g(x) + \lim_{x\to a} h(x) = A + B$
the limit of the sum is the sum of the limits

Difference $\lim_{x\to a}(g(x)-h(x)) = \lim_{x\to a} g(x) - \lim_{x\to a} h(x) = A - B$
the limit of the difference is the difference of the limits

Product $\lim_{x\to a}(g(x)h(x)) = \left(\lim_{x\to a} g(x)\right)\left(\lim_{x\to a} h(x)\right) = AB$
the limit of the product is the product of the limits

Constant Multiple $\lim_{x\to a}(kg(x)) = k\left(\lim_{x\to a} g(x)\right) = kA$, *for any constant* k
special case of Product

Quotient $\lim_{x\to a}\left(\frac{g(x)}{h(x)}\right) = \frac{(\lim_{x\to a} g(x))}{(\lim_{x\to a} h(x))} = \frac{A}{B}$, *provided* $B \neq 0$
the limit of the quotient is the quotient of the limits, provided the denominator is not 0

Power $\lim_{x\to a}\left(g(x)^{h(x)}\right) = \left(\lim_{x\to a} g(x)\right)^{(\lim_{x\to a} h(x))} = A^B$, *provided* $A > 0$
the limit of a varying base to a varying power

EXAMPLE 4 Find $\lim_{x\to 0}\frac{(x^4-16)\sin(5x)}{x^2-2x}$.

SOLUTION Notice that the denominator can be factored to obtain

$$\frac{(x^4-16)\sin(5x)}{x^2-2x} = \frac{x^4-2^4}{x-2}\cdot\frac{\sin(5x)}{x}.$$

This allows the limit to be rewritten as

$$\lim_{x\to 0}\frac{x^4-2^4}{x-2}\cdot\lim_{x\to 0}\frac{\sin(5x)}{x}$$

where we have also used $16 = 2^4$. Now, $\lim_{x\to 0}\frac{x^4-2^4}{x-2} = 4\cdot 2^{4-1} = 32$. Also,

$$\lim_{x\to 0}\frac{\sin(5x)}{x} = 5\lim_{x\to 0}\frac{\sin(5x)}{5x} = 5\cdot 1 = 5.$$

We conclude that

$$\lim_{x\to 0}\frac{(x^4-16)\sin(5x)}{x^2-2x} = \lim_{x\to 0}\frac{x^4-2^4}{x-2}\cdot\lim_{x\to 0}\frac{\sin(5x)}{5x} = 32\cdot 5 = 160.$$

◇

Summary

This section opened with an informal view of continuous functions, expressed in terms of a moving pencil. It then gave the definition, phrased in terms of limits, which we will use throughout the text.

The development concludes in the next section, which describes three important properties of continuous functions.

EXERCISES for 2.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 12, which of these limits can be found at a glance and which require some analysis? That is, is each limit determinant or indeterminant. Do not evaluate the limi

1.[R] $\lim_{x\to 0}(2^x - 1)$

2.[R] $\lim_{x\to\infty}(2^x - 1)$

3.[R] $\lim_{x\to 1}\frac{3^x - 1}{2^x - 1}$

4.[R] $\lim_{x\to 2}\frac{3^x - 1}{2^x - 1}$

5.[R] $\lim_{x\to\infty}\frac{x}{2^x}$

6.[R] $\lim_{x\to 0}\frac{x}{2^x}$

7.[R] $\lim_{x\to 0^+}\frac{x^2}{e^x - 1}$

8.[R] $\lim_{x\to \frac{\pi}{2}^-}(\sin(x))^{\tan(x)}$

9.[R] $\lim_{x\to 0^+} x\log_2(x)$

10.[R] $\lim_{x\to 0^+}(2+x)^{3/x}$

11.[R] $\lim_{x\to\infty}(2+x)^{3/x}$

12.[R] $\lim_{x\to 0^-}\frac{(2+x)^3}{x}$

In Exercises 13 to 16, evaluate the limit.

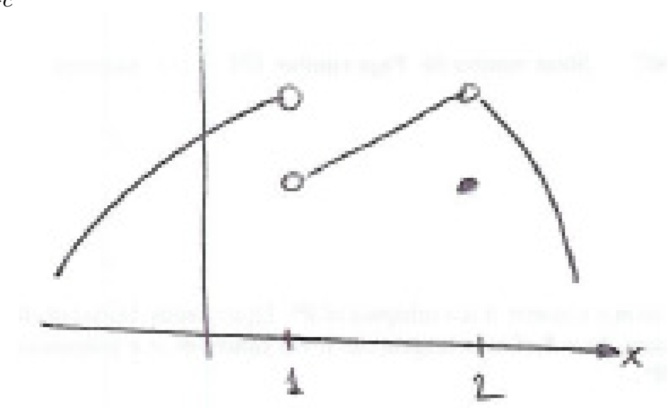
13.[R] $\lim_{x\to\frac{\pi}{2}}\sin(x)\frac{e^x - 1}{x}$

14.[R] $\lim_{x\to 0}\frac{\cos(x)\,(e^x - 1)}{x}$

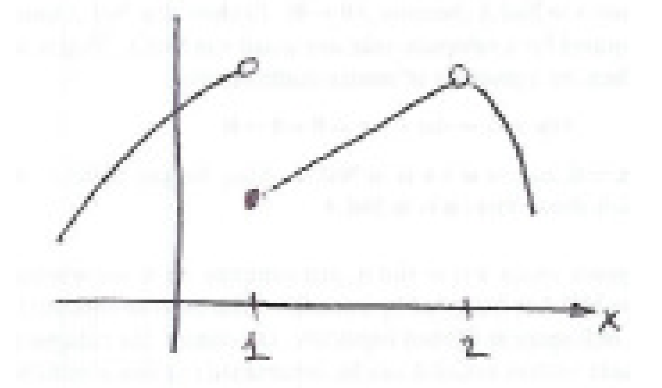
15.[R] $\lim_{x\to 0}\frac{\sin(2x)}{x(\cos(3x))^2}$

16.[R] $\lim_{x\to 1}\frac{(x-1)\cos(x)}{x^3 - 1}$

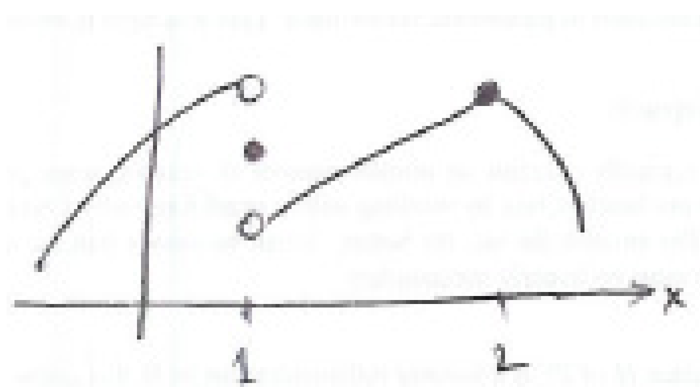
In Exercises 17 to 20 the graph of a function $y = f(x)$ is given. Determine all numbers c for which $\lim_{x \to c} f(x)$ does not exist.



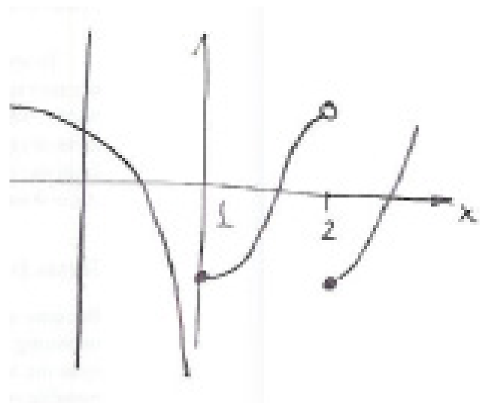
17.[R]



18.[R]

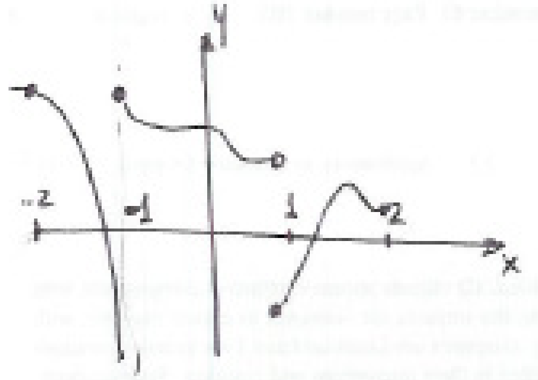


19.[R]



20.[R]

In Exercises 21 and 22 the graph of a function $y = f(x)$ and several intervals are given. For each interval, decide if the function is continuous on that interval.



21.[R]

(a) $[-2, -1]$

(b) $(-2, -1)$

(c) $(-1, 1)$

(d) $[-1, 1)$

(e) $(-1, 1]$

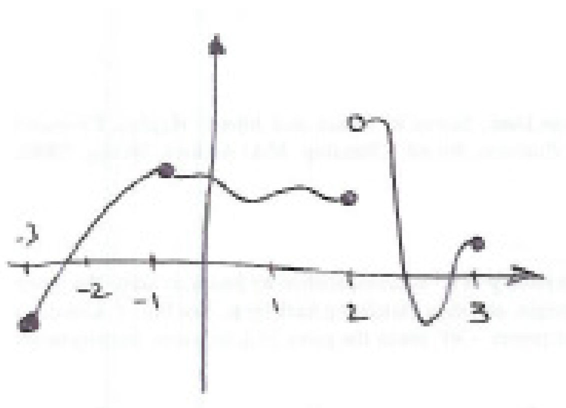
(f) $[-1, 1]$

(g) $(1, 2)$

(h) $[1, 2)$

(i) $(1, 2]$

(j) $[1, 2]$



22.[R]

(a) $[-3, 2]$

(b) $(-1, 3)$

(c) $(-1, 2)$

(d) $[-1, 2)$

(e) $(-1, 2]$

(f) $[-1, 2]$

(g) $(2, 3)$

(h) $[2, 3)$

(i) $(2, 3]$

(j) $[2, 3]$

23.[R] Let $f(x) = x + |x|$.

(a) Graph f.

(b) Is f continuous at -1?

(c) Is f continuous at 0?

24.[M] Let $f(x) = 2^{1/x}$ for $x \neq 0$.

(a) Find $\lim_{x\to\infty} f(x)$.

(b) Find $\lim_{x\to-\infty} f(x)$.

(c) Does $\lim_{x\to 0^+} f(x)$ exist?

(d) Does $\lim_{x\to 0^-} f(x)$ exist?

(e) Graph f, incorporating the information from parts *(a)* to *(d)*.

(f) Is it possible to define $f(0)$ in such a way that f is continuous throughout the x-axis?

25.[M] Let $f(x) = x\sin(1/x)$ for $x \neq 0$.

(a) Find $\lim_{x\to\infty} f(x)$.

(b) Find $\lim_{x\to-\infty} f(x)$.

(c) Find $\lim_{x\to 0} f(x)$.

(d) Is it possible to define $f(0)$ in such a way that f is continuous throughout the x-axis?

(e) Sketch the graph of f.

In Exercises 26 to 28 find equations that the numbers k, p, and/or m must satisfy to make each function continuous.

26.[M] $f(x) = \begin{cases} \frac{\sin(x)}{2x} & x \neq 0 \\ p & x = 0 \end{cases}$

27.[M] $f(x) = \begin{cases} k & x \leq 0 \\ \arcsin(x) & 0 < x \leq \frac{\pi}{2} \\ p & x > \frac{\pi}{2} \end{cases}$

28.[M] $f(x) = \begin{cases} \ln(x) & x > 1 \\ k - m\sqrt{x} & 0 < x \leq 1 \\ pe^{-x} & x \leq 0 \end{cases}$

29.[M]

(a) Let f and g be two functions defined for all numbers. If $f(x) = g(x)$ when x is not 3, must $f(3) = g(3)$?

(b) Let f and g be two continuous functions defined for all numbers. If $f(x) = g(x)$ when x is not 3, must $f(3) = g(3)$?

Explain your answers.

30.[C] *The reason* 0^0 *is not defined.* It might be hoped that if the positive number b and the number x are both close to 0, then b^x might be close to some fixed number. If that were so, it would suggest a definition for 0^0. Experiment with various choices of b and x near 0 and on the basis of your data write a paragraph on the theme, "Why 0^0 is not defined."

2.4 Three Important Properties of Continuous Functions

Continuous functions have three properties important in calculus: the "extreme-value" property, the "intermediate-value" property, and the "permanence" property. All three are quite plausible, and a glance at the graph of a typical continuous function may persuade us that they are obvious. No proofs will be offered: they depend on the precise definitions of limits given in Sections 3.8 and 3.9 and are part of an advanced calculus course.

We will say that a function has a **local or relative maximum** at a point $(c, f(c))$ when $f(c) \geq f(x)$ for x near c. More precisely, there is an open interval I containing c such that if x is in I, and $f(x)$ is defined, then $f(x) \leq f(c)$. Likewise, a function has a **local or relative minimum** at a point $(c, f(c))$ when $f(c) \leq f(x)$ for x near c. Each maximum or minimum is referred to as an **extreme value** or **extremum** of the function.

The plural of extremum is extrema.

Extreme-Value Property

The first property is that a function continuous throughout the closed interval $[a, b]$ takes on a largest value somewhere in the interval.

Theorem (Maximum-Value Property). *Let f be continuous throughout a closed interval $[a, b]$. Then there is at least one number in $[a, b]$ at which f takes on a maximum value. That is, for some number c in $[a, b]$, $f(c) \geq f(x)$ for all x in $[a, b]$.*

To persuade yourself that this is plausible, imagine sketching the graph of a continuous function. (See Figure 2.4.1.)

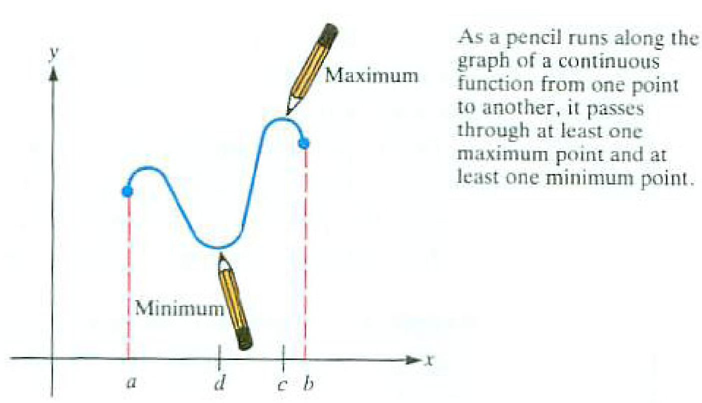


Figure 2.4.1:

The maximum-value property guarantees that a maximum value exists, but it does *not tell how* to find it. The problem of finding it is addressed in Chapter 4.

There is also a **minimum-value property** that states that every continuous function on a closed interval takes on a smallest value somewhere in this interval. See Figure 2.4.1 for an illustration of this property. Combining the two properties, we have:

Theorem (Extreme-Value Property). *Let f be continuous throughout the closed interval $[a, b]$. Then there is at least one number in $[a, b]$ at which f takes on a minimum value and there is at least one number in $[a, b]$ at which f takes on a maximum value. That is, for some numbers c and d in $[a, b]$, $f(d) \leq f(x) \leq f(c)$ for all x in $[a, b]$.*

EXAMPLE 1 Find all numbers in $[0, 3\pi]$ at which the cosine function, $f(x) = \cos(x)$, takes on a maximum value. Also, find all numbers in $[0, 3\pi]$ at which f takes on a minimum value.

SOLUTION Figure 2.4.2 is a graph of $f(x) = \cos(x)$ for x in $[0, 3\pi]$. Inspection of the graph shows that the maximum value of $\cos(x)$ for $0 \leq x \leq 3\pi$ is 1, and it is attained twice: when $x = 0$ and when $x = 2\pi$. The minimum value is -1, which is also attained twice: when $x = \pi$ and when $x = 3\pi$. ◇

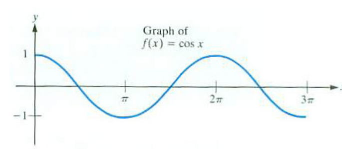


Figure 2.4.2:

The Extreme-Value Property has two assumptions: "f is continuous" and "the domain is a closed interval." If either of these conditions is removed, the conclusion need not hold.

Figure 2.4.3(a) shows the graph of a function that is *not* continuous, is defined on a closed interval, but has no maximum value. On the other hand $f(x) = \frac{1}{1-x^2}$ is continuous on $(-1, 1)$. It has no maximum value, as a glance at Figure 2.4.3(b) shows. This does not violate the Extreme-Value Property, since the domain $(-1, 1)$ is not a closed interval.

Intermediate-Value Property

Imagine graphing a continuous function f defined on the closed interval $[a, b]$. As your pencil moves from the point $(a, f(a))$ to the point $(b, f(b))$ the y-coordinate of the pencil point goes through all values between $f(a)$ and $f(b)$. (Similarly, if you hike all day, starting at an altitude of 5.000 feet and ending at 11,000 feet, you must have been, say, at 7,000 feet at least once during the day. In mathematical terms, not in terms of a pencil (or a hike), "a function that is continuous throughout an interval takes on all values between any two of its values".

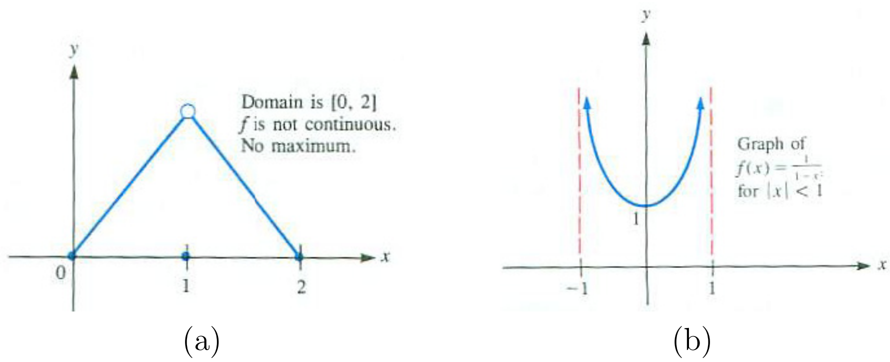


Figure 2.4.3:

Theorem (**Intermediate-Value Property**). *Let f be continuous throughout the closed interval $[a,b]$. Let m be any number such that $f(a) \leq m \leq f(b)$ or $f(a) \geq m \geq f(b)$. Then there is at least one number c in $[a,b]$ such that $f(c) = m$.*

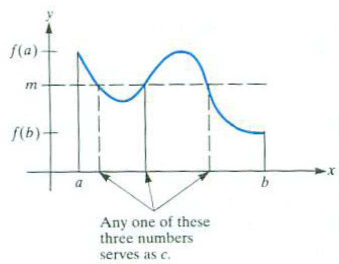


Figure 2.4.4:

Pictorially, the Intermediate-Value Property asserts that, if m is between $f(a)$ and $f(b)$, a horizontal line of height m must meet the graph of f at least once, as shown in Figure 2.4.4.

Even though the property guarantees the existence of a certain number c, it does *not tell how* to find it. To find c we must be able to solve an equation, namely, the equation $f(x) = m$.

EXAMPLE 2 Use the Intermediate-Value Property to show that the equation $2x^3 + x^2 - x + 1 = 5$ has a solution in the interval $[1, 2]$.

SOLUTION Let $P(x) = 2x^3 + x^2 - x + 1$. Then

$$\begin{aligned} P(1) &= 2 \cdot 1^3 + 1^2 - 1 + 1 = 3 \\ \text{and} \quad P(2) &= 2 \cdot 2^3 + 2^2 - 2 + 1 = 19. \end{aligned}$$

Since P is continuous (on $[1, 2]$) and $m = 5$ is between $P(1) = 3$ and $P(2) = 19$, the Intermediate-Value Property says there is at least one number c between 1 and 2 such that $P(c) = 5$.

To get a more accurate estimate for a number c such that $P(c) = 5$, find a shorter interval for which the Intermediate-Value Property can be applied. For instance, $P(1.2) = 4.696$ and $P(1.3) = 5.784$. By the Intermediate-Value Property, there is a number c in $[1.2.1.3]$ such that $P(c) = 5$. ◇

EXAMPLE 3 Show that the equation $-x^5 - 3x^2 + 2x + 11 = 0$ has at least one real root. In other words, the graph of $y = -x^5 - 3x^2 + 2x + 11$ crosses the x-axis.

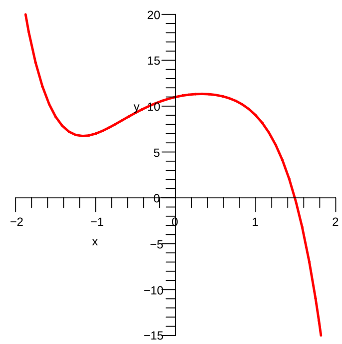


Figure 2.4.5:

SOLUTION Let $f(x) = -x^5 - 3x^2 + 2x + 11$. We wish to show that there is a number c such that $f(c) = 0$. In order to use the Intermediate-Value Property, we need an interval $[a, b]$ for which 0 is between $f(a)$ and $f(b)$, that is, one of $f(a)$ and $f(b)$ is positive and the other is negative. Then we could apply that property, using $m = 0$.

We show that there are numbers a and b with $a < b$, $f(a) > 0$ and $f(b) < 0$. Because $\lim_{x\to\infty} f(x) = -\infty$, for x large and positive, $f(x)$ is negative for x large and positive. Thus, there is a positive number b such that $f(b) < 0$. Similarly, $\lim_{x\to-\infty} f(x) = \infty$, means that when x is negative and of large absolute value, $f(x)$ is positive. So there is a negative number a such that $f(a) > 0$. Thus there are numbers a and b, with $a < b$, such that $f(a) > 0$ and $f(b) < 0$. For instance, $f(-1) = 7$ and $f(2) = -29$.

The number 0 is between $f(a)$ and $f(b)$. Since f is continuous on the interval $[a, b]$, there is a number c in $[a, b]$ such that $f(c) = 0$. (In particular there is a number c in $[-1, 2]$. This number c is a solution to the equation $-x^5 - 3x^2 + 2x + 11 = 0$. ◇

Note that the argument in Example 3 shows that any polynomial of odd degree has a real root. The argument does not hold for polynomials of even degree; the equation $x^2 + 1 = 0$, for instance, has no real solutions.

EXAMPLE 4 Use the Intermediate-Value Property to show that there is a negative number such that $\ln(x + 4) = x^2 - 3$.

SOLUTION We wish to show that there is a negative number c where the function $\ln(x + 4)$ has the same value as the function $x^2 - 3$. The equation $\ln(x+4) = x^2 - 3$ is equivalent to $\ln(x+4) - x^2 + 3 = 0$. The problem reduces to showing that the function $f(x) = \ln(x+4) - x^2 + 3$ has the value 0 for some input c (with $c < 0$).

We will proceed, as we did in the previous example. We want to find numbers a and b (both in $(-\infty, 0]$) such that $f(a)$ and $f(b)$ have opposite signs.

Before beginning the search for a and b, note that $\ln(x + 4)$ is defined only for $x + 4 > 0$, that is, for $x > -4$. To complete the search for a and b, make a table of values of $f(x)$ for some sample arguments in $(-4, 0]$.

x	-3	-2	-1	0
$f(x)$	-6	-0.307	3.099	4.386

We see that $f(-2)$ is negative and $f(-1)$ is positive. Since $m = 0$ lies between $f(-2)$ and $f(-1)$, and f is continuous on $[-2, -1]$, the Intermediate-Value Property asserts that there is a number, in $[-2, -1]$ such that $f(c) = 0$. It follows that $\ln(c + 4) = c^2 - 3$. ◇

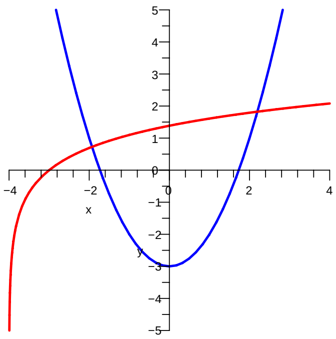


Figure 2.4.6:

In Example 4 the Intermediate-Value Property does not tell what c is. The graphs of $\ln(x+1)$ and x^2-3 in Figure 2.4.6 suggest that there are two points of intersection, but only one with a negative input. The graph, and the table of values, suggest that the intersection point occurs when the input is close to -2. Calculations on a calculator or computer show that $c \approx -1.931$.

Permanence Property

The extrema property as well as the intermediate-value property involve the behavior of a continuous function throughout an interval. The next property concerns the "local" behavior of a continuous function.

Consider a continuous function f on an open interval that contains the number a. Assume that $f(a) = p$ is positive. Then it seems plausible that f remains positive in some open interval that contains a. We can say something stronger:

Theorem 2.4.1 (The Permanence Property)**.** *Assume that the domain of a function f contains an open interval that includes the number a. Assume that f is continuous at a and that $f(a) = p$ is positive. Let q be any number less than p. Then there is an open interval including a such that $f(x) \geq q$ for all x in that interval.*

To persuade yourself that the permanence principle is plausible, imagine what the graph of $y = f(x)$ looks like near $(a, f(a))$, as in Figure 2.4.7.

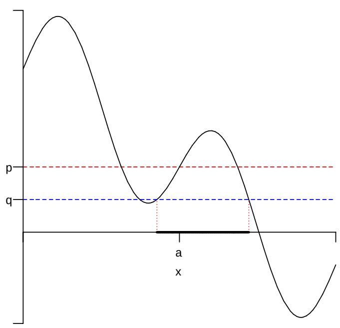


Figure 2.4.7:

Summary

This section stated, without proofs, the Extreme-Value Property, the Intermediate-Value Property, and the Permanence Property. Each will be used several times in later chapters.

EXERCISES for 2.4

Key: R–routine, M–moderate, C–challenging

1.[R] For each of the given intervals, find the maximum value of $\cos(x)$ over that interval and find the value of x at which it occurs.

(a) $[0, \pi/2]$

(b) $[0, 2\pi]$

2.[R] Does the function $\frac{x^3+x^4}{1+5x^2+x^6}$ have (a) a maximum value for x in $[1, 4]$? (b) a minimum value for x in $[1, 4]$? If so, use a graphing device to determine the extreme values?

3.[R] Does the function $2^x - x^3 + x^5$ have (a) a maximum value for x in $[-3, 10]$? (b) a minimum value for x in $[-3, 10]$? If so, use a graphing device to determine the extreme values?

4.[R] Does the function x^3 have a maximum value for x in (a) $[2, 4]$? (b) $[-3, 5]$? (c) $(1, 6)$? If so, where does the maximum occur and what is the maximum value?

5.[R] Does the function x^4 have a minimum value for x in (a) $[-5, 6]$? (b) $(-2, 4)$? (c) $(3, 7)$? (d) $(-4, 4)$? If so, where does the minimum occur and what is the minimum value?

6.[R] Does the function $2 - x^2$ have (a) a maximum value for x in $(-1, 1)$? (b) a minimum value for x in $(-1, 1)$? If so, where?

7.[R] Does the function $2 + x^2$ have (a) a maximum value for x in $(-1, 1)$? (b) a minimum value for x in $(-1, 1)$? If so, where?

8.[R] Show that the equation $x^5 + 3x^4 + x - 2 = 0$ has at least one solution in the interval $[0, 1]$.

9.[R] Show that the equation $x^5 - 2x^3 + x^2 - 3x = -1$ has at least one solution in the interval $[1, 2]$.

In Exercises 10 to 14 verify the Intermediate-Value Property for the specified function f, the interval $[a, b]$, and the indicated value m. Find all c's in each case.

10.[R] $f(x) = 3x + 5$, $[a, b] = [1, 2]$, $m = 10$.

11.[R] $f(x) = x^2 - 2x$, $[a, b] = [-1, 4]$, $m = 5$.

12.[R] $f(x) = \sin(x)$, $[a, b] = [\frac{\pi}{2}, \frac{11\pi}{2}]$, $m = -1$.

13.[R] $f(x) = \cos(x)$, $[a, b] = [0, 5\pi]$, $m = \frac{\sqrt{3}}{2}$.
14.[R] $f(x) = x^3 - x$, $[a, b] = [-2, 2]$, $m = 0$.

15.[R] Use the Intermediate-Value Property to show that the equation $3x^3+11x^2-5x = 2$ has a solution.

16.[M] Show that the equation $2^x = 3x$ has a solution in the interval $[0, 1]$.

17.[M] Does the equation $x + \sin(x) = 1$ have a solution?

18.[M] Does the equation $x^3 = 2^x$ have a solution?

19.[M] Let $f(x) = 1/x$, $a = -1$, $b = 1$, $m = 0$. Note that $f(a) \leq 0 \leq f(b)$. Is there at least one c in $[a, b]$ such that $f(c) = 0$? If so, find c; if not, does this imply the Intermediate-Value Property sometimes does not hold?

20.[M] Use the Intermediate-Value Property to show that there is a positive number such that $\ln(x + 4) = x^2 + 3$.

Exercises 21 and 22 illustrate the Permanence Property.
21.[M] Let $f(x) = 5x$. Then $f(1) = 5$. Find an interval (a, b) containing 1 such that $f(x) \geq 4.9$ for all x in (a, b).

22.[M] Let $f(x) = x^2$. Then $f(2) = 4$. Find an interval (a, b) containing 2 such that $f(x) \geq 3.8$ for all x in (a, b).

23.[C] Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of odd degree n and with positive leading coefficient a_n. Show that there is at least one real number r such that $P(r) = 0$.

24.[C] (This continues Exercise 23.) The **factor theorem** from algebra asserts that the number r is a root of a polynomial $P(x)$ if and only if $x - r$ is a factor of $P(x)$. For instance, 2 is a root of the polynomial $x^2 - 3x + 2$ and $x - 2$ is a factor of it: $x^2 - 3x + 2 = (x - 2)(x - 1)$. NOTE: See also Exercise 47 in Section 8.4.

(a) Use the factor theorem and Exercise 23 to show that every polynomial of odd degree has a factor of degree 1.

(b) Show that none of the polynomials x^2+1, x^4+1, or $x^{100}+1$ has a first-degree factor.

(c) Verify that $x^4 + 1 = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$. (It can be shown using complex numbers that every polynomial with real coefficients is the product of polynomials with real coefficients of degrees at most 2.)

25.[C] Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ where a_n and a_0 have opposite signs.

(a) Show that the $f(x)$ has a positive root, that is, the equation $f(x) = 0$ has a positive solution.

(b) Show that if each of the roots in (a) is simple, there are an odd number of them. HINT: Use a picture.

(c) If the roots in (a) are not simple, what would be the corresponding statement? HINT: Use a picture.

(d) What can you say about the roots of $f(x)$ if a_n and a_0 have the same sign?

Convex Sets and Curves

A set in the plane bounded by a curve is **convex** if for any two points P and Q in the set the line segment joining them also lies in the set. (See Figure 2.4.8(a).) The boundary of a convex set we will call a **convex curve**. (These ideas generalize to a solid and its boundary surface.) The notion of convexity dates back to Archimedes. Disks, triangles, and parallelograms are convex sets. The quadrilateral shown in Figure 2.4.8(b) is not convex. Convex sets will be referred to in the following exercises and occasionally in the exercises in later chapters.

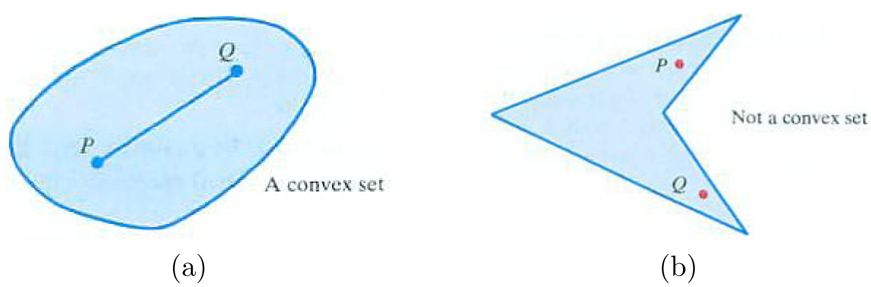


Figure 2.4.8: (a) There are no dents in the boundary of a convex set. (b) Not a convex set.

Exercises 26 to 32 concern convex sets and show how the Intermediate-Value Property gives geometric information. In these exercises you will need to define various functions geometrically. *You may assume these functions are continuous.*

26.[C] Let L be a line in the plane and let K be a convex set. Show that there is a line parallel to L that cuts K into two pieces with equal areas.

Follow these steps.

(a) Introduce an x-axis perpendicular to L with its origin on L. Each line parallel to L and meeting K crosses the x-axis at a number x. Label the line L_x. Let a be the smallest and b the largest of these numbers x. (See Figure 2.4.9.) Let the area of K be A.

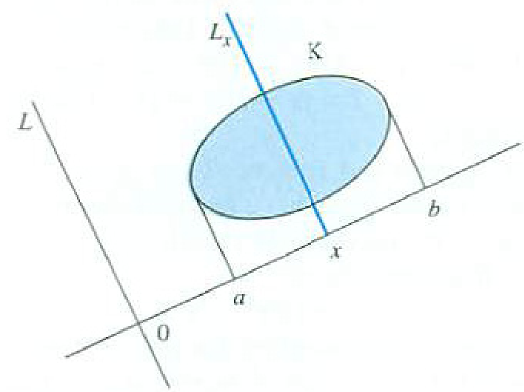


Figure 2.4.9:

(b) Let $A(x)$ be the area of K situated to the left of the line L_x corresponding to x. What is $A(a)$? $A(b)$?

(c) Use the Intermediate-Value Property to show that there is an x in $[a, b]$ such that $A(x) = \frac{A}{2}$.

(d) Why does *(c)* show that there is a line parallel to L that cuts K into two pieces of equal areas?

27.[C] Solve the preceding exercise by applying the Intermediate-Value Property to the function $f(x) = A(x) - B(x)$, where $B(x)$ is the area to the right of L_x.

28.[C] Let P be a point in the plane and let K be a convex set. Is there a line through P that cuts K into two pieces of equal area?

29.[C] Let K_1 and K_2 be two convex sets in the plane. Is there a line that simultaneously cuts K_1 into two pieces of equal areas and cuts K_2 into two pieces of equal areas? NOTE: This is known as the "two pancakes" question.

30.[C] Let K be a convex set in the plane. Show that there is a line that simultaneously cuts K into two pieces of equal area and cuts the boundary of K into two pieces of equal length.

31.[C] Let K be a convex set in the plane. Show that there are two perpendicular lines that cut K into four pieces of equal area. (It is not known whether it is always possible to find two perpendicular lines that divide K into four pieces whose areas are $\frac{1}{8}$, $\frac{1}{8}$, $\frac{3}{8}$, and $\frac{3}{8}$ of the area of K, with the parts of equal area sharing an edge, as in Figure 2.4.10.) What if the parts of equal areas are to be opposite each other, instead?

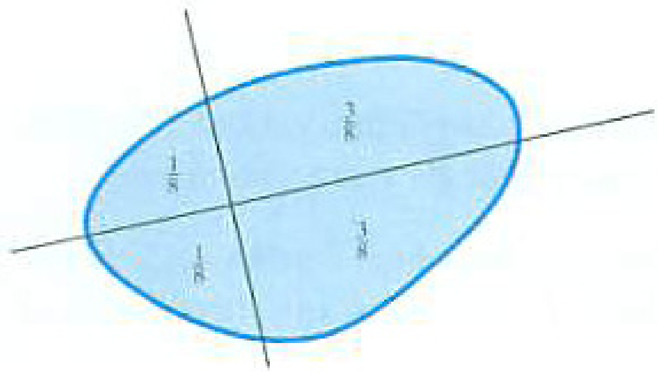


Figure 2.4.10:

32.[C] Let K be a convex set in the plane whose boundary contains no line segments. A polygon is said to **circumscribe** K if each edge of the polygon is tangent to the boundary of K.

(a) Is there necessarily a circumscribing equilateral triangle? If so, how many?

(b) Is there necessarily a circumscribing rectangle? If so, how many?

(c) Is there necessarily a circumscribing square?

33.[C] Let f be a continuous function whose domain is the x-axis and has the property that

$$f(x+y) = f(x) + f(y) \qquad \text{for all numbers } x \text{ and } y.$$

For any constant c, $f(x) = cx$ satisfies this equation since $c(x+y) = cx + cy$. This exercise shows that f must be of the form $f(x) = cx$ for some constant c.

(a) Let $f(1) = c$. Show that $f(2) = 2c$.

(b) Show that $f(0) = 0$.

(c) Show that $f(-1) = -c$.

(d) Show that that for any positive integer n, $f(n) = cn$.

(e) Show that that for any negative integer n, $f(n) = cn$.

(f) Show that $f(\frac{1}{2}) = \frac{c}{2}$.

(g) Show that that for any non-zero integer n, $f(\frac{1}{n}) = \frac{c}{n}$.

(h) Show that that for any intger m and any positive integer n, $f(\frac{m}{n}) = \frac{m}{n}c$.

(i) Show that for any irrational number x, $f(x) = cx$. This is where the continuity of f enters. Parts *(h)* and *(i)* together complete the solution.

34.[C]

(a) Let f be a continuous function defined for all real numbers. Is there necessarily a number x such that $f(x) = x$?

(b) Let f be a continuous function with domain $[0, 1]$ such that $f(0) = 1$ and $f(1) = 0$. Is there necessarily a number x such that $f(x) = x$?

35.[C] Let f be a continuous function defined on $(-\infty, \infty)$ such that $f(0) = 1$ and $f(2x) = f(x)$ for all numbers x.

(a) Give an example of such a function f.

(b) Find all functions satisfying these conditions.

Explain your answers.

2.5 Techniques for Graphing

One way to graph a function $f(x)$ is to compute $f(x)$ at several inputs x, plot the points $(x, f(x))$ that you get, and draw a curve through them. This procedure may be tedious and, if you happen to choose inputs that give misleading information, may result in an inaccurate graph.

Another way is to use a calculator that has a graphing routine built in. However, only a portion of the graph is displayed and, if you have no idea what to expect, you may have asked it to display a part of the graph that is misleading or of little interest. At points with large function values, the graph may be distorted by the calculator's choice of scale.

So it pays to be able to get some idea of the general shape of a graph quickly, without having to compute lots of values. This section describes some shortcuts.

Intercepts

The x-coordinates of the points where the graph of a function meets the x-axis are the x**-intercepts** of the function. The y coordinates of the points where a graph meets the y-axis are the y**-intercepts** of the function.

EXAMPLE 1 Find the intercepts of the graph of $y = x^2 - 4x - 5$.
SOLUTION To find the x-intercepts, set $y = 0$, obtaining

$$0 = x^2 - 4x - 5.$$

Fortunately, this quadratic factors nicely:

$$0 = x^2 - 4x - 5 = (x - 5)(x + 1).$$

The equation is satisfied when $x = 5$ or $x = -1$. There are two x-intercepts, 5 and -1. (If the equation did not factor easily, the quadratic formula could be used.)

To find y-intercepts, set $x = 0$, obtaining

$$y = 0^2 - 4 \cdot 0 - 5 = -5.$$

There is only one y-intercept, namely -5.

The intercepts in this case give us three points on the graph. Tabulating a few more points gives the parabola in Figure 2.5.1, where the intercepts are shown as well. ◇

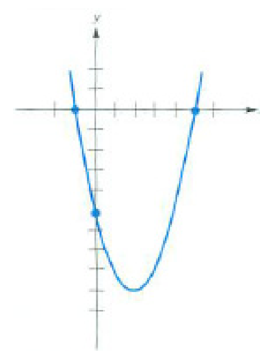


Figure 2.5.1: The graph of $y = x^2 - 4x - 5$, with intercepts.

If $f(x)$ is not defined when $x = 0$, there is no y-intercept. If $f(x)$ is defined when $x = 0$, then it's easy to get the y-intercept; just evaluate $f(0)$. While there is at most one y-intercept, there may be many x-intercepts. To find them, solve the equation $f(x) = 0$. In short,

Finding Asymptotes
To find the y-intercept, compute $f(0)$.
To find the x-intercepts, solve the equation $f(x) = 0$.

Symmetry of Odd and Even Functions

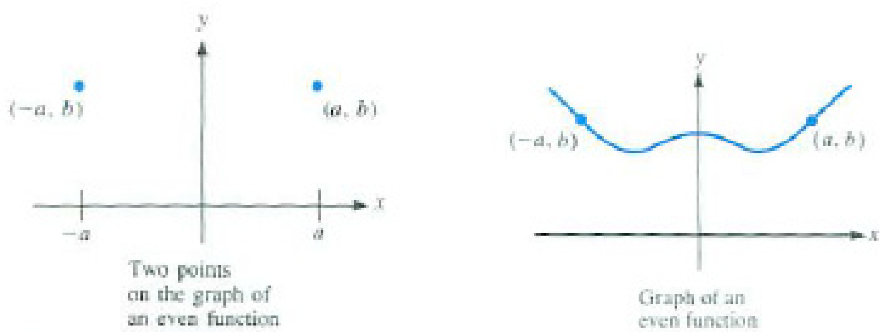
Some functions have the property that when you replace x by $-x$ you get the same value of the function. For instance, the function $f(x) = x^2$ has this property since

$$f(-x) = (-x)^2 = x^2 = f(x).$$

So does the function $f(x) = x^n$ for any *even* integer n. There are fancier functions, such as $3x^4 - 5x^2 + 6x$, $\cos(x)$, and $e^x + e^{-x}$, that also have this property.

DEFINITION *(Even function.)* A function f such that $f(-x) = f(x)$ is called an **even function**.

For an even function f, if $f(a) = b$, then $f(-a) = b$ also. In other words, if the point (a, b) is on the graph of f, so is the point $(-a, b)$, as indicated by Figure 2.5.2(a).



(a) (b)

Figure 2.5.2:

This means that the graph of f is symmetric with respect to the y-axis, as shown in Figure 2.5.2(b). So if you notice that a function is even, you can save half the work in finding its graph. First graph it for positive x and then get the part for negative x free of charge by reflecting across the y-axis. If you wanted to graph $y = x^4/(1 - x^2)$, for example, first stick to $x > 0$, then reflect the result.

DEFINITION *(Odd function.)* A function f such that $f(-x) = -f(x)$ is called an **odd function**.

The function $f(x) = x^3$ is odd since

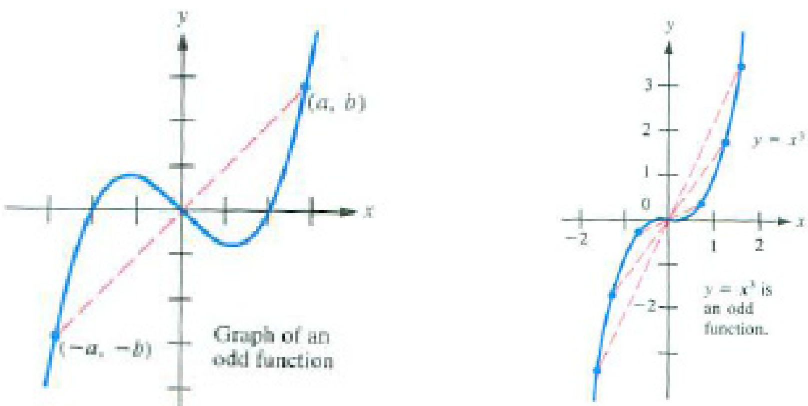
$$f(-x) = (-x)^3 = -(x^3) = -f(x).$$

For any odd integer n, $f(x) = x^n$ is an odd function. The sine function is also odd, since $\sin(-x) = -\sin(x)$.

If the point (a, b) is on the graph of an odd function, so is the point $(-a, -b)$, since

$$f(-a) = -f(a) = -b.$$

(See Figure 2.5.3(a).) Note that the origin $(0, 0)$ is the midpoint of (a, b) and $(-a, -b)$. The graph is said to be "symmetric with respect to the origin."



(a) (b)

Figure 2.5.3:

If you work out the graph of an odd function for positive x, you can obtain the graph for negative x by reflecting it point by point through the origin. For example, if you graph $y = x^3$ for $x \geq 0$, as in Figure 2.5.3(b), you can complete the graph by reflection with respect to the origin, as indicated by the dashed lines.

Most functions are neither even nor odd. For instance, $x^3 + x^4$ is neither even nor odd since $(-x)^3 + (-x)^4 = -x^3 + x^4$, which is neither $x^3 + x^4$ nor $-(x^3 + x^4)$.

Asymptotes

If $\lim_{x\to\infty} f(x) = L$ where L is a real number, the graph of $y = f(x)$ gets arbitrarily close to the horizontal line $y = L$ as x increases. The line $y = L$ is called a **horizontal asymptote** of the graph of f. (See Figure 2.5.4.)

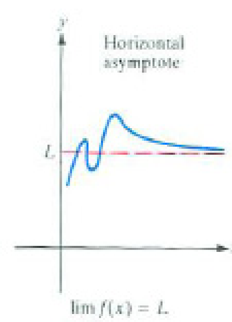


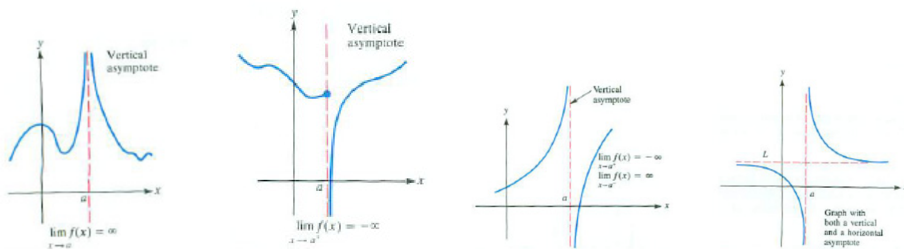
Figure 2.5.4:

If a graph has an asymptote, we can draw it and use it as a guide in drawing the graph.

If $\lim_{x\to a} f(x) = \infty$, then the graph resembles the vertical line $x = a$ for x near a. The line $x = a$ is called a **vertical asymptote** of the graph of $y = f(x)$. The same term is used if

$$\lim_{x\to a} f(x) = -\infty, \qquad \lim_{x\to a^+} f(x) = \infty \text{ or } -\infty, \quad \text{or} \quad \lim_{x\to a^-} f(x) = \infty \text{ or } -\infty.$$

Figure 2.5.5 illustrates these situations.



(a) (b) (c) (d)

Figure 2.5.5:

EXAMPLE 2 Graph $f(x) = 1/(x-1)^2$.

SOLUTION To see if there is any *symmetry*, check whether $f(-x)$ is $f(x)$ of $-f(x)$. We have

$$f(-x) = \frac{1}{(-x-1)^2} = \frac{1}{(x+1)^2}.$$

Since $1/(x+1)^2$ is neither $1/(x-1)^2$ nor $-1/(x-1)^2$, the function $f(x)$ is neither even nor odd. Therefore the graph is *not* symmetric with respect to the y-axis or with respect to the origin.

To determine the y-intercept compute $f(0) = 1/(0-1)^2 = 1$. The y-intercept is 1. To find any x-intercepts, solve the equation $f(x) = 0$, that is,

$$\frac{1}{(x-1)^2} = 0.$$

Since no number has a reciprocal equal to zero, there are no x-intercepts.

To search for a *horizontal asymptote* examine

$$\lim_{x\to\infty} 1/(x-1)^2 \qquad \text{and} \qquad \lim_{x\to-\infty} 1/(x-1)^2.$$

Both limits are 0. The line $y = 0$, that is, the x-axis, is an asymptote both to the right and to the left. Since $1/(x-1)^2$ is positive, the graph lies above the asymptote.

To discover any *vertical asymptotes*, find where the function $1/(x-1)^2$ "blows up" — that is, becomes arbitrarily large (in absolute value). This happens when the denominator $(x-1)^2$ becomes zero. Solving $(x-1)^2 = 0$ we find $x = 1$. The function is not defined for $x = 1$. The line $x = 1$ is a vertical asymptote.

To determine the shape of the graph near the line $x = 1$, we examine the one-sided limits: $\lim_{x\to 1^+} 1/(x-1)^2$ and $\lim_{x\to 1^-} 1/(x-1)^2$. Since the square of a nonzero number is always positive, we see that $\lim_{x\to 1^+} 1/(x-1)^2 = \infty$ and $\lim_{x\to 1^-} 1/(x-1)^2 = \infty$. All this information is displayed in Figure 2.5.6. ◇

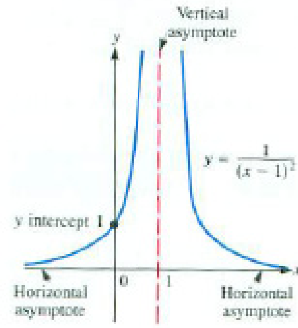


Figure 2.5.6:

Technology-Assisted Graphing

A graphing utility needs to "know" the function and the viewing window. We will show by three examples some of the obstacles you may run into and how to avoid them. More techniques to help overcome these challenges will be presented in Chapter 4.

The **viewing window** is the portion of the xy-plane to be displayed. We will say the viewing window is $[a, b] \times [c, d]$ when the window extends horizontally from $x = a$ to $x = b$ and vertically from $y = c$ to $y = d$. The graph of a function $y = f(x)$ is created by evaluating $f(x)$ for a sample of numbers x between a and b. The point $(x, f(x))$ is added to the plot. It is customary to connect these points to form the graph of $y = f(x)$. The examples in the remainder of this section demonstrate some of the unpleasant messes that can happen, and how you can avoid them.

EXAMPLE 3 Find a viewing window that shows the general shape of the graph of $y = x^4 + 6x^3 + 3x^2 - 12x + 4$. Use graphs to estimate the location of the rightmost x intercept.

SOLUTION Figure 2.5.7(a) is typical of the first plot of a function. Choose a fairly wide x interval, here $[-10, 10]$, and let the graphing software choose an appropriate vertical range. While this view is useless for estimating any specific x intercept, it is tempting to say that any x intercepts will be between $x = -6$ and $x = 3$. Figure 2.5.7(b) is the graph of this function on the viewing window $[-6, 3] \times [-30, 30]$. Now four x intercepts are visible. The rightmost one occurs around $x = 0.8$. Figure 2.5.7(c) is the result of zooming in on this part of the graph. From this view we estimate that the rightmost x intercept is about 0.83.

In fact, using a CAS, the four x intercepts for this function are found to occur at 0.8284, 0.4142, -2.4142, and -4.8284 (to four decimal places). ◇

Generating a collection of points and connecting the dots can sometimes

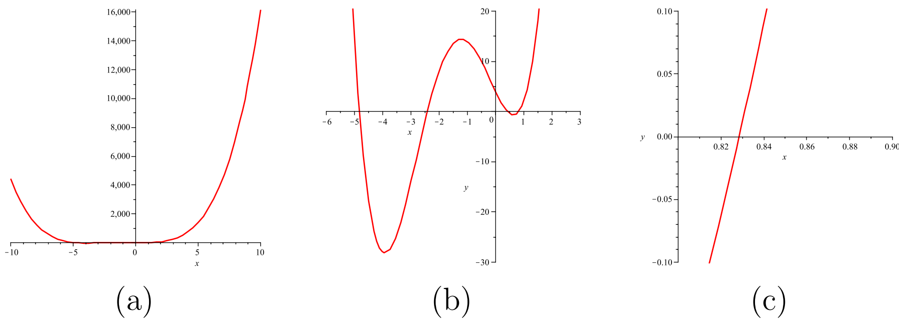


Figure 2.5.7:

lead to ridiculous results, as in Example 4.

EXAMPLE 4 Find a viewing window that clearly shows the general shape and periodicity of the graph of $y = \tan(x)$.
SOLUTION A computer-generated plot of $y = \tan(x)$ for x between -10 and 10 with no vertical height of the viewing window is shown in Figure 2.5.8(a). This graph is not periodic; it looks more like an echocardiogram than the graph of one of the trigonometric functions.

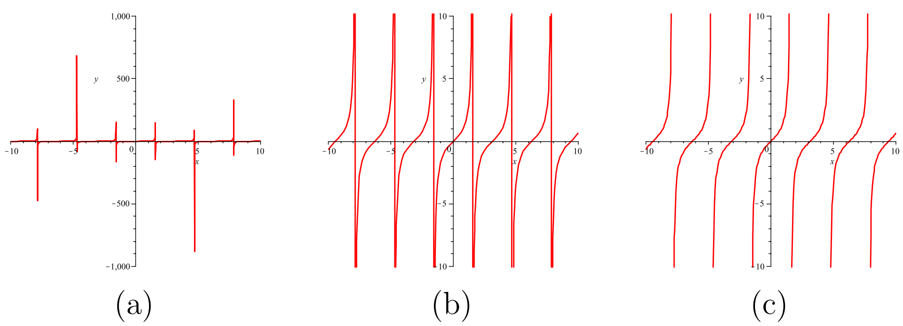


Figure 2.5.8:

Notice that the default vertical height is very long: $[-1000, 1000]$. Reducing this by a factor of 100, that is, to $[-10, 10]$, yields Figure 2.5.8(b). This graph is periodic and exhibits the expected behavior.

To understand this plot you must realize that the software selects a sample of input values from the domain, computes the value of tangent of each input, then connects the points in order of the input values. The tangent of the last input smaller than $\pi/2$ is large and positive and the tangent of the first input larger than $\pi/2$ is large but negative. Neither of these points is in the viewing window, but the line segment connecting these points does pass through the viewing window and appears as the "vertical" line at $x = \pi/2$ in

Figure 2.5.8(b). Because the tangent is not defined for every odd multiple of $\pi/2$, similar reasoning explains the other "vertical" lines at every odd multiple of $\pi/2$

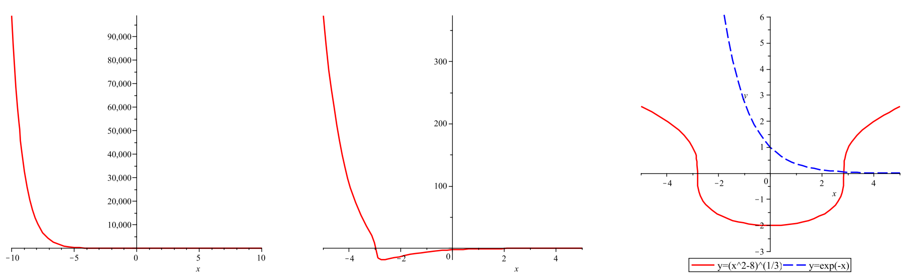
These segments are not really a part of the graph. Figure 2.5.8(c) shows the graph of $y = \tan(x)$ with these extraneous segments removed. ◇

Example 4 illustrates why we must remain alert when using technology. We have to check that the results are consistent with what we already know.

The next example shows that sometimes it is not possible to show all of the important features of a function in a single graph.

EXAMPLE 5 Use one or more graphs to show all major features of the graph $y = e^{-x}\sqrt[3]{x^2 - 8}$.

SOLUTION The graph of this function on the x interval $[-10, 10]$ with the vertical window chosen by the software is shown in Figure 2.5.9(a). In this window, the exponential function dominates the graph.



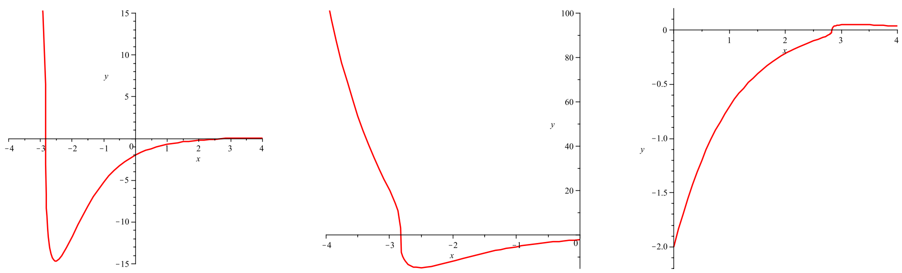
(a) (b) (c)

Figure 2.5.9:

At $x = 0$ the value of the function is $(0 - 8)^{1/3}e^0 = -2$. To get enough detail to see both the positive and negative values of the function, zoom in by reducing the x interval to $[-5, 5]$. The result is Figure 2.5.9(b). Reducing the x interval to $[-4, 4]$ and specifying the y interval as $[-15, 15]$ gives Figure 2.5.9(c).

We could continue to adjust the viewing window until we find suitable views. A more systematic approach is to look at the graphs of $y = \sqrt[3]{x^2 - 8}$ and $y = e^{-x}$ separately, but on the same pair of axes. (See Figure 2.5.10(a).) The exponential growth of e^{-x} for negative values of x stretches (vertically) the graph of $y = \sqrt[3]{x^2 - 8}$ to the left of the y-axis while the exponential decay for $x > 0$ (vertically) compresses the graph of $y = \sqrt[3]{x^2 - 8}$ to the right of the y-axis.

It is prudent to produce two separate plots to represent the sketch of this function. To the left of the y-axis, with a viewing window of $[-4, 0] \times [-15, 100]$, the graph of the function is shown in Figure 2.5.10(b). To the right of the y-axis, with a much shorter viewing window of $[0, 4] \times [-2.2, 0.2]$, the graph is as shown in Figure 2.5.10(c). ◇



(a) (b) (c)

Figure 2.5.10:

Summary

The first half of this section presents three tools for making a quick sketch of the graph of $y = f(x)$ by hand.

1. *Check for intercepts.* Find $f(0)$ to get the y-intercept. Solve $f(x) = 0$ to get the x-intercepts.

2. *Check for symmetry.* Is $f(-x)$ equal to $f(x)$ or $-f(x)$?

3. *Check for asymptotes.* If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ (where L is some real number), then the line $y = L$ is a horizontal asymptote. If $\lim_{x\to a} f(x) = +\infty$ of $-\infty$, then the line $x = a$ is a vertical asymptote. This is also the case whenever $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ is $+\infty$ or $-\infty$.

The second half of the section provides some pointers for using an automatic graphing utility. The key to their use for graphing is to specify an appropriate viewing window.

Computer-Based Mathematics

Graphing calculators provide an easy way to graph a function. Computer algebra systems (CAS) such as Maple, Mathematica, and Derive can perform symbolic operations on mathematical expressions: for example, they can factor a polynomial:

$$x^5 - 2x^4 - 2x^3 + 4x^2 + x - 2 = (x-1)^2(x+1)^2(x-2),$$

express the quotient of two polynomials as the sum of simpler quotients:

$$\frac{36}{x^5 - 2x^4 - 2x^3 + 4x^2 + x - 2} = \frac{-3}{(x+1)^2} - \frac{9}{(x-1)^2} - \frac{4}{x+1} + \frac{4}{x-2},$$

and solve equations, such as

$$\arctan(x^2+1) = \pi/3 \qquad \text{and} \qquad \sin\left(\frac{\pi}{x}\right) - \frac{\pi}{x}\cos\left(\frac{\pi}{x}\right) = 0.$$

Some of these symbolic features are now available on calculators, PDAs, telephones, and other handheld devices.

These tools will continue to develop and you need to be aware that these tools do exist, and can do much more than graph functions. As they become more common, and easier to use, they will change the way mathematics is used in the real world. The ability to factor a polynomial or to solve an equation will be less important than the ability to apply basic principles of mathematics and science to set up and to analyze the equations.

EXERCISES for 2.5 *Key:* R–routine, M–moderate, C–challenging

1.[R] Show that these are even functions.

(a) $x^2 + 2$

(b) $\sqrt{x^4 = 1}$

(c) $1/x^2$

2.[R] Show that these are even functions.

(a) $5x^4 - x^2$

(b) $\cos(2x)$

(c) $7/x^6$

3.[R] Show that these are odd functions.

(a) $x^3 - x$

(b) $x + 1/x$

(c) $\sqrt[3]{x}$

4.[R] Show that these are odd functions.

(a) $2x + \frac{1}{2}x$

(b) $\tan(x)$

(c) $x^{5/3}$

5.[R] Show that these functions are neither odd nor even.

(a) $3 + x$

(b) $(x + 2)^2$

(c) $\frac{x}{x+1}$

6.[R] Show that these functions are neither odd nor even.

(a) $2x - 1$

(b) e^x

(c) $x^2 + 1/x$

7.[R] Label each function as even, odd, or neither.

(a) $x + x^3 + 5x^4$

(b) $7x^4 - 5x^2$

(c) $e^x - e^{-x}$

8.[R] Label each function as even, odd, or neither.

(a) $\frac{1+x}{1-x}$

(b) $\ln(x^2 + 1)$

(c) $\sqrt[3]{x^2 + 1}$

In Exercises 9 to 14 find the x- and y-intercepts, if any.

9.[R] $y = 2x + 3$

10.[R] $y = 3x - 7$

11.[R] $y = x^2 + 3x + 2$

12.[R] $y = 2x^2 + 5x + 3$

13.[R] $y = 2x^2 + 1$

14.[R] $y = x^2 + x + 1$

In Exercises 15 to 20 find all the horizontal and vertical asymptotes.

15.[R] $y = \frac{x+2}{x-2}$

16.[R] $y = \frac{x-2}{x^2-9}$

17.[R] $y = \frac{x}{x^2+1}$

18.[R] $y = \frac{2x+3}{x^2+4}$

19.[R] $y = \frac{x^2+1}{x^2-3}$

20.[R] $y = \frac{x}{x^2+2x+1}$

In Exercises 21 to 28 graph the function.

21.[R] $y = \frac{1}{x-2}$

22.[R] $y = \frac{1}{x+3}$

23.[R] $y = \frac{1}{x^2-1}$

24.[R] $y = \frac{x}{x^2-2}$

25.[R] $y = \frac{x^2}{1+x^2}$

26.[R] $y = \frac{1}{x^3+x^{-1}}$

27.[R] $y = \frac{1}{x(x-1)(x+2)}$

28.[R] $y = \frac{x+2}{x^3+x^2}$

Use a graphing utility to sketch a graph of the functions in Exercise 29 to 47. Be sure to indicate the viewing window used to generate your graph.

29.[R] $(x^2 + x - 6)\ln(x + 2)$

30.[R] $(x^2 - x + 6)\ln(x + 2)$

31.[R] $(x^2 + 4)\ln(x + 1)$

32.[R] $(x^2 - 4)\ln(x + 1)$

33.[R] $\frac{x^3}{x^2-4}\arctan\left(\frac{x}{5}\right)$

34.[R] $\frac{(x^2-4)}{x^3}\arctan\left(\frac{x}{5}\right)$

35.[R] $\frac{x^3-3x}{x^2-4}$

36.[R] $\frac{x^3-2x}{x^2-4}$

37.[R] $\frac{\sin(x)}{x}$

38.[R] $\frac{\sin(2x)}{x}$

39.[R] $\frac{\sin(2x)}{3x}$

40.[R] $\frac{\sin(x)}{3x}$

41.[R] $\frac{x-\arctan(x)}{x^3}$

42.[R] $\frac{x-\arctan(x)}{x^3+x}$

43.[R] $\frac{x-\arctan(x)}{x^3-1}$

44.[R] $\frac{x-\arctan(x)}{x^3+1}$

45.[R] $\frac{5x^3+x^2+1}{7x^3+x+4}$

46.[R] $\frac{x^3-3x}{x^2-4}\arctan\left(\frac{x}{4}\right)$

47.[R] $\frac{x^3-2x}{x^2-4}\arctan\left(\frac{x}{4}\right)$

Exercises 48 to 54 concern even and odd functions.

48.[M] If two functions are odd, what can you say about

(a) their sum?

(b) their product?

(c) their quotient?

49.[M] If two functions are even, what can you say about

(a) their sum?

(b) their product?

(c) their quotient?

50.[M] If f is odd and g is even, what can you say about

(a) $f + g$?

(b) fg?

(c) f/g?

51.[M] What, if anything, can you say about $f(0)$ if

(a) f is an even function?

(b) f is an odd function?

NOTE: Assume 0 is in the domain of f.

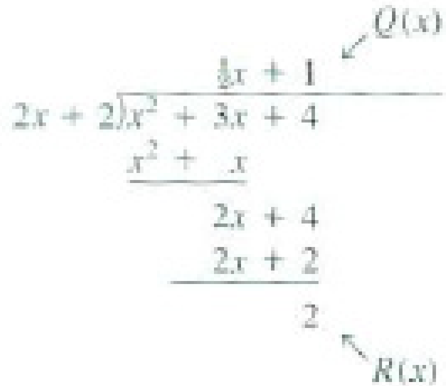
52.[M] Which polynomials are even? Explain.

53.[M] Which polynomials are odd? Explain.

54.[M] Is there a function that is both odd and even? Explain.

Exercises 55 to 58 concern tilted asymptotes. Let $A(x)$ and $B(x)$ be polynomials such that the degree of $A(x)$ is equal to 1 more than the degree of $B(x)$. Then when you divide $B(x)$ into $A(x)$, you get a quotient $Q(x)$, which is a polynomial of degree 1, and a remainder $R(x)$, which is a polynomial of degree less than the degree of $B(x)$.

For example, if $A(x) = x^2 + 3x + 4$ and $B(x) = 2x + 2$,



Thus

$$x^2 + 3x + 4 = (\frac{1}{2}x + 1)(2x + 2) + 2/$$

This tells us that

$$\frac{x^2 + 3x + 4}{2x + 2} = \frac{1}{2}x + 1 + \frac{2}{2x + 2}.$$

When x is large, $2/(2x + 2) \to 0$. Thus the graph of $y = (x^2 + 3x + 4|2x + 2$ is asymptotic to the line $y = \frac{1}{2}x + 1$. (See Figure 2.5.11.)

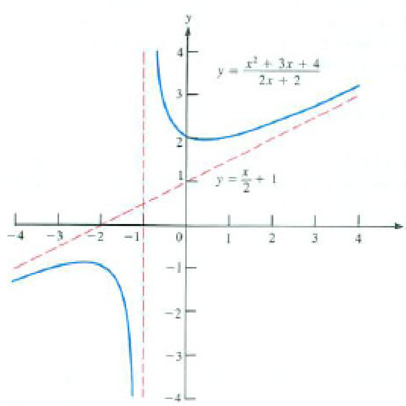


Figure 2.5.11:

Whenever the degree of $A(x)$ exceeds the degree of $B(x)$ by exactly 1, the graph of $y = A(x)/B(x)$ has a tilted asymptote. You find it as we did in the example, by dividing $B(x)$ into $A(x)$, obtaining a quotient $Q(x)$ and a remainder $R(x)$. Then

$$\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)}.$$

The asymptote is $y = Q(x)$. In each exercise graph the function, showing all asymptotes.

55.[M] $y = \frac{x^2}{x-1}$

56.[M] $y = \frac{x^3}{x^2-1}$

57.[M] $y = \frac{x^2-4}{x+4}$

58.[M] $y = \frac{x^2+x+1}{x-2}$

A **piecewise-defined function** is a function that is given by different formulas on different pieces of the domain.

Read the directions for your graphing software to learn how to graph a piecewise-defined function. Then use your graphing utility to sketch a graph of the functions in Exercises 59 and 60.

59.[M] $y = \begin{cases} x^2 - x & x < 1 \\ \sqrt{x-1} & x \geq 1 \end{cases}$

60.[M] $y = \begin{cases} \frac{\sin(x)}{x} & x < 0 \\ \sin x & 0 \leq x \geq \pi \\ x - 2 & x > \pi \end{cases}$

Some graphing utilities have trouble plotting functions with fractional exponents. General rules when graphing $y = x^{p/q}$ where p/q is a positive fraction in lowest terms are:

- If p is even and q is odd, then graph $y = |x|^{p/q}$.
- If p and q are both odd, then graph $y = \frac{|x|}{x}|x|^{p/q}$.

Use that advice and a calculator to sketch the graph of each function in Exercises 61 to 64.

61.[M] $y = x^{1/3}$

62.[M] $y = x^{2/3}$

63.[M] $y = x^{4/7}$

64.[M] $y = x^{3/7}$

65.[C] Let $P(x)$ be a polynomial of degree m and $Q(x)$ a polynomial of degree n. For which m and n does the graph of $y = P(x)/Q(x)$ have a horizontal asymptote?

66.[C] Assume you already have drawn the graph of a function $y = f(x)$. How would you obtain the graph of $y = g(x)$ from that graph if

(a) $g(x) = f(x) + 2$?

(b) $g(x) = f(x) - 2$?

(c) $g(x) = f(x - 2)$?

(d) $g(x) = f(x+2)$?

(e) $g(x) = 2f(x)$?

(f) $g(x) = 3f(x-2)$?

67.[C] Is there a function f defined for all x such that $f(-x) = 1/f(x)$? If so, how many? If not, explain why there are no such functions.

68.[C] Is there a function f defined for all x such that $f(-x) = 2f(x)$? If so, how many? If not, explain why there are no such functions.

69.[C] Is there a constant k such that the function

$$f(x) = \frac{1}{3^x - 1} + k$$

is odd? even?

2.6 Chapter Summary

One concept underlies calculus: the limit of a function. For a function defined near a (but not necessarily at a) we ask, "What happens to $f(x)$ as x gets nearer and nearer to a." If the values get nearer and nearer one specific number, we call that number the limit of the function as x approaches a. This concept, which is not met in arithmetic or algebra or trigonometry, distinguishes calculus.

For instance, when $f(x) = (2^x - 1)/x$, which is not defined at $x = 0$, we conjectured on the basis of numerical evidence that $f(x)$ approaches 0.693 (to three decimals). Later we will see that this limit is a certain logarithm. With that information we found that $(4^x - 1)/x$ must approach 2(0.693), which is larger than 1. We then defined e as that number (between 2 and 4) such that $(e^x - 1)/x$ approaches 1 as x approaches 0. The number e is as important in calculus as π is in geometry or trigonometry. The number e is about 2.718 (again to three decimals) and is called Euler's number. That is why a scientific calculator has a key for e^x, the most convenient exponential for calculus, as will become clear in the next chapter.

When angles are measured in radians,

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$$

These two limits will serve as the basis of the calculus of trigonometric functions developed in the next chapter. The simplicity of the first limit is one reason that in calculus and its applications angles are measured in radians. If angles were measured in degrees, the first limit would be $\pi/180$, which would complicate computations.

Most of the functions of interest in later chapters are "continuous." The value of such a function at a number a in its domain is the same as the limit of the function as x approaches a. However, we will be interested in a few functions that are not continuous.

A continuous function has three properties, which will be referred to often:

- On a closed interval it attains a maximum value and a minimum value. *Extreme-Value Property*
- On a closed interval it takes on all values between its values at the end points of the interval. *Intermediate -Value Property*
- If it is positive at some number and defined at least on an open interval containing that number, then it remains positive at least on some open interval containing that number. More precisely, if $f(a) = p > 0$, and q is less than p, then $f(x)$ remains larger than q, at least on some open interval containing a. A similar statement holds when $f(a)$ is negative. *Permanence Property*

A quick sketch of the graph of a generic continuous function makes the three properties plausible. In advanced calculus they are all established using only the precise definition of continuity and properties of the real numbers — but no pictures. Such strictness is necessary because there are some pretty wild continuous functions. For instance, there is one such that when you zoom in on its graph at any point, the parts of the graph nearer and nearer the point do not look like straight line segments.

The initial steps in the analysis of a function utilize intercepts, symmetry, and asymptotes. The same ideas are also helpful when selecting an appropriate viewing window when using an electronic graphing utility. Additional techniques will be added in Chapter 4, particularly Section 4.3.

EXERCISES for 2.6 *Key:* R–routine, M–moderate, C–challenging

1.[R] Define Euler's constant, e, and give its decimal value to five places.

In Exercises 2 to 4 state the given property in your own words, using as few mathematical symbols as possible.

2.[R] The Maximum-Value Property.

3.[R] The Intermediate-Value Property.

4.[R] The Permanence Property.

5.[R]

(a) Verify that $x^5 - y^5 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$.

(b) Use (a) to find $\lim_{x\to a} \frac{x^5-a^5}{x-a}$.

6.[M] Show that $\lim_{x\to 1} \frac{x^{1/3}-1}{x-1} = \frac{1}{3}$ by first writing the denominator as $\left(x^{1/3}\right)^3 - 1$ and using the factorization $u^3 - 1 = (u-1)(u^2+u+1)$.

7.[M] Use the factorization in Exercise 5 to find $\lim_{x\to a} \frac{x^{-5}-a^{-5}}{x-a}$.

8.[M] Assume $b > 1$. If $\lim_{x\to 0} \frac{b^x-1}{x} = L$, find $\lim_{x\to 0} \frac{(1/b)^x-1}{x}$

9.[M] By sketching a graph, show that if a function is not continuous it may not

(a) have a maximum even if its domain is a closed interval,

(b) satisfy the Intermediate-Value Theorem, even if its domain is a closed interval,

(c) have the Permanence Property, even if its domain is an open interval.

10.[M] Let g be an increasing function such that $\lim_{x\to a} g(x) = L$.

(a) Sketch the graph of a function f whose domain includes an open interval around L such that

$$f\left(\lim_{x\to a} g(x)\right) \text{ and } \lim_{x\to a} f(g(x))$$

both exist but the are not equal

(b) What property of f would assure us that the two limits in (a) would be equal?

We obtained $\lim_{x\to a} \frac{x^n - a^n}{x-a}$ be exploiting the factorization of $x^n - a^n$. Calling $x - a$ simply h, that limit can be written as $\lim_{h\to 0} \frac{(a+h)^n - a^n}{h}$. This limit can be evaluated, but by different algebra, as Exercises 11 and 12 show.

11.[M]

(a) Show that $(a + h)^2 = a^2 + 2ah + h^2$.

(b) Use (a) to evaluate $\lim_{h\to 0} \frac{(a+h)^2 - a^2}{h}$.

12.[M]

(a) Show that $(a + h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$.

(b) Use (a) to evaluate $\lim_{h\to 0} \frac{(a+h)^3 - a^3}{h}$.

13.[M] If you are familiar with the Binomial Theorem, use it to show that for any positive integer n, $\lim_{h\to 0} \frac{(a+h)^n - a^n}{h} = na^{n-1}$. NOTE: The Binomial Theorem expresses $(a + b)^n$, when multiplied out, as the sum of $n + 1$ terms. Using calculus, we will develop it in Section 5.4 (Exercise 31).

In Exercises 14 to 17 find each limit.

14.[M] $\lim_{x\to\infty} \frac{\ln(5x)}{\ln(4x^2)}$

15.[M] $\lim_{x\to\infty} \frac{\ln(5x)}{\ln(4x)}$

16.[M] $\lim_{x\to\infty} \frac{\log_2(x^2)}{\log_4(x)}$

17.[M] $\lim_{x\to\infty} \frac{\log_3(x^5)}{\log_9(x)}$

18.[M] Find $\lim_{h\to 0} \frac{(e^2)^h - 1}{h}$ by factoring the numerator.

19.[M] Define $f(x) = \begin{cases} \frac{h(x)}{x-3} & x \neq 3 \\ p & x = 3 \end{cases}$ What conditions on h must be satisfied to make f continuous?

20.[M] Assuming that $\lim_{x\to 0^+} x^x = 1$ and that $\lim_{x\to\infty} \ln(x) = \infty$, deduce each of the following limits:

(a) $\lim_{x\to 0} x\ln(x)$

(b) $\lim_{x\to\infty} \frac{\ln(x)}{x}$ HINT: Use (a).

(c) $\lim_{x\to\infty} x^{1/x}$

(d) $\lim_{x\to\infty} \frac{\ln(x)}{x^k}$, k a positive constant

(e) $\lim_{x\to\infty} \frac{x}{e^x}$

(f) $\lim_{x\to\infty} \frac{x^n}{e^x}$, n a positive integer

(g) $\lim_{x\to\infty} \frac{\ln(x)^n}{x}$, n a positive integer

21.[M] Define $f(x) = \begin{cases} \frac{x^3-3x^2-4x+k}{x-3} & x \neq 3 \\ p & x = 3 \end{cases}$

(a) For what values of k and p is f continuous? (Justify your answer.)

(b) For these values of k and p, is f an even or odd function? (Justify your answer.)

22.[M] Two points on a circle or sphere will be called "opposite" if they are the ends of a diameter of the circle or sphere.

(a) Assuming the temperature is continuous, show that there are opposite points on the equator that have the same temperature.

(b) Show that there may not be opposite points on the equator where the temperatures are equal and also the barometric pressures are equal.

NOTE: The Borsuk-Ulam theorem in topology implies that there are opposite points on the earth where the temperatures are equal and the pressures are equal.

23.[M] Let $f = g+h$, where g is an even function and f is an odd function. Express g and h in terms of f.

24.[M]

(a) Show that any function f can be written as the sum of an even function and an odd function.

(b) In how many ways can a given function be written that way?

25.[M] If f is an odd function and g is an even function, what, if anything, can be said about (a) fg, (b) f^2, (c) $f + g$, (d) $f + f$, and (e) f/g? Explain.

26.[C] The graph of some function f whose domain is $[2, 4]$ and range is $[1, 3]$ is shown in Figure 2.6.1(a). Sketch the graphs of the following functions and state their domain and range.

(a) $g(x) = 2f(x)$,

(b) $g(x) = f(x + 1)$,

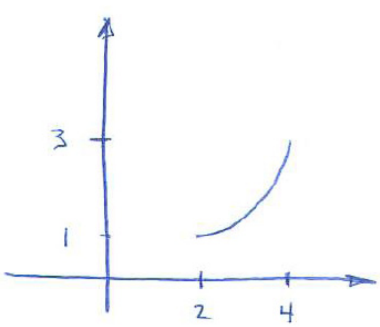
(c) $g(x) = f(x - 1)$,

(d) $g(x) = 3 + f(x)$,

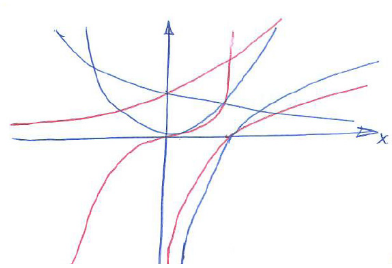
(e) $g(x) = f(2x)$,

(f) $g(x) = f(x/2)$,

(g) $g(x) = f(2x - 1)$.



(a)



(b)

Figure 2.6.1:

27.[C] For a constant k, find $\lim_{h\to 0} \frac{(e^k)^h - 1}{h}$. HINT: Replace h in the denominator by hk, but do it legally.

28.[C]

(a) Calculate $(0.99999)^x$ for various large values of x.

(b) Using the evidence gathered in (a), conjecture the value of $\lim_{x\to\infty}(0.99999)^x$.

(c) Why is $\lim_{x\to\infty}(0.99999)^{x+1}$ the same as $\lim_{x\to\infty}(0.99999)^x$?

(d) Denoting the limit in (b) as L, show that $0.99999L = L$.

(e) Using (d), find L.

29.[C] (Contributed by G. D. Chakerian) This exercise obtains $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}$ without using areas. Figure 2.6.2 shows a circle C of radius 1 with center at the origin and a circle $C(r)$ of radius $r > 1$ that passes through the center of C. Let $S(r)$ be the part of $C(r)$ that lies within C. Its ends are P and Q. Let θ be the angle subtended by the top half of $S(r)$ at the center of $C(r)$. Note that as $r \to \infty$, $\theta \to 0$. Define $A(\theta)$ to be the length of the arc $S(r)$ as a function of θ.

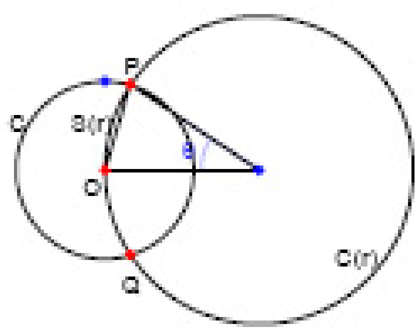


Figure 2.6.2:

(a) Looking at Figure 2.6.2, determine $\lim_{\theta\to 0} A(\theta)$. Hint: What happens to P as $r \to \infty$?

(b) Show that $A(\theta)$ is $\frac{\theta/2}{\sin(\theta/2)}$.

(c) Combining (a) and (b), show that $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$.

Calculus is Everywhere
Bank Interest and the Annual Percentage Yield

The Truth in Savings Act, passed in 1991, requires a bank to post the Annual Percentage Yield (APY) on deposits. That yield depends on how often the bank computes the interest earned, perhaps as often as daily or as seldom as once a year. Imagine that you open an account on January 1 by depositing \$1000. The bank pays interest monthly at the rate of 5 percent a year. How much will there be in your account at the end of the year? For simplicity, assume all the months have the same length. To begin, we find out how much there is in the account at the end of the first month. The account then has the initial amount, \$1000, plus the interest earned during January. Because there are 12 months, the interest rate in each month is 5 percent divided by 12, which is 0.05/12 percent per month. So the interest earned in January is \$1000 times 0.05/12. At the end of January the account then has

$$\$1000 + \$1000(0.05/12) = \$1000(1 + 0.05/12).$$

The initial deposit is "magnified" by the factor $(1 + 0.05/12)$.

The amount in the account at the end of February is found the same way, but the initial amount is $\$1000(1 + 0.05/12)$ instead of \$1000. Again the amount is magnified by the factor $1 + 0.05/12$, to become

$$\$1000(1 + 0.05/12)^2.$$

The amount at the end of March is

$$\$1000(1 + 0.05/12)^3.$$

At the end of the year the account has grown to

$$\$1000(1 + 0.05/12)^{12},$$

which is about \$1051.16.

The deposit earned \$51.16. If instead the bank computed the interest only once, at the end of the year, so-called "simple interest," the deposit would earn only 5 percent of \$1000, which is \$50. The depositor benefits when the interest is computed more than once a year, so-called "compound interest." A competing bank may offer to compute the interest every day. In that case, the account would grow to

$$\$1000(1 + 0.05/365)^{365},$$

n	$(1+1/n)^n$	$(1+1/n)^n$
1	$(1+1/1)^1$	2.00000
2	$(1+1/2)^2$	2.25000
3	$(1+1/3)^3$	2.37037
10	$(1+1/10)^{10}$	2.59374
100	$(1+1/100)^{100}$	2.70481
1000	$(1+1/1000)^{1000}$	2.71692

Table 2.6.1:

which is about \$1051.27, eleven cents more than the first bank offers. More generally, if the initial deposit is A, the annual interest rate is r, and interest is computed n times a year, the amount at the end of the year is

$$A(1+r/n)^n. \tag{2.6.1}$$

In the examples, A is \$1000, r is 0.05, and n is 12 and then 365. Of special interest is the case when A is 1 and r is a generous 100 percent, that is, $r=1$. Then (2.6.1) becomes

$$(1+1/n)^n. \tag{2.6.2}$$

How does (2.6.2) behave as n increase? Table 2.6.1 shows a few values of (2.6.2), to five decimal places. The base, $1+1/n$, approaches 1 as n increases, suggesting that (2.6.2) may approach a number near 1. However, the exponent gets large, so we are multiplying lots of numbers, all of them larger than 1. It turns out that as n increases $(1+1/n)^n$ approaches the number e defined in Section 2.1. One can write

$$\lim_{x\to 0^+}(1+x)^{1/x}=e.$$

Note that the exponent, $1/x$, is the reciprocal of the "small number" x.

With that fact at our disposal, we can figure out what happens when an account opens with \$1000, the annual interest rate is 5 percent, and the interest is compounded more and more often. In that case we would be interested in

$$1000\lim_{n\to\infty}\left(1+\frac{0.05}{n}\right)^n.$$

Unfortunately, the exponent n is not the reciprocal of the small number $0.05/n$. But a little algebra can overcome that nuisance, for

$$\left(1+\frac{0.05}{n}\right)^n=\left(\left(1+\frac{0.05}{n}\right)^{\frac{n}{0.05}}\right)^{0.05}. \tag{2.6.3}$$

The expression in parentheses has the form "(1 + small number) raised to the reciprocal of that small number." Therefore, as n increases, (2.6.3)) approaches $e^{0.05}$, which is about 1.05127. No matter how often interest is compounded, the \$1000 would never grow beyond \$1051.27.

The definition of e given in Section 2.1 has no obvious connection to the fact that $\lim_{x\to 0^+}(1+x)^{1/x}$ equals the number e. It seems "obvious," by thinking in terms of banks, that as n increases, so does $(1+1/n)^n$. Without thinking about banks, try showing that it does increase. (This limit will be evaluated in Section 3.4.)

EXERCISES

1.[R] A dollar is deposited at the beginning of the year in an account that pays an interest rate r of 100% a year. Let $f(t)$, for $0 \leq t \leq 1$, be the amount in the account at time t. Graph the function if the bank pays

(a) only simple interest, computed only at $t = 1$.

(b) compound interest, twice a year computed at $t = 1/2$ and 1.

(c) compound interest, three times a year computed at $t = 1/3$, $2/3$, and 1.

(d) compound interest, four times a year computed at $t = 1/4$, $1/2$, $3/4$, and $1/$

(e) Are the functions in (a), (b), (c), and (d) continuous?

(f) One could expect the account that is compounded more often than another would always have more in it. Is that the case?

Chapter 3

The Derivative

In this chapter we meet one of the two main concepts of calculus, the **derivative of a function**. The derivative tells how rapidly or slowly a function changes. For instance, if the function describes the position of a moving particle, the derivative tells us its velocity.

The definition of a derivative rests on the notion of a limit. The particular limits examined in Chapter 2 are the basis for finding the derivatives of all functions of interest.

A few techniques allow us to find the derivative of almost any function that we will encounter. The goal of this chapter is twofold: to develop those techniques and also an understanding of the meaning of a derivative.

3.1 Velocity and Slope: Two Problems with One Theme

This section discusses two problems which at first glance may seem unrelated. The first one concerns the slope of a tangent line to a curve. The second involves velocity. A little arithmetic will show that they are both just different versions of one mathematical idea: the *derivative*.

Slope

Our first problem is important because it is related to finding the straight line that most closely resembles a given graph near a point on the graph.

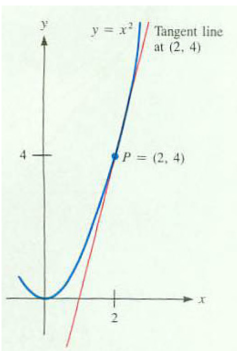


Figure 3.1.1:

EXAMPLE 1 What is the slope of the tangent line to the graph of $y = x^2$ at the point $P = (2, 4)$, as shown in Figure 3.1.1

For the present, the tangent line to a curve at a point P on the curve shall mean the line through P that has the "same direction" as the curve at P. (Look again at Figure 3.1.1.) This will he made precise in the next section.

Appendix B has a further discussion of the slope of a line.

SOLUTION If we know two points (x_1, y_1) and (x_2, y_2) on a straight line, we can compute the slope of that line. The slope is "change in y divided by change in x;" that is,

$$\text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}.$$

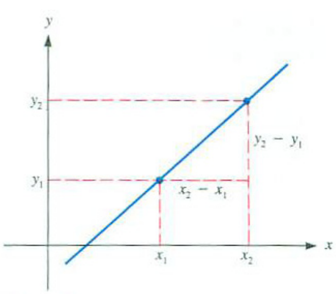


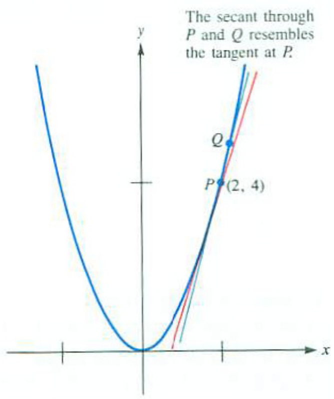
Figure 3.1.2:

See Figure 3.1.2.

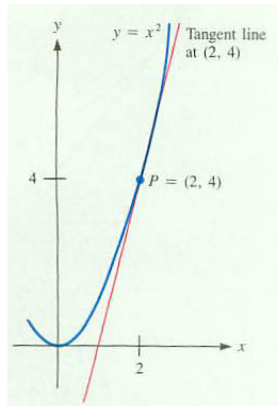
However, we know only one point on the tangent line at $(2, 4)$, namely, just the point $(2, 4)$ itself. To get around this difficulty we will choose a point Q on the parabola $y = x^2$ near P and compute the slope of the line through P and Q. Such a line is called a **secant**. As Figure 3.1.3 suggests, such a secant line resembles the tangent line at $(2, 4)$. For instance, choose $Q = (2.1, 2.1^2)$ and compute the slope of the line through P and Q as shown in Figure 3.1.3(b).

$$\begin{aligned}
\text{Slope of secant} &= \tfrac{\text{Change in } y}{\text{Change in } x} \\
&= \tfrac{2.1^2 - 2^2}{2.1 - 2} \\
&= \tfrac{4.41 - 4}{0.1} \\
&= \tfrac{0.41}{0.1} \\
&= 4.1.
\end{aligned}$$

Thus an estimate of the slope of the tangent line is 4.1. Note that in making this estimate there was no need to draw the curve or the secant.



(a)



(b)

Figure 3.1.3:

We can also choose the point Q on the parabola to be to the left of $P = (2,4)$. For instance, choose $Q = (1.9, 1.9^2)$. (See Figure 3.1.4.) Then

$$\begin{aligned}\text{slope of secant} &= \frac{\text{Change in } y}{\text{Change in } x}\\ &= \frac{1.9^2 - 2^2}{1.9-2}\\ &= \frac{3.61-4}{-0.1}\\ &= \frac{-0.39}{-0.1}\\ &= 3.9.\end{aligned}$$

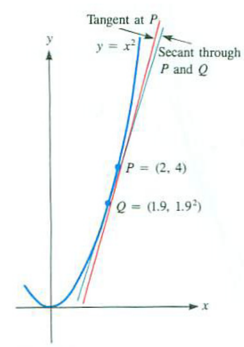


Figure 3.1.4:

To obtain a better estimate, we could repeat the process using, for instance, the line through $P = (2,4)$ and $Q = (2.01, 2.01^2)$. Rather than do this, it is simpler to consider a typical point Q. That is, consider the line through $P = (2,4)$ and $Q = (x, x^2)$ when x is close to 2 but not equal to 2. (See Figures 3.1.3(a) and (b).) This line has slope

$$\frac{x^2 - 2^2}{x - 2}.$$

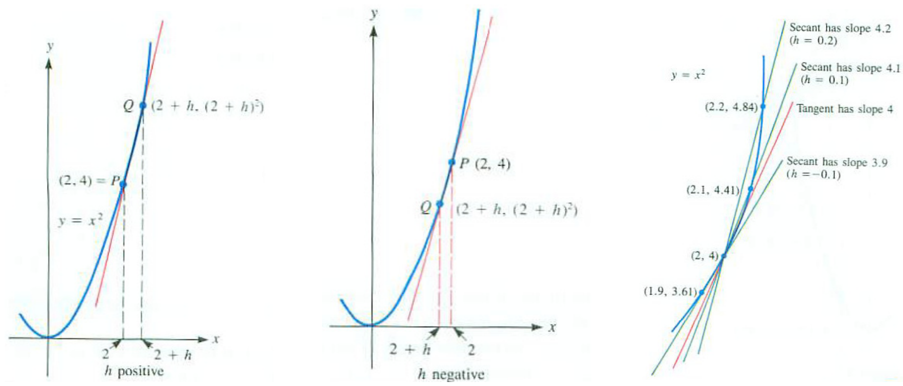
To find out what happens to this quotient as Q moves closer to P (and x moves closer to 2) apply the techniques of limits (see Section 2.1). We have

Recall $a^2 - b^2 = (a+b)(a-b)$.

$$\lim_{x\to 2} \frac{x^2 - 2^2}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

Thus, we expect the tangent line to $y = x^2$ at $(2,4)$ to have slope 4.

Figure 3.1.5(c) shows how secant lines approximate the tangent line. It suggests a blowup of a small part of the curve $y = x^2$. ◇



(a) (b) (c)

Figure 3.1.5:

Velocity

If an airplane or automobile is moving at a constant velocity, we know that "distance traveled equals velocity times time." Thus

$$\text{velocity} = \frac{\text{distance traveled}}{\text{elapsed time}}.$$

If the velocity is *not* constant, we still may speak of its "average velocity," which is defined as

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{elapsed time}}.$$

For instance, if you drive from San Francisco to Los Angeles, a distance of 400 miles, in 8 hours, the average velocity is 400 miles/8 hours or 50 miles per hour.

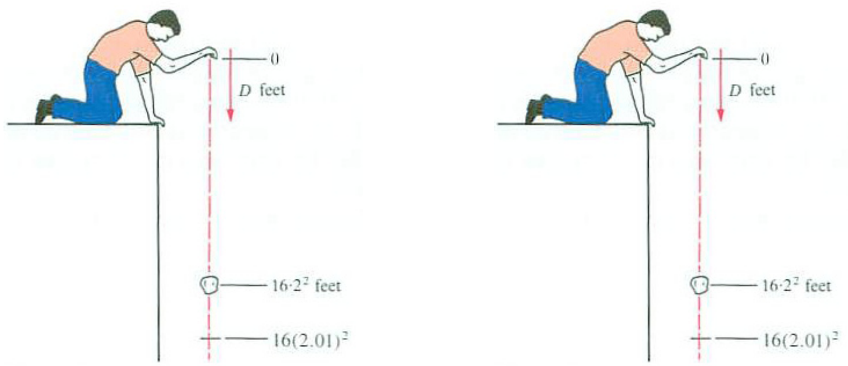
Suppose that up to time t_1 you have traveled a distance D_1, while up to time t_2 you have traveled a distance D_2, where $t_2 > t_1$. Then during the time interval $[t_1, t_2]$ the distance traveled is $D_2 - D_1$. Thus the average velocity during the time interval $[t_1, t_2]$, which has duration $t_2 - t_1$, is

$$\text{average velocity} = \frac{D_2 - D_1}{t_2 - t_1}.$$

The arithmetic of average velocity is the same as that for the slope of a line.

The next problem shows how to find the velocity at any instant for an object whose velocity is not constant.

EXAMPLE 2 A rock initially at rest falls $16t^2$ feet in t seconds. What is its velocity after 2 seconds? Whatever it is, it will be called the **instantaneous velocity**.



(a) (b)

Figure 3.1.6: Note: (b) needs to have 2.01 replaced by t.

SOLUTION

To start, make an estimate by finding the average velocity of the rock during a short time interval, say from 2 to 2.01 seconds. At the start of this interval the rock has fallen $16(2^2) = 64$ feet. By the end it has fallen $16(2.01^2) = 16(4.0401) = 64.6416$ feet. So, during this interval of 0.01 seconds the rock fell 0.6416 feet. Its average velocity during this time interval is

$$\text{average velocity} = \frac{64.6416 - 64}{2.01 - 2} = \frac{0.6416}{0.01} = 64.16 \text{ feet per second.}$$

This is an estimate of the velocity at time $t = 2$ seconds. (See Figure 3.1.6(a).)

Rather than make another estimate with the aid of a still shorter interval of time, let us consider the typical time interval from 2 to t seconds, $t > 2$. (Although we will keep $t > 2$, estimates could just as well be made with $t < 2$.) During this short time of $t - 2$ seconds the rock travels $16(t^2) - 16(2^2) = 16(t^2 - 2^2)$ feet, as shown in Figure 3.1.6(b). The average velocity of the rock during this period is

$$\text{average velocity} = \frac{16t^2 - 16(2^2)}{t - 2} = \frac{16(t^2 - 2^2)}{t - 2} \text{ feet per second.}$$

When t is close to 2, what happens to the average velocity? It approaches

$$\lim_{t \to 2} \frac{16(t^2 - 2^2)}{t - 2} = 16 \lim_{t \to 2} \frac{t^2 - 2^2}{t - 2} = 16 \lim_{t \to 2} (t + 2) = 16 \cdot 4 = 64 \text{ feet per second.}$$

We say that the (instantaneous) velocity at time $t = 2$ is 64 feet per second. ◇

Even though Examples 1 and 2 seem unrelated, their solutions turn out to be practically identical: The slope in Example 1 is approximated by the quotient

$$\frac{x^2 - 2^2}{x - 2}$$

and the velocity in Example 2 is approximated by the quotient

$$\frac{16t^2 - 16(2^2)}{t - 2} = 16 \cdot \frac{t^2 - 2^2}{t - 2}.$$

A variable by any name is a variable.

The only difference between the solutions is that the second quotient has an extra factor of 16 and x is replaced with t. This may not be too surprising, since the functions involved, x^2 and $16t^2$ differ by a factor of 16. (That the independent variable is named t in one case and x in the other does not affect the computations.)

The Derivative of a Function

In both the slope and velocity problems we were lead to studying similar limits. For the function x^2 it was

$$\frac{x^2 - 2^2}{x - 2} \quad \text{as } x \text{ approaches 2.}$$

For the function $16t^2$ it was

$$\frac{16t^2 - 16(2^2)}{t - 2} \quad \text{as } t \text{ approaches 2.}$$

In both cases we formed "change in outputs divided by change in inputs" and then found the limit as the change in inputs became smaller and smaller. This can be done for other functions, and brings us to one of the two key ideas in calculus, the *derivative of a function*.

DEFINITION *(Derivative of a function at a number a)* Let f be a function that is defined at least in some open interval that contains the number a. If

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists, it is called the **derivative of f at a**, and is denoted $f'(a)$. In this case the function f is said to be **differentiable at a**.

Read $f'(a)$ as "f prime at a" or "the derivative of f at a."

EXAMPLE 3 Find the derivative of $f(x) = 16x^2$ at 2.
SOLUTION In this case, $f(x) = 16x^2$ for any input x. By definition, the derivative of this function at 2 is

$$\lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{16x^2-16(2^2)}{x-2} = 16\lim_{x\to 2}\frac{x^2-2^2}{x-2} = 16\lim_{x\to 2}(x+2) = 64.$$

We say that "the derivative of the function $f(x)$ at 2 is 64" and write $f'(2) = 64$. ◇

Now that we have the derivative of f, we can define the slope of its graph at a point $(a, f(a))$ as the value of the derivative, $f'(a)$. Then we define the **tangent line** at $(a, f(a))$ as the line through $(a, f(a))$ whose slope is $f'(a)$.

EXAMPLE 4 Find the derivative of e^x at a.
SOLUTION We must find

$$\lim_{x\to a}\frac{e^x-e^a}{x-a}. \tag{3.1.1}$$

The limit is hard to see. However, it is easy to calculate if we write x as $a+h$, and find what happens as h approaches 0. The denominator $x-a$ is just h. Then (3.1.1) now reads

$$\lim_{h\to 0}\frac{e^{a+h}-e^a}{h}.$$

This form of the limit is more convenient:

$$\begin{aligned}
\lim_{h\to 0}\frac{e^{a+h}-e^a}{h} &= \lim_{h\to 0}\frac{e^ae^h-e^a}{h} && \text{law of exponents}\\
&= e^a\lim_{h\to 0}\frac{e^h-1}{h} && \text{factor out a constant}\\
&= e^a\cdot 1 && \text{Section 2.1}\\
&= e^a.
\end{aligned}$$

So the limit is e^a. In short, "the derivative of e^x is e^x itself." ◇

Differentiability and Continuity

If a function is differentiable at each point in its domain the function is said to be **differentiable**.

A small piece of the graph of a differentiable function at a looks like part of a straight line. You can check this by zooming in on the graph of a function of your choice. Differential calculus can be described as the study of functions whose graphs locally look almost like a line.

It is no surprise that a differentiable function is always continuous. To show that a function is continuous at an argument a in its domain we must show that $\lim_{x\to a} f(x) - f(a) = 0$. To relate this limit to $f'(a)$ we rewrite the limit as

$$\begin{aligned}
\lim_{x\to a} f(x) - f(a) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}(x-a)\right) \\
&= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\lim_{x\to a}(x-a) \\
&= f'(a)\cdot 0 && \text{definition of the derivative} \\
&= 0.
\end{aligned}$$

So, f is continuous at a.

A function can be continuous yet not differentiable. For instance, $f(x) = |x|$ is continuous but not differentiable at 0, as Figure 3.1.7 suggests.

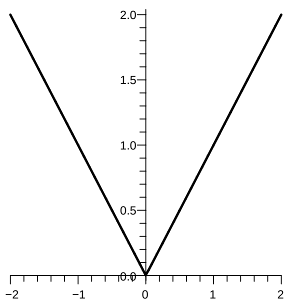


Figure 3.1.7:

Summary

From a mathematical point of view, the problems of finding the slope of the tangent line and the velocity of the rock are the same. In each case estimates lead to the same type of quotient, $\frac{f(x)-f(a)}{x-a}$. The behavior of this *difference quotient* is studied as x approaches a. In each case the answer is a limit, called the derivative of the function at the given number, a. Finding the derivative of a function is called "differentiating" the function.

EXERCISES for 3.1 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 4 review the concept of slope of a line. (See also Appendix B.)
1.[R] What angle does a line make with the x-axis if its slope is

(a) 1?

(b) 2?

(c) 1/2?

NOTE: Use a calculator for (b) and (c).

2.[R] What angle does a line make with the x-axis if its slope is

(a) -1?

(b) -2?

(c) $-1/2$?

HINT: Remember that the angle is in the second quadrant. NOTE: Use a calculator for (b) and (c).

3.[R] Draw x- and y-axes and a line that is neither horizontal nor vertical. Using a ruler, estimate the slope of the line you drew.

4.[R] Draw the line through $(1, 2)$ that has

(a) slope $\frac{3}{2}$.

(b) slope $\frac{-3}{2}$.

Exercises 5 to 16 concern slope. In each case use the technique of Example 1 to find the slope of the tangent line to the curve at the point.
5.[R] $y = x^2$ at the point $(3, 3^2) = (3, 9)$
6.[R] $y = x^2$ at the point $(\frac{1}{2}, (\frac{1}{2})^2) = (\frac{1}{2}, \frac{1}{4})$
7.[R] $y = x^2$ at the point $(-2, (-2)^2) = (-2, 4)$
8.[R] $y = x^2$ at the point $(1, 1^2) = (1, 1)$
9.[R] $y = x^3$ at the point $(2, 2^3) = (2, 8)$
10.[R] $y = x^3$ at the point $(1, 1^3) = (1, 1)$
11.[R] $y = \sin(x)$ at the point $(0, \sin(0)) = (0, 0)$
12.[R] $y = \cos(x)$ at the point $(0, \cos(0)) = (0, 1)$
13.[R] $y = 2^x$ at the point $(0, 2^0) = (0, 1)$

14.[R] $y = 4^x$ at the point $(0, 4^0) = (0, 1)$

15.[R]

(a) $y = x^3$ at the point $(0, 0)$

(b) Sketch the graph of $y = x^3$ and the tangent line at $(0, 0)$.

NOTE: Be particularly careful when sketching the graph near $(0, 0)$. In this case the tangent line crosses the curve.

16.[R]

(a) $y = x^2$ at the point $(0, 0)$

(b) Sketch the graph of $y = x^2$ and the tangent line at $(0, 0)$.

In Exercises 17 to 20 use the method of Example 2 to find the velocity of the rock after

17.[R] 3 seconds

18.[R] $\frac{1}{2}$ second

19.[R] 1 second

20.[R] $\frac{1}{4}$ second

21.[R] A certain object travels t^3 feet in the first t seconds.

(a) How far does it travel during the time interval from 2 to 2.1 seconds?

(b) What is the average velocity during that time interval?

(c) Let h be any positive number. Find the average velocity of the object from time 2 to $2 + h$ seconds. HINT: To find $(2 + h)^3$, just multiply out $(2 + h)(2 + h)(2 + h)$.

(d) Find the velocity of the object at 2 seconds by letting h approach 0 in the result found in (c).

22.[R] A certain object travels t^3 feet in the first t seconds.

(a) Find the average velocity during the time interval from 3 to 3.01 seconds?

(b) Find its average velocity during the time interval from 3 to t seconds, $t > 3$.

(c) By letting t approach 3 in the result found in (b), find the velocity of the object at 3 seconds.

In the slope problem the nearby point Q was usually pictured as being to the right of P. The point Q could just as well have been chosen to the left of P. Exercises 23 and 24 illustrate this case.

23.[R] Consider the parabola $y = x^2$.

(a) Find the slope of the line through $P = (2, 4)$ and $Q = (1.99, 1.99^2)$.

(b) Find the slope of the line through $P = (2, 4)$ and $Q = (2.1, 2.1^2)$.

(c) Find the slope of the line through $P = (2, 4)$ and $Q = (2 + h, (2 + h)^2)$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (c) approaches 4.

24.[R] Consider the curve $y = x^3$.

(a) Find the slope of the line through $P = (2, 8)$ and $Q = (1.9, 1.9^3)$.

(b) Find the slope of the line through $P = (2, 8)$ and $Q = (2.01, 2.01^3)$.

(c) Find the slope of the line through $P = (2, 8)$ and $Q = (2 + h, (2 + h)^3)$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (c) approaches 12.

25.[R] Consider the curve $y = \sin(x)$.

(a) Find the slope of the line through $P = (0, 0)$ and $Q = (-0.1, \sin(-0.1))$.

(b) Find the slope of the line through $P = (0, 0)$ and $Q = (0.01, \sin(0.01))$.

(c) Find the slope of the line through $P = (0, 0)$ and $Q = (h, \sin(h))$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (c) approaches 1.

26.[R] Consider the curve $y = \cos(x)$.

(a) Find the slope of the line through $P = (0, 1)$ and $Q = (-0.1, \cos(-0.1))$.

(b) Find the slope of the line through $P = (0, 1)$ and $Q = (0.01, \cos(0.01))$.

(c) Find the slope of the line through $P = (0, 1)$ and $Q = (h, \cos(h))$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (c) approaches 0.

27.[R] Consider the curve $y = 2^x$.

(a) Find the slope of the line through $P = (2, e^2)$ and $Q = (1.9, e^{1.9})$.

(b) Find the slope of the line through $P = (2, e^2)$ and $Q = (2.1, e^{2.1})$.

(c) Find the slope of the line through $P = (2, e^2)$ and $Q = (2 + h, e^{2+h})$, where $h \neq 0$.

(d) Show that the slope of the curve $y = e^x$ at $(2, e^2)$ is e^2.

28.[R] Consider the curve $y = e^x$.

(a) Find the slope of the line through $P = (0, 1)$ and $Q = (-0.1, e^{-0.1})$.

(b) Find the slope of the line through $P = (0, 1)$ and $Q = (0.01, e^{0.01})$.

(c) Find the slope of the line through $P = (0, 1)$ and $Q = (0 + h, e^{0+h})$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (c) approaches $\ln(e) = 1$.

29.[R] Show that the slope of the curve $y = 2^x$ as $(3, 8)$ is $8 \ln(2)$.

30.[R]

(a) Find the slope of the tangent line to $y = x^2$ at $(4, 16)$.

(b) Use it to draw the tangent line to the curve at $(4, 16)$.

31.[R]

(a) Find the slope of the tangent line to $y = x^2$ at $(-1, 1)$.

(b) Use it to draw the tangent line to the curve at $(-1, 1)$.

32.[R]

(a) Use the method of this section to find the slope of the curve $y = x^3$ at $(1, 1)$.

(b) What does the graph of $y = x^3$ look like near $(1, 1)$?

33.[R]

(a) Use the method of this section to find the slope of the curve $y = x^3$ at $(-1, -1)$.

(b) What does the graph of $y = x^3$ look like near $(-1, -1)$?

34.[R]

(a) Draw the curve $y = e^x$ for x in the interval $[-2, 1]$.

(b) Draw as well as you can, using a straightedge, the tangent line at $(1, e)$.

(c) Estimate the slope of the tangent line by measuring its "rise" and its "run."

(d) Using the derivative of e^x, find the slope of the curve at $(1, e)$.

35.[R]

(a) Sketch the curve $y = e^x$ for x in $[-1, 1]$.

(b) Where does the curve in (a) cross the y-axis?

(c) What is the (smaller) angle between the graph of $y = e^x$ and the y-axis at the point found in (b)?

36.[R] With the aid of a calculator, estimate the slope of $y = 2^x$ at $x = 1$, using the intervals

(a) $[1, 1.1]$

(b) $[1, 1.01]$

(c) $[0.9, 1]$

(d) $[0.99, 1]$

37.[R] With the aid of a calculator, estimate the slope of $y = \frac{x+1}{x+2}$ at $x = 2$, using the intervals

(a) $[2, 2.1]$

(b) $[2, 2.01]$

(c) $[2, 2.001]$

(d) $[1.999, 2]$

38.[M] Estimate the derivative of $\sin(x)$ at $x = \pi/3$

(a) to two decimal places.

(b) to three decimal places.

39.[M] Estimate the derivative of $\ln(x)$ at $x = 2$

(a) to two decimal places.

(b) to three decimal places.

The ideas common to both slope and velocity also appear in other applications. Exercises 40 to 43 present the same ideas in biology, economics, and physics.

40.[M] A certain bacterial culture has a mass of t^2 grams after t minutes of growth.

(a) How much does it grow during the time interval $[2, 2.01]$?

(b) What is the average rate of growth during the time interval $[2, 2.01]$?

(c) What is the "instantaneous" rate of growth when $t = 2$?

41.[M] A thriving business has a profit of t^2 million dollars in its first t years. Thus from time $t = 3$ to time $t = 3.5$ (the first half of its fourth year) it has a profit of $(3.5)^2 - 3^2$ million dollars, giving an annual rate of

$$\frac{(3.5)^2 - 3^2}{0.5} = 6.5 \text{ million dollars per year.}$$

(a) What is its annual rate of profit during the time interval $[3, 3.1]$?

(b) What is its annual rate of profit during the time interval $[3, 3.01]$?

(c) What is its instantaneous rate of profit after 3 years?

Exercises 42 and 43 concern density.

42.[M] The mass of the left-hand x centimeters of a nonhomogeneous string 10 centimeters long is x^2 grams, as shown in Figure 3.1.8. For instance, the string in the interval $[0, 5]$ has a mass of $5^2 = 25$ grams and the string in the interval $[5, 6]$ has mass $6^2 - 5^2 = 11$ grams. The **average density** of any part of the string is its mass divided by its length. ($\frac{\text{total mass}}{\text{length}}$ grams per centimeter) grams.

(a) Consider the leftmost 5 centimeters of the string, the middle 2 centimeters of the string, and the rightmost 2 centimeters of the string. Which piece of the string has the largest mass?

(b) Of the three pieces of the string in (a), which part of the string is denser?

(c) What is the mass of the string in the interval $[3, 3.01]$?

(d) Using the interval $[3, 3.01]$, estimate the density at 3.

(e) Using the interval $[2.99, 3]$, estimate the density at 3.

(f) By considering intervals of the form $[3, 3 + h]$, h positive, find the density at the point 3 centimeters from the left end.

(g) By considering intervals of the form $[3 + h, 3]$, h negative, find the density at the point 3 centimeters from the left end.

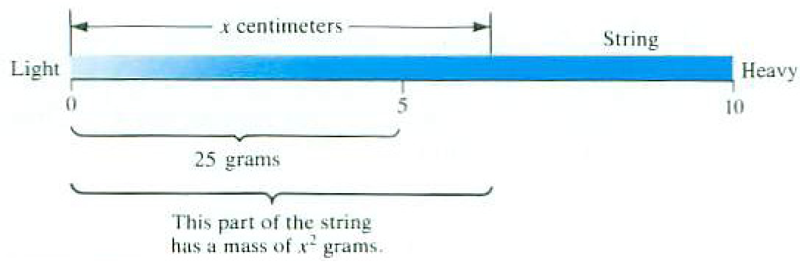


Figure 3.1.8:

43.[M] The left x centimeters of a string have a mass of x^2 grams.

(a) What is the mass of the string in the interval $[2, 2.01]$?

(b) Using the interval $[2, 2.01]$, estimate the density at 2.

(c) Using the interval $[1.99, 2]$, estimate the density at 2.

(d) By considering intervals of the form $[2, 2+h]$, h positive, find the density at the point 2 centimeters from the left end.

(e) By considering intervals of the form $[2+h, 2]$, h negative, find the density at the point 2 centimeters from the left end.

44.[M]

(a) Graph the curve $y = 2x^2 + x$.

(b) By eye, draw the tangent line to the curve at the point $(1, 3)$. Using a ruler, estimate the slope of the tangent line.

(c) Sketch the line that passes through the point $(1, 3)$ and the point $(x, 2x^2 + x)$.

(d) Find the slope of the line in (c).

(e) Letting x get closer and closer to 1, find the slope of the tangent line at $(1, 3)$.

(f) How close was your estimate in (b)?

45.[M] An object travels $2t^2 + t$ feet in t seconds.

(a) Find its average velocity during the interval of time $[1, x]$, where x is positive.

(b) Letting x get closer and closer to 1, find the velocity at time 1.

(c) How close was your estimate in (a)?

46.[M] Find the slope of the tangent line to the curve $y = x^2$ of Example 1 at the typical point $P = (x, x^2)$. To do this, consider the slope of the line through P and the nearby point $Q = (x + h, (x + h)^2)$ and let h approach 0.

47.[M] Find the velocity of the falling rock of Example 2 at any time t. To do this, consider the average velocity during the time interval $[t, t + h]$ and then let h approach 0.

48.[M] Does the tangent line to the curve $y = x^2$ at the point $(1, 1)$ pass through the point $(6, 12)$?

49.[M]

(a) Graph the curve $y = 2^x$ as well as you can for $-2 \leq x \leq 3$.

(b) Using a straight edge, draw as well as you can a tangent to the curve at $(2, 4)$. Estimate the slope of this tangent by using a ruler to draw and measure a "rise-and-run" triangle.

(c) Using a secant through $(2, 4)$ and $(x, 2^x)$, for x near 2, estimate the slope of the tangent to the curve at $(2, 4)$. HINT: Choose particular values of x and use your calculator to create a table of your results.

50.[C]

(a) Using your calculator estimate the slope of the tangent line to the graph of $f(x) = \sin(x)$ at $(0, 0)$.

(b) At what (famous) angle do you think the curve crosses the x-axis at $(0, 0)$?

51.[C]

(a) Sketch the curve $y = x^3 - x^2$.

(b) Using the method of the nearby point, find the slope of the tangent line to the curve at the point $(a, a^3 - a^2)$.

(c) Find all points on the curve where the tangent line is horizontal.

(d) Find all points on the curve where the tangent line has slope 1.

52.[C] Repeat Exercise 51 for the curve $y = x^3 - x$.

53.[C] An astronaut is traveling from left to right along the curve $y = x^2$. When she shuts off the engine, she will fly off along the line tangent to the curve at the point where she is at the moment the engines turn off. At what point should she shut off the engine in order to reach the point

(a) $(4, 9)$?

(b) $(4, -9)$?

54.[C] See Exercise 53. Where can an astronaut who is traveling from left to right along $y = x^3 - x$ shut off the engine and pass through the point $(2, 2)$?

3.2 The Derivatives of the Basic Functions

In this section we use the definition of the derivative to find the derivatives of the important functions x^a (a rational), e^x, $\sin x$, and $\cos x$. We also introduce some of the standard notations for the derivative. For convenience, we begin with by repeating the definition of the derivative.

DEFINITION *(Derivative of a function at a number)* Assume that the function f is defined at least in an open interval containing a. If

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \tag{3.2.1}$$

exists, it is called the **derivative of f at a**.

There are several notations for the quotient that appears in (3.2.1) and also for the derivative. Sometimes it is convenient to use $a+h$ instead of x and let h approach 0. Then, (3.2.1) reads

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}. \tag{3.2.2}$$

Expression (3.2.2) says the same thing as (3.2.1). "See how the quotient, change in output divided by change in input, behaves as the change in input gets smaller and smaller."

Sometimes it is useful to call the change in output "Δf" and the change in input "Δx." That is, $\Delta f = f(x) - f(a)$ and $\Delta x = x - a$. Then

The symbol Δ is Greek for "D"; it is pronounced "delta". So Δf is read 'delta eff." In mathematics, "Δ" generally indicates difference or change.

$$f'(a) = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}. \tag{3.2.3}$$

There is nothing sacred about the letters a, x, and h. One could say

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \tag{3.2.4}$$

or

$$f'(x) = \lim_{u\to x} \frac{f(u)-f(x)}{u-x}. \tag{3.2.5}$$

The symbol "$f'(a)$" is read aloud as "f prime at a" or "the derivative of f at a." The symbol $f'(x)$ is read similarly. However, the notation $f'(x)$ reminds us that f', like f, is a function. For each input x the derivative, $f'(x)$, is the output. The derivative of the function f is also written as $D(f)$.

The derivative of a specific function, such as x^2, is denoted $(x^2)'$ or $D(x^2)$. Then, $D(x^2) = 2x$ is read aloud as "the derivative of x^2 is $2x$." This is shorthand for "the derivative of the function that assigns x^2 to x is the function that assigns $2x$ to x." Since the value of derivative depends on x, it is a function.

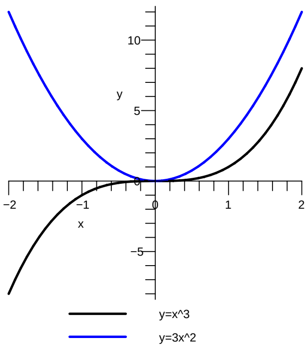


Figure 3.2.1:

EXAMPLE 1 Find the derivative of x^3 at a.
SOLUTION

$$(x^3)' = \lim_{x\to a} \frac{x^3 - a^3}{x - a} = 3a^2$$

This limit was evaluated by noticing that it is one of the four limits in Section 2.1 (page 60). Using (2.1.6), we can write $(x^3)' = 3x^2$ or $D(x^3) = 3x^2$. ⋄

In the same manner, $\lim_{x\to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$ implies that for any positive integer n, the derivative of x^n is nx^{n-1}. The exponent n becomes the coefficient and the exponent of x shrinks from n to $n-1$:

Derivative of x^n

$(x^n)' = nx^{n-1}$ where n is a positive integer.

The next example treats an exponential function with a fixed base.
EXAMPLE 2 Find the derivative of 2^x.

SOLUTION

$$\begin{aligned} D(2^x) &= \lim_{h\to 0} \frac{2^{(x+h)} - 2^x}{h} \\ &= \lim_{h\to 0} \frac{2^x 2^h - 2^x}{h} \\ &= \lim_{h\to 0} 2^x \frac{2^h - 1}{h} \\ &= 2^x \lim_{h\to 0} \frac{2^h - 1}{h}. \end{aligned}$$

In Section 2.1, page 60, we found that $\lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.693$. Thus,

$$D(2^x) \approx (0.693)2^x.$$

⋄

No one wants to remember the (approximate) constant 0.693, which appears when we use base 2. Recall that in Section 3.1 we found that the derivative of e^x is e^x. There is no need to memorize some fancy constant, such as 0.693.

We emphasize this important, and simple, formula

Derivative of e^x

$$D(e^x) = e^x.$$

The function e^x has the remarkable property that it equals its derivative.

Next, we turn to trigonometric functions.

EXAMPLE 3 Find the derivative of $\sin(x)$.

SOLUTION

$$\begin{aligned} D(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\ &= \lim_{h\to 0} \sin x \frac{\cos(h) - 1}{h} + \cos(x)\frac{\sin(h)}{h}. \end{aligned}$$

In Section 2.1 we found: $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0} \frac{1-\cos(h)}{h} = 0$. Thus $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$ and

$$D(\sin x) = (\sin x)(0) + (\cos x)(1) = \cos(x).$$

◇

We have the important formula

Derivative of $\sin(x)$

$$D(\sin(x)) = \cos(x).$$

If we graph $y = \sin(x)$ (see Figure 3.2.2), and consider its shape, the formula $D(\sin(x)) = \cos(x)$, is not a surprise. For instance, for x in $(-\pi/2, \pi/2)$ the slope is positive. So is $\cos(x)$. For x in $(\pi/2, 3\pi/2)$ the slope of the sine curve is negative. So is $\cos(x)$. Since $\sin(x)$ has period 2π, we would expect its derivatve also to have period 2π. Indeed, $\cos(x)$ does have period 2π.

In a similar manner, using the definition of the derivative and the identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, one can show that

Derivative of $\cos(x)$

$$D(\cos(x)) = -\sin(x).$$

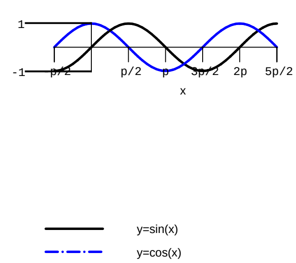


Figure 3.2.2:

Derivatives of Other Power Functions

We showed that if n is a positive integer, $D(x^n) = nx^{n-1}$. Now let us find the derivative of power functions x^n where n is not a positive integer.

EXAMPLE 4 Find the derivative of $x^{-1} = \frac{1}{x}$.
SOLUTION Before we calculate the necessary limit, let's pause to see how the slope of $y = 1/x$ behaves. A glance at Figure 3.2.3 shows that the slope is always negative. Also, for x near 0, the slope is large, but when $|x|$ is large, the slope is near 0.

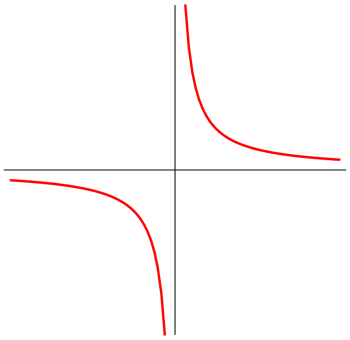


Figure 3.2.3:

Now, let's find the derivative of $1/x$:

$$\begin{aligned} D(1/x) &= \lim_{t\to x} \frac{1/t - 1/x}{t - x} \\ &= \lim_{t\to x} \frac{1}{t-x}\left(\frac{x-t}{xt}\right) \\ &= \lim_{t\to x} \frac{-1}{xt} \\ &= -\frac{1}{x^2}. \end{aligned}$$

As a check, note that $-1/x^2$ is always negative, is large when x is near 0, and is near 0 when $|x|$ is large. ◇

It is worth memorizing that

Derivative of x^{-1}

$$D\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Or, written in exponential notation,

$$D(x^{-1}) = -x^{-2}.$$

The second form fits into the pattern established for positive integers n, $D(x^n) = nx^{n-1}$.

EXAMPLE 5 Find the derivative of $x^{2/3}$.
SOLUTION Once again we use the definition of the derivative:

$$D(x^{2/3}) = \lim_{t\to x} \frac{t^{2/3} - x^{2/3}}{t - x}.$$

A bit of algebra will help us find that limit. We write the four terms $t^{2/3}$, $x^{2/3}$, t, and x as powers of $t^{1/3}$ and $x^{1/3}$. Thus

$$D(x^{2/3}) = \lim_{t\to x} \frac{\left(t^{1/3}\right)^2 - \left(x^{1/3}\right)^2}{\left(t^{1/3}\right)^3 - \left(x^{1/3}\right)^3}.$$

Recalling that $a^2 - b^2 = (a-b)(a+b)$ and $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, we find

If you don't recall these formulas, multiply out $(a-b)(a+b)$ and $(a-b)(a^2+ab+b^2)$.

$$\begin{aligned}
D(x^{2/3}) &= \lim_{t\to x} \frac{\left(\left(t^{1/3}\right) - \left(x^{1/3}\right)\right)\left(\left(t^{1/3}\right) + \left(x^{1/3}\right)\right)}{\left(\left(t^{1/3}\right) - \left(x^{1/3}\right)\right)\left(\left(t^{1/3}\right)^2 + \left(t^{1/3}\right)\left(x^{1/3}\right) + \left(x^{1/3}\right)^2\right)} \\
&= \lim_{t\to x} \frac{\left(t^{1/3}\right) + \left(x^{1/3}\right)}{\left(t^{1/3}\right)^2 + \left(t^{1/3}\right)\left(x^{1/3}\right) + \left(x^{1/3}\right)^2} \\
&= \frac{\left(x^{1/3}\right) + \left(x^{1/3}\right)}{\left(x^{1/3}\right)^2 + \left(x^{1/3}\right)\left(x^{1/3}\right) + \left(x^{1/3}\right)^2} \\
&= \frac{2x^{1/3}}{3x^{2/3}} = \frac{2}{3}x^{-1/3}.
\end{aligned}$$

In short,

$$D(x^{2/3}) = \frac{2}{3}x^{-1/3}.$$

Note that this formula follows the pattern we found for $D(x^n)$ for $n = 1, 2, 3, \ldots$ and -1. The exponent of x becomes the coefficient and the exponent of x is lowered by 1. ◇

The method used in Example 5 applies to any positive rational exponent. In the next two sections we will show how this result extends first to negative rational exponents (Section 3.3) and then to irrational exponents (Section 3.5). In all three cases the formula will be the same. We state the general result here, but remember that — so far — we have justified it only for positive rational exponents and -1.

Derivative of Power Functions x^a

For any fixed number a, $D(x^a) = ax^{a-1}$. (3.2.6)

This formula holds for values of x where both x^a and x^{a-1} are defined. For instance, $x^{1/2} = \sqrt{x}$ is defined for $x \geq 0$, but its derivative $\frac{1}{2}x^{-1/2}$ is defined only for $x > 0$.

The derivative of the square root function occurs so often, we emphasize its formula

Derivative of Square Root Function (as Power Function)

$$D(x^{1/2}) = \frac{1}{2}x^{-1/2}$$

or, in terms of the usual square root sign,

Derivative of Square Root Function (Square Root Sign)

$$D(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

Another Notation for the Derivative

We have used the notations f' and $D(f)$ for the derivative of a function f. There is another notation that is also convenient.

If $y = f(x)$, the derivative is denoted by the symbols

$$\frac{dy}{dx} \text{ or } \frac{df}{dx}.$$

The symbol $\frac{dy}{dx}$ is read as "the derivative of y with respect to x" or "dee y, dee x."

In this notation the derivative of x^3, for instance, is written

$$\frac{d\left(x^3\right)}{dx}.$$

If the function is expressed in terms of another letter, such as t, we would write

$$\frac{d\left(t^3\right)}{dt}.$$

In Section 5.4 a meaning will be given to dx and dy.

Keep in mind that in the notations df/dx and dy/dx, the symbols df, dy, and dx have no meaning by themselves. The symbol dy/dx should be thought of as a signle entity, just like the numeral 8, which we do not think of as formed of two 0's.

In the study of motion, Newton's **dot notation** is often used. If x is a function of time t, then $\dot{x}$ denotes the derivative dx/dt.

Summary

In this section we see why limits are important in calculus. We need them to define the derivative of a function. The definition can be stated in several ways, but each one says, informally, "look at how a small change in input changes the output." Here is the formal definition, in various costumes:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad f'(x) = \lim_{x\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \qquad f'(x) = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}.$$

The following derivatives should be memorized. However, if you forget a formula, you should be able to return to the definition and evaluate the necessary limit.

Function	Derivative
$f(x)$	$f'(x)$
x^a	ax^{a-1}
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

EXERCISES for 3.2 *Key:* R–routine, M–moderate, C–challenging

1.[R] Show that $D(\cos(x)) = -\sin(x)$. HINT: $\cos(A+B) = \cos(a)\cos(b) - \sin(a)\sin(b)$

Using the definition of the derivative, compute the appropriate limit to find the derivatives of the functions in Exercises 2 to 12.

2.[R] $1/(x+2)$

3.[R] $\cos(x)$ HINT: $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

4.[R] 3^x HINT: use your calculator to estimate the messy coefficient that appears

5.[R] $6x^3$

6.[R] $x^{4/3}$

7.[R] $5x^2$

8.[R] $4\sin(x)$

9.[R] $2e^x + \sin(x)$

10.[R] $x^2 + x^3$

11.[R] $1/(2x+1)$

12.[R] $1/x^2$

13.[R] Use the formulas obtained for the derivatives of e^x, x^a, $\sin(x)$, and $\cos(x)$ to evaluate the derivatives of the given function at the given input.

(a) e^x at -1

(b) $x^{1/3}$ at -8

(c) $\sqrt[3]{x}$ at 27

(d) $\cos(x)$ at $\pi/4$

(e) $\sin(x)$ at $2\pi/3$

14.[R] Use the formulas obtained for the derivatives of e^x, x^a, $\sin(x)$, and $\cos(x)$ to evaluate the derivatives of the given function at the given input.

(a) e^x at 0

(b) $x^{2/3}$ at -1

(c) $\sqrt{x}$ at 25

(d) $\cos(x)$ at $-\pi$

(e) $\sin(x)$ at $\pi/3$

15.[R] State the definition of the derivative of a function in words, using no mathematical symbols.

16.[R] State the definition of the derivative of $g(t)$ at b as a mathematical formula, with no words.

In Exercises 17 to 22 use the definition of the derivative to show that the given equation is correct. Later in this chapter we will develop shortcuts for finding these derivatives.

17.[M] $D\left(e^{-x}\right) = -e^{-x}$

18.[M] $D\left(e^{3x}\right) = 3e^{3x}$

19.[M] $D(1/\cos(x)) = \sin(x)/\cos^2(x)$

20.[M] $D(\tan(x)) = 1 + \tan^2(x) = \sec^2(x)$

HINT: use the identity $\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$

21.[M] $D(\sin(2x)) = 2\cos(2x)$

22.[M] $D(\cos(x/2) = -1/2\sin(x/2)$

23.[M]

(a) Graph $y = x^3$ on the interval $[-2, 2]$. NOTE: Be sure to use the same scale on both axes.

(b) Find $D(x^3)$ at $x = 0$.

(c) What does (b) tell about the graph in (a)? HINT: Think about the tangent line to $y = x^3$ at $x = 0$.

(d) In view of (c), redraw the graph in (a) in the vicinity of $(0, 0)$.

24.[M] This Exercise shows why, in calculus, angles are measured in radians. Let $\text{Sin}(x)$ denote the sine of an angle of x degrees and let $\text{Cos}(x)$ denote the cosine of an angle of x degrees.

(a) Graph $y = \text{Sin}(x)$ on the interval $[-180, 360]$, using the same scale on both the x- and y-axes.

(b) Find $\lim_{x\to 0} \frac{\text{Sin}(x)}{x}$.

(c) Find $\lim_{x\to 0} \frac{1-\text{Cos}(x)}{x}$.

(d) Using the definition of the derivative, differentiate $\text{Sin}(x)$.

25.[C] Use the limit process to show that $D\left((x^{-5}\right) = -5x^{-6}$.

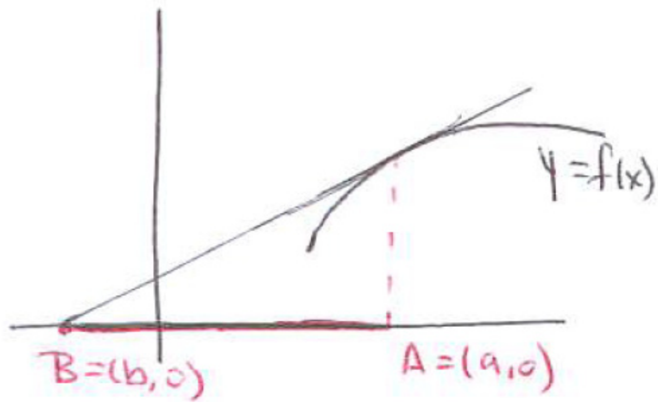


Figure 3.2.4:

Let f be a differentiable function and a a number such that $f'(a)$ is not zero. The tangent to the graph of f at $A = (a, f(a))$ meets the x-axis at a point $B = (b, 0)$, see Figure 3.2.4. The **subtangent** of f is the line AB. Its length is $|a - b|$.

Exercises 26 and 27 involve the subtangent of a function.

26.[C] Show that for the function e^x the length of the subtangent is the same for all values of a.

27.[C] Find the length of the subtangent at $(a, f(a))$ for any differentiable function f. Hint: Assume $f'(a)$ is not zero.

3.3 Shortcuts for Computing Derivatives

This section develops methods for finding the derivative of a function, or what is called **differentiating** a function. With these methods it will he a routine matter to find, for instance, the derivative of

The verb is "differentiate."

$$\frac{(3+4x+5x^2)e^x}{\sin(x)}$$

without going back to the definition of the derivative and (at great effort) finding the limit of a complicated quotient.

Before developing the methods in this and the next two sections, it will be useful to find the derivative of any constant function.

The Derivative of a Constant Function

In other symbols, $\frac{d(C)}{dx} = 0$ and $D(C) = 0$.

Constant Rule

The derivative of a constant function $f(x) = C$ is 0.

$$(C)' = 0$$

Proof

Let C be a fixed number and let f be the constant function, $f(x) = C$ for all inputs x. By the definition of a derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Since the function f has the same output C for all inputs,

Δx is another name for h

$$f(x+\Delta x) = C \text{ and } f(x) = C.$$

Thus

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{C - C}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{0}{\Delta x} \\ &= \lim_{\Delta x \to 0} 0 \qquad \text{since } \Delta x \neq 0 \\ &= 0. \end{aligned}$$

This shows the derivative of any constant function is 0 for all x. •

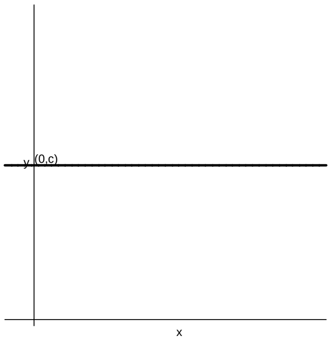


Figure 3.3.1:

From two points of view, the Constant Rule is no surprise: Since the graph of $f(x) = C$ is a horizontal line, it coincides with each of its tangent lines, as can he seen in Figure 3.3.1. Also, if we think of x as time and $f(x)$ as the position of a particle at time x, the Constant Rule implies that a stationary particle has zero velocity.

Derivatives of $f + g$ and $f - g$

The next theorem asserts that if the functions, f and g have derivatives at a certain number, so does their sum $f + g$ and

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}$$

In other words, "the derivative of the sum is the sum of the derivatives." Equivalently, $(f + g)' = f' + g'$ and $D(f + g) = D(f) + D(g)$. A similar formula holds for the derivative of $f - g$.

Sum Rule and Difference Rule

If f and g are differentiable functions, then so are $f + g$ and $f - g$. The **Sum Rule** and **Difference Rule** for computing their derivatives are

$$\begin{aligned} (f+g)' &= f' + g' && \text{Sum Rule} \\ (f-g)' &= f' - g' && \text{Difference Rule} \end{aligned}$$

Proof

To justify this we must go back to the definition of the derivative. To begin, we give the function $f + g$ the name u, that is, $u(x) = f(x) + g(x)$. We have to examine

$$\lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} \tag{3.3.1}$$

or, equivalently,

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}. \tag{3.3.2}$$

In order to evaluate (3.3.2), we will express Δu in terms of Δf and Δg. Here are the details:

$$\begin{aligned} \Delta u &= u(x + \Delta x) - u(x) \\ &= (f(x + \Delta x) + g(x + \Delta x)) - (f(x) + g(x)) && \text{definition of } u \\ &= (f(x) + \Delta f) + (g(x) + \Delta g) - (f(x) + g(x)) && \text{definition of } \Delta f \text{ and } \Delta g \\ &= \Delta f + \Delta g \end{aligned}$$

All told, $\Delta u = \Delta f + \Delta g$. The change in u is the change in f plus the change in g.

The hard work is over. We can now evaluate (3.3.2):

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f + \Delta g}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} = f'(x) + g'(x).$$

Thus, $u = f + g$ is differentiable and

$$u'(x) = f'(x) + g'(x).$$

A similar argument applies to $f - g$. •

The Sum and Difference Rules extend to any finite number of differentiable functions. For example.

$$\begin{aligned} (f+g+h)' &= f' + g' + h' \\ (f-g+h)' &= f' - g' + h' \end{aligned}$$

EXAMPLE 1 Using the Sum Rule, differentiate $x^2 + x^3 + \cos(x) + 3$.
SOLUTION

$$\begin{aligned} D\left(x^2 + x^3 + \cos(x) + 3\right) &= D(x^2) + D(x^3) + D(\cos(x)) + D(3) \\ &= 2x^{2-1} + 3x^{3-1} + (-\sin(x)) + 0 \\ &= 2x + 3x^2 - \sin(x). \end{aligned}$$

◇

EXAMPLE 2 Differentiate $x^4 - \sqrt{x} - e^x$.
SOLUTION

$$\begin{aligned} \tfrac{d}{dx}\left(x^4 - \sqrt{x} - e^x\right) &= \tfrac{d}{dx}\left(x^4\right) - \tfrac{d}{dx}\left(\sqrt{x}\right) - \tfrac{d}{dx}\left(e^x\right) \\ &= 4x^3 - \tfrac{1}{2\sqrt{x}} - e^x \end{aligned}$$

◇

The Derivative of fg

The following theorem, concerning the derivative of the product of two functions, may be surprising, for it turns out that the derivative of the product is *not* the product of the derivatives. The formula is more complicated than the one for the derivative of the sum. It asserts that *"the derivative of the product is the derivative of the first function times the second plus the first function times the derivative of the second."*

Product Rule

If f and g are differentiable functions, then so is their product fg. Its derivative is given by the formula

$$(fg)' = f'g + fg'$$

Proof

The proof is similar to that for the Sum and Difference Rules. This time we give the product fg the name u. Then we express Δu in terms of Δf and Δg. Finally, we determine $u'(x)$ by examining $\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$. These steps are practically forced upon us.

We have

$$u(x) = f(x)g(x) \quad \text{and} \quad u(x + \Delta x) = f(x + \Delta x)g(x + \Delta x).$$

Rather than subtract $u(x)$ from $u(x + \Delta x)$ directly, first write

$$f(x + \Delta x) = f(x) + \Delta f \quad \text{and} \quad g(x + \Delta x) = g(x) + \Delta g.$$

Then

$$\begin{aligned} u(x + \Delta x) &= (f(x + \Delta x))\,(g(x + \Delta x)) \\ &= (f(x) + \Delta f)\,(g(x) + \Delta g) \\ &= f(x)g(x) + (\Delta f)g(x) + f(x)\Delta g + (\Delta f)(\Delta g). \end{aligned}$$

Hence

$$\begin{aligned} \Delta u &= u(x + \Delta x) - u(x) \\ &= f(x)g(x) + (\Delta f)g(x) + f(x)(\Delta g) + (\Delta f)(\Delta g) - f(x)g(x) \\ &= (\Delta f)g(x) + f(x)(\Delta g) + (\Delta f)(\Delta g) \end{aligned}$$

and

$$\begin{aligned} \frac{\Delta u}{\Delta x} &= \frac{(\Delta f)g(x) + f(x)(\Delta g) + (\Delta f)(\Delta g)}{\Delta x} \\ &= \frac{\Delta f}{\Delta x}g(x) + f(x)\frac{\Delta g}{\Delta x} + \Delta f\frac{\Delta g}{\Delta x} \end{aligned}$$

As $\Delta x \to 0$, $\Delta g/\Delta x \to g'(x)$ and $\Delta f/\Delta x \to f'(x)$. Furthermore, because f is differentiable, hence continuous, $\Delta f \to 0$ as $x \to 0$. It follows that

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = f'(x)g(x) + f(x)g'(x) + 0 \cdot g'(x).$$

The formula for $(fg)'$ was discovered by Leibniz in 1676. His first guess was wrong.

Therefore, u is differentiable and

$$u' = f'g + fg'.$$

•

Remark: Figure 3.3.2 illustrates the Product Rule and its proof. With f, Δf, g, and Δg taken to be positive, the inner rectangle has area $u = fg$ and the whole rectangle has area $u + \Delta u = (f + \Delta f)(g + \Delta g)$. The shaded region whose area is Δu is made up of rectangles of areas $f \cdot (\Delta g)$, $(\Delta f) \cdot g$, and $(\Delta f) \cdot (\Delta g)$. The little corner rectangle, of area $(\Delta f) \cdot (\Delta g)$, is negligible in comparison with the other two rectangles. Thus, $\Delta u \approx (\Delta f)g + f(\Delta g)$, which suggests the formula for the derivative of a product.

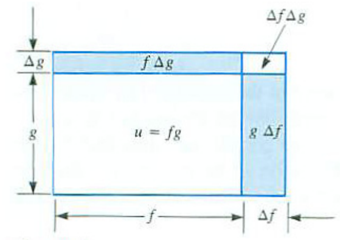


Figure 3.3.2:

EXAMPLE 3 Find $D\left((x^2 + x^3 + \cos(x) + 3)\,(x^4 - \sqrt{x} - e^x)\right)$.
SOLUTION By the Product Rule,

$$\begin{aligned} &D\left(\left(x^2 + x^3 + \cos(x) + 3\right)\left(x^4 - \sqrt{x} - e^x\right)\right) \\ &= \left(D\left(x^2 + x^3 + \cos(x) + 3\right)\right)\left(x^4 - \sqrt{x} - e^x\right) \\ &\quad + \left(x^2 + x^3 + \cos(x) + 3\right)\left(D\left(x^4 - \sqrt{x} - e^x\right)\right) \\ &= \left(2x + 3x^2 - \sin(x)\right)\left(x^4 - \sqrt{x} - e^x\right) \\ &\quad + \left(x^2 + x^3 + \cos(x) + 3\right)\left(4x^3 - \frac{1}{2\sqrt{x}} - e^x\right) \end{aligned}$$

◇

Note that the function to be differentiated is the product of the functions differentiated in Examples 1 and 2.

Derivative of Constant Times f

A special case of the formula for the product rule occurs so frequently that it is singled out in the Constant Multiple Rule.

> **Constant Multiple Rule**
> If C is a constant function and f is a differentiable function, the Cf is differentiable and its derivative is given by the formula
>
> $$(Cf)' = C(f').$$
>
> In other notations, $\frac{d(Cf)}{dx} = C\frac{df}{dx}$ and $D(Cf) = CD(f)$.

The derivative of a constant times a function is the constant times the derivative of the function.

Proof
Because we are dealing with a product of two differentiable functions, C and f, we may use the Product Rule. We have

$$\begin{aligned} (Cf)' &= (C')f + C(f') && \text{derivative of a product} \\ &= 0 \cdot f + Cf' && \text{derivative of constant is 0} \\ &= C(f'). \end{aligned}$$

•

The Constant Multiple Rule asserts that that "it is legal to move a constant factor outside the derivative symbol."

EXAMPLE 4 Find $D(6x^3)$.
SOLUTION

$$\begin{aligned} D(6x^3) &= 6D(x^3) && \text{6 is a constant} \\ &= 6 \cdot 3x^2 && D(x^n) = nx^{n-1} \\ &= 18x^2. \end{aligned}$$

With a little practice, one would simply write $D(6x^3) = 18x^2$. ◇

EXAMPLE 5 Find $D\left(\sqrt{x}/11\right)$.
SOLUTION

$$D\left(\frac{\sqrt{x}}{11}\right) = D\left(\frac{1}{11}\sqrt{x}\right) = \frac{1}{11}D(\sqrt{x}) = \frac{1}{11}\frac{1}{2\sqrt{x}} = \frac{1}{22}x^{-1/2}$$

◇

Example 5 generalizes to the fact that for a nonzero C,

Constant Division Rule

$$\left(\frac{f}{C}\right)' = \frac{f'}{C}, \qquad C \neq 0.$$

The formula for the derivative of the product extends to the product of several differentiable functions. For instance,

$$(fgh)' = (f')gh + f(g')h + fg(h')$$

See Exercise 45.

In each summand only one derivative appears. The next example illustrates the use of this formula.

EXAMPLE 6 Differentiate $\sqrt{x}e^x \sin(x)$.
SOLUTION

$$\begin{aligned} &\left(\sqrt{x}e^x \sin(x)\right)' \\ &= (\sqrt{x})'e^x \sin(x) + \sqrt{x}(e^x)' \sin(x) + \sqrt{x}e^x(\sin(x))' \\ &= \left(\frac{1}{2\sqrt{x}}\right) e^x \sin(x) + \sqrt{x}e^x \sin(x) + \sqrt{x}e^x \cos(x) \end{aligned}$$

◇

Any polynomial can be differentiated by the methods already developed.

EXAMPLE 7 Differentiate $6t^8 - t^3 + 5t^2 + \pi^3$.
SOLUTION Notice that the independent variable in this polynomial is t, and the polynomial is to be differentiated with respect to t.

Differentiate a polynomial "term-by-term". Note that π^3 is a constant.

$$\begin{aligned}\frac{d}{dt}\left(6t^8 - t^3 + 5t^2 + \pi^3\right) &= \frac{d}{dt}\left(6t^8\right) - \frac{d}{dt}\left(t^3\right) + \frac{d}{dt}\left(5t^2\right) + \frac{d}{dt}\left(\pi^3\right)\\ &= 48t^7 - 3t^2 + 10t + 0\\ &= 48t^7 - 3t^2 + 10t\end{aligned}$$

◇

Derivative of $1/g$

Often one needs the derivative of the reciprocal of a function g, that is, $(1/g)'$.

Reciprocal Rule
If g is a differentiable function, then

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}, \qquad \text{where } g(x) \neq 0$$

Proof

Again we must go back to the definition of the derivative.

Assume $g(x) \neq 0$ and let $u(x) = 1/g(x)$. Then $u(x+\Delta x) = 1/g(x+\Delta x) = 1/(g(x)+\Delta g)$. Thus

$$\begin{aligned}\Delta u &= u(x+\Delta x) - u(x)\\ &= \frac{1}{g(x)+\Delta g} - \frac{1}{g(x)}\\ &= \frac{g(x) - (g(x)+\Delta g)}{g(x)(g(x)+\Delta g)} && \text{common denominator}\\ &= \frac{-\Delta g}{g(x)(g(x)+\Delta g)} && \text{cancellation.}\end{aligned}$$

Then

$$
\begin{aligned}
u'(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} &= \lim_{\Delta x \to 0} \frac{-\Delta g / \left(g(x)(g(x)+\Delta g)\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta g / \Delta x}{g(x)(g(x)+\Delta g)} && \text{algebra: } \frac{(a/b)}{c} = \frac{(a/c)}{b} \\
&= \frac{\lim_{\Delta x \to 0} \left(\frac{-\Delta g}{\Delta x}\right)}{\lim_{\Delta x \to 0} \left(g(x)(g(x)+\Delta g)\right)} && \text{quotient rule for limits} \\
&= \frac{-g'(x)}{g(x)^2} && \text{because } g(x) \text{ is continuo} \\
& && \lim_{\Delta x \to 0} \Delta g = 0.
\end{aligned}
$$

•

EXAMPLE 8 Find $D\left(\frac{1}{\cos(x)}\right)$.

SOLUTION In this case, $g(x) = \cos(x)$ and $g'(x) = -\sin(x)$. Therefore,

$$
\begin{aligned}
D\left(\frac{1}{\cos(x)}\right) &= \frac{-(-\sin(x))}{(\cos(x))^2} \\
&= \frac{\sin(x)}{\cos^2(x)} && \text{for all } x \text{ with } \cos(x) \neq 0
\end{aligned}
$$

◇

Example 8 gives a formula for the derivative of $\sec(x)$, which is defined as $1/\cos(x)$.

$$
D(\sec(x)) = D\left(\frac{1}{\cos(x)}\right) = \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)} \frac{1}{\cos(x)} = \tan(x)\sec(x)
$$

Therefore,

Memorize this formula.

Derivative of $\sec(x)$

$$
D(\sec(x)) = \sec(x)\tan(x)
$$

The reciprocal rule allows us to complete the justification of the power rule for exponents that are negative rational numbers.

EXAMPLE 9 Show that the Power Rule, (3.2.6) in Section 3.2, is valid when a is a negative rational number. That is, show that $D(x^{-p/q}) = (-p/q)x^{-p/q-1}$ for any integers p and q, with $q \neq 0$.

SOLUTION The key is to notice that the Reciprocal Rule can be applied to find the derivative of $x^{-p/q} = 1/x^{p/q}$.

$$D\left(x^{-p/q}\right) = D\left(\frac{1}{x^{p/q}}\right) = \frac{-D(x^{p/q})}{(x^{p/q})^2} = \frac{-\frac{p}{q}x^{\frac{p}{q}-1}}{x^{2\frac{p}{q}}} = -\frac{p}{q}x^{\frac{p}{q}-1-2\frac{p}{q}} = -\frac{p}{q}x^{-p/q-1}.$$

◇

The Derivative of f/g

EXAMPLE 10 Derive a formula for the derivative of the quotient f/g.
SOLUTION The quotient f/g can be written as a product $f \cdot \frac{1}{g}$. Assuming f and g are differentiable functions, we may use the product and reciprocal rules to find

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \left(f(x)\frac{1}{g(x)}\right)' && \text{rewrite quotient as product}\\
&= f'(x)\left(\frac{1}{g(x)}\right) + f(x)\left(\frac{1}{g(x)}\right)' && \text{product rule}\\
&= f'(x)\left(\frac{1}{g(x)}\right) + f(x)\left(\frac{-g'(x)}{g(x)^2}\right) && \text{reciprocal rule, assuming } g(x) \neq 0\\
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} && \text{algebra}\\
&= \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2} && \text{algebra: common denominator.}
\end{aligned}$$

◇

Example 10 is the proof of the **quotient rule**. The quotient rule should be committed to memory. A simple case of the quotient rule has already been used to find the derivative of $\sec(x) = \frac{1}{\cos(x)}$ (Example 8). The full quotient rule will be used to find the derivative of $\tan(x) = \frac{\sin(x)}{\cos(x)}$ (Example 11). Because the quotient rule is used so often, it should be memorized.

Quotient Rule

Let f and g be differentiable functions at x, and assume $g(x) \neq 0$. Then the quotient f/g is differentiable at x, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \qquad \text{where } g(x) \neq 0.$$

A memory device for $(f/g)'$

Remark: Because the numerator in the quotient rule is a difference, it is important to get the terms in the numerator in the correct order. Here is an easy way to remember the quotient rule.

Step 1. Write down the parts where g^2 and g appear:

$$\frac{g\_\_\_\_\_}{g^2}.$$

This ensures that you get the denominator correct, and have a good start on the numerator.

Step 2. To complete the numerator, remember that it has a minus sign:

$$\frac{gf' - fg'}{g^2}.$$

EXAMPLE 11 Find the derivative of the tangent function.
SOLUTION

$$\begin{aligned}
(\tan(x))' &= \left(\frac{\sin(x)}{\cos(x)}\right)' \\
&= \frac{\cos(x)(\sin(x))' - \sin(x)(\cos(x))'}{(\cos(x))^2} && \text{quotient rule} \\
&= \frac{(\cos(x))\cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} && \sin^2(x) + \cos^2(x) = 1 \\
&= \sec^2(x) && \sec(x) = 1/\cos(x)
\end{aligned}$$

This result is valid whenever $\cos(x) \neq 0$, and should be memorized. ◇

Derivative of $\tan(x)$

$$D(\tan(x)) = \sec^2(x) \qquad \text{for all } x \text{ in the domain of } \tan(x).$$

EXAMPLE 12 Compute $\left(x^2/(x^3+1)\right)'$, showing each step.

SOLUTION

$$\left(\frac{x^2}{x^3+1}\right)' = \frac{(x^3+1)\cdots}{(x^3+1)^2} \qquad \text{write denominator and start numerator}$$

$$= \frac{(x^3+1)(x^2)' - (x^2)(x^3+1)'}{(x^3+1)^2} \qquad \text{complete numerator, remembering the minus sign}$$

$$= \frac{(x^3+1)(2x) - (x^2)(3x^2)}{(x^3+1)^2} \qquad \text{compute derivatives}$$

$$= \frac{2x^4 + 2x - 3x^4}{(x^3+1)^2} \qquad \text{algebra}$$

$$= \frac{2x - x^4}{(x^3+1)^2} \qquad \text{algebra: collecting}$$

◇

As Example 12 illustrates, the techniques for differentiating polynomials and quotients can be combined to differentiate any **rational function**, that is, any quotient of polynomials.

Summary

Let f and g be two differentiable functions and let C be a constant function. We obtained formulas for differentiating $f+g$, $f-g$, fg, Cf, $1/f$, and f/g.

Rule	Formula	Comment
Constant Rule	$C' = 0$	C a constant
Sum Rule	$(f+g)' = f' + g'$	
Difference Rule	$(f-g)' = f' - g'$	
Product Rule	$(fg)' = (f')g + f(g')$	
Constant Multiple Rule	$(Cf)' = C(f')$	
Reciprocal Rule	$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$	$g(x) \neq 0$
Quotient Rule	$\left(\frac{f}{g}\right)' = \frac{g(f') - f(g')}{g^2}$	$g(x) \neq 0$

Table 3.3.1:

With the aid of the formulas in Table 3.3.1, we can differentiate $\sec(x)$, $\csc(x)$, $\tan(x)$, and $\cot(x)$ using $(\sin(x))' = \cos(x)$ and $(\cos(x))' = -\sin(x)$. We also have shown that $D(x^a) = ax^{a-1}$ for any fixed number a.

Function	Derivative	Comment
x^a	ax^{a-1}	a is a fixed number
$\tan(x)$	$\sec^2(x)$	for all x except odd multiples of $\pi/2$
$\sec(x)$	$\sec(x)\tan(x)$	for all x except odd multiples of $\pi/2$

Table 3.3.2:

EXERCISES for 3.3

Key: R–routine, M–moderate, C–challenging

In Exercises 1 to 15 differentiate the given function. *Use only the formulas presented in this and earlier sections.*

1.[R] $5x^3$

2.[R] $5x^3 - 7x + 2^3$

3.[R] $3\sqrt{x} - \sqrt[3]{x}$

4.[R] $1/\sqrt{x}$

5.[R] $(5+x)(x^2 - x + 7)$

6.[R] $\sin(x)\cos(x)$

7.[R] $3\tan(x)$

8.[R] $3(\tan(x))^2$ Hint: Write $(\tan(x))^2$ as $\tan(x)\tan(x)$

9.[R] $\dfrac{x^3 - 1}{2x + 1}$

10.[R] $\dfrac{\sin(x)}{e^x}$

11.[R] $\dfrac{3x^2 + x + \sqrt{2}}{\cos(x)}$

12.[R] $\dfrac{2}{x^3} + \dfrac{3}{x^4}$

13.[R] $x^2\sin(x)e^x$

14.[R] $\sqrt{x}\sin(x)$

15.[R] $\sqrt{x}/e^x$

16.[R] Differentiate the following functions:

(a) $\dfrac{(1+\sqrt{x})(x^3 + \sin(x))}{x^2 + 5x + 3e^x}$

(b) $\dfrac{(3 + 4x + 5x^2)e^x}{\sin(x)}$

17.[R] Use the quotient rule to obtain the following derivatives.

(a) $D(\tan(x)) = (\sec(x))^2$

(b) $D(\cot(x)) = -(\csc(x))^2$

(c) $D(\sec(x)) = \sec(x)\tan(x)$

(d) $D(\csc(x)) = -\csc(x)\cot(x)$

Note: There is a pattern here. The minus sign goes with each "co" function (cos, cot, csc).

18.[R] Find $\left(e^{2x}\right)'$ by writing e^{2x} as $e^x e^x$.

19.[R] Find $\left(e^{3x}\right)'$ by writing e^{3x} as $e^x e^x e^x$.

20.[R] Find $(e^{-x})'$ by writing e^{-x} as $\frac{1}{e^x}$.

21.[R] Find $\left(e^{-2x}\right)'$ by writing $e^{-2x} = e^{-x} \cdot e^{-x}$. (See Exercise 20.)

22.[R] Find $\left(e^{-2x}\right)'$ by writing $e^{-2x} = \frac{1}{e^{2x}}$. (See Exercise 18.)

In Exercises 23 to 41 find the derivative of the function using formulas from this section.

23.[R] $2^3 - \sqrt{\pi}$

24.[R] $(x - x^{-1})^2$

25.[R] $3\sin(9x) - 5\cos(x)$

26.[R] $5\tan(x)$

27.[R] $u^5 - 6u^3 + u - 7$

28.[R] $t^8/8$

29.[R] $s^{-7}/(-7)$

30.[R] $\sqrt{t}(t+4)$

31.[R] $5/u^5$

32.[R] $(x^3)^{1/2}$

33.[R] $6\tan(x)$

34.[R] $3\sec(x) - 4\cos(x)$

35.[R] $\sec^2(\theta) - \tan^2(\theta)$ NOTE: remember to simplify your answer

36.[R] $(3x)^4$

37.[R] $u^2 e^u$

38.[R] $e^t \sin(t)/\sqrt{t}$

39.[R] $(3 + x^5)e^{-x}\tan(x)$

40.[R] $(x - x^2)^3$ HINT: multiply it out first

41.[R] $\sqrt[3]{x}/\sqrt[5]{x}$

42.[R] In Section 3.1 we showed that $D(1/x) = -1/x^2$. Obtain this same formula by using the Quotient Rule.

43.[R] If you had lots of time, how would you differentiate $(1+2x)^{100}$ using the formulas developed so far? NOTE: In Section 3.5 we will obtain a shortcut for differentiating $(1+2x)^{100}$.

44.[M] At what point on the graph of $y = xe^{-x}$ is the tangent horizontal?

45.[M] Using the formula for the derivative of a product, obtain the formula for $(fgh)'$. HINT: First write fgh as $(f)(gh)$. Then use the Product Rule twice.

46.[M] Obtain the formula for $(f-g)'$ by first writing $f-g$ as $f+(-1)g$.

47.[M] Using the definition of the derivative, show that $(f-g)' = f' - g'$.

48.[M] Using the version of the definition of the derivative that makes use of both x and $x+h$, obtain the formula for differentiating the sum of two functions.

49.[C] Using the version of the definition of the derivative in the form $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, obtain the formula for differentiating the product of two functions.

Exercises 50 to 52 are examples of **proof by mathematical induction**. In this technique the truth of the statement for n is used to prove the truth of the statement of $n+1$.

50.[C] In Section 3.2 we show that $D(x^n) = nx^{n-1}$, when n is a positive integer. Now that we have the formula for the derivative of a product of two functions we can obtain this result much more easily.

(a) Show, using the definition of the derivative, that the formula $D(x^n) = nx^{n-1}$ holds when $n=1$.

(b) Using (a) and the formula for the derivative of a product, show that the formula holds when $n=2$. HINT: $x^2 = x \cdot x$.

(c) Using (b) and the formula for the derivative of a product, show that it holds when $n=3$.

(d) Show that if it holds for some positive integer n, it also holds for the integer $n+1$.

(e) Combine (c) and (d) to show that the formula holds for $n=4$.

(f) Why must it hold for $n=5$?

(g) Why must it hold for all positive integers?

51.[C] Using induction, as in Exercise 50, show that for each positive integer n, $D\left(x^{-n}\right) = -nx^{-n-1}$.

52.[C] Using induction, as in Exercise 50, show that for each positive integer n, $D(\sin^n(x)) = n\sin^{n-1}(x)\cos(x)$.

53.[C] We obtained the formula for $(f/g)'$ by writing f/g as the product of f and $1/g$. Obtain $(f/g)'$ directly from the definition of the derivative. HINT: First review how we obtained the formula for the derivative of a product.

3.4 The Chain Rule

We come now to the most frequently used formula for computing derivatives. For example, it will help us to find the derivative of $(1+x^2)^{100}$ without having to multiply out one hundred copies of $(1+x^2)$. You might be tempted to guess that the derivative of $(1+x^2)^{100}$ would be $100(1+x^2)^{99}$. *This cannot be right.* After all, when you expand $(1+x^2)^{100}$ you get a polynomial of degree 200, so its derivative is a polynomial of degree 199. But when you expand $(1+x^2)^{99}$ you get a polynomial of degree 198. Something is wrong.

At this point we know the derivative of $\sin(x)$, but what is the derivative of $\sin(x^2)$? It is *not* the cosine of x^2. In this section we obtain a way to differentiate these functions easily — and correctly.

The key is that both $(1+x^2)^{100}$ and $\sin(x^2)$ are composite functions. This section shows how to differentiate composite functions.

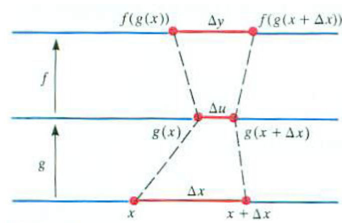


Figure 3.4.1:

How to Differentiate a Composite Function

Recall that $y = (f \circ g)(x) = f(g(x))$ can be built up by setting $u = g(x)$ and $y = f(u)$. The derivative of y with respect to x is the limit of $\Delta y/\Delta x$ as Δx approaches 0. Now, the change in Δx causes a change Δu in u, which, in turn, causes the change Δy in y. (See Figure 3.4.1.) If Δu is not zero, then we may write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}. \qquad (3.4.1)$$

It could happen that $\Delta u = 0$, as it would, for instance, if g were a constant function. This special case is treated in Exercise 75.

Then,

$$(f \circ g)'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}.$$

Since g is continuous, $\Delta u \to 0$ as $\Delta x \to 0$. So we have

$$(f \circ g)'(x) = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = f'(u)g'(x).$$

Which gives us

Chain Rule

Let g be differentiable at x and f be differentiable at $g(x)$, then

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

The Chain Rule is the technique most frequently used in finding derivatives.

This formula tells us how to differentiate a composite function, $f \circ g$:

Step 1. Compute the derivative of the outer function f, evaluated at the inner function. This is $f'(g(x))$.

First term in the formula.

Second term in the formula. Step 2. Compute the derivative of the inner function, $g'(x)$.

Step 3. Multiply the derivatives found in Steps 1. and 2., obtaining $f'(g(x))g'(x)$.

In short, to differentiate $f(g(x))$, think of g as the "inner function" and f as the "outer function." Then the derivative of $f \circ g$ is

$$\underbrace{f'(g(x))}_{\text{derivative of outer function evaluated at inner function}} \quad \text{times} \quad \underbrace{g'(x)}_{\text{derivative of inside function}} = 2x\cos(x^2).$$

Examples

EXAMPLE 1 Find $D\left((1+x^2)^{100}\right)$.

SOLUTION Here $g(x) = 1+x^2$ (the inside function) and $f(u) = u^{100}$ (the outside function). The first step is to compute $f'(u) = 100u^{99}$, which gives us $f'(g(x)) = 100(1+x^2)^{99}$. The second step is to find $g'(x) = 2x$. Then,

$$(f \circ g)'(x) = f'(\underbrace{u}_{u=g(x)})g'(x) = \underbrace{100u^{99}}_{f'(g(x))} \cdot \underbrace{2x}_{g'(x)} = 100(1+x^2)^{99} \cdot 2x = 200x(1+x^2)^{99}.$$

The answer is not just $100(1+x^2)^{99}$: there is an extra factor of $2x$ that comes from the derivative of the inner function, so its degree is 199, as expected. ⋄

The same example, done with Leibniz notation, looks like this:

$$y = (1+x^2)^{100} = u^{100}, \qquad u = 1+x^2.$$

Then the Chain Rule reads simply

$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}\frac{du}{dx}}_{\text{Chain Rule}} = 100u^{99} \cdot 2x = \underbrace{100(1+x^2)^{99}(2x)}_{\text{using } u=1+x^2}$$

$$= 200x(1+x^2)^{99}.$$

George Berkeley, 1734, *The Analyst: A Discourse Addressed to an Infidel Mathematician*. See also `http://muse.jhu.edu/journals/configurations/v004/4.1paxson.html`.

WARNING *(Notation)* We avoided using Leibniz notation earlier, in particular, during the derivation of the Chain Rule, because it tempts the reader to cancel the du's in (3.4.1). However, the expressions dy, du, and dx are meaningless — in themselves. In Leibniz's time in the late seventeenth century their meaning was fuzzy, standing for a quantity that was zero and also vanishingly small at the same time. Bishop Berkeley poked fun at this, asking "may we not call them the ghosts of departed quantities?"

With practice, you will be able to do the whole calculation without introducing extra symbols, such as u, which do not apear in the final answer. You will be writing just

$$D\left((1+x^2)^{100}\right) = 100(1+x^2)^{99} \cdot 2x = 200x(1+x^2)^{99}.$$

But this skill, like playing the guitar, takes practice, which the exercises at the end of this section (and chapter) provide.

When we write $\frac{dy}{du}$ and $\frac{du}{dx}$, the u serves two rolls. In $\frac{dy}{du}$ it denotes an independent variable while in $\frac{du}{dx}$, u is a dependent variable. This double role usually causes no problem in computing derivatives.

EXAMPLE 2 If $y = \sin(x^2)$, find $\frac{dy}{dx}$.
SOLUTION Starting from the outside, let $y = \sin(u)$ and $u = x^2$. Then, using the Chain Rule,

$$\left(\sin(x^2)\right)' = \frac{dy}{dx} = \underbrace{\frac{dy}{du}\frac{du}{dx}}_{\text{by the Chain Rule}} = \cos(u) \cdot 2x = \cos(x^2) \cdot 2x = 2x\cos(x^2).$$

In this case the outside function is the sine and the inside function is x^2. So we have

$$(\underbrace{\sin}_{\text{outside}}\underbrace{(x^2)}_{\text{inside}})' = \underbrace{\cos(x^2)}_{\text{derivative of outside function evaluated at inside function}} \quad \text{times} \quad \underbrace{2x}_{\text{derivative of inside function}} = 2x\cos(x^2).$$

◇

The Chain Rule holds for compositions of more than two functions. We illustrate this in the next example.

EXAMPLE 3 Differentiate $y = \sqrt{\sin(x^2)}$.
SOLUTION In this case the function is the composition of three functions:

$$u = x^2 \qquad v = \sin(u) \qquad y = \sqrt{v} \text{ (provided } v \geq 0).$$

Then

Do this example yourself without introducing any auxiliary symbols (u, v, and y).

$$\frac{dy}{dx} = \underbrace{\frac{dy}{dv}\frac{dv}{dx}}_{\text{Chain Rule}} = \underbrace{\frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}}_{\text{Chain Rule, again}} = \frac{1}{2\sqrt{v}} \cdot \cos(u) \cdot 2x$$

$$= \frac{1}{2\sqrt{\sin(x^2)}} \cdot \cos(x^2) \cdot 2x = \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}$$

◇

EXAMPLE 4 Let $y = 2^x$. Find y'.

$b = e^{\ln(b)}$ for any $b > 0$

SOLUTION As it stands, 2^x is not a composite function. However, we can write $2 = e^{\ln(2)}$ and then 2^x equals $(e^{\ln(2)})^x = e^{\ln(2)x}$. Now we see that 2^x can be expressed as the composite function:

$$y = e^u, \text{ where } u = (\ln(2))x.$$

Then

$$y' = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot \ln(2) = e^{\ln(2)x}\ln(2) = 2^x \ln(2).$$

In Example 2 (Section 3.2), using a calculator, we found $D(2^x) \approx (0.693)2^x$. We have just seen that the exact formula for this derivative is $D(2^x) = 2^x \ln(2)$. This means that 0.693 is an approximation of $\ln(2)$. Does your calculator agree that $\ln(2) \approx 0.693$. ◇

Sometimes it is convenient to introduce an intermediate variable when using the Chain Rule. The next Example illustrates this idea, which will be used extensively in the next section.

EXAMPLE 5 Find $\frac{d}{dx}\cos(2^x)$.

SOLUTION Let $y = 2^x$ so that $\cos(2^x) = \cos(y)$. Observe that in Example 4 we found $\frac{dy}{dx} = 2^x \ln(2)$. Thus, by the Chain Rule,

$$\frac{d}{dx}\cos(2^x) = \frac{d}{dx}\cos(y) = -\sin(y)\frac{dy}{dx} = -\sin(2^x)\, 2^x \ln(2).$$

◇

The next Example shows how the Chain Rule can be combined with other differentiation rules such as the Product and Quotient Rules.

EXAMPLE 6 Find $D(x^3 \tan(x^2))$.

SOLUTION The function $x^3 \tan(x^2)$ is the product of two functions. We first apply the Product Rule to obtain:

Product Rule: $(fg)' = f' \cdot g + f \cdot g'$

$$\begin{aligned} D(x^3 \tan(x^2)) &= (x^3)' \tan(x^2) + x^3 (\tan(x^2))' \\ &= 3x^2 \tan(x^2) + x^3 (\tan(x^2))'. \end{aligned}$$

$(\tan(x))' = \sec^2(x)$

Since "the derivative of the tangent is the square of the secant," the Chain Rule tells us that

$$\left(\tan(x^2)\right)' = \sec^2(x^2)(x^2)' = 2x\sec^2(x^2).$$

Thus,

$$\begin{aligned} D(x^3 \tan(x^2)) &= 3x^2 \tan(x^2) + x^3 (\tan(x^2))' \\ &= 3x^2 \tan(x^2) + x^3 (2x\sec^2(x^2)) \\ &= 3x^2 \tan(x^2) + 2x^4 \sec^2(x^2). \end{aligned}$$

◇

In the computation of $D(\tan(x^2))$ we did not introduce any new symbols. That is how your computations will look, once you get the rhythm of the Chain Rule.

Famous Composite Functions

Certain types of composite functions occur so often that it is worthwhile memorizing their derivatives. Here is a list:

Function	Derivative	Example
$(g(x))^n$	$ng(x)^{n-1}g'(x)$	$((1+x^2)^{100})' = 100(1+x^2)^{99}(2x)$
$\frac{1}{g(x)}$	$\frac{-g'(x)}{(g(x))^2}$	$D\left(\frac{1}{\cos(x)}\right) = \frac{-(-\sin(x))}{(\cos(x))^2}$
$\sqrt{g(x)}$	$\frac{g'(x)}{2\sqrt{g(x)}}$	$\left(\sqrt{\tan(x)}\right)' = \frac{(\sec(x))^2}{2\sqrt{\tan(x)}}$
$e^{g(x)}$	$e^{g(x)}g'(x)$	$\left(e^{x^2}\right)' = e^{x^2}(2x)$

Table 3.4.1:

Summary

This section presented the single most important tool for computing derivatives: the Chain Rule, which says that the derivative of $f \circ g$ at x is

$$\underbrace{f'(g(x))}_{\text{derivative of outer function evaluated at the inner function}} \quad \text{times} \quad \underbrace{g'(x)}_{\text{derivative of inner function}}$$

Introducing the symbol u, we described the Chain Rule for $y = f(u)$ and $u = g(x)$ with the brief notation

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

When the function is built up from more than two functions, such as $y = f(u)$, $u = g(v)$, and $v = h(x)$. Then we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx},$$

a chain of more derivatives.

With practice, applying the chain rule can become second nature.

All that remains to describe how to differentiate $\ln(x)$ and the inverse trigonometric functions. The next section, with the aid of the chain rule, determines their derivatives.

With practice, applying the Chain Rule can become second nature.

EXERCISES for 3.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4, repeat the specified example from this section *without* introducing an extra variable (such as u).

1.[R] Example 1.

2.[R] Example 2.

3.[R] Example 3.

4.[R] Example 4.

In Exercises 5 to 18 find the derivative of each function.

5.[R] $(x^3+2)^5$

6.[R] $(x^2+3x+1)^4$

7.[R] $\sqrt{\cos(x^3)}$

8.[R] $\sqrt{\tan(x^2)}$

9.[R] $\left(\frac{1}{x}\right)^{10}$

10.[R] $\cos(3x)\sin(2x)$

11.[R] $x^2\tan(x^3)$

12.[R] $(1+2x)\sin(x^4)$

13.[R] $5(\tan(x^3))^2$

14.[R] $\frac{\cos^3(2x)}{x^5}$

15.[R] $\sin(2\exp(x))$

16.[R] $e^{\cos(x)}$

17.[R] $\frac{(1+2x)^2}{x^3}$

18.[R] $(\sec(5x))(\cos(5x))$ HINT: simplify your answer

In Exercises 19 to 40 differentiate the given function.

19.[R] $(5x^2+3)^{10}$

20.[R] $(\sin(3x))^3$

21.[R] $\frac{1}{5t^2+t+2}$

22.[R] $\frac{1}{e^{5s}+s}$

23.[R] $\sqrt{4+u^2}$

24.[R] $(\sqrt{\cos(2\theta)})^3$

25.[R] e^{5x^3}

26.[R] $\sin^2(3x)$

27.[R] $e^{\tan(3t)}$

28.[R] $\sqrt{\tan(2u)}$

29.[R] $\sqrt[3]{\tan(s^2)}$

30.[R] $v^3\tan(2v)$

31.[R] $e^{2r}\sin(3r)$

32.[R] $\frac{\sec(2x)}{x^2}$

33.[R] $\exp(\sin(2x))$

34.[R] $\frac{(3t+2)^4}{\sin(2t)}$

35.[R] $e^{-5s}\tan(3s)$

36.[R] e^{x^2}

37.[R] $(\sin(2u))^5(\cos(3u))^6$

38.[R] $\left(x+3^{3x}\right)^2(\sin(\sqrt{x}))^3$

39.[R] $\frac{t^3}{\left(t+\sin^2(3t)\right)}$

40.[R] $\frac{(3x+2)^4}{(x^3+x+1)^2}$

Learning to use the chain rule takes practice. Exercises 41 to 68 offer more opportunities to practice that skill. They also show that sometimes the derivative of a function can be much simpler than the function. In each case show that the derivative of the first function is the second function. The letters a, b, and c denote constants.

41.[M] $\frac{b}{2a^2(ax+b)^2}-\frac{1}{a^2(ax+b)}$, $\frac{x}{(ax+b)^2}$

42.[M] $\frac{-1}{2a(ax+b)^2}$, $\frac{1}{(ax+b)^3}$

43.[M] $\frac{2}{3a}\sqrt{(ax+b)^3}$, $\sqrt{ax+b}$

44.[M] $\frac{2(3ax-2b)}{15a^2}\sqrt{(ax_b)^3}$, $x\sqrt{ax+b}$

45.[M] $\frac{-\sqrt{ax^2+c}}{cx}$, $\frac{1}{x^2\sqrt{ax^2+c}}$

46.[M] $\frac{x}{c\sqrt{ax^2+c}}$, $(ax^2+c)^{-3/2}$

47.[M] $\frac{1}{a}\sin(ax)-\frac{1}{3a}\sin^3(ax)$, $\cos^3(ax)$

48.[M] $\frac{1}{a(n+1)}\sin^{n+1}(ax)$, $\sin^n(ax)\ \cos(ax)$

49.[M] $\frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$, $\frac{x}{\sqrt{ax+b}}$

50.[M] $\frac{2(3a^2x^2-4abx+8b^2)}{15a^3}\sqrt{ax+b}$, $\frac{x^2}{\sqrt{ax+b}}$

51.[M] $\frac{-\sqrt{ax^2+c}}{cx}$, $\frac{1}{x^2\sqrt{ax^2+c}}$

52.[M] $\frac{-x^2}{a\sqrt{ax^2+c}}+\frac{2}{a^2}\sqrt{ax^2+c}$, $\frac{x^3}{(ax^2+c)^{3/2}}$

53.[M] $\frac{-1}{a}\cos(ax)+\frac{1}{3a}\cos^3(ax)$, $\sin^3(ax)$

54.[M] $\frac{3x}{8}-\frac{3\sin(2ax)}{16a}-\frac{\sin^3(ax)\cos(ax)}{4a}$, $\sin^4(ax)$

55.[M] $\frac{\sin((a-b)x)}{2(a-b)}-\frac{\sin((a+b)x)}{2(a+b)}$, $\sin(ax)\sin(bx)$ (Assume $a^2\neq b^2$.)

56.[M] $\frac{x}{2}+\frac{\sin(2ax)}{3a}$, $\cos^3(ax)$

57.[M] $\frac{1}{a}\tan(ax)$, $\frac{1}{\cos^2(ax)}$

58.[M] $\frac{1}{a}\tan\left(\frac{ax}{2}\right)$, $\frac{1}{1+\cos(ax)}$

59.[M] $2\sqrt{2}\sin\left(\frac{x}{2}\right)$, $\sqrt{1+\cos(x)}$ NOTE: You will need to use a trigonometric identity.
60.[M] $\frac{\sin((a-b)x}{2(a-b)}+\frac{\sin((a+b)x)}{2(a+b)}$, $\cos(ax)\cos(bx)$ (Assume $a^2\neq b^2$.)
61.[M] $\frac{1}{a}\left(\tan(ax)-\cot(ax)\right)$, $\frac{1}{\sin^2(ax)\cos^2(ax)}$
62.[M] $\frac{1}{a}\tan(ax)-1$, $\tan^2(ax)$
63.[M] $\frac{\sec^n(ax)}{an}$, $\tan(ax)\sec^n(ax)$ (Assume $n\neq 0$.)
64.[M] $\frac{\sin(ax)}{a^2}-\frac{x\cos(ax)}{a}$, $x\sin(ax)$
65.[M] $\frac{\cos(ax)}{a^2}+\frac{x\sin(ax)}{a}$, $x\cos(ax)$
66.[M] $\frac{1}{a^2}e^{ax}(ax-1)$, xe^{ax}
67.[M] $\frac{1}{a^3}e^ax(a^2x62-2ax+2)$, x^2e^{ax}
68.[M] $\frac{e^{ax}(a\sin(bx)-b\cos(bx))}{a^2+b^2}$, $e^{ax}\sin(bx)$

Exercises 69 and 70 illustrate how differentiation can be used to obtain one trigonometry identity from another.
69.[M]

(a) Differentiate both sides of the identity $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$. What trigonometric identity do you get?

(b) Differentiate the identity found in (a) to obtain another trigonometric identity. What identity is obtained?

(c) Does this process continued forever produce new identities?

70.[M] Let k be a constant. Differentiate both sides of the identity $\sin(x+k)=\sin(x)\cos(k)+\cos(x)\sin(k)$ to obtain the corresponding identity for $\cos(x+k)$.

71.[M] Differentiate $(e^x)^3$

(a) directly, by the Chain Rule

(b) after writing the function as $e^x\cdot e^x\cdot e^x$ and using the product rule

(c) after writing the function as e^{3x} and using the chain rule

(d) Which of these approaches to you prefer? Why?

72.[M] In Section 3.3 we obtain the derivative of $1/g(x)$ by using the definition of the derivative. Obtain that formula for the Reciprocal Rule by using the Chain Rule.

73.[C] In our proof of the Chain Rule we had to assume that Δu is not 0 when Δx is sufficiently small. Show that if the derivative of g is not 0 at the argument x, then the proof is valid.

74.[C] Here is an example of a differentiable g not covered by the proof of the Chain Rule given in the text. Define $g(x)$ to be $x^2 \sin\left(\frac{1}{x}\right)$ for x different from 0 and $g(0)$ to be 0.

(a) Sketch the part of the graph of g near the origin.

(b) Show that there are arbitrarily small values of Δx such that $\Delta u = g(\Delta x) - g(0) = 0$.

(c) Show that g is differentiable at 0.

75.[C] Here is a proof of the Chain Rule that manages to avoid division by $\Delta u = 0$. Let $f(u)$ be differentiable at $g(a)$, where g is differentiable at a. Let $\Delta f = f(g(a) + \Delta u) - f(g(a))$. Then $\frac{\Delta f}{\Delta u} - f'(g(a))$ is a function of Δu, which we call $p(\Delta u)$. This function is defined for $\Delta u \neq 0$. By the definition of f', $p(\Delta u)$ tends to 0 as Δu approaches 0. Define $p(0)$ to be 0. Note that p is continuous at 0.

(a) Show that $\Delta f = f'(g(a))\Delta u + p(\Delta u)\Delta u$ when Δu is different than 0, and also when $\Delta u = 0$.

(b) Define $q(\Delta x) = \frac{\Delta u}{\Delta x} - g'(a)$. Observe that $q(\Delta x)$ approaches 0 as Δx approaches 0. Show that $\Delta u = g'(a)\Delta x + q(\Delta x)\Delta x$ when Δx is not 0.

(c) Combine (a) and (b) to show that

$$\Delta f = f'(g(a))\left(g'(a)\Delta x + q(\Delta x)\Delta x\right) + p(\Delta u)\Delta u.$$

(d) Using (c), show that

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(g(a))g'(a).$$

(e) Why did we have to define $p(0)$ but not $q(0)$?

3.5 Derivative of an Inverse Function

In this section we obtain the derivatives of the inverse functions of e^x and of the six trigonometric functions. This will complete the inventory of basic derivatives. The Chain Rule will be our main tool.

Differentiability of Inverse Functions

As mentioned in Section 1.1, the graph of an inverse function is an exact copy of the graph of the original function. One graph is obtained from the other by reflection across the line $y = x$. If the original function, f, is differentiable at a point (a, b), $b = f(a)$, then the graph of $y = f(x)$ has a tangent line at (a, b). In particular, the reflection of the tangent line to the graph of f is the tangent line to the inverse function at (b, a). Thus, we expect that the inverse function, f^{-1}, is differentiable at (b, a), and we will assume it is.

$b = f(a)$ means $a = f^{-1}(b)$

First, the Chain Rule will be used to find the derivative of $\log_e(x)$.

The Derivative of $\log_e(x)$

Let $y = \log_e(x)$. Figure 3.5.1 shows the graphs of $y = e^x$ and inverse function $y = \log_e(x)$. We want to find $y' = \frac{dy}{dx}$. By the definition of logarithm as the inverse of the exponential function

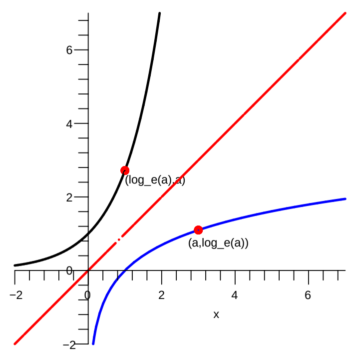


Figure 3.5.1:

$$x = e^y. \tag{3.5.1}$$

We differentiate both sides of (3.5.1) with respect to x:

$\frac{d(x)}{dx}$	$=$	$\frac{d(e^y)}{dx}$	e^y is a function of x, since y is a function of x
1	$=$	$\frac{d(e^y)}{dx}$	observe that $\frac{dx}{dx} = 1$
1	$=$	$e^y \frac{dy}{dx}$	Chain Rule.

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

This is another differentiation rule that should be memorized.

Derivative of e^x

$$(\log_e(x))' = \frac{1}{x}, \qquad x > 0.$$

It may come as a surprise that such a "complicated" function has such a simple derivative. It may also be a surprise that $\log_e(x)$ is one of the most important functions in calculus, mainly because it has the derivative $1/x$.

EXAMPLE 1 Find $(\log_b)'$ for any $b > 0$.
SOLUTION The function $\log_b x$ is just a constant times $\log_e(x)$:

$$\log_b(x) = (\log_b(e)) \log_e(x).$$

Therefore

$$(\log_b(x))' = (\log_b(e)) \frac{1}{x}. \tag{3.5.2}$$

If b is not e, then $\log_b(e)$ is not 1. If e is chosen as the base for logarithms, then the coefficient in front of the $\frac{1}{x}$ becomes $\log_e(e) = 1$. That is why we prefer e as the base for logarithms in calculus ◇

We call $\log_e(x)$ the **natural logarithm**, denoted $\ln(x)$.

> **WARNING** *(Logarithm Notation)* $\ln(x)$ is often written simply as $\log(x)$, with the base understood to be e. All references to the base-10 logarithm will use the notation $\log_{10}$.

Another View of e

For each choice of the base b ($b > 0$), we obtain a certain value for $\lim_{x\to 0} \frac{b^x - 1}{x}$. We defined e to be the base for which that limit is as simple as possible, namely 1: $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$.

Now that we know that the derivative of $\ln x = \log_e x$ is $1/x$, we can obtain a new view of e.

We know that the derivative of $\ln(x)$ at 1 is $1/1 = 1$. By the definition of the derivative, that means

$$\lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h} = 1.$$

Since $\ln(1) = 0$, we have

$$\lim_{h\to 0} \frac{\ln(1+h)}{h} = 1.$$

By a property of logarithms, we may rewrite the limit as

$$\lim_{h\to 0} \ln\left((1+h)^{1/h}\right) = 1.$$

Writing e^x as $\exp(x)$ for convenience, we conclude that

$$\exp\left(\lim_{h\to 0} \ln\left((1+h)^{1/h}\right)\right) = \exp(1) = e.$$

Since exp is a continuous function, we may switch the order of exp and lim, getting

$$\lim_{h\to 0}\left(\exp\left(\ln\left((1+h)^{1/h}\right)\right)\right)=e.$$

But, $\exp(\ln(p))=p$ for any positive number, by the very definition of a logarithm. That tells us that

$$\lim_{h\to 0}(1+h)^{1/h}=e.$$

This is a much more direct view of e than the one we had in Section 2.1. As a check, let $h=1/1000=0.001$. Then $(1+1/1000)^{1000}\approx 2.717$, and values of h that are closer to 0 give even better estimates for e, whose decimal expansion begins 2.718.

The Derivative of $\arcsin(x)$

Inverse trigonometric functions are introduced in Section 1.2.

For x in $[-\pi/2,\pi/2]$ $\sin(x)$ is one-to-one and therefore has an inverse function, $\arcsin(x)$. This function gives the angle, in radians, if you know the sine of the angle. For instance, $\arcsin(1)=\pi/2$, $\arcsin(\sqrt{2}/2)=\pi/4$, $\arcsin(-1/2)=-\pi/6$, and $\arcsin(-1)=-\pi/2$. The domain of $\arcsin(x)$ is $[-1,1]$; its range is $[-\pi/2,\pi/2]$. For convenience we include the graphs of $y=\sin(x)$ and $y=\arcsin(x)$ in Figure 3.5.2, but will not need them as we find $(\arcsin(x))'$.

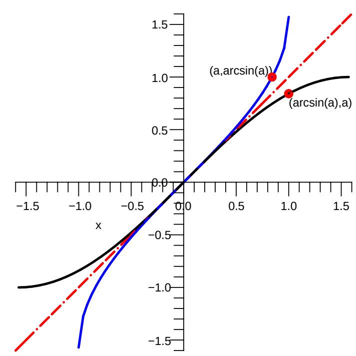


Figure 3.5.2:

To find $(\arcsin(x))'$, we proceed exactly we did when finding $(\log_e(x))'$. Let $y=\arcsin(x)$, then

$$x=\sin(y). \tag{3.5.3}$$

Differentiating with respect to x gives

$$1=\frac{d(x)}{dx}=\frac{d(\sin(y))}{dx}=\cos(y)\frac{dy}{dx}.$$

Thus

$$\frac{dy}{dx}=\frac{1}{\cos(y)}. \tag{3.5.4}$$

All that is left is to use the relationship $\sin(y)=x$ to express $\cos(y)$ in terms of x.

Figure 3.5.3 displays the diagram that defines the sine of an angle. The line segment AB represents $\cos(y)$ and the line segment BC represents $\sin(y)$. Observe that the cosine is positive for angles y in $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$, the first and fourth quadrants. When $x=\sin(y)$, $x^2+\cos^2(y)=1$ gives $\cos(y)=\pm\sqrt{1-x^2}$. We use the positive value: $\cos(y)=\sqrt{1-x^2}$ because arcsin is an increasing function. Consequently, by (3.5.4), we find

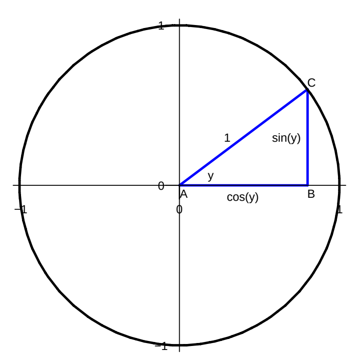


Figure 3.5.3:

Derivative of $\arcsin(x)$

$$\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}}, \qquad |x| < 1.$$

The formula for the derivative of the inverse sine should be committed to memorized.

Note at $x = 1$ or at $x = -1$, the derivative is not defined. However, for x near 1 or -1 the derivative is very large (in absolute value), telling us that the graph of the arcsine function is very steep near its two ends. That is a reflection of the fact that the graph of $\sin(x)$ is horizontal at $x = -\pi/2$ and $x = \pi/2$.

Functions such as $x^3 - x$, $x^{2/7}$, and $\frac{1}{\sqrt{1-x^2}}$ that can be written as a finite number of algebraic operations of addition, subtraction, multiplication, division, raising to a power, and extracting a root are called **algebraic functions**. Functions that cannot be written in this way, including e^x, $\cos(x)$, and $\arcsin(x)$, are known as **transcendental functions**. The derivative of $\arcsin(x)$ shows that the derivative of a transcendental function can be an algebraic function. But, the derivative of an algebraic function will always be algebraic.

An algebraic function always has an algebraic derivative.

EXAMPLE 2 Differentiate $\arcsin\left(x^2\right)$.

SOLUTION This Chain Rule is used to find this derivative:

$$\frac{d}{dx}\left(\arcsin\left(x^2\right)\right) = \frac{1}{\sqrt{1-\left(x^2\right)^2}} \cdot \frac{d}{dx}\left(x^2\right) = \frac{2x}{\sqrt{1-x^4}}.$$

◇

EXAMPLE 3 Differentiate $\frac{1}{2}\left(x\sqrt{x^2-a^2} + a^2 \arcsin\left(\frac{x}{a}\right)\right)$ where a is a constant.

SOLUTION

$$\begin{aligned}
&D\left(\frac{1}{2}\left(x\sqrt{x^2-a^2} + a^2\arcsin\left(\frac{x}{a}\right)\right)\right) \\
&= \tfrac{1}{2} D\left(\left(x\sqrt{x^2-a^2} + a^2\arcsin\left(\tfrac{x}{a}\right)\right)\right) \\
&= \tfrac{1}{2}\left(D\left(x\sqrt{x^2-a^2}\right) + a^2 D\left(\arcsin\left(\tfrac{x}{a}\right)\right)\right) \\
&= \tfrac{1}{2}\left(\left((1)\sqrt{x^2-a^2}\right) + \left(x\left(\frac{-\left(\frac{1}{2}\right)(2x)}{\sqrt{x^2-a^2}}\right)\right) + a^2\left(\frac{\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right)\right) && D(\arcsin(x)) = \tfrac{1}{\sqrt{1-}} \text{ Product and Chain} \\
&= \tfrac{1}{2}\left(\sqrt{x^2-a^2} - \tfrac{-x^2}{\sqrt{x^2-a^2}} + \tfrac{a^2}{\sqrt{a^2-x^2}}\right) && \text{algebra} \\
&= \tfrac{1}{2}\left(\tfrac{a^2-x^2-x^2+a^2}{\sqrt{x^2-a^2}}\right) && \text{common denominat} \\
&= \sqrt{x^2-a^2}
\end{aligned}$$

Note that a rather complicated-looking function can have a simple derivative. ◇

The Derivative of $\arctan(x)$

For x in $(-\pi/2, \pi/2)$ $\tan(x)$ is one-to-one and has an inverse function, $\arctan(x)$. This inverse function tells us the angle, in radians, if we know the tangent of the angle. For instance, $\arctan(1) = \pi/4$, $\arctan(0) = 0$, and $\arctan(-1) = -\pi/4$. When x is a large positive number, $\arctan(x)$ is near, and smaller than, $\pi/2$. When x is a large negative number, $\arctan(x)$ is near, and larger than, $-\pi/2$. Figure 3.5.4 shows the graph of $y = \arctan(x)$ and $y = \tan(x)$. We will not need this graph when differentiating $\arctan(x)$, but it serves as a check on the formula.

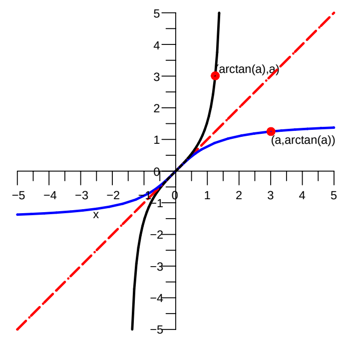


Figure 3.5.4:

See Exercise 81.

To find $(\arctan(x))'$, we again call on the Chain Rule. Starting with

$$y = \arctan(x),$$

we proceed as before:

$$\begin{aligned}
x &= \tan(y). & \\
\frac{d(x)}{dx} &= \frac{d(\tan(y))}{dx} & \text{differentiate } \textit{with respect to } x \\
1 &= \left(\sec^2(y)\right) y' & \text{Chain Rule} \\
y' &= \frac{1}{\sec^2(y)} & \text{algebra} \\
y' &= \frac{1}{1+\tan^2(y)} & \text{trigonometric identity} \\
y' &= \frac{1}{1+x^2} & x = \tan(y).
\end{aligned}$$

This derivation is summarized by a simple formula, which should be memorize.

Derivative of $\arctan(x)$

$$D(\arctan(x)) = \frac{1}{1+x^2} \qquad \text{for all inputs } x$$

EXAMPLE 4 Find $D(\arctan(3x))$.

SOLUTION By the Chain Rule

$$D(\arctan(3x)) = \frac{1}{1+(3x)^2}\frac{d(3x)}{dx} = \frac{3}{1+9x^2}.$$

◇

EXAMPLE 5 Find $D\left(x\tan^{-1}(x) - \frac{1}{2}\ln\left(1+x^2\right)\right)$.
SOLUTION

$$\begin{aligned} D\left(x\tan^{-1}(x) - \tfrac{1}{2}\ln\left(1+x^2\right)\right) &= D\left(x\tan^{-1}(x)\right) - \tfrac{1}{2}D\left(\ln\left(1+x^2\right)\right) \\ &= \left(\tan^{-1}(x) + \frac{x}{1+x^2}\right) - \frac{1}{2}\frac{2x}{1+x^2} \\ &= \tan^{-1}(x). \end{aligned}$$

◇

More on $\ln(x)$

An **antiderivative** of a function, $f(x)$, is another function, $F(x)$, whose derivative is equal to $f(x)$. That is, $F'(x) = f(x)$, and so $\ln(x)$ is an antiderivative of $1/x$. We showed that for $x > 0$, $\ln(x)$ is an antiderivative of $1/x$. But what if we needed an antiderivative of $1/x$ for negative x? The next example answers this question.

Recall that $\ln(x)$ is not defined for $x < 0$.

EXAMPLE 6 Show that for negative x, $\ln(-x)$ is an antiderivative of $1/x$.
SOLUTION Let $y = \ln(-x)$. By the Chain Rule,

$$\frac{dy}{dx} = \left(\frac{1}{-x}\right)\frac{d(-x)}{dx} = \frac{1}{-x}(-1) = \frac{1}{x}.$$

So $\ln(-x)$ is an antiderivative of $1/x$ when x is negative. ◇

In view of Example 6, $\ln|x|$ is an antiderivative of $1/x$, whether x is positive or negative.

Derivative of $\ln|x|$

$$D(\ln|x|) = \frac{1}{x} \qquad \text{for } x \neq 0.$$

We know the derivative of x^a for any rational number a. To extend this result to x^k for any number k, and positive x, we write x as $e^{\ln(x)}$.

EXAMPLE 7 Find $D(x^k)$ for $x > 0$ and any constant $k \neq 0$, rational or irrational.
SOLUTION For $x > 0$ we can write $x = e^{\ln(x)}$. Then

$$x^k = \left(e^{\ln(x)}\right)^k = \left(e^{\ln(x)}\right)^k = e^{k\ln(x)}.$$

Now, $y = e^{k\ln(x)}$ is a composite function, $y = e^u$ where $u = k\ln(x)$. Thus,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\frac{k}{x} = x^k\frac{k}{x} = kx^{k-1}.$$

◇

The preceding example shows that for positive x and any fixed exponent k, $\left(x^k\right)' = kx^{k-1}$. It probably does not come as a surprise. In fact you may wonder why we worked so hard to get the derivative of x^a when a is an integer or rational number when this example covers all exponents. We had two reasons for treating the special cases. First, they include cases when x is negative. Second, they were simpler and helped introduce the derivative.

The Derivatives of the Six Inverse Trigonometric Functions

Of the six inverse trigonometric functions, the most important are arcsin and arctan. The other four are treated in Exercises 70 to 73. Table 3.5.1 summarizes all six derivatives. There is no reason to memorize all six of these formulas. If we need, say, an antiderivative of $\frac{-1}{1+x^2}$, we do not have to use $\operatorname{arccot}(x)$. Instead, $-\arctan(x)$ would do. So, for finding antiderivatives, we don't need arccot — or any of the inverse co-functions. You should memorize the formulas for the derivatives of arcsin, arctan, and arcsec.

Note that the negative signs go with the "co-" functions.

$$\begin{array}{rclrcll} D(\arcsin(x)) &=& \dfrac{1}{\sqrt{1-x^2}} & D(\arccos(x)) &=& -\dfrac{1}{\sqrt{1-x^2}} & (-1 < x < 1) \\ D(\arctan(x)) &=& \dfrac{1}{1+x^2} & D(\operatorname{arccot}(x)) &=& -\dfrac{1}{1+x^2} & (-\infty < x < \infty) \\ D(\operatorname{arcsec}(x)) &=& \dfrac{1}{x\sqrt{x^2-1}} & D(\operatorname{arccsc}(x)) &=& -\dfrac{1}{x\sqrt{x^2-1}} & (x > 1 \text{ or } x < -1) \end{array}$$

Table 3.5.1: Derivatives of the six inverse trigonometric functions.

Summary

A geometric argument suggests that the inverse of every differentiable function is differentiable. The Chain Rule then helps find the derivatives of $\ln(x)$, $\arcsin(x)$, and $\arctan(x)$ and of the other four inverse trigonometric functions.

EXERCISES for 3.5 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 6 evaluate the function and its derivative at the given argument.

1.[R] $\arcsin(x)$; 1/2

2.[R] $\arcsin(x)$; $-1/2$

3.[R] $\arctan(x)$; -1

4.[R] $\arctan(x)$; $\sqrt{3}$

5.[R] $\ln(x)$; e

6.[R] $\ln(x)$; 1

In Exercises 7 to 26 differentiate the function.

7.[R] $\arcsin(3x)\sin(3x)$ NOTE: $\arcsin(3x)$ is *not* the reciprocal of $\sin(3x)$.

8.[R] $\arctan(5x)\tan(5x)$

9.[R] $e^{2x}\ln(3x)$

10.[R] $e^{\left(\ln(3x)x^{\sqrt{2}}\right)}$

11.[R] $x^2\arcsin(x^2)$

12.[R] $(\arcsin(3x))^2$ HINT: Recall that $\arcsin(3x) = \arcsin(3x)$.

13.[R] $\frac{\arctan(2x)}{1+x^2}$

14.[R] $\frac{x^3}{\arctan(6x)}$ NOTE: $\arctan(6x)$ is *not* the reciprocal of $\tan(6x)$.

15.[R] $\log_{10}(x)$ HINT: Express $\log_{10}$ in terms of the natural logarithm.

16.[R] $\log_x(10)$ HINT: Express $\log_x$ in terms of the natural logarithm.

17.[R] $\arcsin(3x)$

18.[R] $\arctan(2x)$

19.[R] $(\tan^{-1}(3x))^2$ NOTE: $\tan^{-1} = \arctan$.

20.[R] $(\cos^{-1}(5x))^3$ NOTE: $\cos^{-1} = \arccos$.

21.[R] $\frac{\arcsin(1+x^2)}{1+3x}$

22.[R] $\operatorname{arcsec}(x^3)$

23.[R] $x^2\arcsin(3x)$

24.[R] $\frac{\arctan(3x)}{\tan(2x)}$

25.[R] $\ln(\sin(3x))$

26.[R] $\ln(\exp(4x))$

In Exercises 27 to 64 check that the derivative of the first function is the second. The letters a, b, and c denote constants.

27.[R]

28.[R] $\frac{1}{cn}\ln\left(\frac{x^n}{ax^n+c}\right)$; $\frac{1}{x(ax^n+c)}$

HINT: To simplify the calculation, first use the fact that $\ln(p/q) = \ln(p) - \ln(q)$.

29.[R] $\frac{1}{nc}\ln\left(\frac{\sqrt{ax^n+c}-\sqrt{c}}{\sqrt{ax^n+c}+\sqrt{c}}\right)$; $\frac{1}{x\sqrt{ax^n+c}}$ (Assume $c>0$.)

30.[R] $\frac{2}{n\sqrt{-c}}\operatorname{arcsec}\left(\sqrt{\frac{ax^n}{-c}}\right)$; $\frac{1}{x\sqrt{ax^n+c}}$ (Assume $c<0$.)

31.[R] $\sqrt{ax^2+c}+\sqrt{c}\ln\left(\frac{\sqrt{ax^2+c}-\sqrt{c}}{x}\right)$; $\frac{\sqrt{ax^2+c}}{x}$ (Assume $c>0$.)

32.[R] $\sqrt{ax^2+c}-\sqrt{-c}\arctan\left(\frac{\sqrt{ax^2+c}}{\sqrt{-c}}\right)$; $\frac{\sqrt{ax^2+c}}{x}$ (Assume $c<0$.)

33.[R] $\frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)$; $\frac{1}{ax^2+bx+c}$ (Assume $b^2<4ac$.)

34.[R] $\frac{-2}{2ax+b}$; $\frac{1}{ax^2+bx+c}$ (Assume $b^2=4ac$.)

35.[R] $\frac{1}{\sqrt{b^2-4ac}}\ln\left(\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right)$; $\frac{1}{ax^2+bx+c}$ (Assume $b^2>4ac$)
HINT: Use properties of ln *before* differentiating.

36.[R] $\frac{1}{2}\left((x-a)\sqrt{2ax-x^2}+a^2\arcsin\left(\frac{x-a}{a}\right)\right)$; $\sqrt{2ax-x^2}$

37.[R] $\arccos\left(\frac{a-x}{a}\right)$; $\frac{1}{\sqrt{2ax-x^2}}$

38.[R] $\arcsin(x)-\sqrt{1-x^2}$; $\sqrt{\frac{1+x}{1-x}}$

39.[R] $2\arcsin\left(\sqrt{\frac{x-b}{a-b}}\right)$; $\frac{1}{\sqrt{x-b}\sqrt{x-a}}$

40.[R] $\frac{1}{a}\ln\left(\tan\left(\frac{ax}{2}\right)\right)$; $\frac{1}{\sin(ax)}$

41.[R] $\ln(\ln(ax))$; $\frac{1}{x\ln(ax)}$

42.[R] $\frac{-1}{(n-1)(\ln(ax))^{n-1}}$; $\frac{1}{x(\ln(ax))^n}$

43.[R] $x\arcsin(ax)+\frac{1}{a}\sqrt{1-a^2x^2}$; $\arcsin(ax)$

44.[R] $x\left(\arcsin(ax)\right)^2-2x+\frac{2}{a}\sqrt{1-a^2x^2}\arcsin(ax)$; $\left(\arcsin(ax)\right)^2$

45.[R] $\frac{1}{ab}\left(ax-\ln\left(b+ce^{ax}\right)\right)$; $\frac{1}{b+ce^{ax}}$

46.[R] $\frac{1}{a\sqrt{bc}}\arctan\left(e^{ax}\sqrt{\frac{b}{c}}\right)$; $\frac{1}{be^ax+ce^{-ax}}$ (Assume b, $c>0$.)

47.[R] $x\left(\ln(ax)\right)^2-2x\ln(ax)+2x$; $\ln^2(ax)=\left(\ln(ax)\right)^2$

48.[R] $-\frac{1}{2}\ln\left(\frac{1+\cos(x)}{1-\cos(x)}\right)$; $\frac{1}{\sin(x)}=\csc(x)$

49.[R] $\frac{1}{b^2}\left(a+bx-a\ln(a+bx)\right)$; $\frac{x}{ax+b}$ (Assume $a+bx>0$.)

50.[R] $\frac{1}{b^3}\left(a+bx-2a\ln(a+bx)-\frac{a^2}{a+bx}\right)$; $\frac{x^2}{(a+bx)^2}$, (Assume $a+bx>0$.)

51.[R] $\frac{1}{ab}\arctan\left(\frac{bx}{a}\right)$; $\frac{1}{a^2+b^2x^2}$

52.[R] $\frac{x}{2a^2(a^2+x^2)}+\frac{1}{2a^2}\arctan\left(\frac{x}{a}\right)$; $\frac{1}{(a^2+x^2)^2}$

53.[R] $\frac{1}{2a^2}\arctan\left(\frac{x^2}{a^2}\right)$; $\frac{x}{a^4+x^4}$

54.[R] $\frac{2\sqrt{x}}{b^2-\frac{2a}{b}^2}\arctan\left(\frac{b\sqrt{x}}{a}\right)$; $\frac{\sqrt{x}}{a^2+b^2x}$

55.[R] $x\,\text{arcsec}(ax) - \frac{1}{a}\ln\left(ax + \sqrt{a^2x^2-1}\right)$; $\text{arcsec}(ax)$

56.[R] $x\arctan(ax) - \frac{1}{2a}\ln\left(1+a^2x^2\right)$; $\arctan(ax)$

57.[R] $x\arccos(ax) - \frac{1}{a}\sqrt{1-a^2x^2}$; $\arccos(ax)$

58.[R] $\frac{x^2}{2}\arcsin(ax) - \frac{1}{4a^2}\arcsin(ax) + \frac{x}{2a}\sqrt{1-a^2x^2}$; $x\arcsin(ax)$

59.[R] $x\left(\arcsin(ax)\right)^2 - 2x + \frac{2}{a}\sqrt{1-a^2x^2}\arcsin(ax)$; $\left(\arcsin(ax)\right)^2$

60.[R] $\frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$; $x\cos(ax)$

61.[R] $\frac{1}{a^3}e^{ax}\left(a^2x^2 - 2ax + 2\right)$

62.[R] $\frac{1}{ab}\left(ax - \ln\left(b + ce^{ax}\right)\right)$; $\frac{1}{b+ce^{ax}}$

63.[R] $\frac{1}{a^2+b^2}e^{ax}\left(a\sin(bx) - b\cos(bx)\right)$; $e^{ax}\sin(bx)$

64.[R] $\ln\left(\sec(x) + \tan(x)\right)$; $\sec(x)$

65.[M] Find $D(\ln^3(x))$

(a) by the Chain Rule and

(b) by first writing $\ln^3(x)$ as $\ln(x)\cdot\ln(x)\cdot\ln(x)$.

Which method do you prefer? Why?

66.[M] We have used the equation $\sec^2(x) = 1 + \tan^2(x)$.

(a) Derive this equation from the equation $\cos^2(x) + \sin^2(x) = 1$.

(b) Derive the equation $\cos^2(x) + \sin^2(x) = 1$ from the Pythagorean Theorem.

67.[M] Find two functions whose derivatives are

(a) $2x$

(b) x^2

(c) $1/x$

(d) $\sqrt{x}$

68.[M] Find two functions whose derivatives are

(a) e^{3x}

(b) $\cos(x)$

(c) $\sin(x)$

(d) $1/(1+x^2)$

69.[M] This problem provides some additional experience with the development of the formula $\log_b(x) = \log_b(e)\log_e(x)$. Let $b > 0$. Recall that $\log_b(a) = \frac{\log_e(a)}{\log_e(b)}$.

(a) Show that $\log_b(e) = 1/\log_e(b)$.

(b) Conclude that $\log_b(x) = \log_b(e)\log_e(x)$.

NOTE: This result is used in Example 1.

In Exercises 70 to 73 use the Chain Rule to obtain the given derivative.

70.[M] $(\arccos(x))' = \frac{-1}{\sqrt{1-x^2}}$

71.[M] $(\text{arcsec}(x))' = \frac{1}{x\sqrt{x^2-1}}$

72.[M] $(\text{arccot}(x))' = \frac{-1}{1+x^2}$

73.[M] $(\text{arccsc}(x))' = \frac{-1}{x\sqrt{x^2-1}}$

74.[M] Verify that $D\left(2(\sqrt{x}-1)e^{\sqrt{x}}\right) = e^{\sqrt{x}}$.

75.[M]

Sam: I say that $D(\log_b(x)) = \frac{1}{x\ln(b)}$. It's simple. Let $y = \log_b(x)$. That tells me $x = b^y$. I differentiate both sides of that, getting $1 = b^y(\ln(b))y'$. So $y' = \frac{1}{b^y\ln(b)} = \frac{1}{x\ln(b)}$.

Jane: Well, not so fast. I start with the equation $\log_b(x) = (\log_b(e))\ln(x)$. So $D(\log_b(x)) = \frac{\log_b(e)}{x}$.

Sam: Something is wrong. Where did you get that equation you started with?

Jane: Just take $\log_b$ of both sides of $x = e^{\ln(x)}$.

Sam: I hope this won't be on the next midterm.

Settle this argument.

We did not need the Chain Rule to find the derivatives of inverse functions. Instead, we could have taken a geometric approach, using the "slope of the tangent line" interpretation of the derivative. When we reflect the graph of f around the line $y = x$ to obtain the graph of f^{-1}, the reflection of the tangent line to the graph of f with slope m is the tangent line to the graph of f^{-1} with slope $1/m$. (See Section 1.1.) Exercises 76 to 80 use this approach to develop formulas obtained in this section.

76.[C] Let $f(x) = \ln(x)$. The slope of the graph of $y = \ln(x)$ at $(a, \ln(a))$, $a > 0$, is the reciprocal of the slope of the graph of $y = e^x$ at $(\ln(a), a)$. Use this fact to show that the slope of the graph of $y = \ln(x)$ when $x = a$ is $1/a$.

In Exercises 77 to 80 use the technique illustrated in Exercise 76 to differentiate the given function.

77.[C] $f(x) = \arctan(x)$.

78.[C] $f(x) = \arcsin(x)$.

79.[C] $f(x) = \operatorname{arcsec}(x)$.

80.[C] $f(x) = \arccos(x)$.

81.[M]

(a) Evaluate $\lim_{x\to\infty} \frac{1}{1+x^2}$ and $\lim_{x\to-\infty} \frac{1}{1+x^2}$.

(b) What do these results tell you about the graph of the arctangent function?

82.[C] Assume you do not know the formula for $D(fg)$. Obtain it by computing the derivative of $\ln(fg)$ in two ways. First, directly. Then, using the identity $\ln(fg) = \ln(f) + \ln(g)$. Assume f and g are positive and f, g and fg are differentiable.

83.[C] Use the ideas in Exercise 82 to find $D(f/g)$. NOTE: Assume f, g, and f/g are differentiable.

84.[C] Use the approach described before Exercise 76 to find $D(x^a)$ for positive x.

85.[C]

Sam: In Exercise 82 they assumed that fg is differentiable if f and g are. I can get around that by using the fact that exp and ln are differentiable.

Jane: How so?

Sam: I write fg as $\exp(\ln(fg))$.

Jane: So?

Sam: But $\ln(fg) = \ln(f) + \ln(g)$, and that does it.

Jane: I'm lost.

Sam: Well, $fg = \exp(\ln(f) + \ln(g))$ and just use the chain rule. It's good for more than grinding out derivatives. In fact, when you differentiate both sides of my equation, you get that fg is differentiable and $(fg)'$ is $f'g + fg'$.

Jane: Why wouldn't the authors use this approach?

Sam: It would make things too easy and reveal that calculus is all about e, exponentials, and logarithms. (I peeked at Chapter 12 and saw that you can even get sine and cosine out of e^x.)

Is Sam's argument correct? If not, identify where it is incorrect.

3.6 Antiderivatives and Slope Fields

So far in this chapter we have started with a function and found its derivative. In this section we will go in the opposite direction: Given a function f, we will be interested in finding a function F whose derivative is f. Why? Because this procedure of going from the derivative back to the function plays a central role in **integral calculus**. Chapter 6 describes several ways to find antiderivatives.

Some Antiderivatives

EXAMPLE 1 Find an antiderivative of x^6.

SOLUTION When we differentiate x^a we get ax^{a-1}. The exponent in the derivative, $a-1$, is one less than the original exponent, a. So we expect an *anti*derivative of x^6 to involve x^7.

Now, $(x^7)' = 7x^6$. This means x^7 is an antiderivative of $7x^6$, not of x^6. We must get rid of that coefficient of 7 in front of x^6. To accomplish this, divide x^7 by 7. We then have

$$\begin{aligned} \left(\tfrac{x^7}{7}\right)' &= \tfrac{7x^6}{7} && \text{because } \left(\tfrac{f}{C}\right)' = \tfrac{f'}{C} \\ &= x^6 && \text{canceling common factor 7 from numerator and denominator.} \end{aligned}$$

We can state that $\frac{1}{7}x^7$ is an antiderivative of x^6.

However, $\frac{1}{7}x^7$ is not the only antiderivative of x^6. For instance,

$$\left(\frac{1}{7}x^7 + 2011\right)' = \frac{1}{7}7x^6 + 0 = x^6.$$

We can add any constant to $\frac{1}{7}x^7$ and the result is always an antiderivative of x^6. ◇

A constant added to any antiderivative of a function f gives another antiderivative of f.

As Example 1 suggests, if $F(x)$ is an antiderivative of $f(x)$ so is $F(x)+C$ for any constant C.

The reasoning in this example suggests that $\frac{1}{a+1}x^{a+1}$ is an antiderivative of x^a. This formula is meaningless when $a+1=0$. We have to expect a different formula for antiderivatives of $x^{-1} = \frac{1}{x}$. In Section 3.5 we saw that $(\ln(x))' = 1/x$. That's one reason the function $\ln(x)$ is so important: it provides an antiderivative for $1/x$.

Power Rule for Antiderivatives

For any number a, except -1, the antiderivatives of x^a are

$$\frac{1}{a+1}x^{a+1}+C \quad \text{for any constant } C.$$

The antiderivatives of $x^{-1}=\frac{1}{x}$ are, when $x>0$,

$$\ln(x)+C \quad \text{for any constant } C.$$

Every time you compute a derivative, you are also finding an antiderivative. For instance, since $D(\sin(x)) = \cos(x)$, $\sin(x)$ is an antiderivative of $\cos(x)$. So is $\sin(x)+C$ for any constant C. There are tables of antiderivatives that go on for hundreds of pages. Here is a miniature table with entries corresponding to the derivatives that we have found so far.

Search Google for "antiderivative table".

Function (f)	Antiderivative (F)	Comment
x^a	$\frac{1}{a+1}x^{a+1}$	for $a \neq -1$
$x^{-1}=\frac{1}{x}$	$\ln(x)$	
e^x	e^x	
$\cos(x)$	$\sin(x)$	
$\sin(x)$	$-\cos(x)$	
$\sec^2(x)$	$\tan(x)$	see Example 8 in Section 3.3
$\sec(x)\tan(x)$	$\sec(x)$	see Example 11 in Section 3.3
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$	see Section 3.4
$\frac{1}{1+x^2}$	$\arctan(x)$	see Section 3.4

Table 3.6.1: Miniature table of antiderivatives ($F'=f$).

An **elementary function** is a function that can be expressed in terms of polynomials, powers, trigonometric functions, exponentials, logarithms, and compositions. The derivative of an elementary function is elementary. We might expect that every elementary function would have an antiderivative that is also elementary.

Joseph Liouville (1809–1882)

In 1833 Joseph Liouville proved beyond a shadow of a doubt that there are elementary functions that do not have elementary antiderivatives. Elementary functions that do not have elementary antiderivatives include

e^{-x^2} is important in statisticians' **bell curve**

$$e^{x^2} \qquad \frac{\sin(x)}{x} \qquad x\tan(x) \qquad \sqrt{x}\sqrt[3]{1+x} \qquad \sqrt[4]{1+x^2}$$

There are two types of elementary functions: the **algebraic** and the **transcendental**. Algebraic functions consist of polynomials, quotients of polynomials (the rational functions), and all functions that can be built up by the four operations of algebra and taking roots. For instance, $\frac{\sqrt{x+\sqrt[3]{x}+x^2}}{(1+2x)^5}$ is algebraic; while functions such as $\sin(x)$ and 2^x are not algebraic. These functions are called **transcendental**.

The four operations of algebra are $+$, $-$, $\times$ and $/$.

The table of antiderivatives will continue to expand as more derivatives are obtained in the rest of Chapter 3. The importance of antiderivatives will be revealed in Chapter 5. Specific techniques for finding them are developed in Chapter 8. (See Exercise 1.)

It is difficult to tell whether a given elementary function has an elementary antiderivative. For instance, $x\sin(x)$ does, namely $-x\cos(x)+\sin(x)$, as you may readily check; but $x\tan(x)$ does not. The function e^{x^2} does not, as mentioned earlier. However, $e^{\sqrt{x}}$, which looks more frightening, does have an elementary antiderivative. (See Exercise 74.)

Reference: http://en.wikipedia.org/wiki/Risch_algorithm

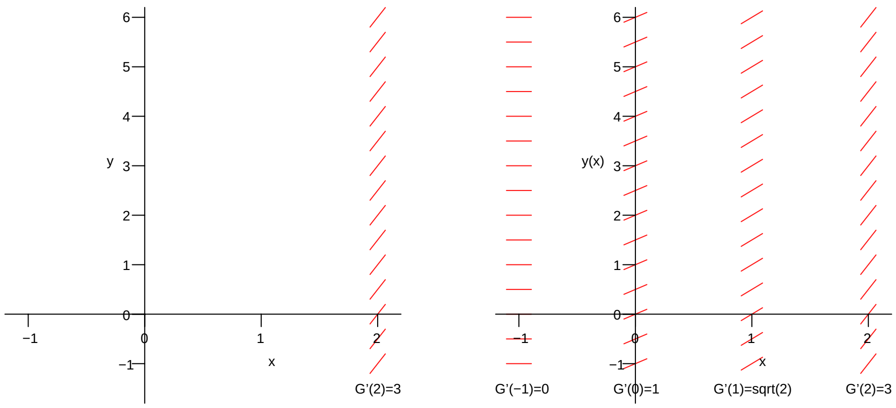
There are algorithms implemented in software on computers, hand-held devices, and calculators that can answer this question. The most well-known is the **Risch algorithm**, developed in 1968, based on differential equations and abstract algebra. A Google search for "risch antiderivative elementary symbolic" produces links related to the Risch algorithm.

Picturing Antiderivatives

If it is not possible to find an explicit formula for the antiderivative of many (most) elementary functions, why do we believe that these functions have antiderivatives? This section puts the answer directly in front of your eyes.

The **slope field** for a function $f(x)$ is made of short line segments with slope $f(x)$ at selected points (x, y). By drawing a slope field you can get a feel for the graph of the antiderivatives of $f(x)$. The fact that an antiderivative can be graphed is strong evidence that the antiderivative does exist. In Chapter 5 we will show that every continuous function has an antiderivative — which may not be elementary.

EXAMPLE 2 Imagine that you are looking for an antiderivative $F(x)$ of $\sqrt{1+x^3}$. You want $F'(x)$ to be $\sqrt{1+x^3}$. Or, to put it geometrically, you want the slope of the curve $y = F(x)$ to be $\sqrt{1+x^3}$. For instance, when $x = 2$, you want the slope to be $\sqrt{1+x^3} = 3$. We do not know what $F(2)$ is, but at least we can draw a short piece of the tangent line at all points for which $x = 2$; they all have slope 3. (See Figure 3.6.1(a).) When $x = 1$, $\sqrt{1+x^3} = \sqrt{2} \approx 1.4$. So we draw short lines with slope $\sqrt{2}$ on the vertical line $x = 1$. When $x = 0$, $\sqrt{1+x^3} = 1$; the tangent lines for $x = 0$ all have slope 1. When $x = -1$, the slopes are $\sqrt{1+x^3} = 0$ so the tangent lines are all horizontal. (See Figure 3.6.1(b).)

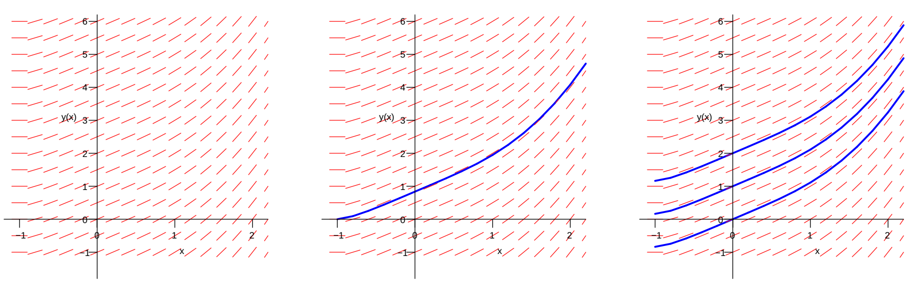


(a) (b)

Figure 3.6.1:

The plot of a slope field is most commonly made with the aid of a graphing device. Some graphing calculators have this facility and there are a number of software products for creating a computer-generated plot of a slope field. These automatic plotters have the patience to plot many line segments. A typical slope field is shown in Figure 3.6.2(a).

For a sample of available resources, search Google for "calculus slope field plot".



(a) (b) (c)

Figure 3.6.2: (a) Slope field for $f(x) = \sqrt{1 + x^3}$. (b) Includes the antiderivative with $F(-1) = 0$. (c) Shows two more antiderivatives of $f(x)$.

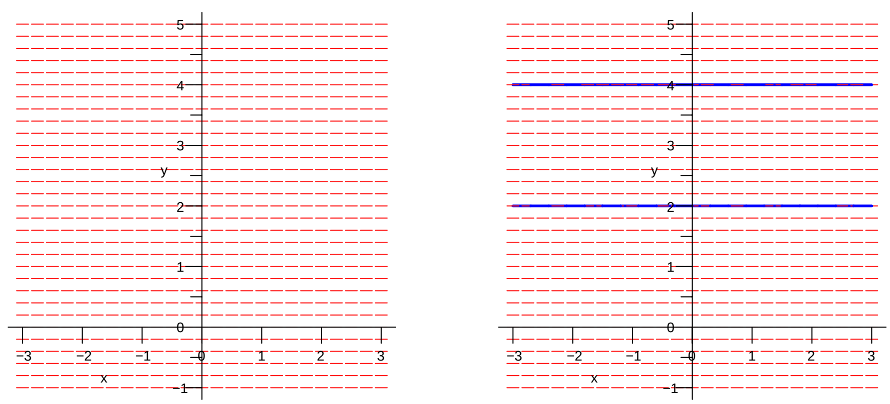
You can almost see the curves that follow the slope field for $f(x) = \sqrt{1 + x^3}$. Start at a point, say $(-1, 0)$. At this point the slope is $F'(-1) = f(-1) = 0$ so the curve starts moving horizontally to the right. As soon as the curve leaves this initial point the slope, as given by $F'(x) = f(x)$, becomes slightly positive. This pushes the curve upward. The slope continues to increase as x increases. The curve in Figure 3.6.2(b) is the graph of the antiderivative of $f(x) = \sqrt{1 + x^3}$ which equals 0 when x is -1.

If you start from a different initial point, you will obtain a different antiderivative. Three antiderivatives are shown in Figure 3.6.2(c). Many other antiderivatives for $f(x) = \sqrt{1 + x^3}$ are visible in the slope field. None of these functions is elementary. ◇

Example 2 suggests that different antiderivatives of a function differ by a constant: the graph of one is simply the graph of the other raised or lowered by their constant difference. In particular, the constant functions are the only antiderivatives of the zero function. In Section 4.1 these assertions will be justified, using only properties of continuous functions and derivatives.

In fact, the constant functions are the only antiderivatives of the zero function. Example 3 presents a good case for this, but the complete justification of this result will not be developed until Section 4.1.

EXAMPLE 3 Draw the slope field for $\frac{dy}{dx} = 0$.
SOLUTION Since the slope is 0 everywhere, each of the tangent lines is represented by a horizontal line segment, as in Figure 3.6.3(a). In Figure 3.6.3(b)



(a) (b)

Figure 3.6.3:

two possible antiderivatives of 0 are shown, namely the constant functions $f(x) = 2$ and $g(x) = 4$. ◇

The derivative of every constant function is zero. Example 3 suggests that constant functions are the only It appears that the only functions whose derivatives are 0 everywhere are the constant functions. This basic result will be established using the definitions and theorems of calculus in Section 3.7.

Summary

The *antiderivative* was introduced as the inverse operation of differentiation. If $F' = f$, then F is an antiderivative of f; so is $F + C$ for any constant C.

We introduced the notion of an *elementary function*. Such a function is built up from polynomials, logarithms, exponentials, and the trigonometric functions by the four operations $+$, $-$, $\times$, $/$, and the most important operation, composition. While the derivative of an elementary function is elementary, its antiderivative does not need to be elementary. Each elementary function is either algebraic or transcendental. Algebraic functions, such as x^2 and $(x+3)/(x^3-2x+4)$, are built from the operations of algebra, starting with the function x; transcendental functions, such as 2^x and $\sin(x)$ are the non-algebraic functions.

We showed how a *slope field* can help analyze an antiderivative even though we may not know a formula for it. Slope fields appear later, in Section 6.4 when we discover one of the most important theorems of calculus and when we study differential equations in Chapter 13.

EXERCISES for 3.6 *Key:* R–routine, M–moderate, C–challenging

1.[R]

(a) Verify that $-x\cos(x)+\sin(x)$ is an antiderivative of $x\sin(x)$.

(b) Spend at least one minute and at most ten minutes trying to find an antiderivative of $x\tan(x)$.

In Exercises 2 to 11 give two antiderivatives for each given function.

2.[R] x^3

3.[R] x^4

4.[R] x^{-2}

5.[R] $\frac{1}{x^3}$

6.[R] $\sqrt[3]{x}$

7.[R] $\frac{2}{x}$

8.[R] $\sec(x)\tan(x)$

9.[R] $\sin(x)$

10.[R] e^{-x}

11.[R] $\sin(2x)$

In Exercises 12 to 20

(a) draw the slope field for the given derivative,

(b) then draw the curves that they suggest for two functions $F(x)$.

12.[R] $f'(x)=2$

13.[R] $F'(x)=x$

14.[R] $F'(x)=\frac{-x}{2}$

15.[R] $F'(x)=\frac{1}{x}$, $x>0$

16.[R] $F'(x)=\cos(x)$

17.[R] $F'(x)=\sqrt{x}$

18.[R] $F'(x)=e^{-x}$, $x>0$

19.[R] $F'(x)=1/x^2$, $x\neq 0$

20.[R] $F'(x)=1/(x-1)$, $x\neq 1$

In Exercises 21 to 30 use differentiation to check that the first function is an antiderivative of the second function.

21.[R] $2x\sin(x) - (x^2 - 2)\cos(x)$; $x^2\sin(x)$

22.[R] $(4x^3 - 24x)\sin(x) - (x^4 - 12x^2 + 24)\cos(x)$; $x^4\sin(x)$

23.[R] $\frac{1}{x^3}$; $\frac{-1}{2x^2}$

24.[R] $\frac{1}{x^{3/2}}$; $\frac{-2}{\sqrt{x}}$

25.[R] $(x-1)e^x$; xe^x

26.[R] $(x^2 - 2x + 2)e^x$; x^2e^x

27.[R] $\frac{1}{2}e^u(\sin(u) - \cos(u))$; $e^u\sin(u)$

28.[R] $\frac{1}{2}e^u(\sin(u) + \cos(u))$; $e^u\cos(u)$

29.[R] $\frac{x}{2} - \frac{\sin(x)\cos(x)}{2}$; $\sin^2(x)$

30.[R] $2x\cos(x) - (x^2 - 2)\sin(x)$; $x^2\cos(x)$

31.[M]

(a) Draw the slope field for $\frac{dy}{dx} = e^{-x^2}$.

(b) Draw the graph of the antiderivative of e^{-x^2} that passes through the point $(0,1)$.

32.[M]

(a) Draw the slope field for $\frac{dy}{dx} = \frac{\sin(x)}{x}$, $x \neq 0$, and $\frac{dy}{dx} = 1$ for $x = 0$.

(b) What is the slope for any point on the y-axis?

(c) Draw the graph of the antiderivative of $f(x)$ that passes through the point $(0,1)$.

33.[C] A table of antiderivatives lists two antiderivatives of $\frac{1}{x^2(a+bx)}$, where a and b are constants, namely

$$\frac{-1}{a^2}\left(\frac{a+bx}{x} - b\ln\left(\frac{a+bx}{x}\right)\right) \quad \text{and} \quad -\frac{1}{ax}+\frac{b}{a^2}\ln\left(\frac{a+bx}{x}\right).$$

Assume $\frac{a+bx}{x} > 0$.

(a) By differentiating both expressions, show that both are correct.

(b) Show that the two expressions differ by a constant, by finding their difference.

34.[C] If $F(x)$ is an antiderivative of $f(x)$, find a function that is an antiderivative of

(a) $g(x) = 2f(x)$,

(b) $h(x) = f(2x)$.

35.[C]

(a) Draw the slope field for $dy/dx = -y$.

(b) Draw the graph of the function $y = f(x)$ such that $f(0) = 1$ and $dy/dx = -y$.

(c) What do you think $\lim_{x\to\infty} f(x)$ is?

3.7 Motion and the Second Derivative

In an official drag race Melanie Troxel reached a speed of 324 miles per hour, which is about 475 feet per second, in a mere 4.539 seconds. By comparison, a 1968 Fiat 850 Idromatic could reach a speed of 60 miles per hour in 25 seconds and a 1997 Porsche 911 Turbo S in a mere 3.6 seconds.

Source: `http://web.missouri.edu/~apcb20/times.html`. Numerical acceleration data for other cars can be found with a web search for "automobile acceleration."

Since Troxel increased her speed from 0 feet per second to 475 feet per second in 4.539 seconds her speed was increasing at the rate of $\frac{475}{4.539} \approx 105$ feet per second per second, assuming she kept the motor at maximum power throughout the time interval. That acceleration is more than three times the acceleration due to gravity at sea level (32 feet per second per second). Ms. Troxel must have felt quite a force as her seat pressed against her back.

This brings us to the formal definition of **acceleration** and an introduction to higher derivatives.

In Sections 3.1 and 3.2 we saw that the velocity of an object moving on a line is represented by a derivative. In this section we examine the acceleration mathematically.

Acceleration

The sign of the velocity indicates direction. **Speed**, the absolute value of velocity, does not indicate direction.

Velocity is the rate at which position changes. The rate at which velocity changes is called **acceleration**, denoted a. Thus if $y = f(t)$ denotes position on a line at time t, then the derivative $\frac{dy}{dt}$ equals the velocity, and the derivative of the derivative equals the acceleration. That is,

$$v = \frac{dy}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

The derivative of the derivative of a function $y = f(x)$ is called the **second derivative**. It is denoted many different ways, including:

$$\frac{d^2y}{dx^2}, \quad D^2y, \quad y'', \quad f'', \quad D^2f, \quad f^{(2)}, \quad \text{or} \quad \frac{d^2f}{dx^2}.$$

If $y = f(t)$, where t denotes the time, the first and second derivatives dy/dt, and d^2y/dt^2 are sometimes denoted $\dot{y}$ and $\ddot{y}$, respectively.

For instance, if $y = x^3$,

$$\frac{dy}{dx} = 3x^2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x$$

Other ways of denoting the second derivative of this function are

$$D^2(x^3) = 6x, \ \frac{d^2(x^3)}{dx^2} = 6x, \quad \text{and} \quad (x^3)'' = 6x.$$

The table in the margin lists dy/dx, the first derivative, and d^2y/dx^2, the second derivative, for a few functions. Most functions f met in applications of calculus can be differentiated repeatedly in the sense that Df exists, the derivative of Df, namely, D^2f, exists, the derivative of D^2f exists, and so on.

y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
x^3	$3x^2$	$6x$
$\frac{1}{x}$	$\frac{-1}{x^2}$	$\frac{2}{x^3}$
$\sin(5x)$	$5\cos(5x)$	$-25\sin(5x)$

The derivative of the second derivative is called the **third derivative** and is denoted many ways, such as

$$\frac{d^3y}{dx^3}, \quad D^3y, \quad y''', \quad f''', \quad f^{(3)}, \quad \text{or} \quad \frac{d^3f}{dx^3}.$$

The fourth derivative is defined similarly, as the derivative of the third derivative. In the same way we can define the n^{th} derivative for any positive integer n and denote this by such symbols as

$$\frac{d^ny}{dx^n}, \quad D^ny, \quad f^{(n)}, \quad \text{or} \quad \frac{d^nf}{dx^n}.$$

It is read as "the n^{th} derivative with respect to x." For instance, if $f(x) = 2x^3 + x^2 - x + 5$, we have

$$\begin{aligned} f^{(1)}(x) &= 6x^2 + 2x - 1 \\ f^{(2)}(x) &= 12x + 2 \\ f^{(3)}(x) &= 12 \\ f^{(4)}(x) &= 0 \\ f^{(n)}(x) &= 0 \qquad\qquad \text{for } n \geq 5. \end{aligned}$$

EXAMPLE 1 Find $D^n(e^{-2x})$ for each positive integer n.

SOLUTION

$$\begin{aligned} D^1\left(e^{-2x}\right) &= D\left(e^{-2x}\right) = -2e^{-2x} \\ D^2\left(e^{-2x}\right) &= D\left(-2e^{-2x}\right) = (-2)^2e^{-2x} \\ D^3\left(e^{-2x}\right) &= D\left((-2)^2e^{-2x}\right) = (-2)^3e^{-2x} \end{aligned}$$

At each differentiation another (-2) becomes part of the coefficient. Thus

$$D^n\left(e^{-2x}\right) = (-2)^ne^{-2x}.$$

This can also be written

The power $(-1)^n$ records a "plus" if n is even and a "minus" if n is odd.

$$D^n\left(e^{-2x}\right) = (-1)^n2^ne^{-2x}.$$

◇

Finding Velocity and Acceleration from Position

EXAMPLE 2 A falling rock drops $16t^2$ feet in the first t seconds. Find its velocity and acceleration.

SOLUTION Place the y-axis in the usual position, with 0 at the beginning of the fall and the part with positive values above 0, as in Figure 3.7.1. At time t the object has the y coordinate

$$y = -16t^2.$$

The velocity is $v = (-16t^2)' = -32t$ feet per second, and the acceleration is $a = (-32t)' = -32$ feet per second per second. The velocity changes at a constant rate. That is, the acceleration is constant. ◇

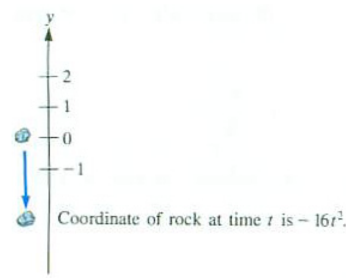


Figure 3.7.1:

Finding Position from Velocity and Acceleration

To calculate the position of a moving object at any time it is enough to know the object's acceleration at all times, its initial position, and its initial velocity. This will be demonstrated in the next two examples in the special case that the acceleration is constant. In the first example, the acceleration is 0.

EXAMPLE 3 In the simplest motion, no forces act on a moving particle, hence its acceleration is 0. Assume that a particle is moving on the x-axis and no forces act on it. Let its location at time t seconds be $x = f(t)$ feet. See Figure 3.7.2. If at time $t = 0$, $x = 3$ feet and the velocity is 5 feet per second, determine $f(t)$.

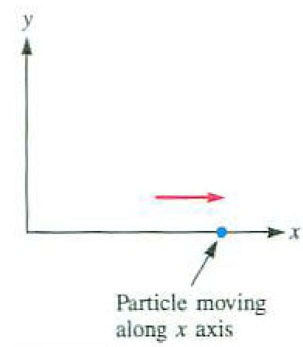


Figure 3.7.2:

SOLUTION The assumption that no force operates on the particle tells us that there is no acceleration: $d^2x/dt^2 = 0$. Call the velocity v. Then

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = 0$$

Now, v is a function of time whose derivative is 0. At the end of Section 3.6 we saw that constant functions are the antiderivatives of 0. Thus, v must be constant:

$$v(t) = C \qquad \text{for some constant } C.$$

Since $v(0) = 5$, the constant C must be 5.

To find the position x as a function of time, note that its derivative is the velocity. Hence

$$\frac{dx}{dt} = 5$$

Similar reasoning tells us that $x = f(t)$ has the form

$$x = 5t + K \qquad \text{for some constant } K.$$

Now, when $t = 0$, $x = 3$. Thus $K = 3$. In short, at time t seconds, the particle is at $x = 5t + 3$ feet. ◇

The next example concerns the case in which the acceleration is constant, but not zero.

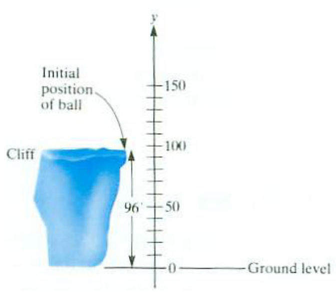


Figure 3.7.3:

EXAMPLE 4 A ball is thrown straight up, with an initial speed of 64 feet per second, from a cliff 96 feet above a beach. Where is the ball t seconds later? When does it reach its maximum height? How high above the beach does the ball rise? When does the ball hit the beach? Assume that there is no air resistance and that the acceleration due to gravity is constant.

SOLUTION Introduce a vertical coordinate axis to describe the position of the ball. It is more natural to call it the y-axis, and so the velocity is dy/dt and acceleration is d^2y/dt^2. Place the origin at ground level and let the positive part of the y-axis be above the ground, as in Figure 3.7.3. At time $t = 0$, the velocity dy/dt is 64, since the ball is thrown up at a speed of 64 feet per second. As time increases, dy/dt decreases from 64 to 0 (when the ball reaches the top of it path and begins its descent) and continues to decrease through larger and larger negative values as the ball falls to the ground. Since v is decreasing, the acceleration dv/dt is negative. The (constant) value of dv/dt, gravitational acceleration, is approximately -32 feet per second per second.

If it had been thrown down dy/dt would be -64.

From the equation

$$a = \frac{dv}{dt} = -32,$$

Velocity is an antiderivative of acceleration.

it follows that

$$v = -32t + C,$$

where C is some constant. To find C, recall that $v = 64$ when $t = 0$. Thus

$$64 = -32 \cdot 0 + C,$$

and $C = 64$. Hence $v = -32t + 64$ for any time t until the ball hits the beach.

So we have

$$\frac{dy}{dt} = v = -32t + 64.$$

Since the position function y is an antiderivative of the velocity, $-32t + 64$, we have

$$y(t) = -16t^2 + 64t + K,$$

where K is a constant. To find K, make use of the fact that $y = 96$ when $t = 0$. Thus

$$96 = -16 \cdot 0^2 + 64 \cdot 0 + K,$$

and $K = 96$.

We have obtained a complete description of the position of the ball at any time t while it is in the air:

$$y = -16t^2 + 64t + 96.$$

This, together with $v = -32t + 64$, provides answers to many questions about the ball's flight. (As a check, note that when $t = 0$, $y = 96$, the initial height.)

When does it reach its maximum height? When it is neither rising nor falling. In other words, the velocity is neither positive nor negative, but must be 0. The velocity is zero when $-32t + 64 = 0$, which is when $t = 2$ seconds.

How high above the ground does the ball rise? Compute y when $t = 2$. This gives $-16 \cdot 2^2 + 64 \cdot 2 + 96 = 160$ feet. (See Figure 3.7.4.)

When does the ball hit the beach? When $y = 0$. Find t such that

$$y = -16t^2 + 64t + 96 = 0$$

Division by -16 yields the simpler equation $t^2 - 4t - 6 = 0$, which has the solutions

$$t = \frac{4 \pm \sqrt{16 + 24}}{2} = 2 \pm \sqrt{10}.$$

Since $2 - \sqrt{10}$ is negative and the ball cannot hit the beach before it is thrown, the only physically meaningful solution is $2 + \sqrt{10}$. The ball lands $2 + \sqrt{10}$ seconds after it is thrown; it is in the air for about 5.2 seconds.

The graphs of position, velocity, and acceleration as functions of time provide another perspective on the motion of the ball, as shown in Figure 3.7.4.

◇

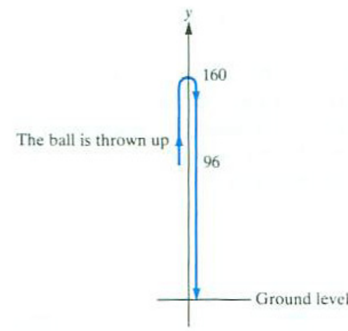


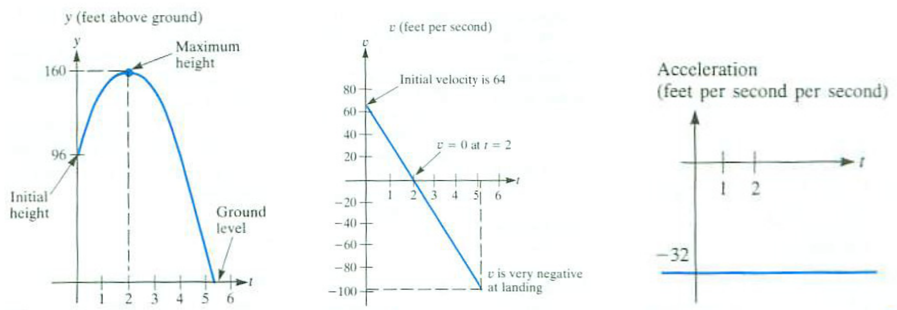
Figure 3.7.4:

Reasoning like that in Examples 3 and 4 establishes the following description of motion in all cases where the acceleration is constant.

OBSERVATION *(Motion Under Constant Acceleration)* Assume that a particle moving on the y-axis has a constant acceleration a at any time. Assume that at time $t = 0$ it has an initial velocity v_0 and has the initial y-coordinate y_0. Then at any time $t \geq 0$ its y-coordinate is

$$y = \frac{a}{2}t^2 + v_0 t + y_0.$$

In Example 3, $a = 0$, $v_0 = 5$, and $y_0 = 3$; in Example 4, $a = -32$ $v_0 = 64$,



(a) (b) (c)

Figure 3.7.5: (a) Position, (b) velocity, and (c) acceleration for the object in Example 4.

and $y_0 = 96$. Note that the data must be given in consistent units, for instance, all in meters or all in feet.

Summary

We defined the higher derivatives of a function. They are obtained by repeatedly differentiating. The second derivative is the derivative of the derivative, the third derivative being the derivative of the second derivative, and so on. The first and second derivatives, $D(f)$ and $D^2(f)$, are used in many applications. We used these two derivatives to analyze motion under constant acceleration. The higher derivatives will be used in Section 5.4 to describe the error in approximately a function by a polynomial, and in Section 8.4 in estimating the error when calculating areas.

EXERCISES for 3.7 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 16 find the first and second derivatives of the given functions.

1.[R] $y = 2x + 3$

2.[R] $y = e^{-x^3}$

3.[R] $y = x^5$

4.[R] $y = \ln(6x + 1)$

5.[R] $y = \sin(\pi x)$

6.[R] $y = 4x^3 - x^2 + x$

7.[R] $y = \frac{x}{x+1}$

8.[R] $y = \frac{x^2}{x-1}$

9.[R] $y = x\cos(x^2)$

10.[R] $y = \frac{x}{\tan(3x)}$

11.[R] $y = (x - 2)^4$

12.[R] $y = (x + 1)^3$

13.[R] $y = e^{3x}$

14.[R] $y = \tan(x^2)$

15.[R] $y = x^2 \arctan(3x)$

16.[R] $y = -\frac{\arcsin(2x)}{x^2}$

17.[R] Use calculus, specifically derivatives, to restate the following reports about the Leaning Tower of Pisa.

(a) "Until 2001, the tower's angle from the vertical was increasing more rapidly."

(b) "Since 2001, the tower's angle from the vertical has not changed."

HINT: Let $\theta = f(t)$ be the angle of deviation from the vertical at time t. NOTE: Incidently, the tower, begun in 1174 and completed in 1350, is 179 feet tall and leans about 14 feet from the vertical. Each day it leaned on the average, another $\frac{1}{5000}$ inch until the tower was propped up in 2001.

Exercises 18 to 20 concern Example 4.

18.[R]

(a) How long after the ball in Example 4 is thrown does it pass by the top of the hill?

(b) What are its speed and velocity at this instant?

19.[R] Suppose the ball in Example 4 had simply been dropped from the cliff. Find the position y as a function of time. How long would it take the ball to reach the beach?

20.[R] In view of the result of Exercise 19, provide a physical interpretation of the three terms on the right-hand side of the formula $y = -16t^2 + 64t + 96$.

21.[R] At time $t = 0$ a particle is at $y = 3$ feet and has a velocity of -3 feet per second; it has a constant acceleration of 6 feet per second per second. Find its position at any time t.

22.[R] At time $t = 0$ a particle is at $y = 10$ feet and has a velocity of 8 feet per second; it has a constant acceleration of -8 feet per second per second.

(a) Find its position at any time t.

(b) What is its maximum y coordinate.

23.[R] At time $t = 0$ a particle is at $y = 0$ feet and has a velocity of 0 feet per second. Find its position at any time t if its acceleration is always -32 feet per second per second.

24.[R] At time $t = 0$ a particle is at $y = -4$ feet and has a velocity of 6 feet per second; it has a constant acceleration of -32 feet per second per second.

(a) Find its position at any time t.

(b) What is its largest y coordinate.

In Exercises 25 to 34 find the given derivatives.

25.[R] $D^3\left(5x^2 - 2x + 7\right)$.

26.[R] $D^4(\sin(2x))$.

27.[R] $D^n(e^x)$.

28.[R] $D(\sin(x))$, $D^2(\sin(x))$, $D^3(\sin(x))$, and $D^4(\sin(x))$.

29.[R] $D(\cos(x))$, $D^2(\cos(x))$, $D^3(\cos(x))$, and $D^4(\cos(x))$.

30.[R] $D(\ln(x))$, $D^2(\ln(x))$, $D^3(\ln(x))$, and $D^4(\ln(x))$.

31.[R] $D^4(x^4)$ and $D^5(x^4)$.

32.[M] $D^{200}(\sin(x))$

33.[M] $D^{200}(e^x)$

34.[M] $D^2(5^x)$

35.[M] Find all functions f such that $D^2(f) = 0$ for all x.

36.[M] Find all functions f such that $D^3(f) = 0$ for all x.

37.[M] A jetliner begins its descent 120 miles from the airport. Its velocity when the descent begins is 500 miles per hour and its landing velocity is 180 miles per hour. Assuming a constant deceleration, how long does the descent take?

38.[M] Let $y = f(t)$ describe the motion on the y-axis of an object whose acceleration has the constant value a. Show that

$$y = \frac{a}{2}t^2 + v_0 t + y_0$$

where v_0 is the velocity when $t = 0$ and y_0 is the position when $t = 0$.

39.[M] Which has the highest acceleration? Melanie Troxel's dragster, a 1997 Porsche 911 Turbo S, or an airplane being launched from an aircraft carrier? The plane reaches a velocity of 180 miles per hour in 2.5 seconds, within a distance of 300 feet. HINT: Assume each acceleration is constant.

40.[M] Why do engineers call the third derivative of position with respect to time the **jerk**?

41.[C] Give two functions f such that $D^2(f) = 9f$. Neither should be a constant multiple of the other.

42.[C] Give two functions f such that $D^2(f) = -4f$. Neither should be a constant multiple of the other.

43.[C] A car accelerates with constant acceleration from 0 (rest) to 60 miles per hour in 15 seconds. How far does it travel in this period? NOTE: Be sure to do your computations either all in seconds, or all in hours; for instance, 60 miles per hour is 88 feet per second.

44.[C] Show that a ball thrown straight up from the ground takes as long to rise as to fall back to its initial position. How does the velocity with which it strikes the ground compare with its initial velocity? How do the initial and landing speeds compare?

3.8 Precise Definition of Limits at Infinity: $\lim_{x\to\infty} f(x) = L$

One day a teacher drew on the board the graph shown in Figure 3.8.1. It is the graph of $x/2 + \sin(x)$. Then the class was asked my class whether they thought that

$$\lim_{x\to\infty} f(x) = \infty.$$

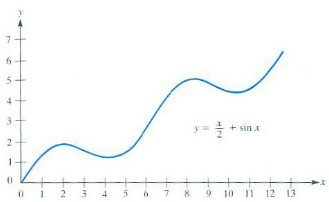


Figure 3.8.1:

A third of the class voted "No" because "it keeps going up and down." A third voted "Yes" because "the function tends to get very large as x increases." A third didn't vote. Such a variety of views on such a fundamental concept suggests that we need a more precise definition of a limit than the ones developed in Sections 2.1 and 2.2. (How would you vote?)

The definitions of the limits considered in Chapter 2 used such phrases as "x approaches a," "$f(x)$ approaches a specific number," "as x gets larger," and "$f(x)$ becomes and remains arbitrarily large." Such phrases, although appealing to the intuition and conveying the sense of a limit, are not precise. The definitions seem to suggest moving objects and call to mind the motion of a pencil point as it traces out the graph of a function.

This informal approach was adequate during the early development of calculus, from Leibniz and Newton in the seventeenth century through the Bernoullis, Euler, and Gauss in the eighteenth and early nineteenth centuries. But by the mid-nineteenth century, mathematicians, facing more complicated functions and more difficult theorems, no longer could depend solely on intuition. They realized that glancing at a graph was no longer adequate to understand the behavior of functions — especially if theorems covering a broad class of functions were needed.

It was Weierstrass who developed, over the period 1841–1856, a way to define limits without any hint of motion or pencils tracing out graphs. His approach, on which he lectured after joining the faculty at the University of Berlin in 1859, has since been followed by pure and applied mathematicians throughout the world. Even an undergraduate advanced calculus course depends on Weierstrass's approach.

In this section we examine how Weierstrass would define the "limits at infinity:"

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = L.$$

In the next section we consider limits at finite points:

$$\lim_{x\to a} f(x) = L.$$

The Precise Definition of $\lim_{x\to\infty} f(x) = \infty$

Recall the definition of $\lim_{x\to\infty} f(x) = \infty$ given in Section 2.1.

Informal definition of $\lim_{x\to\infty} f(x) = \infty$

1. $f(x)$ is defined for all x beyond some number
2. As x gets large through positive values, $f(x)$ becomes and remains arbitrarily large and positive.

To take us part way to the precise definition, let us reword the informal definition, paraphrasing it in the following definition, which is still informal.

Reworded informal definition of $\lim_{x\to\infty} f(x) = \infty$

1. Assume that $f(x)$ is defined for all x greater than the number c.
2. If x is sufficiently large and positive, then $f(x)$ is necessarily large and positive.

The precise definition parallels the reworded definition.

DEFINITION *(Precise definition of* $\lim_{x\to\infty} f(x) = \infty$*)*

1. Assume the $f(x)$ is defined for all x greater than some number c.
2. For each number E there is a number D such that for all $x > D$ it is true that $f(x) > E$.

The "challenge and reply" approach to limits. Think of E as the "enemy" and D as the "defense."

Think of the number E as a challenge and D as the reply. The *larger* E is, the *larger* D must usually be. Only if a number D (which depends on E) can he found for *every* number E can we make the claim that $\lim_{x\to\infty} f(x) = \infty$. In other words, D could be expressed as a function of E. To picture the idea behind the precise definition, consider the graph in Figure 3.8.2(a) of a function f for which $\lim_{x\to\infty} f(x) = \infty$. For each possible choice of a horizontal line, say, at height E, if you are far enough to the right on the graph of f, you *stay above* that horizontal line. That is, there is a number D such that if $x > D$, then $f(x) > E$, as illustrated in Figure 3.8.2(b).

The number D in Figure 3.8.3 is not a suitable reply. It is too small since there are some values of $x > D$ such that $f(x) \leq E$.

Examples 1 and 2 illustrate how the precise definition is used.

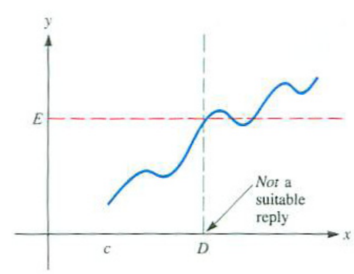
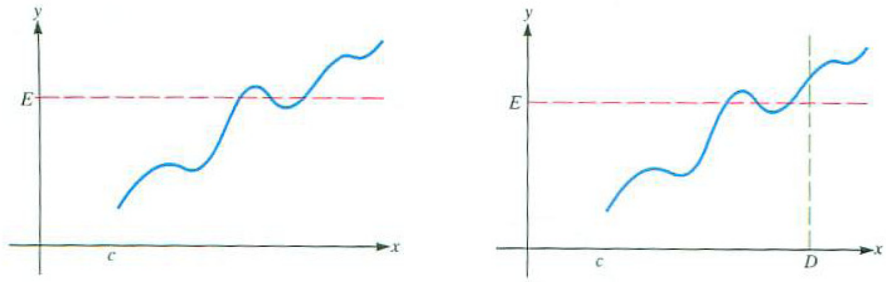


Figure 3.8.3:

EXAMPLE 1 Using the precise definition, show that $\lim_{x\to\infty} 2x = \infty$.
SOLUTION Let E be any positive number. We must show that there is a



(a) (b)

Figure 3.8.2:

number D such that whenever $x > D$ it follows that $2x > E$. (For example, if $E = 100$, then $D = 50$ would do because it is indeed the case that if $x > 50$, then $2x > 100$.) The number D will depend on E. Our goal is find a formula for D for any value of E.

Now, the inequality $2x > E$ is equivalent to

$$x > \frac{E}{2}.$$

D depends on E

In other words, if $x > E/2$, then $2x > E$. So choosing $D = E/2$ will suffice. To verify this: when $x > D$ $(= E/2)$, $2x > 2D = 2\frac{E}{2} = E$. This allows us to conclude that

$$\lim_{x\to\infty} 2x = \infty.$$

◇

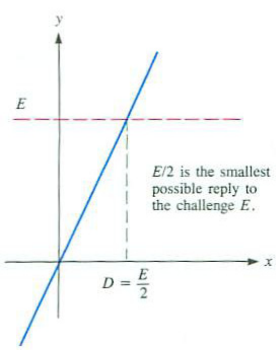


Figure 3.8.4:

In Example 1 a formula was provided for a suitable D in terms of E, namely, $D = E/2$ (see Figure 3.8.4. For instance, when challenged with $E = 1000$, the response $D = 500$ suffices. In fact, any larger value of D also is suitable. If $x > 600$, it is still the case that $2x > 1000$ (since $2x > 1200$). If one value of D is a satisfactory response to a given challenge E, then any larger value of D also is a satisfactory response.

Now that we have a precise definition of $\lim_{x\to\infty} f(x) = \infty$ we can settle the question, "Is $\lim_{x\to\infty}(x/2 + \sin(x)) = \infty$?"

EXAMPLE 2 Using the precise definition, show that $\lim_{x\to\infty} \frac{x}{2} + \sin(x) = \infty$.

SOLUTION Let E be any number. We must exhibit a number D, depending on E, such that $x > D$ forces

$$\frac{x}{2} + \sin(x) > E. \tag{3.8.1}$$

Now, $\sin(x) \geq -1$ for all x. So, if we can force

$$\frac{x}{2} + (-1) > E \tag{3.8.2}$$

then it will follow that

$$\frac{x}{2} + \sin(x) > E.$$

The smallest value of x that satisfies inequality (3.8.1) can be found as follows:

$$\begin{array}{rcll} \frac{x}{2} & > & E+1 & \text{add 1 to both sides} \\ x & > & 2(E+1) & \text{multiply by a positive constant.} \end{array}$$

Thus $D = 2(E+1)$ will suffice. That is,

D depends on E

$$\text{If } x > 2(E+1), \text{ then } \frac{x}{2} + \sin(x) > E.$$

To verify this assertion we must check that $D = 2(E+1)$ is a satisfactory reply to E. Assume that $x > D = 2(E+1)$. Then

$$\begin{array}{lrcl} & \frac{x}{2} & > & E+1 \\ \text{and} & \sin(x) & \geq & -1. \end{array}$$

Adding these last two inequalities gives

If $a > b$ and $c \geq d$, then $a + c > b + d$.

$$\begin{array}{lrcl} & \frac{x}{2} + \sin(x) & > & (E+1) + (-1) \\ \text{or simply} & \frac{x}{2} + \sin(x) & > & E, \end{array}$$

which is inequality (3.8.1). Therefore we can conclude that

$$\lim_{x\to\infty} \left(\frac{x}{2} + \sin(x)\right) = \infty.$$

As x increases, the function does *become* and *remain* large, despite the small dips downward. ◇

The Precise Definition of $\lim_{x\to\infty} f(x) = L$

Next, recall the definition of $\lim_{x\to\infty} f(x) = L$ given in Section 2.1.

L is a finite number.

Informal definition of $\lim_{x\to\infty} f(x) = L$

1. $f(x)$ is defined for all x beyond some number
2. As x gets large through positive values, $f(x)$ approaches L.

Again we reword this definition before offering the precise definition.

Reworded informal definition of $\lim_{x\to\infty} f(x) = L$

1. Assume that $f(x)$ is defined for all x greater than some number c.
2. If x is sufficiently large, then $f(x)$ is necessarily near L.

Once again, the precise definition parallels the reworded definition. In order to make precise the phrase "$f(x)$ is necessarily near L," we shall use the absolute value of $f(x) - L$ to measure the distance from $f(x)$ to L. The following definition says that "if x is large enough, then $|f(x) - L|$ is as small as we please".

DEFINITION *(Precise definition of* $\lim_{x\to\infty} f(x) = L$*)*

1. Assume the $f(x)$ is defined for all x beyond some number c.
2. For each positive number ϵ there is a number D such that for all $x > D$ it is true that

$$|f(x) - L| < \epsilon.$$

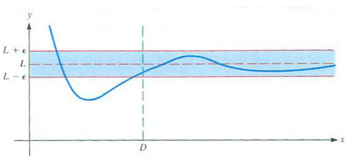


Figure 3.8.5:

Draw two lines parallel to the x-axis, one of height $L + \epsilon$ and one of height $L - \epsilon$. They are the two edges of an endless band of width 2ϵ and centered at $y = L$. Assume that for each positive ϵ, a number D can be found such that the part of the graph to the right of $x = D$ lies within the band. Then we say that "as x approaches ∞, $f(x)$ approaches L" and write

$$\lim_{x\to\infty} f(x) = L.$$

The positive number ϵ is the challenge, and D is a reply. The smaller ϵ is, the narrower the band is, and the larger D usually must be chosen. The geometric meaning of the precise definition of $\lim_{x\to\infty} f(x) = L$ is shown in Figure 3.8.5.

"ϵ" (epsilon) is the Greek letter corresponding to the English letter "e". Because mathematicians think of ϵ as being small, the number theorist, Paul Erdös, called children "epsilons."

EXAMPLE 3 Use the precise definition of "$\lim_{x\to\infty} f(x) = L$" to show that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right) = 1.$$

SOLUTION Here $f(x) = 1+1/x$, which is defined for all $x \neq 0$. The number L is 1. We must show that for each positive number ϵ, however small, there is a number D such that, for all $x > D$,

$$\left|\left(1 + \frac{1}{x}\right) - 1\right| < \epsilon. \qquad (3.8.3)$$

Inequality (3.8.3) reduces to

$$\left|\frac{1}{x}\right| < \epsilon.$$

Since we may consider only $x > 0$, this inequality is equivalent to

$$\frac{1}{x} < \epsilon. \tag{3.8.4}$$

Multiplying inequality (3.8.4) by the positive number x yields the equivalent inequality

$$1 < x\epsilon. \tag{3.8.5}$$

Division of inequality (3.8.5) by the positive number ϵ yields

$$\frac{1}{\epsilon} < x \quad \text{or} \quad x > \frac{1}{\epsilon}.$$

D depends on ϵ.

These steps are reversible. This shows that $D = 1/\epsilon$ is a suitable reply to the challenge ϵ. If $x > 1/\epsilon$, then

$$\left|\left(1 + \frac{1}{x}\right) - 1\right| < \epsilon.$$

That is, inequality (3.8.3) is satisfied.

According to the precise definition of "$\lim_{x\to\infty} f(x) = L$", we conclude that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right) = 1.$$

◇

The graph of $f(x) = 1+1/x$, shown in Figure 3.8.6, reinforces the argument. It seems plausible that no matter how narrow a band someone may place around the line $y = 1$, it will always be possible to find a number D such that the part of the graph to the right of $x = D$ stays within that band. In Figure 3.8.6 the typical band is shown shaded.

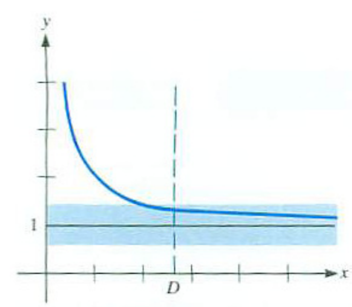


Figure 3.8.6:

The precise definitions can also be used to show that some claim about an alleged limit is false. The next example illustrates how this is done.

EXAMPLE 4 Show that the claim that $\lim_{x\to\infty} \sin(x) = 0$ is false.

SOLUTION To show that the claim is *false*, we must exhibit a challenge $\epsilon > 0$ for which no response D can be found. That is, we must exhibit a positive number ϵ such that no D exists for which $|\sin(x) - 0| < \epsilon$ for all $x > D$.

Recall that $\sin(\pi/2) = 1$ and that $\sin(x) = 1$ whenever $x = \pi/2 + 2n\pi$ for any integer n. This means that there are arbitrarily large values of x for which

$\sin(x) = 1$. This suggests how to exhibit an $\epsilon > 0$ for which no response D can be found. Simply pick the challenge ϵ to be some positive number less than or equal to 1. For instance, $\epsilon = 0.7$ will do.

For any number D there is always a number $x^* > D$ such that we have $\sin(x^*) = 1$. This means that $|\sin(x^*) - 0| = 1 > 0.7$. Hence no response can he found for $\epsilon = 0.7$. Thus the claim that $\lim_{x\to\infty} \sin(x) = 0$ is false. ◇

To conclude this section, we show how the precise definition of the limit can be used to obtain information about new limits.

EXAMPLE 5 Use the precise definition of "$\lim_{x\to\infty} f(x) = L$" to show that if f and g are defined everywhere and $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to\infty} g(x) = 3$, then $\lim_{x\to\infty}(f(x) + g(x)) = 5$.

SOLUTION The objective is to show that for each positive number ϵ, however small, there is a number D such that, for all $x > D$,

$$|(f(x) + g(x)) - 5| < \epsilon.$$

Observe that $|(f(x) + g(x)) - 5|$ can be written as $|(f(x) - 2) + (g(x) - 3))|$, and this is no larger than the sum $|f(x) - 2| + |g(x) - 3|$. If we can show that for all x sufficiently large that both $|f(x)-2| < \epsilon/2$ and $|g(x)-3| < \epsilon/2$, then there sum will be no larger than $\epsilon/2 + \epsilon/2 = \epsilon$.

Here is how this plan can be implemented.

The fact that $\lim_{x\to\infty} f(x) = 2$ means for any given $\epsilon > 0$ there exists a number D_1 with the property that $|f(x) - 2| < \epsilon/2$ for all $x > D_1$. Likewise, the fact that $\lim_{c\to\infty} g(x) = 3$ means for any given $\epsilon > 0$ there exists a number D_2 with the property that $|g(x) - 2| < \epsilon/2$ for all $x > D_2$.

Let D refer to the larger of D_1 and D_2. For any x greater than D we know that

$D = \max\{D_1, D_2\}$

$$|f(x) + g(x) - 5| < |f(x) - 2| + |g(x) - 3| < \epsilon/2 + \epsilon/2 = \epsilon.$$

According to the precise definition of a limit at infinity, we conclude that

$$\lim_{x\to\infty}(f(x) + g(x)) = 2 + 3 = 5.$$

◇

Summary

We developed a precise definition of the limit of a function as the argument becomes arbitrarily large: $\lim_{x\to\infty} f(x)$. The definition involves being able to respond to a challenge. In the case of an infinite limit, the challenge is a large number. In the case of a finite limit, the challenge is a small number used to describe a narrow horizontal band.

EXERCISES for 3.8

Key: R–routine, M–moderate, C–challenging

1.[R] Let $f(x) = 3x$.

(a) Find a number D such that, for $x > D$, it follows that $f(x) > 600$.

(b) Find another number D such that, for $x > D$, it follows that $f(x) > 600$.

(c) What is the smallest number D such that, for all $x > D$, it follows that $f(x) > 600$?

2.[R] Let $f(x) = 4x$.

(a) Find a number D such that, for $x > D$, it follows that $f(x) > 1000$.

(b) Find another number D such that, for $x > D$, it follows that $f(x) > 1000$.

(c) What is the smallest number D such that, for all $x > D$, it follows that $f(x) > 1000$?

3.[R] Let $f(x) = 5x$. Find a number D such that, for all $x > D$,

(a) $f(x) > 2000$,

(b) $f(x) > 10,000$.

4.[R] Let $f(x) = 6x$. Find a number D such that, for all $x > D$,

(a) $f(x) > 1200$,

(b) $f(x) > 1800$.

In Exercises 5 to 12 use the precise definition of the assertion "$\lim_{x\to\infty} f(x) = \infty$" to establish each limit.

5.[R] $\lim_{x\to\infty} 3x = \infty$

6.[R] $\lim_{x\to\infty} 4x = \infty$

7.[R] $\lim_{x\to\infty} (x + 5) = \infty$

8.[R] $\lim_{x\to\infty} (x - 600) = \infty$

9.[R] $\lim_{x\to\infty} (2x + 4) = \infty$

10.[R] $\lim_{x\to\infty}(3x-1200)=\infty$

11.[R] $\lim_{x\to\infty}(4x+100\cos(x))=\infty$

12.[R] $\lim_{x\to\infty}(2x-300\cos(x))=\infty$

13.[R] Let $f(x)=x^2$.

(a) Find a number D such that, for all $x>D$, $f(x)>100$.

(b) Let E be any nonnegative number. Find a number D such that, for all $x>D$, it follows that $f(x)>E$.

(c) Let E be any negative number. Find a number D such that, for all $x>D$, it follows that $f(x)>E$.

(d) Using the precise definition of "$\lim_{x\to\infty}f(x)=\infty$", show that $\lim_{x\to\infty}x^2=\infty$.

14.[R] Using the precise definition of "$\lim_{x\to\infty}f(x)=\infty$", show that $\lim_{x\to\infty}x^3=\infty$. HINT: See Exercise 13.

Exercises 15 to 22 concern the precise definition of "$\lim_{x\to\infty}f(x)=L$".

15.[R] Let $f(x)=3+1/x$ if $x\neq 0$.

(a) Find a number D such that, for all $x>D$, it follows that $|f(x)-3|<\frac{1}{10}$.

(b) Find another number D such that, for all $x>D$, it follows that $|f(x)-3|<\frac{1}{10}$.

(c) What is the smallest number D such that, for all $x>D$, it follows that $|f(x)-3|<\frac{1}{10}$?

(d) Using the precise definition of "$\lim_{x\to\infty}f(x)=L$", show that $\lim_{x\to\infty}(3+1/x)=3$.

16.[R] Let $f(x)=2/x$ if $x\neq 0$.

(a) Find a number D such that, for all $x>D$, it follows that $|f(x)-0|<\frac{1}{100}$.

(b) Find another number D such that, for all $x>D$, it follows that $|f(x)-0|<\frac{1}{100}$.

(c) What is the smallest number D such that, for all $x>D$, it follows that $|f(x)-0|<\frac{1}{100}$?

(d) Using the precise definition of "$\lim_{x\to\infty}f(x)=L$", show that $\lim_{x\to\infty}(2/x)=0$.

In Exercises 17 to 22 use the precise definition of "$\lim_{x\to\infty} f(x) = L$" to establish each limit.

17.[M] $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$ HINT: $|\sin(x)| \leq 1$ for all x.

18.[M] $\lim_{x\to\infty} \frac{x+\cos(x)}{x} = 1$

19.[M] $\lim_{x\to\infty} \frac{4}{x^2} = 0$

20.[M] $\lim_{x\to\infty} \frac{2x+3}{x} = 2$

21.[M] $\lim_{x\to\infty} \frac{1}{x-100} = 0$

22.[M] $\lim_{x\to\infty} \frac{2x+10}{3x-5} = \frac{2}{3}$

23.[M] Using the precise definition of "$\lim_{x\to\infty} f(x) = \infty$," show that the claim that $\lim_{x\to\infty} x/(x+1) = \infty$ is false.

24.[M] Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that the claim that $\lim_{x\to\infty} \sin(x) = \frac{1}{2}$ is false.

25.[M] Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that the claim that $\lim_{x\to\infty} 3x = 6$ is false.

26.[M] Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that for every number L the assertion that $\lim_{x\to\infty} 2x = L$ is false.

In Exercises 27 to 30 develop precise definitions of the given limits. Phrase your definitions in terms of a challenge number E or ϵ and a reply D. Show the geometric meaning of your definition on a graph.

27.[M] $\lim_{x\to\infty} f(x) = -\infty$

28.[M] $\lim_{x\to-\infty} f(x) = \infty$

29.[M] $\lim_{x\to-\infty} f(x) = -\infty$

30.[M] $\lim_{x\to-\infty} f(x) = L$

31.[M] Let $f(x) = 5$ for all x. (See Exercise 30

(a) Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that $\lim_{x\to\infty} f(x) = 5$.

(b) Using the precise definition of "$\lim_{x\to-\infty} f(x) = L$," show that $\lim_{x\to-\infty} f(x) = 5$.

32.[C] Is this argument correct? "I will prove that $\lim_{x\to\infty}(2x+\cos(x))=\infty$. Let E be given. I want

$$\begin{array}{lrcl} & 2x+\cos(x) & > & E \\ \text{or} & 2x & > & E-\cos(x) \\ \text{so} & x & > & \frac{E-\cos(x)}{2}. \end{array}$$

Thus, if $D=\frac{E-\cos(x)}{2}$, then $2x+\cos(x)>E$."

33.[M] Use the precise definition of "$\lim_{x\to\infty} f(x)=L$," to prove this version of the sum law for limits: if $\lim_{x\to\infty} f(x)=A$ and $\lim_{x\to\infty} g(x)=B$, then $\lim_{x\to\infty}(f(x)+g(x))=A+B$. HINT: See Example 5.

34.[C] Use the precise definition of "$\lim_{x\to\infty} f(x)=L$," to prove this version of the product law for limits: if $\lim_{x\to\infty} f(x)=A$, then $\lim_{x\to\infty}(f(x)^2)=A^2$. HINT: $f(x)^2-A^2=(f(x)-A)(f(x)+A)$, and control the size of each factor.

35.[C] Use the precise definition of "$\lim_{x\to\infty} f(x)=L$," to prove this version of the product law for limits: if $\lim_{x\to\infty} f(x)=A$ and $\lim_{x\to\infty} g(x)=B$, then $\lim_{x\to\infty}(f(x)g(x))=AB$. HINT: To make use of the two given limits, write $f(x)$ as $A+(f(x)-A)$ and $g(x)$ as $B+(g(x)-B)$.

3.9 Precise Definition of Limits at a Finite Point: $\lim_{x\to a} f(x) = L$

To conclude the discussion of limits, we extend the ideas developed in Section 3.8 to limits of a function at a number a.

Informal definition of $\lim_{x\to a} f(x) = L$

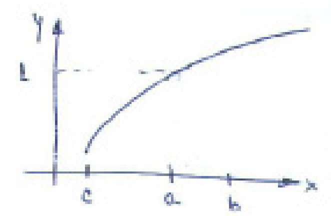


Figure 3.9.1:

Let f be a function and a some fixed number. (See Figure 3.9.1.)

1. Assume that the domain of f contains open intervals (c, a) and (a, b) for some number $c < a$ and some number $b > a$.
2. If, as x approaches a, either from the left or from the right, $f(x)$ approaches a specific number L, then L is called the **limit** of $f(x)$ as x approaches a. This is written

$$\lim_{x\to a} f(x) = L.$$

Keep in mind that a need not be in the domain of f. Even if it happens to be in the domain of f, the value of $f(a)$ plays no role in determining whether $\lim_{x\to a} f(x) = L$.

Reworded informal definition of $\lim_{x\to a} f(x) = L$

Let f be a function and a some fixed number.

1. Assume that the domain of f contains open intervals (c, a) and (a, b) for some number $c < a$ and some number $b > a$.
2. If x is is sufficiently close to a but not equal to a, then $f(x)$ is necessarily near L.

The following precise definition parallels the reworded informal definition.

DEFINITION *(Precise definition of* $\lim_{x\to a} f(x) = L$*)* Let f be a function and a some fixed number.

1. Assume that the domain of f contains open intervals (c, a) and (a, b) for some number $c < a$ and some number $b > a$.

"δ" (delta) is the lower case version of the Greek letter "Δ"; it corresponds to the English letter "d."

2. For each positive number ϵ there is a positive number δ such that for all x that satisfy the inequality

$$0 < |x - a| < \delta$$

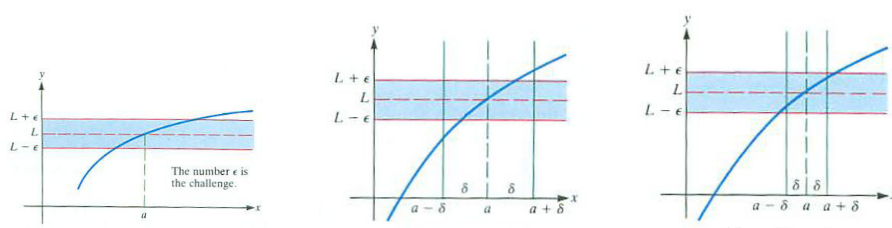
it is true that

$$|f(x) - L| < \epsilon$$

The meaning of $0 < |x - a| < \delta$

The inequality $0 < |x-a|$ that appears in the definition is just a fancy way of saying "x is not a." The inequality $|x - a| < \delta$ asserts that x is within a distance δ of a. The two inequalities may be combined as the single statement $0 < |x - a| < \delta$, which describes the open interval $(a - \delta, a + \delta)$ from which a is deleted. This deletion is made since the $f(a)$ pIays no role in the definition of $\lim_{x\to a} f(x)$.

Once again ϵ is the challenge. The reply is δ. Usually, the smaller ϵ is, the *smaller* δ will have to be.



(a) (b) (c)

Figure 3.9.2: (a) The number ϵ is the challenge. (b) δ is not small enough. (c) δ is small enough.

The geometric significance of the precise definition of "$\lim_{x\to a} f(x) = L$" is shown in Figure 3.9. The narrow horizontal band of width 2ϵ is again the challenge (see Figure 3.9(a)). The desired response is a sufficiently narrow vertical band, of width 2δ, such that the part of the graph within that vertical band (except perhaps at $x = a$) also lies in the horizontal band of width 2ϵ. In Figure 3.9(b) the vertical band shown is not narrow enough to meet the challenge of the horizontal band shown. But the vertical band shown in Figure 3.9(c) is sufficiently narrow.

Assume that for each positive number ϵ it is possible to find a positive number δ such that the parts of the graph between $x = a - \delta$ and $x = a$ and between $x = a$ and $x = a + \delta$ lie within the given horizontal band. Then we say that "as x approaches a, $f(x)$ approaches L". The narrower the horizontal band around the line $y = L$, the smaller δ usually must be.

EXAMPLE 1 Use the precise definition of "$\lim\limits_{x\to a} f(x) = L$" to show that $\lim\limits_{x\to 2}(3x + 5) = 11$.

SOLUTION Here $f(x) = 3x + 5$, $a = 2$, and $L = 11$. Let ϵ be a positive number. We wish to find a number $\delta > 0$ such that for $0 < |x - 2| < \delta$ we have $|(3x + 5) - 11| < \epsilon$.

So let us find out for which x it is true that $|(3x + 5) - 11| < \epsilon$. This inequality is equivalent to

$$\begin{array}{lrcl} & |3x-6| & < & \epsilon \\ \text{or} & 3|x-2| & < & \epsilon \\ \text{or} & |x-2| & < & \frac{\epsilon}{3}. \end{array}$$

Thus $\delta = \epsilon/3$ is a suitable response. If $0 < |x-2| < \epsilon/3$, then $|(3x+5)-11| < \epsilon$. ◇

Any positive number less than $\epsilon/3$ is also a suitable response.

The algebra of finding a response δ can be much more involved for other functions, such as $f(x) = x^2$. The precise definition of limit can actually be easier to apply in more general situations where f and a are not given explicitly. To illustrate, we present a proof of the Permanence Property.

When the Permanence Property was introduced in Section 2.4, the only justification we provided was a picture and an appeal to your intuition that a continuous function cannot jump instantaneously from a positive value to zero or a negative value — the function has to remain positive on some open interval. Mathematicians call this a "proof by handwaving".

EXAMPLE 2 Prove the Permanence Property: Assume that f is continuous in an open interval that contains a and that $f(a) = p > 0$. Then for any number $q < p$, there is an open interval I containing a such that $f(x) > q$ for all x in I.

SOLUTION Let $p = f(a) > 0$ and let q be any positive number less than p. Pick $\epsilon = p - q$. Because f is continuous at a there is a positive number δ such that

The reason for this choice for ϵ will become clear in a moment.

$$|f(a) - f(x)| < p - q \qquad \text{for } a - \delta < x < a + \delta.$$

Thus

$$-(p - q) < f(a) - f(x) < p - q.$$

In particular,

$$f(a) - f(x) < p - q \tag{3.9.1}$$

Because $f(a) = p$, (3.9.1) can be rewritten as

$$p - f(x) < p - q$$

or

$$f(x) > q.$$

Thus $f(x)$ is greater than q if x is in the interval $I = (a - \delta, a + \delta)$.

◇

One of the common uses of the Permanence Property is to say that if a continuous function is positive at a number, a, then there is an interval containing a on which the function is strictly positive. (This corresponds to $p = f(a) > 0$ and $q = 0$.)

Summary

This section developed a precise definition of the limit of a function as the argument approaches a fixed number: $\lim_{x\to a} f(x)$. This definition involves being able to respond to an arbitrary challenge number. In the case of a finite limit, the challenge is a small positive number. The smaller that number, the harder it is to meet the challenge.

In addition, it also gave a rigorous proof of the permanence principle.

EXERCISES for 3.9

Key: R–routine, M–moderate, C–challenging

In Exercises 1 to 4 use the precise definition of "$\lim_{x\to a} f(x) = L$" to justify each statement.

1.[R] $\lim_{x\to 2} 3x = 6$

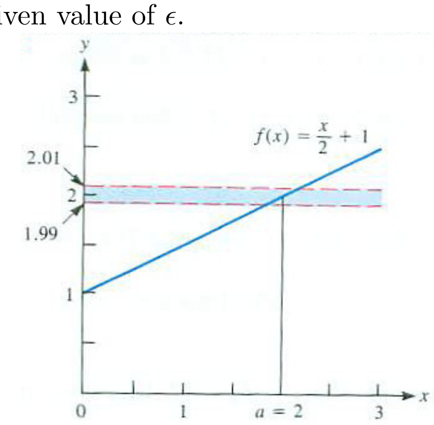
2.[R] $\lim_{x\to 3}(4x - 1) = 11$

3.[R] $\lim_{x\to 1}(x + 2) = 3$

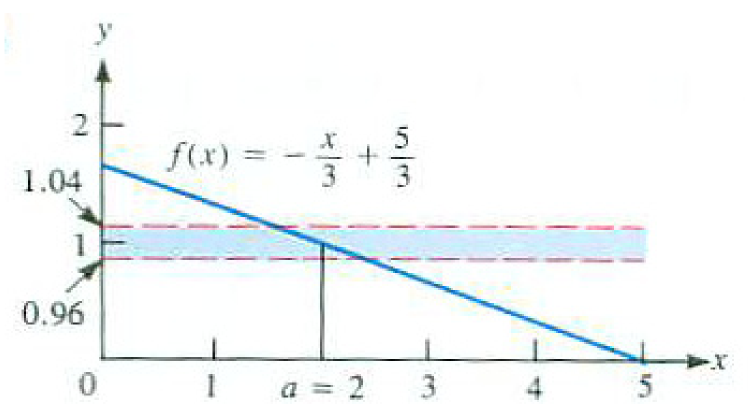
4.[R] $\lim_{x\to 5}(2x - 3) = 7$

In Exercises 5 and 8 find a number δ such that the point $(x, f(x))$ lies in the shaded band for all x in the interval $(a - \delta, a + \delta)$. HINT: Draw suitable vertical band for the given value of ϵ.

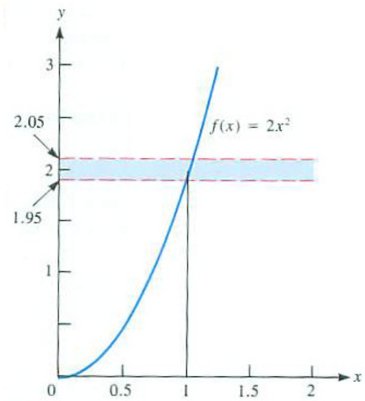
5.[R]



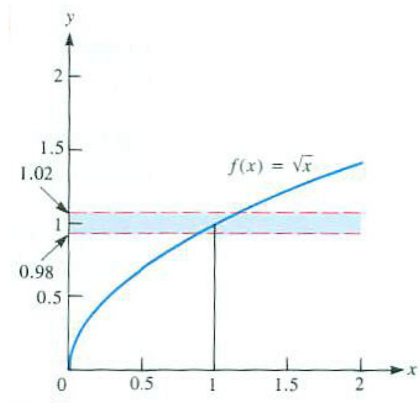
6.[R]



7.[R]



8.[R]



In Exercises 9 and 12 use the precise definition of "$\lim_{x\to a} f(x) = L$" to justify each statement.

9.[R] $\lim_{x\to 1}(3x+5) = 8$

10.[R] $\lim_{x\to 1} \frac{5x+3}{4} = 2$

11.[M] $\lim_{x\to 0} \frac{x^2}{4} = 0$

12.[M] $\lim_{x\to 0} 4x^2 = 0$

13.[M] Give an example of a number $\delta > 0$ such that $|x^2-4| < 1$ if $0 < |x-2| < \delta$.

14.[M] Give an example of a number $\delta > 0$ such that $|x^2+x-2| < 0.5$ if $0 < |x-1| < \delta$.

Develop precise definitions of the given limits in Exercises 15 to 20. Phrase your definitions in terms of a challenge, E or ϵ, and a response, δ.

15.[M] $\lim_{x\to a^+} f(x) = L$

16.[M] $\lim_{x\to a^-} f(x) = L$

17.[M] $\lim_{x\to a} f(x) = \infty$

18.[M] $\lim_{x\to a} f(x) = -\infty$

19.[M] $\lim_{x\to a^+} f(x) = \infty$

20.[M] $\lim_{x\to a^-} f(x) = \infty$

21.[M] Let $f(x) = 9x^2$.

(a) Find $\delta > 0$ such that, for $0 < |x - 0| < \delta$, it follows that $|9x^2 - 0| < \frac{1}{100}$.

(b) Let ϵ be any positive number. Find a positive number δ such that, for $0 < |x - 0| < \delta$ we have $|9x^2 - 0| < \epsilon$.

(c) Show that $\lim_{x\to 0} 9x^2 = 0$.

22.[M] Let $f(x) = x^3$.

(a) Find $\delta > 0$ such that, for $0 < |x - 0| < \delta$, it follows that $|x^3 - 0| < \frac{1}{1000}$.

(b) Show that $\lim_{x\to 0} x^3 = 0$.

23.[M] Show that the assertion "$\lim_{x\to 2} 3x = 5$" is false. To do this, it is necessary to exhibit a positive number ϵ such that there is no response number $\delta > 0$
HINT: Draw a picture.

24.[M] Show that the assertion"$\lim_{x\to 2} x^2 = 3$" is false.

25.[C] Review the proof of the Permanence Property given in Example 2. Recall that $p = f(a) > 0$ and q is chosen so that $p > q > 0$.

(a) Would the argument have worked if we had used $\epsilon = 2(p - q)$?

(b) Would the argument have worked if we had used $\epsilon = \frac{1}{2}(p - q)$?

(c) Would the argument have worked if we had used $\epsilon = q$?

(d) What is the largest value of ϵ for which the proof of the Permanence Property works?

26.[C] The Permanence Property discussed in Example 2 and Exercise 25 pertains to limits at a finite point a. State, and prove, a version of the Permanence Property that is valid "at ∞."

27.[M]

(a) Show that, if $0 < \delta < 1$ and $|x - 3| < \delta$, then $|x^2 - 9| < 7\delta$. HINT: Factor $x^2 - 9$.

(b) Use (a) to deduce that $\lim_{x\to 3} x^2 = 9$.

28.[C]

(a) Show that, if $0 < \delta < 1$ and $|x - 4| < \delta$, then

$$|\sqrt{x} - 2| < \frac{\delta}{\sqrt{3} + 2}.$$

HINT: Rationalize $\sqrt{x} - 2$.

(b) Use (a) to deduce that $\lim_{x\to 4} \sqrt{x} = 2$.

29.[C]

(a) Show that, if $0 < \delta < 1$ and $|x - 3| < \delta$, then $|x^2 + 5x - 24| < 12\delta$. HINT: Factor $x^2 + 5x - 24$.

(b) Use (a) to deduce that $\lim_{x\to 3}(x^2 + 5x) = 24$.

30.[C]

(a) Show that, if $0 < \delta < 1$ and $|x - 2| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{\delta}{2}.$$

(b) Use (a) to deduce that $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

31.[C] Use the precise definitions of limits to prove: if f is defined in an open interval including a and f is continuous at a, so is $3f$.

32.[C] Use the precise definitions of limits to prove: if f and g are both defined in an open interval including a and both functions are continuous at a, so is $f + g$.

33.[C] Use the precise definitions of limits to prove: if f and g are both continuous at a, then their product, fg, is also continuous at a. NOTE: Assume that both functions are defined at least in an open interval around a.

34.[C] Assume that $f(x)$ is continuous at a and is defined at least on an open interval containing a. Assume that $f(x) = p > 0$. Using the precise definition of a limit, show that there is an open interval, I, containing a such that $f(x) > \frac{2}{11}p$ for all x in I.

3.10 Chapter Summary

In this chapter we defined the derivative of a function, developed ways to compute derivatives, and applied them to graphs and motion.

The derivative of a function f at a number $x = a$ is defined as the limit of the slopes of secant lines through the points $(a, f(a))$ and $(b, f(b))$ as the input b is taken closer and closer to the input a.

Algebraically, the derivative is the limit of a quotient, "the change in the output divided by the change in the input". The limit is usually written in one of the following forms:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}, \quad \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}, \quad \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}.$$

The derivative is denoted in several ways, such as f', $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, and $D(f)$.

For the functions most frequently encountered in applications, this limit exists. Geometrically speaking, the derivative exists whenever the graph of the function on a very small interval looks almost like a straight line.

The derivative records how fast something changes. For instance, the velocity of a moving object is defined as the derivative of the object's position. Also, the derivative gives the slope of the tangent line to the graph of a function.

We then developed ways to compute the derivative of functions expressible in terms of the functions met in algebra and trigonometry, including exponentials with a fixed base and logarithms; the so-called "elementary functions". That development was based on three limits:

$$\begin{aligned}
\lim_{x\to a} \frac{x^n - a^n}{x - a} &= na^{n-1}, \qquad n \text{ a positive integer} \\
\lim_{x\to 0} \frac{e^x - 1}{x} &= 1 \\
\lim_{x\to 0} \frac{\sin(x)}{x} &= 1.
\end{aligned}$$

Using these limits, we obtained the derivatives of x^n, e^x, and $\sin(x)$. We showed, if we knew the derivatives of two functions, how to compute the derivatives of their sum, difference, product, and quotient. Naturally, this was based on the definition of the derivative as a limit.

The next step was the development of the most important computational tool: the Chain Rule. This enables us to differentiate a composite function, such as $\cos^3(x^2)$. It tells us that its derivative is $3\cos^2(x^2)(-\sin(x^2))(2x)$.

Differentiating inverse functions enabled us to show that the derivative of $\ln|x|$ is $\frac{1}{x}$ and the derivative of $\arcsin(x)$ is $\frac{1}{\sqrt{1-x^2}}$. The following list of derivatives of key functions should be memorized.

function	derivative
x^a (a constant)	ax^{a-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	e^x
a^x (a constant)	write $a^x = e^{x(\ln(a))}$
$\ln(x)$ $(x > 0)$	$1/x$
$\ln\|x\|$ $(x \neq 0)$	$1/x$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$
$\frac{1}{x}$	$-1/x^2$

Figure 3.10.1: Table of Common Functions and Derivatives.

As you work with derivatives you may begin to think of them as slope or velocity or rate of change, and forget their underlying definition as a limit. However, we will from time to time return to the definition in terms of limits as we develop more applications of the derivative.

We also introduced the antiderivative and, closely related to it, the slope field. While the derivative of an elementary function is again elementary, an antiderivative often is not. For instance, $\sqrt{1+x^3}$ does not have an elementary antiderivative. However, as we will see in Chapter 6, it does have an antiderivative. Chapter 8 will present a few ways to find antiderivatives.

The derivative of the derivative is the second derivative. In the case of motion, the second derivative describes acceleration. It is denoted several ways, such as D^2f, $\frac{d^2f}{dx^2}$, f'', and $f^{(2)}$. While the first and second derivatives suffice for most applications, higher derivatives of all orders are used in Chapter 5, where we estimate the error when approximating a function by a polynomial.

The final two sections returned to the notion of a limit, providing a precise definition of that concept.

EXERCISES for 3.10 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 19 find the derivative of the given function.

1.[R] $\exp(x^2)$

2.[R] 2^{x^2}

3.[R] $x^3 \sin(4x)$

4.[R] $\frac{1+x^2}{1+x^3}$

5.[R] $\ln(x^3)$

6.[R] $\ln(x^3+1)$

7.[R] $\cos^4(x^2)\tan(2x)$

8.[R] $\sqrt{5x^2+x}$

9.[R] $\arcsin(\sqrt{3+2x})$

10.[R] $x^2 \arctan(2x)e^{3x}$

11.[R] $\sec^2(3x)$

12.[R] $\sec^2(3x) - \tan^2(3x)$

13.[R] $\left(\frac{3+2x}{4+5x}\right)^3$

14.[R] $\frac{1}{1+2e^{-x}}$

15.[R] $\frac{x}{\sqrt{x^2+1}}$

16.[R] $(\arcsin(3x))^2$

17.[R] $x^2 \arctan(3x)$

18.[R] $\sin^5(3x^2)$

19.[R] $\frac{1}{(2^x+3^x)^{20}}$

In Exercises 20 to 29 give an antiderivative for the given function. Use differentiation to check each answer. (Chapter 8 presents techniques for finding antiderivatives, but the ones below do not require these methods.)

20.[R] $4x^3$

21.[R] x^3

22.[R] $3/x^2$

23.[R] $\cos(x)$

24.[R] $\cos(2x)$

25.[R] $\sin^{100}(x)\cos(x)$

26.[R] $1/(x+1)$

27.[R] $5e^{4x}$

28.[R] $1/e^x$

29.[R] 2^x

In Exercises 30 to 51 carry out the differentiation to check each equation. The letters a, b, and c denote constants. NOTE: These problems provide good practice in differentiation and algebra. Each differentiation formula has a corresponding antiderivative formula. In fact, these exercises are based on several tables of antiderivatives.

30.[R] $\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{a^2+x^2}$

31.[R] $D\left(\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)\right) = \frac{1}{a^2-x^2}$

32.[R] $\left(\ln\left(x+\sqrt{a^2+x^2}\right)\right)' = \frac{1}{\sqrt{a^2+x^2}}$

33.[R] $\frac{d}{dx}\left(\frac{1}{a}\ln\left(\frac{x+\sqrt{a^2-x^2}}{x}\right)\right) = \frac{1}{x\sqrt{a^2+x^2}}$

34.[R] $D\left(\frac{-1}{b(a+bx)}\right) = \frac{1}{(a+bx)^2}$

35.[R] $\left(\frac{1}{b^2}\left(a+bx-a\ln(a+bx)\right)\right)' = \frac{x}{a+bx}$

36.[R] $\frac{d}{dx}\left(\frac{1}{b^2}\left(\frac{a}{2(a+bx)^2}-\frac{1}{a+bx}\right)\right) = \frac{x}{(a+bx)^3}$

37.[R] $D\left(\frac{1}{ab'-a'b}\ln\left(\frac{a'+b'x}{a+bx}\right)\right) = \frac{1}{(a+bx)(a'+b'x)}$ (a, b, a', b' constants)

38.[R] $\left(\frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2cx+b}{\sqrt{4ac-b^2}}\right)\right)' = \frac{1}{a+bx+cx^2}$ $(4ac > b^2)$

39.[R] $\frac{d}{dx}\left(\frac{-2}{\sqrt{b^2-4ac}}\ln\left(\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}}\right)\right) = \frac{1}{a+bx+cx^2}$ $(4ac < b^2)$

40.[R] $D\left(\frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right)\right) = \frac{1}{x\sqrt{x^2-a^2}}$

41.[R] $\left(\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\left(\frac{x}{a}\right)\right)\right)' = \sqrt{a^2-x^2}$ $(|x| < |a|)$

42.[R] $\frac{d}{dx}\left(\frac{-x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)\right) = \frac{x^2}{\sqrt{a^2-x^2}}$ $(|x| < |a|)$

43.[R] $D\left(-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\left(\frac{x}{a}\right)\right) = \frac{\sqrt{a^2-x^2}}{x^2}$ $(|x| < |a|)$

44.[R] $\left(\arcsin(x)-\sqrt{1-x^2}\right)' = \sqrt{\frac{1+x}{1-x}}$ $(|x| < 1)$

45.[R] $\frac{d}{dx}\left(\frac{x}{2}-\frac{1}{2}\cos(x)\sin(x)\right) = \sin^2(x)$

46.[R] $D\left(x\arcsin x+\sqrt{1-x^2}\right) = \arcsin(x)$ $(|x| < 1)$

47.[R] $\left(x\tan^{-1}(x)-\frac{1}{2}\ln(1+x^2)\right)' = \arctan(x)$

48.[R] $\frac{d}{dx}\left(\frac{e^{ax}}{a^2}\left(a^2-1\right)\right) = xe^{ax}$

49.[R] $D\left(x-\ln(1+e^x)\right) = \frac{1}{1+e^x}$

50.[R] $\left(\frac{x}{2}\left(\sin(\ln(ax))-\cos(\ln(ax))\right)\right)' = \sin(\ln(ax))$

51.[R] $\left(\frac{e^{ax}(a\sin(bx)-b\cos(bx))}{a^2+b^2}\right)' = e^{ax}\sin(bx)$

In Exercises 52 to 55 give two antiderivatives for each given function.

52.[M] xe^{x^2}

53.[M] $(x^2+x)e^{x^3+3x}$

54.[M] $\cos^3(x)\sin(x)$

55.[M] $\sin(2x)$

56.[M] Verify that $2(\sqrt{x}-1)e^{\sqrt{x}}$ is an antiderivative of $e^{\sqrt{x}}$.

In Exercises 57 to 60 (a) sketch the slope field and (b) draw the solution curve through the point $(0,1)$.

57.[R] $dy/dx = 1/(x+1)$

58.[R] $dy/dx = e^{-x^2}$

59.[R] $dy/dx = -y$

60.[R] $dy/dx = y - x$

61.[R] Sam threw a baseball straight up and caught it 6 seconds later.

(a) How high above his head did it rise?

(b) How fast was it going as it left his hand?

(c) How fast was it going when he caught it?

(d) Translate the answers in (b) and (c) to miles per hour. (Recall: 60 mph = 88 fps.)

62.[M] Assuming that $D(x^4) = 4x^3$ and $D(x^7) = 7x^6$, you could find $D(x^3)$ directly by viewing x^3 as x^7/x^4 and using the formula for differentiating a quotient. Show how you could find directly $D(x^{11})$, $D(x^{-4})$, $D(x^{28})$, and $D(x^8)$.

63.[M] Let $y = x^{m/n}$, where $x > 0$ and m and $n \neq 0$ are integers. Assuming that y is differentiable, show that $\frac{dy}{dx} = \frac{m}{n}x^{\frac{m}{n}-1}$ by starting with $y^n = x^m$ and differentiating both y^n and x^m with respect to x. HINT: Think of y as $y(x)$ and remember to use the chain rule when differentiating y^n with respect to x.

64.[M] A spherical balloon is being filled with helium at the rate of 3 cubic feet per minute. At what rate is the radius increasing when the radius is (a) 2 feet? (b) 3 feet? HINT: The volume of a ball of radius r is $\frac{4}{3}\pi r^3$.

65.[M] An object at the end of a vertical spring is at rest. When you pull it down it goes up and down for a while. With the origin of the y-axis at the rest position, the position of the object t seconds later is $3e^{-2t}\cos(2\pi t)$ inches.

(a) What is the physical significance of 3 in the formula?

(b) What does e^{-2t} tell us?

(c) What does $\cos(2\pi t)$ tell us?

(d) How long does it take the object to complete a full cycle (go from its rest position, down, up, then down to its rest position)?

(e) What happens to the object after a long time?

66.[M] The motor on a moving motor boat is turned off. It then coasts along the x-axis. Its position, in meters, at time t (seconds) is $500 - 50e^{-3t}$.

(a) Where is it at time $t = 0$?

(b) What is its velocity at time t?

(c) What is its acceleration at time t?

(d) How far does it coast?

(e) Show that its acceleration is proportional to its velocity. NOTE: This means the force of the water slowing the boat is proportional to the velocity of the boat.

67.[M] It is safe to switch the "sin" and "lim" in $\sin\left(\lim_{x\to 0}\frac{e^x-1}{x}\right) = \lim_{x\to 0}\left(\sin\left(\frac{e^x-1}{x}\right)\right)$. However, such a switch sometimes is not correct. Consider f defined by $f(x) = 2$ for $x \neq 1$ and $f(1) = 0$.

(a) Show that $f\left(\lim\limits_{x\to 0}\dfrac{e^x-1}{x}\right)$ is not equal to $\lim_{x\to 0} f\left(\frac{e^x-1}{x}\right)$.

(b) What property of the function $\sin(x)$ permits us to switch it with "lim"?

The preceding exercises offered an opportunity to practice computing derivatives. However, it is important to keep in mind the definition of a derivative as a limit. Exercises 68 to 72 will help to reinforce this definition.

68.[R] Define the derivative of the function $g(x)$ at $x = a$ in (a) the x and $x + h$ notation, (b) the x and a notation, and (c) the Δy and Δx notation.

69.[M] We obtained the derivative of $\sin(x)$ using the x and $x + h$ notation and the addition identity for $\sin(x + h)$. Instead, obtain the derivative of $\sin(x)$ using the x and a notation. That is, find

$$\lim_{x\to a} \frac{\sin(x) - \sin(a)}{x - a}.$$

(a) Show that $\sin(x) - \sin(a) = 2\sin\left(\frac{1}{2}(x - y)\right)\cos\left(\frac{1}{2}(x + y)\right)$.

(b) Use the identity in (a) to find the limit.

70.[M] We obtained the derivative for $\tan(x)$ by writing it as $\sin(x)/\cos(x)$. Instead, obtain the derivative directly by finding

$$\lim_{h\to 0} \frac{\tan(x + h) - \tan(x)}{h}.$$

HINT: The identity $\tan(a + b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$ will help.

71.[C] Show that $\frac{\tan(a)}{\tan(b)} > \frac{a}{b} > \frac{\sin(a)}{\sin(b)}$ for all angles a and b in the first quadrant with $a > b$. HINT: Be ready to make use of the two inequalities that squeezed $\sin(x)/x$ toward 1.

72.[C] We obtained the derivative of $\ln(x)$, $x > 0$, by viewing it as the inverse of $\exp(x)$. Instead, find the derivative directly from the definition. HINT: Use the x and h notation.

Exercises 73 and 74 show how we could have predicted that $\ln(x)$ would provide an antiderivative for $1/x$.

73.[C] The antiderivative of $1/x$ that passes through $(1,0)$ is $\ln(x)$. One would expect that for t near 1, the antiderivative of $1/x^t$ that passes through $(1,0)$ would look much like $\ln(x)$ when x is near 1. To verify that this is true

(a) graph the slope field for $1/x^t$ with $t = 1.1$

(b) graph the antiderivative of $1/x^t$ that passes through $(1,0)$ for $t = 1.1$

(c) repeat (a) and (b) for $t = 0.9$

(d) repeat (a) and (b) for $t = 1.01$

(e) repeat (a) and (b) for $t = 0.99$

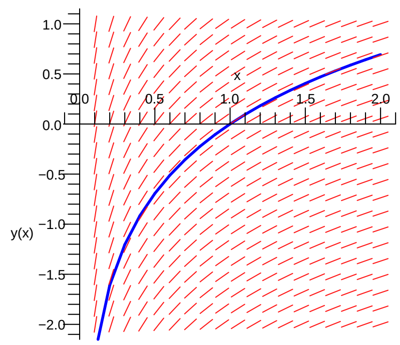


Figure 3.10.2:
The slope field for $1/x$ and the antiderivative of $1/x$ passing through $(1,0)$ are shown in Figure 3.10.2.

74.[C] (See Exercise 73.)

(a) Verify that for $t \neq 1$ the antiderivative of $1/x^t$ that passes through $(1,0)$ is $\frac{x^{1-t}-1}{1-t}$.

(b) Holding x fixed and letting t approach 1, show that

$$\lim_{t\to 1} \frac{x^{1-t}-1}{1-t} = \ln(x).$$

HINT: Recognize the limit as the derivative of a certain function at a certain input. Keep in mind that x is constant in this limit.

75.[C] Define f as follows:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$$

(a) What does the graph of f look like? NOTE: A dotted curve would indicate that points are missing.

(b) Does $\lim_{x\to 0} f(x)$ exist?

(c) Does $\lim_{x\to 1} f(x)$ exist?

(d) Does $\lim_{x\to\sqrt{2}} f(x)$ exist?

(e) For which numbers a does $\lim_{x\to a} f(x)$ exist?

76.[C] Define f as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ x^3 & \text{if } x \text{ is irrational.} \end{cases}$$

(a) What does the graph of f look like? NOTE: A dotted curve may be used to indicate that points are missing.

(b) Does $\lim_{x\to 0} f(x)$ exist?

(c) Does $\lim_{x\to 1} f(x)$ exist?

(d) Does $\lim_{x\to\sqrt{2}} f(x)$ exist?

(e) For which numbers a does $\lim_{x\to a} f(x)$ exist?

77.[C] A heavy block rests on a horizontal table covered with thick oil. The block, which is at the origin of the x-axis is given an initial velocity v_0 at time $t = 0$. It then coasts along the positive x-axis.
Assume that its acceleration is of the form $-k\sqrt{v(t)}$, where $v(t)$ is the velocity at time t and k is a constant. (That means it meets a resistance force proportional to the square root of its velocity.)

(a) Show that $\frac{dv}{dt} = -kv^{1/2}$.

(b) Is k positive or negative? Explain.

(c) Show that $2v^{1/2}$ and $-kt$ have the same derivative with respect to t.

(d) Show that $2v^{1/2} = -kt + 2v_0^{1/2}$.

(e) When does the block come to a rest? (Express that time in terms of v_0 and k.)

(f) How far does the block slide? (Express that distance in terms of v_0 and k.)

78.[C] A motorboat traveling along the x-axis at the speed v_0 stops its motor at time $t = 0$ when it is at the origin. It then coasts along the positive x-aixis.
Assuming the resistance force of the water is proportional to the velocity. That implies the acceleration of the boat is proportional to its velocity, $v(t)$.

(a) Show that there is a constant k such that $\frac{dv}{dt} = -kv(t)$.

(b) Is k positive or negative? Explain.

(c) Deduce that $\ln(v)$ and $-kt$ have the same derivative with respect to t.

(d) Deduce that $\ln(v(t)) = -kt + \ln(v_0)$.

(e) Deduce that $v(t) = v_0 e^{-kt}$.

(f) According to (e), how long does the boat continue to move? (Express that time in terms of v_0 and k.)

(g) How far does it move during that time? (Express that distance in terms of v_0 and k.)

79.[C] Archimedes used the following property of a parabola in his study of the equilibrium of floating bodies. Let P be any point on the parabola $y = x^2$ other than the origin. The normal to the parabola at P meets the y-axis in a point Q. The line through P and parallel to the x-axis meets the y-axis in a point R. Show that the length of QR is constant, independent of the choice of P. NOTE: This problem introduces the **subnormal** of the graph; compare this with Exercises 26 and 27 in Section 3.2.

Calculus is Everywhere

Solar Cookers

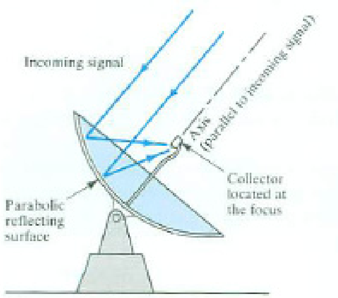


Figure 3.10.1:

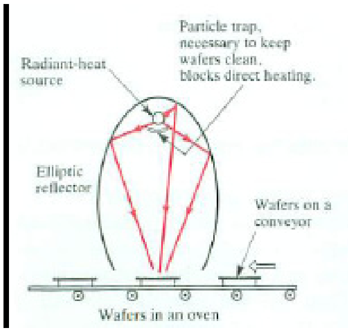


Figure 3.10.2:

A satellite dish is parabolic in shape. It is formed by rotating a parabola about its axis The reason is that all radio waves parallel to the axis of the parabola, after bouncing off the parabola, pass through a common point. This point is called the **focus** of the parabola. (See Figure 3.10.1.) Similarly, the reflector behind a flashlight bulb is parabolic.

An ellipse also has a reflection property. Light, or sound, or heat radiating off one focus, after bouncing off the ellipse, goes through the other focus. This is applied, for instance, in the construction of computer chips where it is necessary to bake a photomask onto the surface of a silicon wafer. The heat is focused at the mask by placing a heat source at one focus of an ellipse and positioning the wafer at the other focus, as in Figure 3.10.2.

The reflection property is used in wind tunnel tests of aircraft noise. The test is run in an elliptical chamber, with the aircraft model at one focus and a microphone at the other.

Whispering rooms, such as the rotunda in the Capitol in Washington, D.C., are based on the same principle. A person talking quietly at one focus can be heard easily at the other focus and not at other points between the foci. (The whisper would be unintelligible except for the additional property that all the paths of the sound from one focus to the other have the same length.)

An ellipsoidal reflector cup is used for crushing kidney stones. (An ellipsoid is formed by rotating an ellipse about the line through its foci.) An electrode is placed at one focus and an ellipsoid positioned so that the stone is at the other focus. Shock waves generated at the electrode bounce off the ellipsoid, concentrate on the other focus, and pulverize the stones without damaging other parts of the body. The patient recovers in three to four days instead of the two to three weeks required after surgery. This advance also reduced the mortality rate from kidney stones from 1 in 50 to 1 in 10,000.

The reflecting property of the ellipse also is used in the study of air pollution. One way to detect air pollution is by light scattering. A laser is aimed through one focus of a shiny ellipsoid. When a particle passes through this focus, the light is reflected to the other focus where a light detector is located. The number of particles detected is used to determine the amount of pollution in the air.

The Angle Between Two Lines

To establish the reflection properies just mentioned we will use the principle that the angle of reflection equals the angle of incidence, as in Figure 3.10.3, and work with the angle between two lines, given their slopes.

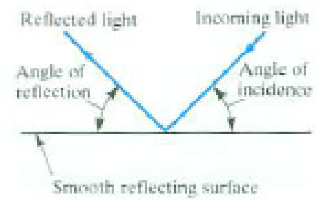
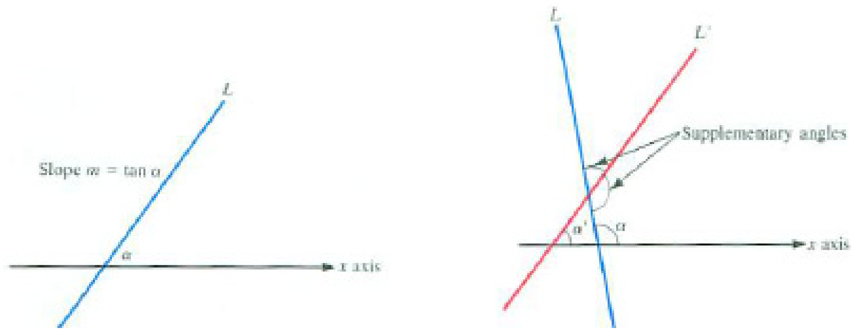


Figure 3.10.3:

Consider a line L in the xy-plane. It forms an **angle of inclination** α, $0 \leq \alpha < \pi$, with the positive x-axis. The slope of L is $\tan(\alpha)$. (See Figure 3.10.4(a).) If $\alpha = \pi/2$, the slope is not defined.



(a) (b)

Figure 3.10.4:

Consider two lines L and L' with angles of inclination α and α' and slopes m and m', respectively, as in Figure 3.10.4(b). There are two (supplementary) angles between the two lines. The following definition serves distinguishes one of these two angles as *the* angle between L and L'.

DEFINITION *(Angle between two lines.)* Let L and L' be two lines in the xy-plane, named so that L has the larger angle of inclination, $\alpha > \alpha'$. The angle θ between L and L' is defined to be

$$\theta = \alpha - \alpha'.$$

If L and L' are parallel, define θ to be 0.

Note that θ is the counterclockwise angle from L' to L and that $0 \leq \theta < \pi$. The tangent of θ is easily expressed in terms of the slopes m of L and m' of L'. We have

$$\begin{aligned} \tan(\theta) &= \tan(\alpha - \alpha') && \text{definition of } \theta \\ &= \frac{\tan(\alpha)-\tan(\alpha')}{1+\tan(\alpha)\tan(\alpha')} && \text{by the identity for } \tan(A-B) \\ &= \frac{m-m'}{1+mm'}. \end{aligned}$$

Thus

$$\tan(\theta) = \frac{m - m'}{1 + mm'}. \tag{3.10.1}$$

The Reflection Property of a Parabola

Consider the parabola $y = x^2$. (The geometric description of this parabola is the set of all points whose distance from the point $(0, \frac{1}{4})$ equals its distance from the line $y = \frac{-1}{4}$, but this information is not needed here.)

In Figure 3.10.5 we wish to show that angles A and B at the typical point (a, a^2) on the parabola are equal. We will do this by showing that $\tan(A) = \tan(B)$.

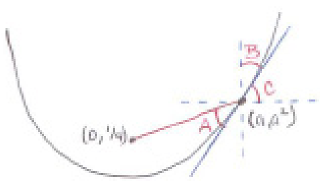


Figure 3.10.5:

First of all, $\tan(C) = 2a$, the slope of the parabola at (a, a^2). Since A is the complement of C, $\tan(A) = 1/(2a)$.

The slope of the line through the focus $(0, \frac{1}{4})$ and a point on the parabola (a, a^2) is

$$\frac{a^2 - \frac{1}{4}}{a - 0} = \frac{4a^2 - 1}{4a}.$$

Therefore,

$$\tan(B) = \frac{2a - \frac{4a^2-1}{4a}}{1 + 2a\left(\frac{4a^2-1}{4a}\right)}.$$

Exercise 1 asks you to supply the algebraic steps to complete the proof that $\tan(B) = \tan(A)$.

The Reflection Property of an Ellipse

An ellipse consists of every point such that the sum of the distances from the point to two fixed points is constant. Let the two fixed points, called the **foci** of the ellipse, be a distance $2c$ apart, and the fixed sum of the distances be $2a$, where $a > c$. If the foci are at $(c, 0)$ and $(-c, 0)$ and $b^2 = a^2 - c^2$, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b^2 = a^2 - c^2$. (See Figure 3.10.6.)

As in the case of the parabola, one shows $\tan(A) = \tan(B)$.

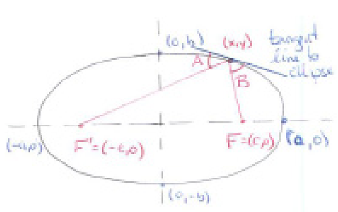


Figure 3.10.6: Diocles, *On Burning Mirrors*, edited by G. J. Toomer, Springer, New York, 1976.

One reason to do Exercise 2 is to appreciate more fully the power of vector calculus, developed later in Chapter 14, for with that tool you can establish the reflection property of either the parabola or the ellipse in one line.

Diocles, in his book *On Burning Mirrors*, written around 190 B.C., studied spherical and parabolic reflectors, both of which had been considered by earlier scientists. Some had thought that a spherical reflector focuses incoming light at a single point. This is false, and Diocles showed that a spherical reflector subtending an angle of 60° reflects light that is parallel to its axis of symmetry to points on this axis that occupy about one-thirteenth of the radius. He proposed an experiment, "Perhaps you would like to make two examples of a burning-mirror, one spherical, one parabolic, so that you can measure the burning power of each." Though the reflection property of a parabola was already known, *On Burning Mirrors* contains the first known proof of this property.

Exercise 3 shows that a spherical oven is fairly effective. After all, a potato or hamburger is not a point.

EXERCISES

1.[R] Do the algebra to complete the proof that $\tan(A) = \tan(B)$.

2.[R] This exercise establishes the reflection property of an ellipse. Refer to Figure 3.10.6 for a description of the notation.

(a) Find the slope of the tangent line at (x, y).

(b) Find the slope of the line through $F = (c, 0)$ and (x, y).

(c) Find $\tan(B)$.

(d) Find the slope of the line through $F' = (c', 0)$ and (x, y).

(e) Find $\tan(A)$.

(f) Check that $\tan(A) = \tan(B)$.

3.[M] Use trigonometry to show that a spherical mirror of radius r and subtending an angle of 60° causes light parallel to its axis of symmetry to reflect and meet the axis in an interval of length $\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right) r \approx r/12.9$.

Chapter 4

Derivatives and Curve Sketching

When you graph a function you typically plot a few points and connect them with (generally) straight line segments. Most electronic graphing devices use the same approach, and obtain better results by plotting more points and using shorter segments. The more points used, the smoother the graph will appear. This chapter will show you how to choose key points when sketching a graph.

Three properties of the derivative developed in Section 4.1, and proved in Section 4.4, will be used in Section 4.2 to help graph a function. In Section 4.3 we see what the second derivative tells about a graph.

4.1 Three Theorems about the Derivative

This section is based on plausible observations about the graphs of differentiable functions, which we restate as theorems. These ideas will then be combined, in Section 4.2, to sketch graphs of functions.

An effective approach to sketching graphs of functions is to find the extreme values of the function, that is, where the function takes on its largest and smallest values.

OBSERVATION *(Tangent Line at an Extreme Value)* Suppose that a function $f(x)$ attains its largest value when $x = c$, that is, $f(c)$ is the largest value of $f(x)$ over a given open interval that contains c. Figure 4.1.1 illustrates this. The maximum occurs at a point $(c, f(c))$, which we call P. If $f(x)$ is differentiable, at c, then the tangent line at P will exist. What can we say about it?

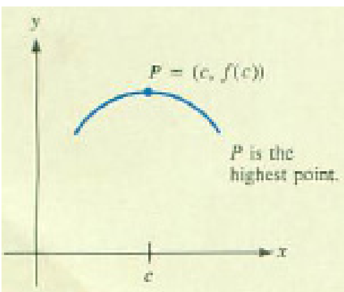
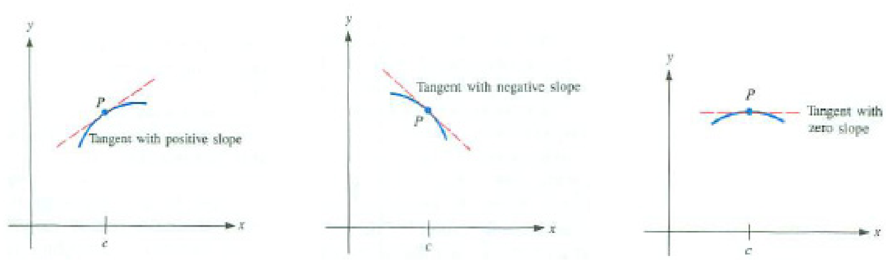


Figure 4.1.1:



(a) (b) (c)

Figure 4.1.2:

If the tangent at P were *not* horizontal (that is, not parallel to the x-axis), then it would be tilted. So a small piece of the graph around P which appears to be almost straight — would look as shown in Figure 4.1.2(a) or (b).

In the first case P could not be the highest point on the curve because there would be higher points *to the right* of P. In the second case P could not be the highest point because there would be higher points *to the left* of P. Therefore the *tangent at P must be horizontal*, as shown in Figure 4.1.2(c). That is, $f'(c) = 0$.

This observation is the foundation for a simple criterion for identifying local extrema.

Theorem of the Interior Extremum

Theorem 4.1.1 (Theorem of the Interior Extremum). *Let f be a function defined at least on the open interval (a, b). If f takes on an extreme value at* *a number c in this interval, then either*

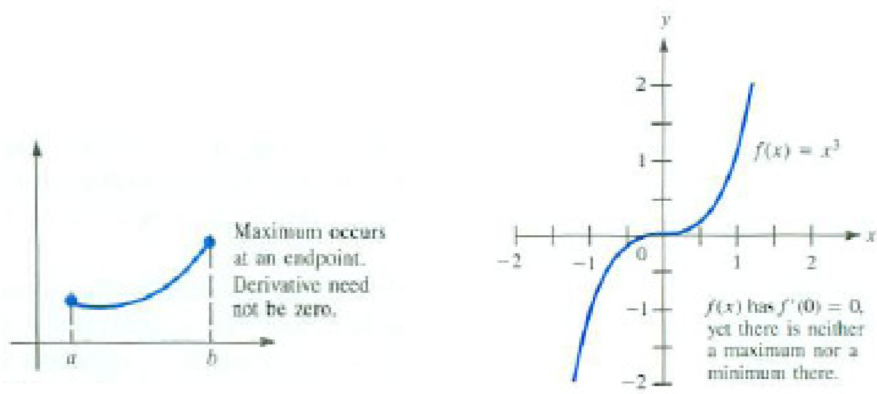
1. $f'(c) = 0$ *or*

2. $f'(c)$ *does not exist.*

If an extreme value occurs within an open interval and the derivative exists there, the derivative must be 0 there. This idea will be used in Section 4.2 to find the maximum and minimum values of a function.

WARNING *(Two Cautions about Theorem 4.1.1)*

1. If in Theorem 4.1.1 the open interval (a, b) is replaced by a closed interval $[a, b]$ the conclusion may not hold. A glance at Figure 4.1.3(a) shows why — the extreme value could occur at an endpoint ($x = a$ or $x = b$).



(a) (b)

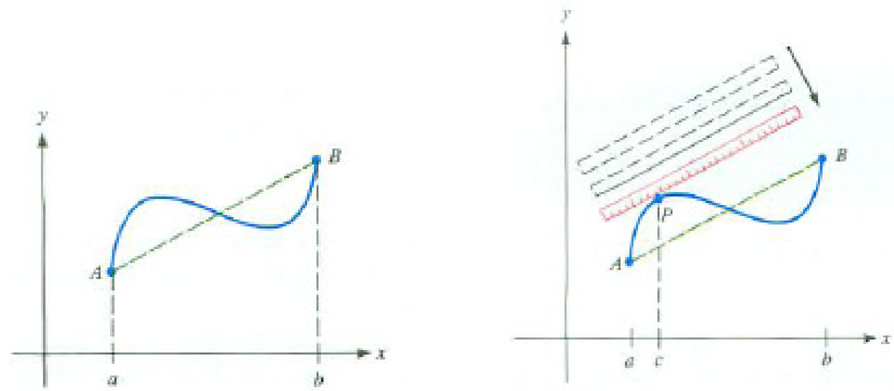
Figure 4.1.3:

2. The converse of Theorem 4.1.1 is not true. Having the derivative equal to 0 at a point does *not* guarantee that there is an extremum at this point. The graph of $y = x^3$, Figure 4.1.3(b), shows why. Since $f'(x) = 3x^2$, $f'(0) = 0$. While the tangent line is indeed horizontal at $(0, 0)$, it crosses the curve at this point. The graph has neither a maximum nor a minimum at the origin.

Though the next observation is phrased in terms of slopes, we will see that it has implications for velocity and any changing quantity.

A line segment that joins two points on the graph of a function f is called a **chord** of f.

OBSERVATION *(Chord and Tangent Line with Same Slope)* Let $A = (a, f(a))$ and $B = (b, f(b))$ be two points on the graph of a differentiable function f defined at least on the interval $[a, b]$, as shown in Figure 4.1.4(a). Draw the line segment AB joining A and B. Assume part of the graph lies above that line. Imagine holding a ruler parallel to AB and lowering it until it just touches the graph of $y = f(x)$, as in Figure 4.1.4(b). The ruler touches the



(a) (b)

Figure 4.1.4:

curve at a point P and lies along the tangent at P. At that point $f'(c)$ is equal to the slope of AB. (In Figure 4.1.4(b) there are two such numbers between a and b.)

It is customary to state two separate theorems based on the observation about chords and tangent lines. The first, Rolle's Theorem, is a special case of the second, the Mean-Value Theorem.

Rolle's Theorem

The next theorem is suggested by a special case of the second observation. When the points A and B in Figure 4.1.4(a) have the same y coordinate, the chord AB has slope 0. (See Figure 4.1.5.) In this case, the observation tells us there must a horizontal tangent to the graph. Expressed in terms of derivatives, this gives us Rolle's Theorem[1]

[1]Michel Rolle (1652–1719) was a French mathematician. and an early critic of calculus before later changing his opinion. In addition to his discovery of Rolle's Theorem in 1691, he is the first person known to have placed the index in the opening of a radical to denote the n^{th} root of a number: $\sqrt[n]{x}$. Source: Cajori, A History of Mathematical Notation, Dover Publ., 1993 and `http://en.wikipedia.org/wiki/Michel_Rolle`.

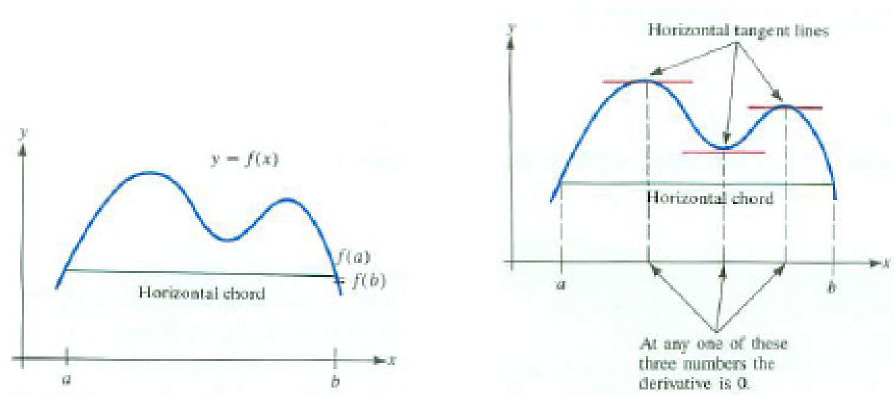


Figure 4.1.5:

Theorem 4.1.2 (Rolle's Theorem)**.** *Let f be a continuous function on the closed interval $[a, b]$ and have a derivative at all x in the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.*

EXAMPLE 1 Verify Rolle's Theorem for the case with $f(t) = (t^2-1)\ln\left(\frac{t}{\pi}\right)$ on $[1, \pi]$.

SOLUTION The function $f(t)$ is defined for $t > 0$ and is differentiable. In particular, $f(t)$ is differentiable on the closed interval $[1, \pi]$. Notice that $f(1) = 0$ and, because $\ln(1) = 0$, $f(\pi) = 0$. Therefore, by Rolle's Theorem, there must be a value of c between 1 and π where $f'(c) = 0$.

The derivative $f'(t) = 2t\ln\left(\frac{t}{\pi}\right) + \frac{t^2-1}{t}$ is a pretty complicated function. Even though it is not possible to find the exact value of c with $f'(c) = 0$, Rolle's Theorem guarantees that there is at least one such value of c. Figure 4.1.6 confirms that there is only one solution to $f'(c) = 0$ on $[1, \pi]$. In Exercise 6 (at the end of Chapter 10 on page 896) you will find that this critical number is approximately 2.128. ◇

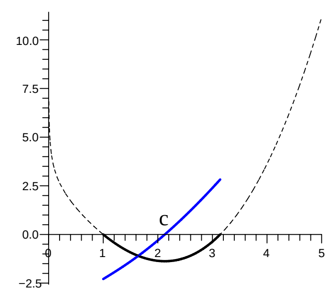


Figure 4.1.6: Graph of $y = f(t)$ (black) and $y = f'(t)$ (blue).

Remark: Assume that $f(x)$ is a differentiable function such that $f'(x)$ is never 0 for x in an interval. Then the equation $f(x) = 0$ *can have at most one solution in that interval.* (If it had two solutions, a and b, then $f(a) = 0$ and $f(b) = 0$, and we could apply Rolle's Theorem on $[a, b]$. (See Figure 4.1.7.)

This justifies the observation:

In an interval in which the derivative $f'(x)$ is never 0, the graph of $y = f(x)$ can have no more than one x-intercept.

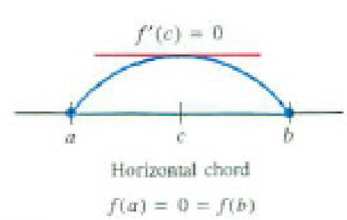


Figure 4.1.7:

Example 2 applies this.

EXAMPLE 2 Use Rolle's Theorem to determine how many real roots there are for the equation

$$x^3 - 6x^2 + 15x + 3 = 0. \tag{4.1.1}$$

SOLUTION Recall that the Intermediate Value Theorem guarantees that an odd degree polynomial, such as $f(x) = x^3 - 6x^2 + 15x + 3$, has at least one real solution to $f(x) = 0$. Call it r. Could there be another root, s? If so, by Rolle's Theorem, there would be a number c (between r and s) at which $f'(c) = 0$.

To check, we compute the derivative of $f(x)$ and see if it is ever equal to 0. We have $f'(x) = 3x^2 - 12x + 15$. To find when $f'(x)$ is 0, we solve the equation $3x^2 - 12x + 15 = 0$ by the quadratic formula, obtaining

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(15)}}{6} = \frac{12 \pm \sqrt{-36}}{6} = 2 \pm \sqrt{-1}.$$

Thus the equation $x^3 - 6x^2 + 15x + 3$ has only one real root. In Exercise 7 (at the end of Chapter 10) you will find that the sole real solution to (4.1.1) in approximately -0.186. ◇

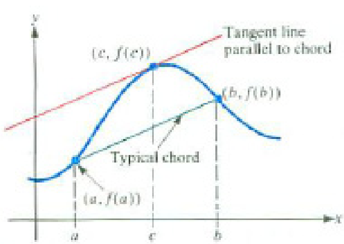


Figure 4.1.8:

Mean-Value Theorem

The "mean-value" theorem, is a generalization of Rolle's Theorem in that it applies to any chord, not just horizontal chords.

In geometric terms, the theorem asserts that if you draw a chord for the graph of a well-behaved function (as in Figure 4.1.8), then somewhere above or below that chord the graph has at least one tangent line parallel to the chord. (See Figure 4.1.4(a).) Let us translate this geometric statement into the language of functions. Call the ends of the chord $(a, f(a))$ and $(b, f(b))$. The slope of the chord is

$$\frac{f(b) - f(a)}{b - a}.$$

Since the tangent line and the chord are parallel, they have the same slopes. If the tangent line is at the point $(c, f(c))$, then

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Specifically, we have

Theorem 4.1.3 (Mean-Value Theorem)**.** *Let f be a continuous function on the closed interval $[a, b]$ and have a derivative at every x in the open interval (a, b). Then there is at least one number c in the open interval (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

EXAMPLE 3 Verify the Mean-Value Theorem for $f(t) = \sqrt{4 - t^2}$ on the interval $[0, 2]$.
SOLUTION Because $4 - t^2 \geq 0$ for t between -2 and 2 (including these two endpoints), f is continuous on $[0, 2]$ and is differentiable on $(0, 2)$. The slope of the chord through $(a, f(a)) = (0, 2)$ and $(b, f(b)) = (2, 0)$ is

$$\frac{f(b) - f(a)}{b - a} = \frac{0 - 2}{2 - 0} = -1.$$

According to the Mean-Value Theorem, there is at least one number c between 0 and 2 where $f'(c)$ is -1.

Let us try to find c. Since $f'(t) = \frac{-2t}{2\sqrt{4 - t^2}}$, we need to solve the equation

$$\begin{aligned} \frac{-c}{\sqrt{4 - c^2}} &= -1 \\ -c &= -\sqrt{4 - c^2} && \text{multiply both sides by } \sqrt{4 - t^2} \\ c^2 &= 4 - c^2 && \text{square both sides} \\ 2c^2 &= 4 \\ c^2 &= 2. \end{aligned}$$

There are two solutions: $c = \sqrt{2}$ and $c = -\sqrt{2}$. Only $c = \sqrt{2}$ is in $(0, 2)$. This is the number whose existence is guaranteed by the Mean-Value Theorem. (The MVT says nothing about the existence of other numbers satisfying the MVT.) ◇

The interpretation of the derivative as slope suggested the Mean-Value Theorem. What does the Mean-Value Theorem say when the function describes the position of a moving object, and the derivative, its velocity? This is answered in Example 4.

EXAMPLE 4 A car moving on the x-axis has the x-coordinate $x = f(t)$ at time t. At time a its position is $f(a)$. At some later time b its position is $f(b)$. What does the Mean-Value Theorem assert for this car?
SOLUTION In this case the quotient

$$\frac{f(b) - f(a)}{b - a} \quad \text{equals} \quad \frac{\text{Change in position}}{\text{Change in time}}.$$

The Mean-Value Theorem asserts that at some time c, $f'(c)$ is equal to the quotient $\frac{f(b)-f(a)}{b-a}$. This says that the velocity at time c is the same as the average velocity during the time interval $[a,b]$. To be specific, if a car travels 210 miles in 5 hours, then at some time its speedometer must read 42 miles per hour. ◇

Consequences of the Mean-Value Theorem

There are several ways of writing the Mean-Value Theorem. For example, the equation

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

is equivalent to

$$f(b) - f(a) = (b-a)f'(c)$$

and hence to

$$f(b) = f(a) + (b-a)f'(c).$$

In this last form, the Mean-Value Theorem asserts that $f(b)$ is equal to $f(a)$ plus a quantity that involves the derivative f' at some number c between a and b. The following important corollaries are based on this alternative view of the Mean-Value Theorem.

Corollary 4.1.4. *If the derivative of a function is 0 throughout an interval I, then the function is constant on the interval.*

Proof

Let a and b be any two numbers in the interval I and let the function be denoted by f. To prove this corollary, it suffices to prove that $f(a) = f(b)$, for that is the defining property of a constant function.

By the Mean-Value Theorem in the form (1), there is a number c between a and b such that

$$f(b) = f(a) + (b-a)f'(c).$$

But $f'(c) = 0$, since $f'(x) = 0$ for all x in I. Hence

$$f(b) = f(a) + (b-a)(0)$$

which proves that $$f(b) = f(a).$$

•

When Corollary 4.1.4 is interpreted in terms of motion, it is quite plausible. It asserts that if an object has zero velocity for a period of time, then it does not move during that time.

EXAMPLE 5 Use calculus to show that $f(x) = (e^x + e^{-x})^2 - e^{2x} - e^{-2x}$ is a constant. Find the constant.

SOLUTION The function f is differentiable for all numbers x. Its derivative is

$$\begin{aligned} f'(x) &= 2(e^x + e^{-x})(e^x - e^{-x}) - 2e^{2x} + 2e^{-2x} \\ &= 2(e^{2x} - e^{-2x}) - 2e^{2x} + 2e^{-2x} \\ &= 0 \end{aligned}$$

Because $f'(x)$ is always zero, f must be a constant.

To find the constant, just evaluate $f(x)$ for any convenient value of x. For simplicity we choose $x = 0$: $f(0) = (e^0 + e^0)^2 - e^0 - e^0 = 2^2 - 2 = 2$. Thus,

$$(e^x + e^{-x})^2 - e^{2x} - e^{-2x} = 2 \qquad \text{for all numbers } x.$$

This result can also be obtained by squaring $e^x + e^{-x}$. ◇

Corollary 4.1.5. *If two functions have the same derivatives throughout an interval, then they differ by a constant. That is, if $F'(x) = G'(x)$ for all x in an interval, then there is a constant C such that $F(x) = G(x) + C$.*

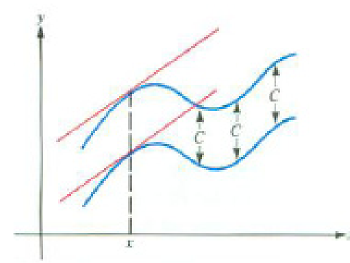


Figure 4.1.9:

Proof

Define a third function h by the equation $h(x) = F(x) - G(x)$. Then

$$h'(x) = F'(x) - G'(x) = 0. \qquad \text{since } F'(x) = G'(x)$$

Since the derivative if h is 0, Corollary 4.1.4 implies that h is constant, that is, $h(x) = C$ for some fixed number C. Thus

$$F(x) - G(x) = C \quad \text{or} \quad F(x) = G(x) + C,$$

and Corollary 4.1.5 is proved. •

Is Corollary 4.1.5 plausible when the derivative is interpreted as slope? In this case, the corollary asserts that if the graphs of two functions have the property that their tangent lines at points with the same x coordinate are parallel, then one graph can be obtained from the other by raising (or lowering) it by a constant amount C. If you sketch two such graphs (as in Figure 4.1.9, you will see that the corollary is reasonable.

EXAMPLE 6 What functions have a derivative equal to $2x$ everywhere?

SOLUTION One such solution is x^2; another is $x^2 + 25$. For any constant

In the language of Section 3.5, any antiderivative of $2x$ must be of the form $x^2 + C$.

C, $D(x^2 + C) = 2x$. Are there any other possibilities? Corollary 4.1.5 tells us there are not, for if F is a function such that $F'(x) = 2x$, then $F'(x) = (x^2)'$ for all x. Thus the functions F and x^2 differ by a constant, say C, that is,

$$F(x) = x^2 + C.$$

The only antiderivatives of $2x$ are of the form $x^2 + C$. ⋄

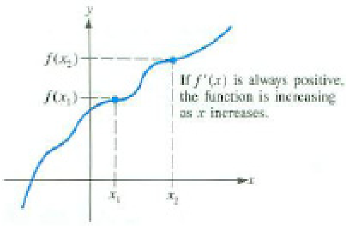


Figure 4.1.10:

Corollary 4.1.4 asserts that if $f'(x) = 0$ for all x, then f is a constant. What can be said about f if $f'(x)$ is *positive* for all x in an interval? In terms of the graph of f, this assumption implies that all the tangent lines slope upward. It is reasonable to expect that as we move from left to right on the graph in Figure 4.1.10, the y-coordinate increases, that is, the function is increasing. (See Section 1.1).)

Corollary 4.1.6. *If f is continuous on the closed interval $[a, b]$ and has a positive derivative on the open interval (a, b), then f is increasing on the interval $[a, b]$.*

If f is continuous on the closed interval $[a, b]$ and has a negative derivative on the open interval (a, b), then f is decreasing on the interval $[a, b]$.

Proof

We prove the "increasing" case; the other case is handled in Exercise 44. Take two numbers x_1 and x_2 such that

$$a \leq x_1 < x_2 \leq b.$$

The goal is to show that $f(x_2) > f(x_1)$.

By the Mean-Value Theorem, there is some number c between x_1 and x_2 such that

$$f(x_2) = f(x_1) + (x_2 - x_1)f'(c).$$

Now, since $x_2 > x_1$, we know $x_2 - x_1$ is positive. Since $f'(c)$ is assumed to be positive, and the product of two positive numbers is positive, it follows that

$$(x_2 - x_1)f'(c) > 0.$$

Thus, $f(x_2) > f(x_1)$, and so $f(x)$ is an increasing function. •

EXAMPLE 7 Determine whether $2x + \sin(x)$ is an increasing function, a decreasing function, or neither.

SOLUTION The funcion $2x + \sin(x)$ is the sum of two simpler functions: $2x$ and $\sin(x)$. The "$2x$" part is an increasing function. The second term, "$\sin(x)$", increases for x between 0 and $\pi/2$ and decreases for x between $\pi/2$

and π. It is not clear what type of function you will get when you add $2x$ and $\sin(x)$. Let's see what Corollary 4.1.6 tells us.

The derivative of $2x + \sin(x)$ is $2 + \cos(x)$. Since $\cos(x) \geq -1$ for all x,

$$(2x + \sin(x))' = 2 + \cos(x) \geq 2 + (-1) = 1.$$

Because $(2x+\sin(x))'$ is positive for all numbers x, $2x+\sin(x)$ is an increasing function. Figure 4.1.11 shows the graph of $2x+\sin(x)$ together with the graphs of $2x$ and $\sin(x)$. ◇

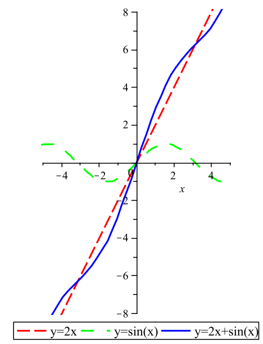


Figure 4.1.11:

Remark: Increasing/Decreasing at a Point

1. Corollary 4.1.6, and the definitions of increasing and decreasing, are stated in terms of intervals. When we talk about a function f increasing (or decreasing) "at a point c," here is what we mean: there is an interval (a, b) with $a < c < b$ where f is increasing.
2. When $f'(c) > 0$ and f' is continuous, the Permanence Property in Section 2.4) tells us there is an interval (a, b) containing c where $f'(x)$ remains positive for all numbers x in (a, b). Thus, f is increasing on (a, b), and hence increasing at c.

"A function is increasing at c" is shorthand for "a function is increasing in an interval that contains c."

More generally, if $f'(x)$ is never negative, that is $f'(x) \geq 0$ for all inputs x, then f is non-decreasing. In the same manner, if $f'(x) \leq 0$ for all inputs x, then f is a non-increasing function.

Summary

This section focused on three theorems, which we state informally. For the assumptions on the functions, see the formal statements in this section.

The Theorem of the Interior Extremum says that at a local extreme the derivative must be zero. (The converse is not true.)

Rolle's Theorem asserts that if a function has equal values at two inputs, its derivative must equal zero at least at one number between them. The Mean-Value Theorem, a generalization of Rolle's Theorem, asserts that for any chord on the graph of a function, there is a tangent line parallel to it. This means that for $a < b$ there is c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, or in a more useful form $f(b) = f(a) + f'(c)(b - a)$.

From the Mean-Value Theorem it follows that where a derivative is positive, a function is increasing; where it is negative it is decreasing; and where it stays at the value zero, it is constant. The last assertion implies that two antiderivatives of the same integrand differ by a constant (which may be zero).

EXERCISES for 4.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] State Rolle's Theorem in words, using as few mathematical symbols as you can.

2.[R] Draw a graph illustrating Rolle's Theorem. Be sure to identify the critical parts of the graph.

3.[R] Draw a graph illustrating the Mean-Value Theorem. Be sure to identify the critical parts of the graph.

4.[R] Express the Mean-Value Theorem in words, using no symbols to denote the function or the interval.

5.[R] Express the Mean-Value Theorem in symbols, where the function is denoted g and the interval is $[e, f]$.

6.[R] Which of the corollaries to the Mean-Value Theorem implies that

(a) if two cars on a straight road have the same velocity at every instant, they remain a fixed distance apart?

(b) If all tangents to a curve are horizontal, the curve is a horizontal line.

Explain each answer.

Exercises 7 to 12 concern the Theorem of the Interior Extremum.

7.[R] Consider the function $f(x) = x^2$ only for x in $[-1, 2]$.

(a) Graph the function $f(x)$ for x in $[-1, 2]$.

(b) What is the maximum value of $f(x)$ for x in the interval $[-1, 2]$?

(c) Does $f'(x)$ exist at the maximum?

(d) Does $f'(x)$ equal zero at the maximum?

(e) Does $f'(x)$ equal zero at the minimum?

8.[R] Consider the function $f(x) = \sin(x)$ only for x in $[0, \pi]$.

(a) Graph the function $f(x)$ for x in $[0, \pi]$.

(b) What is the maximum value of $f(x)$ for x in the interval $[0, \pi]$?

(c) Does $f'(x)$ exist at the maximum?

(d) Does $f'(x)$ equal zero at the maximum?

(e) Does $f'(x)$ equal zero at the minimum?

9.[R]

(a) Repeat Exercise 7 on the interval $[1, 2]$.

(b) Repeat Exercise 7 on the interval $(-1, 2)$.

(c) Repeat Exercise 7 on the interval $(1, 2)$.

(d) Repeat Exercise 8 on the interval $[0, 2\pi]$.

(e) Repeat Exercise 8 on the interval $(0, \pi)$.

(f) Repeat Exercise 8 on the interval $(0, 2\pi)$.

10.[R]

(a) Graph $y = -x^2 + 3x + 2$ for x in $[0, 2]$.

(b) Looking at the graph, estimate the x coordinate where the maximum value of y occurs for x in $[0, 2]$.

(c) Find where $dy/dx = 0$.

(d) Using (c), determine exactly where the maximum occurs.

11.[R]

(a) Graph $y = 2x^2 - 3x + 1$ for x in $[0, 1]$.

(b) Looking at the graph, estimate the x coordinate where the maximum value of y occurs for x in $[0, 1]$. At which value of x does it occur?

(c) Looking at the graph, estimate the x coordinate where the minimum value of y occurs for x in $[0, 12]$.

(d) Find where $dy/dx = 0$.

(e) Using (d), determine exactly where the minimum occurs.

12.[R] For each of the following functions, (a) show that the derivative of the given function is 0 when $x = 0$ and (b) decide whether the function has an extremum at $x = 0$.

(a) $x^2 \sin(x)$

(b) $1 - \cos(x)$

(c) $e^x - x$

(d) $x^2 - x^3$

Exercises 13 to 21 concern Rolle's Theorem.

13.[R]

(a) Graph $f(x) = x^{2/3}$ for x in $[-1, 1]$.

(b) Show that $f(-1) = f(1)$.

(c) Is there a number c in $(-1, 1)$ such that $f'(c) = 0$?

(d) Why does this not contradict Rolle's Theorem?

14.[R]

(a) Graph $f(x) = 1/x^2$ for x in $[-1, 1]$.

(b) Show that $f(-1) = f(1)$.

(c) Is there a number c in $(-1, 1)$ such that $f'(c) = 0$?

(d) Why does this not contradict Rolle's Theorem?

In Exercises 15 to 20, verify that the given function satisfies Rolle's Theorem for the given interval. Find all numbers c that satisfy the conclusion of the theorem.

15.[R] $f(x) = x^2 - 2x - 3$ and $[0, 2]$

16.[R] $f(x) = x^3 - x$ and $[-1, 1]$

17.[R] $f(x) = x^4 - 2x^2 + 1$ and $[-2, 2]$

18.[R] $f(x) = \sin(x) + \cos(x)$ and $[0, 4\pi]$

19.[R] $f(x) = e^x + e^{-x}$ and $[-2, 2]$

20.[R] $f(x) = x^2 e^{-x^2}$ and $[-2, 2]$

21.[M] Let $f(x) = \ln(x^2)$. Note that $f(-1) = f(1)$. Is there a number c in $(-1, 1)$ such that $f'(c) = 0$? If so, find at least one such number. If not, why is this not a contradiction of Rolle's Theorem?

Exercises 22 to 27 concern the Mean-Value Theorem.
In Exercises 22 to 25, find explicitly all values of c which satisfy the Mean-Value Theorem for the given functions and intervals.
22.[R] $f(x) = x^2 - 3x$ and $[1, 4]$
23.[R] $f(x) = 2x^2 + x + 1$ and $[-2, 3]$
24.[R] $f(x) = 3x + 5$ and $[1, 3]$
25.[R] $f(x) = 5x - 7$ and $[0, 4]$

26.[R]

(a) Graph $y = \sin(x)$ for x in $[\pi/2, 7\pi/2]$.

(b) Draw the chord joining $(\pi/2, f(\pi/2))$ and $(7\pi/2, f(7\pi/2))$.

(c) Draw all tangents to the graph parallel to the chord drawn in (b).

(d) Using (c), determine how many numbers c there are in $(\pi, 7\pi/2)$ such that

$$f'(c) = \frac{f(7\pi/2) - f(\pi/2)}{7\pi/2 - \pi/2}.$$

(e) Use the graph to estimate the values of the c's.

27.[R]

(a) Graph $y = \cos(x)$ for x in $[0, 9\pi/2]$.

(b) Draw the chord joining $(0, f(0))$ and $(9\pi/2, f(9\pi/2))$.

(c) Draw all tangents to the graph that are parallel to the chord drawn in (b).

(d) Using (c), determine how many numbers c there are in $(0, 9\pi/2)$ such that

$$f'(c) = \frac{f(9\pi/2) - f(0)}{9\pi/2 - 0}.$$

(e) Use the graph to estimate the values of the c's.

28.[R] At time t seconds a thrown ball has the height $f(t) = -16t^2 + 32t + 40$ feet.

(a) What is the initial height? That is, the height when t is zero.

(b) Show that after 2 seconds it returns to its initial height.

(c) What does Rolle's Theorem imply about the velocity of the ball?

(d) Verify Rolle's Theorem in this case by computing the numbers c which it asserts exist.

29.[R] Find all points where $f(x) = 2x^3(x - 1)$ can have an extreme value on the following intervals

(a) $(-1/2, 1)$

(b) $[-1/2, 1]$

(c) $[-1/2, 1/2]$

(d) $(-1/2, 1/2)$

30.[R] Let $f(x) = |2x - 1|$.

(a) Explain why $f'(1/2)$ does not exist.

(b) Find $f'(x)$. Hint: Write the absolute value in two parts, one for $x < 1/2$ and the other for $x > 1/2$.

(c) Does the Mean-Value Theorem apply on the interval $[-1, 2]$?

31.[R] The year is 2015. Because a gallon of gas costs six dollars and Highway 80 is full of tire-wrecking potholes, the California Highway Patrol no longer patrols the 77 miles between Sacramento and Berkeley. Instead it uses two cameras. One, in Sacramento, records the license number and time of a car on the freeway, and another does the same in Berkeley. A computer processes the data instantly. Assume that the two cameras show that a car that was in Sacramento at 10:45 reached Berkeley at 11:40. Show that the Mean-Value Theorem justifies giving the driver a ticket for exceeding the 70 mile-per-hour speed limit. (Of course, intuition justifies the ticket, but mentioning the Mean-Value Theorem is likely to impress a judge who studied calculus.) Note: While it makes a nice story to suggest that mentioning the Mean-Value Theorem will impress a judge who studied calculus, reality is that the California Vehicle Code forbids this way to catch speeders. It reads, "No speed trap shall be used in securing evidence as to the speed of any vehicle. A 'speed trap' is a particlar section of highway measured as to distance in order that the speed of a vehicle may be calculated by securing the time it takes the vehicle to travel the known distance. It sounds as though the lawmakers who wrote this law studied calculus.

32.[M] What is the shortest time for the trip from Berkeley to Sacramento for which the Mean-Value Theorem does not convict the driver of speeding? NOTE: See Exercise 31.

33.[R] Verify the Mean-Value Theorem for $f(t) = x^2e^{-x/3}$ on $[1, 10]$. NOTE: See Example 1.

34.[R] Find all antiderivatives of each of the following functions. Check your answer by differentiation.

(a) $3x^2$

(b) $\sin(x)$

(c) $\frac{1}{1+x^2}$

(d) e^x

35.[R] Find all antiderivatives of each of the following functions. Check your answer by differentiation.

(a) $\cos(x)$

(b) $\sec(x)\tan(x)$

(c) $1/x$ $(x > 0)$

(d) $\sqrt{x}$ $(x > 0)$

36.[R]

(a) Differentiate $\sec^2(x)$ and $\tan^2(x)$.

(b) The derivatives in (a) are equal. Corollary 4.1.5 then asserts that there exists a constant C such that $\sec^2(x) = \tan^2(x) + C$. Find the constant.

37.[R] Show by differentiation that $f(x) = \ln(x/5) - \ln(5x)$ is a constant for all values of x. Find the constant.

38.[M] Find all functions whose second derivative is 0 for all x in $(-\infty, \infty)$.

39.[M] Use Rolle's Theorem to determine how many real roots there are for the equation $x^3 - 6x^2 + 15x + 3 = 0$.

40.[M] Use Rolle's Theorem to determine how many real roots there are for the equation $3x^4 + 4x^3 - 12x^2 + 4 = 0$. Give intervals on which there is exactly one root.

41.[M] Use Rolle's Theorem to determine how many real roots there are for the polynomial $f(x) = 3x^4 + 4x^3 - 12x^2 + A$. That number may depend on A. For which A is there exactly one root? Are there any values of A for which there is an odd number of real roots? NOTE: Exercise 40 uses this equation with $A = 4$.

42.[M] Consider the equation $x^3 - ax^2 + 15x + 3 = 0$. The number of real roots to this equation depends on the value of a.

(a) Find all values of a when the equation has 3 real roots.

(b) Find all values of a when the equation has 1 real root.

(c) Are there any values of a with exactly two real roots?

NOTE: Exercise 39 uses this equation with $a = 6$.

43.[M] If f is differentiable for all real numbers and $f'(x) = 0$ has three solutions, what can be said about the number of solutions of $f(x) = 0$? of $f(x) = 5$?

44.[M] Prove the "decreasing" case of Corollary 4.1.6.

45.[M] For which values of the constant k is the function $7x + k\sin(2x)$ always increasing?

46.[C] If two functions have the same second derivative for all x in $(-\infty, \infty)$, what can be said about the relation between them?

47.[C] If a function f is differentiable for all x and c is a number, is there necessarily a chord of the graph of f that is parallel to the tangent line at $(c, f(c))$? Explain.

48.[C] Sketch a graph of a continuous function $f(x)$ defined for all numbers such that $f'(1)$ is 2, yet there is no open interval around 1 on which f is increasing.

Exercises 49 to 52 involve the **hyperbolic functions**. The **hyperbolic sine** function is $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and the **hyperbolic cosine** function is $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Hyperbolic functions are discussed in greater detail in Section 5.7.

49.[R]

(a) Show that $\frac{d}{dx}\sinh(x) = \cosh(x)$.

(b) Show that $\frac{d}{dx}\cosh(x) = \sinh(x)$.

50.[M] Define $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{2}{e^x + e^{-x}}$ and $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

(a) Show that $\frac{d}{dx}\tanh(x) = (\operatorname{sech}(x))^2$.

(b) Show that $\frac{d}{dx}\operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x)$.

51.[M] Use calculus to show that $(\cosh(x))^2 - (\sinh(x))^2$ is a constant. Find the constant.

52.[M] Use calculus to show that $(\operatorname{sech}(x))^2 + (\tanh(x))^2$ is a constant. Find the constant.

4.2 The First-Derivative and Graphing

Section 4.1 showed the connection between extrema and the places where the derivative is zero. In this section we use this connection to find high and low points on a graph.

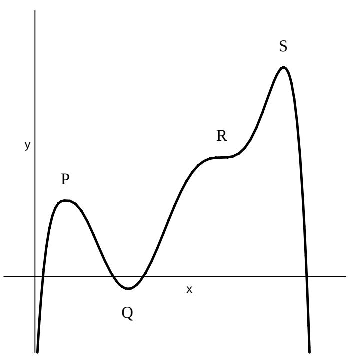


Figure 4.2.1:

The graph of a differentiable function f defined for all real numbers x is shown in Figure 4.2.1. The points P, Q, R, and S are of special interest. S is the highest point on the graph *for all x in the domain*. We call it a *global maximum* or *absolute maximum*. The point P is higher than all points *near it* on the graph; it is called a *local maximum* or *relative maximum*. Similarly, Q is called a *local minimum* or *relative minimum*. The point R is neither a relative maximum nor a relative minimum.

A point that is either a maximum or minimum is called an **extremum**. The plural of extremum is extrema.

If you were to walk left to right along the graph in Figure 4.2.1, you would call P the top of a hill, Q the bottom of a valley, and S the highest point on your walk (it is also a top of a hill). You might notice R, for you get a momentary break from climbing from Q to S. For just this one instant it would be like walking along a horizontal path.

These important aspects of a function and its graph are made precise in the following definitions. These definitions are phrased in terms of a general domain. In most cases the domain of the function will be an interval — open, closed, or half-open.

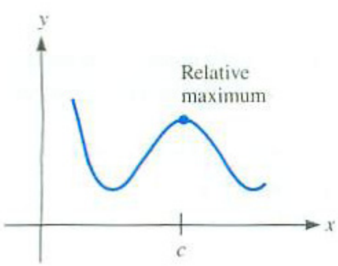


Figure 4.2.2:

DEFINITION *(Relative Maximum (Local Maximum))* The function f has a **relative maximum** (or **local maximum**) at a number c if there is an open interval around c such that $f(c) \geq f(x)$ for all x in that interval that lie in the domain of f.

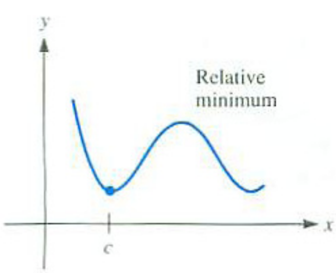


Figure 4.2.3:

DEFINITION *(Relative Minimum (Local Minimum))* The function f has a **relative minimum** (or **local minimum**) at a number c if there is an open interval around c such that $f(c) \leq f(x)$ for all x in that interval that lie in the domain of f.

Each global extremum is also a local extremum.

DEFINITION *(Absolute Maximum (Global Maximum))* The function f has a **absolute maximum** (or **global maximum**) at a number c if $f(c) \geq f(x)$ for all x in the domain of f.

DEFINITION *(Absolute Minimum (Global Minimum))* The function f has a **absolute minimum** (or **global minimum**) at a number c if $f(c) \leq f(x)$ for all x in the domain of f.

A local extremum is like the summit of a single mountain or an individual valley. A global maximum corresponds to Mt. Everest at more than 29,000 feet above sea level; a global minimum corresponds to the Mariana Trench in the Pacific Ocean 36,000 feet below sea level, the lowest point on the Earth's crust.

In this section it is assumed that the functions are differentiable. If a function is not differentiable at an isolated point, this point will need to be considered separately.

DEFINITION *(Critical Number and Critical Point)* A number c at which $f'(c) = 0$ is called a **critical number** for the function f. The corresponding point $(c, f(c))$ on the graph of f is a **critical point**.

Remark: Some texts define a critical number as a number where the derivative is 0 or else is not defined. Since we emphasize differentiable functions, a critical number is defined to be a number where the derivative is 0.

The Theorem of the Interior Extremum, in Section 4.1, says that every local maximum and minimum of a function f occurs where the tangent line to the curve either is horizontal or does not exist.

Some functions have extreme values, and others do not. The following theorem gives simple conditions under which both a global maximum and a global minimum are guaranteed to exist. To convince yourself that this is plausible, imagine drawing the graph of the function. At some point your pencil will reach a highest point and at another point a lowest point.

> **Theorem 4.2.1** (Extreme Value Theorem)**.** *Let f be a continuous function on a closed interval $[a, b]$. Then f attains an absolute maximum value $M = f(c)$ and an absolute minimum value $m = f(d)$ at some numbers c and d in $[a, b]$.*

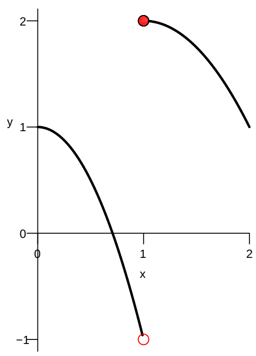


Figure 4.2.4:

EXAMPLE 1 Find the absolute extrema on the interval $[0, 2]$ of the function whose graph is shown in Figure 4.2.4.
SOLUTION The function has an absolute maximum value of 2 but no absolute minimum value. The range is $(-1, 2]$. This function takes on values that are arbitrarily close to -1, but -1 is not in the range of this function. This can occur only because the function is not continuous at $x = 1$. ◇

Recall that Corollary 4.1.6 provides a convenient test to determine if a function is increasing or decreasing at a point: if $f'(c) > 0$ then f is increasing at $x = c$ and if $f'(c) < 0$ then f is decreasing at $x = c$.

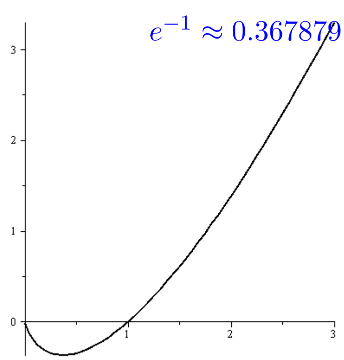
> **WARNING** *D*ifferentiable implies continuous, so "not continuous" implies "not differentiable."

EXAMPLE 2 Let $f(x) = x\ln(x)$ for all $x > 0$. Determine the intervals on which f is increasing, decreasing, or neither.
SOLUTION The function is increasing at numbers x where $f'(x) > 0$ and decreasing where $f'(x) < 0$. More effort is needed to determine the behavior at points where $f'(x) = 0$ (or does not exist). (Observe that the natural domain of f is $x > 0$.) The Product Rule allows us to find

$$f'(x) = \ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1.$$

In order to find where $f'(x)$ is positive or is negative, we first find where it is zero. At such numbers the derivative may switch sign, and the function switch between increasing and decreasing. So we solve the equation:

$$\begin{aligned} f'(x) &= 0 \\ \ln(x) + 1 &= 0 \\ \ln(x) &= -1 \\ e^{\ln(x)} &= e^{-1} \\ x &= e^{-1}. \end{aligned}$$



When x is larger than e^{-1}, $\ln(x)$ is larger than -1 so that $f'(x) = \ln(x)+1$ is positive and f is increasing. Finally, f is decreasing when x is between 0 and e^{-1} because $\ln(x) < -1$, which makes $f'(x) = \ln(x)+1$ negative. The graph of $y = x\ln(x)$ in Figure 4.2.5 confirms these findings. In addition, observe that $x = e^{-1}$ is a minimum of this function. ◇

Using Critical Numbers to Identify Local Extrema

The previous examples show there is a close connection between critical points and local extrema. Notice that, generally, just to the left of a local maximum the function is increasing, while just to the right it is decreasing. The opposite holds for a local minimum. The First-Derivative Test for a Local Extreme Value at $x = c$ gives a precise statement of this result.

First-Derivative Test for a Local Extreme Value at $x = c$

Theorem 4.2.2. *Let f be a function and let c be a number in its domain. Suppose f is continuous on an open interval that contains c and is differentiable on that interval, except possibly at c. Then:*

1. *If f' changes from positive to negative as x moves from left to right through the value c,*
   then f has a local maximum at c.
2. *If f' changes from negative to positive as x moves from left to right through the value c,*
   then f has a local minimum at c.
3. *If f' does not change sign at c,*
   then f does not have a local extremum at $x = c$.

EXAMPLE 3 Classify all critical numbers of $f(x) = 3x^5 - 20x^3 + 10$ as a local maximum, local minimum, or neither.
SOLUTION To identify the critical numbers of f, we find and factor the derivative:

$$f'(x) = 15x^4 - 60x^2 = 15x^2(x^2 - 4) = 15x^2(x - 2)(x + 2).$$

The critical numbers of f are $x = 0$, $x = 2$, and $x = -2$. To determine if any of these numbers provide local extrema it is necessary to know where f is increasing and where it is decreasing.

Because f' is continuous the three critical numbers are the only places the sign of f' can possibly change. All that remains is to determine if f is increasing or decreasing on the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$. This is easily answered from table of function values shown in the first two rows of Table 4.2.1. Observe that $f(-2) = 74 > 10 = f(0)$; this means f is decreasing on $(-2, 0)$. Likewise, f must be decreasing on $(0, 2)$ because $f(0) = 10 > -54 = f(2)$. For the two unbounded intervals, limits at $\pm\infty$ must be used but the overall idea is the same. Since $\lim_{x\to-\infty} f(x) = -\infty$,

x	$\to -\infty$	-2	0	2	$\to \infty$
$f(x)$	$-\infty$	74	10	-54	∞
$f'(x)$		0	0	0	

Table 4.2.1:

the function must be increasing on $(-\infty, -2)$. Likewise, in order to have $\lim_{x\to\infty} f(x) = +\infty$, f must be increasing on $(2, \infty)$. (See Figure 4.2.6.)

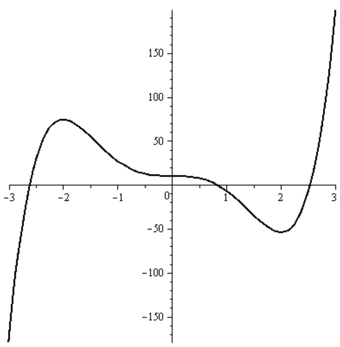


Figure 4.2.6:

To conclude, because the graph of f changes from increasing to decreasing at $x = -2$, there is a local maximum at $(-2, 74)$. At $x = 2$ the graph changes from decreasing to increasing, so a local minimum occurs at $(2, -54)$. Because the derivative does not change sign at $x = 0$, this critical number is not a local extreme. ◇

EXAMPLE 4 Find all local extrema of $f(x) = (x+1)^{2/7}e^{-x}$.
SOLUTION (Observe that the domain of f is $(-\infty, \infty)$.) The Product and Chain Rules for derivatives can be used to obtain

$$\begin{aligned} f'(x) &= \tfrac{2}{7}(x+1)^{-5/7}e^{-x} + (x+1)^{2/7}e^{-x}(-1) \\ &= \tfrac{2}{7}(x+1)^{-5/7}e^{-x} - (x+1)^{2/7}e^{-x} \\ &= (x+1)^{-5/7}e^{-x}\left(\tfrac{2}{7} - (x+1)\right) \\ &= (x+1)^{-5/7}e^{-x}\left(-x - \tfrac{5}{7}\right) \\ &= \frac{-x-\frac{5}{7}}{(x+1)^{5/7}e^{x}}. \end{aligned}$$

The only solution to $f'(x) = 0$ is $x = -5/7$, so $c = -5/7$ is the only critical number. In addition, because the denominator of $f'(x)$ is zero when $x = -1$, f is not differentiable for $x = -1$. Using the information in Table 4.2.2, we

x	$\to -\infty$	-1	$-5/7$	$\to \infty$
$f(x)$	∞	0	$(2/7)^{(2/7)}e^{5/7} \approx 1.42811$	0
$f'(x)$		dne	0	

Table 4.2.2: Note that dne means the limit does not exist.

conclude f is decreasing on $(-\infty, -1)$, increasing on $(-1, -5/7)$, and decreasing on $(-5/7, \infty)$. By the First-Derivative Test, f has a local minimum at $(-1, 0)$ and a local maximum at $(-5/7, (2/7)^{(2/7)}e^{5/7}) \approx (-0.71429, 1.42811)$.

Notice that the First-Derivative Test applies at $x = -1$ even though f is not differentiable for $x = -1$. A graph of $y = f(x)$ is shown in Figure 4.2.7. (See also Exercise 27 in Section 4.3.) ◇

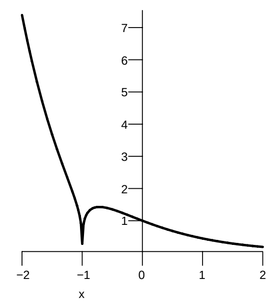


Figure 4.2.7:

Extreme Values on a Closed Interval

Many applied problems involve a continuous function only on a closed interval $[a, b]$. (See Section 4.1.)

The Extreme Value Theorem guarantees the function attains both a maximum and a minimum at some point in the interval. The extreme values occur either at

1. an endpoint ($x = a$ or $x = b$),
2. a critical number ($x = c$ where $f'(c) = 0$), or
3. where f is not differentiable ($x = c$ where $f'(c)$ is not defined).

EXAMPLE 5 Find the absolute maximum and minimum values of $f(x) = x^4 - 8x^2 + 1$ on the interval $[-1, 3]$.

SOLUTION The function is continuous on a closed and bounded interval. The absolute maximum and minimum values occur either at a critical point or at an endpoint of the interval. The endpoints are $x = -1$ and $x = 3$. To find the critical points we solve $f'(x) = 0$:

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2) = 0.$$

There are three critical numbers, $x = 0$, 2, and -2, but only $x = 0$ and $x = 2$ are in the interval. The intervals where the graph of $y = f(x)$ is increasing and decreasing can be determined from the information in Table 4.2.3.

x	-1	0	2	3
$f(x)$	-6	1	-15	10
$f'(x)$		0	0	0

Table 4.2.3:

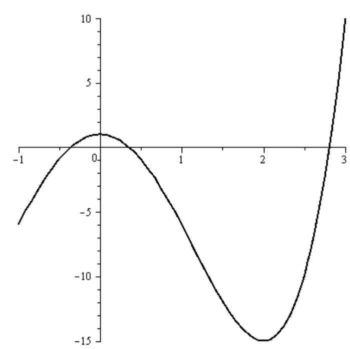


Figure 4.2.8:

Since we are looking only for global extrema on a closed interval, it is unnecessary to determine these intervals or to classify critical points as local extrema. Instead, we simply scan the list of function values at the endpoints and at the critical numbers – row 2 of Table 4.2.3 – for the largest and smallest values of $f(x)$. The largest value is 10, so the global maximum occurs at $x = 3$. The smallest value is -15, so the global minimum occurs at $x = 2$. (See Figure 4.2.8.) ◇

In Example 5 it was not necessary to determine the intervals on which the function is increasing and decreasing, nor did we need to identify the local extreme values. (See also Exercise 5.)

Summary

This section shows how to use the first derivative to find extreme values of a function. Namely, identify when the derivative is zero, positive, and negative, and where it changes sign.

A continuous function on a closed and bounded interval always has a maximum and a minimum. All extrema occur either at an endpoint, a critical number (where $f'(c) = 0$), or where f is not differentiable.

EXERCISES for 4.2

Key: R–routine, M–moderate, C–challenging

In Exercises 1 to 28, sketch the graph of the given function. Find all intercepts and critical points, determine the intervals where the function is increasing and where it is decreasing, and identify all local extreme values.

1.[R] $f(x) = x^5$

2.[R] $f(x) = (x-1)^4$

3.[R] $f(x) = 3x^4 + x^3$

4.[R] $f(x) = 2x^3 + 3x^2$

5.[R] $f(x) = x^4 - 8x^2 + 1$

6.[R] $f(x) = x^3 - 3x^2 + 3x$

7.[R] $f(x) = x^4 - 4x + 3$

8.[R] $f(x) = 2x^2 + 3x + 5$

9.[R] $f(x) = x^4 + 2x^3 - 3x^2$

10.[R] $f(x) = 2x^3 + 3x^2 - 6x$

11.[R] $f(x) = xe^{-x/2}$

12.[R] $f(x) = xe^{x/3}$

13.[R] $f(x) = e^{-x^2}$

14.[R] $f(x) = xe^{-x^2/2}$

15.[R] $f(x) = x\sin(x) + \cos(x)$

16.[R] $f(x) = x\cos(x) - \sin(x)$

17.[R] $f(x) = \frac{\cos(x)-1}{x^2}$

18.[R] $f(x) = x\ln(x)$

19.[R] $f(x) = \frac{\ln(x)}{x}$

20.[R] $f(x) = \frac{e^x-1}{x}$

21.[R] $f(x) = \frac{e^{-x}}{x}$

22.[R] $f(x) = \frac{x-\arctan(x)}{x^3}$

23.[R] $f(x) = \dfrac{3x+1}{3x-1}$

24.[R] $f(x) = \dfrac{x}{x^2+1}$

25.[R] $f(x) = \dfrac{x}{x^2-1}$

26.[R] $f(x) = \dfrac{1}{2x^2-x}$

27.[R] $f(x) = \dfrac{1}{x^2-3x+2}$

28.[R] $f(x) = \dfrac{\sqrt{x^2+1}}{x}$

In Exercises 29 to 36 sketch the general shape of the graph, using the given information. Assume the function and its derivative are defined for all x and are continuous. Explain your reasoning.

29.[R] Critical point $(1,2)$, $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$.

30.[R] Critical point $(1,2)$ and $f'(x) < 0$ for all x except $x = 1$.

31.[R] x intercept -1, critical points $(1,3)$ and $(2,1)$, $\lim_{x\to\infty} f(x) = 4$, $\lim_{x\to-\infty} f(x) = -1$.

32.[R] y intercept 3, critical point $(1,2)$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 4$.

33.[R] x intercept -1, critical points $(1,5)$ and $(2,4)$, $\lim_{x\to\infty} f(x) = 5$, $\lim_{x\to-\infty} f(x) = -\infty$.

34.[R] x intercept 1, y intercept 2, critical points $(1,0)$ and $(4,4)$, $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to-\infty} f(x) = \infty$.

35.[R] x intercepts 2 and 4, y intercept 2, critical points $(1,3)$ and $(3,-1)$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 1$.

36.[R] No x intercepts, y intercept 1, no critical points, $\lim_{x\to\infty} f(x) = 2$, $\lim_{x\to-\infty} f(x) = 0$.

Exercises 37 to 52 concern functions whose domains are restricted to closed intervals. In each, find the maximum and minimum values for the given function on the given interval.

37.[R] $f(x) = x^2 - x^4$ on $[0,1]$

38.[R] $f(x) = 4x - x^2$ on $[0,5]$

39.[R] $f(x) = 2x^2 - 5x$ on $[-1,1]$

40.[R] $f(x) = x^3 - 2x^2 + 5x$ on $[-1,3]$

41.[R] $f(x) = \dfrac{x}{x^2+1}$ on $[0,3]$

42.[R] $f(x) = x^2 + x^4$ on $[0,1]$

43.[R] $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$ on $[0,3]$

44.[R] $f(x) = \sin(x) + \cos(x)$ on $[0,\pi]$

45.[R] $f(x) = \sin(x) - \cos(x)$ on $[0,\pi]$

46.[R] $f(x) = x + \sin(x)$ on $[-\pi/2,\pi/2]$

47.[R] $f(x) = x + \sin(x)$ on $[-\pi,2\pi]$

48.[R] $f(x) = x/2 + \sin(x)$ on $[-\pi,2\pi]$

49.[R] $f(x) = 2\sin(x) - \sin(2x)$ on $[-\pi,\pi]$

50.[R] $f(x) = \sin(x^2) + \cos(x^2)$ on $[0,\sqrt{2\pi}]$

51.[R] $f(x) = \sin(x) - \cos(x)$ on $[-2\pi,2\pi]$

52.[R] $f(x) = \sin^2(x) - \cos^2(x)$ on $[-2\pi,2\pi]$

In Exercises 53 to 59 graph the function.

53.[R] $f(x) = \dfrac{\sin(x)}{1 + 2\cos(x)}$

54.[R] $f(x) = \dfrac{\sqrt{x^2 - 1}}{x}$

55.[R] $f(x) = \dfrac{1}{(x-1)^2(x-2)}$

56.[R] $f(x) = \dfrac{3x^2 + 5}{x^2 - 1}$

57.[R] $f(x) = 2x^{1/3} + x^{4/3}$

58.[R] $f(x) = \dfrac{3x^2 + 5}{x^2 + 1}$

59.[R] $f(x) = \sqrt{3}\sin(x) + \cos(x)$

60.[M] Graph $f(x) = (x^2 - 9)^{1/3} e^{-x}$. Hint: This function is difficult to graph in one picture. Instead, create separate sketches for $x > 0$ and for $x < 0$. Watch out for the points where f is not differentiable.

61.[M] A certain differentiable function has $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$. Moreover, $f(0) = 3$, $f(1) = 1$, and $f(2) = 2$.

(a) What is the minimum value of $f(x)$ for x in $[0, 2]$? Why?

(b) What is the maximum value of $f(x)$ for x in $[0, 2]$? Why?

In Exercises 62 to 64 decide if there is a function that meets all of the stated conditions. If you think there is such a function, sketch its possible graph. Otherwise, explain why a function cannot meet all of the conditions.

62.[M] $f(x) > 0$ for all x, $f'(x) < 0$ for all x

63.[M] $f(3) = 1$, $f(5) = 1$, $f'(x) > 0$ for x in $[3, 5]$

64.[M] $f'(x) \neq 0$ for all x except $x = 3$ and 5, when $f'(x) = 0$ and $f(x) = 0$ for $x = -2$, 4, and 5

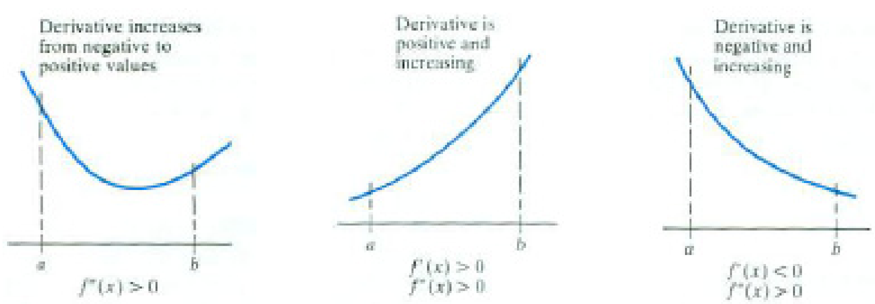
65.[M] What is the minimum value of $y = (x^3 - x)/(x^2 - 4)$ for $x > 2$?

4.3 The Second Derivative and Graphing

The sign of the first derivative tells whether a function is increasing or decreasing. In this section we examine what the sign of the second derivative tells us about a function and its graph. This information will be used to help graph functions and also to provide an additional way to test whether a critical point is a maximum or minimum.

Concavity and Points of Inflection

The second derivative is the derivative of the first derivative. Thus, the sign of the second derivative determines if the first derivative is increasing or decreasing. For example, if $f''(x)$ is positive for all x in an interval (a, b), then f' is an increasing function throughout the interval (a, b). In other words, the slope of the graph of $y = f(x)$ increases as x increases from left to right on that part of the graph corresponding to (a, b). The slope may increase from



(a) (b) (c)

Figure 4.3.1:

As you drive along it, going from left to right, you keep turning the steering wheel counterclockwise.

negative values to zero to positive values, as in Figure 4.3.1(a). Or the slope may be positive throughout (a, b), as in Figure 4.3.1(b). Or the slope may be negative throughout (a, b), as in Figure 4.3.1(c).

In the same way, if $f''(x)$ is negative on the interval (a, b) then f' is decreasing on (a, b). The slope of the graph of $y = f(x)$ decreases as x increases from left to right on that part of the graph corresponding to (a, b).

DEFINITION *(Concave Up and Concave Down)*

A function f whose first derivative is increasing throughout the open interval (a, b) is called **concave up** in that interval.

A function f whose first derivative is decreasing throughout the open interval (a, b) is called **concave down** in that interval.

When a curve is concave up, it lies above its tangent lines and below its chords. The graph of a concave up function is shaped like a cup. See Figure 4.3.2.

When a curve is concave down, it lies below its tangent lines and above its chords. The graph of a concave *down* function is shaped like a *frown*. See Figure 4.3.3.

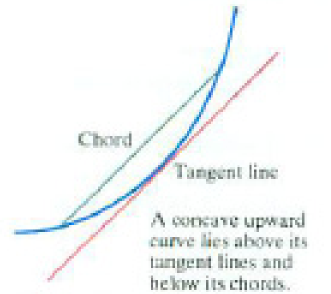


Figure 4.3.2:

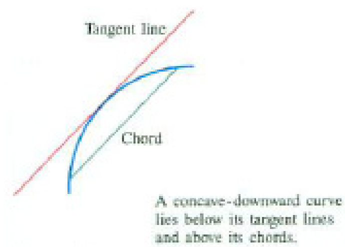


Figure 4.3.3:

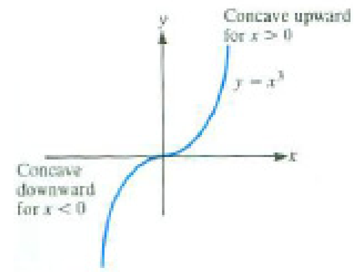


Figure 4.3.4:

Convex and Concave Sets

In more advanced courses "concave up" is called "convex." This is because the set in the xy-plane above this part of a graph is a convex set. (A convex set is a set with the property that any two points P and Q in the set the line segment joining them also lies in the set. See also Exercises 26 to 32 in Section 2.4.) In the same way "concave down" is called "concave." For instance, the part of the graph of $y = x^3$ to the right of the x-axis is convex and the part to the left is concave.

EXAMPLE 1 Where is the graph of $f(x) = x^3$ concave up? concave down?

SOLUTION First, compute the second derivative: $f''(x) = 6x$. Clearly, $6x$ is positive when x is positive and negative when x is negative. Thus, the graph is concave up for $x > 0$ and is concave down for $x < 0$. Note that the sense of concavity changes at $x = 0$, where $f''(x) = 0$. (See Figure 4.3.4.) ◇

In an interval where $f''(x)$ is positive, the function $f'(x)$ is increasing, and so the function f is concave up. However, if a function is concave up, $f''(x)$ need not be positive for all x in the interval. For instance, consider $y = x^4$. Even though the second derivative $12x^2$ is zero for $x = 0$, the first derivative $4x^3$ is increasing on any interval, so the graph is concave up over any interval.

Any point where the graph of a function changes concavity is important.

DEFINITION *(Inflection Number and Inflection Point)* Let f be a function and let a be a number. Assume there are numbers b and c such that $b < a < c$ and

1. f is continuous on the open interval (b, c)
2. f is concave up on (b, a) and concave down on (a, c)
   or
   f is concave down on (b, a) and concave up on (a, c).

Then, the point $(a, f(a))$ is called an **inflection point** or **point of inflection** of f. The number a is called an **inflection number** of f.

Notice that having $f''(a) = 0$ does not automatically make a an inflection number of f. To be an inflection number, the concavity has to change at a.

Observe that if the second derivative changes sign at the number a, then a is an inflection number. If the second derivative exists at an inflection number, it must be 0. But there can be an inflection point if $f''(a)$ is not defined. This is illustrated in the next example.

EXAMPLE 2 Examine the concavity of the graph of $y = x^{1/3}$.
SOLUTION Here $y' = \frac{1}{3}x^{-2/3}$ and $y'' = \frac{1}{3}\left(\frac{-2}{9}\right)x^{-5/3}$. Althought $x = 0$ is in the domain of this function, neither y' nor y'' is defined for $x = 0$. When x is negative, y'' is positive; when x is positive, y'' is negative. Thus, the concavity changes from concave up to concave down at $x = 0$. This means $x = 0$ is an inflection number and $(0, 0)$ is an inflection point. See Figure 4.3.5. ◇

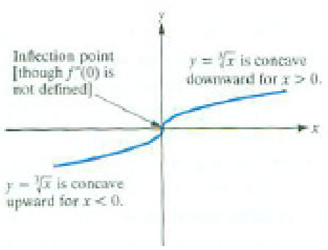


Figure 4.3.5:

The simplest way to look for inflection points is to use both the first and second derivatives:

To find inflection points of $y = f(x)$:

1. Compute $f'(x)$ and $f''(x)$.
2. Look for numbers a such that f'' is not defined at a.
3. Look for numbers a such that $f''(a) = 0$
4. For each interval defined by the numbers found in Steps 2 and 3, determine the sign of $f''(x)$.

This process can be implemented using the same ideas used in Section 4.2 to identify critical points, as Example 3 shows.

EXAMPLE 3 Find the inflection point(s) of $f(x) = x^4 - 8x^3 + 18x^2$.
SOLUTION First, $f'(x) = 4x^3 - 24x^2 + 36x$ and

$$f''(x) = 12x^2 - 48x + 36 = 12(x^2 - 4x + 3) = 12(x - 1)(x - 3).$$

Because f'' is defined for all real numbers, the only candidate for inflection numbers are the solutions to $f''(x) = 0$. Solving $f''(x) = 0$ yields:

$$0 = 12(x - 1)(x - 3).$$

Hence $x - 1 = 0$ or $x - 3 = 0$, and $x = 1$ or $x = 3$.

To decide whether 1 or 3 are inflection numbers of f, look at the sign of $f''(x) = 12(x-1)(x-3)$. For $x > 3$ both $x-1$ and $x-3$ are positive, so $f''(x)$ is positive. For x in $(1, 3)$, $x - 1$ is positive and $x - 3$ is negative, so $f''(x)$ is

x	$(-\infty, 1)$	1	$(1, 3)$	3	$(3, \infty)$
$f''(x)$	$+$	0	$-$	0	$+$

Table 4.3.1:

negative. For $x < 1$, both $x - 1$ and $x - 3$ are negative, so $f''(x)$ is positive. This is recorded in Table 4.3.1. Since sign changes in $f''(x)$ correspond to changes in concavity of the graph of f, this function has two inflection points: $(1, 11)$ and $(3, 27)$. (See Figure 4.3.6.)

◇

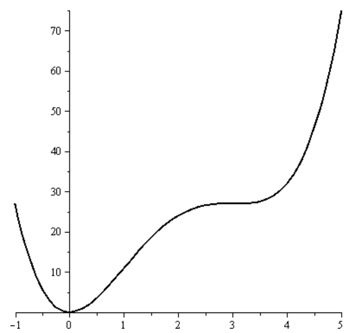


Figure 4.3.6:

Using Concavity in Graphing

The second derivative, together with the first derivative and the other tools of graphing, can help us sketch the graph of a function. Example 4 continues Example 3.

EXAMPLE 4 Graph $f(x) = x^4 - 8x^3 + 18x^2$.

SOLUTION Because f is defined for all real numbers and $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty$, it has no asymptotes. Since $f(0) = 0^4 - 8(0^3) + 18(0^2)$, its y intercept is 0. To find its x intercepts we look for solutions to the equation

$$\begin{aligned} x^4 - 8x^3 + 18x^2 &= 0 \\ x^2(x^2 - 8x + 18) &= 0. \end{aligned}$$

Thus $x = 0$ or $x^2 - 8x + 18 = 0$. The quadratic equation can be solved by the quadratic formula. The discriminant is $(-8)^2 - 4(1)(18) = -8$ which is negative, so there are no real solutions of $x^2 - 8x + 18 = 0$. The only x intercept of $y = f(x)$ is $x = 0$.

The **discriminant** of $ax^2 + bx + c$ is $b^2 - 4ac$.

In Example 3 the first derivative was found:

Analysis based on $f'(x)$

$$f'(x) = 4x^3 - 24x^2 + 36x = 4x(x^2 - 6x + 9) = 4x(x - 3)^2.$$

Thus, $f'(x) = 0$ only when $x = 0$ and $x = 3$. The two critical points are $(0, f(0)) = (0, 0)$ and $(3, f(3)) = (3, 27)$. The information in Table 4.3.2 allows us to conclude that the function f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ with a local minimum at $(0, 0)$.

x	$(-\infty, 0)$	0	$(0, 3)$	3	$(3, \infty)$
$f'(x)$	$-$	0	$+$	0	$+$

Table 4.3.2:

By Example 3, the graph is concave up on $(-\infty, 1)$ and $(3, \infty)$ and concave down on $(1, 3)$.

To begin to sketch the graph of $y = f(x)$, plot the three points $(0, f(0)) = (0, 0)$, $(1, f(1)) = (1, 11)$, and $(3, f(3)) = (3, 27)$. These three points divide the domain into four intervals. On $(-\infty, 0)$ the function is decreasing and concave up; on $(0, 1)$ it is increasing and concave up; on $(1, 3)$ it is increasing and concave down; and on $(3, \infty)$ it is once again increasing and concave up. The final graph is shown in Figure 4.3.7. ◇

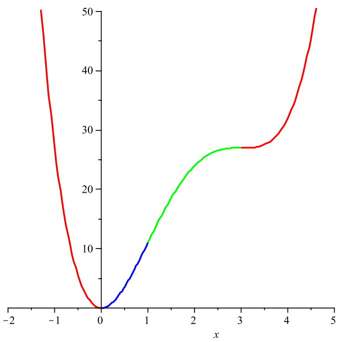


Figure 4.3.7:

(a) (b) (c) (d)

Figure 4.3.8: The general shape of a function that is (a) increasing and concave up, (b) increasing and concave down, (c) decreasing and concave up, and (d) decreasing and concave down

The procedure demonstrated in Example 4 has several advantages. Note that it was necessary to evaluate $f(x)$ only at a few "important" inputs x. These inputs cut the domain into intervals where neither the first derivative nor the second derivative changes sign. On each of these intervals the graph of the function will have one of the four shapes shown in Figure 4.3.8. A graph usually is made up of these four shapes.

Local Extrema and the Second-Derivative Test

The second derivative is also useful in testing whether a critical number corresponds to a relative minimum or relative maximum. For this, we will use the relationships between concavity and tangent lines shown in Figures 4.3.2 and 4.3.3.

Let a be a critical number for the function f. Assume, for instance, that $f''(a)$ is negative. If f'' is continuous in some open interval that contains a, then (by the Permanence Property) $f''(x)$ remains negative for a suitably small open interval that contains a. This means the graph of f is concave down near $(a, f(a))$, hence it lies below its tangent lines. In particular, it lies below the horizontal tangent line at the critical point $(a, f(a))$, as illustrated in Figure 4.3.9. Thus the function f has a *relative maximum* at the critical

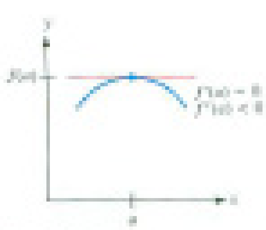


Figure 4.3.9:

number a. Similarly, if $f'(a) = 0$ and $f''(a) > 0$, the critical point $(a, f(a))$ is a relative minimum because the graph of f is concave up and lies above the horizontal tangent line at $(a, f(a))$. These observations suggest the following test for a relative extremum.

Theorem 4.3.1. *Second-Derivative Test for Relative Extrema Let f be a function such that $f'(x)$ is defined at least on some open interval containing the number a. Assume that $f''(x)$ is continuous and $f''(a)$ is defined.*

If $f'(a) = 0$ and $f''(a) < 0$, then f has a relative minimum at $(a, f(a))$.

If $f'(a) = 0$ and $f''(a) > 0$, then f has a relative maximum at $(a, f(a))$.

EXAMPLE 5 Use the Second-Derivative Test to classify all local extrema of the function $f(x) = x^4 - 8x^3 + 18x^2$.

Compare with Examples 3 and 4.

SOLUTION This is the same function analyzed in Examples 3 and 4. The two critical points are $(0, 0)$ and $(3, 27)$. The second derivative is $f''(x) = 12x^2 - 48x + 36$. At $x = 0$ we have

$$f''(0) = 12(0^2) - 48(0) + 36 = 36,$$

which is positive. Since $f'(0) = 0$ and $f''(0) > 0$, f has a local minimum at $(0, 0)$. At $x = 3$ we have

$$f''(3) = 12(3^2) - 48(3) + 36 = 0.$$

Since $f''(3) = 0$, the Second-Derivative Test tells us nothing about the critical number 3.

This is consistent with our previous findings. The point at $(3, 27)$ is an inflection point and not a local extreme point. ⋄

Summary

Table 4.3.3 shows the meaning of the signs of $f(x)$, $f'(x)$, and $f''(x)$ in terms of the graph of $y = f(x)$.

The graph has a critical point at $(a, f(a))$ whenever $f'(a) = 0$ (or $f'(a)$ does not exist). This critical point is an extremem of f if the first derivative changes sign at $x = a$; a maximum if the first derivative changes from positive

	is positive (> 0).	is negative (< 0).	changes sign.
Where the ordinate $f(x)$	the graph is above the x-axis.	the graph is below the x-axis.	the graph crosses th x-axis.
Where the slope $f'(x)$	the graph slopes upward.	the graph slopes downward.	the graph has a hori zontal tangent and relative extremum.
Where $f''(x)$	the graph is concave up (like a cup).	the graph is concave down (like a frown).	the graph has an in flection point.

Table 4.3.3: EDITOR: This table should appear after the first, short, paragraph of the Summary.

to negative and a minimum if the first derivative changes from negative to positive.

Keep in mind that the graph has an inflection point at $(a, f(a))$ when the sign of $f''(x)$ changes at $x = a$. This can occur when either $f''(a) = 0$ or when $f''(a)$ is not defined. Similarly, a graph can have a maximum or minimum at $(a, f(a))$ when either $f'(a) = 0$ or $f'(a)$ is not defined.

EXERCISES for 4.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 16 describe the intervals where the function is concave up and concave down and give any inflection points.

1.[R] $f(x) = x^3 - 3x^2 + 2$

2.[R] $f(x) = x^3 - 6x^2 + 1$

3.[R] $f(x) = x^2 + x + 1$

4.[R] $f(x) = 2x^2 - 5x$

5.[R] $f(x) = x^4 - 4x^3$

6.[R] $f(x) = 3x^5 - 5x^4$

7.[R] $f(x) = \frac{1}{1+x^2}$

8.[R] $f(x) = \frac{1}{1+x^4}$

9.[R] $f(x) = x^3 + 6x^2 - 15x$

10.[R] $f(x) = \frac{x^2}{2} + \frac{1}{x}$

11.[R] $f(x) = e^{-x^2}$

12.[R] $f(x) = xe^x$

13.[R] $f(x) = \tan(x)$

14.[R] $f(x) = \sin(x) + \sqrt{3}\cos(x)$

15.[R] $f(x) = \cos(x)$

16.[R] $f(x) = \cos(x) + \sin(x)$

In Exercises 17 to 29 graph the polynomials, showing critical points, inflection points, and intercepts.

17.[R] $f(x) = x^3 + 3x^2$

18.[R] $f(x) = 2x^3 + 9x^2$

19.[R] $f(x) = x^4 - 4x^3 + 6x^2$

20.[R] $f(x) = x^4 + 4x^3 + 6x^2 - 2$

21.[R] $f(x) = x^4 - 6x^3 + 12x^2$

22.[R] $f(x) = 2x^6 - 10x^4 + 10$

23.[R] $f(x) = 2x^6 + 3x^5 - 10x^4$

24.[R] $f(x) = 3x^4 + 4x^3 - 12x^2 + 4$

25.[R] $f(x) = xe^{-x}$

26.[R] $f(x) = e^{x^3}$

27.[R] $f(x) = 3x^5 - 20x^3 + 10$ NOTE: This function was first encountered in Example 3 in Section 4.2.

28.[R] $f(x) = 3x^4 + 4x^3 - 12x^2 + 4$

29.[R] $f(x) = 2x^6 - 15x^4 + 20x^3 - 20x + 10$

In each of Exercises 30 to 37 sketch the general appearance of the graph of the given function near $(1, 1)$ on the basis of the information given. Assume that f, f', and f'' are continuous.

30.[R] $f(1) = 1$, $f'(1) = 0$, $f''(1) = 1$

31.[R] $f(1) = 1$, $f'(1) = 0$, $f''(1) = -1$

32.[R] $f(1) = 1$, $f'(1) = 0$, $f''(1) = 0$ NOTE: Sketch four quite different possibilities.

33.[R] $f(1) = 1$, $f'(1) = 0$, $f''(1) = 0$, $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$

34.[R] $f(1) = 1$, $f'(1) = 0$, $f''(1) = 1$ and $f''(x) < 0$ for x near 1

35.[R] $f(1) = 1$, $f'(1) = 1$, $f''(1) = -1$

36.[R] $f(1) = 1$, $f'(1) = 1$, $f''(1) = 0$, $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$

37.[R] $f(1) = 1$, $f'(1) = 1$, $f''(1) = 0$ and $f''(x) > 0$ for x near 1

38.[R] Find all inflection points of $f(x) = x\ln(x)$. On what intervals is the graph of $y = f(x)$ concave up? concave down? Graph $y = f(x)$ on an interval large enough to clearly show all interesting features of the graph. On what intervals is the graph increasing? decreasing? NOTE: This graph was first encountered in Example 2.

39.[R] Find all inflection points of $f(x) = x + \ln(x)$. On what intervals is the graph of $y = f(x)$ concave up? concave down? Graph $y = f(x)$ on an interval large enough to show all interesting features of the graph. On what intervals is the function increasing? decreasing?

40.[R] Find all inflection points of $f(x) = (x+1)^{2/7}e^{-x}$. On what intervals is the graph of $y = f(x)$ concave up? concave down? On what intervals is the function increasing? decreasing? NOTE: This function was first encountered in Example 4.

41.[R] Find the critical points and inflection points of $f(x) = x^2e^{-x/3}$. NOTE: See Example 1.

In Exercises 42 to 43 sketch a graph of a hypothetical function that meets the given conditions. Assume f' and f'' are continuous. Explain your reasoning.

42.[R] Critical point $(2, 4)$; inflection points $(3, 1)$ and $(1, 1)$; $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$

43.[R] Critical points $(-1, 1)$ and $(3, 2)$; inflection point $(4, 1)$; $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to 0^-} f(x) = \infty$ $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = \infty$

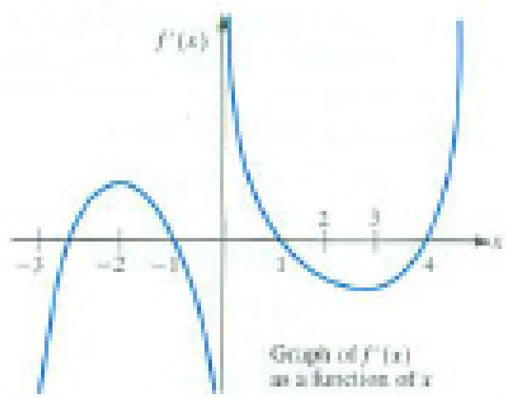


Figure 4.3.10:

44.[M] *(Contributed by David Hayes)* Let f be a function that is continuous for all x and differentiable for all x other than 0. Figure 4.3.10 is the graph of its derivative $f'(x)$ as a function of x.

(a) Answer the following questions about f (*not* about f'). Where is f increasing? decreasing? concave up? concave down? What are the critical numbers? Where do any relative extrema occur? Explain.

(b) Assuming that $f(0) = 1$, graph a hypothetical function f that satisfies the conditions given.

(c) Graph $f''(x)$.

45.[M] Graph $y = 2(x-1)^{5/3} + 5(x-1)^{2/3}$, paying particular attention to points where y' does not exist.

46.[M] Graph $y = x + (x+1)^{1/3}$.

47.[M] Find the critical points and inflection points in $[0, 2\pi]$ of $f(x) = \sin^2(x)\cos(x)$.

48.[M] Can a polynomial of degree 6 have (a) no inflection points? (b) exactly one inflection point? Explain.

49.[M] Can a polynomial of degree 5 have (a) no inflection points? (b) exactly one inflection point? Explain.

50.[C] In the theory of **inhibited growth** it is assumed that the growing quantity y approaches some limiting size M. Specifically, one assumes that the rate of growth is proportional both to the amount present and to the amount left to grow:

$$\frac{dy}{dt} = ky(M - y),$$

where k is a positive number. Prove that the graph of y as a function of time has an inflection point when the amount y is exactly half the limiting amount M.

51.[M] Let f be a function such that $f''(x) = (x - 1)(x - 2)$.

(a) For which x is f concave up?

(b) For which x is f concave down?

(c) List its inflection number(s).

(d) Find a specific function f whose second derivative is $(x - 1)(x - 2)$.

52.[C] A certain function $y = f(x)$ has the property that

$$y' = \sin(y) + 2y + x.$$

Show that at a critical number the function has a local minimum.

53.[C] Assume that the domain of $f(x)$ is the entire x-axis, and $f'(x)$ and $f''(x)$ are continuous. Assume that $(1, 1)$ is the only critical point and that $\lim_{x\to\infty} f(x) = 0$.

(a) Can $f(x)$ be negative for some $x > 1$?

(b) Must $f(x)$ be decreasing for $x > 1$?

(c) Must $f(x)$ have an inflection point?

4.4 Proofs of the Three Theorems

In Section 4.1 two observations about tangent lines led to the Theorem of the Interior Extremum, Rolle's Theorem, and the Mean-Value Theorem. Now, using the definition of the derivative, and no pictures, we prove them. That the proofs go through based only on the definition of the derivative as a limit reassures us that this definition is suitable to serve as part of the foundation of calculus.

Proof of Theorem 4.1.1:

Proof of the Theorem of the Interior Extremum

Suppose the maximum of f on the open interval (a, b) occurs at the number c. This means that $f(c) \geq f(x)$ for each number x between a and b.

$f'(c) = 0$ at the maximum or minimum on an open interval.

Our challenge is to use only this information and the definition of the derivative as a limit to show that $f'(c) = 0$.

Assume that f is differentiable at c. We will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, forcing $f'(c)$ to be zero.

Recall that

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

The assumption that f is differentiable on (a, b) means that $f'(c)$ exists. Consider the difference quotient

$$\frac{f(c + \Delta x) - f(c)}{\Delta x}. \qquad (4.4.1)$$

when Δx is so small that $c+\Delta x$ is in the interval (a, b). Then $f(c+\Delta x) \leq f(c)$. Hence $f(c + \Delta x) - f(c) \leq 0$. Therefore, when Δx is positive, the difference quotient in (4.4.1) will be negative, or 0. Consequently, as $\Delta x \to 0$ through *positive* values,

$\frac{\text{negative}}{\text{positive}} = \text{negative}$

$$f'(c) = \lim_{\Delta x \to 0^+} \frac{f(c + \Delta x) - f(c)}{\Delta x} \leq 0. \qquad (4.4.2)$$

If, on the other hand, Δx is negative, then the difference quotient in (4.4.3) will be positive, or 0. Hence, as $\Delta x \to 0$ through *negative* values,

$\frac{\text{negative}}{\text{negative}} = \text{positive}$

$$f'(c) = \lim_{\Delta x \to 0^-} \frac{f(c + \Delta x) - f(c)}{\Delta x} \geq 0. \qquad (4.4.3)$$

The only way $f'(c) \leq 0$ and $f'(c) \geq 0$ can both hold is when $f'(c) = 0$. This proves that if f has a maximum on (a, b), then $f'(c) = 0$.

See Exercise 12.

The proof for the case when f has a minimum on (a, b) is essentially the same. •

The proofs of Rolle's Theorem and the Mean-Value Theorem are related. Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

Proof of Rolle's Theorem

Proof of Theorem 4.1.2:

If $f(a) = f(b)$, then $f'(c) = 0$ for at least one number between a and b.

The goal here is to use the facts that f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$ to conclude that there must a number c in (a, b) with $f'(c) = 0$.

Since f is continuous on the closed interval $[a, b]$, it has a maximum value M and a minimum value m on that interval. There are two cases to consider: $m < M$ and $m = M$.

Case 1: If $m = M$, f is constant and $f'(x) = 0$ for all x in $[a, b]$. Then any number in (a, b) will serve as the desired number c.

Case 2: Suppose $m < M$. Because $f(a) = f(b)$ the minimum and maximum cannot both occur at the ends of the interval. At least one of the extrema occurs at a number c strictly between a and b. By assumption, f is differentiable at c, so $f'(c)$ exists. Thus, by the Theorem of the Interior Extremum, $f'(c) = 0$. This completes the proof of Rolle's Theorem. •

The idea behind the proof of the Mean-Value Theorem is to define a function to which Rolle's Theorem can be applied.

Proof of the Mean-Value Theorem

Proof of Theorem 4.1.3:

Let $y = L(x)$ be the equation of the chord through the two points $(a, f(a))$ and $(b, f(b))$. The slope of this line is $L'(x) = \dfrac{f(b) - f(a)}{b - a}$. Define $h(x) = f(x) - L(x)$. Note that $h(a) = h(b) = 0$ because $f(a) = L(a)$ and $f(b) = L(b)$.

$f'(c) = \frac{f(b)-f(a)}{b-a}$ for at least one number between a and b.

By assumption, f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). So h, being the difference of f and L, is also continuous on $[a, b]$ and differentiable on (a, b).

Rolle's Theorem applies to h on the interval $[a, b]$. Therefore, there is at least one number c in (a, b) where $h'(c) = 0$. Now, $h'(c) = f'(c) - L'(c)$, so that

$$f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a}.$$

•

Summary

Using only the definition of the derivative and the assumption that a continuous function defined on a closed interval assumes maximum and minimum values, we proved the Theorem of the Interior Extremum, Rolle's Theorem, and the Mean-Value Theorem. Note that we did not appeal to any pictures or to our geometric intuition.

EXERCISES for 4.4 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 3 sketch a graph of a differentiable function that meets the given conditions. (Just draw the graph; there is no need to come up with a formula for the function.)

1.[R] $f'(x) < 0$ for all x

2.[R] $f'(3) = 0$ and $f'(x) < 0$ for x not equal to 3

3.[R] $f'(x) = 0$ only when $x = 1$ or 4; $f(1) = 3$, $f(4) = 1$; $f'(x) > 0$ for $x < 1$ and for $x > 4$

In Exercises 4 to 5 explain why no differentiable function satisfies all the conditions.

4.[M] $f(1) = 3$, $f(2) = 4$, $f'(x) < 0$ for all x

5.[M] $f(x) = 2$ only when $x = 0$, 1, and 3; $f'(x) = 0$ only when $x = \frac{1}{4}$, $\frac{3}{4}$, and 4.

6.[M] In "Surely You're Joking, Mr. Feynmann!," Norton, New York, 1985, Nobel laureate Richard P. Feynmann writes:

> I often liked to play tricks on people when I was at MIT. One time, in mechanical drawing class, some joker picked up a French curve (a piece of plastic for drawing smooth curves — a curly funny-looking thing) and said, "I wonder if the curves on that thing have some special formula?"
>
> I thought for a moment and said, "Sure they do. The curves are very special curves. Lemme show ya," and I picked up my French curve and began to turn it slowly. "The French curve is made so that at the lowest point on each curve, no matter how you turn it, the tangent is horizontal."
>
> All the guys in the class were holding their French curve up at different angles, holding their pencil up to it at the lowest point and laying it down, and discovering that, sure enough, the tangent is horizontal.

How was Feynmann playing a trick on his classmates?

7.[M] What can be said about the number of solutions of the equation $f(x) = 3$ for a differentiable function if

(a) $f'(x) > 0$ for all x?

(b) $f'(x) > 0$ for $x < 7$ and $f'(x) < 0$ for $x > 7$?

8.[M] Consider the function $f(x) = x^3 + ax^2 + c$. Show that if $a < 0$ and $c > 0$, then f has exactly one negative root.

9.[M] With the book closed, obtain the Mean-Value Theorem from Rolle's Theorem.

10.[M]

(a) Recall the definition of $L(x)$ in the proof of the Mean-Value Theorem, and show that
$$L(x) = f(a) + \frac{x-a}{b-a}\left(f(b) - f(a)\right).$$

(b) Using (a), show that
$$L'(x) = \frac{f(b) - f(a)}{b-a}.$$

11.[M] Show that Rolle's Theorem is a special case of the Mean-Value Theorem.

12.[C] Prove the Theorem of the Interior Extremum when the minimum of f on (a, b) occurs at c.

13.[C] Show that a polynomial $f(x)$ of degree n, $n \geq 1$, can have at most n distinct real roots, that is, solutions to the equation $f(x) = 0$.

(a) Use algebra to show that the statement holds for $n = 1$ and $n = 2$.

(b) Use calculus to show that the statement then holds for $n = 3$.

(c) Use calculus to show that the statement continues to hold for $n = 4$ and $n = 5$.

(d) Why does it hold for all positive integers n?

14.[C] Is this proposed proof of the Mean-Value Theorem correct?
Proof

Tilt the x and y axes and the graph of the function until the x-axis is parallel to the given chord. The chord is now "horizontal," and we may apply Rolle's Theorem. •

15.[C] Is there a differentiable function f whose domain is the x-axis such that f is increasing and yet the derivative is *not* positive for all x?

16.[C] Prove: If f has a negative derivative on (a, b) then f is decreasing on the interval $[a, b]$.

Exercises 17 to 19 provide analytic justification for the statement in Section 4.3 that "[W]hen a curve is concave up, it lies above its tangent lines and below its chords."

17.[C] Show that in an open interval in which f'' is positive, tangents to the graph of f lie below the curve. HINT: Why do you want to show that if a and x are in the interval, then $f(x) > f(a) + f'(a)(x-a)$? Treat the cases $a < x$ and $x > a$ separately.

18.[C] Assume that $f''(x)$ is positive for x in an open interval. Let $a < b$ be two numbers in the interval. Show that the chord joining $(a, f(a))$ and $(b, f(b))$ lies above the graph of f. HINT: Consider the following three questions:

1. Why does one want to prove that $f(x) < f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$?
2. How does it help to know that $\frac{f(b)-f(a)}{b-a} < \frac{f(x)-f(a)}{x-a}$?
3. Show that the function on the right-hand side of the inequality in (b) is increasing for $a < x < b$. Why does this show that the chords lie above the curve?

19.[C]

Sam: I can do Exercise 18 more easily. I'll show that (b) is true. By the Mean-Value Theorem, I can write the left side as $f'(c)$ where c is in $[a, b]$ and the right side as $f'(d)$ where d is in $[a, x]$. Since $b > x$, I know $c > d$, hence $f'(c) > f'(d)$. Nothing to it.

Is Sam's reasoning correct?

20.[C] We stated, in Section 4.3, that if $f(x)$ is defined in an open interval around the critical number a and $f''(a)$ is negative, then $f(x)$ has a relative maximum at a. Explain why this is so, following these steps.

(a) Why is $\lim_{\Delta x \to 0} \frac{f'(a+\Delta x) - f'(a)}{\Delta x}$ negative?

(b) Deduce that if Δx is small and positive, then $f'(a+\Delta x)$ is negative.

(c) Show that if Δx is small and negative, then $f'(a+\Delta x)$ is positive.

(d) Show that $f'(x)$ changes sign from positive to negative at a. By the First-Derivative Test for a Relative Maximum, $f(x)$ has a relative maximum at a.

Skill Drill

21.[M] To keep your differentiation skills sharp, differentiate each of the following expressions:

(a) $\sqrt{1-x^2}\sin(3x)$

(b) $\dfrac{\sqrt[3]{x}}{x^2+1}$

(c) $\tan\left(\dfrac{1}{(2x+1)^2}\right)$

(d) $\ln\left(\dfrac{(x^2+1)^3\sqrt{1-x^2}}{\sec^2(x)}\right)$

(e) e^{x^4}

4.5 Chapter Summary

In this chapter we saw that the sign of the function and of its first and second derivatives influenced the shape of its graph. In particular the derivatives show where the function is increasing or decreasing and is concave up or down. That enabled us to find extreme points and inflection points. (See Table 4.3.3 on page 298.)

We state here the main ideas informally for a function with continuous first and second derivatives.

If a function has an extremum at a number, then the derivative there is zero, or is not defined, or the number may be an end point of the domain. This narrows the search for extrema. If the derivative is zero and the second derivative is not zero, the function has an extremum there.

The rationale for these tests rests on Rolle's theorem, which says that if a differentiable function vanishes at two inputs on an interval in its domain, its derivative must be zero somewhere between them.

The Mean Value Theorem generalizes this idea. It says that between any two points on its graph there is a point on the graph where the tangent is parallel to the chord through those two points. We used this to show that: If a and b are two numbers, then $f(b) = f(a) + f'(c)(b - a)$ for some number c between a and b.

If $f'(a)$ is positive and if f' is continuous in some open interval containing a, then, by the permanence principle, $f'(x)$ remains positive for some open interval containing a. Typically, if the derivative is positive at some number, then the function is increasing for inputs near that number. (A similar statement holds when $f'(a)$ is negative.)

Sam: Why bother me with limits? The authors say we need them to define derivatives.

Jane: Aren't you curious about why the formula for the derivative of a product is what it is?

Sam: No. It's been true for over three centuries. Just tell me what it is. If someone says the speed of light is 186,000 miles per second am I supposed to find a meter stick and clock and check it out?

Jane: But what if you forget the formula during a test?

Sam: That's not much of a reason.

Jane: But my physics class uses derivatives and limits to define basic concepts.

Sam: Oh?

Jane: Density of mass at a point or density of electric charge are defined as limits. And it uses derivatives all over the place. You will be lost if you don't know their definitions. Just look at the applications in Chapter 5.

Sam: O.K., O.K. enough. I'll look.

EXERCISES for 4.5 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 13 decide if it is possible for a single function to have all of the properties listed. If it is possible, sketch a graph of a differentiable function that meets the given conditions. (Just draw the graph; there is no need to come up with a formula for the function.) If it is not possible, explain why no differentiable function satisfies all of the conditions.

1.[R] $f(0) = 1$, $f(x) > 0$, and $f'(x) < 0$ for all positive x

2.[R] $f(0) = -1$, $f'(x) < 0$ for all x in $[0, 2]$, and $f(2) = 0$

3.[R] x intercepts at 1 and 5; y intercept at 2; $f'(x) < 0$ for $x < 4$; $f'(x) > 0$ for $x > 4$

4.[R] x intercepts at 2 and 5; y intercept at 3; $f'(x) > 0$ for $x < 1$ and for $x > 3$; $f'(x) < 0$ for x in $(1, 3)$

5.[R] $f(0) = 1$, $f'(x) < 0$ for all positive x, and $\lim_{x\to\infty} f(x) = 1/2$

6.[R] $f(2) = 5$, $f(3) = -1$, $f'(x) \geq 0$ for all x

7.[R] x intercepts only at 1 and 2; $f(3) = -1$, $f(4) = 2$

8.[R] $f'(x) = 0$ only when $x = 1$ or 4; $f(1) = 3$, $f(4) = 1$; $f'(x) < 0$ for $x < 1$; $f'(x) > 0$ for $x > 4$

9.[R] $f(0) = f(1) = 1$ and $f'(0) = f'(1) = 1$

10.[R] $f(0) = f(1) = 1$, $f'(0) = f'(1) = 1$, and $f(x) \neq 0$ for all x in $[0, 1]$

11.[R] $f(0) = f(1) = 1$, $f'(0) = f'(1) = 1$, and $f(x) = 0$ for exactly one number x in $[0, 1]$

12.[R] $f(0) = f(1) = 1$, $f'(0) = f'(1) = 1$, and $f(x)$ has exactly two inflection numbers in $[0, 1]$

13.[R] $f(0) = f(1) = 1$, $f'(0) = f'(1) = 1$, and $f(x)$ has exactly two extrema in $[0, 1]$

14.[R] State the assumptions and conclusions of the Theorem of the Interior Extremum for a function F defined on (a, b).

15.[R] State the assumptions and conclusions of the Mean-Value Theorem for a function g defined on $[c, d]$.

16.[R] The following discussion on higher derivatives in economics appears on page 124 of the College Mathematics Journal **37** (2006):

> Charlie Marion of Shrub Oak, NY, submitted this excerpt from "Curses! The Second Derivative" by Jeremy J. Siegel in the October 2004 issue of *Kiplinger's* (p. 73):
> "... I think what is bugging the market is something that I have seen happen many times before: the Curse of the Second Derivative. The second derivative, for all those readers who are a few years away from their college calculus class, is the rate of change of the rate of change — or, in this case, whether corporate earnings, which are still rising, are rising at a faster or slower pace."
> In the October 1996 issue of the *Notices of the American Mathematical Society*, Hugo Rossi wrote, "IN the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection."

Explain why the third derivative is involved in President Nixon's statement.

17.[M] If you watch the tide come in and go out, you will notice at high tide and at low tide, the height of the tide seems to change very slowly. The same holds when you watch an outdoor thermometer: the temperature seems to change the slowest when it is at its highest or at its lowest. Why is that?

18.[R]

(a) Graph $y = \sin^2(2\theta)\cos(2\theta)$ for θ in $[-\pi/2, \pi/2]$.

(b) What is the maximum value of y?

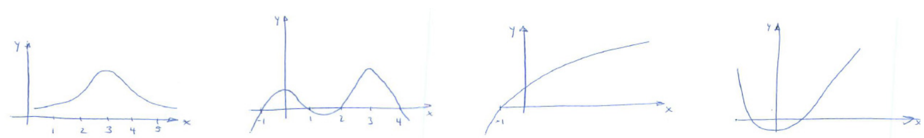
Exercises 19 to 22 display the graph of a function f with continuous f' and f''. Sketch a possible graph of f' and a possible graph of f''.

19.[R] Figure 4.5.1(a)

20.[R] Figure 4.5.1(b)

21.[R] Figure 4.5.1(c)

22.[R] Figure 4.5.1(d)



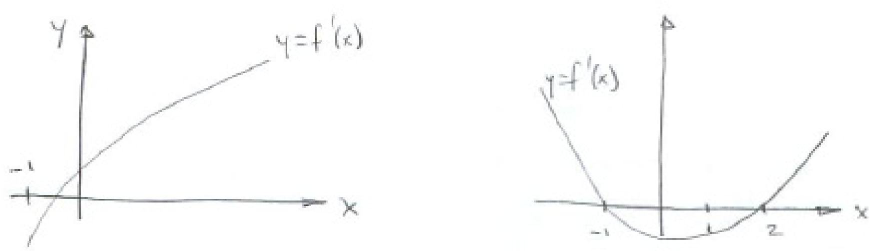
(a) (b) (c) (d)

Figure 4.5.1:

In Exercises 23 and 24 sketch the graphs of two possible functions f whose derivative f' is graphed in the given figure.

23.[R] Figure 4.5.2(a)

24.[R] Figure 4.5.2(b)



(a) (b)

Figure 4.5.2:

25.[R] Sketch the graph of a function f whose second derivative is graphed in Figure 4.5.3.

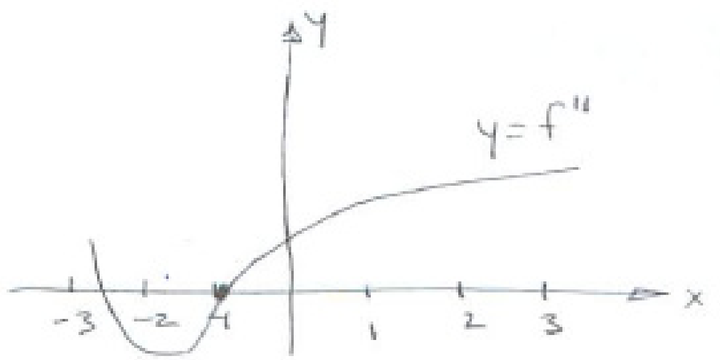
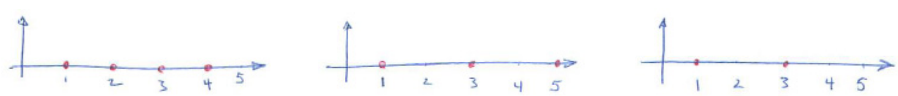


Figure 4.5.3:

26.[R] Figure 4.5.4(a) shows the only x-intercepts of a function f. Sketch the graph of possible f' and f''.

27.[R] Figure 4.5.4(b) shows the only arguments at which $f'(x) = 0$. Sketch the graph of possible f and f''.

28.[R] Figure 4.5.4(c) shows the only arguments at which $f''(x) = 0$. Sketch the graph of possible f and f'.



(a) (b) (c)

Figure 4.5.4:

In Exercises 29 to 36 graph the given functions, showing extrema, inflection points, and asymptotes.

29.[R] $e^{-2x}\sin(x)$, x in $[0, 4\pi]$

30.[R] $\frac{e^x}{1-e^x}$

31.[R] $x^3 - 9x^2$

32.[R] $x\sqrt{3-x}$

33.[R] $\frac{x-1}{x-2}$

34.[R] $\cos(x) - \sin(x)$, x in $[0, 2\pi]$

35.[R] $x^{1/2} - x^{1/4}$

36.[R] $\frac{x}{4-x^2}$

37.[R] Figure 4.5.5 shows the graph of a function f. Estimate the arguments where

(a) f changes sign,

(b) f' changes sign,

(c) f'' changes sign.

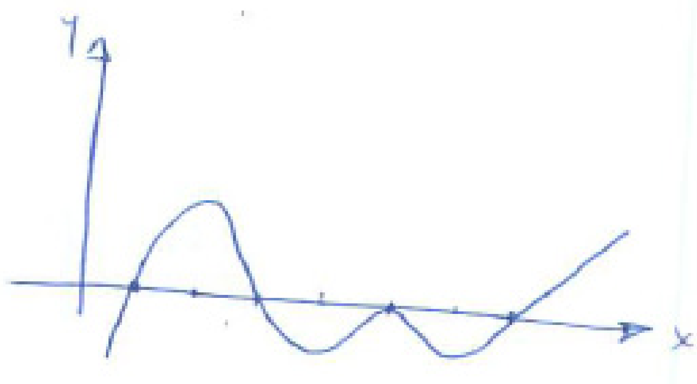


Figure 4.5.5:

38.[R] Assume the function f has continuous f' and f'' defined on an open interval.

(a) If $f'(a) = 0$ and $f''(a) = 0$, does f necessarily have an extrema at a? Explain.

(b) If $f''(a) = 0$, does f necessarily have an inflection point at $x = a$?

(c) If $f'(a) = 0$ and $f''(a) = 3$, does f necessarily have an extremum at a?

39.[R] Find the maximum value of $e^{2\sqrt{3}x}\cos(2x)$ for x in $[0, \pi/4]$.

40.[M]

(a) Show that the equation $5x - \cos(x) = 0$ has exactly one solution.

(b) Find a specific interval which contains the solution.

41.[M] Consider the function f given by the formula $f(x) = x^3 - 3x$.

(a) At which numbers x is $f'(x) = 0$?

(b) Use the theorem of the Interior Extremum to show that the maximum value of $x^3 - 3x$ for x in $[1, 5]$ occurs either at 1 or at 5.

42.[M] Let f and g be polynomials without a common root.

(a) Show that if the degree of g is odd, the graph of f/g has a vertical asymptote.

(b) Show that if the degree of f is less than or equal to the degree of g, then f/g has a horizontal asymptote.

43.[M] If $\lim_{x\to\infty} f'(x) = 0$, does it follow that f has a horizontal asymptote? Explain.

44.[M] Let f be a positive function on $(0, \infty)$ with f' and f'' both continuous. Let $g = f^2$.

(a) If f is increasing, is g?

(b) If f is concave up, is g?

45.[M] Give an example of a positive function on $(0, \infty)$ that is concave down but f^2 is concave up.

46.[M] Graph $\cos(2\theta) + 4\sin(\theta)$ for θ in $[0, 2\pi]$.

47.[M] Graph $\cos(2\theta) + 2\sin(\theta)$ for θ in $[0, 2\pi]$.

48.[M] Figure 4.5.3(b) shows part of a unit circle. The line segment CD is tangent to the circle and has length x. This exercise uses calculus to show that $AB < BC < CD$. (BC is the length of arc joining B and C.)

(a) Express AB and BC in terms of x.

(b) Using (a) and calculus, show that for $x > 0$, $AB < BC < CD$.

49.[M] Show that in an open interval in which f'' is positive, tangents to the graph of f lie below the curve. HINT: Why do you want to show that if a and x are in the interval, then $f(x) > f(a) + f'(a)(x - a)$? It is still necessary to treat the cases $x > a$ and $x < a$ separately. NOTE: This problem appears again as Exercise 96 in Section 5.8, when you have more tools to solve it.

50.[M] Assume that $f''(x)$ is positive for x in an open interval. Let $a < b$ be in the interval. In this exercise you will show that the chord joining $(a, f(a))$ to $(b, f(b)))$ lies above the graph of f. ("A concave up curve has chords that lie above the curve.")

(a) Why does one want to prove that

$$f(a) + \frac{f(b) - f(a)}{b - a}(x - a) > f(x), \qquad \text{for } a < x < b?$$

(b) Why does one want to prove that

$$\frac{f(b) - f(a)}{b - a} > \frac{f(x) - f(a)}{x - a}?$$

(c) Show that the function on the right-hand side of the inequality in (b) is increasing for $a < x < b$. Why does this show that chords lie above the curve?

51.[M]

(a) Graph $y = \frac{\sin(x)}{x}$ showing intercepts and asymptotes.

(b) Graph $y = x$ and $y = \tan(x)$ relative to the same axes.

(c) Use (b) to find how many solutions there are to the equation $x = \tan(x)$.

(d) Write a short commentary on the critical points of $\sin(x)/x$. HINT: Part (c) may come in handy.

(e) Refine the graph produced in (a) to show several critical points.

52.[M] Let $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

(a) Show that the graph of $y = f(x)$ always has exactly one inflection point.

(b) Show that the inflection point separates the graph of the cubic polynomial into two parts that are congruent. HINT: Show the graph is symmetric with respect to the inflection point. NOTE: Why can one assume it is enough to show this for $a = 1$ and $d = 0$?

53.[M] Find all functions $f(x)$ such that $f'(x) = 2$ for all x and $f(1) = 4$.

54.[M] Find all differentiable functions such that $f(1) = 3$, $f'(1) = -1$, and $f''(1) = e^x$.

55.[C]

(a) Graph $y = 1/(1 + 2^{-x})$.

(b) The point $(0, 1/2)$ is on the graph and divides it into two pieces. Are the two pieces congruent?

(Curves of this type model the depletion of a finite resource; x is time and y is the fraction used up to time x. See also Exercise 71 in Section 5.8.)

56.[C]

(a) If the graph of f has a horizontal asymptote (say, $\lim_{x\to\infty} f(x) = L$), does it follow that $\lim_{x\to\infty} f'(x)$ exists?

(b) If $\lim_{x\to\infty} f'(x)$ exists in (a), most it be 0?

57.[C] Assume that f is continuous on $[1, 3]$, $f(1) = 5$, $f(2) = 4$, and $f(3) = 5$. Show that the graph of f has a horizontal chord of length 1.

58.[C] A function f defined on the whole x-axis has continuous first- and second-derivatives and exactly one inflection point. In at most how many points can a straight line intersect the graph of f? Explain. (x^n, n an odd integer greater than 1, are examples of such functions.)

59.[C] Let f be an increasing function with continuous f' and f''. What, if anything, can be said about the concavity of $f \circ f$ if

(a) f is concave up?

(b) f is concave down?

60.[C] Assume f has continuous f' and f''. Show that if f and $g = f^2$ have inflection points at the same argument a, then $f'(a) = 0$.

61.[C] Graph $y = x^2 \ln(x)$, showing extrema and inflection points. NOTE: Use the fact that $\lim_{x\to 0^+} x^2 \ln(x) = 0$; see Exercise 20 of Section 5.5.

62.[C] Assume $\lim\limits_{x\to\infty} f'(x) = 3$. Show that for x sufficiently large, $f(x)$ is greater than $2x$. HINT: Review the Mean-Value Theorem.

63.[C] Assume that f is differentiable for all numbers x.

(a) If f is an even function, what, if anything, can be said about $f'(0)$?

(b) If f is an odd function, what, if anything, can be said about $f'(0)$?

Explain your answers.

64.[M] Graph $y = \sin(x^2)$ on the interval $[-\sqrt{\pi}, \sqrt{\pi}]$. Identify the extreme points and the inflection points.

65.[M] Assume that $f(x)$ is a continuous function not identically 0 defined on $(-\infty, \infty)$ and that $f(x + y) = f(x) \cdot f(y)$ for all x and y.

(a) Show that f(0)=1.

(b) Show that $f(x)$ is never 0.

(c) Show that $f(x)$ is positive for all x.

(d) Letting $f(1) = a$, find $f(2)$, $f(1/2)$, and $f(-1)$.

(e) Show that $f(x) = a^x$ for all x.

66.[C] Can a straight line meet the curve $y = x^5$ four times?

67.[C] Assume $y = f(x)$ is a twice differentiable function with $f(0) = 1$ and $f''(x) < -1$ for all x. Is it possible that $f(x) > 0$ for all x in $(1, \infty)$?

68.[C] If $\lim_{x\to\infty} f'(x) = 3$, does it follow that the graph of $y = f(x)$ is asymptotic to some line of the form $y = a + 3x$ for some constant a?

Calculus is Everywhere
Calculus Reassures a Bicyclist

Both authors enjoy bicycling for pleasure and running errands in our flat towns. One of the authors (SS) often bicycles to campus through a parking lot. On each side of his route is a row of parked cars. At any moment a car can back into his path. Wanting to avoid a collision, he wonders where he should ride. The farther he rides from a row, the safer he is. However, the farther he rides from one row, the closer he is to the other row. Where should he ride?

Instinct tells him to ride midway between the two rows, an equal distance from both. But he has second thoughts. Maybe it's better to ride, say, one-third of the way from one row to the other, which is the same as two-thirds of the way from the other row. That would mean he has two safest routes, depending on which row he is nearer. Wanting a definite answer, he resorted to calculus.

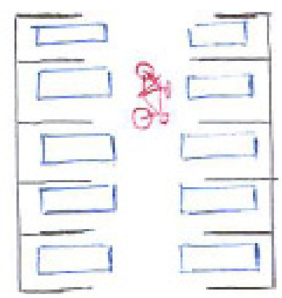


Figure 4.5.1: ARTIST:picture of two rows of parked cars, with bicycle

He introduced a function, $f(x)$, which is the probability that he gets through safely when his distance from one row is x, considering only cars in that row. Then he calls the distance between the two rows be d. When he was at a distance x from one row, he was at a distance $d - x$ from the other row. The probability that he did not collide with a car backing out from either row is then the product, $f(x)f(d - x)$. His intuition says that this is maximized when $x = d/2$, putting him midway between the two rows.

What did he know about f? First of all, the farther he rode from one line of cars, the safer he is. So f is an increasing function; thus f' is positive. Moreover, when he was very far from the cars, the probability of riding safely through the lot approached 1. So he assumed $\lim_{x\to\infty} f(x) = 1$ (which it turned out he did not need).

The derivative of f' measured the rate at which he gained safety as he increased his distance from the cars. When x is small, and he rode near the cars, $f'(x)$ was large: he gained a great deal of safety by increasing x. However, when he was far from the cars, he gained very little. That means that f was a decreasing function. In other words f' is negative.

Does that information about f imply that midway is the safest route?

In other words, does the maximum of $f(x)f(d - x)$ occur when $x = d/2$? Symbolically, is

$$f(d/2)f(d/2) \geq f(x)f(d - x)?$$

To begin, he took the logarithm of that expression, in order to replace a product by something easier, a sum. He wanted to see if

$$2\ln(f(d/2)) \geq \ln(f(x)) + \ln(f(d - x)).$$

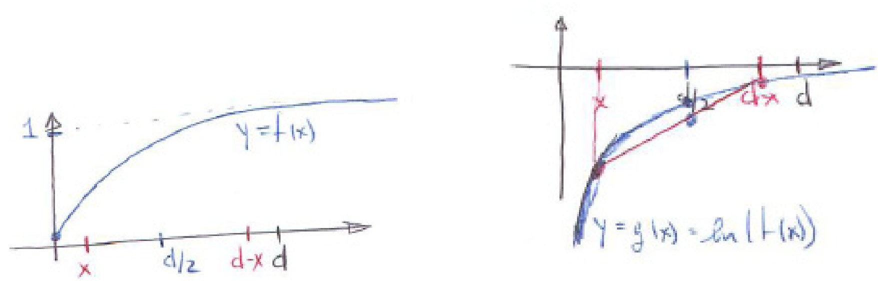
Letting $g(x)$ denote the composite function $\ln(f(x))$, he faced the inequality,

$$2g(d/2) \geq g(x) + g(d-x),$$

or

$$g(d/2) \geq \frac{1}{2}(g(x) + g(d-x)).$$

This inequality asserts that the point $(d/2, g(d/2))$ on the graph of g is at least



(a) (b)

Figure 4.5.2:

as high as the midpoint of the chord joining $(x, g(x))$ to $(d-x, g(d-x))$. This would be the case if the second derivative of g were negative, and the graph of g were concave down. He had to compute g'' and hope it is negative. First of all, $g'(x)$ is $f'(x)/f(x)$. Then $g''(x)$ is

$$\frac{f(x)f''(x) - (f'(x))^2}{f(x)^2}.$$

The denominator is positive. Because $f(x)$ is positive and concave down, the numerator is negative. So the quotient is negative. That means that the safest path is midway between the two rows. The bicyclist continues to follow that route, but, after these calculations, with more confidence that it is indeed the safest way.

Calculus is Everywhere
Graphs in Economics

Elementary economics texts are full of graphs. They provide visual images of a variety of concepts, such as production, revenue, cost, supply, and demand. Here we show how economists use graphs to help analyze production as a function of the amount of labor, that is, the number of workers.

Let $P(L)$ be the amount of some product, such as cell phones, produced by a firm employing L workers. Since both workers and wireless network cards come in integer amounts, the graph of $P(L)$ is just a bunch of dots. In practice, these dots suggest a curve, and the economists use that curve in their analysis. So $P(L)$ is viewed as a differentiable function defined for some interval of the form $[0, b]$.

If there are no workers, there is no production, so $P(0) = 0$. When the first few workers are added, production may increase rapidly, but as more are hired, production may still increase, but not as rapidly. Figure 4.5.1 is a typical **production curve**. It seems to have an inflection point when the gain from adding more workers begins to decline. The inflection point of $P(L)$ occurs at L_2 in Figure 4.5.2.

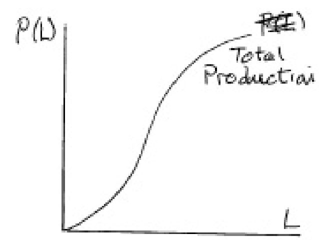


Figure 4.5.1:

When the firm employs L workers and adds one more, production increases by $P(L+1) - P(L)$, the marginal production. Economists manage to relate this to the derivative by a simple trick:

$$P(L+1) - P(L) = \frac{P(L+1) - P(L)}{(L+1) - L} \qquad (4.5.1)$$

The right-hand side of (4.5.1) is "change in output" divided by "change in input", which is, by the definition of the derivative, an approximation to the derivative, $P'(L)$. For this reason economists define the **marginal production** as $P'(L)$, and think of it as the extra product produced by the L plus first worker. We denote the marginal product as $m(L)$, that is, $m(L) = P'(L)$.

The **average production** per worker when there are L workers is defined as the quotient $P(L)/L$, which we denote $a(L)$. We have three functions: $P(L)$, $m(L) = P'(L)$, and $a(L) = P(L)/L$.

Now the fun begins.

At what point on the graph of the production function is the average production a maximum?

Since $a(L) = P(L)/L$, it is the slope of the line from the origin to the point $(L, P(L))$ on the graph. Therefore we are looking for the point on the graph where the slope is a maximum. One way to find that point is to rotate

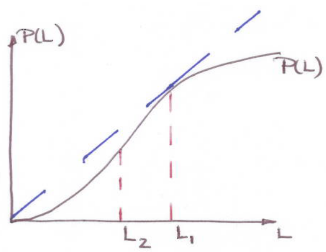


Figure 4.5.2:

a straightedge around the origin, clockwise, starting at the vertical axis until it meets the graph, as in Figure 4.5.2. Call the point of tangency $(L_1, P(L_1))$. For L less than L_1 or greater than L_q, average productivity is less than $a(L_1)$.

Note that at L_1 the average product is the same as the marginal product, for the slope of the tangent at $(L_1, P(L_1))$ is both the quotient $P(L_1)/L_1$ and the derivative $P'(L_1)$. We can use calculus to obtain the same conclusion:

Since $a(L)$ has a maximum when the input is L_1, its derivative is 0 then. The derivative of $a(L)$ is

$$\frac{d}{dL}\left(\frac{P(L)}{L}\right) = \frac{LP'(L) - P(L)}{L^2}. \tag{4.5.2}$$

At L_1 the quotient in (4.5.2) is 0. Therefore, its numerator is 0, from which it follows that $P'(L_1) = P(L_1)/L_1$. (You might take a few minutes to see why this equation should hold, without using graphs or calculus.)

In any case, the graphs of $m(L)$ and $a(L)$ cross when L is L_1. For smaller values of L, the graph of $m(L)$ is above that of $a(L)$, and for larger values it is below, as shown in Figure 4.5.3.

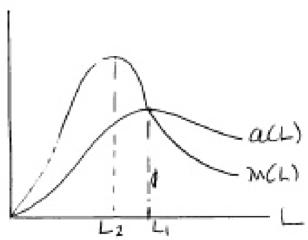


Figure 4.5.3:

What does the maximum point on the marginal product graph tell about the production graph?

Assume that $m(L)$ has a maximum at L_2. For smaller L than L_2 the derivative of $m(L)$ is positive. For L larger than L_2 the derivative of $m(L)$ is negative. Since $m(L)$ is defined as $P'(L)$, the second derivative of $P(L)$ switches from positive to negative at L_2, showing that the production curve has an inflection point at $(L_2, P(L_2))$.

Economists use similar techniques to deal with a variety of concepts, such as marginal and average cost or marginal and average revenue, viewed as functions of labor or of capital.

Chapter 5

More Applications of Derivatives

Chapter 2 constructed the foundation for derivatives, namely the concept of a limit. Chapters 3 and 4 developed the derivative and applied it to graphs of functions. The present chapter will apply the derivative in a variety of ways, such as: finding the most efficient way to accomplish a task (Section 5.1), connecting the rate one variable changes to the rate another changes (Section 5.2), the approximation of functions such as e^x by polynomials (Sections 5.3 and 5.4), the evaluation of certain limits (Section 5.5), natural growth and decay (Section 5.6), and to certain special functions (Section 5.7).

5.1 Applied Maximum and Minimum Problems

In Chapter 4, we saw how the derivative and second derivative are of use in finding the maxima and minima of a given function – the locally high and low points on its graph. Now we will use these same techniques to find extrema in applied problems. Though the examples will be drawn mainly from geometry they illustrate the general procedure. The main challenge in these situations is figuring out the formula for the function that describes the quantity to be maximized (or minimized).

The General Procedure

The general procedure runs something along these lines.

1. Get a feel for the problem (experiment with particular cases.)
2. Devise a formula for the function whose maximum or minimum you want to find.
3. Determine the domain of the function – that is, the inputs *that make sense in the application.*
4. Find the maximum or minimum of the function found in Step 2 for inputs that are in the domain identified in Step 3.

Additional worked examples can be found on the website for this book.

The most important step is finding a formula for the function. To become skillful at doing this takes practice. First, carefully read and study the three examples that comprise the remainder of this section.

A Large Garden

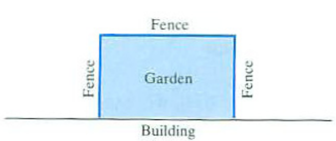
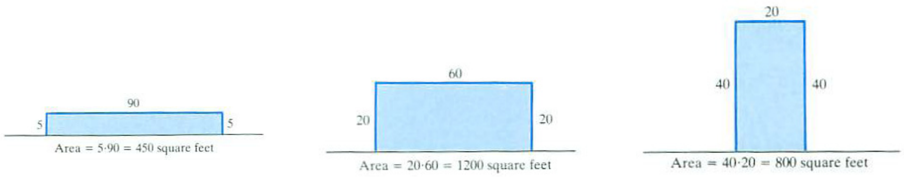


Figure 5.1.1:

EXAMPLE 1 A couple have enough wire to construct 100 feet of fence. They wish to use it to form three sides of a rectangular garden, one side of which is along a building, as shown in Figure 5.1.1. What shape garden should they choose in order to enclose the largest possible area?

SOLUTION *Step* 1. First make a few experiments. Figures 5.1.2 show some possible ways of laying out the 100 feet of fence. In the first case the side parallel to the building is very long, in an attempt to make a large area. However, doing this forces the other sides of the garden to be small. The area is $90 \times 5 = 450$ square feet. In the second case, the garden has a larger area, $60 \times 20 = 1200$ square feet. In the third case, the side parallel to the building



(a) (b) (c)

Figure 5.1.2:

is only 20 feet long, but the other sides are longer. The area is $20 \times 40 = 800$ square feet.

In all three cases, once the length of the side parallel to the building is set, the other side lengths are known and the area can be computed.

Clearly, we may think of the *area of the garden as a function of the length of the side parallel to the building.*

Step 2. Let $A(x)$ be the area of the garden when the length of the side parallel to the building is x feet, as in Figure 5.1.3. The other sides of the garden have length y. But y is completely determined by x since the total length of the fence is 100 feet:

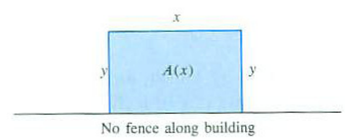


Figure 5.1.3:

$$x + 2y = 100.$$

Thus $y = (100 - x)/2$.

Since the area of a rectangle is its length times its width,

$$A(x) \quad = \quad xy = x\left(\frac{100 - x}{2}\right) = 50x - \frac{x^2}{2}.$$

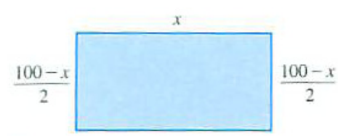


Figure 5.1.4:

(See Figure 5.1.4.) We now have the function.

Step 3. Which values of x in (5.1.1) correspond to possible gardens?

Since there is only 100 feet of fence, $x \leq 100$. Furthermore, it makes no sense to have a negative amount of fence; hence $x \geq 0$. Therefore the domain on which we wish to consider the function (5.1.1) is the closed interval $[0, 100]$.

Step 4. To maximize $A(x) = 50x - x^2/2$ on $[0, 100]$ we examine $A(0)$, $A(100)$, and the value of $A(x)$ at any critical numbers.

To find critical numbers, differentiate $A(x)$:

$$A(x) = 50x - \frac{x^2}{2} \qquad \text{so} \qquad A'(x) = 50 - x$$

and solve $A'(x) = 0$ to find:

$$0 = 50 - x \qquad \text{or} \qquad x = 50.$$

There is one critical number, 50.

All that is left is to find the largest of $A(0)$, $A(100)$, and $A(50)$. We have

$$\begin{aligned} A(0) &= 50 \cdot 0 - \frac{0^2}{2} = 0, \\ A(100) &= 50 \cdot 100 - \frac{100^2}{2} = 0, \\ \text{and} \qquad A(50) &= 50 \cdot 50 - \frac{50^2}{2} = 1250. \end{aligned}$$

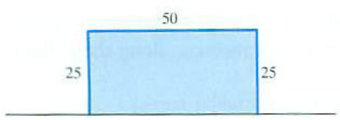
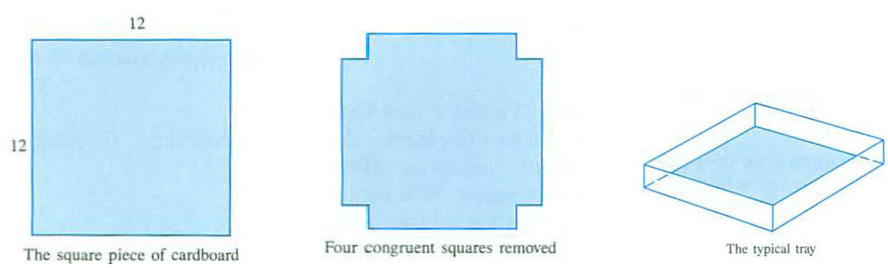


Figure 5.1.5:

The maximum possible area is 1250 square feet, and the fence should be laid out as shown in Figure 5.1.5. ◇

A Large Tray

EXAMPLE 2 Four congruent squares are cut out of the corners of a square piece of cardboard 12 inches on each side and the four remaining flaps can be folded up to obtain a tray without a top. (See Figure 5.1.6.) What size squares should be cut in order to maximize the volume of the tray?



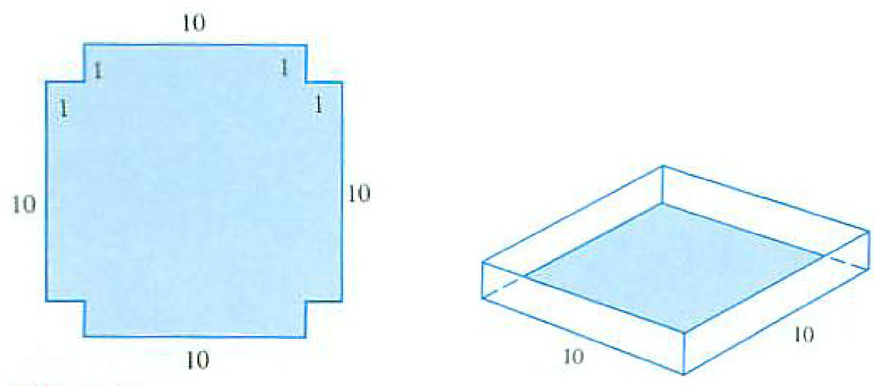
(a) (b) (c)

Figure 5.1.6:

Step 1. *SOLUTION* First we get a feel for the problem. Let us make a couple of experiments.

Say that we remove small squares that are 1 inch by 1 inch, as in Figure 5.1.7(a). When we fold up the flaps we obtain a tray whose base is a 10-inch by 10-inch square and whose height is 1 inch, as in Figure 5.1.7(b). The volume of the tray is

$$\text{Area of base} \times \text{height} = \underbrace{10 \times 10}_{\text{base area}} \times \underbrace{1}_{\text{height}} = 100 \text{ cubic inches.}$$



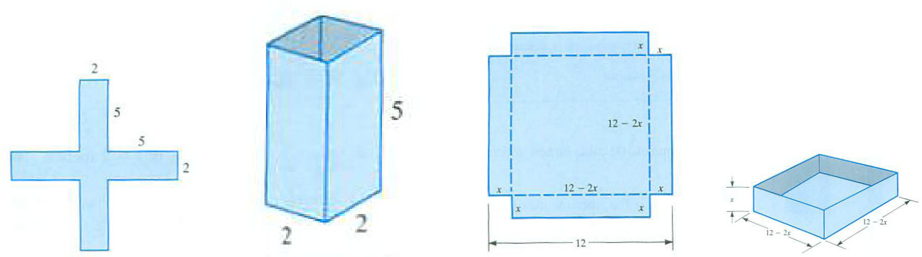
(a) (b)

Figure 5.1.7:

For our second experiment, let's try cutting out a large square, say 5 inches by 5 inches, as in Figure 5.1.8(a). When we fold up the flaps, we get a very tall tray with a very small base, as in Figure 5.1.8(b). It volume is

$$\text{Area of base} \times \text{height} = 2 \times 2 \times 5 = 20 \text{ cubic inches.}$$

Clearly *volume depends on the size of the cut-out squares.* The function



(a) (b) (c) (d)

Figure 5.1.8:

we will investigate is $V(x)$, the volume of the tray formed by removing four squares whose sides all have length x.

To find the formula for $V(x)$ we make a *large*, clear diagram of the typical case, as in Figure 5.1.8(c) and Figure 5.1.8(d). Now Step 2.

$$\text{Volume of tray} = \underbrace{(12-2x)}_{\text{length}}\underbrace{(12-2x)}_{\text{width}}\underbrace{x}\text{height} = (12-2x)^2x,$$

hence

$$V(x) = (12 - 2x)^2 x = 4x^3 - 48x^2 + 144x. \tag{5.1.1}$$

We have obtained a formula for volume as a function of the length of the sides of the cut-out squares.

Step 3. Next determine the domain of the function $V(x)$ that is *meaningful* in the problem.

The smallest that x can be is 0. In this case the tray has height 0 and is just a flat piece of cardboard. (Its volume is 0.) The size of the cut is not more than 6 inches, since the cardboard has sides of length 12 inches. The cut can be as near 6 inches as we please, and the nearer it is to 6 inches, the smaller is the base of the tray. For convenience of our calculations, we allow cuts with $x = 6$, when the area of the base is 0 square inches and the height is 6 inches. (The volume in each of these cases is 0 cubic inches.) Therefore the domain of the volume function $V(x)$ is the closed interval $[0, 6]$.

Step 4. To maximize $V(x) = 4x^3 - 48x^2 + 144x$ on $[0, 6]$, evaluate $V(x)$ at critical numbers in $[0, 6]$ and at the endpoints of $[0, 6]$.

We have

$$V'(x) = 12x^2 - 96 + 144 = 12(x^2 - 8x + 12) = 12(x-2)(x-6).$$

A critical number is a solution to the equation

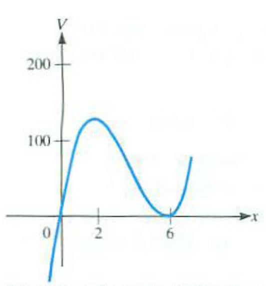
$$0 = 12(x-2)(x-6).$$

Hence $x - 2 = 0$ or $x - 6 = 0$. The critical numbers are 2 and 6.

The endpoints of the interval $[0, 6]$ are 0 and 6. Therefore the maximum value of $V(x)$ for x in $[0, 6]$ is the largest of $V(0)$, $V(2)$, and $V(6)$. Since $V(0) = 0$ and $V(6) = 0$, the largest value is

$$V(2) = 4(2^3) - 48(2^2) - 144 \cdot 2 = 128 \text{ cubic inches.}$$

The cut that produces the tray with the largest volume is $x = 2$ inches. ◇



Only values of x in the portion above [0, 6] correspond to physically realizable trays.

Figure 5.1.9:

As a matter of interest, let us graph the function V, showing its behavior for all x, not just for values of x significant in the problem. Note in Figure 5.1.9 that at $x = 2$ and $x = 6$ the tangent is horizontal.

Remark: In Example 2 you might say $x = 0$ and $x = 6$ don't really correspond to what you would call a tray. If so, you would restrict the domain of $V(x)$ to the open interval $(0, 6)$. You would then have to examine the behavior of $V(x)$ for x *near* 0 and for x *near* 6. By making the domain $[0, 6]$ from the start, you avoid the extra work of examining $V(x)$ for x *near* the ends of the interval.

The key step in these two examples, and in any applied problem, is Step 2: findng a formula for the quantity whose extremum you are seeking. In case the problem is geometrical, the following chart may be of aid.

Setting Up the Function

1. Draw and label the appropriate diagrams.
   (Make them large enough so that there is room for labels.)
2. Label the various quantities by letters, such as x, y, A, V.
3. Identify the quantity to be maximized (or minimized).
4. Express the quantity to be maximized (or minimized) in terms of one or more of the other variables.
5. Finally, express that quantity in terms of only one variable.

An Economical Can

EXAMPLE 3 Of all the tin cans that enclose a volume of 100π cubic centimeters, which requires the least metal?

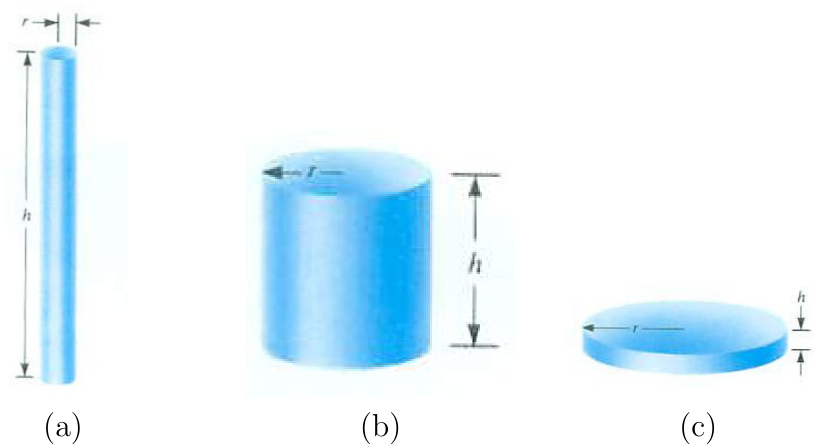
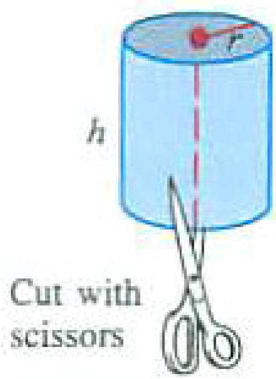


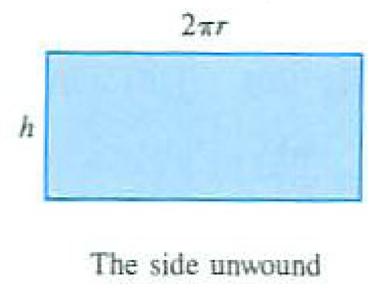
Figure 5.1.10:

SOLUTION The can may be flat or tall. If the can is flat, the side uses little metal, but then the top and bottom bases are large. If the can is shaped like a mailing tube, then the two bases require little metal, but the curved side requires a great deal of metal. (See Figure 5.1.10, where r denotes the radius and h the height of the can.) What is the ideal compromise between these two extremes? Step 1

The surface area S of the can is the sum of the area of the top, side, and bottom. The top and bottom are circles with radius r so their total area is Step 2



(a)



(b)

Figure 5.1.11:

$2\pi r^2$. Figure 5.1.11 shows why the area of the side is $2\pi rh$. The total surface area of the can is given by

$$S = 2\pi r^2 + 2\pi rh. \tag{5.1.2}$$

Since the amount of metal in the can is proportional to S, it suffices to minimize S.

Equation (5.1.2) gives S as a function of two variables, but we can express one of the variables in terms of the other. The radius and height are related by the equation

$$V = \pi r^2 h = 100\pi, \tag{5.1.3}$$

since their volume is 100π cubic centimeters. In order to express S as a function of one variable, use (5.1.3) to eliminate either r or h. Choosing to eliminate h, we solve (5.1.3) for h,

$$h = \frac{100}{r^2}.$$

Substitution into (5.1.2) yields

$$S = 2\pi r^2 + 2\pi r\frac{100}{r^2} \qquad \text{or} \qquad S = 2\pi r^2 + \frac{200}{r}\pi. \tag{5.1.4}$$

Equation (5.1.4) expresses S as a function of just one variable, r.

Step 3 The cans have a positive radius as large as you please. The function $S(r)$ is continuous and differentiable on $(0, \infty)$.

Step 4 Compute dS/dr:

$$\frac{dS}{dr} = 4\pi r - \frac{200\pi}{r^2}\pi = \frac{4\pi r^3 - 200\pi}{r^2}. \tag{5.1.5}$$

Set the derivative equal to 0 to find any critical numbers. We have

$r = 0$ is *not* a critical number because it is not in the domain of V.

$$\begin{aligned} 0 &= \frac{4\pi r^3 - 200}{r^2}, \\ \text{hence} \quad 0 &= 4\pi r^3 - 200\pi \\ \text{or} \quad 4\pi r^3 &= 200\pi \\ r^3 &= \frac{200}{4} \\ r &= \sqrt[3]{50} \approx 0.7071. \end{aligned}$$

There is only one critical number. Does it provide a minimum? Let's check it two ways, first by the first-derivative test, then by the second-derivative test.

The first derivative is

$$\frac{dS}{dr} = \frac{4\pi r^3 - 200\pi}{r^2}. \tag{5.1.6}$$

When $r = \sqrt[3]{50}$, the numerator in (5.1.6) is 0. When $r < \sqrt[3]{50}$ the numerator is negative and when $r > \sqrt[3]{50}$ the numerator is positive. (The denominator is always positive.) Since $dS/dr < 0$ for $r < \sqrt[3]{50}$, and $dS/dr > 0$ for $r > \sqrt[3]{50}$, the function $S(r)$ decreases for $r < \sqrt[3]{50}$ and increases for $r > \sqrt[3]{50}$. That shows that a global minimum occurs at $\sqrt[3]{50}$. (See Figure 5.1.12(a).)

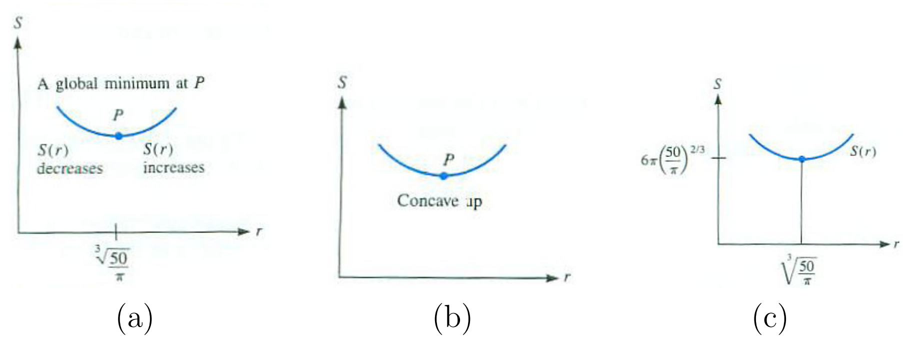


Figure 5.1.12:

Let us instead use the second-derivative test. Differentiation of (5.1.5) gives

$$\frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}\pi. \tag{5.1.7}$$

Inspection of (5.1.7) shows that for all meaningful values of r, that is r in $(0, \infty)$, d^2S/dr^2 is positive. (The function is concave up as shown in Figure 5.1.12(b).) Not only is P a relative minimum, it is a global minimum, since the graph lies above its tangents, in particular, the tangent at P.

The minimum of $S(r)$ is shown in Figure 5.1.12(c).

To find the height of the most economical can, solve (5.1.7) for h:

$$\begin{aligned} h = \tfrac{100}{r^2} &= \tfrac{100}{\pi(\sqrt[3]{50})^2} \\ &= \tfrac{100}{\pi(\sqrt[3]{50})^2}\tfrac{\sqrt[3]{50}}{\sqrt[3]{50}} && \text{rationalize the denominator} \\ &= \tfrac{100}{\pi(50)}\sqrt[3]{50} = 2\sqrt[3]{50}. \end{aligned}$$

The height of the can is equal to twice its radius, that is, its diameter. The total surface area of the can is

$$S = 2\pi r^3 + \frac{200\pi}{r}\bigg|_{r=50^{1/3}} = (100 + 4 \cdot 50^{2/3}) \approx 154.288 \text{ square centimeters.}$$

◇

Summary

We showed how to use calculus to solve applied problems: experiment, set up a function, find its domain, and its critical points. Then test the critical points and endpoints of the domain to determine the extrema.

1. Draw and label appropriate diagrams.
2. Express the quantity to be optimized in terms of one other variable.
3. Determine the domain of the function.
4. Use the first or second derivative test to determine the maximum or minimum of the function in its domain.

If the interval is closed, the maximum or minimum will occur at a critical point or an endpoint. If the interval is not closed, a little more care is needed to confirm that a critical number provides an extremum.

With practice this process becomes second nature.

EXERCISES for 5.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] A gardener wants to make a rectangular garden with 100 feet of fence. What is the largest area the fence can enclose?

2.[R] Of all rectangles with area 100 square feet, find the one with the shortest perimeter.

3.[R] Solve Example 1, expressing A in terms of y instead of x.

4.[R] A gardener is going to put a rectangular garden inside one arch of the cosine curve, as shown in Figure 5.1.13. What is the garden with the largest area.

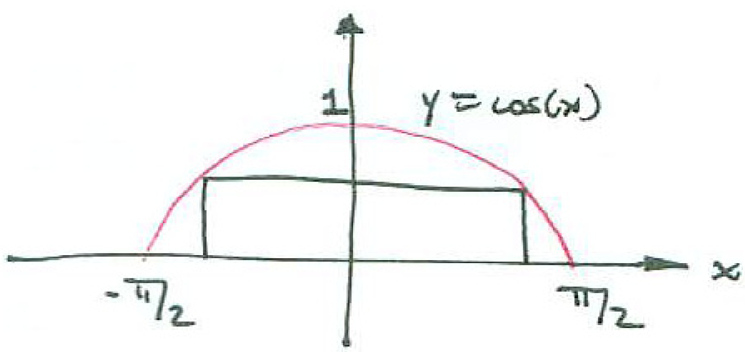


Figure 5.1.13:

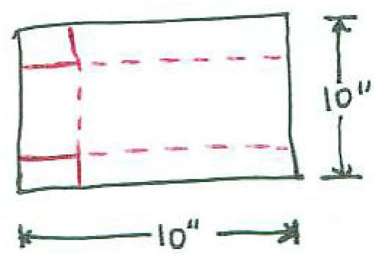
Exercises 5 to 8 are related to Example 2. In each case find the length of the cut that maximizes the volume of the tray. The dimensions of the cardboard are given.

5.[R] 5 inches by 5 inches

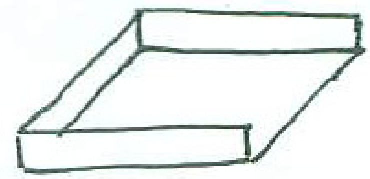
6.[R] 5 inches by 7 inches

7.[R] 4 inches by 8 inches,

8.[R] 6 inches by 10 inches,



(a)



(b)

Figure 5.1.14:

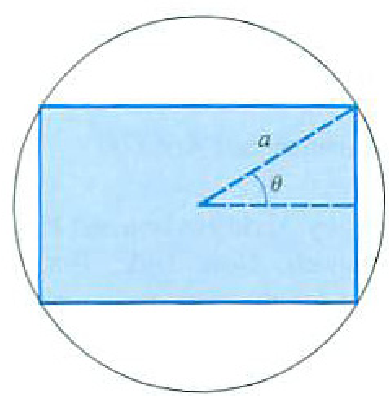
9.[R] Starting with a square piece of paper 10″ on a side, Sam wants to make a paper holder with three sides. The pattern he will use is shown in Figure 5.1.14 along with the tray. He will remove two squares and fold up three flaps.

(a) What size square maximizes the volume of the tray?

(b) What is that volume?

10.[C] A chef wants to make a cake pan out of a circular piece of aluminum of radius 12 inches. To do this he plans to cut the circular base from the center of the piece and then cut the side from the remainder. What should the radius and height be to maximize the volume of the pan? (See Figure 5.1.15(a).)

(a)



(b)

Figure 5.1.15:

11.[R] Solve Example 3, expressing S in terms of h instead of r.

12.[R] Of all cylindrical tin cans *without a top* that contains 100 cubic inches, which requires the least material?

13.[R] Of all enclosed rectangular boxes with square bases that have a volume of 1000 cubic inches, which uses the least material?

14.[R] Of all topless rectangular boxes with square bases that have a volume of 1000 cubic inches, which uses the least material?

15.[M] Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius a. The typical rectangle is shown in Figure 5.1.15(b). HINT: Express the area in terms of the angle θ shown.

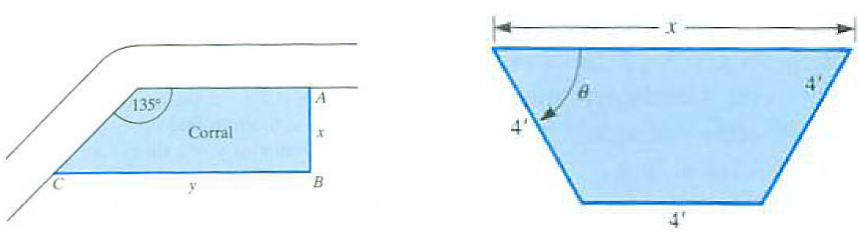
16.[M] Solve Exercise 15, expressing the area in terms of half the width of the rectangle, x. HINT: Square the area to avoid square roots.

17.[M] Find the dimensions of the rectangle of largest perimeter that can be inscribed in a circle of radius a.

18.[M] Show that of all rectangles of a given area, the square has the shortest perimeter. *Suggestion:* Call the fixed area A and keep in mind that it is a constant.

19.[M] A rancher wants to construct a rectangular corral. He also wants to divide the corral by a fence parallel to one of the sides. He has 240 feet of fence. What are the dimensions of the corral of largest area he can enclose?

20.[M] A river has a 45° turn, as indicated in Figure 5.1.16(a). A rancher wants to construct a corral bounded on two sides by the river and on two sides by 1 mile of fence ABC, as shown. Find the dimensions of the corral of largest area.



(a) (b)

Figure 5.1.16:

21.[M]

(a) How should one choose two nonnegative numbers whose sum is 1 in order to maximize the sum of their squares?

(b) To minimize the sum of their squares?

22.[M] How should one choose two nonnegative numbers whose sum is 1 in order to maximize the product of the square of one of them and the cube of the other?

23.[M] An irrigation channel made of concrete is to have a cross section in the form of an isosceles trapezoid, three of whose sides are 4 feet long. See Figure 5.1.16(b). How should the trapezoid be shaped if it is to have the maximum possible area? HINT: Consider the area as a function of x and solve.

24.[R]

(a) Solve Exercise 23 expressing the area as a function of θ instead of x.

(b) Do the answers in (a) and Exercise 23 agree? Explain.

In Exercises 25 to 28 use the fact that the combined length and girth (distance around) of a package to be sent through the mail by the United States Postal Service (USPS) cannot exceed 108 inches. NOTE: The combined length and girth of a packages sent as "parcel post" is 130 inches. The United Parcel Service (UPS) limit is 165 inches for combined length and girth with the length not exceeding 108 inches. Why do you think they have this restriction?

25.[R] Find the dimensions of the right circular cylinder of largest volume that can be sent through the mail.

26.[R] Find the dimensions of the right circular cylinder of largest surface area that can be sent through the USPS.

27.[R] Find the dimensions of the rectangular box with square base of largest volume that can be sent through the USPS.

28.[R] Find the dimensions of the rectangular box with square base of largest surface area that can be sent through the USPS.

29.[M]

(a) Repeat Exercise 25 with for a package sent by UPS.

(b) Generalize your solutions to Exercise 25 for a packages subject to a combined length and girth that does not exceed M inches.

30.[M]

(a) Repeat Exercise 26 with for a package sent by UPS.

(b) Generalize your solutions to Exercise 26 for a packages subject to a combined length and girth that does not exceed M inches.

Exercises 31 to 38 concern "minimal cost" problems.

31.[MR] A cylindrical can is to be made to hold 100 cubic inches. The material for its top and bottom costs twice as much per square inch as the material for its side. Find the radius and height of the most economical can. *Warning:* This is not the same as Example 3.

(a) Would you expect the most economical can in this problem to be taller or shorter than the solution to Example 3? (Use common sense, not calculus.)

(b) For convenience, call the cost of 1 square inch of the material for the side k cents. Thus the cost of 1 square inch of the material for the top and bottom is $2k$ cents. (The precise value of k will not affect the answer.) Show that a can of radius r and height h costs

$$C = 4k\pi r^2 + 2k\pi rh \text{ cents.}$$

(c) Find r that minimizes the functions C in (b). Keep in mind during any differentiation that k is constant.

(d) Find the corresponding h.

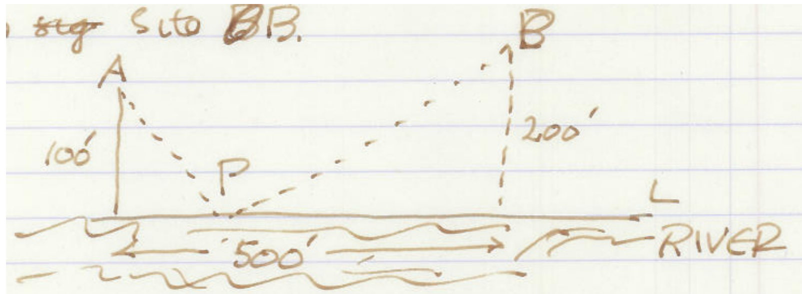


Figure 5.1.17: Sketch of situation in Exercise 32.

32.[M] A camper at A will walk to the river, put some water in a pail at P, and take it to the campsite at B.

(a) Express $\bar{AP} + \bar{PB}$ as a function of x.

(b) Where should P be located to minimize the length of the walk, $AP + PB$? (See Figure 5.1.17.) HINT: Reflect B across the line L.

NOTE: This exercise was first encountered as Exercise 34 in Section 1.1.

33.[M] Sam is at the edge of a circular lake of radius one mile and Jane is at the edge, directly opposite. Sam wants to visit Jane. He can walk 3 miles per hour and he has a canoe. What mix of paddling and walking should Sam use to minimize the time needed to reach Jane if

(a) he paddles at least three miles an hour?

(b) he paddles at 1.5 miles per hour?

(c) he paddles at 2 miles per hour?

34.[M] Consider a right triangle ABC, with C being at the right angle. There are two routes from A to B. One is direct, along the hypotenuse. The other is along the two legs, from A to C and then to B. Now, the shortest path between two points is the straight one. That raises this question: What is the largest percentage saving possible by walking along the hypotenuse instead of along the two legs? For which shape right triangle does this savings occur?

35.[M] A rectangular box with a square base is to hold 100 cubic inches. Material for the top of the box costs 2 cents per square inch; material for the sides costs 3 cents per square inch; material for the bottom costs 5 cents per square inch. Find the dimensions of the most economical box.

36.[M] The cost of operating a certain truck (for gasoline, oil, and depreciation) is $(20 + s/2)$ cents per mile when it travels at a speed of s miles per hour. A truck driver earns \$18 per hour. What is the most economical speed at which to operate the truck during a 600-mile trip?

(a) If you considered only the truck, would you want s to be small or large?

(b) If you, the employer, considered only the expense of the driver's wages, would you want s to be small or large?

(c) Express cost as a function of s and solve. (Be sure to put the costs all in terms of cents or all in terms of dollars.)

(d) Would the answer be different for a 1000-mile trip?

37.[R] A government contractor who is removing earth from a large excavation can route trucks over either of two roads. There are $10,000$ cubic yards of earth to move. Each truck holds 10 cubic yards. On one road the cost per truckload is $1+2x^2$ cents, when x trucks use that raod; the function records the cost of congestion. On the other road the cost is $2 + x^2$ cents per truckload when x trucks use that road. How many trucks should be dispatched to each of the two roads?

38.[R] On one side of a river 1 mile wide is an electric power station; on the other side, s miles upstream, is a factory. (See Figure 5.1.18.) It costs 3 dollars per foot to run cable over land and 5 dollars per foot under water. What is the most economical way to run cable from the station to the factory?

(a) Using no calculus, what do you think would be (approximately) the best route if s were very small? if s were very large?

(b) Solve with the aid of calculus, and draw the routes for $s = \frac{1}{2}$, $\frac{3}{4}$, 1, and 2.

(c) Solve for arbitrary s.

Warning: Minimizing the length of cable is *not* the same as minimizing its cost.

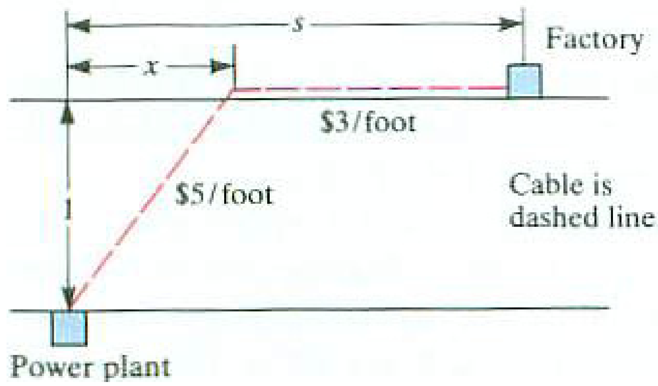


Figure 5.1.18:

39.[R] (From a text on the dynamics of airplanes.) "Recalling that

$$I = A\cos^2\theta + C\sin^2\theta - 2E\cos\theta\sin\theta,$$

we wish to find θ when I is a maximum or a minimum." Show that at an extremum of I,

$$\tan 2\theta = \frac{2\ E}{C - A}.\ (\text{assume that} A \neq C)$$

40.[R] (From a physics text.) "By differentiating the equation for the horizontal range,

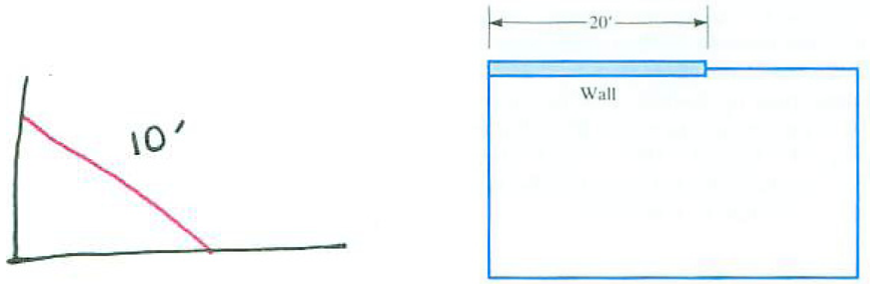
$$R = \frac{v_0^2\sin(2\theta)}{g},$$

show that the initial elevation angle θ for maximum range is 45°." In the formula for R, v_0 and g are constants. (R is the horizontal distance a baseball covers if you throw it at an angle θ with speed v_0. Air resistance is disregarded.)

(a) Using calculus, show that the maximum range occurs when $\theta = 45°$.

(b) Solve the same problem without calculus.

41.[R] A gardener has 10 feet of fence and wishes to make a triangular garden next to two buildings, as in Figure 5.1.19(a). How should he place the fence to enclose the maximum area?



(a) (b)

Figure 5.1.19:

42.[R] Fencing is to be added to an existing wall of length 20 feet, as shown in Figure 5.1.19(b). How should the extra fence be added to maximum the area of the enclosed rectangle if the additional fence is

(a) 40 feet long?

(b) 80 feet long?

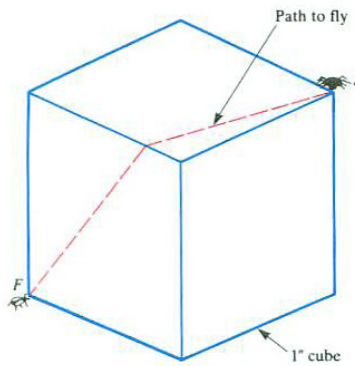
(c) 60 feet long?

43.[R] Let A and B be constants. Find the maximum and mimimum values of $A\cos t + B\sin t$.

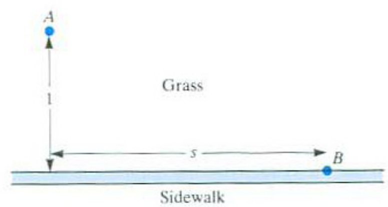
44.[R] A spider at corner S of a cube of side 1 inch wishes to capture a fly at the opposite corner F. (See Figure 5.1.20(a).) The spider, who must walk on the surface of the solid cube, wishes to find the shortest path.

(a) Find a shortest path without the aid of calculus.

(b) Find a shortest path with calculus.



(a)



(b)

Figure 5.1.20:

45.[R] A ladder of length b leans against a wall of height a, $a < b$. What is the maximal horizontal distance that the ladder can extend beyond the wall if its base rests on the horizontal ground?

46.[R] A woman can walk 3 miles per hour on grass and 5 miles per hour on sidewalk. She wishes to walk from point A to point B, shown in Figure 5.1.20(b), in the least time. What route should she follow if s is

(a) $\frac{1}{2}$?

(b) $\frac{3}{4}$?

(c) 1?

47.[R] The potential energy in a diatomic molecule is given by the formula

$$U(r) = u_0 \left(\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6} \right),$$

where U_0 and r_0 are constants and r is the distance between the atoms. For which value of r is $U(r)$ a minimum?

48.[R] What are the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius a?

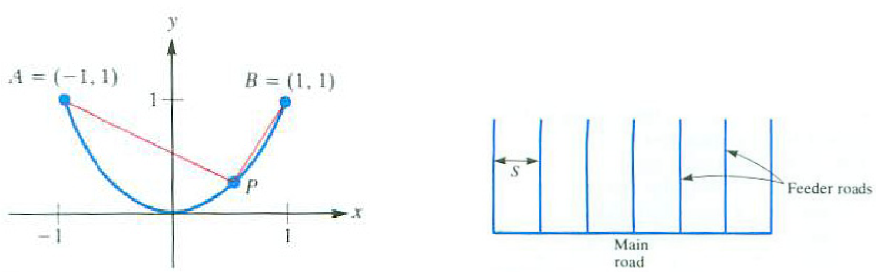
49.[R] The stiffness of a rectangular beam is proportional to the product of the width and the cube of the height of its cross section. What shape beam should be cut from a log in the form of a right circular cylinder of radius r in order to maximize its stiffness.

50.[R] A rectangular box-shaped house is to have a square floor. Three times as much heat per square foot enters through the roof as through the walls. What shape should the house be if it is to enclose a volume of 12,000 cubic feet and minimize heat entry. (Assume no heat enters through the floor.)

51.[R] (See Figure 5.1.21(a).) Find the coordinates of the points $P = (x, y)$, with $y \leq 1$, on the parabola $y = x^2$, that

(a) minimize $\overline{PA}^2 + \overline{PB}^2$,

(b) maximize $\overline{PA}^2 + \overline{PB}^2$.



(a) (b)

Figure 5.1.21:

52.[R] The speed of traffic through the Lincoln Tunnel in New York City depends on the amount of traffic. Let S be the speed in miles per hour and let D be the amount of traffic measured in vehicles per mile. The relation between S and D was seen to be approximated closely, for $D \leq 100$, by the formula

$$S = 42 - \frac{D}{3}.$$

(a) Express in terms of S and D the total number of vehicles that enter the tunnel in an hour.

(b) What value of D will maximize the flow in (a)?

53.[R] When a tract of timber is to be logged, a main logging road is built from which small roads branch off as feeders. The question of how many feeders to build arises in practice. If too many are built, the cost of construction would be prohibitive. If too few are built, the time spent moving the logs to the roads would be prohibitive. The formula for total cost,

$$y = \frac{CS}{4} + \frac{R}{VS},$$

is used in a logger's manual to find how many feeder roads are to be built. R, C, and V are known constants: R is the cost of road at "unit spacing"; C is the cost of moving a log a unit distance; V is the value of timber per acre. S denotes the distance between the regularly spaced feeder roads. (See Figure 5.1.21(b).) Thus the cost y is a function of S, and the object is to find that value of S that minimizes y. The manual says, "To find the desired S set the two summands equal to each other and solve

$$\frac{CS}{4} = \frac{r}{VS}."$$

Show that the method if valid.

54.[R] A delivery service is deciding how many warehouses to set up in a large city. The warehouses will serve similarly shaped regions of equal area A and, let us assume, an equal number of people.

(a) Why would transportation costs per item presumably be proportional to $\sqrt{A}$?

(b) Assuming that the warehouse cost per item is inversely proportional to A, show that C, the cost of transportation and storage per item, is of the form $t\sqrt{A} + w/A$, where t and w are appropraite constants.

(c) Show that C is a minimum when $A = (2w/t)^{2/3}$.

Exercises 55 and 56 are related.

55.[R] A pipe of length b is carried down a long corridor of width $a < b$ and then around corner C. (See Figure 5.1.22.) During the turn y starts out at 0, reaches a maximum, and then returns to 0. (Try this with a short stick.) Find that maximum in terms of a and b. *Suggestion:* Express y in terms of a, b, and θ; θ is a variable, while a and b are constants.

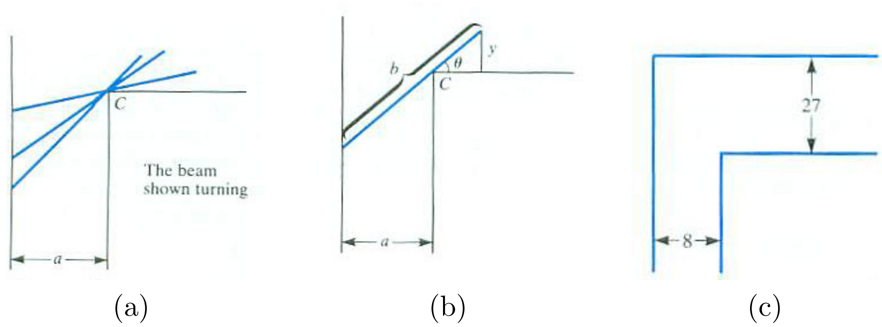


Figure 5.1.22:

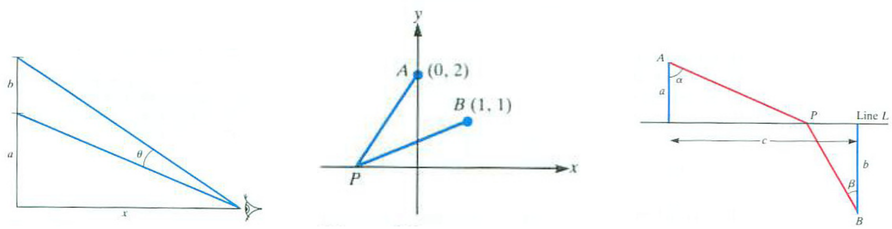
56.[M] Figure 5.1.22(c) shows two corridors meeting at right angle. One has width 8; the other, width 27. Find the length of the longest pipe that can be carried horizontally from one hall, around the corner and into the other hall. *Suggestion:* Do Exercise 55 first.

57.[M] Two houses, A and B, are a distance p apart. They are distances q and r, respectively, from a straight road, and on the same side of the road. Find the length of the shortest path that goes from A to the road, and then on to the other house B.

(a) Use calculus.

(b) Use only elementary geometry. *Hint:* Introduce an imaginary house C such that the midpoint of B and C is on the road and the segment BC is perpendicular to the road; that is, "reflect" B across the road to become C.

58.[R] The base of a painting on a wall is a feet above the eye of an observer, as shown in Figure 5.1.23(a). The vertical side of the painting is b feet long. How far from the wall should the ovserver stand to maximize the angle that the painting subtends? *Hint:* It is more convenient to maximize $\tan\theta$ than θ itself. HINT: Recall that $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$.



(a) (b) (c)

Figure 5.1.23:

59.[R] Find the point P on the x-axis such that the angle APB in Figure 5.1.23(b) is maximal. Hint: See the hint in Exercise 58.

60.[R] *(Economics)* Let p denote the price of some commodity and y the number sold at that price. To be concrete, assume that $y = 250 - p$ for $0 \leq p \leq 250$. Assume that it costs the producer $100 + 10y$ dollars to manufacture y units. What price p should the producer choose in order to maximize total profit, that is, "revenue minus cost"?

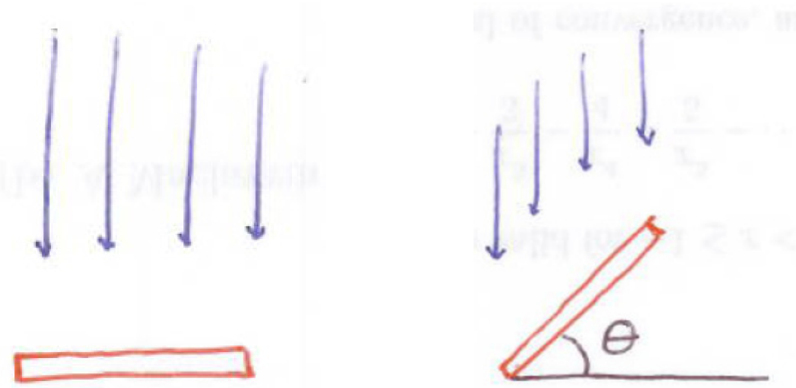
61.[R] *(Leibniz on light)* A ray of light travels from point A to point B in Figure 5.1.23(c) in minimal time. The point A is in one medium, such as air or a vacuum. The point B is in another medium, such as water or glass. In the first medium, light travels at velocity v_1 and in the second at velocity v_2. The media are separated by line L. Show that for the path APB of minimal time,

$$\frac{\sin\alpha}{v_1} = \frac{\sin(\beta)}{v_2}.$$

Leibniz solved this problem with calculus in a paper published in 1684. (The result is called **Snell's law of refraction**.)

Leibniz then wrote, "other very learned men have sought in many devious ways what someone versed in this calculus can accomplish in these lines as by magic." (See C. H. Edwards Jr., *The Historical Development of the Calculus*, p. 259, Springer-Verlag, New York, 1979.)

Exercises 62 to 65 concern the intensity of light.



(a) (b)

Figure 5.1.24:

62.[R] Why is it reasonable to assume that the intensity of light from a lamp is inversely proportional to the square of the distance from the lamp? HINT: Imagine the light spreading out in all directions.

63.[R] A solar panel perpendicular to the sun's rays catches more light than when it is tilted at any other angle, as shown in Figure 5.1.24(a). Let θ be the angle the panel is tilted, as in Figure 5.1.24(b). Show that it then receives $\cos(\theta)$ times the light the panel would receive when perpendicular to the sun's rays.

64.[M] In view of the preceding introduction and exercises, the intensity of light on a small (flat) surface is inversely proportional to the square of the distance from the source and proportional to the angle between the surface and a surface perpendicular to the source.

(a) A person wants to put a light at a horizontal distance of ten feet from his address, which is on a wall (a vertical surface). At what height should the lamp be placed to maximize the intensity of light at the address? HINT: No calculus is needed for this.

(b) Now the person paints the address on the horizontal surface of the curb. Again the lamp will be placed at a horizontal distance of ten feet from the address. Without doing any calculations sketch what the graph of "intensity of light on the address versus height of lamp" might look like.

(c) Find the height the lamp should have to maximize the light on the address. HINT: Use height as the independent variable.

65.[M] Solve Exercise 64(c) using an angle as the independent variable.

66.[M] The following calculation occurs in an article concerning the optimum size of new cities: "The net utility to the total client-centered system is

$$U = \frac{RLv}{A}n^{1/2} - nK - \frac{ALc}{v}n^{-1/2}.$$

All symbols except U and n are constant; n is a measure of decentralization. Regarding U as a differentiable function of n, we can determine when $dU/dn = 0$. This occurs when

$$\frac{RLv}{2A}n^{-1/2} - K + \frac{ALc}{2v}n^{-3/2} = 0.$$

This is a cubic equation for $n^{-1/2}$."

(a) Check that the differentiation is correct.

(b) Of what cubic polynomial is $n^{-1/2}$ a root?

67.[C] Consider the curve $y = x^{-2}$ in the first quadrant. A tangent to this curve, together with axes, determine a triangle.

(a) What is the largest area of such a triangle?

(b) The smallest area?

68.[C] Let f be a differentiable function that is never zero on its domain. Let $g(x) = (f(x))^2$. Show that the functions f and g have the same critical numbers. NOTE: This is useful for getting rid of square roots.

69.[C] Let f be a differentiable function. Define the function g by $g(x) = \tan(f(x))$. Show that the functions f and g have the same critical numbers.

5.2 Implicit Differentiation and Related Rates

Sometimes a function $y = f(x)$ is given indirectly by an equation that links y and x. This section shows how to differentiate y without solving for y explicitly in terms of x.

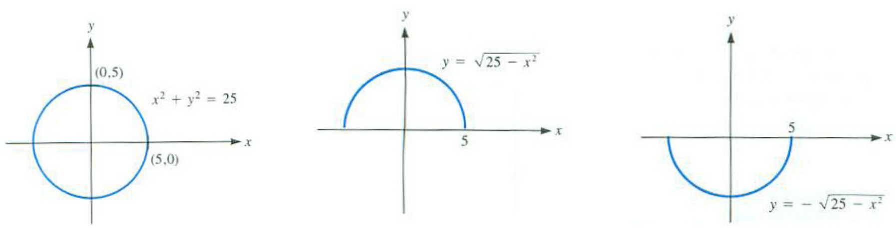
We will apply this technique to determine how the rate at which one quantity changes influences the rate at which another changes.

A Function Given Implicitly

The equation

$$x^2 + y^2 = 25 \tag{5.2.1}$$

describes a circle of radius 5 and center at the origin, as in Figure 5.2.1(a). This circle is not the graph of a function, since some vertical lines meet the



(a) (b) (c)

Figure 5.2.1:

circle in two points. However, the top half is the graph of a function and so is the bottom half. To find these functions explicitly, solve (5.2.1) for y:

$$\begin{aligned} y^2 &= 25 - x^2 \\ y &= \pm\sqrt{25 - x^2}. \end{aligned}$$

So either $y = \sqrt{25 - x^2}$ or $y = -\sqrt{25 - x^2}$. The graph of $y = \sqrt{25 - x^2}$ is the top semicircle (see Figure 5.2.1(b)); the graph of $y = -\sqrt{25 - x^2}$ is the bottom semicircle (see Figure 5.2.1(c)). There are two continuous functions that satisfy (5.2.1).

The equation $x^2 + y^2 = 25$ is said to describe the function $y = f(x)$ **implicitly**. The equations

$$y = \sqrt{25 - x^2} \qquad \text{and} \qquad y = -\sqrt{25 - x^2}$$

describe the function $y = f(x)$ **explicitly**.

Differentiating an Implicit Function

It is possible to differentiate a function given implicitly without having to solve for it and express it explicitly. An example will illustrate the method, which is to differentiate both sides of the equation that defines the function implicitly. This procedure is called **implicit differentiation**.

EXAMPLE 1 Let $y = f(x)$ be the continuous function that satisfies the equation

$$x^2 + y^2 = 25$$

such that $y = 4$ when $x = 3$. Find dy/dx when $x = 3$ and $y = 4$.

SOLUTION (We could, of course, solve for y, $y = \sqrt{25 - x^2}$, and differentiate directly. However, the algebra would be more involved since square roots would appear.) Differentiating both sides of the equation

$$x^2 + y^2 = 25$$

with respect to x yields

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(25), \\ 2x + \frac{d(y^2)}{dx} &= 0. \end{aligned}$$

To differentiate y^2 with respect to x, write $w = y^2$, where y is a function of x.

By the chain rule $\quad \dfrac{dw}{dx} = \dfrac{dw}{dy}\dfrac{dy}{dx},$

which gives us $\quad \dfrac{d(y^2)}{dx} = 2y\dfrac{dy}{dx}.$

Thus $\quad 2x + 2y\dfrac{dy}{dx} = 0,$

or $\quad x + y\dfrac{dy}{dx} = 0.$

In particular, when $x = 3$ and $y = 4$, $\quad 3 + 4\dfrac{dy}{dx} = 0,$

and therefore, $\quad \dfrac{dy}{dx} = -\dfrac{3}{4}.$

◇

Observe that the algebra involves no square roots.

If you look back at Section 3.5, you will see that we already used implicit differentiation to find derivatives of inverse functions. For instance, we differentiated both sides of $y = e^x$ with respect to y, obtaining $1 = e^x(dx/dy)$. Then $dx/dy = 1/e^x = 1/y$. In short, $D(\ln(y)) = 1/y$.

In the next example implicit differentiation is the only way to find the derivative, for in this case there is no formula expressible in terms of trigonometric and algebraic functions giving y explicitly in terms of x.

EXAMPLE 2 Assume that the equation

$$2xy + \pi \sin(y) = 2\pi$$

defines a function $y = f(x)$. Find dy/dx when $x = 1$ and $y = \pi/2$.

Verify that the point $(1, \pi/2)$ is on the graph of $y = f(x)$ by checking that the equation is satisfied when $x = 1$ and $y = \pi/2$.

SOLUTION Implicit differentiation yields

$$\begin{aligned} \frac{d}{dx}(2xy + \pi \sin y) &= \frac{d(2\pi)}{dx}, \\ \left(2\frac{dx}{dx}y + 2x\frac{dy}{dx}\right) + \pi(\cos y)\frac{dy}{dx} &= 0, \end{aligned}$$

by the formula for the derivative of a product and the chain rule. Hence

$$2y + 2x\frac{dy}{dx} + \pi(\cos y)\frac{dy}{dx} = 0.$$

Solving for the derivative, dy/dx, we get

$$\frac{dy}{dx} = \frac{-2y}{2x + \pi \cos y}.$$

In particular, when $x = 1$ and $y = \pi/2$,

$$\frac{dy}{dx} = -\frac{2 \cdot \frac{\pi}{2}}{2 \cdot 1 + \pi \cos \frac{\pi}{2}} = -\frac{\pi}{2 + \pi \cdot 0} = -\frac{\pi}{2}.$$

◇

Implicit Differentiation and Extrema

Example 3 of Section 5.1 answered the question, "Of all the tin cans that enclose a volume of 100 cubic inches, which requires the least metal?" The radius of the most economical can is $\sqrt[3]{50/\pi}$. From this and the fact that its volume is 100 cubic inches, its height was found to be $2\sqrt[3]{50/\pi}$, exactly twice the radius. In the next example implicit differentiation is used to answer the same question. Not only will the algebra be simpler but it will provide the shape – the proportion between height and radius – easily.

EXAMPLE 3 Of all the tin cans that enclose a volume of 100 cubic inches, which requires the least metal?

SOLUTION The height h and radius r of any can of volume 100 cubic inches are related by the equation

$$\pi r^2 h = 100. \tag{5.2.2}$$

The surface area S of the can is

$$S = 2\pi r^2 + 2\pi r h \tag{5.2.3}$$

Consider h, and hence S, as functions of r. It is *not* necessary to find h and S explicitly in terms of r. Differentiation of (5.2.2) and (5.2.3) with respect to r yields

$$\pi r^2 \frac{dh}{dr} + 2\pi r h = \frac{d(100)}{dr} = 0 \tag{5.2.4}$$

and

$$\frac{dS}{dr} = 4\pi r + 2\pi r \frac{dh}{dr} + 2\pi h. \tag{5.2.5}$$

When S is a minimum, $dS/dr = 0$, so we have

$$0 = 4\pi r + 2\pi r \frac{dh}{dr} + 2\pi h. \tag{5.2.6}$$

Equations (5.2.4) and (5.2.6) yield, with a little algebra, a relation between h and r, as follows:

Factoring πr out of (5.2.4) and 2π out of (5.2.6) shows that

$$r\frac{dh}{dr} + 2h = 0 \qquad \text{and} \qquad 2r + r\frac{dh}{dr} + h = 0. \tag{5.2.7}$$

Elimination of dh/dr from (5.2.7) yields

$$2r + r\left(\frac{-2h}{r}\right) + h = 0,$$

which simplifies to

$$2r = h. \tag{5.2.8}$$

We have obtained the shape before the specific dimensions. Equation (5.2.8) asserts that the height of the most economical can is the same as its diameter. Moreover, this is the ideal shape, no matter what the prescribed volume happens to be.

The specific dimensions of the most economical can are found by eliminating h from equations (5.2.2) and (5.2.4). This shows that

$$\pi r^2 (2r) = 100 \qquad \text{or} \qquad r^3 = \frac{50}{\pi}.$$

Hence

$$r = \sqrt[3]{\frac{50}{\pi}} \qquad \text{and} \qquad h = 2r = 2\sqrt[3]{\frac{50}{\pi}}$$

◇

The procedure illustrated in Example 3 is quite general. It may be of use when maximizing (or minimizing) a quantity that at first is expressed as a function of two variable which are linked by an equation. The equation that links them is called the **constraint**. In Example 3, the constraint is $\pi r^2 h = 100$.

Using Implicit Differentiation in an Extremum Problem

1. Name the various quantities in the problem by letters, such as x, y, h, r, A, V.
2. Identify the quantity to be maximized (or minimized).
3. Express that quantity in terms of other quantities, such as x and y.
4. Obtain an equation relating x and y. (This equation is called a constraint.)
5. Differentiate implicitly both the constraint and the quantity to be maximized (or minimized), interpreting all quantities to be functions of a single variable (which you choose).
6. Set the derivative of the quantity to be maximized (or minimized) equal to 0 and combine with the derivative of the constraint to obtain an equation relating x and y at a maximum (or minimum).
7. Step 6 gives only a relation between x and y at an extremum. If the explicit values of x and y are desired, find them by using the fact that x and y also satisfy the constraint.

Warning: Sometimes an extremum occurs where a derivative, such as dy/dx, is not defined.

Exercise 22 illustrates this possibility.

Related Rates

Implicit differentiation also comes in handy when showing how the rate of change of one quantity affects the rate of change of another.

EXAMPLE 4 An angler has a fish at the end of his line, which is reeled in at 2 feet per second from a bridge 30 feet above the water. At what speed

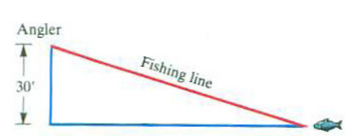


Figure 5.2.2:

is the fish moving through the water when the amount of line out is 50 feet? 31 feet? Assume the fish is at the surface of the water. (See Figure 5.2.2.)

SOLUTION Our first impression might be that since the line is reeled in at a constant speed, the fish at the end of the line moves through the water at a constant speed. As we will see, this is not the case.

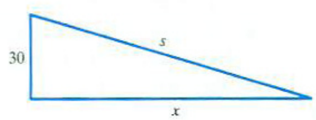


Figure 5.2.3:

Let s be the length of the line and x the horizontal distance of the fish from the bridge. (See Figure 5.2.3.)

Since the line is reeled in at the rate of 2 feet per second, s is shrinking, and

$$\frac{ds}{dt} = -2.$$

The rate at which the fish moves through the water is given by the derivative, dx/dt. The problem is to find dx/dt when $s = 50$ and also when $s = 31$.

We need an equation that relates s and x at *any* time, not just when $x = 50$ or $x = 31$. If we consider only $x = 50$ or $x = 31$, there would be no motion, and no chance to use derivatives.

This equation is the heart of the example.

The quantities x and s are related by the Pythagorean Theorem:

$$x^2 + 30^2 = s^2.$$

Both x and s are functions of time t. Thus both sides of the equation may be differentiated with respect to t, yielding

$$\begin{aligned} \frac{d(x^2)}{dt} + \frac{d(30^2)}{dt} &= \frac{d(s^2)}{dt} \\ \text{or} \qquad 2x\frac{dx}{dt} + 0 &= 2s\frac{ds}{dt}. \\ \text{Hence} \qquad x\frac{dx}{dt} &= s\frac{ds}{dt}. \end{aligned}$$

This last equation provides the tool for answering the questions.

Since $ds/dt = -2$,

$$\begin{aligned} x\frac{dx}{dt} &= (s)(-2). \\ \text{Hence} \qquad \frac{dx}{dt} &= \frac{-2s}{x}. \\ \text{When } s = 50, \qquad x^2 + 30^2 &= 50^2, \end{aligned}$$

so $x = 40$. Thus when 50 feet of line is out, the speed is

$$\left|\frac{dx}{dt}\right| = \frac{2s}{x} = \frac{2 \cdot 50}{40} = 2.5 \text{ feet per second.}$$

$$\text{When } s = 31, \qquad x^2 + 30^2 = 31^2.$$

$$\text{Hence} \qquad x = \sqrt{31^2 - 30^2} = \sqrt{961 - 900} = \sqrt{61}.$$

Thus when 31 feet of line is out, the fish is moving at the speed of

$$\frac{dx}{dt} = \frac{2s}{x} = \frac{2 \cdot 31}{\sqrt{61}} = \frac{62}{\sqrt{61}} \approx 7.9 \text{ feet per second.}$$

Let us look at the situation from the fish's point of view. When it is x feet from the point in the water directly below the bridge, its speed is $2s/x$ feet per second. Since s is larger than x, its speed is always greater than 2 feet per second. When x is very large, s/x is near 1 so the fish is moving through the water only a little faster than the line is reeled in. However, when the fish is almost at the point under the bridge, x is very small; then $2s/x$ is huge, and the fish finds itself moving at huge speeds, but according to Einstein, not faster than the speed of light. ◇

In Example 4 it would be a tactical mistake to indicate in Figure 5.2.3 that the hypotenuse of the triangle is 50 feet long, for if one leg is 30 feet and the hypotenuse is 50 feet, the triangle is determined; there is nothing left free to vary with time.

In general, label all the lengths or quantities that can change with letters x, y, s, and so on, even if not all are needed in the solution. Only after you finish differentiating do you determine what the rates are at a specified value of the variable.

The General Procedure

The method used in Example 4 applies to many related rate problems. This is the general procedure, broken into steps:

Procedure for Finding a Related Rate

1. Find an equation that relates the varying quantities.
   (If the quantities are geometric, draw a picture and label the varying quantities with letters.)
2. Differentiate both sides of the equation with respect to time, obtaining an equation that relates the various rates of change.
3. Solve the equation obtained in Step 2 for the unknown rate.
   (Only at this step do you substitute constants for variable.)

WARNING Differentiate, then substitute the specific numbers for the variables. If you reversed the order, you would just be differentiating constants.

Finding an Acceleration

The method described in Example 4 for determining unknown rates from known ones extends to finding an unknown acceleration. Just differentiate another time. Example 5 illustrates the procedure.

EXAMPLE 5 Water flows into a conical tank at the constant rate of 3 cubic meters per second. The radius of the cone is 5 meters and its height is 4 meters. Let $h(t)$ represent the height of the water above the bottom of the cone at time t. Find dh/dt (the rate at which the water is rising in the tank) and d^2h/dt^2 (the rate at which that rate changes) when the tank is filled to a height of 2 meters. (See Figure 5.2.4.)

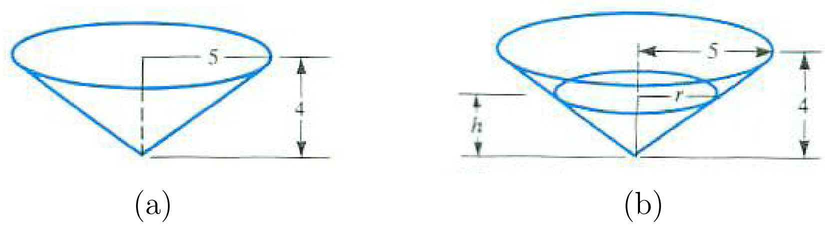


Figure 5.2.4:

SOLUTION Let $V(t)$ be the volume of water in the tank at time t. The fact that water flows into the tank at 3 cubic meters per second is expressed as

$$\frac{dV}{dt} = 3,$$

and, since this rate is constant,

$$\frac{d^2V}{dt^2} = 0.$$

To find dh/dt and d^2h/dt^2, first obtain an equation relating V and h.

When the tank is filled to the height h, the water forms a cone of height h and radius r. (See Figure 5.2.4(b).) By similar triangles,

$$\frac{r}{h} = \frac{5}{4} \qquad \text{or} \qquad r = \frac{5h}{4}.$$

Thus

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{4}h\right)^2 h = \frac{25}{48}\pi h^3.$$

So the equation relating V and h is

$$V = \frac{25\pi}{48}h^3. \tag{5.2.9}$$

From here on, just differentiate as often as needed.

Differentiating both sides of (5.2.9) once (using the chain rule) yields

$$\frac{dV}{dt} = \frac{25\pi}{48}\frac{d(h^3)}{dh}\frac{dh}{dt}$$

or

$$\frac{dV}{dt} = \frac{25\pi}{16}h^2\frac{dh}{dt}.$$

Since $dV/dt = 3$ all the time,

$$3 = \frac{25\pi h^2}{16}\frac{dh}{dt},$$

from which it follows that

$$\frac{dh}{dt} = \frac{48}{25\pi h^2} \text{ meters per second.} \tag{5.2.10}$$

As (5.2.10) shows, the larger h is, the slower the water rises. (Why is this to be expected?)

Even though the water enters the tank at a constant rate, it does not rise at a constant rate.

To find dh/dt when $h = 2$ meters, substitute 2 for h in (5.2.10), obtaining

$$\frac{dh}{dt} = \frac{48}{25\pi 2^2} = \frac{12}{25\pi} \approx 0.15279 \text{ meters per second.}$$

Now we turn to the acceleration, d^2h/dt^2. *We do not* differentiate the equation $dh/dt = 12/(25\pi)$ since this equation holds only when $h = 2$. We must go back to (5.2.10), which holds at any time.

Differentiating (5.2.10) with respect to t yields

$$\frac{d^2h}{dt^2} = \frac{48}{25\pi}\frac{d}{dt}\left(\frac{1}{h^2}\right) = \frac{48}{25\pi}\frac{-2}{h^3}\frac{dh}{dt} = \frac{-96}{25\pi h^3}\frac{dh}{dt}. \tag{5.2.11}$$

The last equation expresses the acceleration in terms of h and dh/dt. Substituting (5.2.10) into (5.2.11) gives

$$\frac{d^2h}{dt^2} = \frac{-96}{25\pi h^3}\frac{48}{25\pi h^2}$$

or

$$\frac{d^2h}{dt^2} = \frac{-(96)(48)}{(25\pi)^2 h^5} \text{ meters per second per second.} \tag{5.2.12}$$

Equation (5.2.12) tells us that, since d^2h/dt^2 is negative, the rate at which the water rises in the tank is decreasing.

The problem also asked for the value of d^2h/dt^2 when $h = 2$. To find it, replace h by 2 in (5.2.12), obtaining

$$\frac{d^2h}{dt^2} = \frac{-(96)(48)}{(25\pi)^2 2^5}$$

or

$$\frac{d^2h}{dt^2} = \frac{-144}{625\pi^2} \approx -0.02334 \text{ meters per second per second.}$$

◇

Logarithmic Differentiation

If $\ln(f(x))$ is simpler than $f(x)$, there is a technique for finding $f'(x)$ that saves labor. Example 6 illustrates this method, which depends on implicit differentiation.

EXAMPLE 6 Let $y = \frac{\cos(3x)}{\left(\sqrt[3]{x^2+5}\right)^4}$. Find $\frac{dy}{dx}$.

SOLUTION The solution to this problem by **logarithmic differentiation** begins by simplifying $\ln(y)$ using the properties of logarithms:

$$\begin{aligned} \ln(y) &= \ln(\cos(3x)) - \ln\left(\left(\sqrt[3]{x^2+5}\right)^4\right) && [\ln(A/B) = \ln(A) - \ln(B)] \\ &= \ln(\cos(3x)) - \tfrac{4}{3}\ln(x^2+5) && [\ln(A^B) = B\ln(A)]. \end{aligned}$$

Next, since $\frac{d}{dx}(\ln(y)) = \frac{1}{y}\frac{dy}{dx}$ by the Chain Rule, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(\ln(\cos(3x)) - \frac{4}{3}\ln\left(x^2+5\right)\right) = \frac{-3\sin(3x)}{\cos(3x)} - \frac{4}{3}\frac{2x}{x^2+5}.$$

Therefore

$$\frac{dy}{dx} = (y)\left(-3\tan(3x) - \frac{4}{3}\frac{2x}{x^2+5}\right).$$

Finally, replace y by its formula, getting

$$\frac{dy}{dx} = \frac{\cos(3x)}{\left(\sqrt[3]{x^2+5}\right)^4}\left(-3\tan(3x) - \frac{4}{3}\frac{2x}{x^2+5}\right).$$

To appreciate logarithmic differentiation, find the derivative directly, as requested in Exercise 53. ◇

If you want to differentiate $\ln(f(x))$ for some function f, first see if you can simplify the expression by using the properties of a logarithm.

Properties of Logarithms

$$\ln(AB) = \ln(A) + \ln(B) \qquad \ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B) \qquad \ln\left(A^B\right) = B\ln(A)$$

Summary

We described "implicit differentiation," in which you differentiate a function without having an explicit formula for it. The function appears in an equation linking it and another variable. To find its derivative, just differentiate both sides of the equation, using the chain rule.

We applied these techniques in findng extrema and the relation between the rates of change of quantities linked by an equation. We also saw how the properties of logarithms can simplify finding the derivatives of some functions, particularly those involving products, quotients, and powers.

EXERCISES for 5.2 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4 find dy/dx at the indicated values of x and y in two ways: explicitly (solving for y first) and implicitly.

1.[R] $xy = 4$ at $(1, 4)$

2.[R] $x^2 - y^2 = 3$ at $(2, 1)$

3.[R] $x^2y + xy^2 = 12$ at $(3, 1)$

4.[R] $x^2 + y^2 = 100$ at $(6, -8)$

In Exercises 5 to 8 find dy/dx at the given points by implicit differentiation.

5.[R] $\frac{2xy}{\pi} + \sin y = 2$ at $(1, \pi/2)$

6.[R] $2y^3 + 4xy + x^2 = 7$ at $(1, 1)$

7.[R] $x^5 + y^3x + yx^2 + y^5 = 4$ at $(1, 1)$

8.[R] $x + \tan(xy) = 2$ at $(1, \pi/4)$

9.[R] Solve Example 3 by implicit differentiation, but differentiate (5.2.2) and (5.2.3) with respect to h instead of r.

10.[R] What is the shape of the cylindrical can of largest volume that can be constructed with a given surface area? Do not find the radius and height of the largest can; find the ratio between them. *Suggestion:* Call the surface area S and keep in mind that it is constant.

11.[M] Using implicit differentiation, find $D(\arctan x)$. *Hint:* Start with $x = \tan(y)$.

12.[M] Using implicit differentiation, find $D(\arcsin x)$. *Hint:* Start with $x = \sin(y)$.

In Exercises 13 to 16 find dy/dx at a general point (x, y) on the given curve.

13.[R] $xy^3 + \tan(x + y) = 1$

14.[R] $\sec(x + 2y) + \cos(x - 2y) + y = 2$

15.[R] $-7x^2 + 48xy + 7y^2 = 25$

16.[R] $\sin^3(xy) + \cos(x + y) + x = 1$

In Exercises 17 to 20 implicit differentiation is used to find a second derivative.

17.[R] Assume that $y(x)$ is a differentiable function of x and that $x^3y + y^4 = 2$. Assume that $y(1) = 1$. Find $y''(1)$, following these steps.

(a) Show that $x^3y' + 3x^2y + 4y^3y' = 0$.

(b) Use (a) to find $y'(1)$.

(c) Differentiate the equation in (a) and show that $x^3y'' + 6x^2y' + 6xy + 4y^3y'' + 12y^2(y')^2 = 0$.

(d) Use the equation in (c) to find $y''(1)$. [*Hint:* $y(1)$ and $y'(1)$ are known.]

18.[R] Find $y''(1)$ if $y(1) = 2$ and $x^5 + xy + y^5 = 35$.

19.[R] Find $y'(1)$ and $y''(1)$ if $y(1) = 0$ and $\sin y = x - x^3$.

20.[R] Find $y''(2)$ if $y(2) = 1$ and $x^3 + x^2y - xy^3 = 10$.

21.[R] Use implicit differentiation to find the highest and lowest points on the ellipse $x^2 + xy + y^2 = 12$. HINT: What do you know about dy/dx at the highest and lowest points on the graph of a function?

22.[M]

(a) What difficulty arises when you use implicit differentiation to maximize x^2+y^2 subject to $x^2 + 4y^2 = 16$?

(b) Show that a maximum occurs when dy/dx is not defined. What is the maximum of $x^2 + y^2$ subject to $x^2 + 4y^2 = 16$?

(c) The problem can be viewed geometrically as "Maximize the square of the distance from the origin for points on the ellipse $x^2 + 4y^2 = 16$." Sketch the ellipse and interpret (b) in terms of it.

23.[R] How fast is the fish in Example 4 moving through the water when it is 1 foot horizontally from the bridge?

24.[R] The angler in Example 4 decides to let the line out as the fish swims away. The fish swims away at a constant speed of 5 feet per second relative to the water. How fast is the angler paying out his line when the horizontal distance from the bridge to the fish is

(a) 1 foot?

(b) 100 feet?

25.[R] A 10-foot ladder is leaning against a wall. A person pulls the base of the ladder away from the wall at the rate of 1 foot per second.

(a) Draw a neat picture of the situation and label the varying lengths by letters and the fixed lengths by numbers.

(b) Obtain an equation involving the variables in (a).

(c) Differentiate it with respect to time.

(d) How fast is the top going down the wall when the base of the ladder is 6 feet from the wall? 8 feet from the wall? 9 feet from the wall?

26.[R] A kite is flying at a height of 300 feet in a horizontal wind.

(a) Draw a neat picture of the situation of label the varying lengths by letters and the fixed lengths by numbers.

(b) When 500 feet of string is out, the kite is pulling the string out at a rate of 20 feet per second. What is the kite's velocity? (Assume the string remains straight.)

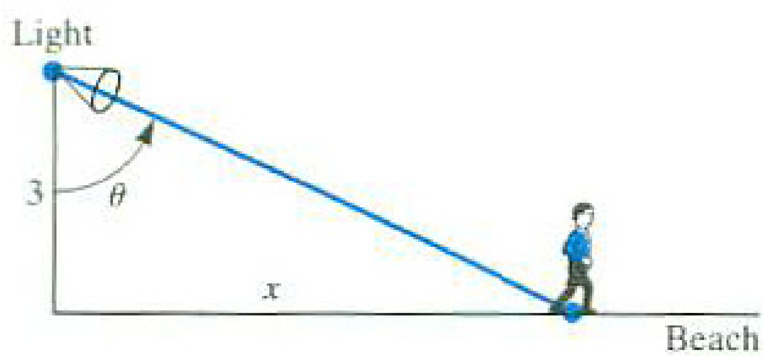


Figure 5.2.5:

27.[R] A beachcomber walks 2 miles per hour along the shore as the beam from a rotating light 3 miles offshore follows him. (See Figure 5.2.5.)

(a) Intuitively, what do you think happens to the rate at which the light rotates as the beachcomber walks further and further along the shore away from the lighthouse?

(b) Let x describe the distance of the beachcomber from the point on the shore nearest the light and θ the angle of the light, obtain an equation relating θ and x.

(c) With the aid of (b), show that $d\theta/dt = 6/(9 + x^2)$ (radians per hour).

(d) Does the formula in (c) agree with your guess in (a)?

28.[R] A man 6 feet tall walks at the rate of 5 feet per second away from a street lamp that is 20 feet high. At what rate is his shadow lengthening when he is

(a) 10 feet from the lamp?

(b) 100 feet from the lamp?

29.[R] A large spherical balloon is being inflated at the rate of 100 cubic feet per minute. At what rate is the radius increasing when the radius is

(a) 10 feet?

(b) 20 feet?

(The volume of a sphere of radius r is $V = 4\pi r^3/3$.)

30.[R] A shrinking spherical balloon loses air at the rate of 1 cubic inch per second. At what rate is its radius changing when the radius is

(a) 2 inches

(b) 1 inch?

31.[R] Bulldozers are moving earth at the rate of $1,000$ cubic yards per hour onto a conically shaped hill whose height of the hill increasing when the hill is

(a) 20 yards high?

(b) 100 yards high?

(The volume of a cone of radius r and height h is $V = \pi r^2 h/3$.)

32.[R] The lengths of the two legs of a right triangle depend on time. One leg, whose length is x, increaes at the rate of 5 feet per second, while the other, of length y, decreases at the rate of 6 feet per second. At what rate is the hypotenuse changing when $x = 3$ feet and $y = 4$ feet? Is the hypotenuse increasing or decreasing then?

33.[R] Two sides of a triangle and their included angle are changing with respect to time. The angle increases at the rate of 1 radian per second, one side increases at the rate of 3 feet per second, and the other side decrease at the rate of 2 feet per second. Find the rate at which the area is changing when the angle is $\pi/4$, the first side is 4 feet long, and the second side is 5 long. Is the area decreasing or increasing then?

34.[R] The length of a rectangle is increasing at the rate of 7 feet per second, and the width is decreasing at the rate of 3 feet per second. When the length is 12 feet and the width is 5 feet, find the rate of change of

(a) the area,

(b) the perimeter

(c) the length of the diagonal.

Exercises 35 to 39 concern acceleration.

35.[R] What is the acceleration of the fish described in Example 4 when the length of line is

(a) 300 feet?

(b) 31 feet?

NOTE: The notation $\dot{x}$ for dx/dt, $\dot{\theta}$ for $d\theta/dt$, $\ddot{x}$ for d^2x/dt^2, and $\ddot{\theta}$ for $d^2\theta/dt^2$ was introduced by Newton and is still common in physics.

36.[R] A woman on the ground is watching a jet through a telescope as it approaches at a speed of 10 miles per minute at an altitude of 7 miles. At what rate (in radians per minute) is the angle of the telescope changing when the horizontal distance of the jet from the woman is 24 miles? When the jet is directly above the woman?

37.[R] Find $\ddot{\theta}$ in Example 36 when the horizontal distance from the jet is

(a) 7 miles,

(b) 1 mile.

38.[R] A particle moves on the parabola $y = x^2$ in such a way that $\dot{x} = 3$ throughout the journey. Find the formulas for (a) $\dot{y}$ and (b) $\ddot{y}$.

39.[R] Call one acute angle of a right triangle θ. The adjacent leg has length x and the opposite leg has length y.

40.[R] Call one acute angle of a right triangle θ. The adjacent leg has length x and the opposite leg has length y.

(a) Obtain an equation relating x, y and θ.

(b) Obtain an equation involving $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ (and other variables).

(c) Obtain an equation involving $\ddot{x}$, $\ddot{y}$, and $\ddot{\theta}$ (and other variables).

41.[R] A two-piece extension ladder leaning against a wall is collapsing at the rate of 2 feet per second and the base of the ladder is moving away from the wall at the rate of 3 feet per second. How fast is the top of the ladder moving down the wall when it is 8 feet from the ground and the foot is 6 feet from the wall? (See Figure 5.2.6.)

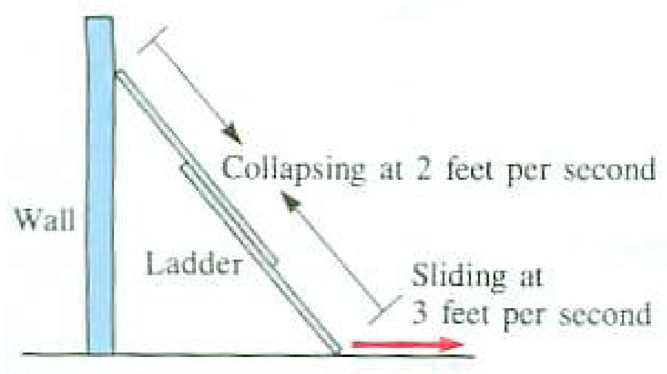


Figure 5.2.6:

42.[R] At an altitude of x kilometers, the atmospheric pressure decreases at a rate of $128(0.88)^x$ millibars per kilometer. A rocket is rising at the rate of 5 kilometers per second vertically. At what rate is the atmospheric pressure changing (in millibars per second) when the altitude of the rocket is (a) 1 kilometer? (b) 50 kilometers?

43.[R] A woman is walking on a bridge that is 20 feet above a river as a boat passes directly under the center of the bridge (at a right angle to the bridge) at 10 feet per second. At that moment the woman is 50 feet from the center and approaching it at the rate of 5 feet per second.

(a) At what rate is the distance between the boat and woman changing at that moment?

(b) Is the rate at which they are approaching or separating increasing or is it decreasing?

44.[R] A spherical raindrop evaporates at a rate proportional to its surface area. Show that the radius shrinks at a constant rate.

45.[R] A couple is on a Ferris wheel when the sun is directly overhead. The diameter of the wheel is 50 feet, and its speed is 0.01 revolution per second.

(a) What is the speed of their shadows on the ground when they are at a two-o'clock position?

(b) A one-o'clock position?

(c) Show that the shadow is moving its fastest when they are at the top or bottom, and its slowest when they are at the three-o'clock or nine-o'clock position.

46.[R] Does the tangent line to the curve $x^3 + xy^2 + x^3y^5 = 3$ at the point $(1, 1)$ pass through the point $(-2, 3)$? (Explain.)

Exercises 47 and 48 obtain by implicit differentiation the formulas for differentiating $x^{1/n}$ and $x^{m/n}$ with the assumption that they are differentiable functions. Here m and n are integers.

47.[M] Let n be a positive integer. Assume that $y = x^{1/n}$ is a differentiable function of x. From the equation $y^n = x$ deduce by implicit differentiation that $y' = (1/n)x^{1/n-1}$.

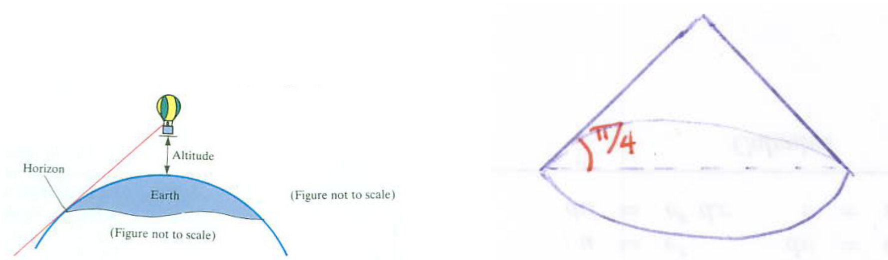
48.[M] Let m be a nonzero integer and n a positive interger. Assume that $y = x^{m/n}$ is a differentiable function of x. From the equation $y^n = x^m$ deduce by implicit differentiation that $y' = (m/n)x^{m/n-1}$.

49.[R] Water is flowing into a hemispherical bowl of radius 5 feet at the constant rate of 1 cubic foot per minute.

(a) At what rate is the top surface of the water rising when it height above the bottom of the bowl is 3 feet? 4 feet? 5 feet?

(b) If $h(t)$ is the depth in feet at time t, find $\ddot{h}$ when $h = 3, 4$, and 5.

50.[R] A man in a hot-air balloon is ascending at the rate of 10 feet per second. How fast is the distance from the balloon to the horizon (that is, the distance the man can see) increasing when the balloon is $1,000$ feet high? Assume that the earth is a ball of radius $4,000$ miles. (See Figure 5.2.7(a).)



(a) (b)

Figure 5.2.7:

51.[R] The Clean Waste company adds 100 cubic yards of debris to a landfill each day. The operator decides to keep piling it up in the form of a cone whose base angle is $\pi/4$. See Figure 5.2.7(b). (He plans either to turn it into a ski run or put an observation restaurant on top.) At what rate is the height of the cone increasing when it is

(a) 10 yards?

(b) 20 yards ?

(c) 100 yards?

(d) How long will it take to make a cone 30 yards high?

(e) How long to make one 300 yards high, which is the operator's goal?

52.[R] (*Contributed by Keith Sollers, when an undergraduate at the University of California at Davis.*) We quote from his note. "The numbers are ugly, but I think it's a good problem nevertheless. I didn't think it up myself. The Medical Center eye group gave me the problem and asked me to solve it. They were going to put a gas bubble in someone's eye."
The volume of a gas bubble changes from 0.4 cc to 1.6 cc in 74 hours. Assuming that the rate of change of the radius is constant, find,

(a) The rate at which the radius changes;

(b) The rate at which the volume of the bubble is increasing at any volume V;

(c) The rate at which the volume is increasing when the volume is 1 cc.

53.[R] Differentiate the function in Example 6 directly, without taking logarithms first.

In Exercises 54 to 59 differentiate the given function by logarithmic differentiation.

54.[R] $y = x^3 \sin^2(2x)$

55.[R] $y = \sqrt{\sin(2x)}\,\sqrt[3]{1+x^3}$

56.[R] $y = \frac{x^3\cos(2x)}{(1+x^2)^4}$

57.[R] $y = \frac{\tan^3(5x)}{\sqrt[3]{e^{x^2}}\arcsin(5x)}$

58.[R] $y = \frac{(x^3+2x)(\arctan(3x)}{1+e^{2x}}$

59.[R] $y = \frac{\left(\sqrt{\ln(2x)}\right)^3(\sin(3x))^5}{(x^3+x)^2}$

In Exercises 60 to 64 first simplify the formula for the function with the aid of properties of logarithms. Then, find dy/dx.

60.[M] $y = \ln\left(\frac{\left(\sqrt{1+x^2}\right)^3\left(e^{3x}+1\right)}{1+\sin(2x)}\right)$.

61.[M] $y = \ln\left(\left(\sqrt{1+\sin(2x)}\right)^3\right)$

62.[M] $y = \ln\left(\frac{(x^3+2)^5}{(x^2+5)^2}\right)$

63.[M] $y = \ln\left((\sin(2x))^3\sqrt{\arctan(3x)}\right)$

64.[M] $y = \ln\left(\frac{(\ln(x^2))^5(\arcsin(3x))^5}{(\tan(5x)^2}\right)$

65.[M] Find $D(x^k)$, $x > 0$, by logarithmic differentiation of $y = x^k$.

66.[M] Let $y = x^x$.

(a) Find y' by logarithmic differentiation. That is, first take the logarithm of both sides.

(b) Find y' by first writing the base as $e^{\ln(x)}$. That is, write $y = x^x = \left(e^{\ln(x)}\right)^x = e^{x\ln(x)}$.

67.[M] Find the first and second derivatives of $y = \sec(x^2)\frac{\sin(x^2)}{x}$.

5.3 Higher Derivatives and the Growth of a Function

The only higher derivative we've used so far is the second derivative. In the study of motion, if y denotes position then y'' is acceleration. In the study of graphs, the second derivative determines whether the graph is concave up ($y'' > 0$) or down ($y'' < 0$). Later, in Section 9.6, the second derivative will appear in a formula that measures the curviness of a curve.

Now we will see how the higher derivatives (including the second derivative) influence the growth of a function. In the next section this will be applied to estimate the error in approximating a function by a polynomial.

Introduction

Imagine that you are in a car motionless at the origin of the x-axis. Then you put your foot to the gas pedal and accelerate. The greater the acceleration, the faster the speed increases; the greater the speed, the further you travel in a given time. So the acceleration, which is the second derivative of the position function, influences the function itself. This illustrates how a higher derivative of a function influences the growth of a function. In this section we examine this influence in more detail.

The following lemma is the basis for our analysis. In terms of daily life, it says, "The faster runner wins the race."

If $a > b$, then $f(x) \geq g(x)$. See Exercise 31.

Lemma 5.3.1. *Let $f(x)$ and $g(x)$ be differentiable functions on an interval I. Let a be a number in I where $f(a) = g(a)$. Assume that $f'(x) \leq g'(x)$ for x in I. Then $f(x) \leq g(x)$ for all x in I to the right of a and $f(x) \geq g(x)$ for all x in I to the left of a.*

Figure 5.3.1 makes this plausible, when the graphs of f and g are straight lines. To the right of $x = a$ the steeper line lies above the other line. To the left of $x = a$ the steeper line lies below the other line.

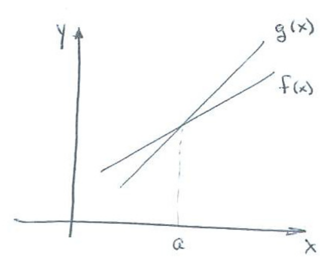


Figure 5.3.1:

Proof of Lemma 5.3.1

Consider the case when $x > a$. Let $h(x) = f(x) - g(x)$. Then $h(a) = 0$ and $h'(x) = f'(x) - g'(x) \leq 0$. Thus, h is a non-increasing function. Since $h(a) = 0$, it follows that $h(x) \leq 0$ for $x \geq a$. That is, $f(x) - g(x) \leq 0$, hence $f(x) \leq g(x)$ for $x > a$. •

Repeated application of Lemma 5.3.1 will enable us to establish a connection between higher derivatives and the function itself.

Higher Derivatives and the Growth of a Function

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$

In the following theorem we name the function $R(x)$ because that will be the notation in the next section when $R(x)$ is the "remainder" function. The notation $n!$ (read: "n factorial") for a positive integer n is shorthand for the product of all integers from 1 through n: $n! = n(n-1)\cdots 3\cdot 2\cdot 1$. The symbol $0!$ is usually defined to be 1.

Theorem 5.3.2 (Growth Theorem). *Assume that at a the function R and its first n derivatives are zero,*

$$R(a) = R'(a) = R''(a) = R^{(3)}(a) = \cdots = R^{(n)}(a) = 0.$$

Assume also that $R(x)$ has continuous derivatives up through the derivative of order $n+1$ in some open interval I containing the numbers a and x. Then there is a number c_n in the interval $[a, x]$ such that

$$R(x) = R^{(n+1)}(c_n)\frac{(x-a)^{n+1}}{(n+1)!}. \tag{5.3.1}$$

Before giving the straightforward proof, we illustrate the theorem by several examples.

EXAMPLE 1 Assume that $R(5) = R'(5) = R''(5) = 0$ and $\left|R^{(3)}(x))\right| \le 4$ for x in the interval $(3, 7)$. Show that $|R(x)| \le 2|x-5|^3/3$ for x in $(3, 7)$.
SOLUTION By the Growth Theorem, with $a = 5$ and $n = 2$,

$$R(x) = R^{(3)}(c_3)\frac{(x-5)^3}{3!} \qquad \text{for some number } c_3 \text{ between 5 and } x.$$

Though we do not know c_3, we do know that $|R^{(3)}(c_3)| \le 4$. So

$$|R(x)| = |R^{(3)}(c_3)|\frac{|x-5|^3}{3!} \le 4\frac{|x-5|^3}{3!} = \frac{2}{3}|x-5|^3.$$

◇

The Growth theorem with $n = 1$ and $a = 0$ describes the position of an accelerating car. One has $R(0) = 0$ (at time 0 the car is at position 0), $R'(0) = 0$ (at time 0 the car is not moving) and R'' describes the acceleration. If that acceleration is constant, equal to k, then (5.3.1) gives the car's position at time x as $R(x) = k\frac{x^2}{2!}$. If the acceleration is not constant, it says that $R(x)$ equals the acceleration at some time multiplied by $x^2/2$.

EXAMPLE 2 Show that $|e^x - 1 - x| \le \frac{e}{2}x^2$ for x in $(-1, 1)$.
SOLUTION Let $R(x) = e^x - 1 - x$. Then $R(0) = e^0 - 1 - 0 = 0$. And, since

$R'(x) = e^x - 1$, $R'(0) = e^0 - 1 = 0$ also. $R''(x) = e^x$. By the Growth Theorem, with $a = 0$ and $n = 1$, there is a number c_1 in $(-1, 1)$ such that

$$e^x - 1 - x = e^{c_1}\frac{(x-0)^2}{2!}.$$

We do not know c_1, but, since it is less than 1, $e^{c_1} < e$. Thus

$$|e^x - 1 - x| \le e\frac{x^2}{2}. \tag{5.3.2}$$

◇

The inequality (5.3.2) in the preceding example provides a way to estimate e^x when x is small. For instance, $|e^{0.1} - 1 - 0.1| \le \frac{e}{2}(0.1)^2 = e/200$. The estimate 1.1 for $e^{0.1}$ is off by at most $e/200 \approx 0.013591$.

EXAMPLE 3 Let $R(x) = \cos(x) - 1 + \frac{x^2}{2}$. Show that $|R(x)| \le \frac{|x^3|}{6}$.
SOLUTION As in Example 2 we use the Growth Theorem with $a = 0$ and $x > 0$. Since powers of $x = (x - 0)$ appear in $R(x)$, this suggests examining $R(x)$ at $a = 0$:

$$\begin{aligned} R(x) &= \cos(x) - 1 - \frac{x^2}{2}, \quad \text{so} \quad R(0) = 1 - 1 + 0 = 0; \\ R'(x) &= -\sin(x) + x, \quad \text{so} \quad R'(0) = 0 + 0 = 0; \\ R''(x) &= -\cos(x) + 1, \quad \text{so} \quad R''(0) = -1 + 1 = 0; \textit{and} \\ R^{(3)}(x) &= \sin(x). \end{aligned}$$

By the Growth theorem, with $a = 0$ and $n = 2$,

$$R(x) = \sin(c_2)\frac{x^3}{3!} \qquad \text{for some number } c_2 \text{ between 0 and } x.$$

Because $|\sin(x)| \le 1$,

$$|R(x)| \le \left|(1)\frac{x^3}{6}\right| = \frac{|x|^3}{6}.$$

◇

Example 3 provides a good estimate for values of the cosine function for small angles. For instance, if $x = 0.1$ radians, we have

0.1 radians $= 0.1\frac{180^\circ}{\pi}{}^\circ \approx 5.7^\circ$

$$\left|\cos(0.1) - 1 + \frac{0.1^2}{2}\right| \le \frac{0.1^3}{6} = 0.00016667 = 1.6667 \times 10^{-4}.$$

Thus, $1 - \frac{0.1^2}{2} = 1 - 0.005 = 0.995$ is an estimate of $\cos(0.1) \approx 0.9950041653$ with an error less than $0.00016667 - \frac{1}{6} \times 10^{-3}$.

Remark: An even better bound on the growth of $R(x)$ in Example 3 is possible. In addition to $R(0) = R'(0) = R''(0) = 0$, notice that $R^{(3)}(0) = \sin(0) = 0$. This means that $|R(x)| \leq \left| M_4 \frac{(x-0)^4}{4!} \right|$ where M_4 is the maximum value of $R^{(4)}(t) = \cos(t)$ in the interval $[0, x]$. As in Example 3, $M \leq 1$. Thus,

$$|R(x)| \leq \left| (1)\frac{x^4}{4!} \right| = \frac{x^4}{24}.$$

In fact, $|\cos(0.1) - 0.995| \approx 4.16528 \times 10^{-6}$.

This means the difference between the exact value of $\cos(0.1)$ and the estimate $1 - \frac{0.1^2}{2} = 0.995$ is no more than $\frac{0.1^4}{24} = 4.16667 \times 10^{-6}$. This shows the estimate in Example 3 is accurate to five decimal places.

In any case, $1 - \frac{x^2}{2}$ is a good estimate of $\cos(x)$ for small values of x.

A Refinement of the Growth Theorem

When proving the Growth theorem we will establish something stronger:

Theorem 5.3.3. *Refined Growth Theorem If $m \leq R^{(n+1)}(t) \leq M$ and all earlier derivatives of R are 0 at a, then*

$$R(x) \text{ is between } m\frac{(x-a)^{n+1}}{(n+1)!} \text{ and } M\frac{(x-a)^{n+1}}{(n+1)!}. \tag{5.3.3}$$

This statement holds even if x is less than a and $(x - a)$ is negative.

EXAMPLE 4 Let $R(x) = e^x - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})$. Show that $\frac{1}{1152} \leq R(\frac{1}{2}) \leq \frac{1}{128}$. Use this estimate to obtain approximations, with error bounds, for $\sqrt{e} = e^{1/2}$ and e.

SOLUTION

$$\begin{aligned} R(0) &= e^0 - 1 - 0. \\ R'(x) &= e^x - (1 + x + \frac{x^2}{2!}), &\text{so}\quad& R'(0) = 0. \\ R''(x) &= e^x - (1 + x), &\text{so}\quad& R''(0) = 0. \\ R^{(3)}(x) &= e^x - 1, &\text{so}\quad& R^{(3)}(0) = 0. \\ R^{(4)}(x) &= e^x, &\text{and}\quad& R^{(4)}(0) = 1 \neq 0. \end{aligned}$$

But, for x in $I = (-1, 1)$, $\frac{1}{3} \le e^{-1} \le e^x \le e^1 < 3$. Theorem 5.3.3, with $a = 0$, $n = 3$, $m = \frac{1}{3}$, $M = 3$, and $x = \frac{1}{2}$ gives

$$\begin{array}{llcl} & \frac{1}{3}\frac{(1/2)^4}{4!} & \le R(1/2) \le & 3\frac{(1/2)^4}{4!} \\ \text{Then,} & \frac{1}{1152} & \le \sqrt{e} - \left(1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!}\right) \le & \frac{1}{128} \\ \text{or} & \frac{79}{48} + \frac{1}{1152} & \le \sqrt{e} \le & \frac{79}{48} + \frac{1}{128} \\ \text{so} & 1.64670 & \le \sqrt{e} \le & 1.65365. \end{array}$$

As you can check with your calculator, $\sqrt{e} \approx 1.64872$ to five decimal places. ⋄

As Example 4 shows, the Growth Theorem provides not only upper bounds on the error in approximating a function by certain polynomials, but lower bounds on that error as well.

Proof of the Growth Theorem

Proof of the Growth Theorem

We illustrate the proof in the case $n = 2$. For convenience, we take the case $x > a$. The case with $x < a$ is complicated by the fact that $x - a$ is then negative and the sign of $(x - a)^n$ depends on whether n is odd or even.

Assume $R(a) = R'(a) = R''(a) = 0$ and $R^{(3)}(x)$ is continuous in the interval $[a, x]$. We want to show there is a number c_2 in $[a, x]$ such that

$$R(x) = R^{(3)}(c_2)\frac{(x-a)^3}{3!}.$$

Let M be the maximum of $R^{(3)}(t)$ and m be the minimum of $R^{(3)}(t)$ on the closed interval $[a, x]$. Thus

$$m \le R^{(3)}(t) \le M \qquad \text{for all } t \text{ in } [a, x].$$

We will see first what the inequality $R^{(3)}(t) \le M$ implies about $R(x)$.

We rewrite that inequality as

$$\frac{d}{dt}\left(R^{(2)}(t)\right) \le \frac{d}{dt}\left(M(t-a)\right). \tag{5.3.4}$$

Now apply Lemma 5.3.1 with $f(t) = R^{(2)}(t)$ and $g(t) = M(t - a)$. Note that $f(a) = 0$ and $g(a) = M(a - a) = 0$. (That is why we used the antiderivative $M(t - a)$ rather than the expected Mt.) Also $f''(a) = 0 = g''(a)$. By the lemma

$$R^{(2)}(t) \le M(t-a). \tag{5.3.5}$$

Next, rewrite (5.3.5) as

$$\frac{d}{dt}\left(R'(t)\right) \leq \frac{d}{dt}\left(M\frac{(t-a)^2}{2}\right).$$

Applying the lemma again shows that

$$R'(t) \leq M\frac{(t-a)^2}{2}. \tag{5.3.6}$$

Finally, rewrite (5.3.6) as

$$\frac{d}{dt}\left(R(t)\right) \leq \frac{d}{dt}\left(M\frac{(t-a)^3}{3\cdot 2}\right).$$

The lemma asserts that

$$R(t) \leq M\frac{(t-a)^3}{3!}. \tag{5.3.7}$$

Similar reasoning, starting with $m \leq R^{(3)}(t)$ shows that

$$m\frac{(t-a)^3}{3!} \leq R(t). \tag{5.3.8}$$

Combining (5.3.7) and (5.3.8) gives two bounds on $R(t)$; in particular on $R(x)$:

$$m\frac{(x-a)^3}{3!} \leq R(x) \leq M\frac{(x-a)^3}{3!}.$$

Because $R^{(3)}$ is continuous on $[a, x]$ it assumes all values between m and M. Thus there is a number c_2 in $[a, x]$ such that

$$R(x) = R^{(3)}(c_2)\frac{(x-a)^3}{3!}.$$

•

Summary

We showed that the bound on the size of the derivative of a function limits the growth of the function itself. When this observation is applied repeatedly we showed that if a function $R(x)$ and its first n derivatives are all zero at a, then

$$R(x) = R^{(n+1)}(c_n)\frac{(x-a)^{n+1}}{(n+1)!} \qquad \text{for some } c_n \text{ between } a \text{ and } x.$$

The number c_n depends on n, not just on a, x, and the function $R(x)$.

EXERCISES for 5.3

Key: R–routine, M–moderate, C–challenging

1.[R] If $f'(x) \geq 3$ for all $x \in (-\infty, \infty)$ and $f(0) = 0$, what can be said about $f(2)$? about $f(-2)$?

2.[R] If $f'(x) \geq 2$ for all $x \in (-\infty, \infty)$ and $f(1) = 0$, what can be said about $f(3)$? about $f(-3)$?

3.[R] What can be said about $f(2)$ if $f(1) = 0$, $f'(1) = 0$, and $2.5 \leq f''(x) \leq 2.6$ for all x?

4.[R] What can be said about $f(4)$ if $f(1) = 0$, $f'(1) = 0$, and $2.9 \leq f''(x) \leq 3.1$ for all x?

5.[R] A car starts from rest and travels for 4 hours. Its acceleration is always at least 5 miles per hour per hour, but never exceeds 12 miles per hour per hour. What can you say about the distance traveled during those 4 hours?

6.[R] A car starts from rest and travels for 6 hours. Its acceleration is always at least 4.1 miles per hour per hour, but never exceeds 15.5 miles per hour per hour. What can you say about the distance traveled during those 6 hours?

7.[R] State the Growth Theorem for $x \geq a$ in the case where R has at least five continuous derivatives and $R(a) = R'(a) = R''(a) = R^{(3)}(a) = R^{(4)}(a) = 0$.

8.[R] State the Growth Theorem in words, using as little math notation as possible.

9.[R] If $R(1) = R'(1) = R''(1) = 0$ and $R^{(3)}(x)$ is continuous on an interval that includes 1 and $R^{(3)}(x) \leq 2$, what can be said about $R(4)$?

10.[R] If $R(3) = R'(3) = R''(3) = R^{(3)}(3) = R^{(4)}(3) = 0$ and $R^{(5)}(x) \leq 6$, what can be said about $R(3.5)$?

11.[R] Let $R(x) = \sin(x) - \left(x - \frac{x^3}{6}\right)$. Show that

(a) $R(0) = R'(0) = R''(0) = R^{(3)}(0) = 0$.

(b) $R^{(4)}(x) = \sin(x)$.

(c) $|R(x)| \leq \frac{x^4}{24}$.

(d) Use $x - \frac{x^3}{6}$ to approximate $\sin(x)$ for $x = 1/2$.

(e) Use (c) to estimate the difference between the exact value for $\sin\left(\frac{1}{2}\right)$ and the approximation obtained in (d).

(f) Explain why $|R(x)| \leq \frac{|x|^5}{120}$. How can this be used to obtain a better estimate of the difference between the exact value for $\sin\left(\frac{1}{2}\right)$ and the approximation obtained in (d)?

(g) By how much does the estimate in (d) differ from $\sin\left(\frac{1}{2}\right)$?

Incidentally, an angle of $\frac{1}{2}$ radian is about 29°.

12.[R] Let $R(x) = \cos(x) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)$. Show that

(a) $R(0) = R'(0) = R''(0) = R^{(3)}(0) = R^{(4)}(0) = R^{(5)}(0) = 0$.

(b) $R^{(6)}(x) = -\cos(x)$.

(c) $|R(x)| \leq \frac{x^6}{6!}$.

(d) Use $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ to estimate $\cos(x)$ for $x = 1$.

(e) By how much does the estimate in (d) differ from $\cos(1)$?

Incidentally, an angle of 1 radian is about 57°.

13.[R] Let $R(x) = (1 + x)^5 - (1 + 5x + 10x^2)$. Show that

(a) $R(0) = R'(0) = R''(0) = 0$.

(b) $R^{(3)}(x) = 60(1 + x)^2$.

(c) $|R(x)| \leq 80x^3$ (on $[-1, 1]$)

(d) Use $1 + 5x + 10x^2$ to estimate $(1 + x)^5$ for $x = 0.2$.

(e) By how much does the estimate in (d) differ from $(1.2)^5$?

14.[M] If $f(3) = 0$ and $f'(x) \geq 2$ for all $x \in (-\infty, \infty)$, what can be said about $f(1)$? Explain.

15.[M] If $f(0) = 3$ and $f'(x) \geq -1$ for all $x \in (-\infty, \infty)$, what can be said about $f(2)$ and about $f(-2)$? Explain.

In Example 3 the polynomial $1-\frac{x^2}{2}$ was shown to be a good approximation to $\cos(x)$ for x near 0. You may wonder how that polynomial was chosen. Exercise 16 shows how.

16.[M] Let $P(x) = a_0 + a_1x + a_2x^2$ be an arbitrary quadratic polynomial. For which values of a_0, a_1, and a_2 is:

(a) $\cos(0) - P(0) = 0$?

(b) $\cos'(0) - P'(0) = 0$?

(c) $\cos''(0) - P''(0) = 0$?

(d) Let $R(x) = \cos(x) - P(x)$. For which $P(x)$ is $R(0) = R'(0) = R''(0) = 0$?

17.[M] Find constants a_0, a_1, a_2, and a_3 such that if $R(x) = \tan(x)-\left(a_0 + a_1x + a_2x^2 + a_3x^3\right)$ then $R(0) = R'(0) = R''(0) = R^{(3)}(0) = 0$.

18.[M] Find constants a_0, a_1, a_2, and a_3 such that if $R(x) = \sqrt{1+x}-\left(a_0 + a_1x + a_2x^2 + a_3x^3\right)$ then $R(0) = R'(0) = R''(0) = R^{(3)}(0) = 0$.

19.[M] Find constants a_0, a_1, a_2, and a_3 such that if

$$R(x) = \sin x - \left(a_0 + a_1\left(x-\frac{\pi}{6}\right) + a_2\left(x-\frac{\pi}{6}\right)^2 + a_3\left(x-\frac{\pi}{6}\right)^3\right)$$

then $R\left(\frac{\pi}{6}\right) = R'\left(\frac{\pi}{6}\right) = R''\left(\frac{\pi}{6}\right) = R^{(3)}\left(\frac{\pi}{6}\right) = 0$.

Exercises 20 to 24 are related.

20.[M] Because $e > 1$, it is known that $e^x \geq 1$ for every $x \geq 0$.

(a) Use Lemma 5.3.1 to deduce that $e^x > 1 + x$, for $x > 0$.

(b) Use (a) and Lemma 5.3.1 to deduce that, for $x > 0$, $e^x > 1 + x + \frac{x^2}{2!}$.

(c) Use (b) and Lemma 5.3.1 to deduce that, for $x > 0$, $e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

(d) In view of (a), (b), and (c), what is the general inequality that can be proved by this approach?

21.[M] Let k be a fixed positive number. For x in $[0, k]$, $e^x \leq e^k$.

(a) Deduce that $e^x \leq 1 + e^kx$ for x in $[0, k]$.

(b) Deduce that $e^x \leq 1 + x + e^k\frac{x^2}{2!}$ for x in $[0, k]$.

(c) Deduce that $e^x \le 1 + x + \frac{x^2}{2!} + e^k \frac{x^3}{3!}$ for x in $[0, k]$.

(d) In view of (a), (b), and (c), what is the general inequality that can be proved by this approach?

22.[M] Combine the results of Exercises 20 and 21 to estimate $e = e^1$ to two decimal places. NOTE: Assume $e \le 3$.

23.[M] What properties of e^x did you use in Exercises 20 and 21?

24.[M] Let $E(x)$ be a function such that $E(0) = 1$ and $E'(x) = E(x)$ for all x.

(a) Show that $E(x) \ge 1$ for all $x \ge 0$.

(b) Use (a) to show that $E(x)$ is an increasing function for all $x \ge 0$. HINT: Show that $E'(x) \ge 1$, for all $x \ge 0$.

(c) Show $E(x) \ge 1 + x + \frac{x^2}{2}$ for all $x \ge 0$.

Exercises 25 to 30 show that $\lim_{x\to\infty} \frac{x}{e^x}$, $\lim_{x\to\infty} \frac{\ln(y)}{y}$, $\lim_{x\to 0^+} x\ln(x)$, $\lim_{x\to\infty} \frac{x^k}{b^x}$ $(b > 1)$, and $\lim_{x\to 0^+} x^x$ are closely connected. (If you know one of them you can deduce the other three.)

Exercises 25 to 26 use the fact that $e^x > 1 + x + \frac{x^2}{2}$ for all $x > 0$ (see Exercise 20).
25.[M] Evaluate $\lim_{x\to\infty} \frac{x}{e^x}$.

26.[M] Evaluate $\lim_{y\to\infty} \frac{\ln(y)}{y}$. HINT: Let $y = e^x$ and compare with Exercise 25.

Exercise 27 provides a proof of the fact that the exponential function grows faster than any power of x. **27.**[M]

(a) Let n be a positive integer. Write $\frac{x^n}{e^x} = \left(\frac{x}{e^{x/n}}\right)\left(\frac{x}{e^{x/n}}\right)\cdots\left(\frac{x}{e^{x/n}}\right)$. Let $y = x/n$ so that $\frac{x}{e^{x/n}} = \frac{ny}{e^y}$. Use Exercise 25 ($n$ times) to show that $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$.

(b) Deduce that for any fixed number k, $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$.

28.[M] Evaluate $\lim_{x\to 0^+} x\ln(x)$ as follows: Let $x = 1/t$, where $t \to \infty$. Then $x\ln(x) = \frac{1}{t}\ln\left(\frac{1}{t}\right) = \frac{-\ln(t)}{t}$. and refer to Exercise 26.

29.[M] Evaluate $\lim_{x\to 0^+} x^x$ as follows: Let $y = x^x$. Then $\ln(y) = x\ln(x)$, a limit that was evaluated in Exercise 28. Explain why $\ln(y) \to 0$ implies $y \to 1$.

30.[M] Evaluate $\lim_{x\to\infty} \frac{x^k}{b^x}$ for any $b > 1$ and k is a positive integer, HINT: Use the result obtained in Exercise 27.

31.[M] Explain why $f(a) = g(a)$ and $f'(x) \le g'(x)$ on $[a, b]$ with $a > b$ implies $f(x) \ge g(x)$ for all x in $[a, b]$.

32.[M] In Example 2 it is shown that $|e^x - 1 - x| \le \frac{e}{2}x^2$ for all x in $(-1, 1)$. Find a bound for

(a) $R(x) = e^x - 1 - x - \frac{x^2}{2}$ on $(-1, 1)$.

(b) $R(x) = e^x - 1 - x$ on $(-2, 1)$.

(c) $R(x) = e^x - 1 - x$ on $(-1, 2)$.

(d) $R(x) = e^x - 1 - x - \frac{x^2}{2}$ on $(-2, 1)$.

(e) $R(x) = e^x - 1 - x - \frac{x^2}{2}$ on $(-1, 2)$.

33.[C] Apply Lemma 5.3.1 for $x > a$ to the case when $R(a) = R''(a) = 0$, $R^{(3)}(t) \le M$, (for all t in $[a, x]$) but $R'(a) = 5$.

34.[C] Consider the following proposal by Sam: "As usual, I can do things more simply than the text. For instance, say $R(a) = R'(a) = R''(a) = 0$ and $R^{(3)}(x) \le M$. I'll show how M affects the size of $R(x)$, for $x > a$.
By the Mean-Value Theorem, $R(x) = R(x) - R(a) = R'(c_1)(x - a)$ for some c_1 in $[a, x]$. Then I just use the MVT again, this time finding $R'(c_1) = R'(c_1) - R'(a) = R''(c_2)(c_1 - a)$ for some c_2 in $[a, c_1]$. One more application of this idea then gives $R''(c_2) = R''(c_2) - R''(a) = R^{(3)}(c_3)(c_3 - a)$.
Then I put these all together, getting

$$R(x) \le M(x - a)(c_2 - a)(c_3 - a).$$

Since c_1, c_2, and c_3 are in $[a, x]$, I can certainly say that

$$R(x) \le M(x - a)^3.$$

I didn't need that lemma about two functions."
Is Sam correct? Is this a valid substitute for the text's treatment? Explain.

35.[C] The proof of the Growth Theorem when x is less than a is slightly different than the proof when x is greater than a. Prove it for the case $n = 4$. Note that in this case $(x - a)^3$ and $(x - a)$ are negative $x < a$.

5.4 Taylor Polynomials and Their Errors

We spend years learning how to add, subtract, multiply, and divide. These same operations are built into any calculator or computer. Both we and machines can evaluate a polynomial, such as

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

when x and the coefficients a_0, a_1, a_2, ..., a_n are given. Only multiplication and addition are needed. But how do we evaluate e^x? We resort to our calculators or look in a table that lists values of e^x. If e^x were a polynomial in disguise, then it would be easy to evaluate it by finding the polynomial and evaluating it instead. But e^x cannot be a polynomial, as tje reasons in the margin show.

Three different reasons:

1. Because e^x equals its own derivative and no polynomial equals its own derivative (other than the polynomial that has constant value 0).
2. When you differentiate a non-constant polynomial, you get a polynomial with a lower degree.
3. Also, $e^x \to 0$ as $x \to -\infty$ and no non-constant polynomial has this property.

Since we cannot write e^x as a polynomial, we settle for the next best thing. Let's look for a polynomial that closely *approximates* e^x. However, no polynomial can be a good approximation of e^x for *all* x, since e^x grows too fast as $x \to \infty$. We search, instead, for a polynomial that is close to e^x for x in some short interval.

In this section we develop a method to construct polynomial approximations to functions. The accuracy of these approximations can be determined using the Growth Theorem from the previous section. Higher derivatives play a pivotal role.

Fitting a Polynomial, Near 0

Suppose we want to find a polynomial that closely approximates a function $y = f(x)$ for x near the input 0. For instance, what polynomial $p(x)$ of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ might produce a good fit?

First we insist that

$$p(0) = f(0) \tag{5.4.1}$$

so the approximation is exact when $x = 0$.

Second, we would like the slope of the graph of $p(x)$ to be the same as that of $f(x)$ when x is 0. Therefore, we require

$$p'(0) = f'(0). \tag{5.4.2}$$

There are many polynomials that satisfy these two conditions. To find the best choices for the four numbers a_0, a_1, a_2, and a_3 we need four equations. To get them we continue the pattern started by (5.4.1) and (5.4.2). So we also insist that

$$p''(0) = f''(0) \tag{5.4.3}$$

and

$$p^{(3)})(0) = f^{(3)}(0). \tag{5.4.4}$$

Equation (5.4.3) forces the polynomial $p(x)$ to have the same sense of concavity as the function $f(x)$ at $x = 0$. We expect the graphs of $f(x)$ and such a polynomial $p(x)$ to resemble each other for x close to a.

To find the unknowns a_0, a_1, a_2, and a_3 we first compute $p(x)$, $p'(x)$, $p''(x)$, and $p^{(3)}(x)$ at 0. Table 5.4.1 displays the computations that express the unknowns, a_0, a_1, a_2, and a_3, in terms of $f(x)$ and its derivatives. For example, note how we compute $p''(x) = 2a_2 + 3 \cdot 2a_3x$ and evaluate it at 0 to obtain $p''(0) = 2a_2 + 3 \cdot 2a_3 \cdot 0 = 2a_2$. Then we obtain an equation for a_2 by equating $p''(0)$ and $f''(0)$; that is, $2a_2 = f''(0)$, so $a_2 = \frac{1}{2}f''(0)$.

$p(x)$ and its derivatives	Their values at 0	Equation for a_k	Formula for a_k
$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$	$p(0) = a_0$	$a_0 = f(0)$	$a_0 = f(0)$
$p^{(1)}(x) = a_1 + 2a_2x + 3a_3x^2$	$p^{(1)}(0) = a_1$	$a_1 = f^{(1)}(0)$	$a_1 = f^{(1)}(0)$
$p^{(2)}(x) = 2a_2 + 3 \cdot 2a_3x$	$p^{(2)}(0) = 2a_2$	$2a_2 = f^{(2)}(0)$	$a_2 = \frac{1}{2}f^{(2)}(0)$
$p^{(3)}(x) = 3 \cdot 2a_3$	$p^{(3)}(0) = 3 \cdot 2a_3$	$3 \cdot 2a_3 = f^{(3)}(0)$	$a_3 = \frac{1}{3\cdot 2}f^{(3)}(0)$

Table 5.4.1:

We can write a general formula for a_k if we let $f^{(0)}(x)$ denote $f(x)$ and recall that $0! = 1$ (by definition), $1! = 1$, $2! = 2 \cdot 1 = 2$, and $3! = 3 \cdot 2$. According to Table 5.4.1,

Factorials appear in the denominators.

$$a_k = \frac{f^{(k)}(0)}{k!}, \qquad k = 0, 1, 2, 3.$$

Therefore

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)})(0)}{3!}x^3.$$

The coefficient of x^k is *completely determined* by the k^{th} derivative of f evaluated at 0. It equals the k^{th} derivative of f at 0 divided by $k!$.

DEFINITION *(Taylor Polynomials at 0)* Let n be a non-negative integer and let f be a function with derivatives at 0 of all orders through n. Then the polynomial

The n^{th}-order Taylor polynomial has degree at most n.

$$f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \tag{5.4.5}$$

is called the **n^{th}-order Taylor polynomial of f centered at 0** and is denoted $P_n(x; 0)$. It is also called a **Maclaurin polynomial**.

Whether $P_n(x;0)$ approximates $f(x)$ for x near 0 is not obvious. We will show that the Macaurin polynomials for e^x do provide good approximations of the functions when x is not too large.

EXAMPLE 1 Find the Maclaurin polynomial $P_4(x;0)$ that agrees with $1/(1-x)$ and its first four derivatives at 0.

SOLUTION The first step is to compute $1/(1-x)$ and its first four derivatives, then evaluate them at $x=0$. Dividing them by suitable factorials gives the coefficients of the Maclaurin polynomial. Table 5.4.2 records the computations.

at x			at 0
$f(x)$	$=$	$\frac{1}{1-x}$	1
$f'(x)$	$=$	$\frac{1}{(1-x)^2}$	1
$f''(x)$	$=$	$\frac{2}{(1-x)^3}$	2
$f^{(3)}(x)$	$=$	$\frac{3\cdot 2}{(1-x)^4}$	$3\cdot 2$
$f^{(4)}(x)$	$=$	$\frac{4\cdot 3\cdot 2}{(1-x)^5}$	$4\cdot 3\cdot 2$

Table 5.4.2:

So the fourth-degree Maclaurin polynomial is

$$P_4(x;0) = 1 + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{3\cdot 2}{3!}x^3 + \frac{4\cdot 3\cdot 2}{4!}x^4,$$

which simplifies to

$$P_4(x;0) = 1 + x + x^2 + x^3 + x^4.$$

Figure 5.4.1 suggests that $P_4(x;0)$ does a fairly good job of approximating $1/(1-x)$ for x near 0. ◇

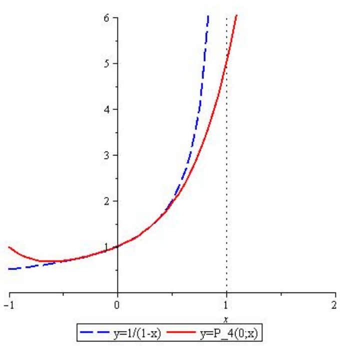


Figure 5.4.1:

The calculations in Example 1 suggest that

> The Maclaurin polynomial $P_n(x;0)$ associated with $1/(1-x)$ is
> $$1 + x + x^2 + x^3 + \cdots + x^n.$$

Because all the derivatives of e^x at 0 are 1,

> The Maclaurin polynomial $P_n(x;0)$ associated with e^x is
> $$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

EXAMPLE 2 Find the Maclaurin polynomial of degree 5 for $f(x) = \sin(x)$.

SOLUTION Again we make a table for computing the coefficients of the Taylor polynomial centered at 0. (See Table 5.4.3.)

Thus

$$\begin{aligned} P_4(x;0) &= f^{(0)}(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \frac{f^{(5)}(0)}{5!}x^5 \\ &= 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!}. \end{aligned}$$

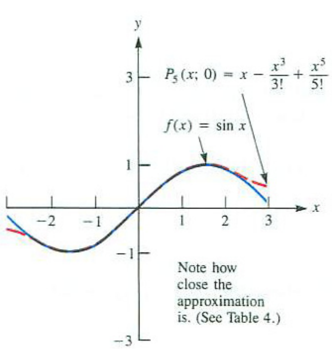


Figure 5.4.2:

at x			at 0		
$f^{(0)}(x)$	$=$	$\sin(x)$	$f^{(0)}(0)$	$=$	$\sin(0) = 0$
$f^{(1)}(x)$	$=$	$\cos(x)$	$f^{(1)}(0)$	$=$	$\cos(0) = 1$
$f^{(2)}(x)$	$=$	$-\sin(x)$	$f^{(2)}(0)$	$=$	$-\sin(0) = 0$
$f^{(3)}(x)$	$=$	$-\cos(x)$	$f^{(3)}(0)$	$=$	$-\cos(0) = -1$
$f^{(4)}(x)$	$=$	$\sin(x)$	$f^{(4)}(0)$	$=$	$\sin(0) = 0$
$f^{(5)}(x)$	$=$	$\sin(x)$	$f^{(5)}(0)$	$=$	$\sin(0) = 0$

Table 5.4.3:

Figure 5.4.2 illustrates the graphs of $P_5(x;1)$ and $\sin(x)$ near 0. ◇

Having found the fifth-order Maclaruin polynomial for $\sin(x)$, let us see how good an approximation it is of $\sin(x)$. Table 5.4.4 compares their values to six-decimal-place accuracy for inputs both near 0 and far from 0. As we see, the closer x is to 0, the better the Taylor approximation is. When x is large, $P_5(x;0)$ gets very large, but the value of $\sin(x)$ stays between -1 and 1.

x	$\sin(x)$	$P_5(x;0)$
0.0	0.000000	0.000000
0.1	0.099833	0.099833
0.5	0.479426	0.479427
1.0	0.841471	0.841667
2.0	0.909297	0.933333
π	0.000000	0.524044
2π	0.000000	46.546732

Table 5.4.4:

A Shorthand Notation

The Maclaurin polynomials associated with $\sin(x)$ have only odd powers and its terms alternate in sign. For m odd,

$$P_m(x;0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \pm \frac{x^m}{m!}.$$

The $\pm$ in front of $x^m/m!$ indicates the coefficient is either positive or negative. For the terms involving x, x^5, x^9, ..., the coefficient is $+1$. For x^3, x^7, x^{11}, ... it is -1. Because m is odd, it can be written as $2n+1$. If n is even, the coefficient of x^{2n+1} is $+1$. If n is odd, the coefficient of x^{2n+1} is -1. The shorthand notation to write the typical summand is

$$(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

So we may write

$$P_{2n+1}(x;0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1_!}.$$

Taylor Polynomials Centered at a

We may be interested in estimating a function $f(x)$ near a number a, not just near 0. In that case, we express the approximating polynomial in terms of powers of $x - a$ instead of powers of $x = x - 0$ and make the derivatives of the approximating polynomial, evaluated at a, coincide with the derivatives of the function at a. Calculuations similar to those that gave us the polynomial (5.4.5) produce the polynomial called a "Taylor polynomial centered at a". (If a is not 0, it is not called a Maclaurin polynomial.)

The n^{th}-order Taylor polynomial of f centered at a is denoted $P_n(x;a)$. It's degree is at most n.

DEFINITION *(Taylor Polynomials of degree n, $P_n(x;a)$)* If the function f has derivatives through order n at a, then the **n^{th}-order Taylor polynomial of f centered at a** is defined as

$$f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and is denoted $P_n(x;a)$.

EXAMPLE 3 Find the n^{th}-order Taylor polynomial centered at a for $f(x) = e^x$.

SOLUTION All the derivatives of e^x evaluated at a are e^a. Thus

$$P_n(x;a) = e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + \frac{e^a}{3!}(x-a)^3 + \cdots + \frac{e^a}{n!}(x-a)^n.$$

◇

The Error in Using A Taylor Polynomial

There is no point using $P_n(x;a)$ to estimate a function $f(x)$ if we have no idea how large the difference between $f(x)$ and $P_n(x;a)$ may be. So let us take a look at the difference.

Define the **remainder** to be the difference between the function, $f(x)$, and the Taylor polynomial, $P_n(x;a)$. Denote the remainder as $R_n(x;a)$. Then,

$$f(x) = P_n(x;a) + R_n(x;a).$$

We will be interested in the absolute value of the remainder. We call $|R_n(x;a)|$ the **error** in using $P_n(x;a)$ to approximate $f(x)$. We do not care whether $P_n(x;a)$ is larger or smaller than the exact value.

Theorem 5.4.1 (The Lagrange Form of the Remainder)**.** *Assume that a function $f(x)$ has continuous derivatives of orders through $n+1$ in an interval that includes the numbers a and x. Let $P_n(x;a)$ be the n^{th}-order Taylor polynomial associated with $f(x)$ in powers of $x-a$. Then there is a number c_n between a and x such that*

$$R_n(x;a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}.$$

Proof of Theorem 5.4.1

For simplicity, we denote the remainder $R_n(x; a) = f(x) - P_n(x; a)$ by $R(x)$. Since $P_n(a; a) = f(a)$,

$$R(a) = f(a) - P_n(a; a) = f(a) - f(a) = 0.$$

Similarly, repeated differentiation of $R(x)$, leads to

$$R^{(k)}(x) = f^{(k)}(x) - P_n^{(k)}(x; a), \tag{5.4.6}$$

for each integer k, $1 \le k \le n$. From the definition of $P_n(x; a)$,

$P_n^{(k)}(a; a) = f^{(k)}(a)$, $k = 0$, $1, \ldots, n$.

$$R^{(k)}(a) = f^{(k)}(a) - P_n^{(k)}(a; a) = 0.$$

$R^{(n+1)}(x) = f^{(n+1)}(x)$

Since $P_n(x; a)$ is a polynomial of degree at most n, its $(n+1)^{\text{st}}$ derivative is 0. As a result, the $(n+1)^{\text{st}}$ derivative of $R(x)$ is the same as the $(n+1)^{\text{st}}$ derivative of $f(x)$. Thus, $R(x)$ satisfies all the assumptions of the Growth Theorem. Recalling (5.3.1) from Section 5.3, we see

See Theorem 5.3.2 in Section 5.3.

Lagrange Form of the Remainder

There is a number c_n between a and x such that

$$R_n(x; a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}.$$

•

EXAMPLE 4 Discuss the error in using $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ to estimate $\sin(x)$ for $x > 0$.

SOLUTION Example 2 showed that $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is the Maclaurin polynomial, $P_5(x; 0)$, associated with $\sin(x)$. In this case $f(x) = \sin(x)$ and each derivative of $f(x)$ is either $\pm\sin(x)$ or $\pm\cos(x)$. Therefore, $|f^{n+1}(c_n)|$ is at most 1, and we have

$$\frac{|f^{5+1}(c_5)|}{6!}x^6 \le \frac{x^6}{6!}.$$

Then

$$\left|\sin(x) - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right| \le \frac{|x|^6|}{6!} = \frac{x^6}{720}.$$

For instance, with $x = 1/2$,

$$\left|\sin\left(\frac{1}{2}\right) - \left(\left(\frac{1}{2}\right) - \frac{\left(\frac{1}{2}\right)^3}{6} + \frac{\left(\frac{1}{2}\right)^5}{120}\right)\right| \le \frac{\left(\frac{1}{2}\right)^6}{720} = \frac{1}{(64)(720)} = \frac{1}{46,080} \approx 0.0000217 = 2.17\times 10^{-5}$$

So the approximation

$$P_5(\frac{1}{2};0) = \frac{1}{2} - \frac{1}{3!}\left(\frac{1}{2}\right)^3 + \frac{1}{5!}\left(\frac{1}{2}\right)^5 = \frac{1}{2} - \frac{1}{48} + \frac{1}{3840} = \frac{1841}{3840} \approx 0.4794271$$

differs from $\sin(1/2)$ (the sine of half a radian) by less than 2.17×10^{-5}; this means at least the first four decimal places are correct. The exact value of $\sin(1/2)$, to ten decimal places is 0.4794255386 and our estimate is correct to five decimal places. By comparison, a calculator gives $\sin(1/2) \approx 0.479426$, which is also correct to five decimal places. ◇

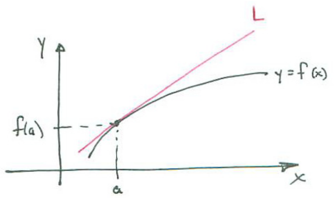


Figure 5.4.3: (Insert label for point $(a, f(a))$.)

The Linear Approximation $P_1(x;a)$

The graph of the Taylor polynomial $P_1(x;a) = f(a) + f'(a)(x-a)$ is a line that passes through the point $(a, f(a))$ and has the same slope as f does at a. That means that the graph of $P_1(x;a)$ is the *tangent line* to the graph of f at $(a, f(a))$. It is customary to call $P_1(x;a) = f(a) + f'(a)(x-a)$ the **linear approximation** to $f(x)$ for x near a. It is often denoted $L(x)$. Figure 5.4.3 shows the graphs of f and L near the point $(a, f(a))$.

Let x be a number close to a and define $\Delta x = x - a$ and $\Delta y = f(a+\Delta x) - f(a)$, quantities used in the definition of the derivative: $f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Often Δx is denoted by dx and $f'(a)dx$ is defined to be "dy", as shown in Figure 5.4.4. Note that dy is an approximation to Δy, and $f(a) + dy$ is an approximation to $f(a + \Delta x) = f(a) + \Delta y$.

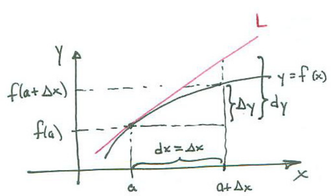


Figure 5.4.4:

In Section 8.2 we will use $dy = f'(x)dx$ and dx as bookkeeping tools to simplify the search for antiderivatives.

The expressions "dx" and "dy" are called **differentials**. In the seventeenth century, dx and dy referred to "infinitesimals", infinitely small numbers. Leibniz viewed the derivative as the quotient $\frac{dy}{dx}$, and that notation for the derivative persists more than three centuries later.

> **WARNING** *(The derivative is not a quotient.)* The derivative is the limit of a quotient.

The next example uses the linear approximation to estimate $\sqrt{x}$ near $x = 1$.

EXAMPLE 5 Use $P_1(x;1)$ to estimate $\sqrt{x}$ for x near 1. Then discuss the error.

SOLUTION In this case $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, and $f'(1) = 1/2$. The linear approximation of $f(x)$ near $a = 1$ is

$$P_1(x;1) = f(1) + f'(1)(x-1) = 1 + \frac{1}{2}(x-1)$$

and the remainder is

$$R_1(x;1) = \sqrt{x} - \left(1 + \frac{1}{2}(x-1)\right).$$

Table 5.4.5 shows how rapidly $R_1(x;1)$ approaches 0 as $x \to 1$ and compares

x	$R_1(x;1)$	$(x-1)^2$	$R_1(x;1)/(x-1)^2$
2.0	$\sqrt{2} - \left(1+\frac{1}{2}(2-1)\right) \approx -0.08578643$	1	-0.08579
1.5	$\sqrt{1.5} - \left(1+\frac{1}{2}(1.5-1)\right) \approx -0.02525512$	0.25	-0.10102
1.1	$\sqrt{1.1} - \left(1+\frac{1}{2}(1.1-1)\right) \approx -0.00119115$	0.01	-0.11912
1.01	$\sqrt{1.01} - \left(1+\frac{1}{2}(1.01-1)\right) \approx -0.00001243$	0.0001	-0.12438

Table 5.4.5:

this difference with $(x-1)^2$.

The final column in Table 5.4.5 shows that $\frac{R_1(x;1)}{(x-1)^2}$ is nearly constant. Because $(x-1)^2 \to 0$ as $x \to 0$, this means $R_1(x;1)$ approaches 0 at the same rate as the square of $(x-1)$.

Since the Lagrange form for $R_1(x;1)$ is approximately $\frac{1}{2}f''(1)(x-1)^2$ when x is near 1, $\frac{R_1(x;1)}{(x-1)^2}$ should be near $\frac{1}{2}f''(1)$ when x is near 1. Just as a check, compute $\frac{1}{2}f''(1)$. We have $f''(x) = \frac{-1}{4}x^{-3/2}$. Thus $\frac{1}{2}f''(1) = \frac{1}{2}\left(\frac{-1}{4}\right) = \frac{-1}{8} = -0.125$. This is consistent with the final column of Table 5.4.5. ⋄

Summary

Given a function f with n derivatives on an interval that contains the number a we defined the n^{th}-order Taylor polynomial at a, $P_n(x;a)$. The first n derivatives of the Taylor polynomial of degree n coincide with the first n derivatives of the given function f at a. Also, $P_n(x;a)$ has the same function value at a that f does.

We define the "zeroth derivative" of a function to be the function itself *and* start counting from 0. This allows us to say simply that the derivatives $P_n^{(k)}(x;a)$ coincide with $f^{(k)}(a)$ for $k = 0, 1, \ldots, n$.

$$P_n(x;a) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $a = 0$, $P_n(x;0)$ is call a Maclaurin polynomial. The general Maclaurin polynomial associated with

$$\begin{array}{rcl} e^x & \text{is} & 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \\ \sin(x) & \text{is} & x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(n+1)!} \\ \cos(x) & \text{is} & 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \\ 1/(1-x) & \text{is} & 1 + x + x^2 + x^3 + \cdots + x^n \end{array}$$

The remainder in using the Taylor polynomial of degree n to estimate a function involves the $(n+1)^{\text{st}}$ derivative of the function:

$$R_n(x;a) = f(x) - P_n(x;a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}$$

where c_n is a number between a and x . The error is the absolute value of the remainder, $|R_n(x;a)|$.

The linear approximation to a function near a is

$$L(x) = P_1(x;a) = f(a) + f'(a)(x-a).$$

The differentials are $dx = x - a$ and $dy = f'(a)dx$. While $dx = \Delta x$, $dy \approx \Delta y = f(x + \Delta x) - f(x)$.

Finite Geometric Series

Let b and r be numbers and n a positive integer. An expression of the form

$$b + br + br^2 + \cdots + br^n \tag{5.4.7}$$

is called a **finite geometric series**. For instance, when $b = 1$ and $r = x$, it takes the form

$$1 + x + x^2 + \cdots + x^n,$$

the Maclaurin polynomial of order n associated with $1/(1-x)$.
If $b = x^n$ and $r = a/x$, (5.4.7) becomes

$$x^n + x^n\left(\frac{a}{x}\right) + x^n\left(\frac{a}{x}\right)^2 + \cdots + x^n\left(\frac{a}{x}\right)^n,$$

which reduces to

$$x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^n.$$

We met this in Section 2.1 when we factored $x^n - a^n$. It is easy to find a short formula for the sum in (5.4.7), which we call S_n. We have

$$\begin{array}{rrcccccccccl} & S_n & = & b & + & br & + & br^2 & + & \cdots & + & br^n \\ \text{and} & rS_n & = & & & br & + & br^2 & + & \cdots & + & br^n + br^{n+1}. \end{array}$$

Subtracting rS_n from S_n yields

$$(1-r)S_n = b - br^{n+1},$$

and we have, if r is not 1,

$$S_n = \frac{b\left(1 - r^{n+1}\right)}{1-r}.$$

EXERCISES for 5.4 *Key:* R–routine, M–moderate, C–challenging

Use a graphing calculator or computer algebra computer algebra system to assist with the computations and with the graphing.

1.[R] Give at least three reasons $\sin(x)$ cannot be a polynomial.

In Exercises 2 to 13 compute the Taylor polynomials. Graph $f(x)$ and $P_n(x;a)$ on the same axes on a domain centered at a. Keep in mind that the graph of $P_1(x;a)$ is the tangent line at the point $(a, f(a))$.

2.[R] $f(x) = 1/(1+x)$, $P_1(x;0)$ and $P_2(x;0)$

3.[R] $f(x) = 1/(1+x)$, $P_1(x;1)$ and $P_2(x;1)$

4.[R] $f(x) = \ln(1+x)$, $P_1(x;0)$, $P_2(x;0)$ and $P_3(x;0)$

5.[R] $f(x) = \ln(1+x)$, $P_1(x;1)$, $P_2(x;1)$ and $P_3(x;1)$

6.[R] $f(x) = e^x$, $P_1(x;0)$, $P_2(x;0)$, $P_3(x;0)$, and $P_4(x;0)$

7.[R] $f(x) = e^x$, $P_1(x;2)$, $P_2(x;2)$, $P_3(x;2)$, and $P_4(x;2)$

8.[R] $f(x) = \arctan(x)$, $P_1(x;0)$, $P_2(x;0)$, and $P_3(x;0)$

9.[R] $f(x) = \arctan(x)$, $P_1(x;-1)$, $P_2(x;-1)$, and $P_3(x;-1)$

10.[R] $f(x) = \cos(x)$, $P_2(x;0)$ and $P_4(x;0)$

11.[R] $f(x) = \sin(x)$, $P_7(x;0)$

12.[R] $f(x) = \cos(x)$, $P_6(x;\pi/4)$

13.[R] $f(x) = \sin(x)$, $P_7(x;\pi/4)$

14.[R] Can there be a polynomial $p(x)$ such that $\sin(x) = p(x)$ for all x in the interval $[1, 1.0001]$? Explain.

15.[R] Can there be a polynomial $p(x)$ such that $\ln(x) = p(x)$ for all x in the interval $[1, 1.0001]$? Explain.

16.[R] State the Lagrange formula for the error in using a Taylor polynomial as an estimate of the value of a function. Use as little mathematical notation as you can.

In Exercises 17 to 22 obtain the Maclaurin polynomial of order n associated with the given function.

17.[R] $1/(1-x)$

18.[R] e^x

19.[R] e^{-x}

20.[R] $\sin(x)$

21.[R] $\cos(x)$

22.[R] $1/(1+x)$

23.[R] Let $f(x) = \sqrt{x}$.

(a) What is the linear approximation, $P_1(x; 4)$, to $\sqrt{x}$ at $x = 4$?

(b) Fill in the following table.

x	$R_1(x;4) = f(x) - P_1(x;4)$	$(x-4)^2$	$\frac{R_1(x;4)}{(x-4)^2}$
5.0			
4.1			
4.01			
3.99			

(c) Compute $f''(4)/2$. Explain the relationship between this number and the entries in the fourth column of the table in (b).

24.[R] Repeat Exercise 23 for the linear approximation to $\sqrt{x}$ at $a = 3$. Use $x = 4$, 3.1, 3.01, and 2.99.

25.[R] Assume $f(x)$ has continuous first and second derivatives and that $4 \leq f''(x) \leq 5$ for all x.

(a) What can be said in general about the error in using $f(2) + f'(2)(x-2)$ to approximate $f(x)$?

(b) How small should $x - 2$ be to be sure that the error — the absolute value of the remainder — is less than or equal to 0.005? NOTE: This ensures the approximate value is correct to 2 decimal places.

26.[R] Let $f(x) = 2 + 3x + 4x^2$.

(a) Find $P_2(x; 0)$.

(b) Find $P_3(x; 0)$.

(c) Find $P_2(x; 5)$.

(d) Find $P_3(x; 5)$.

27.[R]

(a) What can be said about the degree of the polynomial $P_n(x;0)$?

(b) When is the degree of $P_n(x;0)$ less than n?

(c) When is the degree of $P_n(x;a)$ less than n? $(a \neq 0)$

28.[M] In the case of $f(x) = 1/(1-x)$ the error $R_n(x;0)$ in using a Maclaurin polynomial $P_n(x;0)$ to estimate the function can be calculated exactly. Show that it equals $\left|x^{n+1}/(1-x)\right|$.

Exercises 29 to 32 are related.

29.[R] Let $f(x) = (1+x)^3$.

(a) Find $P_3(x;0)$ and $R_3(x;0)$.

(b) Check that your answer to (a) is correct by multiplying out $(1+x)^3$.

30.[R] Let $f(x) = (1+x)^4$.

(a) Find $P_4(x;0)$ and $R_4(x;0)$.

(b) Check that your answer to (a) is correct by multiplying out $(1+x)^4$.

31.[R] Let $f(x) = (1+x)^5$. Using $P_5(x;0)$, show that

$$(1+x)^5 = 1 + 5x + \frac{5\cdot 4}{1\cdot 2}x^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}x^3 + \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5.$$

For a positive integer n and a non-negative integer k, with $k \leq n$, the symbol $\binom{n}{k}$ denotes the **binomial coefficient**:

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k} = \frac{n!}{k!(n-k)!}.$$

Thus

$$(1+x)^5 = \binom{5}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5.$$

Using $P_n(x;0)$ one can show that, for any positive integer n,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n = \sum_{k=0}^{n}\binom{n}{k}x^k.$$

This is the basis for the **Binomial Theorem**,

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}.$$

NOTE: Recall that $\binom{n}{0} = \frac{n!}{0!n!} = 1$ and $\binom{n}{n} = \frac{n!}{n!0!} = 1$.

32.[M]

(a) Using algebra (no calculus) derive the binomial theorem for $(a+b)^3$ from the binomial theorem for $(1+x)^3$.

(b) Obtain the binomial theorem for $(a+b)^{12}$ from the special case $(1+x)^{12} = \sum_{k=0}^{12}\binom{12}{k}x^k$.

In Exercises 33 and 34, use a calculator or computer to help evaluate the Taylor polynomials

33.[M] Let $f(x) = e^x$.

(a) Find $P_{10}(x;0)$.

(b) Compute $f(x)$ and $P_{10}(x;0)$ at $x = 1$, $x = 2$, and $x = 4$.

34.[M] Let $f(x) = \ln(x)$.

(a) Find $P_{10}(x;1)$.

(b) Compute $f(x)$ and $P_{10}(x;1)$ at $x = 1$, $x = 2$, and $x = 4$.

Exercises 35 to 38 involve even and odd functions Recall, from Section 2.5, that a function is even if $f(-x) = f(x)$ and is odd if $f(-x) = -f(x)$.

35.[M] Show that if f is an odd function, f' is an even function.

36.[M] Show that if f is an even function, f' is an odd function.

37.[M]

(a) Which polynomials are even functions?

(b) If f is an even function, are its associated Maclaurin polynomials necessarily even functions? Explain.

38.[M]

(a) Which polynomials are odd functions?

(b) If f is an odd function, are its associated Maclaurin polynomials necessarily odd functions? Explain.

39.[C] This exercise constructs Maclaurin polynomials that do not approximate the associated function. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$.

(a) Find $f'(0)$.

(b) Find $f''(0)$.

(c) Find $P_2(x; 0)$.

(d) What is $P_{100}(x; 0)$.

HINT: Recall the definition of the derivative.

40.[C] Show that in an open interval in which f''' is positive, that $f(x) > f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$. HINT: Treat the cases $a < x$ and $x > a$ separately. NOTE: See also Exercise 17 in Section 4.4.

41.[C]

(a) Show that in an open interval in which $f^{(n+1)}$ is positive (n a positive integer), that $f(x)$ is greater than $P_n(x;0)$.

(b) What additional information is needed to make this a true statement for $x < a$?

NOTE: See also Exercise 40.

42.[C] The quantity $\sqrt{1 - v^2/c^2}$ occurs often in the theory of relativity. Here v is the velocity of an object and c the velocity of light. Justify the following approximations that physicists use:

(a) $\sqrt{1-\frac{v^2}{c^2}} \approx 1 - \frac{1}{2}\frac{v^2}{c^2}$

(b) $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$g

NOTE: Even for a rocket v/c is very small.

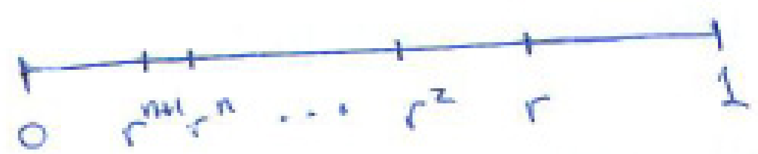


Figure 5.4.5:

43.[C] Let r be a number in $(0, 1)$. Figure 5.4.5 indicates the location of r, r^2, r^3, $\dots$, r^{n+1}. The length of $[r^{n+1}, 1]$ is $1 - r^{n+1}$.

(a) Show geometrically why

$$1 - r^{n+1} = (1-r) + (r - r^2) + (r^2 - r^3) + \cdots + (r^n - r^{n+1}).$$

(b) Deduce that

$$\frac{1 - r^{n+1}}{1 - r} = 1 + r + \cdots + r^n.$$

44.[C] Using the formula for the sum of a finite geometric series, justify the factorization used in Section 2.1. (See Exercise 39, Section 2.1. on page 76.)

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1}).$$

45.[C] If $P_n(x;0)$ is the Maclaurin polynomial associated with $f(x)$, is $P_n(-x;0)$ the Maclaurin polynomial associated with $f(-x)$? Explain.

46.[C] Let $P(x)$ be the Maclaurin polynomial of the second-order associated with $f(x)$. Let $Q(x)$ be the Maclaurin polynomial of the second-order associated with $g(x)$. What part, if any, of $P(x)Q(x)$ is a Maclaurin polynomial associated with $f(x)g(x)$? Explain.

5.5 L'Hôpital's Rule for Finding Certain Limits

There are two types of limits in calculus: those that you can evaluate at a glance, and those that require some work to evaluate. In Section 2.3 we learned to call the limits that can be evaluated easily **determinate** and those that require some work to evaluate are called **indetermnate**.

For instance $\lim_{x\to\pi/2} \frac{\sin(x)}{x}$ is clearly $1/(\pi/2) = 2/\pi$. That's easy. But $\lim_{x\to 0}(\sin(x))/x$ is not obvious. Back in Section 2.1 we used a diagram of circles, sectors, and triangles, to show that this limit is 1.

In this section we describe a technique for evaluating more indeterminate limits, for instance

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

when both $f(x)$ and $g(x)$ approach 0 as x approaches a. The numerator is trying to drag $f(x)/g(x)$ toward 0, at the same time as the denominator is trying to make the quotient large. L'Hôpital's rule helps determine which term wins or whether there is a compromise.

L'Hôpital is pronounced lope-ee-tall.

Indeterminate Limits

The following limits are called **indeterminate** because you can't determine them without knowing more about the functions of f and g.

$$\lim_{x\to a} \frac{f(x)}{g(x)}, \text{ where } \lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

$$\lim_{x\to a} \frac{f(x)}{g(x)}, \text{ where } \lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty$$

L'Hôpital's Rule provides a way for dealing with these limits (and limits that can be transformed to those forms.) In short, l'Hôpital's rule applies only when you need it.

Theorem 5.5.1 (L'Hôpital's Rule (zero-over-zero case))**.** *Let a be a number and let f and g be differentiable over some open interval that contains a. Assume also that $g'(x)$ is not 0 for any x in that interval except perhaps at a. If*

$$\lim_{x\to a} f(x) = 0, \ \lim_{x\to a} g(x) = 0, \text{ and } \lim_{x\to a} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

In short, "to evaluate the limit of a quotient that is indeterminant, evaluate the limit of the quotient of their derivatives." You evaluate the limit of the quotient of the derivatives, not the derivative of the quotient. We will discuss the proof after some examples.

EXAMPLE 1 Find $\lim_{x\to 1}(x^5-1)/(x^3-1)$.
SOLUTION In this case,

$$a=1,\ f(x)=x^5-1,\ \text{and } g(x)=x^3-1.$$

Remember to check that the hypotheses of l'Hôpital's Rule are satisfied.

All the assumptions of l'Hôpital's rule are satisfied. In particular,

$$\lim_{x\to 1}(x^5-1)=0 \text{ and } \lim_{x\to 1}(x^3-1)=0.$$

According to l'Hôpital's rule,

$$\lim_{x\to 1}\frac{x^5-1}{x^3-1}\overset{\text{l'H}}{=}\lim_{x\to 1}\frac{(x^5-1)'}{(x^3-1)'}$$

if the latter limit exists. Now,

$$\begin{aligned}\lim_{x\to 1}\tfrac{(x^5-1)'}{(x^3-1)'} &= \lim_{x\to 1}\tfrac{5x^4}{3x^2} && \text{differentiation of numerator and differentiation of denominator}\\ &= \lim_{x\to 1}\tfrac{5}{3}x^2 && \text{algebra}\\ &= \tfrac{5}{3}.\end{aligned}$$

Thus

$$\lim_{x\to 1}\frac{x^5-1}{x^3-1}=\frac{5}{3}.$$

◇

Sometimes it may be necessary to apply l'Hôpital's Rule more than once, as in the next example.

EXAMPLE 2 Find $\lim_{x\to 0}(\sin(x)-x)/x^3$.
SOLUTION As $x\to 0$, both numerator and denominator approach 0. By l'Hôpital's Rule,

$$\lim_{x\to 0}\frac{\sin(x)-x}{x^3}\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{(\sin(x)-x)'}{(x^3)'}=\lim_{x\to 0}\frac{\cos(x)-1}{3x^2}.$$

But as $x\to 0$, both $\cos(x)-1\to 0$ and $3x^2\to 0$. So use l'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{\cos(x)-1}{3x^2}\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{(\cos(x)-1)'}{(3x^2)'}=\lim_{x\to 0}\frac{-\sin(x)}{6x}.$$

Both $\sin(x)$ and $6x$ approach 0 as $x \to 0$. Use l'Hôpital's Rule yet another time:

Or recall from Section 2.1 that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

$$\lim_{x\to 0} \frac{-\sin(x)}{6x} \overset{\text{l'H}}{=} \lim_{x\to 0} \frac{(-\sin(x))'}{(6x)'} = \lim_{x\to 0} \frac{-\cos(x)}{6} = \frac{-1}{6}.$$

So after three applications of l'Hôpital's Rule we find that

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = -\frac{1}{6}.$$

◇

Sometimes a limit may be simplified before l'Hôpital's Rule is applied. For instance, consider

$$\lim_{x\to 0} \frac{(\sin(x) - x)\cos^5(x)}{x^3}.$$

Since $\lim_{x\to 0} \cos^5(x) = 1$, we have

$$\lim_{x\to 0} \frac{(\sin(x) - x)\cos^5(x)}{x^3} = \left(\lim_{x\to 0} \frac{\sin(x) - x}{x^3}\right) \cdot 1,$$

which, by Example 2, is $-\frac{1}{6}$. This shortcut saves a lot of work, as may be checked by finding the limit using l'Hôpital's Rule without separating $\cos^5(x)$.

Theorem 5.5.1 concerns limits as $x \to a$. L'Hôpital's Rule also applies if $x \to \infty$, $x \to -\infty$, $x \to a^+$, or $x \to a^-$. In the first case, we would assume that $f(x)$ and $g(x)$ are differentiable in some interval (c, ∞) and $g'(x)$ is not zero there. In the case of $x \to a^+$, assume that $f(x)$ and $g(x)$ are differentiable in some open interval (a, b) and $g'(x)$ is not 0 there.

Infinity-over-Infinity Limits

"Infinity-over-infinity" is indeterminate.

Theorem 5.5.1 concerns the limit of $f(x)/g(x)$ when both $f(x)$ and $g(x)$ approach 0. But a similar problem arises when both $f(x)$ and $g(x)$ get arbitrarily large as $x \to a$ or as $x \to \infty$. The behavior of the quotient $f(x)/g(x)$ will be influenced by how rapidly $f(x)$ and $g(x)$ become large.

In short, if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a}(f(x)/g(x))$ is an indeterminate form.

The next theorem presents a form of l'Hôpital's Rule that covers the case in which $f(x) \to \infty$ and $g(x) \to \infty$.

Theorem 5.5.2 (L'Hôpital's Rule (infinity-over-infinity case)**.** *Let f and g be defined and differentiable for all x larger than some number. Then, if $g'(x)$ is not zero for all x larger*

$$\lim_{x\to\infty} f(x) = \infty, \ \lim_{x\to\infty} g(x) = \infty, \ \textit{and} \ \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L,$$

it follows that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

A similar result holds for $x \to a$, $x \to a^-$, $x \to a^+$, or $x \to -\infty$. Moreover, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ could both be $-\infty$, or one could be ∞ and the other $-\infty$.

EXAMPLE 3 Find $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$.

SOLUTION Since $\ln(x) \to \infty$ and $x^2 \to \infty$ as $x \to \infty$, we may use l'Hôpital's Rule in the "infinity-over-infinity" form.

We have

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{(\ln(x))'}{(x^2)'} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Hence $\lim_{x\to\infty} ((\ln(x))/x^2) = 0$. This says that $\ln(x)$ grows much more slowly than x^2 does as x gets large. ◇

EXAMPLE 4 Find

$$\lim_{x\to\infty} \frac{x - \cos(x)}{x}. \tag{5.5.1}$$

SOLUTION Both numerator and denominator approach ∞ and $x \to \infty$. Trying l'Hôpital's Rule, we obtain

$$\lim_{x\to\infty} \frac{x - \cos(x)}{x} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{(x - \cos(x))'}{x'} = \lim_{x\to\infty} \frac{1 + \sin(x)}{1}.$$

L'Hôpital's Rule may fail to provide an answer.

But $\lim_{x\to\infty}(1+\sin(x))$ does not exist, since $\sin(x)$ oscillates back and forth from -1 to 1 as $x \to \infty$

What can we conclude about the limit in (5.5.1)? Nothing at all. L'Hôpital's Rule says that if $\lim_{x\to\infty} f'(x)/g'(x)$ exists, then $\lim_{x\to\infty} f(x)/g(x)$ exists and has the same value. It say nothing about the case when $\lim_{x\to\infty} f'(x)/g'(x)$ does not exist.

It is not difficult to evaluate (5.5.1) directly, as follows:

$$\begin{aligned} \lim_{x\to\infty} \frac{x - \cos(x)}{x} &= \lim_{x\to\infty} \left(1 - \frac{\cos(x)}{x}\right) && \text{algebra} \\ &= 1 - 0 && \text{since } |\cos(x)| \le 1 \\ &= 1. \end{aligned}$$

◇

Moral: Look carefully at a limit before you decide to use l'Hôpital's Rule.

Two cars can help make Theorem 5.5.2 plausible. Imagine that $f(t)$ and $g(t)$ describe the locations on the x-axis of two cars at time t. Call the cars the f-car and the g-car. See Figure 5.5.1. Their velocities are therefore $f'(t)$ and $g'(t)$. These two cars are on endless journeys. But assume that as time $t \to \infty$ the f-car tends to travel at a speed closer and closer to L times the speed of the g-car. That is, assume that

$$\lim_{t\to\infty} \frac{f'(t)}{g'(t)} = L.$$

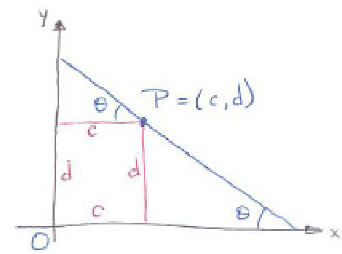


Figure 5.5.1:

No matter how the two cars move in the short run, it seems reasonable that in the long run the f-car will tend to travel about L times as far as the g-car; that is,

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = L.$$

Transforming Limits So You Can Use l'Hôpital's Rule

Many limits can be transformed to limits to which l'Hôpital's Rule applies. For instance, the problem of finding

"zero-times-infinity" is indeterminate

$$\lim_{x\to 0^+} x\ln(x)$$

does not fit into l'Hôpital's Rule, since it does not involve the quotient of two functions. As $x \to 0^+$, one factor, x, approaches 0 and the other factor $\ln(x)$, approaches $-\infty$. So this is another type of indeterminate limit, involving a small number times a large number ("zero-times-infinity"). It is not obvious how this product, $x\ln(x)$, behaves as $x \to 0^+$. (Such a limit can turn out to be "zero, medium, large, or infinite"). A little algebra transforms the zero-times-infinity case into a problem to which l'Hôpital's Rule applies, as the next example illustrates.

EXAMPLE 5 Find $\lim_{x\to 0^+} x\ln(x)$.
SOLUTION Rewrite $x\ln(x)$ as a quotient, $\frac{\ln(x)}{(1/x)}$. Note that

$$\lim_{x\to 0^+} \ln(x) = -\infty \text{ and } \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

By l'Hôpital's Rule,

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} \overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

Thus

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = 0,$$

The factor x, which approaches 0, dominates the factor $\ln(x)$ which "slowly grows towards $-\infty$."

from which it follows that $\lim_{x\to 0^+} x\ln(x) = 0$. ◇

The final example illustrates another type of limit that can be found by first relating it to limits to which l'Hôpital's Rule applies.

Try this on your calculator first.

EXAMPLE 6 $\lim_{x\to 0^+} x^x$.

SOLUTION Since this limit involves an exponential, not a quotient, it does not fit directly into l'Hôpital's Rule. But a little algebra changes the problem to one covered by l'Hôpital's Rule.

Let	y	$=$	x^x.
Then	$\ln(y)$	$=$	$\ln(x^x) = x\ln(x)$
By Example 5,	$\lim_{x\to 0^+} x\ln(x)$	$=$	0.

Therefore, $\lim_{x\to 0^+}\ln(y) = 0$. By the definition of $\ln(y)$ and the continuity of $e^x = \exp(x)$,

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+}\exp(\ln(y)) = \exp(\lim_{x\to 0^+}(\ln(y)) = e^0 = 1.$$

Hence $x^x \to 1$ as $x \to 0^+$. ◇

Concerning the Proof

A complete proof of Theorem 5.5.1 may be found in Exercises 71 to 73. The following argument is intended to make the theorem plausible. To do so, consider the *special case* where f, f', g, and g' are all continuous throughout an open interval containing a — in particular, all four functions are defined at a. Assume that $g'(x) \neq 0$ throughout the interval. Since we have $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, it follows by continuity that $f(a) = 0$ and $g(a) = 0$.

Assume that $\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$. Then

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && \text{since } f(a)=0 \text{ and } g(a)=0\\
&= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} && \text{algebra}\\
&= \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} && \text{limit of quotient equals quotient of limits}\\
&= \frac{f'(a)}{g'(a)} && \text{definitions of } f'(a) \text{ and } g'(a)\\
&= \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} && f' \text{ and } g' \text{ are continuous, by assumption}\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)} && \text{quotient of limits equals limit of quotients}\\
&= L && \text{by assumption.}
\end{aligned}$$

Consequently,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Summary

We described l'Hôpital's Rule, which is a technique for dealing with limits of the indeterminate form "zero-over-zero" and "infinity-over-infinity" . In both of these cases

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if the latter limit exists. Note that it concerns the quotient of two derivatives, *not* the derivative of the quotient.

Table 5.5.1 shows how some limits of other indeterminate forms can be converted into either of these two forms.

L'Hôpital's rule comes in handy during our study of a uniform sprinkler in the Calculus is Everywhere section at the end of this chapter.

Indeterminate Forms	Name	Conversion Method	New Form
$f(x)g(x)$; $f(x) \to 0$, $g(x) \to 0$	Zero-times-infinity $(0 \cdot \infty)$	Write as $\frac{f(x)}{1/g(x)}$ or $\frac{g(x)}{1/f(x)}$	$\frac{0}{0}$ or $\frac{\infty}{\infty}$
$f(x)^{g(x)}$; $f(x) \to 1$, $g(x) \to \infty$	One-to-infinity (1^∞)	Let $y = f(x)^{g(x)}$; take $\ln(y)$, find limit of $\ln(y)$, and then find limit of $y = e^{\ln(y)}$	$\ln(y)$ has form $\infty \cdot 0$
$f(x)^{g(x)}$; $f(x) \to 0$, $g(x) \to 0$	Zero-to-zero (0^0)	Same as for 1^∞	$\ln(y)$ has form $0 \cdot \infty$.

Table 5.5.1:

EXERCISES for 5.5

Key: R–routine, M–moderate, C–challenging

In Exercises 1 to 16 check that l'Hôpital's Rule applies and use it to find the limits. *Identify all uses of l'Hôpital's Rule, including the type of indeterminant form.*

1.[R] $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$

2.[R] $\lim_{x \to 1} \frac{x^7 - 1}{x^3 - 1}$

3.[R] $\lim_{x \to 0} \frac{\sin(3x)}{\sin(2x)}$

4.[R] $\lim_{x \to 0} \frac{\sin(x^2)}{(\sin(x))^2}$

5.[R] $\lim_{x \to 0} \frac{\sin(5x)\cos(3x)}{x}$

6.[R] $\lim_{x \to 0} \frac{\sin(5x)\cos(3x)}{x - \frac{\pi}{2}}$

7.[R] $\lim_{x \to \frac{\pi}{2}} \frac{\sin(5x)\cos(3x)}{x}$

8.[R] $\lim_{x \to \frac{\pi}{2}} \frac{\sin(5x)\cos(3x)}{x - \frac{\pi}{2}}$

9.[R] $\lim_{x \to \infty} \frac{x^3}{e^x}$

10.[R] $\lim_{x \to \infty} \frac{x^5}{3^x}$

11.[R] $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$

12.[R] $\lim_{x \to 0} \frac{\sin(x) - x}{(\sin(x))^3}$

13.[R] $\lim_{x \to 0} \frac{\tan(3x)}{\ln(1 + x)}$

14.[R] $\lim_{x \to 1} \frac{\cos(\pi x/2)}{\ln(x)}$

15.[R] $\lim_{x \to 2} \frac{(\ln(x))^2}{x}$

16.[R] $\lim_{x \to 0} \frac{\arcsin(x)}{e^{2x} - 1}$

In each of Exercises 17 to 22 transform the problem into one to which l'Hôpital's Rule applies; then find the limit. *Identify all uses of l'Hôpital's Rule, including the type of indeterminant form.*

17.[R] $\lim_{x\to 0}(1-2x)^{1/x}$

18.[R] $\lim_{x\to 0}(1+\sin(2x))^{\csc(x)}$

19.[R] $\lim_{x\to 0^+}(\sin(x))^{(e^x-1)}$

20.[R] $\lim_{x\to 0^+} x^2\ln(x)$

21.[R] $\lim_{x\to 0^+}(\tan(x))^{\tan(2x)}$

22.[R] $\lim_{x\to 0^+}(e^x-1)\ln(x)$

> **WARNING** *(Do Not Overuse l'Hôpital's Rule)* Remember that l'Hôpital's Rule, carelessly applied, may give a wrong answer or no answer.

In Exercises 23 to 51 find the limits. Use l'Hôpital's Rule only if it applies. *Identify all uses of l'Hôpital's Rule, including the type of indeterminant form.*

23.[R] $\lim_{x\to\infty}\frac{2^x}{3^x}$

24.[R] $\lim_{x\to\infty}\frac{2^x+x}{3^x}$

25.[R] $\lim_{x\to\infty}\frac{\log_2(x)}{\log_3(x)}$

26.[R] $\lim_{x\to 1}\frac{\log_2(x)}{\log_3(x)}$

27.[R] $\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)$

28.[R] $\lim_{x\to\infty}\left(\sqrt{x^2+3}-\sqrt{x^2+4x}\right)$

29.[R] $\lim_{x\to\infty}\frac{x^2+3\cos(5x)}{x^2-2\sin(4x)}$

30.[R] $\lim_{x\to\infty}\frac{e^x-1/x}{e^x-1/x}$

31.[R] $\lim_{x\to 0}\frac{3x^3+x^2-x}{5x^3+x^2+x}$

32.[R] $\lim_{x\to\infty}\frac{3x^3+x^2-x}{5x^3+x^2+x}$

33.[R] $\displaystyle\lim_{x\to\infty} \frac{\sin(x)}{4+\sin(x)}$

34.[R] $\displaystyle\lim_{x\to\infty} x\sin(3x)$

35.[R] $\displaystyle\lim_{x\to 1^+} (x-1)\ln(x-1)$

36.[R] $\displaystyle\lim_{x\to\pi/2} \frac{\tan(x)}{x-(\pi/2)}$

37.[R] $\displaystyle\lim_{x\to 0} (\cos(x))^{1/x}$

38.[R] $\displaystyle\lim_{x\to 0^+} x^{1/x}$

39.[R] $\displaystyle\lim_{x\to 0} (1+x)^{1/x}$

40.[R] $\displaystyle\lim_{x\to 0} \left(1+x^2\right)^x$

41.[R] $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x^3-1}$

42.[R] $\displaystyle\lim_{x\to 0} \frac{xe^x(1+x)^3}{e^x-1}$

43.[R] $\displaystyle\lim_{x\to 0} \frac{xe^x\cos^2(6x)}{e^{2x}-1}$

44.[R] $\displaystyle\lim_{x\to 0} (\csc(x)-\cot(x))$

45.[R] $\displaystyle\lim_{x\to 0} \frac{\csc(x)-\cot(x)}{\sin(x)}$

46.[R] $\displaystyle\lim_{x\to 0} \frac{5^x-3^x}{\sin(x)}$

47.[R] $\displaystyle\lim_{x\to 0} \frac{(\tan(x))^5-(\tan(x))^3}{1-\cos(x)}$

48.[R] $\displaystyle\lim_{x\to 2} \frac{x^3+8}{x^2+5}$

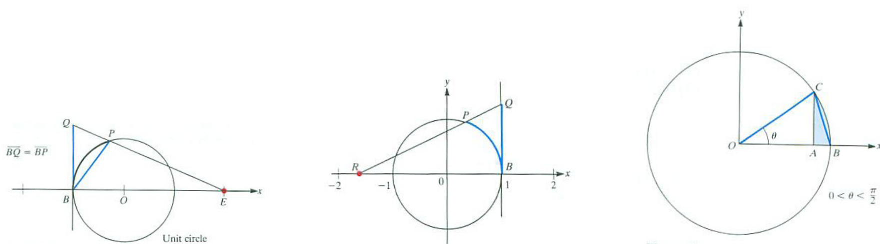
49.[R] $\displaystyle\lim_{x\to\pi/4} \frac{\sin(5x)}{\sin(3x)}$

50.[R] $\displaystyle\lim_{x\to 0} \left(\frac{1}{1-\cos(x)}-\frac{2}{x^2}\right)$

51.[R] $\displaystyle\lim_{x\to 0} \frac{\arcsin(x)}{\arctan(2x)}$

52.[M] In Figure 5.5.2(a) the unit circle is centered at O, BQ is a vertical tangent line, and the length of BP is the same as the length of BQ. What happens to the point E as $Q \to B$?

53.[M] In Figure 5.5.2(b) the unit circle is centered at the origin, BQ is a vertical tangent line, and the length of BQ is the same as the arc length $\widehat{BP}$. Show that the x-coordinate of R approaches -2 as $P \to B$.



(a) (b) (c)

Figure 5.5.2:

54.[M] Exercise 41 of Section 2.1 asked you to guess a certain limit. Now that limit will be computed.

> **WARNING** *(Common Sense)* As Albert Einstein observed, "Common sense is the deposit of prejudice laid down in the mind before the age of 18."

In Figure 5.5.2(c), which shows a circle, let $f(\theta)$ be the area of triangle ABC and $g(\theta)$ be the area of the shaded region formed by deleting triangle OAC from sector OBC.

(a) Why is $f(\theta)$ smaller than $g(\theta)$?

(b) What would you guess is the value of $\lim_{\theta\to 0} f(\theta)/g(\theta)$?

(c) Find $\lim_{\theta\to 0} f(\theta)/g(\theta)$.

55.[M] The following argument appears in an economics text:
"Consider the production function

$$y = k\left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-1/\rho},$$

where k, α, x_1, and x_2 are positive constants and $\alpha < 1$. Taking the limit as $\rho \to 0^+$, we find that

$$\lim_{\rho\to 0^+} y = k x_1^{\alpha} x_2^{1-\alpha},$$

which is the Cobb-Douglas function, as expected."
Fill in the details.

56.[M] Sam proposes the following proof for Theorem 5.5.1: "Since

$$\lim_{x\to a^+} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a+} g(x) = 0,$$

I will define $f(a) = 0$ and $g(a) = 0$. Next I consider $x > a$ but near a. I now have continuous functions f and g defined on the closed interval $[a, x]$ and differentiable on the open interval (a, x). So, using the Mean-Value Theorem, I conclude that there is a number c, $a < c < x$, such that

$$\frac{f(x) - f(a)}{x - a} = f'(c) \qquad \text{and} \qquad \frac{g(x) - g(a)}{x - a} = g'(c).$$

Since $f(a) = 0$ and $g(a) = 0$, these equations tell me that

$$f(x) = (x - a)f'(c) \qquad \text{and} \qquad g(x) = (x - a)g'(c)$$

$$\text{Thus} \qquad \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

$$\text{Hence} \qquad \lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(c)}{g'(c)}.$$

Sam made one error. What is it?

57.[C] Find $\lim_{x\to 0} \left(\frac{1+2^x}{x}\right)^{1/x}$.

58.[C] R. P. Feynman, in *Lectures in Physics*, wrote: "Here is the quantitative answer of what is right instead of kT. This expression

$$\frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

should, of course, approach kT as $\omega \to 0$. . . . . See if you can prove that it does — learn how to do the mathematics."
Do the mathematics. NOTE: All symbols, except T, denote constants.

59.[M] Graph $y = x^x$ for $0 < x \leq 1$, showing its minimum point.

In Exercises 60 to 62 graph the specified function, being sure to show (a) where the function is increasing and decreasing, (b) where the function has any asymptotes, and (c) how the function behaves for x near 0.
60.[M] $f(x) = (1 + x)^{1/x}$ for $x > -1$, $x \neq 0$

61.[M] $y = x\ln(x)$

62.[M] $y = x^2 \ln(x)$

63.[M] In which cases below is it possible to determine $\lim_{x\to a} f(x)^{g(x)}$ without further information about the functions?

(a) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = 7$

(b) $\lim_{x\to a} f(x) = 2$; $\lim_{x\to a} g(x) = 0$

(c) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = 0$

(d) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = \infty$

(e) $\lim_{x\to a} f(x) = \infty$; $\lim_{x\to a} g(x) = 0$

(f) $\lim_{x\to a} f(x) = \infty$; $\lim_{x\to a} g(x) = -\infty$

64.[M] In which cases below is it possible to determine $\lim_{x\to a} f(x)/g(x)$ without further information about the functions?

(a) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = \infty$

(b) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = 1$

(c) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = 0$

(d) $\lim_{x\to a} f(x) = \infty$; $\lim_{x\to a} g(x) = -\infty$

65.[M] Sam is angry. "Now I know why calculus books are so long. They spend all of page 65 showing that $\lim_{x\to 0} \frac{\sin(x)}{x}$ is 1. They could have saved space (and me a lot of trouble) if they had just used l'Hôpital's approach."
Is Sam right, for once?

66.[M] Jane says, "I can get $\lim_{x\to 0} \frac{e^x-1}{x}$ easily. It's just the derivative of e^x evaluated at 0. I don't need l'Hôpital's Rule." Is Jane right, or has Sam's influence affected her ability to reason?

67.[M]

$$\begin{array}{lrcl} \text{If} & \lim_{t\to\infty} f(t) = & \infty & = \lim_{t\to\infty} g(t) \\ \text{and} & \lim_{t\to\infty} \frac{f(t)}{g(t)} & = & 3, \end{array}$$

what can be said about

$$\lim_{t\to\infty} \frac{\ln(f(t))}{\ln(g(t))}?$$

NOTE: Do *not* assume f and g are differentiable.

68.[C] Give an example of a pair of functions f and g such that we have $\lim_{x\to 0} f(x) = 1$, $\lim_{x\to 0} g(x) = \infty$, and $\lim_{x\to 0} f(x)^{g(x)} = 2$.

69.[C] Obtain l'Hôpital's Rule for $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ from the case $\lim_{t\to 0^+} \frac{f(t)}{g(t)}$. HINT: Let $t = 1/x$.

70.[C] Find the limit of $(1^x + 2^x + 3^x)^{1/x}$ as

(a) $x \to 0$

(b) $x \to \infty$

(c) $x \to -\infty$.

The proof of Theorem 5.5.1, to be outlined in Exercise 73, depends on the following generalized mean-value theorem.

Generalized Mean-Value Theorem. Let f and g be two functions that are continuous on $[a, b]$ and differentiable on (a, b). Furthermore, assume that $g'(x)$ is never 0 for x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

71.[M] During a given time interval one car travels twice as far as another car. Use the Generalized Mean-Value Theorem to show that there is at least one instant when the first car is traveling exactly twice as fast as the second car.

72.[C] To prove the Generalized Mean-Value Theorem, introduce a function h defined by

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)). \tag{5.5.2}$$

Show that $h(b) = 0$ and $h(a) = 0$. Then apply Rolle's Theorem to h on (a, b). NOTE: Rolle's Theorem is Theorem 4.1.2 in Section 4.1.

> **Remark:** The function h in (5.5.2) is similar to the function h used in the proof of the Mean-Value Theorem (Theorem 4.1.3 in Section 4.1). Check that $h(x)$ is the vertical distance between the point $(g(x), f(x))$ and the line through $(g(a), f(a))$ and $(g(b), f(b))$.

73.[C] Assume the hypotheses of Theorem 5.5.1. Define $f(a) = 0$ and $g(a) = 0$, so that f and g are continuous at a. Note that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)},$$

and apply the Generalized Mean-Value Theorem from Exercise 71. NOTE: This Exercise proves Theorem 5.5.1, l'Hôpital's Rule in the zero-over-zero case.

74.[C]

$$\begin{array}{lrcl} \text{If} & \lim_{t\to\infty} f(t) = & \infty & = \lim_{t\to\infty} g(t) \\ \text{and} & \lim_{t\to\infty} \frac{\ln(f(t))}{\ln(g(t))} & = & 1, \\ \text{must} & \lim_{t\to\infty} \frac{f(t)}{g(t)} & = & 1? \end{array}$$

Explain.

75.[C] Assume that f, f', and f'' are defined in $[-1, 1]$ and are continuous. Also, $f(0) = 0$, $f'(0) = 0$, and $f''(0) > 0$.

(a) Sketch what the graph of f may look like for x in $[0, a]$, where a is a small positive number.

(b) Interpret the quotient

$$Q(a) = \frac{\int_0^a f(x)\,dx}{af(a) - \int_0^a f(x)\,dx}$$

in terms of the graph in (a).

(c) What do you think happens to $Q(a)$ as $a \to 0$?

(d) Find $\lim_{a\to 0} Q(a)$.

HINT: Because f''' might not be continuous at 0, you need to use $\lim_{a\to 0} \frac{f'(a)}{a} = f''(0)$.

76.[C]

Sam: I bet I can find $\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$ by using the Taylor polynomial $P_2(x; 0)$ for e^x and paying attention to the error.

Is Sam right?

5.6 Natural Growth and Decay

In 2009 the population of the United States was about 306 million and growing at a rate of about 1% (roughly 3 million people) a year. The world population was about 6.79 billion and growing at a rate of about 1.5% (roughly 100 million people) a year.

The United States population has been increasing at about 1% a year for years. It is an example of natural growth.

Natural Growth

Let $P(t)$ be the size of a population at time t. If its rate of growth is proportional to its size, there is a positive constant k such that

$$\frac{dP(t)}{dt} = kP(t). \tag{5.6.1}$$

To find an explicit formula for $P(t)$ as a function of t, rewrite (5.6.1) as

$$\frac{\frac{dP(t)}{dt}}{P(t)} = k. \tag{5.6.2}$$

The left-hand side can be rewritten as the derivative of $\ln(P(t))$ and so (5.6.2) can be rewritten as

$$\frac{d(\ln(P(t))}{dt} = \frac{d(kt)}{dt}.$$

Therefore there is a constant C such that

$$\ln(P(t)) = kt + C. \tag{5.6.3}$$

From (5.6.3) it follows, by the definition of a logarithm, that

$$P(t) = e^{kt+C},$$

hence

$$P(t) = e^C e^{kt}.$$

Since C is a constant, so is e^C, which we give a simpler name: A. We have the following simple explicit formula for $P(t)$:

The equation for **natural growth** is

$$P(t) = Ae^{kt}$$

where k is a positive constant. Because $P(0) = Ae^{k(0)} = A$, the coefficient A is the **initial population**.

Because of the presence of the exponential e^{kt}, natural growth is also called **exponential growth**.

EXAMPLE 1 The size of the world population at the beginning of 1988 was approximately 5.14 billion. At the beginning of 1989 it was 5.23 billion. Assume that the growth rate remains constant.

(a) What is the growth constant k?

(b) What would the population be in 2009?

(c) When will the population double is size?

SOLUTION Let $P(t)$ be the population in billions at time t. For convenience, measure time starting in the year 1988; that is, $t = 0$ corresponds to 1988 and $t = 1$ to 1989. Thus $P(0) = 5.14$ and $P(1) = 5.23$. The natural growth equation describing the population in billions at time is

$$P(t) = 5.14e^{kt}. \tag{5.6.4}$$

(a) To find k, we note that

$$P(1) = 5.14e^{k\cdot 1},$$

so

$$\begin{aligned} 5.14e^k &= 5.23 \\ e^k &= \frac{5.23}{5.14} \\ k &= \ln\left(\frac{5.23}{5.14}\right) \approx 0.174. \end{aligned}$$

Hence (5.6.4) takes the form

$$P(t) = 5.14e^{0.174t}.$$

This equation is all that we need to answer the remaining questions.

(b) The year 2009 corresponds to $t = 21$, so in the year 2009 the population, in billions, would be

$$P(21) = 5.14e^{0.174\cdot 21} = 5.14e^{0.3654} \approx 5.14(1.441) \approx 7.41.$$

The population would be approximately 7.41 billion in 2009. (Recall from the introduction of this section that the actual estimate of the world population in 2009 is about 6.79 billion. This suggests that the actual growth rate has not been constant; it has increased during the past 21 years.)

(c) The population will double when it reaches $2(5.14) = 10.28$ billion. We need to solve for t in the equation $P(t) = 10.28$. We have

$$\begin{aligned} 5.14e^{kt} &= 10.28 \\ e^{kt} &= 2 \\ kt &= \ln(2) \\ t &= \frac{\ln(2)}{k} \approx \frac{0.6931}{0.0174} \approx 39.8360. \end{aligned}$$

The world population will double approximately 40 years after 1988, which corresponds to the year 2028.

◇

The time it takes for a population to double is called the **doubling time** and is denoted t_2. Exponential growth is often described by its doubling time t_2 rather than by its growth constant k. However, if you know either t_2 or k you can figure out the other, as they are related by the equation

$$t_2 = \frac{\ln(2)}{k}$$

which appeared during part (c) of the solution to Example 1.

Exponential growth may also be described in terms of an annual percentage increase, such as "The population is growing 6 percent per year." That is, each year the population is multiplied by the factor 1.06: $P(t+1) = P(t)(1.06)$.

On the other hand, from the exponential growth function, we see that

$$P(t+1) = P(0)e^{k(t+1)} = P(0)e^{kt}e^k = P(t)e^k.$$

That is, during each unit of time the population increases by a factor of e^k. Now, when k is small, $e^k \approx 1 + k$. Consequently we can approximate 6 percent annual growth by letting $k = 0.06$. This approximation is valid whenever the growth rate is only a few percent. Since population figures are themselves only an approximation, setting the growth constant k equal to the annual percentage rate is a reasonable tactic.

EXAMPLE 2 Find the doubling time if the growth rate is 2 percent per year.

SOLUTION The growth rate is 2 percent, so we set $k = 0.02$. Then

$$t_2 = \frac{\ln(2)}{k} \approx \frac{0.693}{0.02} = 34.65 \text{ years.}$$

◇

The Mathematics of Natural Decay

As Glen Seaberg observes in the conversation given on page 417, some radioactive elements decay at a rate proportional to the amount present. The time it takes for half the initial amount to decay is denoted $t_{1/2}$ and is called the element's **half-life**.

Similarly, in medicine one speaks of the half-life of a drug administered to a patient: the time required for half the drug to be removed from the body. This half-life depends both on the drug and the patient, and can from 20 minutes for penicillin to 2 weeks for quinacrine, an antimalarial drug. This half-life is critical to determining how frequently a drug can be administered. Some elderly patients died from overdoses before it was realized that the half-life of some drugs is longer in the elderly than in the young.

Letting $P(t)$ again represent the amount present at time t, we have

Now k is negative.

$$P'(t) = kP(t) \qquad k < 0$$

where k is the **decay constant**. This is the same equation as (5.6.1), so

$$P(t) = P(0)e^{kt},$$

as before, except now k is a *negative number*. Since k is negative, the factor e^{kt} is a decreasing function of t.

Just as the doubling time is related to (positive) k by the equation $t_2 = (\ln(2))/k$, the half-life is related to (negative) k by the equation $t_{1/2} = (\ln(1/2))/k$, which can be rewritten as $t_{1/2} = -(\ln(2))/k$.

EXAMPLE 3 The Chernobyl nuclear reactor accident, in April 1986, released radioactive cesium 137 into the air. The half-life of ^{137}Cs is 27.9 years.

(a) Find the decay constant k of ^{137}Cs.

(b) When will only one-fourth of an initial amount remain?

(c) When will only 20 percent of an initial amount remain?

SOLUTION

(a) The formula for the half-life can be solved for k to give:

$$k = \frac{-\ln(2)}{t_{1/2}} \approx \frac{-0.693}{27.9} \approx -0.0248.$$

(b) This can be done without the aid of any formulas. Since $\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$, in two half-lives only one-quarter of an initial amount remains. The answer is $2(27.9) = 55.8$ years.

(c) We want to find t such that only 20 percent remains. While we know the answer is greater than 55.8 years (since 20% is less than 25%), finding the exact time requires using the formula for $P(t)$.

We want
$$P(t) = 0.20P(0).$$

That is, we want to solve

$$\begin{aligned} P(0)e^{kt} &= 0.20P(0). \\ \text{Then} \qquad e^{kt} &= 0.20 \\ kt &= \ln(0.20) \\ t &= \frac{\ln(0.20)}{k}. \end{aligned}$$

Since $k \approx -0.0248$, this gives
$$t \approx \frac{-1.609}{-0.0248} = 64.9 \text{ years.}$$

After 64.9 years (that is, 2051) only 20% of the original amount remains.

◇

Summary

We developed the mathematics of growth or decay that is proportional to the amount present. This required solving the differential equation
$$\frac{dP}{dt} = kP$$
where k is a constant, positive in the case of growth and negative in the case of decay. The solution is
$$P(t) = Ae^{kt}$$
where A is $P(0)$, the amount of the substance present when $t = 0$.

In the case of growth, the time for the quantity to double (the "doubling time") is denoted t_2. In the case of decay, the time when only half the original amount survives is denoted $t_{1/2}$, the "half-life." One has

$$t_2 = \frac{\ln(2)}{k} \qquad \text{and} \qquad t_{1/2} = \frac{\ln(1/2)}{k} = -\frac{\ln(2)}{k}.$$

The Scientist, The Senator, and Half-Life

During the hearings in 1963 before the Senate Foreign Relations Committee on the nuclear test ban treaty, this exchange took place between Glen Solborg, winner of the Nobel prize for chemistry in 1951, and Senator James W. Fulbright.

Seaborg: Tritium is used in a weapon, and it decays with a half-life of about 12 years. But the plutonium and uranium have such long half-lives that there is no detectable change in a human lifetime.

Fulbright: I am sure this seems to be a very naive question, but why do you refer to half-life rather than whole life? Why do you measure by half-lives?

Seaborg: Here is something that I could go into a very long discussion on.

Fulbright: I probably wouldn't benefit adequately from a long discussion. It seems rather odd that you should call it a half-life rather than its whole life.

Seaborg: Well, I will try. If we have, let us say, one million atoms of a material like tritium, in 12 years half of those will be transformed into a decay product and you will have 500,000 atoms.

Then, in another 12 years, half of what remains transforms, so you have 250,000 atoms left. And so forth.

On that basis it never all decays, because half is always left, but of course you finally get down to where your last atom is gone.

EXERCISES for 5.6 *Key:* R–routine, M–moderate, C–challenging

1.[R]

(a) Show that exponential growth can be expressed as $P = Ab^t$ for some constants A and b.

(b) What can be said about b?

2.[R]

(a) Show that exponential decay can be expressed as $P = Ab^t$ for some constants A and b.

(b) What can be said about b?

3.[R] If $P(t) = 30e^{0.2t}$ what are the initial size and the doubling time?

4.[R] If $P(t) = 30e^{-0.2t}$ what are the initial size and the half life?

5.[R] What is the doubling time for a population always growing at 1% a year?

6.[R] What is the half life for a population always shrinking at 1% a year?

7.[R] A quantity is increasing according to the law of natural growth. The amount present at time $t = 0$ is A. It will double when $t = 10$.

(a) Express the amount at time t in the form Ae^{kt} for a suitable k.

(b) Express the amount at time t in the form Ab^t for a suitable b.

8.[R] The mass of a certain bacterial culture after t hours is $10 \cdot 3^t$ grams.

(a) What is the initial amount?

(b) What is the growth constant k?

(c) What is the percent increase in any period of 1 hour?

9.[R] Let $f(t) = 3 \cdot 2^t$.

(a) Solve the equation $f(t) = 12$.

(b) Solve the equation $f(t) = 5$.

(c) Find k such that $f(t) = 3e^{kt}$.

10.[R] In 1988 the world population was about 5.1 billion and was increasing at the rate of 1.7 percent per year. If it continues to grow at that rate, when will it (a) double? (b) quadruple? (c) reach 100 billion?

11.[R] The population of Latin America has a doubling time of 27 years. Estimate the percent it grows per year.

12.[R] At 1:00 P.M. a bacterial culture weighed 100 grams. At 4:30 P.M. it weighed 250 grams. Assuming that it grows at a rate proportional to the amount present, find (a) at what time it will grow to 400 grams, (b) its growth constant.

13.[R] A bacterial culture grows from 100 to 400 grams in 10 hours according to the law of natural growth.

(a) How much was present after 3 hours?

(b) How long will it take the mass to double? quadruple? triple?

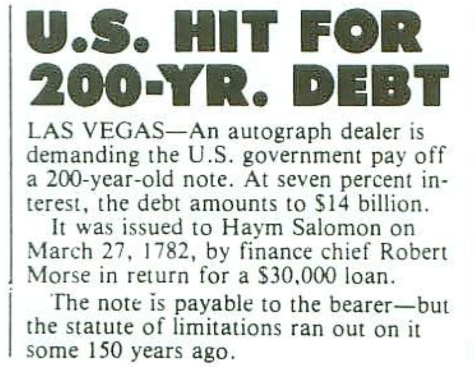
14.[R] A radioactive substance disintegrates at the rate of 0.05 grams per day when its mass is 10 grams.

(a) How much of the substance will remain after t days if the initial amount is A?

(b) What is its half-life?

15.[R] In 2009 the population of Mexico was 111 million and of the United States 308 million. If the population of Mexico increases at 1.15% per year and the population of the United States at 1.0% per year, when would the two nations have the same size population?

16.[R] The size of the population in India was 689 million in 1980 and 1,027 million in 2007. What is its doubling time t_2?



U.S. HIT FOR 200-YR. DEBT

LAS VEGAS—An autograph dealer is demanding the U.S. government pay off a 200-year-old note. At seven percent interest, the debt amounts to $14 billion.

It was issued to Haym Salomon on March 27, 1782, by finance chief Robert Morse in return for a $30,000 loan.

The note is payable to the bearer—but the statute of limitations ran out on it some 150 years ago.

Figure 5.6.1:

17.[R] The newspaper article shown in Figure 5.6.1 illustrates the rapidity of exponential growth.

(a) Is the figure of $14 billion correct? Assume that the interest is compounded annually.

(b) What interest rate would be required to produce an account of $14 billion if interest were compounded once a year?

(c) Answer (b) for "continuous compounding," which is another term for natural growth (a bank account increases at a rate proportional to the amount in the account at any instant).

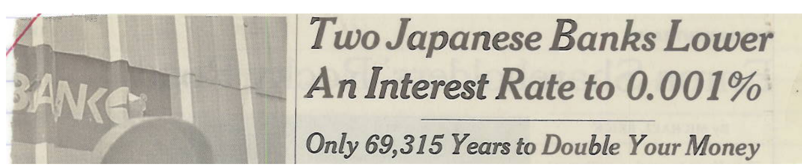


Figure 5.6.2:

18.[R] The headline shown in Figure 5.6.2 appeared in 2002. Is the number 69,315 correct? Explain.

19.[R] Carbon 14 (chemical symbol ^{14}C), an isotope of carbon, is radioactive and has a half-life of approximately 5,730 years. If the ^{14}C concentration in a piece of wood of unknown age is half of the concentration in a present-day live specimen, then it is about 5,730 years old. (This assumes that ^{14}C concentrations in living objects remain about the same.) This gives a way of estimating the age of an undated specimen. Show that if A_C is the concentration of ^{14}C in a live (contemporary) specimen and A_u is the concentration of ^{14}C in a specimen of unknown age, then the age of the undated material is about $8,300\ln(A_C/A_u)$ years. NOTE: This method, called **radiocarbon dating** is reliable up to about 70,000 years.

20.[R] From a letter to an editor in a newspaper:

> **I've been hearing bankers and investment advisers talk about something called the "rule of 72." Could you explain what it means?**
>
> How quickly would you like to double your money? That's what the "rule of 72" will tell you. To find out how fast your money will double at any given interest rate or yield, simply divide that yield into 72. This will tell you how many years doubling will take.
>
> Let's say you have a long-term certificate of deposit paying 12 percent [annually]. At that rate your money would double in six years. A money-market fund paying 10 percent would take 7.2 years to double your investment.

(a) Explain the rule of 72 and what number should be used instead of 72.

(b) Why do you think 72 is used?

21.[R] Benjamin Franklin conjectured that the population of the United States would double every 20 years, beginning in 1751, when the population was 1.3 million.

(a) If Franklin's conjecture were right, what would the population of the United States be in 2010?

(b) In 2010 the population was 310 million. Assuming natural growth, what would the doubling time be?

22.[M] (Doomsday equation) A differential equation of the form $dP/dt = kP^{1.01}$ is called a **doomsday equation**. The rate of growth is just slightly higher than that for natural growth. Solve the differential equation to find $P(t)$. How does $P(t)$ behave as t increases? Does $P(t)$ increase forever?

23.[M] The following situations are all mathematically the same:

1. A drug is administered in a dose of A grams to a patient and gradually leaves the system through excretion.
2. Initially there is an amount A of smoke in a room. The air conditioner is turned on and gradually the smoke is removed.
3. Initially there is an amount A of some pollutant in a lake, when further dumping of toxic materials is prohibited. The rate at which water enters the lake equals the rate at which it leaves. (Assume the pollution is thoroughly mixed.)

In each case, let $P(t)$ be the amount present at time t (whether drug, smoke, or pollution).

(a) Why is it reasonable to assume that there is a constant k such that for small intervals of time, Δt, $\Delta P \approx kP(t)\Delta t$?

(b) From (a) deduce that $P(t) = Ae^{kt}$.

(c) Is k positive or negative?

24.[M] **Newton's law of cooling** assumes that an object cools at a rate proportional to the difference between its temperature and the room temperature. Denote the room temperature as A. The differential equation for Newton's law of cooling is $dy/dt = k(y - A)$ where k and A are constants.

(a) Explain why k is negative.

(b) Draw the slope field for this differential equation when $k = -1/2$.

(c) Use (b) to conjecture the behavior of $y(t)$ as $t \to \infty$.

(d) Solve for y as a function of t.

(e) Draw the graph of $y(t)$ on the slope field produced in (b).

(f) Find $\lim_{t\to\infty} y(t)$.

25.[M] Let $I(x)$ be the intensity of sunlight at a depth of x meters in the ocean. As x increases, $I(x)$ decreases.

(a) Why is it reasonable to assume that there is a constant k (negative) such that $\Delta I \approx kI(x)\Delta x$ for small Δx?

(b) Deduce that $I(x) = I(0)e^{kx}$, where $I(0)$ is the intensity of sunlight at the surface. Incidentally, sunlight at a depth of 1 meter is only one-fourth as intense as at the surface.

26.[M] A particle moving through a liquid meets a "drag" force proportional to the velocity; that is, its acceleration is proportional to its velocity. Let x denote its position and v its velocity at time t. Assume $v > 0$.

(a) Show that there is a positive constant k such that $dv/dt = -kv$.

(b) Show that there is a constant A such that $v = Ae^{-kt}$.

(c) Show that there is a constant B such that $x = -\frac{1}{k}Ae^{-kt} + B$.

(d) How far does the particle travel as t goes from 0 to ∞? (Is this a finite or infinite distance?)

27.[M]

(a) Show that the natural growth function $P(t) = Ae^{kt}$ can be written in terms of A and t_2 as $P(t) = A \cdot 2^{t/t_2}$.

(b) Check that the function found in (a) is correct when $t = 0$ and $t = t_2$.

28.[M]

(a) Express the natural decay function $P(t) = Ae^{kt}$ in terms of A and $t_{1/2}$.

(b) Check that the function found in (a) is correct when $t = 0$ and $t = t_{1/2}$.

29.[M] A population is growing exponentially. Initially, at time 0, it is P_0. Later, at time u it is P_u.

(a) Show that at time t it is $P_0(P_u/P_0)^{t/u}$.

(b) Check that the formula in (a) gives the correct population when $t = 0$ and $t = u$.

30.[M] Let $P(t) = Ae^{kt}$. Then $\frac{P(t+1)-P(t)}{P(t)} = e^k - 1$. Show that when k is small, $e^k - 1 \approx k$. That means the relative change in one unit of time is approximately k.

31.[C] A certain fish population increases in number at a rate proportional to the size of the population. In addition, it is being harvested at a constant rate. Let $P(t)$ be the size of the fish population at time t.

(a) Show that there are positive constants h and k such that for small Δt, $\Delta P \approx kP\Delta t - h\Delta t$.

(b) Find a formula for $P(t)$ in terms of $P(0)$, h, and k. HINT: First divide by Δt in (a) and then take limits as $\Delta t \to 0$.

(c) Describe the behavior of $P(t)$ in the three cases $h = kP(0)$, $h > kP(0)$, and $h < kP(0)$

32.[C] The half-life of a drug administered to a certain patient is 8 hours. It is given in a 1-gram dose every 8 hours.

(a) How much is there in the patient just after the second dose is administered?

(b) How much is there in the patient just after the third dose? The fourth dose?

(c) Let $P(t)$ be the amount in the patient at t hours after the first dose. Graph $P(t)$ for a period of 48 hours. NOTE: $P(t)$ has meaning for all values of t, not just at the integers.

(d) Does the amount in the patient get arbitrarily large as time goes on?

33.[C] The half-life of the drug in Exercise 32 is 16 hours when administered to a different patient. Answer, for this patient, the questions in Exercise 32.

34.[C] The half-life of a drug in a certain patient is $t_{1/2}$ hours. It is administered every h hours. Can it happen that the concentration of the drug gets arbitrarily high? Explain your answer.

Exercises 35 to 37 introduce and analyze the **inhibited or logistic growth** model. This model will be encountered in the CIE for Chapter 10.

35.[C] In many cases of growth there is obviously a finite upper bound M which the population cannot exceed. Why is it reasonable to assume (or to take as a model) that

$$\frac{dP}{dt} = kP(t)(M - P(t)) \qquad 0 < P(t) < M \tag{5.6.5}$$

for some constant k?

36.[C]

(a) Solve the differential equation in Exercise 35. HINT: You will need the partial fraction identity

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

and the property of logarithms: $\ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$. After simplification, your answer should have the form

$$P(t) = \frac{M}{1 + ae^{-Mkt}}$$

for a suitable constant a.

(b) Find $\lim_{t\to\infty} P(t)$. Is this reasonable?

(c) Express a in terms of $P(0)$, M, and k.

37.[C] By considering (5.6.5) in Exercise 35 directly (not the explicit formula in Exercise 36), show that

(a) P is an increasing function.

(b) The maximum rate of change of P occurs when $P(t) = M/2$.

(c) The graph of $P(t)$ has an inflection point.

38.[C] A salesman, trying to persuade a tycoon to invest in Standard Coagulated Mutual Fund, shows him the accompanying graph which records the value of a similar investment made in the fund in 1965. "Look! In the first 5 years the investment increased \$1,000," the salesman observed, "but in the past 5 years it increased by \$2,000. It's really improving. Look at the graph of the graph from 1985 to 1990, which you can see clearly in Figure 5.6.3."

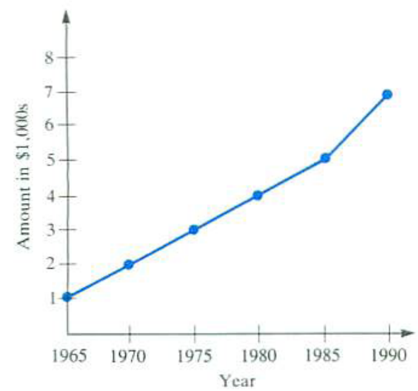


Figure 5.6.3:

The tycoon replied, "Hogwash. Though your graph is steeper from 1985 to 1990, in fact, the rate of return is less than from 1965 to 1970. Indeed, that was your best period."

(a) If the percentage return on the accumulated investment remains the same over each 5-year period as the first 5-year period, sketch the graph.

(b) Explain the tycoon's reasoning.

39.[C] Each of two countries is growing exponentially but at different rates. One is describe by the function $A_1e^{k_1t}$, the other by $A_2e^{k_2t}$, and k_1 is not equal to k_2. Is their total population growing exponentially? That is, are there constants A and k such that the formula describing their total population has the form Ae^{kt}. Explain your answer.

40.[C] Assume c_1, c_2, and c_3 are distinct constants. Can there be constants A_1, A_2, and A_3, not all 0, such that $A_1e^{c_1x} + A_2e^{c_2x} + A_3e^{c_3x} = 0$ for all x?

41.[C] If each of two functions describes natural growth does their (a) product? (b) quotient? (c) sum?

5.7 The Hyperbolic Functions and Their Inverses

Certain combinations of the exponential functions e^x and e^{-x} occur often in differential equations and engineering — for instance, in the study of the shape of electrical transmission or suspension cables — to be given names. This section defines these **hyperbolic functions** and obtains their basic properties. Since the letter x will be needed later for another purpose, we will use the letter t when writing the two preceding exponentials, namely, e^t and e^{-t}.

The Hyperbolic Functions

DEFINITION *(The hyperbolic cosine.)* Let t be a real number. The **hyperbolic cosine** of t, denoted $\cosh(t)$, is given by the formula

$$\cosh(t) = \frac{e^t + e^{-t}}{2}.$$

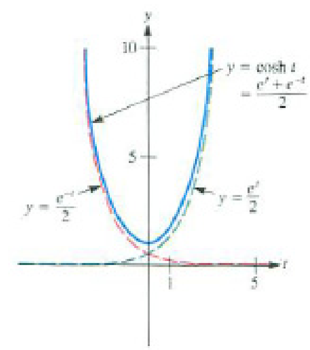


Figure 5.7.1:

To graph $\cosh(t)$, note first that

$$\cosh(-t) = \frac{e^{-t} + e^{-(-t)}}{2} = \frac{e^t + e^{-t}}{2} = \cosh(t).$$

Since $\cosh(-t) = \cosh(t)$, the cosh function is even, and so its graph is symmetric with respect to the vertical axis. Furthermore, $\cosh(t)$ is the sum of the two terms

$$\cosh(t) = \frac{e^t}{2} + \frac{e^{-t}}{2}.$$

Pronounced as written, "cosh," rhyming with "gosh."

As $t \to \infty$, the second term, $e^{-t}/2$, is positive and approaches 0. Thus, for $t > 0$ and large, the graph of $\cosh(t)$ is just a little above the graph of $e^t/2$. This information, together with the fact that $\cosh(0) = (e^0 + e^{-0})/2 = 1$, is the basis for Figure 5.7.1.

For $|t| \to \infty$, the graph of $y = \cosh(t)$ is asymptotic to the graph of $y = e^t/2$ or $y = e^{-t}/2$.

The curve $y = \cosh(t)$ in Figure 5.7.1 is called a **catenary** (from the Latin *catena* meaning "chain"). It describes the shape of a free-hanging chain. (See the CIE on the Suspension Bridge and the Hanging Cable for Chapter 15.)

DEFINITION *(The hyperbolic sine.)* Let t be a real number. The **hyperbolic sine** of t, denoted $\sinh(t)$, is given by the formula

"sinh" is pronounced "sinch," rhyming with "pinch."

$$\sinh(t) = \frac{e^t - e^{-t}}{2}.$$

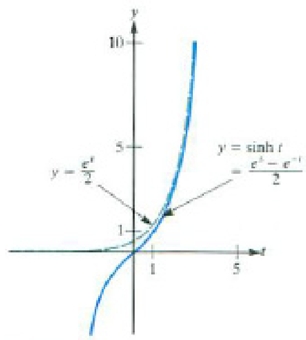


Figure 5.7.2:

It is a simple matter to check that $\sinh(0) - 0$ and $\sinh(-t) = -\sinh(t)$, so that the graph of $\sinh(t)$ is symmetric with respect to the origin. Moreover, it lies below the graph of $e^t/2$. However, the graphs of $\sinh(t)$ and $e^t/2$ approach each other since $e^{-t}/2 \to 0$ as $t \to \infty$. Figure 5.7.2 shows the graph of $\sinh(t)$.

Note the contrast between $\sinh(t)$ and $\sin(t)$. As $|t|$ becomes large, the hyperbolic sine becomes large, $\lim_{t\to\infty} \sinh(t) = \infty$ and $\lim_{t\to-\infty} \sinh(t) = -\infty$. There is a similar contrast between $\cosh(t)$ and $\cos(t)$. While the trigonometric functions are periodic, the hyperbolic functions are not.

Example 1 shows why the functions $(e^t + e^{-t})/2$ and $(e^t - e^{-t})/2$ are called **hyperbolic**.

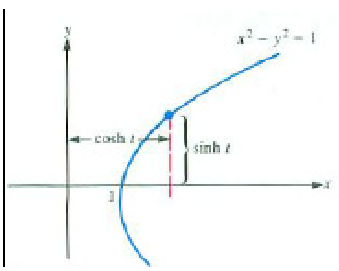


Figure 5.7.3:

EXAMPLE 1 Show that for any real number t the point with coordinates

$$x = \cosh(t), \qquad y = \sinh(t)$$

lie on the hyperbola $x^2 - y^2 = 1$.

SOLUTION Compute $x^2 - y^2 = \cosh^2(t) - \sinh^2(t)$ and see whether it simplifies to 1. We have

$$\begin{aligned} \cosh^2(t) - \sinh^2(t) &= \left(\tfrac{e^t+e^{-t}}{2}\right)^2 - \left(\tfrac{e^t-e^{-t}}{2}\right)^2 \\ &= \tfrac{e^{2t}+2e^te^{-t}+e^{-2t}}{4} - \tfrac{e^{2t}-2e^te^{-t}+e^{-2t}}{4} \\ &= \tfrac{2+2}{4} && \text{cancellation} \\ &= 1. \end{aligned}$$

Observe that since $\cosh(t) \geq 1$, the point $(\cosh(t), \sinh(t))$ is on the right half of the hyperbola $x^2 - y^2 = 1$, as shown in Figure 5.7.3. ◇

Example 1 proves a fundamental identity for hyperbolic functions: $\cosh^2(t) - \sinh^2(t) = 1$

By contrast, $(\cos(\theta), \sin(\theta))$ lies on the circle $x^2 + y^2 = 1$, so the trigonometric functions are also called **circular functions**.

There are four more hyperbolic functions, namely, the hyperbolic tangent, hyperbolic secant, hyperbolic cotangent, and hyperbolic cosecant. They are defined as follows:

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} \qquad \text{sech}(t) = \frac{1}{\cosh(t)} \qquad \coth(t) = \frac{\cosh(t)}{\sinh(t)} \qquad \text{csch}(t) = \frac{1}{\sinh(t)}.$$

Each can be expressed explicitly in terms of exponentials. For instance,

$$\tanh(t) = \frac{(e^t - e^{-t})/2}{(e^t + e^{-t})/2} = \frac{e^t - e^{-t}}{e^t + e^{-t}}.$$

As $t \to \infty$, $e^t \to \infty$ and $e^{-t} \to 0$. Thus $\lim_{t\to\infty} \tanh(t) = 1$. Similarly, $\lim_{t\to-\infty} \tanh(t) = -1$. Figure 5.7.4 is a graph of $y = \tanh(t)$.

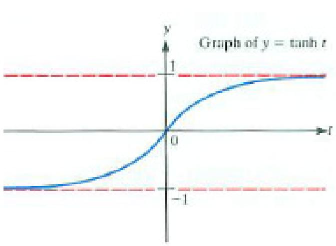


Figure 5.7.4:

The Derivatives of the Hyperbolic Functions

The derivatives of the six hyperbolic functions can be computed directly. For instance,

$$(\cosh(t))' = \left(\frac{e^t + e^{-t}}{2}\right)' = \frac{e^t - e^{-t}}{2} = \sinh(t).$$

Table 5.7.1 lists the derivatives of the six hyperbolic functions. Notice that the formulas, except for the signs, are like those for the derivatives of the trigonometric functions.

Function	Derivative
$\cosh(t)$	$\sinh(t)$
$\sinh(t)$	$\cosh(t)$
$\tanh(t)$	$\operatorname{sech}^2(t)$
$\coth(t)$	$-\operatorname{csch}^2(t)$
$\operatorname{sech}(t)$	$-\operatorname{sech}(t)\tanh(t)$
$\operatorname{csch}(t)$	$-\operatorname{csch}(t)\coth(t)$

Table 5.7.1:

The Inverses of the Hyperbolic Functions

Inverse hyperbolic functions appear on some calculators and in integral tables. Just as the hyperbolic functions are expressed in terms of the exponential function, each inverse hyperbolic function can be expressed in terms of a logarithm. They provide useful antiderivatives as well as solutions to some differential equations.

Consider the inverse of $\sinh(t)$ first. Since $\sinh(t)$ is increasing, it is one-to-one; there is no need to restrict its domain. To find its inverse, it is necessary to solve the equation

$$x = \sinh(t)$$

for t as a function of x. The steps are straightforward:

Finding the inverse of the hyperbolic sine

$$\begin{array}{rrcll} & x &=& \frac{e^t - e^{-t}}{2}, & \text{definition of } \sinh(t) \\ & 2x &=& e^t - \frac{1}{e^t}, & e^{-t} = 1/e^t \\ & 2xe^t &=& (e^t)^2 - 1, & \text{multiply by } e^t \\ \text{or} & (e^t)^2 - 2xe^t - 1 &=& 0. & \end{array}$$

Equation (5.7) is quadratic in the unknown e^t. By the quadratic formula,

$$e^t = \frac{2x \pm \sqrt{(2x)^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since $e^t > 0$ and $\sqrt{x^2 + 1} > x$, the plus sign is kept and the minus sign is rejected. Thus

$$e^t = x + \sqrt{x^2 + 1} \qquad \text{and} \qquad t = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Consequently, the inverse of the function $\sinh(t)$ is given by the formula

$$\boxed{\operatorname{arcsinh}(x) = \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2 + 1}\right).}$$

Formula for $\operatorname{arcsinh}(x)$

Computation of $\text{arctanh}(x)$ is a little different. Since the derivative of $\tanh(t)$ is $\text{sech}^2(t)$, the function $\tanh(t)$ is increasing and has an inverse. However, $|\tanh(t)| < 1$, and so the inverse function will be defined only for $|x| < 1$. Computations similar to those for $\text{arcsinh}(x)$ show that

Formula for $\text{arctanh}(x)$

$$\text{arctanh}(x) = \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad |x| < 1.$$

Inverses of the other four hyperbolic functions are computed similarly. The functions $\text{arccosh}(x)$ and $\text{arcsech}(x)$ are chosen to be positive. Their formulas are included in Table 5.7.2.

The derivatives are found by differentiating the formulas in the second column.

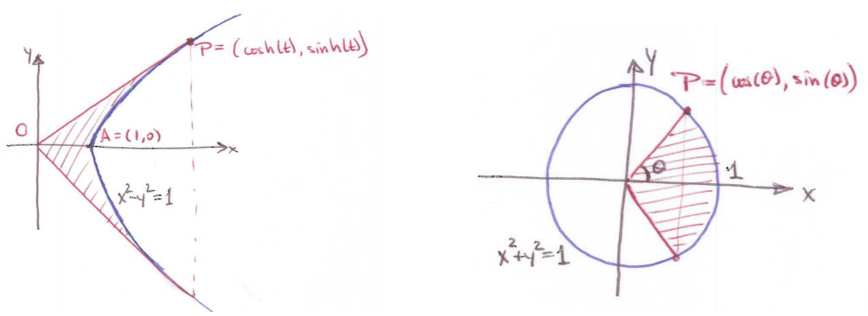
Function	Formula	Derivative	Domain
$\text{arccosh}(x)$	$\ln(x+\sqrt{x^2-1})$	$\frac{1}{\sqrt{x^2-1}}$	$x \geq 1$
$\text{arcsinh}(x)$	$\ln(x+\sqrt{x^2+1})$	$\frac{1}{\sqrt{x^2+1}}$	x-axis
$\text{arctanh}(x)$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$\frac{1}{1-x^2}$	$\|x\| < 1$
$\text{arccoth}(x)$	$\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$	$\frac{1}{1-x^2}$	$\|x\| > 1$
$\text{arcsech}(x)$	$\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$\frac{-1}{x\sqrt{1-x^2}}$	$0 < x \leq 1$
$\text{arccsch}(x)$	$\ln\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right)$	$\frac{-1}{\|x\|\sqrt{1+x^2}}$	$x \neq 0$

Table 5.7.2:

Summary

We introduced the six hyperbolic functions and their inverses, including $\sinh(x)$ (pronounced *sinch*), $\cosh(x)$ (pronounced *c$\bar{o}$sh*), $\tanh(x)$ (pronounced *tanch* or rhymes with "ranch") and their inverses $\text{arcsinh}(x)$, arccosh, and arctanh. Because they are all expressible in terms of exponentials, square roots, and logarithms, they do not add to the collection of elementary functions. However, some of them are especially convenient.

The points $(\cosh(t), \sinh(t))$ lie on the graph of the hyperbola $x^2 - y^2 = 1$. (See Example 1.) The parameter t, which can be any number, has a geometric interpretation: it is the area of the shaded region in Figure 5.7.5(a). This corresponds to the fact that a sector of the unit circle with angle 2θ has area θ, as shown in Figure 5.7.5(b). (See Exercise 78.)



(a) (b)

Figure 5.7.5:

EXERCISES for 5.7 *Key:* R–routine, M–moderate, C–challenging

1.[R]

(a) Compute $\cosh(t)$ and $e^t/2$ for $t = 0$, 1, 2, 3, and 4.

(b) Using the data in (a), graph $y = \cosh(t)$ and $y = e^t/2$ relative to the same axes.

2.[R]

(a) Compute $\tanh(t)$ for $t = 0$, 1, 2, and 3.

(b) Using the data in (a), and the fact that $\tanh(-t) = \tanh(t)$, graph $y = \tanh(t)$.

In Exercises 3 to 5 obtain the derivatives of the given functions and express them in terms of hyperbolic functions.

3.[R] $\tanh(x)$

4.[R] $\sinh(x)$

5.[R] $\cosh(x)$

6.[R]

(a) Compute $\sinh(t)$ and $\cosh(t)$ for $t = -3$, -2, -1, 0, 1, 2, and 3.

(b) Plot the seven points $(x, y) = (\cosh(t), \sinh(t))$ found in (a).

(c) Explain why the point plotted in (b) lie on the hyperbola $x^2 - y^2 = 1$.

7.[R]

(a) Show that $\text{sech}^2(x) + \tanh^2(x) = 1$.

(b) What equation links $\sec(\theta)$ and $\tan(\theta)$?

In Exercises 8 to 16 use the definitions of the hyperbolic functions to verify the given identities. Notice how they differ from the corresponding identities for the trigonometric functions. In Section 12.6, with the aid of complex numbers, the hyperbolic functions are expressed in terms of the trigonometric functions.

8.[R] $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

9.[R] $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

10.[R] $\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}$

11.[R] $\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$

12.[R] $\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$

13.[R] $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$

14.[R] $\sinh(2x) = 2\sinh(x)\cosh(x)$

15.[R] $2\sinh^2(x/2) = \cosh(x) - 1$

16.[R] $2\cosh^2(x/2) = \cosh(x) + 1$

In Exercises 17 to 19 obtain a formula for the given function.

17.[M] $\text{arctanh}(x)$

18.[M] $\text{arcsech}(x)$

19.[M] $\text{arccosh}(x)$

In Exercises 20 to 23 show that the derivative of the first function is the second function.

20.[M] $\text{arccosh}(x)$; $1/\sqrt{x^2-1}$

21.[M] $\text{arcsinh}(x)$; $1/\sqrt{x^2+1}$

22.[M] $\text{arcsech}(x)$; $1/(x\sqrt{1-x^2})$

23.[M] $\text{arccsch}(x)$; $1/(x\sqrt{1+x^2})$

24.[M] Find the inflection points on the curve $y = \tanh(x)$.

25.[M] Graph $y = \sinh(x)$ and $y = \text{arcsinh}(x)$ relative to the same axes. Show any inflection points.

26.[C] One of the applications of hyperbolic functions is to the study of motion in which the resistance of the medium is proportional to the square of the velocity. Suppose that a body starts from rest and falls x meters in t seconds. Let g (a constant) be the acceleration due to gravity. It can be shown that there is a constant $V > 0$ such that

$$x = \frac{V^2}{g} \ln\left(\cosh\left(\frac{gt}{V}\right)\right).$$

(a) Find the velocity $v(t) = dx/dt$ as a function of t.

(b) Show that $\lim_{t\to\infty} v(t) = V$.

(c) Compute the acceleration $a(t) = dv/dt$ as a function of t.

(d) Show that the acceleration equals $g - g(v/V)^2$.

(e) What is the limit of the acceleration as $t \to \infty$?

27.[C] In this exercise you will discover two different formulas for an antiderivative of $f(x) = \frac{1}{\sqrt{ax+b}\sqrt{cx+d}}$. The correct formula to use depends on the signs of a and c.

(a) Show that $\frac{2}{\sqrt{-ac}} \arctan\sqrt{\frac{-c(ax+b)}{a(cx+d)}}$ is an antiderivative of $f(x)$ when $a > 0$ and $c < 0$.

(b) Show that $\frac{2}{\sqrt{ac}} \operatorname{arctanh}\sqrt{\frac{c(ax+b)}{a(cx+d)}}$ is an antiderivative of $f(x)$ when $a > 0$ and $c > 0$.

5.8 Chapter Summary

This chapter show the derivative at work; applying it to practical problems, estimating errors, and evaluating some limits.

To determine the extrema of some quantity one must determine a function that represents how the quantity depends on other quantities. Then, finding the extrema is like finding the highest or lowest points on the graph of the function.

When two varying quantities are related by an equation, the derivative can tell the relation between the rates at which they change: just differentiate both sides of the equation that relates them. That differentiation depends on the chain rule and is called implicit differentiation because one differentiates a function without having an explicit formula for it.

The next two sections form a unit that rests one of the main uses of higher derivatives: to estimate errors when approximating a function by a polynomial and later, in Section 6.5, to estimate errors in approximating area under a curve by trapezoids and parabolas.

The key to the Growth Theorem is that if R is a function such that

$$0 = R(a) = R'(a) = R''(a) = R^{(n)}(a)$$

and in some interval around a we know $R^{(n+1)}(x)$ is continuous, then there is a number c_n in $[a, x]$ such that

$$|R(x)| \leq R^{(n+1)}(c_n)\frac{(x-a)^{n+1}}{(n+1)!} \qquad \text{for all } x \text{ in that interval.}$$

That means we have information on how rapidly $R(x)$ can grow for x near a. This information was used to control the error when using a polynomial to approximate a function.

A likely candidate for the polynomial of degree n that closely resembles a given function f near $x = a$ is the one whose derivatives at a, up through order n, agree with those of f there. That polynomial is

$$P(x) = P_n(x; a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Because the polynomial was chosen so that $P^{(k)}(a) = f^{(k)}(a)$ for all k up through n, the remainder function function $R(x) = f(x) - P(x)$ has all its derivatives up through order n at a equal to 0. Moreover, since the $(n+1)^{\text{st}}$ derivative of any polynomial of degree at most n is identically 0, $R^{(n+1)}(x) = f^{(n+1)}(x)$. Thus the error $|f(x) - P(x)|$ is at most $M\frac{|x-a|^{n+1}}{(n+1)!}$, if $|f^{(n+1)}(t)|$ stays less than or equal to M for t between a and x. A similar conclusion holds if

$|f^{n+1}(t)|$ stays larger than a fixed number. Using these facts we obtained Lagrange's formula for the error:

$$\frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c_n \text{ between } a \text{ and } x.$$

The case $n = 1$ reduces to the linear approximation of a curve by the tangent line at $(a, f(a))$. In this case the error is controlled by the second derivative.

We return to Taylor polynomials in Chapter 12, where we express e^x, $\sin(x)$, and $\cos(x)$ as "polynomials of infinite degree," and use them and complex numbers to express $\sin(x)$ and $\cos(x)$ in terms of exponential functions.

Section 5.5 concerns l'Hôpital's rule, a tool for computing certain limits, such as the limit of a quotient whose numerator and denominator both approach zero.

The final two sections, on natural growth and decay and the hyperbolic functions, conclude the chapter. While these sections are not needed in future chapters of this book, they are important applications in a wide variety of disciplines, including biology and engineering.

EXERCISES for 5.8 *Key:* R–routine, M–moderate, C–challenging

1.[R] Arrange the following numbers by order of increasing size as $x \to \infty$.

(a) $1000x$

(b) $\log_2(x)$

(c) $\sqrt{x}$

(d) $(1.0001)^x$

(e) $\log_{1000}(x)$

(f) $0.01x^3$

In Exercises 2 to 28 find the limits, if they exist.

2.[R] $\lim_{u\to\infty} \left(\frac{u+1}{u}\right)^{u+1} \frac{1}{\sqrt{u}}$

3.[R] $\lim_{x\to\infty} \left(\frac{x+2}{x+1}\right)^{x+3}$

4.[R] $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{x+1}$

5.[R] $\lim_{x\to 3} \frac{x-2}{\cos(\pi x)}$

6.[R] $\lim_{x\to 3} \frac{x-2}{\sin(\pi x)}$

7.[R] $\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x}$

8.[R] $\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$

9.[R] $\lim_{x\to\infty} \frac{(1+x^2)^{1/2}}{(2+x^2)^{1/3}}$

10.[R] $\lim_{x\to\infty} \frac{1+x+x^2}{2+3x+4x^2}$

11.[R] $\lim_{x\to 1} \frac{\ln(x)\tan\left(\frac{\pi x}{4}\right)}{\cos\left(\frac{\pi x}{2}\right)}$

12.[R] $\lim_{x\to 0} \frac{f(3+x)-f(3)}{x}$ where $f(x) = (x^2+5)\sin^2(3x)$.

13.[R] $\lim_{x\to\infty} \frac{\ln(6x)-\ln(5x)}{\ln(7x)-\ln(6x)}$

14.[R] $\lim_{x\to\infty} \frac{\ln(6x)-\ln(5x)}{x\ln(7x)-x\ln(6x)}$

15.[R] $\lim_{x\to\pi} \frac{e^{-x^2}\sin(x)}{x^2-\pi^2}$

16.[R] $\lim_{x\to\pi} \frac{\ln(x^3-\sin(x))-3\ln(\pi)}{x-\pi}$

17.[R] $\lim_{x\to 0} \frac{(x+2)}{(x+3)}\frac{(\cos(5x)-1)}{\cos(7x)-1)}$

18.[R] $\lim_{x\to\infty} \left(\frac{x+2}{x+1}\right)^{2x}$

19.[R] $\lim_{x\to\pi} \frac{\sin^4(x)}{(\pi^4-x^4)^2}$

20.[R] $\lim_{x\to\infty} \frac{\sec^4(x)\tan(3x)}{\sin(2x)}$

21.[R] $\lim_{x\to 1} \frac{e^{3x}(x^2-1)}{\cos(\sqrt{2}x)\tan(3x-3)}$

22.[R] $\lim_{x\to 0}(1+0.005x)^{20x}$

23.[R] $\lim_{t\to 0} \frac{e^{3(x+t)}-e^{3x}}{5t}$

24.[R] $\lim_{t\to 0} \frac{e^{3(x+t)}-e^{3x}}{5t}$

25.[R] $\lim_{x\to 0} \left(\frac{1+2^x}{2}\right)^{1/x}$

26.[R] $\lim_{x\to 0} \left(\frac{1+2^x}{1+3^x}\right)^{1/x}$

27.[R] $\lim_{x\to\infty}(1+0.003x)^{20/x}$

28.[R] $\lim_{x\to\infty}(1+0.003x)^{20/x}$

In Exercises 29 to 36 find the derivative of the given function.

29.[R] $(\cos(x))^{1/x^2}$

30.[R] $\ln\left(\sec^2(3x)\sqrt{1+x^2}\right)$

31.[R] $\ln\left(\sqrt{e^{x^3}}\right)$

32.[R] $\frac{5+3x+7x^2}{58-4x+x^2}$

33.[R] $\frac{\tan^2(2x)}{(1+\cos(2x))^4}$

34.[R] $(\cos^2(3x))^{\cos^2(2x)}$

35.[R] $f(x) = \begin{cases} x^2\sin(\pi/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ HINT: Use the definition of the derivative to find $f'(0)$.

36.[R] $f(x) = \begin{cases} \frac{\sin(\pi x)}{x} & \text{if } x \neq 0 \\ \pi & \text{if } x = 0 \end{cases}$

37.[R]

(a) Find $P_1(x;64)$ for $f(x) = \sqrt{x}$.

(b) Use $P_1(x;64)$ to estimate $\sqrt{67}$.

(c) Put bounds on the error in the estimate in (b).

38.[R]

(a) Show that when x is small $\sqrt[3]{1+x}$ is approximately $1 + x/3$.

(b) Use (a) to estimate $\sqrt[3]{0.94}$ and $\sqrt[3]{1.06}$.

39.[R]

(a) Show that when x is small $1/\sqrt[3]{1+x}$ is approximately $1 - x/3$.

(b) Use (a) to estimate $\sqrt[3]{0.94}$ and $\sqrt[3]{1.06}$.

40.[R]

(a) Find the Maclaurin polynomial of degree 6 associated with $\cos(x)$.

(b) Use (a) to estimate $\cos(\pi/4)$.

(c) What is the error between the estimate found in (b) and the exact value, $\sqrt{2}/2$.

(d) What is the Lagrange bound for the error?

In Exercises 41 to 52, examine the limit, determine whether it exists, and, if it does exist, find its value.

41.[R] $\lim_{x \to 1} \frac{1 - e^x}{1 - e^{2x}}$

42.[R] $\lim_{x \to 0} \frac{x}{\sqrt{1 + x^2}}$

43.[R] $\lim_{x \to 0} \frac{1 - e^x}{1 - e^{2x}}$

44.[R] $\lim_{x \to \infty} \frac{x^2}{(1 + x^3)^{2/3}}$

45.[R] $\lim_{x \to \infty} x^2 \sin(x)$

46.[R] $\lim_{x \to 8} \frac{2^x - 2^8}{x - 8}$

47.[R] $\lim_{x \to 1} \frac{e^{x^2} - e^x}{x - 1}$

48.[R] $\lim_{x \to 4} \frac{2^x + 2^4}{x + 4}$

49.[R] $\lim_{x \to 0} \frac{\sin(x) - e^{2x}}{x}$

50.[R] $\lim_{x \to 0} \frac{e^{3x} \sin(2x)}{\tan(3x)}$

51.[R] $\lim_{x \to 0} \frac{\sqrt{1 + x^2} - 1}{\sqrt[3]{1 + x^2} - 1}$

52.[R] $\lim_{x \to \pi/2} \frac{\sin 9x) \cos(x)}{x - \pi/2}$

53.[R] If $\lim_{x \to \infty} f'(x) = 3$ and $\lim_{x \to \infty} g'(x) = 3$, what, if anything, can be said about

(a) $\lim_{x \to \infty} \frac{f(x)}{3x}$

(b) $\lim_{x \to \infty} (g(x) - f(x))$

(c) $\lim_{x \to \infty} \frac{f(x)}{g(x)}$

(d) $\lim_{x \to \infty} (f(x) - 3x)$

(e) $\lim_{x \to \infty} \frac{(f(x))^3}{(g(x))^3}$

54.[M] The point $P = (c, d)$ lies in the first quadrant. Each line through P of negative slope determines a triangle whose vertices are the intercepts of the line on the axes, and the origin.

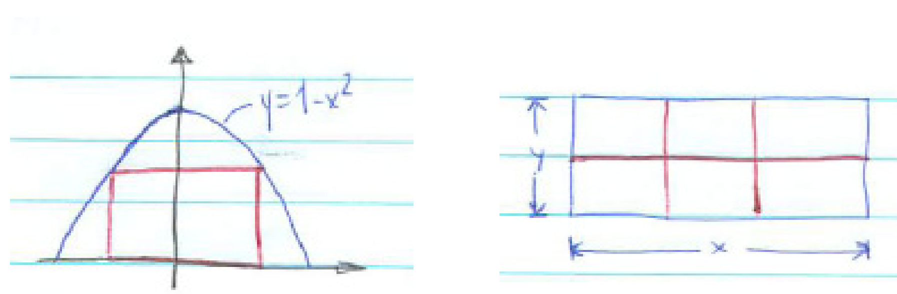
(a) Find the slope of the line that minimizes the area.

(b) Find the minimum area.

55.[M] Figure 5.8.1(a) shows a typical rectangle whose base is the x-axis, inscribed in the parabola $y = 1 - x^2$.

(a) Find the rectangle of largest perimeter.

(b) Find the rectangle of largest area.



(a) (b)

Figure 5.8.1:

56.[M] A rectangle of perimeter 12 inches is spun around one of its edges to produce a circular cylinder.

(a) For which rectangle is the area of the curved surface of the cylinder a maximum?

(b) For which rectangle is the volume of the cylinder a maximum?

57.[M] Consider isosceles triangles whose equal sides have length a and the angle where these two sides meet is θ. For which angle θ is the area of the triangle a maximum?

(a) Solve this problem using calculus.

(b) Solve the same problem without calculus.

58.[M] A farmer has 200 feet of fence which he wants to use to enclose a rectangle divided into six congruent rectangles, as shown in Figure 5.8.1(b). He wishes to enclose a maximum area.

(a) If x is near 0, what is the area, approximately?

(b) How large can x be?

(c) In the case that produces the maximum area, which do you think will be larger x or y? Why?

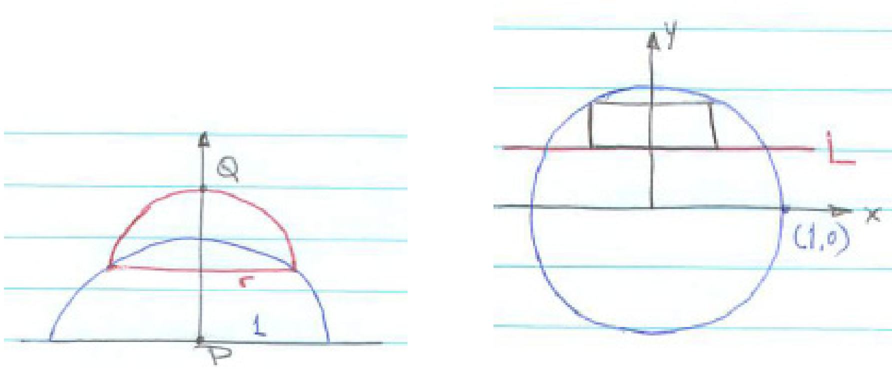
(d) Find the dimensions x and y that maximizes the area.

59.[M] A semicircle of radius $a < r \leq 1$ rests upon a semicircle of radius 1, as shown in Figure 5.8.2(a). The length of PQ, the segment from the origin of the lower circle to the top of the upper circle is a function of r, $f(r)$.

(a) Find $f(0)$ and $f(1)$.

(b) Find $f(r)$.

(c) Maximize $f(r)$, testing the maximum by the second derivative.



(a) (b)

Figure 5.8.2:

Exercises 60 to 62 are independent, but related. They contain a surprise.

60.[M] Figure 5.8.2(b) shows the unit circle $x^2+y^2=1$, the line L whose equation is $y=1/3$, and a typical rectangle with base on L, inscribed in the circle. Find the rectangle with base on L that has (a) minimum perimeter and (b) maximum perimeter.

61.[M] Like Exercise 60 but this time the line L has the equation $y = 1/2$.

62.[M] The analyses in Exercises 60 to 61 are different. Let the line L have the equation $y = c$, $0 < c < 1$. For which values of c is the analysis like that for (a) Exercise 60? (b) Exercise 61?

63.[M] A. Bellemans, in "Power Demand in Walking and Pace Optimization," Amer. J. Physics 49(1981) pp. 25–27, modeling the work spent on walking writes "$H = L(1 - \cos(\gamma)$ or, to a sufficient approximation for the present purpose, $H = L\gamma^2/2$." Justify this approximation.

64.[C] Let k be a constant. Determine $\lim_{x\to\infty} x\left(e^{-k} - \left(1 - \frac{k}{x}\right)^x\right)$.

65.[C] Let k be a constant. Determine $\lim_{x\to\infty} x\left(e^{k} - \left(1 + \frac{k}{x}\right)^x\right)$.

66.[M] Let $p_n(x)$ be the Maclaurin polynomial of degree n associated with e^x. Because $e^x \cdot e^{-x} = 1$, we might expect that $p_n(x)p_n(-x)$ would also be 1. But that cannot be because the degree of the product is $2n$.

(a) Compute $p_2(x)p_2(-x)$ and $p_3(x)p_3(-x)$.

(b) Make a conjecture about $p_n(e^x)p_n(e^{-x})$ based on (a).

67.[M] Let $p_n(x)$ be the Maclaurin polynomial of degree n associated with e^x. Because $e^{2x} = e^x \cdot e^x$, we might expect that $p_{2n}(x) = p_n(x)p_n(x)$.

(a) Why is that equation false for $n \geq 1$?

(b) To what extent does $p_2(x)p_2(x)$ resemble $p_2(2x)$ and $p_3(x)p_3(x)$ resemble $p_3(2x)$?

(c) Make a conjecture based on (a) and (b).

68.[M] Let $p_n(x)$ be the Maclaurin polynomial of degree n associated with e^x. The equation $e^{x+y} = e^x \cdot e^y$ suggests that $p_n(x + y)$ might equal $p_n(x)p_n(y)$.

(a) Why is that hope not realistic?

(b) To what extent does $p_2(x)p_2(y)$ resemble $p_2(x + y)$?

69.[M] What can be said about $f(10)$ if $f(1) = 5$, $f'(1) = 3$ and $2, f''(x) < 4$ for x in $(-10, 20)$?

70.[M] The demand for a product is influenced by its price. In one example an economics text links the amount sold (x) to the price (P) by the equation $x = b - aP$, where b and a are positive constants. As the price increases the sales go down. The cost of producing x items is an increasing function $C(x) = c + kx$, where c and k are positive constants.

(a) Express P in terms of x.

(b) Express the total revenue $R(x)$ in terms of x.

(c) Note that $C(0) = c$. So what is the economic significance of c?

(d) What is the economic significance of k?

(e) Let $E(x)$ be the profit, that is, the revenue minus the cost. Express $E(x)$ as a function of x.

(f) Which value of x produces the maximum profit?

(g) The marginal revenue is defined as dR/dx and the marginal cost as dC/dx. Show that for the value of x that produces the maximum profit, $dR/dx = dC/dx$.

(h) What is the economic significance of $dR/dx = dC/dx$ in (g)?

71.[M] This exercise concerns a function used to describe the consumption of a finite resource, such as petroleum. Let Q be the amount initially available. Let a be a positive constant and b be a negative constant. Let $y(t)$ be the amount used up by the time t. The function $Q/(1 + ae^{bt})$ is often used to represent $y(t)$.

(a) Show that $\lim_{t\to\infty} y(t) = Q$ and $\lim_{t\to-\infty} y(t) = 0$. Why are these realistic?

(b) Show that $y(t)$ has an inflection point when $t = -\ln(a)/b$.

(c) Show that at the inflection point, $y(t) = Q/2$, that is, half the resource has been used up.

(d) Sketch the graph of $y(t)$.

(e) Where is $y'(t)$, the rate of using the resource, greatest?

NOTE: The same function describes limited growth that is bounded by Q, so called **logistic growth**.

72.[M] About 100 cubic yards are added to a land fill every day. The operator decides to pile the debris up in the form of a cone whose base angle is $\pi/4$. (He hopes to make a ski run where it never snows.) At what rate is the height of the cone increasing when the height is (a) 10 yards? (b) 20 yards? (c) 100 yards? (d) How long will it take to make a cone 100 yards high? 300 yards high? NOTE: The volume of a circular cone is one third the product of its height and the area of its base.

73.[M] A wine dealer has a case of wine that he could sell today for \$100. Or, he could decide to store it, letting it mellow, and sell later for a higher price. Assume he could sell in t years for \$ $100e^{\sqrt{t}}$. In order to decide which option to choose he computes the present value of the sale. If the interest rate is r, the present value of one dollar t years hence is e^{-rt}. When should he sell the wine?

74.[M]

(a) Estimate $\int_0^1 \frac{\sin(x)}{x}dx$ by using the Maclaurin polynomial $P_6(x;0)$ associated with $\sin(x)$ to approximate $\sin(x)$.

(b) Use the Lagrange form of the error to put an upper bound on the error in (a).

75.[M] A differentiable function is defined throughout $(-\infty,\infty)$. Its derivative is 0 at exactly two inputs.

(a) Can there be exactly one relative extremum?

(b) Could it have two relative maxima?

(c) What is the maximum number of relative extrema possible?

(d) What is the minimum number?

HINT: Sketch graphs, then explain.

76.[M] A differentiable function is defined throughout $(-\infty,\infty)$. Its derivative is 0 at exactly three inputs. and the function approaches 0 as x approaches ∞

(a) Can there be exactly two relative extremum?

(b) Could it have three relative maxima?

(c) What is the maximum number of relative extrema possible?

(d) What is the minimum number?

HINT: Sketch graphs, then explain.

77.[M] A differentiable function is defined throughout $(-\infty, \infty)$. Its derivative is 0 at exactly two inputs. and the function approaches the same finite limit as x approaches ∞ and $-\infty$.

(a) Can there be exactly one relative extremum?

(b) Could it have two relative maxima?

(c) What is the greatest number of relative extrema possible?

(d) What is the least number?

HINT: Sketch graphs, then explain.

78.[C] In the paper cited in the Exercise 63, Bellemans writes "The total mechanical power required for walking is $P(v,a) = \alpha M v^3/a + (\beta M g v)/L)a$. Enlarging the pace, a, at a constant speed v, lowers the first term and increases the second one so that the formula predicts an optimal pace $a^*(v)$, minimizing $P(v,a)$." In the formula, α, M, v, β, g, and L are constants.

(a) Show that $a^*(v) = \left(\frac{\alpha}{\beta}\right)^{1/2} \left(\frac{L}{g}\right)^{1/2} v$

(b) Verify that the "corresponding minimum power" is

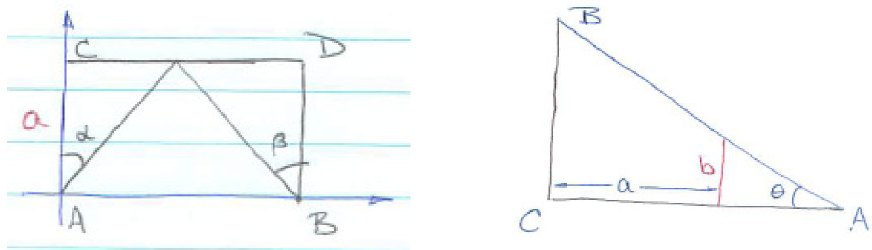
$$P(v, a^*(v)) = 2(\alpha\beta)^{1/2} \left(\frac{g}{L}\right)^{1/2} M v^2.$$

"One would therefore expect that, when walking naturally on the flat at a fixed velocity, a subject will adjust its pace automatically to the optimum value corresponding to the minimum work expenditure. This has indeed been verified experimentally."

79.[C] Figure 5.8.3(a) shows two points A and B a mile apart and both at a distance a from the river CD. Sam is at A. He will walk in a straight line to the river at 4 mph, fill a pail, then continue on to B at 3 mph. He wishes to do this in the shortest time.

(a) For the fastest route which angle in Figure 5.8.3 do you expect to be larger, α or β?

(b) Show that for the fastest route $\sin(\alpha)/\sin(\beta)$ equals 4/3.



(a) (b)

Figure 5.8.3:

80.[C] A fence b feet high is a feet from a tall building, whose wall contains BC. as shown in Figure 5.8.3(b). Find the angle θ that minimizes the length of AB. (That angle produces the shortest ladder to reach the building and stay above the fence.)

81.[C]

(a) Show that if a differentiable function f is even, then f' is odd, by differentiating both sides of the equation $f(-x) = f(x)$.

(b) Explain why the conclusion in (a) is to be expected by interpreting it in terms of the graph of f.

82.[C] Show that if a differentiable function is odd, then its derivative is even.

83.[C] What do the previous two exercises imply about a Maclaurin polynomial associated with an odd function? associated with an even function?

84.[C] Show that

(a) If $p_n(x)$ is a Maclaurin polynomial associated with $f(x)$, then $p_n'(x)$ is a Maclaurin polynomial associated with $f'(x)$.

(b) Use (a) to find the 6$^{\text{th}}$-order Maclaurin polynomial for $1/(1-x)^2$.

85.[C] (Assume $e < 3$.) Let $P_1(x)$ be $P_1(x; 0)$ for e^x. For how large an x can you be sure that

(a) $|e^x - P_1(x)| < 0.01$?

(b) $|e^x - P_2(x)| < 0.01$?

(c) $|e^x - P_3(x)| < 0.01$?

86.[C] A number b is **algebraic** if there is a non-zero polynomial $\sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, with coefficients a_i that are rational numbers, such that $\sum_{i=0}^{n} a_i b^i = 0$. In other words, b is algebraic if there is a function f that satisfies (a) $f(b) = 0$, (b) all derivatives of f at 0 are rational, but not all zero, and (c) there is a positive integer m such that $D^m(f) = 0$. (Recall that D is the differentiation operator.)

We call a number b **almost algebraic** if (a) b is not algebraic and there is a function f with (b) $f(b) = 0$, (c) all derivatives of f at 0 are rational, but not all zero, and (d) there is a non-zero polynomial $p(D)$ such that $p(D)(f) = 0$. For example, if $p(x) = x^2 + 1$ then $p(D)(f) = D^2(f) + f = f'' + f$.

Show that π is almost algebraic. (Assume it is not algebraic.)

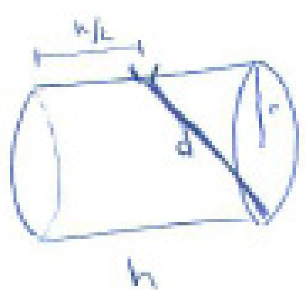


Figure 5.8.4: ARTIST: Show wine level inside the barrel.

87.[M] Kepler, the astrologer and astronomer, to celebrate his wedding in 1613, ordered some wine, which was available in cylindrical barrels of various shapes. He was surprised by the way the merchant measure the volume of a barrel. A ruler was pushed through the opening in the side of the barrel (used to fill the barrel) until it came to a stop at the edge of a circular base. The merchant used the length of the part of the ruler inside the barrel to determine the volume of the barrel. Figure 5.8.4 shows the method.

The barrel in Figure 5.8.4 has radius r, height h, and volume V. The length of the ruler inside the barrel is d.

(a) Using common sense, show that d does not determine V.

(b) How small can V be for a given value of d?

(c) Using calculus, show that the maximum volume for a given d occurs when $h = 2\sqrt{2}d/\sqrt{6}$ and $r = d/\sqrt{6}$.

(d) Show that to maximize the volume the height must be $\sqrt{2}$ times the diameter. (This is what Kepler showed.)

NOTE: Try to solve this problem two different ways. One without implicit differentiation and the other with implicit differentiation.

88.[M] Let m and n be positive numbers. Find the maximum and minimum values of $m\sin(x) + n\cos(x)$.

89.[M] Let m and n be positive integers. Let $f(x) = \sin^m(x)\cos^n(x)$ for x in $[0, \pi/2]$.

(a) For which x is $f(x)$ a minimum?

(b) For which x is $f(x)$ a maximum?

(c) What is the maximum value of $f(x)$?

90.[M]

(a) Let $P(x)$ be a polynomial such that $D^2(x^2P(x)) = 0$. Show that $P(x) = 0$.

(b) Does the same conclusion follow if instead we assume $D^2(xP(x)) = 0$?

HINT: If $P(x)$ has degree n, what are the degrees of $xP(x)$ and $x^2P(x)$?

91.[M] Translate this news item into the language of calculus: "The one positive sign during the quarter was a slowing in the rate of increase in home foreclosures."

92.[M] In May 2009 it was reported that "the nation's industrial production fell in April by the smallest amount in six months, fresh evidence that the pace of the economy's decline is slowing."
Let $P(t)$ denote the total production up to time t with t representing the number of months since January 2000 ($t = 0$).

(a) Translate the above statement into the language of calculus, that is, in terms of $P(t)$ and its derivatives (evaluated at appropriate values of t).

(b) Sketch a possible graph of $P(t)$ for November 2008 through April 2009.

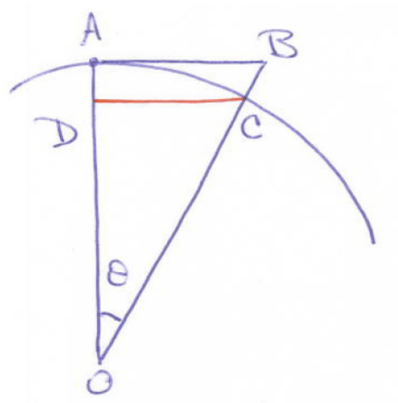


Figure 5.8.5:

93.[M] (A challenge to your intuition.) In Figure 5.8.5 AB is tangent to an arc of a circle, OA is a radius and DC is parallel to AB.

(a) What do you think happens to the ratio of the area of ABC to the area of ADC as $\theta \to 0$?

(b) Using calculus, find the limit of that ratio as $\theta \to 0$.

(c) In view of (b), which provides a better estimate of the area of a disk, the circumscribed regular n-gon or the inscribed regular n-gon?

(d) In view of the limit in (b), what combination of the estimates by the inscribed regular n-gon and the circumscribed regular n-gon, would likely provide a very good estimate of the area of the disk?

94.[M] Let $f(x)$ be a function having a second derivative at a. Supply all the steps to show that the second-order polynomial $g(x)$ such that $g(a) = f(a)$, $g'(a) = f'(a)$, and $g''(a) = f''(a)$ is given by $g(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$.

95.[M] Let f and g be differentiable.

(a) If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 3$, must $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exist and be 3?

(b) If the second limit in (a) exists, can it have a value other than 3?

96.[M] Use Taylor polynomials, and their errors, to show that in an open interval in which f'' is positive, tangents to the graph of f lie below the curve. As in Exercise 49 in Section 4.5, you want to show that if a and x are in the interval, then $f(x) > f(a) + f'(a)(x - a)$. It is necessary to treat the cases $x > a$ and $x < a$ separately.

97.[M] Evaluate each limit, indicating the indeterminate form each time l'Hôpital's Rule is applied.

(a) $\lim_{x\to 0} \left(\frac{1+2^x}{2}\right)^{1/x}$

(b) $\lim_{x\to 0} \left(\frac{1+2^x}{1+3^x}\right)^{1/x}$

98.[C]

Sam: I can use Taylor polynomials to get l'Hôpital's theorem.

Jane: How so?

Sam: I write $f(x) = f(0)+f'(0)x+f''(c)x^2/2$ and $g(x) = g(0)+g'(0)x+g''(d)x^2/2$.

Jane: O.K.

Sam: Since $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ are both zero I have $f(0) = g(0) = 0$. I can write, after canceling some x's

$$\frac{f(x)}{g(x)} = \frac{f'(0)+f''(c)x/2}{g'(0)+g''(d)x/2}.$$

Jane: But you don't know the second derivatives.

Sam: It doesn't matter. I just take limits and get

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(0)+f''(c)x/2}{g'(0)+g''(d)x/2}.$$

So

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}.$$

There you have it.

Jane: Let me check your steps.

Check the steps and comment on Sam's proof.

When you throw a fair six-sided die many times, you would expect a 5 to show about 1/6 of the times. That is, if you throw it n times and get k 5's, you would expect k/n to be near 1/6.
More generally, if a certain trial has probability p of success and $q = 1-p$ of failure, and is repeated n times, with k successes, you would expect k/n to be near p. That means that if n is large you would expect $(k/n) - p$ to be small. In other words, let $\epsilon = (k/n) - p$, where ϵ approaches 0 as $n \to \infty$. This means that in most cases $= np + \epsilon n$, or $k = np + z$, where $z/n \to 0$ as $n \to \infty$.
The probability of exactly k successes (and $n-k$ failures) in n trials is

$$\frac{n!}{k!(n-k)!} p^k q^{n-k}. \tag{5.8.1}$$

Exercises 99 to 103 show that for large n (and k) (5.8.1) is approximately

$$\frac{1}{\sqrt{2\pi npq}} \exp\left(\frac{-z^2}{2npq}\right). \tag{5.8.2}$$

Note that (5.8.2) involves $\exp(-x^2)$, whose graph has the shape of the famous bell curve associated with the normal (or Gaussian) distribution in probability and statistics.

99.[C] In Exercise 9 in Section 11.7 we will derive Stirling's formula for an approximation to $n!$:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Use Stirling's formula to show that (5.8.1) is approximately

$$\left(\frac{n}{2\pi k(n-k)}\right)^{1/2} \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k} \tag{5.8.3}$$

in the sense that (5.8.2) divided by (5.8.3) approaches 1 as $n \to \infty$.

100.[M] Show that as $n \to \infty$, the first factor in (5.8.3) is asymptotic to

$$\left(\frac{1}{2\pi pqn}\right)^{1/2} \tag{5.8.4}$$

in the sense that the ratio between it and (5.8.4) approaches 1 as $n \to \infty$.

101.[M] To relate the rest of (5.8.3) to the exponential function, $\exp(x)$, we take its logarithm. Show that

$$\ln\left(\left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k}\right) = -(np+z)\ln\left(1+\frac{z}{np}\right) - (nq-z)\ln\left(1-\frac{z}{nq}\right). \tag{5.8.5}$$

102.[M] Using the Maclaurin polynomial of degree two to approximate $\ln(1+t)$, show that for large n, (5.8.5) is approximately

$$\frac{-z^2}{2pqn}.$$

103.[M] Conclude that for large n, (5.8.1) is approximately (5.8.2).

104.[M] When studying the normal distribution in statistics one will meet an equation that amounts to

$$\frac{\int_{-\infty}^{\infty} x\exp(-(x-\mu)^2)\ dx}{\int_{-\infty}^{\infty} \exp(-(x-\mu)^2)\ dx} = \mu,$$

where μ is a constant. Show that the equation is correct. HINT: Make the substitution $t = x - \mu$.

105.[M] Show that $\int_1^{\infty} x\exp(-x^2)\ dx$ is less than $\int_0^1 x\exp(-x^2)\ dx$. This implies that the area in the "tail" of the bell curve is fairly small in spite of the growth of the coefficient x. As a result, economic predictions based on the bell curve may downplay the likelihood of rare events. This bias may have been one of the several factors that combined to produce the credit crisis and recession that began in 2007.

106.[M] If $P(x)$ is a Maclaurin polynomial associated with $f(x)$, is $P(-x)$ a Maclaurin polynomial associated with $f(-x)$?

107.[M] If $P(x)$ is a Maclaurin polynomial associated with $f(x)$, what is the Maclaurin polynomial of the same degree associated with $f(2x)$?

108.[M] Find the Maclaurin polynomial of degree 6 associated with $1/e^x$.

109.[M] Find the Maclaurin polynomial of degree 6 associated with $\sin(x)\cos(x)$.

110.[M] The center $(x, 0)$, $x > 0$, of a circle C_1 of radius 1 is at a distance x from the center $(0, 0)$ of a circle C_2 of radius 2. AB is the chord joining their two points in common. Let A_1 be the area within C_1 to the left of that chord and A_2 the area within C_2 to the right of that chord.

(a) Which is larger, A_1 or A_2? HINT: Sketch a diagram of these circles and the chord.

(b) If $\lim_{x\to 3^-} A_2/A_1$ exists, what do you think it is?

(c) Determine whether the limit in (b) exists. If it does, find it.

111.[M] In the setup of Exercise 110, let O_1 be the center of C_1 and O_2 the center of C_2. What happens to the ratio of the area common to the two disks and the area of the quadrilateral AO_1BO_2 as $x \to 3^-$?

112.[M] Let $g(x) = f(x^2)$.

(a) Express the Maclaurin polynomial for $g(x)$ up through the term of degree 4 in terms of f and its derivatives.

(b) How is the answer in (a) related to a Maclaurin polynomial associated with f?

113.[M] Find $\lim_{x\to\pi/2^-}(\sec(x) - \tan(x))$

(a) Using l'Hôpital's rule

(b) Without using l'Hôpital's rule

114.[M] Assume that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$.

(a) If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, what, if anything, can be said about $\lim_{x\to\infty} \frac{ln(f(x))}{ln(g(x))}$?

(b) If $\lim_{x\to\infty} \frac{ln(f(x))}{ln(g(x))} = 1$, what, if anything, can be said about $\lim_{x\to\infty} \frac{f(x)}{g(x)}$?

115.[C] Assume that the function $f(x)$ is defined on $[0, \infty)$, has a continuous positive second derivative and $\lim_{x\to\infty} f(x) = 0$.

(a) Can $f(x)$ ever be negative?

(b) Can $f'(x)$ ever be positive?

(c) What are the possible general shapes for the graph of f?

(d) Give an explicit formula for an example of such a function.

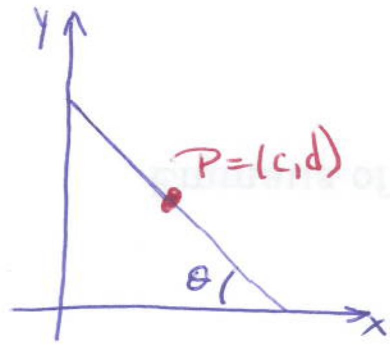


Figure 5.8.6:

116.[C] Let c and d be fixed positive numbers. Consider line segments through $P = (c, d)$ whose ends are on the positive x- and y-axes, as in Figure 5.8.6.
Let θ be the acute angle between the line and the x-axis. Show that the angle α that produces the shortest line segment through P has $\tan^3(\alpha) = d/c$.

117.[C] (See Exercise 116.)

(a) Show that for the angle β such that the the area of the triangle determined by the line segment and the two axes is a minimum, $\tan(\beta) = d/c$.

(b) Show that for β as in (a), OP bisects the line into two parts of equal length.

118.[C] An adventurous bank decides to compound interest twice a year, at time x $(0 < x < 1)$ and at time 1 (instead of at the usual 1/2 and 1). Assume that the annual interest rate is r. Is there a time, x, such that the account grows to more than if the interest was computed at 1/2 and 1?

119.[C] Every six hours a patient takes an amount A of a medicine. Once in the patient, the medicine decays exponentially. In six hours the amount declines from A to kA, where k is less than 1 (and positive). Thus, in 12 hours, the amount in the system is $kA + k^2A$. At exactly 12 hours, the patient takes another pill and the amount in her system is $A + kA + k^2A$.

(a) Graph the general shape of the graph showing the amount of medicine in the patient as a function of time.

(b) When a pill is taken at the end of n six-hour periods how much is in the system?

(c) Does the amount in the system become arbitrarily large? (If so, this could be dangerous.)

The constant k depends on many factors, such as the age of the patient. For this reason, a dosage tested on a 20-year old may be lethal on a 70-year.

Skill Drill: Derivatives

The remaining exercises offer an opportunity to practice differentiating. In each case show that the derivative of the first function is the second function.

120.[M] $\arctan\left(\frac{x}{a}\right)$; $\frac{a}{x^2+a^2}$.

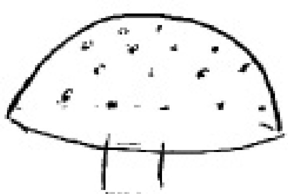
121.[M] $\frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$; $x\sqrt{ax+b}$.

122.[M] $\sin(ax) - \frac{1}{3}\sin^3(ax)$; $a\cos^3(ax)$.

123.[M] $e^{ax}(a\cos(bx) + b\sin(bx))$; $(a^2+b^2)e^{ax}\cos(bx)$.

124.[M] Let $f(x) = (5x^3 + x + 2)^{20}$. Find (a) $f^{(60)}(4)$ and (b) $f^{(61)}(2)$.

Calculus is Everywhere
The Uniform Sprinkler



One day one of the authors (S.S.) realized that the sprinkler did not water his lawn evenly. Placing empty cans throughout the lawn, he discovered that some places received as much as nine times as much water as other places. That meant some parts of the lawn were getting too much water and other parts not enough water.

The sprinkler, which had no moving parts, consisted of a hemisphere, with holes distributed uniformly on its surface, as in Figure 5.8.1. Even though the holes were uniformly spaced, the water was not supplied uniformly to the lawn. Why not?

A little calculus answered that question and advised how the holes should be placed to have an equitable distribution. For convenience, it was assumed that the radius of the spherical head was 1, that the speed of the water as it left the head was the same at any hole, and air resistance was disregarded.

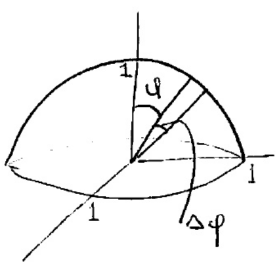


Figure 5.8.2:

Consider the water contributed to the lawn by the uniformly spaced holes in a narrow band of width $d\phi$ near the angle ϕ, as shown in Figure 5.8.2. To be sure the jet was not blocked by the grass, the angle ϕ is assumed to be no more than $\pi/4$.

Water from this band wets the ring shown in Figure 5.8.3.

The area of the band on the sprinkler is roughly $2\pi \sin(\phi)\, d\phi$. As shown in Section 9.3, see Exercises 25 and 26, water from this band lands at a distance from the sprinkler of about

$$x = kv^2 \sin(2\phi).$$

Here k is a constant and v is the speed of the water as it leaves the sprinkler. The width of the corresponding ring on the lawn is roughly

$$dx = 2kv^2 \cos(2\phi) d\phi.$$

Since its radius is approximately $kv^2 \sin(2\phi)$, its area is approximately

$$2\pi \left(kv^2 \sin(2\phi)\right) \left(2kv^2 \cos(2\phi)\ d\phi\right),$$

which is proportional to $\sin(2\phi)\cos(2\phi)$, hence to $\sin(4\phi)$.

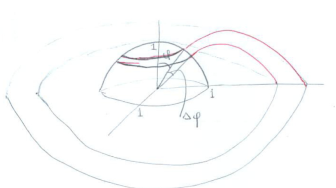


Figure 5.8.3:

Thus the water supplied by the band was proportional to $\sin(\phi)$ but the area watered by that band was proportional to $\sin(4\phi)$. The ratio

$$\frac{\sin(4\phi)}{\sin(\phi)}$$

is the key to understanding both why the distribution was not uniform and to finding out how the holes should be placed to water the lawn uniformly.

By l'Hôpital's rule, this fraction approaches 4 as ϕ approaches zero:

$$\lim_{\phi\to 0} \frac{\sin(4\phi)}{\sin(\phi)} = 4. \tag{5.8.1}$$

This means that for angles ϕ near 0 that ratio is near 4. When ϕ is $\pi/4$, that ratio is $\frac{\sin(\pi)}{\sin(\pi/4)} = 0$, and water was supplied much more heavily far from the sprinkler than near it. To compensate for this bias the number of holes in the band should be proportional to $\sin(4\phi)/\sin(\phi)$. Then the amount of water is proportional to the area watered, and watering is therefore uniform.

Professor Anthony Wexler of the Mechanical Engineering Department of UC-Davis calculated where to drill the holes and made a prototype, which produced a beautiful fountain and a much more even supply of water. Moreover, if some of the holes were removed, it would water a rectangular lawn.

We offered the idea to the firm that made the biased sprinkler. After keeping the prototype for half a year, it turned it down because "it would compete with the product we have."

Perhaps, when water becomes more expensive our uniform sprinkler may eventually water many a lawn.

EXERCISES

1.[R] Show that the limit (5.8.1) is 4

(a) using only trigonometric identities.

(b) using l'Hôpital's rule.

2.[R] Show that $\sin(4x)/\sin(x)$ is a decreasing function for x in the interval $[0, \pi/4]$. HINT: Use trigonometric identities and no calculus. (However, you may be amused if you also do this by calculus.)

3.[R] An oscillating sprinkler goes back and forth at a fixed angular speed.

(a) Does it water a lawn uniformly?

(b) If not, how would you modify it to provide more uniform coverage?

Chapter 6

The Definite Integral

Up to this point we have been concerned with the derivative, which provides local information, such as the slope at a particular point on a curve or the velocity at a particular time. Now we introduce the second major concept of calculus, the definite integral. In contrast to the derivative, the definite integral provides global information, such as the area under a curve.

Section 6.1 motivates the definite integral through three of its applications. Section 6.2 defines the definite integral and Section 6.3 presents ways to estimate it. Sections 6.4 and 6.5 develop the connection between the derivative and the definite integral, which culminates in the Fundamental Theorems of Calculus. The derivative turns out to be essential for evaluating many definite integrals.

Chapters 2 to 6 form the core of calculus. Later chapters are mostly variations or applications of the key ideas in those chapters.

6.1 Three Problems That Are One Problem

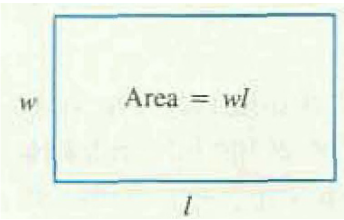


Figure 6.1.1:

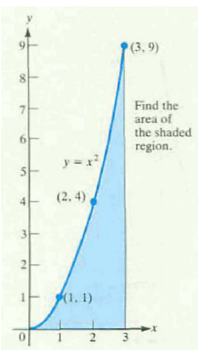


Figure 6.1.2:

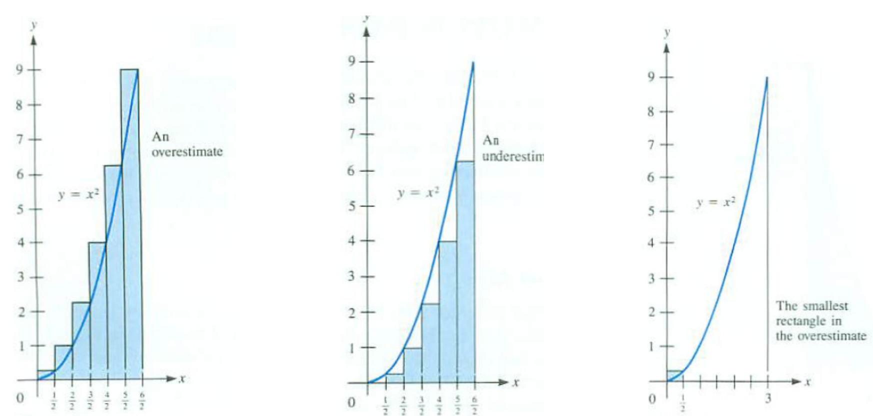
The definite integral is introduced with three problems. At first glance these problems may seem unrelated, but by the end of the section it will be clear that they represent one basic problem in various guises. They lead up to the concept of the definite integral, defined in the next section.

Estimating an Area

It is easy to find the exact area of a rectangle: multiply its length by its width (see Figure 6.1.1). But how do you find the area of the region in Figure 6.1.2? In this section we will show how to make accurate *estimates* of that area. The technique we use will lead up in the next section to the definition of the definite integral of a function.

PROBLEM 1 Estimate the area of the region bounded by the curve $y = x^2$, the x-axis, and the vertical line $x = 3$, as shown in Figure 6.1.2.

Since we know how to find the area of a rectangle, we will use rectangles to approximate the region. Figure 6.1.3(a) shows an approximation by six rectangles whose total area is more than the area under the parabola. Figure 6.1.3(b) shows a similar approximation whose area is less than the area under the parabola.



(a) (b) (c)

Figure 6.1.3:

In each case we break the interval $[0, 3]$ into six short intervals, all of width $\frac{1}{2}$. In order to find the areas of the overestimate and of the underestimate, we must find the height of each rectangle. That height is determined by the curve $y = x^2$. Let us examine only the overestimate, leaving the underestimate for the Exercises.

There are six rectangles in the overestimate shown in Figure 6.1.3(a). The smallest rectangle is shown in Figure 6.1.3(c). The height of this rectangle is equal to the value of x^2 when $x = \frac{1}{2}$. Its height is therefore $\left(\frac{1}{2}\right)^2$ and its area is $\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)$, the product of its height and its width. The areas of the other five rectangles can be found similarly. In each case evaluate x^2 at the right end of the rectangle's base in order to find the height. The total area of the six rectangles is

$$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{2}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{6}{2}\right)^2\left(\frac{1}{2}\right).$$

This equals

$$\frac{1}{8}\left(1^2+2^2+3^2+4^2+5^2+6^2\right)=\frac{91}{8}=11.375. \qquad (6.1.1)$$

The area under the parabola is therefore less than 11.375.

To get a closer estimate we should use more rectangles. Figure 6.1.4 shows an overestimate in which there are 12 rectangles. Each has width $\frac{3}{12}=\frac{1}{4}$. The total area of the overestimate is

$$\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{2}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)+\cdots+\left(\frac{12}{4}\right)^2\left(\frac{1}{4}\right).$$

This equals

$$\frac{1}{4^3}\left(1^2+2^2+3^2+\cdots+12^2\right)=\frac{650}{64}=10.15625. \qquad (6.1.2)$$

Now we know the area under the parabola is less than 10.15625.

To get closer estimates we would cut the interval $[0,3]$ into more sections, maybe 100 or 10,000 or more, and calculate the total area of the corresponding rectangles. (This is an easy computation on a computer.)

In general, we would divide $[0,3]$ into n sections of equal length. The length of each section is then $\frac{3}{n}$. Their endpoints are shown in Figure 6.1.5.

Then, for each integer $i = 1, 2, \ldots, n$, the i^{th} section from the left has endpoints $(i-1)\left(\frac{3}{n}\right)$ and $i\left(\frac{3}{n}\right)$, as shown in Figure 6.1.6.

To make an overestimate, observe that x^2 is increasing for $x > 0$ and evaluate x^2 at the right endpoint of each interval. Then multiply the result by the width of the interval, getting

$$\left(i\left(\frac{3}{n}\right)\right)^2\frac{3}{n}=3^3\frac{i^2}{n^3}.$$

Then, sum these overestimates for all n intervals:

$$3^3\frac{1^2}{n^3}+3^3\frac{2^2}{n^3}+3^3\frac{3^2}{n^3}+\cdots+3^3\frac{(n-1)^2}{n^3}+3^3\frac{n^2}{n^3}$$

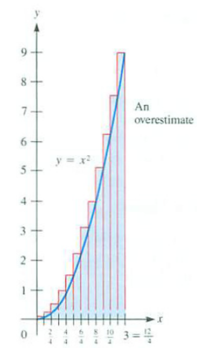


Figure 6.1.4:

Figure 6.1.5:

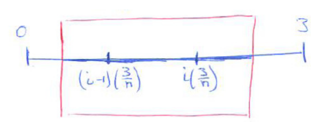


Figure 6.1.6: [ARTIST: Redraw Figure 6.1.6 to give effect of zooming in on ith interval]

which simplifies to

$$3^3\left(\frac{1^2+2^2+3^2+\cdots+(n-1)^2+n^2}{n^3}\right). \qquad (6.1.3)$$

In the summation notation described in Appendix C, this equals

$$\frac{3^3}{n^3}\sum_{i=1}^{n} i^2.$$

We have already seen that these overestimates become more and more accurate as the number of intervals increases. We would like to know what happens to the overestimate as n gets larger and larger. More specifically, does

$$\lim_{n\to\infty}\frac{1^2+2^2+3^2+\cdots+(n-1)^2+n^2}{n^3} \qquad (6.1.4)$$

exist? If it does exist, call it L. (Then the area would be 3^3L.)

The numerator gets large, tending to make the fraction large. But the denominator also gets large, which tends to make the fraction small. Once again we encounter one of the "limit battles" that occurs in the foundation of calculus.

Archimedes, some 2200 years ago, found a short formula for the numerator in (6.1.3), enabling him to find the limit in (6.1.4). See, for instance, S. Stein, "Archimedes: What did he do besides cry Eureka?".

To estimate L, use, say, $n = 6$. Then we have

$$\frac{1}{6^3}\left(1^2+2^2+3^2+4^2+5^2+6^2\right)=\frac{91}{216}\approx 0.42130.$$

Try a larger value of n to get a closer estimate of L.

If we knew L we would know the area under the parabola and above the interval $[0,3]$, for the area is 3^3L. Since we do not know L, we don't know the area. Be patient. We will find L indirectly in this section. You may want to compute the quotient in (6.1.4) for some n and guess what L is. For example, with $n = 12$, the estimate is $\frac{650}{12^3}=\frac{650}{1728}\approx 0.37616$.

Estimating a Distance Traveled

The units simplify: $\frac{\text{mi}}{\text{hr}}\times\text{hr}=\text{mi}$.

If you drive at a constant speed of v miles per hour for a period of t hours, you travel vt miles:

$$\text{Distance} = \text{Speed}\times\text{Time} = vt \text{ miles}.$$

But how would you compute the total distance traveled if your speed were not constant? (Imagine that your odometer, which records distance traveled, was broken. However, your speedometer is still working fine, so you know your speed at any instant.) The next problem illustrates how you could make accurate estimates of the total distance traveled.

PROBLEM 2 A snail is crawling about for three minutes. This remarkable snail knows that she is traveling at the rate of t^2 feet per minute at time t minutes. For instance, after half a minute, she is slowly moving at the rate of $\left(\frac{1}{2}\right)^2$ feet per minute. At the end of her journey she is moving along at 3^2 feet per minute. Estimate how far she travels during the three minutes.

The speed during the three-minute trip increases from 0 to 9 feet per minute. During shorter time intervals, such a wide fluctuation does not occur. As in Problem 1, cut the three minutes of the trip into six equal intervals each 1/2 minute long, and use them to estimate the total distance covered. Represent time by a line segment cut into six parts of equal length, as in Figure 6.1.7.

Figure 6.1.7:

Speed increases as t increases.

Consider the distance she travels during one of the six half-minute intervals, say during the interval $[\frac{3}{2}, \frac{4}{2}]$. At the beginning of this time interval her speed was $\left(\frac{3}{2}\right)^2$ feet per minute; at the end she was going $\left(\frac{4}{2}\right)^2$ feet per minute. The highest speed during this half hour was $\left(\frac{4}{2}\right)^2$ feet per minute. Therefore, she traveled at most $\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)$ feet during the time interval $[3/2, 4/2]$. Similar reasoning applies to the other five half-minute periods. Adding up these upper estimates for the distance traveled during each interval of time, we get the total distance traveled is less than

$$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{2}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{6}{2}\right)^2\left(\frac{1}{2}\right).$$

If we divide the time interval into n equal sections of duration $\frac{3}{n}$, the right endpoint of the i^{th} interval is $i\left(\frac{3}{n}\right)$. At that time the speed is $(3i/n)^2$ feet per minute. So the distance covered during the i^{th} interval of time is less than

$$\underbrace{\left(\frac{3i}{n}\right)^2}_{\text{max speed}} \underbrace{\frac{3}{n}}_{\text{time}} = \frac{3^3 i^2}{n^3}.$$

The total overestimate is then

$$3^3\frac{1^2}{n^3} + 3^3\frac{2^2}{n^3} + 3^3\frac{3^2}{n^3} + \cdots + 3^3\frac{(n-1)^2}{n^3} + 3^3\frac{n^2}{n^3}$$

or

$$3^3\left(\frac{1^2+2^2+3^2+\cdots+(n-1)^2+n^2}{n^3}\right). \tag{6.1.5}$$

The calculations in the area problem, (6.1.3), and in the distance problem, (6.1.5), are the same. Thus, the area and distance have the same upper estimates. Their lower estimates are also the same, as you may check. The limit of (6.1.5) is 3^3L. The two problems are really the same problem.

Estimating a Volume

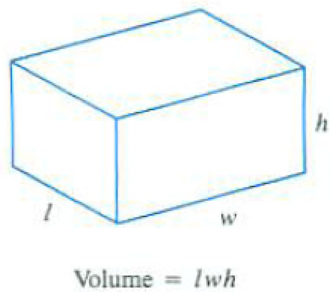
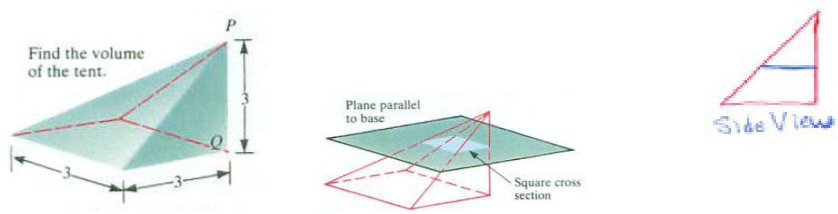


Figure 6.1.8:

The volume of a rectangular box is easy to compute; it is the product of its length, width, and height. See Figure 6.1.8. But finding the volume of a pyramid or ball requires more work. The next example illustrates how we can estimate the volume inside a certain tent.

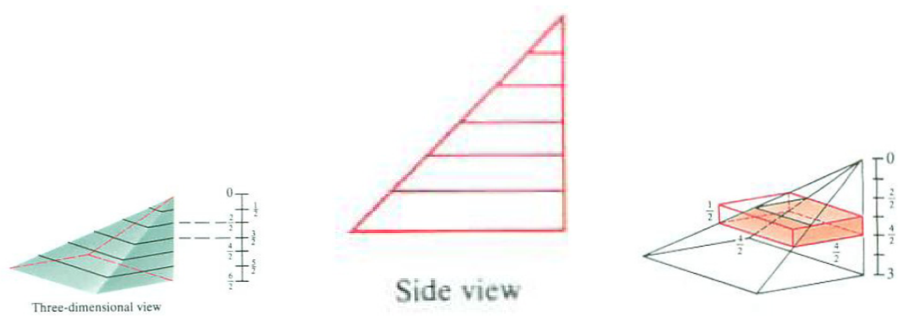
PROBLEM 3 Estimate the volume inside a tent with a square floor of side 3 feet, whose vertical pole, 3 feet long, is located *above one corner* of the floor. The tent is shown in Figure 6.1.9(a).



(a) (b) (c)

Figure 6.1.9:

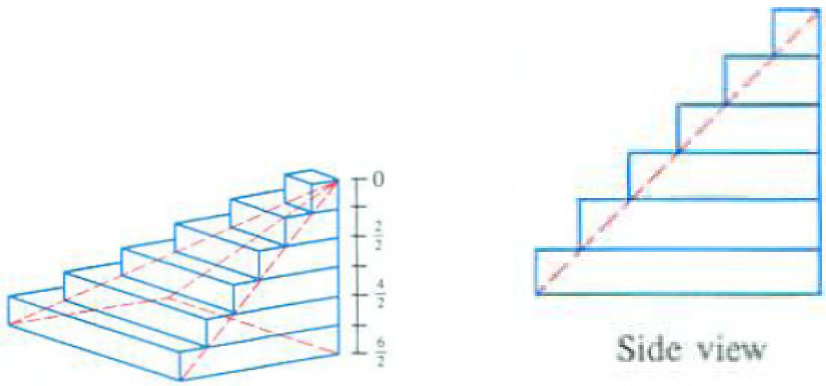
The cross-section of the tent made by any plane parallel to the base is a square, as shown in Figure 6.1.9(b). The width of the square equals its distance from the top of the pole, as shown in Figure 6.1.9(c). Using this fact, we can approximate the volume inside the tent with rectangular boxes with square cross-sections. Begin by cutting a vertical line, representing the pole, into six



(a) (b) (c)

Figure 6.1.10:

sections of equal length, each $\frac{1}{2}$ foot long. Draw the corresponding square cross section of the tent, as in Figure 6.1.10(a). Use these square cross-sections to



(a) (b)

Figure 6.1.11:

form rectangular boxes. Consider the part of the tent corresponding to the interval $[\frac{3}{2}, \frac{4}{2}]$ on the pole. The base of this section is a square with sides $\frac{4}{2}$ feet. The box with this square as a base and height $\frac{1}{2}$ foot encloses completely the part of the tent corresponding to $[\frac{3}{2}, \frac{4}{2}]$. (See Figure 6.1.10(c).) The volume of this box is $\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)$ cubic feet. Figure 6.1.11(a) shows six such boxes, whose total volume is greater than the volume of the tent.

Since the volume of each box is the area of its base times its height, the total volume of the six boxes is

$$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{2}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{6}{2}\right)^2\left(\frac{1}{2}\right) \quad \text{cubic feet.}$$

This sum, which we have encountered twice before, equals 11.375. It is an *overestimate* of the volume of the tent. Better (over)estimates can be obtained by cutting the pole into shorter pieces.

We now know that the number describing the volume of the tent is the same as the number describing the area under the parabola and also the length of the snail's journey. That number is 3^3L. The arithmetic of the estimates is the same in all three cases.

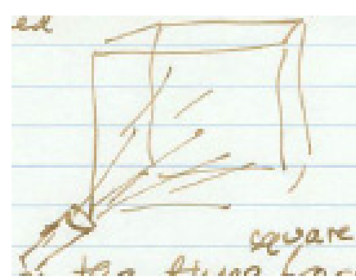


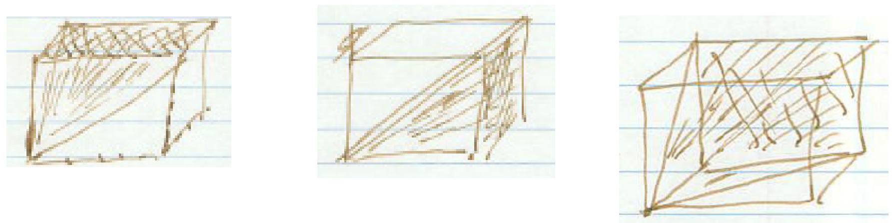
Figure 6.1.12:

A Neat Bit of Geometry

If we knew the limit L in (6.1.3), we would then find the answers to all three problems. But we haven't found L. Luckily, there is a way to find the volume of the tent without knowing L.

The key is that *three identical copies of the tent* fill up a cube of side 3 feet. To see why, imagine a flashlight at one corner of the cube, aimed into the cube, as in Figure 6.1.12.

This trick is similar to the way the area of a triangle is found by arranging two copies of the triangle to form a parallelogram.



(a) (b) (c)

Figure 6.1.13:

The flashlight illuminates the three square faces not meeting the corner at the flashlight. The rays from the flashlight to each of the faces fill out a copy of the tent, as shown in Figure 6.1.13.

Since three copies of the tent fill a cube of volume $3^3 = 27$ cubic feet, the tent has volume 9 cubic feet. From this, we see that the area under the parabola above $[0, 3]$ is 9 and the snail travels 9 feet. Incidentally, the limit L must be $\frac{1}{3}$, since the area under the parabola is both 9 and 3^3L. In short,

$$\lim_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2. = \frac{1}{3}$$

Summary

Using upper estimates, we showed that problems concerning area, distance traveled, and volume were the same problem in various disguises. We were really studying a problem concerning a particular function, x^2, over a particular interval $[0, 3]$. We solved this problem by cutting a cube into three congruent pieces. By the end of this chapter you will learn general techniques that will make such a special device unnecessary.

EXERCISES for 6.1 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 21 concern estimates of areas under curves.

1.[R] In Problem 1 we broke the interval $[0, 3]$ into six sections. Instead, break $[0, 3]$ into four sections of equal lengths and estimate the area under $y = x^2$ and above $[0, 3]$ as follows.

(a) Draw the four rectangles whose total area is larger than the area under the curve. The value of x^2 at the right endpoint of each section determines the height of each rectangle.

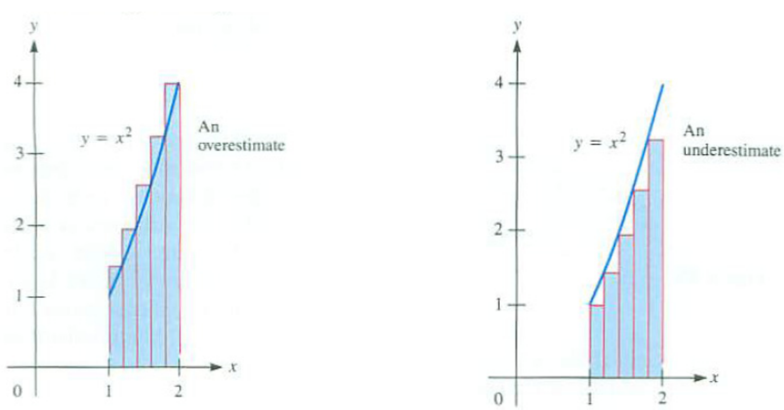
(b) On the diagram in (a), show the height and width of each rectangle.

(c) Find the total area of the four rectangles.

2.[R] Like Exercise 1, but this time obtain an underestimate of the area by using the value of x^2 at the left endpoint of each section to determine the height of the rectangles.

3.[R] Estimate the area under $y = x^2$ and above $[1, 2]$ using the five rectangles with equal widths shown in Figure 6.1.14(a).

4.[R] Repeat Exercise 3 with the five rectangles in Figure 6.1.14(b).



(a) (b)

Figure 6.1.14:

5.[R] Evaluate

(a) $\sum_{i=1}^{4} i^2$

(b) $\sum_{i=1}^{4} 2^i$

(c) $\sum_{n=3}^{4} (n-3)$

6.[R] Evaluate

(a) $\sum_{i=1}^{4} i^3$

(b) $\sum_{i=2}^{5} 2^i$

(c) $\sum_{k=1}^{4}(k^3 - k^2)$

7.[R] Figure 6.1.15(a) shows the curve $y = \frac{1}{x}$ above the interval $[1, 2]$ and an approximation to the area under the curve by five rectangles of equal width.

(a) Make a large copy of Figure 6.1.15(a).

(b) On your diagram show the height and width of each rectangle.

(c) Find the total area of the five rectangles.

(d) Find the total area of the five rectangles in Figure 6.1.15(b).

(e) On the basis of (c) and (d), what can you say about the area under the curve $y = 1/x$ and above $[1, 2]$?

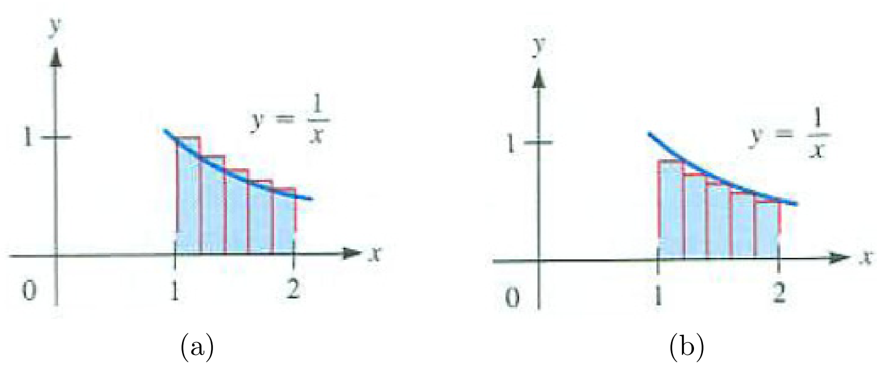


Figure 6.1.15:

Exercises 8 and 9 develop underestimates for each of the problems considered in this section.

8.[R] In Problem 1 we found overestimates for the area under the parabola x^2 over the interval $[0, 3]$. Here we obtain underestimates for this area as follows.

(a) Break $[0, 3]$ into six sections of equal lengths and draw the six rectangles whose total area is smaller than the area under the curve.

(b) Because x^2 is increasing on $[0, 3]$, the left endpoint of each section determines the height of each rectangle. Show the height and width of each rectangle you drew in (a).

(c) Find the total area of the six rectangles.

9.[R] Repeat Exercise 8 with 12 sections of equal lengths.

10.[R] Consider the area under $y = 2^x$ and above $[-1, 1]$.

(a) Graph the curve and estimate the area by eye.

(b) Make an overestimate of the area, using four sections of equal width.

(c) Make an underestimate of the area, using four sections of equal width.

11.[R] Use the information found in Exercises 3 and 4 to complete this sentence: The area in Problem 1 is certainly less than _______ but larger than _______.

12.[R] Estimate the area in Problem 1, using the division of $[0, 3]$ into four sections with endpoints 0, 1, $\frac{5}{3}$, $\frac{11}{4}$, and 3 (see Figure 6.1.16(a)).

(a) Estimate the area when the right-hand endpoints of each section are used to find the heights of the rectangles.

(b) Repeat (a), using the left-hand endpoints of each section to find the heights of the rectangles.

(c) Repeat (a) computing the heights of the rectangles at the points $\frac{1}{2}$, $\frac{3}{2}$, 2, and $\frac{14}{5}$.

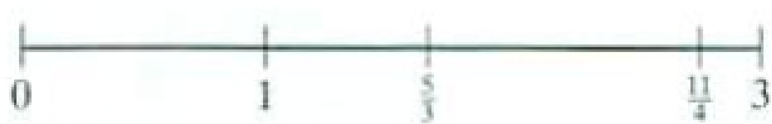


Figure 6.1.16:

In each of Exercises 13 to 18

(a) Draw the region.

(b) Draw six rectangles of equal widths whose total area overestimates the area of the region.

(c) On your diagram indicate the height and width of each rectangle.

(d) Find the total area of the six rectangles. (Give this answer accurate to two decimal places.)

13.[R] Under $y = x^2$, above $[2, 3]$.
14.[R] Under $y = \frac{1}{x}$, above $[2, 3]$.
15.[R] Under $y = x^3$, above $[0, 1]$.
16.[R] Under $y = \sqrt{x}$, above $[1, 4]$.
17.[M] Under $y = \sin(x)$, above $[0, \pi/2]$.
18.[M] Under $y = \ln(x)$, above $[1, e]$.
19.[M] Estimate the area under $y = x^2$ and above $[-1, 2]$ by dividing the interval into six sections of equal lengths.

(a) Draw the six rectangles that form an overestimate for the area under the curve. Note that you cannot do this using only left-endpoints or only right-endpoints.

(b) Find the total area of all six rectangles.

(c) Repeat (a) and (b) to find an underestimate for this area.

20.[M] Estimate the area between the curve $y = x^3$, the x-axis, and the vertical line $x = 6$ using a division into

(a) six sections of equal lengths with left endpoints;

(b) six sections of equal lengths with right endpoints;

(c) three sections of equal lengths with midpoints;

(d) six sections of equal lengths with midpoints.

21.[M] Estimate the area below the curve $y = \frac{1}{x^2}$ and above $[1, 7]$ following the directions in Exercise 20.

22.[M] To estimate the area in Problem 1 you divide the interval $[0, 3]$ into n sections of equal lengths. Using the right-hand endpoint of each of the n sections you then obtain an overestimate. Using the left-hand endpoint, you obtain an underestimate.

(a) Show that these two estimates differ by $\frac{27}{n}$.

(b) How large should n be chosen in order to be sure the difference between the upper estimate and the area under the parabola is less than 0.01?

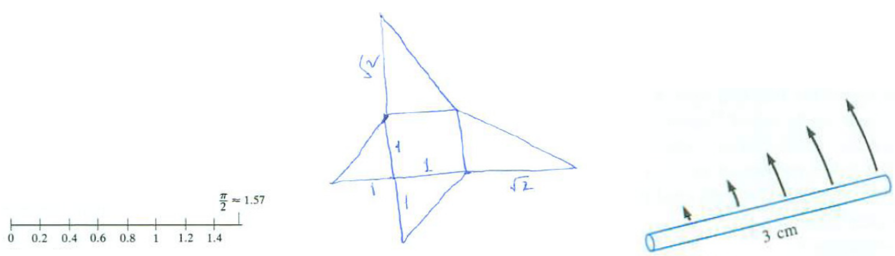
23.[M] Estimate the area of the region under the curve $y = \sin(x)$ and above the interval $[0, \frac{\pi}{2}]$, cutting the interval as shown in Figure 6.1.17(a) and using

(a) left endpoints

(b) right endpoints

(c) midpoints.

(All but the last section are of the same length.)



(a) (b) (c)

Figure 6.1.17:

24.[M] Make three copies of the tent in Problem 3 by folding a pattern as shown in Figure 6.1.17(b). Check that they fill up a cube.

25.[M] An electron is being accelerated in such a way that its velocity is t^3 kilometers per second after t seconds. Estimate how far it travels in the first 4 seconds, as follows:

(a) Draw the interval $[0, 4]$ as the time axis and cut it into eight sections of equal length.

(b) Using the sections in (a), make an estimate that is too large.

(c) Using the sections in (a), make an estimate that is too small.

26.[M] A business which now shows no profit is to increase its profit flow gradually in the next 3 years until it reaches a rate of 9 million dollars per year. At the end of the first half year the rate is to be $\frac{1}{4}$ million dollars per year; at the end of 2 years, 4 million dollars per year. In general, at the end of t years, where t is any number between 0 and 3, the rate of profit is to be t^2 million dollars per year. Estimate the total profit during its first 3 years if the plan is successful using

(a) using six intervals and left endpoints;

(b) using six intervals and right endpoints;

(c) using six intervals and midpoints.

27.[M] Oil is leaking out of a tank at the rate of 2^{-t} gallons per minute after t minutes. Describe how you would estimate how much oil leaks out during the first 10 minutes. Illustrate your procedure by computing one estimate.

28.[C] Archimedes showed that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. You can prove this as follows:

(a) Check that the formula is correct for $n = 1$.

(b) Show that if the formula is correct for the integer n, it is also correct for the next integer, $n + 1$.

(c) Why do (a) and (b) together show that Archimedes' formula holds for all positive integers n?

NOTE: This type of proof is known as **mathematical induction**.

29.[C]

(a) Explain why the area of the region under the curve $y = x^2$ and above the interval $[0, b]$ is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{bi}{n}\right)^2 \frac{b}{n}.$$

(b) Use Exercise 28 to find this limit.

(c) Give an explicit formula for the area of the region under $y = x^2$ and above $[0, b]$.

(d) What is the area under the curve $y = x^2$ and above the interval $[a, b]$. (Assume $0 < a < b$.)

30.[C] The function $f(x)$ is increasing for x in the interval $[a, b]$ and is positive. To estimate the area under the graph of $y = f(x)$ and above $[a, b]$ you divide the interval $[a, b]$ into n sections of equal lengths. You then form an overestimate B (for "big") using right-hand endpoints of the sections and an underestimate S (for "small") using left-hand endpoints. Express the difference between the two estimates, $B - S$, as simply as possible.

31.[C] A right circular cone has a height of 3 feet and a radius of 3 feet, as shown in Figure 6.1.18. Estimate its volume by the sum of the volumes of six cylindrical slabs, just as we estimated the volume of the tent with the aid of six rectangular slabs.

(a) Make a large and neat diagram that shows the six cylinders used in making an overestimate.

(b) Compute the total volume of the six cylinders in (a).

(c) Make a separate diagram showing a corresponding underestimate.

(d) Compute the total volume of the six cylinders in (c). (Note: One of the cylinders has radius 0.)

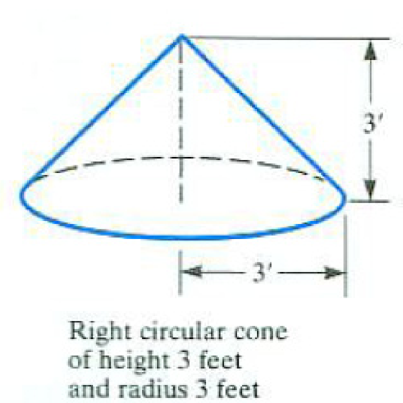


Figure 6.1.18:

32.[C] The kinetic energy of an object, for example, a baseball or car, of mass m grams and speed v centimeters per second is defined as $\frac{1}{2}mv^2$ ergs. Now, in a certain machine a uniform rod 3 centimeters long and weighing 32 grams rotates once per second around one of its ends as shown in Figure 6.1.17(c). Estimate the kinetic energy of this rod by cutting it into six sections, each $\frac{1}{2}$ centimeter long, and taking as the "speed of a section" the speed of its midpoint.

33.[C] Express the sum $\sum_{i=1}^{n} \ln\left(\frac{i+1}{i}\right)$ as simply as possible. (So that you could compute the sum in the fewest steps.)

Skill Drill

In Exercises 34 to 39 differentiate the expression.

34.[R] $(1+x^2)^{4/3}$

35.[R] $\frac{(1+x^3)\sin(3x)}{\sqrt[3]{5x}}$

36.[R] $\frac{3x}{8} + \frac{3x\sin(4x)}{32} + \frac{\cos^3(2x)\sin(2x)}{8}$

37.[R] $\frac{3}{8(2x+3)^2} - \frac{1}{4(2x+3)}$

38.[R] $\frac{\cos^3(2x)}{6} - \frac{\cos(2x)}{2}$

39.[R] $x^3\sqrt{x^2-1}\tan(5x)$

In Exercises 40 to 50 give an antiderivative of the expression.

40.[R] $(x+2)^3$

41.[R] $(x^2+1)^2$

42.[R] $x\sin(x^2)$

43.[R] $x^3 + \frac{1}{x^3}$

44.[R] $\frac{1}{\sqrt{x}}$

45.[R] $\frac{3}{x}$

46.[R] e^{3x}

47.[R] $\frac{1}{1+x^2}$

48.[R] $\frac{1}{x^2}$

49.[R] 2^x

50.[R] $\frac{4}{\sqrt{1-x^2}}$

6.2 The Definite Integral

We now introduce the other main concept in calculus, the "definite integral of a function over an interval."

The preceding section was not really about area under a parabola, distance a snail traveled, or volume of a tent. The common theme of all three was a procedure we carried out with the function x^2 and the interval $[0, 3]$: Cut the interval into small pieces, evaluate the function somewhere in each section, form certain sums, and then see how those sums behave as we choose the sections smaller and smaller.

Here is the general procedure. We have a function f defined at least on an interval $[a, b]$. We cut, or "partition," the interval into n sections by the numbers $x_0 = a$, x_1, x_2, $\ldots$, x_{n-1}, $x_n = b$, as in Figure 6.2.1. They need not all be of the same length, though usually, for convenience, they will be.

The sections $[a, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, b]$ form a **partition** of $[a, b]$.

Figure 6.2.1:

Then we pick a **sampling number** in each interval, c_1 in $[x_0, x_1]$, c_2 in $[x_1, x_2]$, $\ldots$, c_i in $[x_{i-1}, x_i]$, $\ldots$, c_n in $[x_{n-1}, x_n]$ (as in Figure 6.2.1). In Section 6.1, the c_i's were mostly either right-hand or left-hand endpoints or midpoints. However, they can be anywhere in each section.

Next we bring in the particular function f. (In Section 6.1 the function was x^2.) We evaluate that function at each c_i and form the sum

$$f(c_1)(x_1 - x_0) + f(c_2)(x_2 - x_1) + \cdots + f(c_i)(x_i - x_{i-1}) \\ + \cdots + f(c_{n-1})(x_{n-1} - x_{n-2}) + f(c_n)(x_n - x_{n-1}).$$

Rather than continue to write out such a long expression, we choose to take advantage of the fact that each term in (6.2.1) follows the same general pattern: for each of the n sections, multiply the function value at the sampling number by the length of the section. This pattern is easily expressed in the shorthand Σ-notation:

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}). \tag{6.2.1}$$

If the length of section i is written as $\Delta x_i = x_i - x_{i-1}$, the expression for the sum becomes even shorter:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i. \tag{6.2.2}$$

If all the sections have the same length, each Δx_i equals $(b-a)/n$, since the length of $[a, b]$ is $b - a$. Let Δx denote $\frac{b-a}{n}$. We can write (6.2.1) and (6.2.2) also as

$$\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) \qquad \text{or as} \qquad \sum_{i=1}^{n} f(c_i)\Delta x \tag{6.2.3}$$

where $\Delta x = \frac{b-a}{n}$.

The final step is to investigate what happens to the sums of the form (6.2.2) (or (6.2.3)) as the lengths of all the sections approach 0. That is, we try to find

$$\lim_{\text{all } \Delta x_i \text{ approach } 0} \sum_{i=1}^{n} f(c_i)\Delta x_i. \tag{6.2.4}$$

The sums in (6.2.1)–(6.2.3) are called **Riemann sums** in honor of the nineteenth century mathematician, Bernhard Riemann.

Berhnard Riemann, 1826–1866, http://en.wikipedia.org/wiki/Bernhard_Riemann.

In advanced mathematics it is proved that if f is continuous on $[a, b]$ then the sums in (6.2.4) do approach a single number. This brings us to the definition of the definite integral.

The Definite Integral

DEFINITION *(Definite Integral)* Let f be a continuous function defined at least on the interval $[a, b]$. The limit of sums of the form $\sum_{i=1}^{n} f(c_i)\Delta x_i$, as all Δx_i approach 0, exists. The limiting value is called the **definite integral of f over the interval** $[a, b]$ and is denoted

$$\int_a^b f(x)\, dx.$$

Gottfried Liebniz, 1646–1716, http://en.wikipedia.org/wiki/Gottfried_Leibniz.

NOTE: The symbol $\int$ comes from "S," for "sum". The "dx," strictly speaking, is not needed. Both symbols were introduced by Liebniz.

EXAMPLE 1 Express the area under $y = x^2$ and above $[0, 3]$ as a definite integral.

SOLUTION Here the function is $f(x) = x^2$ and the interval is $[0, 3]$. As we saw in the previous section, the area equals the limit of Riemann sums

$$\lim_{\Delta x \to 0^+} \sum_{i=1}^{n} c_i^2 \Delta x = \int_0^3 x^2 \, dx. \tag{6.2.5}$$

◇

The dx traditionally suggests the length of a small section of the x-axis and denotes the **variable of integration** (usually x, as in this case). The function $f(x)$ is called the **integrand**, while the numbers a and b are called the **limits of integration**; a is the **lower limit of integration** and b is the **upper limit of integration**.

Limits of integration refer to the endpoints of the interval $[a, b]$ and are not limits in the sense of Chapter 2. This terminology is traditional and hard to avoid.

The symbol $\int_a^b x^2 \, dx$ is read as "the integral from a to b of x^2". Freeing ourselves from the variable x, we could say, "the integral from a to b of the squaring function". There is nothing special about the symbol x in "x^2". We could just as well have used the letter t — or any other letter. (We would typically pick a letter near the end of the alphabet, since letters near the beginning are customarily used to denote constants.) The notations

$$\int_a^b x^2 \, dx, \qquad \int_a^b t^2 \, dt, \qquad \int_a^b z^2 \, dz, \qquad \int_a^b u^2 \, du, \qquad \int_a^b \theta^2 \, d\theta$$

all denote the same number, that is, "the definite integral of the squaring function from a to b". Taken to the extreme, we could express (6.2.5) as

$$\int_a^b (\ )^2 \, d(\ ).$$

Usually, however, we find it more convenient to use some letter to name the independent variable. Since the letter chosen to represent the variable has no significance of its own, it is called a **dummy variable**. Later in this chapter there will be cases where the interval of integration is $[a, x]$ instead of $[a, b]$. Were we to write $\int_a^x x^2 \, dx$, it would be easy to think there is some relation between the x in x^2 and the x in the upper limit of integration. To avoid possible confusion, we prefer to use a different dummy variable and write, for example, $\int_a^x t^2 \, dt$ in such cases.

It is important to realize that area, distance traveled, and volume are merely applications of the definite integral. (It is a mistake to link the definite integral too closely with one of its applications, just as it narrows our understanding of the number 2 to link it always with the idea of two fingers.) The definite integral $\int_a^b f(x) \, dx$ is also call the **Riemann integral**.

derivatives are limits

Slope and velocity are particular interpretations or applications of the derivative, which is a purely mathematical concept defined as a limit:

$$\text{derivative of } f \text{ at } x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

definite integrals are also limits

Similarly, area, total distance, and volume are just particular interpretations of the definite integral, which is also defined as a limit:

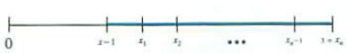
$$\text{definite integral of } f \text{ over } [a,b] = \lim_{\text{as all } \Delta x_i \to 0^+} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}).$$

The Definite Integral of a Constant Function

To bring the definition down to earth, let us use it to evaluate the definite integral of a constant function.

EXAMPLE 2 Let f be the function whose value at any number x is 4; that is, f is the constant function given by the formula $f(x) = 4$. *Use only the definition* of the definite integral to compute

$$\int_1^3 f(x)\, dx.$$



A typical partition of [1, 3]

Figure 6.2.2:

SOLUTION In this case, every partition of the interval $[1,3]$ has $x_0 = 1$ and $x_n = 3$. See Figure 6.2.2. Since, no matter how the sampling number c_i is chosen, $f(c_i) = 4$, the approximating sum equals

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} 4(x_i - x_{i-1})$$

Now

$$\sum_{i=1}^{n} 4(x_i - x_{i-1}) = 4\sum_{i=1}^{n}(x_i - x_{i-1}) = 4 \cdot (x_n - x_0) = 4 \cdot 2 = 8.$$

This is true because the sum of the widths of the sections is the width of the interval $[1,3]$, namely 2. All approximating sums have the same value, namely, 8. For every partition,

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = 8.$$

Thus, as all sections are chosen smaller, the values of the sums are always 8. This number must be the limit:

$$\int_1^3 4\ dx = 8.$$

◇

We could have guessed the value of $\int_1^3 4\ dx$ by interpreting the definite integral as as area. To do so, draw a rectangle of height 4 and base coinciding with the interval $[1,3]$. (See Figure 6.2.3.) Since the area of a rectangle is its base times its height, it follows again that $\int_1^3 4\ dx = 8$.

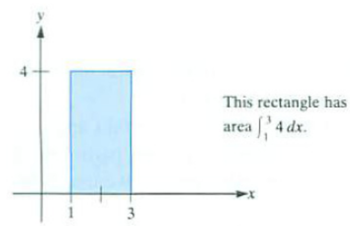


Figure 6.2.3:

Similar reasoning shows that for any constant function that has the fixed value c,

$$\int_a^b c\ dx = c(b-a) \qquad (c \text{ is a constant function})$$

The Definite Integral of x

Exercise 34 shows us how to find $\int_a^b x\ dx$ directly from the definition. Alternatively, let us use the "area" interpretation of the definite integral to predict the value of $\int_a^b x\ dx$.

When the integrand is positive, that is, $0 < a < b$, the area in question then lies above the x-axis, as shown in Figure 6.2.4(a). Two copies of this region form a rectangle of width $b-a$ and height $a+b$, as shown in Figure 6.2.4(b). Thus, the area shown in Figure 6.2.4(a) is half of $(b-a)(b+a) = b^2 - a^2$. Hence,

$$\int_a^b x\ dx = \frac{b^2}{2} - \frac{a^2}{2}.$$

The Definite Integral of x^2

We will find $\int_0^b x^2\ dx$ by examining the approximating sums when all the sections have the same length, as they did in Section 6.1.

Pick a positive integer n and cut the interval $[0,b]$ into n sections of length $\Delta x = b/n$ as in Figure 6.2.5. Then the points of subdivision are 0, Δx, $2\Delta x$, $\ldots$, $(n-1)\Delta x$, and $n\Delta x = b$.

In the typical section $[(i-1)\Delta x, i\Delta x]$ we pick the right-hand endpoint as the sampling number. Thus the approximating sum is

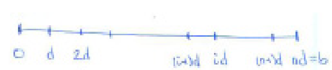


Figure 6.2.5:

$$\sum_{i=1}^{n} (i\Delta x)^2\ (\Delta x) = (\Delta x)^3 \sum_{i=1}^{n} i^2.$$

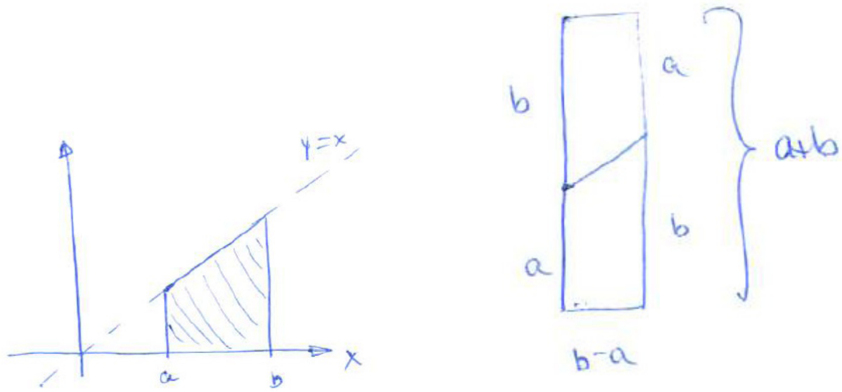


Figure 6.2.4:

Since $\Delta x = b/n$, these overestimates can be written as

$$\frac{b^3}{n^3}\sum_{i=1}^{n} i^2. \tag{6.2.6}$$

Or, see Exercise 29 in Section 6.1.

In Section 6.1 we used geometry to find that

$$\lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{3}.$$

Thus, (6.2.6) approaches $b^3/3$ as n increases, and we conclude that

$$\int_0^b x^2\, dx = \frac{b^3}{3}.$$

Note that when $b = 3$, we have $b^3/3 = 9$, agreeing with the three problems in Section 6.1.

A little geometry suggests the value of $\int_a^b x^2\, dx$, for $0 \le a < b$. Interpret $\int_a^b x^2\, dx$ as the area under $y = x^2$ and above $[a, b]$. This area is equal to the area under $y = x^2$ and above $[0, b]$ minus the area under $y = x^2$ and above $[0, a]$, as shown in Figure 6.2.6. Then

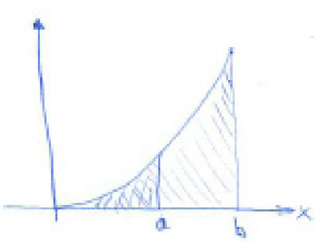


Figure 6.2.6:

$$\int_a^b x^2\, dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

The Definite Integral of 2^x

EXAMPLE 3 Use the definition of the definite integral to evaluate $\int_0^b 2^x\,dx$. (Assume $b > 0$.)

SOLUTION Divide the interval $[0, b]$ into n sections of equal length, $d = b/n$. This time let's evaluate the integrand at the left-hand endpoint of each section. Call this number c_i, $c_i = (i-1)d$. The approximating sum has one term for each section. The contribution from the i^{th} section is

d = width of section

$$2^{c_i} d = 2^{(i-1)d} d.$$

The total estimate is the sum

$$2^0 d + 2^d d + 2^{2d} d + \cdots + 2^{(i-1)d} d + \cdots + 2^{(n-1)d} d.$$

This equals

$$d\left(1 + 2^d + (2^d)^2 + \cdots + (2^d)^i + \cdots + (2^d)^{n-1}\right). \tag{6.2.7}$$

The terms inside the large parentheses in (6.2.7) form a geometric series with n terms, whose first term is 1 and whose ratio is 2^d. Thus, its sum is

Sum of geometric series: $a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}$.

$$\frac{1-(2^d)^n}{1-2^d}.$$

Therefore this typical underestimate is

$$\frac{d(1-(2^d)^n)}{1-2^d} = \frac{d(1-2^{dn})}{1-2^d} = \frac{d(1-2^b)}{1-2^d}. \tag{6.2.8}$$

In the last step we used the fact that $nd = b$. We can rewrite (6.2.8) as

$$\frac{d}{2^d-1}\left(2^b - 1\right). \tag{6.2.9}$$

It still remains to take the limit as n increases without bound. To find what happens to (6.2.9) as $n \to \infty$, we must investigate how $\frac{d}{2^d-1}$ behaves as d approaches 0 (from the right). Though we haven't met this quotient before, we have met its reciprocal, $\frac{2^d-1}{d}$. This quotient occurs in the definition of the derivative of 2^x at $x = 0$:

$$\lim_{x\to 0} \frac{2^x - 2^0}{x} = \lim_{x\to 0} \frac{2^x - 1}{x}.$$

As we saw in Section 3.5, the derivative of 2^x is $2^x \ln(2)$. Thus $D(2^x)$ at $x = 0$ is $\ln(2)$. Hence

$$\lim_{d\to 0^+} \frac{d}{2^d-1}(2^b-1) = \lim_{d\to 0^+} \frac{1}{\left(\frac{2^d-1}{d}\right)}\left(2^b - 1\right) = \frac{2^b-1}{\ln(2)}.$$

Incidentally, $\frac{1}{\ln(2)} \approx 1.443$.

We conclude that

$$\int_0^b 2^x \, dx = \frac{1}{\ln(2)} \left(2^b - 1\right).$$

◇

To evaluate $\int_a^b 2^x \, dx$ with $b > a \geq 0$, we reason as we did when we generalized $\int_0^b x^2 \, dx$ to $\int_a^b x^2 \, dx$. Namely,

$$\int_a^b 2^x \, dx = \int_0^b 2^x \, dx - \int_0^a 2^x \, dx = \frac{2^b - 1}{\ln(2)} - \frac{2^a - 1}{\ln(2)} = \frac{2^b}{\ln(2)} - \frac{2^a}{\ln(2)}.$$

Summary

We defined the definite integral of a function $f(x)$ over an interval $[a, b]$. It is the limit of sums of the form $\sum_{i=1}^n f(c_i)\Delta x_i$ created from partitions of $[a, b]$. It is a purely mathematical idea. You could estimate $\int_a^b f(x) \, dx$ with your calculator – even without having any application in mind. However, the definite integral has many applications: three of them are "area under a curve," "distance traveled" and "volume."

The following table contains a great deal of information. Compare the first three cases with the fourth, which describes the fundamental definition of integral calculus. In this table, all the functions, whether cross-sectional length, velocity, or cross-sectional area, are denoted by the same symbol $f(x)$.

Spend some time studying this table. The concepts it summarizes will be used often.

$f(x)$	$\sum_{i=1}^n f(c_i)(x_i - x_{i-1})$	$\int_a^b f(x) \, dx$
Variable length of cross section of set in plane	Approximate area of set in the plane	The area of set in the plane
Variable velocity	Approximation to the distance traveled	The distance traveled
Variable cross section of a solid	Approximate of volume	The volume of a solid
A function	Just a certain sum	The limit of the sums as the $\Delta x_i \to 0$

Underlying these three applications is one purely mathematical concept, the definite integral, $\int_a^b f(x) \, dx$. The definite integral is defined as a certain limit; it is a number. It is essential to keep the definition of the number $\int_a^b f(x)\,dx$ clear. *It is a limit of certain sums.*

EXERCISES for 6.2

Key: R–routine, M–moderate, C–challenging

1.[R] Using the formula for $\int_a^b x^2\,dx$, find the area under the curve $y = x^2$ and above the interval

(a) $[0, 5]$

(b) $[0, 4]$

(c) $[4, 5]$

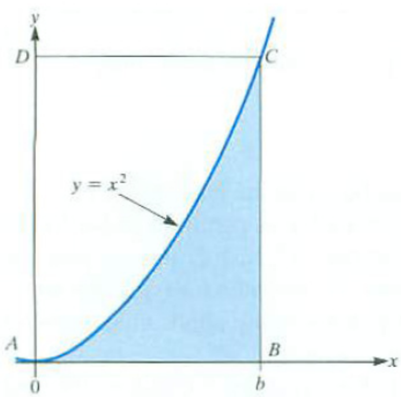


Figure 6.2.7:

2.[R] Figure 6.2.7 shows the curve $y = x^2$. What is the ratio between the shaded area under the curve and the area of the rectangle $ABCD$?

3.[R]

(a) Define "the definite integral of $f(x)$ from a to b, $\int_a^b f(x)\,dx$."

(b) Define the definite integral, using as few mathematical symbols as you can.

(c) Give three applications of the definite integral.

4.[R] Assume $f(x)$ is decreasing for x in $[a, b]$. When you form an approximating sum for $\int_a^b f(x)\,dx$ with left-hand endpoints as sampling points, is your estimate too large or too small? Explain (in one or more complete sentneces).

In Exercises 5 to 8 evaluate the sum

5.[R]

(a) $\sum_{i=1}^{3} i$

(b) $\sum_{i=3}^{7} (2i + 3)$

(c) $\sum_{d=1}^{3} d^2$

6.[R]

(a) $\sum_{i=2}^{4} i^2$

(b) $\sum_{j=2}^{4} j^2$

(c) $\sum_{i=1}^{3}(i^2+i)$

7.[R]

(a) $\sum_{i=1}^{4} 1^i$

(b) $\sum_{k=2}^{6}(-1)^k$

(c) $\sum_{j=1}^{150} 3$

8.[R]

(a) $\sum_{i=3}^{5} \frac{1}{i}$

(b) $\sum_{i=0}^{4} \cos(2\pi i)$

(c) $\sum_{i=1}^{3} 2^{-i}$

In Exercises 9 to 12 write each sum in Σ-notation. (Do not evaluate the sum.)

9.[R]

(a) $1+2+2^2+2^3+\cdots+2^{100}$

(b) $x^3+x^4+x^5+x^6+x^7$

(c) $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{102}+\frac{1}{103}$

10.[R]

(a) $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}$

(b) $\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}110+\frac{1}{14}$

(c) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\frac{1}{101^2}$

11.[R]

(a) $x_0^2(x_1 - x_0) + x_1^2(x_2 - x_1) + x_2^2(x_3 - x_2)$

(b) $x_1^2(x_1 - x_0) + x_2^2(x_2 - x_1) + x_3^2(x_3 - x_2)$

12.[R]

(a) $8t_0^2(t_1 - t_0) + 8t_1^2(t_2 - t_1) + \cdots + 8t_{99}^2(t_{100} - t_{99})$

(b) $8t_1^2(t_1 - t_0) + 8t_2^2(t_2 - t_1) + \cdots + 8t_n^2(t_n - t_{n-1})$

13.[R]

(a) Use the definition of definite integral to evaluate $\int_0^b e^x\ dx$. (See Example 3.)

(b) From (a), deduce that, for $0 \le a < b$, $\int_a^b e^x\ dx = e^b - e^a$.

14.[R]

(a) Use the definition of definite integral to evaluate $\int_0^b 3^x\ dx$.

(b) From (a), deduce that, for $0 \le a < b$, $\int_a^b 3^x\ dx = (3^b - 3^a)/\ln(3)$.

15.[R] The fact that $\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^n f(c_i)\Delta x$ provides another way to evaluate some limits of sums that would otherwise be very challenging to evaluate. Use this idea to write each of the following limits as a definite integral. (Do not evaluate the definite integrals.)

(a) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n e^{i/n} \frac{1}{n}$

(b) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n \frac{1}{1 + \left(1 + \frac{2i}{n}\right)^2} \frac{2}{n}$

(c) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n \sin\left(\frac{i\pi}{n}\right) \frac{\pi}{n}$

(d) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n \left(2 + \frac{3i}{n}\right)^4 \frac{3}{n}$

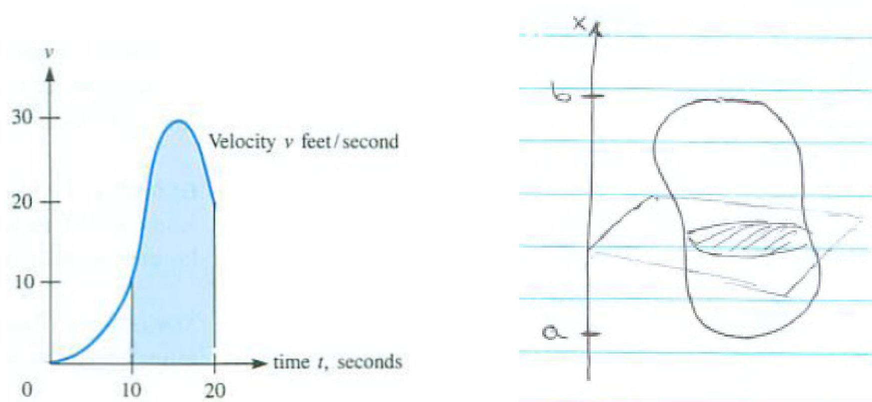
In Exercises 16 to 18 evaluate $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$ for the given function, partition, and sampling numbers.

16.[R] $f(x) = \sqrt{x}$, $x_0 = 1$, $x_1 = 3$, $x_2 = 5$, $c_1 = 1$, $c_2 = 4$ $(n = 2)$

17.[R] $f(x) = \sqrt[3]{x}$, $x_0 = 0$, $x_1 = 1$, $x_2 = 4$, $x_3 = 10$, $c_1 = 0$, $c_2 = 1$, $c_3 = 8$ $(n = 3)$

18.[R] $f(x) = 1/x$, $x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$, $c_1 = 1$, $c_2 = 1.25$, $c_3 = 1.6$, $c_4 = 2$ $(n = 4)$

19.[M] The velocity of an automobile at time t is $v(t)$ feet per second. [Assume $v(t) \geq 0$.] The graph of v for t in $[0, 20]$ is shown in Figure 6.2.8(a). Explain, in complete sentences, why the shaded area under the curve equals the change in position.



(a) (b)

Figure 6.2.8:

20.[M] Show that the volume of a right circular cone of radius a and height h is $\frac{\pi a^3 h}{3}$. Hint: First show that a cross section by a plane perpendicular to the axis of the cone and a distance x from the vertex is a circle of radius ax/h. Note: See Exercise 32.

In Exercises 21 to 24 partition the interval into 4 sections of equal lengths. Estimate the definite integral using sampling numbers chosen to be (a) the left endpoints and (b) the right endpoints.

21.[M] $\int_1^2 (1/x^2)\, dx$.

22.[M] $\int_1^5 \ln(x)\, dx$.

23.[M] $\int_1^5 \frac{2^x}{x}\, dx$.

24.[M] $\int_0^1 \sqrt{1 + x^3}\, dx$.

25.[M] Write the following expression using summation notation.

$$c^{n-1} + c^{n-2}d + c^{n-3}d^2 + \cdots + cd^{n-2} + d^{n-1}.$$

26.[M] Assume that $f(x) \leq -3$ for all x in $[1,5]$. What can be said about the value of $\int_1^5 f(x)\ dx$? Explain, in detail, using the definition of the definite integral.

27.[M] A rocket moving with a varying speed travels $f(t)$ miles per second at time t seconds. Let $t_0, \ldots, t_n$ be a partition of $[a,b]$, and let $T_1, \ldots, T_n$ be sampling numbers. What is the physical interpretation of each of the following quantities?

(a) $t_i - t_{i-1}$

(b) $f(T_i)$

(c) $f(T_i)(t_i - t_{i-1})$

(d) $\sum_{i=1}^{n} f(T_i)(t_i - t_{i-1})$

(e) $\int_a^b f(t)\ dt$

28.[M]

(a) Sketch $y = \cos(x)$, for x in $[0, \pi/2]$.

(b) Estimate, by eye, the area under the curve and above $[0, \pi/2]$.

(c) Partition $[0, \pi/2]$ into three equal sections and use them to provide an overestimate of the area under the curve.

(d) Use the same partition to provide an underestimate of the area under the curve.

29.[M] Repeat Exercise 28 for the area under the curve $y = e^{-x}$ above $[0, 3]$.

30.[M] For x in $[a, b]$, let $A(x)$ be the area of the cross section of a solid perpendicular to the x-axis at x (think of slicing a potato). Let $x_0, x_1, \ldots, x_n$ be a partition of $[a, b]$. Let $c_1, \ldots, c_n$ be the corresponding sampling numbers. What is the geometric interpretation of each of the following quantities? HINT: Refer to Figure 6.2.8(b).

(a) $x_i - x_{i-1}$

(b) $A(c_i)$

(c) $A(c_i)(x_i - x_{i-1})$

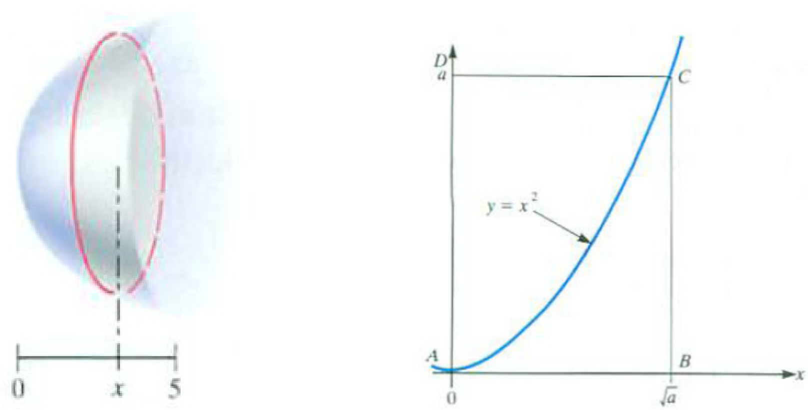
(d) $\displaystyle\sum_{i=1}^{n} A(c_i)(x_i - x_{i-1})$

(e) $\int_a^b A(x)\ dx$

31.[M]

(a) Set up an appropriate definite integral $\int_a^b f(x)\ dx$ which equals the volume of the headlight in Figure 6.2.9(a) whose cross section by a typical plane perpendicular to the x-axis at x is a disk whose radius is $\sqrt{x/\pi}$. NOTE: A circle is a curve and a disk is a flat region inside a circle.

(b) Evaluate the definite integral found in (a).



(a) (b)

Figure 6.2.9:

32.[M]

(a) By considering Figure 6.2.9(b), in particular the area of region ACD, show that $\int_0^a \sqrt{x}\, dx = \frac{2}{3}a^{3/2}$.

(b) Use (a) to evaluate $\int_a^b \sqrt{x}\, dx$ when $0 < a < b$.

Exercises 33 to 36 involve "telescoping sums". Let f be a function defined at least for positive integers. A sum of the form $\sum_{i=1}^n (f(i+1) - f(i))$ is called telescoping. To show why, write the sum out in longhand:

$$(f(2)-f(1))+(f(3)-f(2))+(f(4)-f(3))+\cdots+(f(n)-f(n-1))+(f(n+1)-f(n)).$$

Everything cancels except $-f(1)$ and $f(n+1)$. The whole sum shrinks like a collapsible telescope, with value $f(n+1) - f(1)$.

33.[C]

(a) Show that $\sum_{i=1}^n \left((i+1)^2 - i^2\right) = (n+1)^2 - 1$. HINT: This is a telescoping sum.

(b) From (a), show that $\sum_{i=1}^n (2i+1) = (n+1)^2 - 1$.

(c) From (b), show that $n + 2\sum_{i=1}^n i = (n+1)^2 - 1$.

(d) From (c), show that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

34.[C] Exercise 33 showed that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$. Use this information to find $\int_0^b x\, dx$ directly from the definition of the definite integral (not by interpreting it as an area). No picture is needed.

35.[C]

(a) Starting with the telescoping sum $\sum_{i=1}^{n}\left((i+1)^3 - i^3\right)$ show that

$$n + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i = (n+1)^3 - 1.$$

(b) Use (a) to show that $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

(c) Use (b) to show that $\int_0^b x^2\ dx = \frac{b^3}{3}$.

NOTE: See Exercise 34.

36.[C]

(a) Using the techniques of Exercises 33 to 35, find a short formula for the sum $\sum_{i=1}^{n} i^3$.

(b) Use the formula found in (a) to show that $\int_0^b x^3\ dx = \frac{b^4}{4}$.

37.[C] The function $f(x) = 1/x$ has a remarkable property, namely, for a and b greater than 1,

$$\int_1^a \frac{1}{x}\ dx = \int_b^{ab} \frac{1}{x}\ dx.$$

In other words, "magnifying the interval $[1, a]$ by a positive number b does not change the value of the definite integral." The following steps show why this is so.

(a) Let $x_0 = 1, x_1, x_2, \ldots, x_n = a$ divide the interval $[1, a]$ into n sections. Using left endpoints write out an approximating sum for $\int_1^a \frac{1}{x}\ dx$.

(b) Let $bx_0 = b, bx_1, bx_2, \ldots, bx_n = ab$ divide the interval $[b, ab]$ into n sections. Using left endpoints write out an approximating sum for $\int_b^{ab} \frac{1}{x}\ dx$.

(c) Explain why $\int_1^a \frac{1}{x}\ dx = \int_b^{ab} \frac{1}{x}\ dx$.

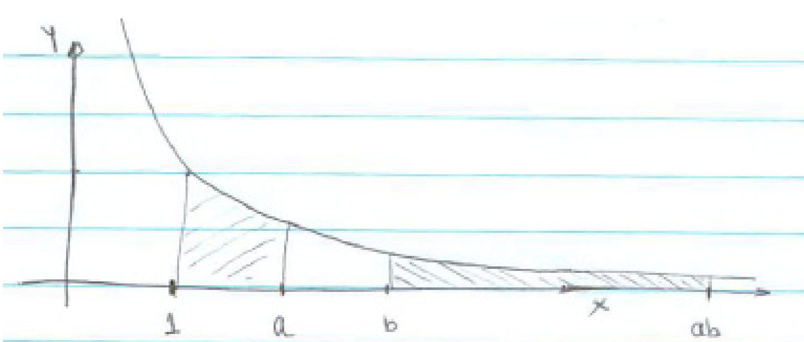


Figure 6.2.10:

38.[C] Let $L(t) = \int_1^t \frac{1}{x}\, dx$, $t > 1$.

(a) Show that $L(a) = L(ab) - L(b)$.

(b) By (a), conclude that $L(ab) = L(a) + L(b)$.

(c) What familiar function has the property listed in (b)?

Gregory St. Vincent noticed the property (a) in 1647, and his friend A.A. de Sarasa saw that (b) followed. Euler, in the 18^{th} century, recognized that $L(x)$ is the logarithm of x to the base e. In short, the area under the hyperbola $y = 1/x$ and above $[1, a]$, $a > 1$, is $\ln(a)$. It can be shown that for a in $(0, 1)$, the negative of the area below that curve and above $[a, 1]$ is $\ln(a)$. (See C. H. Edwards Jr., The Historical development of the Calculus, pp. 154–158.)

39.[C] In Exercise 13 it was shown that for $0 \le a \le b$, $\int_a^b e^x\, dx = e^b - e^a$.

(a) Use the information to show that $\int_{e^a}^{e^b} \ln(x)\, dx = e^b(b-1) - e^a(a-1)$.

(b) From (a), deduce that for $1 \le c \le d$, $\int_c^d \ln(x)\, dx = (d\ln(d) - d) - (c\ln(c) - c)$.

(c) By differentiating $x\ln(x) - x$, show that it is an antiderivative of $\ln(x)$.

40.[C]

(a) To estimate $\int_1^2 \frac{1}{x}\, dx$ divide $[1, 2]$ into n sections of equal lengths and use right endpoints as the sampling points.

(b) Deduce from (a) that

$$\lim_{n\to\infty} \sum_{i=n+1}^{2n} \frac{1}{i} = \lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \text{area under } y = 1/x \text{ and above [}$$

(c) Let $g(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$. Show that $\frac{1}{2} \le g(n) < 1$ and $g(n+1) < g(n)$.

41.[C] (This Exercise is used in Exercise 42.) Consider $b > 1$ and n a positive integer. Define $r(n)$ by the equation $(r(n))^n = b$.

(a) In the case $b = 5$, find $r(n)$ for $n = 1$, 2, 3, and 10. (Note that $r = b^{1/n}$, so you could use the x^y key on a calculator.)

(b) The calculations in (a) suggest that $\lim_{n\to\infty} r(n) = 1$. Show that this conjecture is correct. HINT: Start by taking ln of both sides of the equation $(r(n))^n = b$.

42.[C] For $b > 1$ and k and number, Pierre Fermat (1601–1665) found the area under $y = x^k$ and above $[1, b]$ by using approximating sums. However, he did not cut the interval $[1, b]$ into n sections of equal widths. Instead, for a given positive integer n, he introduced the number r such that $r^n = b$. As n increases, r approaches 1, as Exercise 41 shows. Then he divided the interval $[0, b]$ into sections using the number r, r^2, r^3, ..., r^{n-1}, as shown in Figure 6.2.11. The n sections are $[1, r]$, $[r, r^2]$, ..., $[r^{n-1}, r^n] = [r^{n-1}, b]$.

(a) Show that the width of the i^{th} section, $[r^{i-1}, r^i]$, is $r^{i-1}(r - 1)$.

(b) Using the left endpoints of each section, obtain an underestimate of $\int_1^b x^2\, dx$.

(c) Show that the estimate in (b) is equal to

$$\frac{b^3 - 1}{1 + r + r^2}.$$

(d) Find $\lim_{n\to\infty} \frac{b^3-1}{1+r+r^2}$. Hint: Remember that r depends on n.

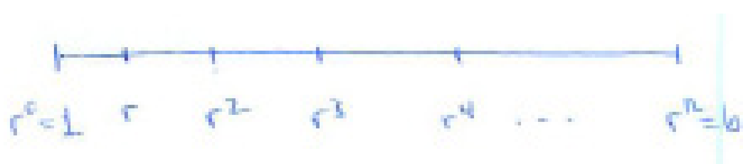


Figure 6.2.11:

43.[C] Use Fermat's approach outlined in Exercise 42, but with right endpoints as the sampling points, to obtain an overestimate of the area under x^2, above $[1, b]$, and then find its limit as $n \to \infty$.

44.[C]

(a) Obtain an underestimate and an overestimate of $\int_0^{\pi/2} \cos(x)\, dx$ that differ by at most 0.1. Note: Remember that the angles are measured in radians.

(b) Average the two estimates in (a).

(c) If $\int_0^{\pi/2} \cos(x)\, dx$ is a famous number, what do you think it is?

45.[C] Is $\int_1^2 \frac{1}{x^2}\,dx$ equal to $1/\int_1^2 x^2\,dx$? HINT: Use Fermat's formula from Exercise 42.

46.[C] By considering the approximating sums in the definition of a definite integral, show that $\int_3^4 \frac{dx}{(x+5)^3}$ equals $\int_2^3 \frac{dx}{(x+6)^3}$.

47.[C] For a continuous function f defined for all x, is $\int_a^b f(x+1)\,dx$ equal to $\int_{a+1}^{b+1} f(x)\,dx$?

48.[C] For continuous functions f and g defined for all x, is $\int_a^b f(x)g(x)\,dx$ equal to $\int_a^b f(x)\,dx \int_a^b g(x)\,dx$?

49.[C] If f is an increasing function such that $f(1) = 3$ and $f(6) = 7$, what can be said about $\int_2^4 f(x)\,dx$? Explain.

50.[C]

(a) Using formulas already developed, evaluate $G(x) = \int_1^x t^2\,dt$.

(b) Find $G'(x)$.

(c) Repeat (a) and (b) for $G(x) = \int_1^x 2^t\,dt$.

(d) Do you noice what appears to be a coincidence in (b) and (c)?

Skill Drill

In Exercises 51 to 58 give two antiderivatives for the given functions.

51.[R] x^2

52.[R] $1/x^3$

53.[R] e^{-4x}

54.[R] $1/(2x+1)$

55.[R] 2^x

56.[R] $\sin(3x)$

57.[R] $\frac{3}{1+9x^2}$

58.[R] $\frac{4}{\sqrt{1-x^2}}$

6.3 Properties of the Antiderivative and the Definite Integral

In Section 3.6 we defined an antiderivative of a function $f(x)$. It is any function $F(x)$ whose derivative is $f(x)$. For instance, x^3 is an antiderivative of $3x^2$. So is $x^3 + 2011$. Keep in mind that *an antiderivative is a function*.

F is an antiderivative of f when $F'(x) = f(x)$

In this section we discuss various properties of antiderivatives and definite integrals. These properties will be needed in Section 6.4 where we obtain a relation between antiderivatives and definite integrals. That relation will be a great time-saver in evaluating many (but not all) definite integrals.

We have not yet introduced a symbol for an antiderivative of a function. We will adopt the following standard notation:

Warning: If a function has an antiderivative, then it has lots of antiderivatives.

Notation: Any antiderivative of f is denoted $\int f(x)\ dx$.

For instance, $x^3 = \int 3x^2\ dx$. This equation is read "x^3 is an antiderivative of $3x^2$". That means simply that "the derivative of x^3 is $3x^2$". It is true that $x^3 + 2011 = \int 3x^2\ dx$, since $x^3 + 2011$ is also an antiderivative of $3x^2$. That does *not* mean that the functions x^3 and $x^3 + 2011$ are equal. All it means is that these two functions both have the same derivative, $3x^2$. The symbol $\int 3x^2\ dx$ refers to *any function* whose derivative is $3x^2$.

$\int f(x)\ dx$ is a function
$\int_a^b f(x)\ dx$ is a number.

If $F'(x) = f(x)$ we write $F(x) = \int f(x)\ dx$. The function $f(x)$ is called the **integrand**. The function $F(x)$ is called an antiderivative of $f(x)$. The symbol for an antiderivative, $\int f(x)\ dx$, is similar to the symbol for a definite integral, $\int_a^b f(x)\ dx$, but they denote vastly different concepts. An antiderivative is often called an "integral" or "indefinite integral," but should not be confused with a definite integral. The symbol $\int f(x)dx$ denotes a *function* — any function whose derivative is $f(x)$. The symbol $\int_a^b f(x)\ dx$ denotes a *number* — one that is defined by a limit of certain sums. The value of the definite integral may vary as the interval $[a, b]$ changes.

We apologize for the use of such similar notations, $\int f(x)\ dx$ and $\int_a^b f(x)\ dx$, for such distinct concepts. However, it is not for us to undo over three centuries of custom. Rather, it is up to you to read the symbols $\int f(x)\ dx$ and $\int_a^b f(x)\ dx$ carefully. You distinguish between such similar-looking words as "density" and "destiny" or "nuclear" and "unclear". Be just as careful when reading mathematics.

Properties of Antiderivatives

The tables inside the covers of this book list many antiderivatives. One example is $\int \sin(x)\ dx = -\cos(x)$. Of course, $-\cos(x) + 17$ also is an antiderivative of $\sin(x)$. In Section 4.1 it was shown that if F and G have the same derivative on an interval, they differ by a constant, C. So $F(x) - G(x) = C$ or $F(x) = G(x) + C$. For emphasis, we state this as a theorem.

The following theorem asserts that if you find an antiderivative $F(x)$ for a function $f(x)$, then any other antiderivative of $f(x)$ is of the form $F(x) + C$ for some constant C.

This result was anticipated back in Section 3.6.

Theorem. *If F and G are both antiderivatives of f on some interval, then there is a constant C such that*

$$F(x) = G(x) + C.$$

Many tables of integrals, including the ones in the cover of this book, omit the $+C$.

When using an antiderivative, it is best to include the constant C. (It was needed in the study of differential equations in Section 5.2.) For example,

$$\begin{aligned} \int 5\ dx &= 5x + C \\ \int e^x\ dx &= e^x + C \\ \text{and} \qquad \int \sin(2x)\ dx &= \frac{-1}{2}\cos(2x) + C. \end{aligned}$$

Observe that

$$\frac{d}{dx}\left(\int x^3\ dx\right) = x^3 \qquad \text{and} \qquad \frac{d}{dx}\left(\int \sin(2x)\ dx\right) = \sin(2x). \quad (6.3.1)$$

Are these two equations profound or trivial? Read them aloud and decide.

The first says, "The derivative of an antiderivative of x^3 is x^3." *It is true simply because that is how we defined the antiderivative.* We know that

We know that the square of the square root of 7 is 7 and that $e^{\ln(3)} = 3$, both by the definition of inverse functions.

$$\frac{d}{dx}\left(\int \frac{\ln(1+x^2)}{(\sin(x))^2}\ dx\right) = \frac{\ln(1+x^2)}{(\sin(x))^2}$$

even though we cannot write out a formula for an antiderivative of $\frac{\ln(1+x^2)}{(\sin(x))^2}$. In other words, by the very definition of the antiderivative,

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Any property of derivatives gives us a corresponding property of antiderivatives. Three of the most important properties of antiderivatives are recorded in the next theorem.

Properties of antiderivatives

Theorem 6.3.1 (Properties of Antiderivatives)**.** *Assume that f and g are functions with antiderivatives $\int f(x)\ dx$ and $\int g(x)\ dx$. Then the following hold:*

A. $\int cf(x)\ dx = c\int f(x)\ dx$ *for any constant* c*.*

B. $\int (f(x)+g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx$*.*

C. $\int (f(x)-g(x))\ dx = \int f(x)\ dx - \int g(x)\ dx$*.*

Proof

(A) Before we prove that $\int cf(x)\ dx = c\int f(x)\ dx$, we stop to see what it means. This equation says that "c times an antiderivative of $f(x)$ is an antiderivative of $cf(x)$". Let $F(x)$ be an antiderivative of $f(x)$. Then the equation says "c times $F(x)$ is an antiderivative of $cf(x)$". To determine if this statement is true we must differentiate $cF(x)$ and check that we get $cf(x)$. So, we compute $(cF(x))'$:

$$\begin{aligned} (cF(x))' &= cF'(x) && [c \text{ is a constant}] \\ &= cf(x). && [F \text{ is antiderivative of } f] \end{aligned}$$

Thus $cF(x)$ is indeed an antiderivative of $cf(x)$. Therefore, we may write

$$cF(x) = \int cf(x)\ dx.$$

Since $F(x) = \int f(x)\ dx$, we conclude that

$$c\int f(x)\ dx = \int cf(x)\ dx.$$

(B) The proof is similar. We show that $\int f(x)\ dx + \int g(x)\ dx$ is an antiderivative of $f(x)+g(x)$. To do this we compute the derivative of the sum $\int f(x)\ dx + \int g(x)\ dx$:

$$\begin{aligned} \tfrac{d}{dx}\left(\int f(x)\ dx + \int g(x)\ dx\right) &= \tfrac{d}{dx}\left(\int f(x)\ dx\right) + \tfrac{d}{dx}\left(\int g(x)\ dx\right) && [\text{derivative of a sum}] \\ &= f(x)+g(x). && [\text{definition of antiderivatives}] \end{aligned}$$

(C) The proof is similar to the one for (b). •

EXAMPLE 1 Find (a) $\int 6\cos(x)\ dx$, (b) $\int (6\cos(x)+3x^2)\ dx$, and (c) $\int (6\cos(x)-\frac{5}{1+x^2}\ dx$.

SOLUTION (a) Part (a) of the theorem is used to move the "6" (a constant) past the integral sign, "$\int$". We then have:

$$\int 6\cos(x)\ dx = 6\int \cos(x)\ dx = 6\sin(x) + C.$$

Notice that the "C" is added as the last step in finding an antiderivative.

(b)

$$\begin{aligned} \int (6\cos(x) + 3x^2)\ dx &= \int 6\cos(x)\ dx + \int 3x^2\ dx && \text{[part (b) of the theorem]} \\ &= 6\sin(x) + x^3 + C. \end{aligned}$$

Here, notice that separate constants are not needed for each antiderivative; again only one "C" is needed for the overall antiderivative.

(c)

$$\begin{aligned} \int \left(6\cos(x) - \tfrac{5}{1+x^2}\right)\ dx &= \int 6\cos(x)\ dx - \int \tfrac{5}{1+x^2}\ dx && \text{[part (c) of the theorem]} \\ &= 6\sin(x) - 5\int \tfrac{1}{1+x^2}\ dx && \text{[part (a) of the theorem]} \\ &= 6\sin(x) - 5\arctan(x) + C && [(\arctan(x))' = \tfrac{1}{1+x^2}] \end{aligned}$$

◇

The last two parts of Theorem 6.3.1 extend to any finite number of functions. For instance,

$$\int (f(x) - g(x) + h(x))\ dx = \int f(x)\ dx - \int g(x)\ dx + \int h(x)\ dx.$$

Theorem. *Let a be a number other than -1. Then*

$$\int x^a\ dx = \frac{x^{a+1}}{a+1} + C.$$

Proof

$$\left(\frac{x^{a+1}}{a+1}\right)' = \frac{(a+1)x^{(a+1)-1}}{a+1} = x^a.$$

•

EXAMPLE 2 Find $\int \left(\frac{3}{\sqrt{1-x^2}} - \frac{2}{x} + \frac{1}{x^3}\right)\, dx$, $0 < x < 1$.

SOLUTION

$$\begin{aligned}\int \left(\frac{3}{\sqrt{1-x^2}} - \frac{2}{x} + \frac{1}{x^3}\right) dx &= 3\int \frac{1}{\sqrt{1-x^2}}\, dx - 2\int \frac{1}{x}\, dx + \int x^{-3}\, dx \\ &= 3\arcsin(x) - 2\ln(x) + \frac{x^{-2}}{-2} + C \\ &= 3\arcsin(x) - 2\ln(x) - \frac{1}{2x^2} + C.\end{aligned}$$

If $-1 < x < 0$, we would write the antiderivative of $1/x$ as $\ln|x|$.

◇

Properties of Definite Integrals

Some of the properties of definite integrals look like properties of antiderivatives. However, they are assertions about numbers, not about functions. In the notation for the definite integral, $\int_a^b f(x)\, dx$, b is larger than a. It will be useful to be able to speak about "the definite integral from a to b" even if b is less than or equal to a. The following two definitions meet this need and we will use them in the proofs of the two fundamental theorems of calculus in the next section.

DEFINITION *(Integral from a to b, where $b < a$.)* If b is less than a, then

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx.$$

EXAMPLE 3 Compute $\int_3^0 x^2\, dx$, the integral from 3 to 0 of x^2.
SOLUTION The symbol $\int_3^0 x^2\, dx$ is defined as $-\int_0^3 x^2\, dx$. As was shown in Section 6.2, $\int_0^3 x^2\, dx = 9$. Thus

$$\int_3^0 x^2\, dx = -9.$$

◇

DEFINITION *(Integral from a to a.)*

$$\int_a^a f(x)\ dx = 0$$

Remark: The definite integral is defined with the aid of partitions of an interval. Rather than permit partitions to have sections of length 0, it is simpler just to make this definition.

The point of making these two definitions is that now the symbol $\int_a^b f(x)\ dx$ is defined for any numbers a and b and any continuous function f, assuming $f(x)$ is defined for x in $[a, b]$. It is no longer necessary that a be less than b.

The definite integral has several properties, some of which we will be using in this section and some in later chapters. Justifications of these properties are provided immediately after the following table.

Properties of antiderivatives

Theorem (Properties of the Definite Integral)**.** *Let f and g be continuous functions, and let c be a constant. Then*

1. ***Moving a Constant Past*** $\int_a^b$
   $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$
2. ***Definite Integral of a Sum***
   $\int_a^b (f(x)+g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
3. ***Definite Integral of a Difference***
   $\int_a^b (f(x)-g(x))\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$
4. ***Definite Integral of a Non-Negative Function***
   If $f(x) \geq 0$ *for all* x *in* $[a, b]$, $a < b$, *then* $\displaystyle\int_a^b f(x)\ dx \geq 0$.
5. ***Definite Integrals Preserve Order***
   If $f(x) \geq g(x)$ *for all* x *in* $[a, b]$, $a < b$, *then*
   $$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx.$$
6. ***Sum of Definite Integrals Over Adjoining Intervals***
   If a, b, *and* c *are numbers, then*
   $$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx.$$
7. ***Bounds on Definite Integrals***
   If m *and* M *are numbers and* $m \leq f(x) \leq M$ *for all* x *between* a *and* b, *then*
   $$m(b-a) \quad \leq \quad \int_a^b f(x)\ dx \leq M(b-a) \qquad \text{if } a < b$$
   and
   $$m(b-a) \quad \geq \quad \int_a^b f(x)\ dx \geq M(b-a) \qquad \text{if } a > b$$

Proof of Property 1

Take the case $a < b$. The equation $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$ resembles part (a) of Theorem 6.3.1 about antiderivatives: $\int cf(x)\ dx = c\int f(x)\ dx$. However, its proof is quite different, since $\int_a^b cf(x)\ dx$ is defined as a limit of sums.

We have

$$\begin{aligned}
\int_a^b cf(x)\ dx &= \lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^{n} cf(c_i)\Delta x_i && \text{definition of definite integral}\\
&= \lim_{\text{all } \Delta x_i \to 0} c\sum_{i=1}^{n} f(c_i)\Delta x_i && \text{algebra (distributive law)}\\
&= c\lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i && \text{property of limits}\\
&= c\int_a^b f(x)\ dx. && \text{definition of definite integral}
\end{aligned}$$

•

Similar approaches can be used to justify each of the other properties. However, we pause only to make them plausible by giving an intuitive interpretation of each property in terms of area.

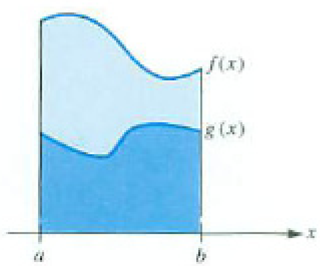


Figure 6.3.1:

Plausibility of Argument for Property 5

This amounts to the assertion that when the graph of $y = f(x)$ is always at least as high as the graph of $y = g(x)$, then the area of a region under the curve $y = f(x)$ is greater than or equal to the area under the curve $y = g(x)$ above a given interval. (See Figure 6.3.) •

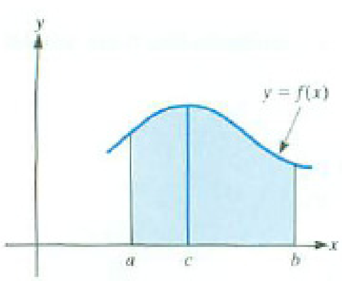


Figure 6.3.2:

Plausibility of Argument for Property 6

In the case that $a < c < b$ and $f(x)$ assumes only positive values, this property asserts that the area of the region below the graph of $y = f(x)$ and above the interval $[a, b]$ is the sum of the areas of the regions below the graph and above the smaller intervals $[a, c]$ and $[c, b]$. Figure 6.3.2 shows that this is certainly plausible. •

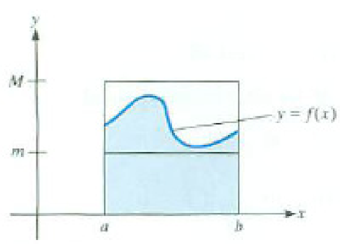


Figure 6.3.3:

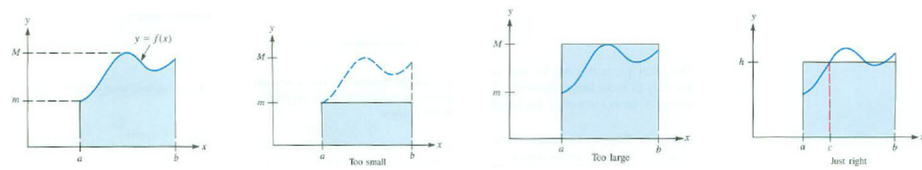
Plausibility of Argument for Property 7

The inequalities in this property compare the area under the graph of $y = f(x)$ with the areas of two rectangles, one of height M and one of height m. (See Figure 6.3.3.) In the case $a < b$, the area of the larger rectangle is $M(b - a)$ and the area of the smaller rectangle is $m(b - a)$. •

The Mean-Value Theorem for Definite Integrals

The mean-value theorem for *derivatives* says that (under suitable hypotheses) $f(b)-f(a)=f'(c)(b-a)$ for some number c in $[a,b]$. The mean-value theorem for *definite integrals* has a similar flavor. First, we state it geometrically.

If $f(x)$ is positive and $a<b$, then $\int_a^b f(x)\,dx$ can be interpreted as the area of the shaded region in Figure 6.3.4(a).



(a) (b) (c) (d)

Figure 6.3.4:

What can you say about the case when $m=M$?

Let m be the minimum and M the maximum values of $f(x)$ for x in $[a,b]$. We assume that $m<M$. The area of the rectangle of height M is larger than the shaded area; the area of the rectangle of height m is smaller than the shaded area. (See Figures 6.3.4(b) and (c).) Therefore, there is a rectangle whose height h is somewhere between m and M, whose area is the same as the shaded area under the curve $y=f(x)$. (See Figure 6.3.4(d).) Hence $\int_a^b f(x)\,dx=(b-a)h$.

Now, h is a number between m and M. By the Intermediate-Value Property for continuous functions, in Section 2.4 there is a number c in $[a,b]$ such that $f(c)=h$. (See Figure 6.3.4(d).) Hence,

$$\text{Area of shaded region under curve} = f(c)(b-a).$$

This suggests the mean-value theorem for definite integrals.

Mean-Value Theorem for Definite Integrals

Theorem (Mean-Value Theorem for Definite Integrals)**.** *Let a and b be numbers, and let f be a continuous function defined at least on the interval $[a,b]$. Then there is a number c in $[a,b]$ such that*

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Proof of the Mean-Value Theorem for Definite Integrals, using only properties

of the definite integral
Consider the case when $a < b$. Let M be the maximum and m the minimum of $f(x)$ on $[a, b]$. Property 7, combined with division by $b - a$, gives

$$m \le \frac{\int_a^b f(x)\, dx}{b-a} \le M,$$

Because f is continuous on $[a, b]$, by the Intermediate-Value Property of Section 2.4 there is a number c in $[a, b]$ such that

$$f(c) = \frac{\int_a^b f(x)\, dx}{b-a},$$

The case $b < a$ can be obtained from the case $a < b$. (see Exercise 35).

and the theorem is proved (without depending on a picture). •

EXAMPLE 4 Verify the mean-value theorem for definite integrals when $f(x) = x^2$ and $[a, b] = [0, 3]$.
SOLUTION In Section 6.2 it was shown that $\int_0^3 x^2\, dx = 9$. Since $f(x) = x^2$, we are looking for c in $[0, 3]$ such that

$$\int_0^3 x^2\, dx = 9 = c^2(3 - 0)$$

$-\sqrt{3}$ is not in $[0, 3]$.

That is, $9 = 3c^2$, so $c^2 = \frac{9}{3} = 3$, $c = \sqrt{3}$. (See Figure 6.3.5.) The rectangle with height $f(\sqrt{3}) = (\sqrt{3})^2 = 3$ and base $[0, 3]$ has the same area as the region under the curve $y = x^2$ and above $[0, 3]$. ◇

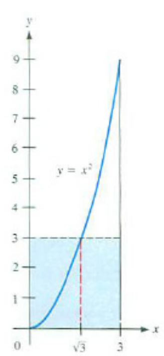


Figure 6.3.5:

The Average Value of a Function

Let $f(x)$ be a continuous function defined on $[a, b]$. What shall we mean by the "average value of $f(x)$ over $[a, b]$"? We cannot add up all the values of $f(x)$ for all x's in $[a, b]$ and divide by the number of x's, since there are an infinite number of such x's. However, we can work with the average (or mean) of n numbers $a_1, a_2, \ldots, a_n$, which is their sum divided by n: $\frac{1}{n}\sum_{i=1}^{n} a_i$. For example, the average of 1, 2, and 6 is $\frac{1}{3}(1 + 2 + 6) = \frac{9}{3} = 3$.

This suggests how to define the "average value of $f(x)$ over $[a, b]$". Choose a large integer n and partition $[a, b]$ into n sections of equal length, $\Delta x = (b - a)/n$. Let the sampling points c_i be the left endpoint of each section, $c_1 = a$, $c_2 = a + \Delta x$, $\ldots$, $c_n = a + (n - 1)\Delta x = b - \Delta x$. Then an estimate of the "average" would be

$$\frac{1}{n}\left(f(c_1) + f(c_2) + \cdots + f(c_n)\right). \tag{6.3.2}$$

Since $\Delta x = (b-a)/n$, it follows that $\frac{1}{n} = \frac{\Delta x}{b-a}$. Therefore, (6.3.2) can be rewritten as

$$\frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x.$$

But, $\sum_{i=1}^{n} f(c_i)\Delta x$ is an estimate of $\int_a^b f(x)\,dx$. It follows that, as $n \to \infty$, this average of the n function values approaches $\frac{1}{b-a}\int_a^b f(x)\,dx$. This motivates the following definition:

DEFINITION *(Average Value of a Function over an Interval)* Let $f(x)$ be defined on the interval $[a,b]$. Assume that $\int_a^b f(x)\,dx$ exists. The **average value** or **mean value** of f on $[a,b]$ is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

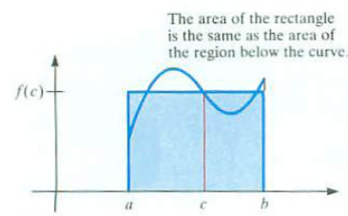


Figure 6.3.6:

Geometrically speaking (if $f(x)$ is positive), this average value is the height of the rectangle that has the base $[a,b]$ and the same area as the area of the region under the curve $y = f(x)$, above $[a,b]$. (See Figure 6.3.6.) Observe that the average value of $f(x)$ over $[a,b]$ is between its maximum and minimum values for x in $[a,b]$. However, it is not necessarily the average of these two numbers.

EXAMPLE 5 Find the average value of 2^x over the interval $[1,3]$.
SOLUTION The average value of 2^x over $[1,3]$ by definition equals

$$\frac{1}{3-1}\int_1^3 2^x\,dx.$$

First, by Example 3 in Section 6.2,

$$\int_1^3 2^x\,dx = \frac{1}{\ln(2)}(2^3 - 2^1) = \frac{6}{\ln(2)}.$$

Hence,

$$\text{average value of } 2^x \text{ over } [1,3] = \frac{1}{3-1}\frac{6}{\ln(2)} = \frac{3}{\ln(2)} \approx 4.2381.$$

◇

The average of the maximum and minimum values of 2^x on $[1,3]$ is $\frac{1}{2}(2^3 + 2^1) = 5$. It's not the same as the average value.

WARNING *(Antiderivative Terminology)* As mentioned earlier, in the real world an antiderivative is most often called an "integral" or "indefinite integral". If you stay alert, the context will always reveal whether the word "integral" refers to an antiderivative (a function) or to a definite integral (a number). They are two wildly different beasts. Even so, the next section will show that there is a very close connection between them. This connection ties the two halves of calculus — differential calculus and integral calculus — into one neat package.

Summary

We introduced the notation $\int f(x)\ dx$ for an **antiderivative** of $f(x)$. Using this notation we stated several properties of antiderivatives.

We defined the symbol $\int_a^b f(x)\ dx$ in the special case when $b \leq a$, and stated various properties of definite integrals.

The mean-value theorem for definite integrals asserts that for a continuous function $f(x)$, $\int_a^b f(x)\ dx$ equals $f(c)$ times $(b-a)$ for at least one value of c in $[a, b]$.

The quantity $\frac{1}{b-a}\int_a^b f(x)\ dx$ is called the **average value** (or **mean value**) of $f(x)$ over $[a, b]$. It can be thought of as the height of the rectangle whose area is the same as the area of the region under the curve $y = f(x)$.

EXERCISES for 6.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 12 evaluate each antiderivative. Remember to add a constant to each answer. Check each answer by differentiating it.

1.[R] $\int 5x^2\ dx$

2.[R] $\int \left(7/x^2\right)\ dx$

3.[R] $\int (2x - x^3 + x^5)\ dx$

4.[R] $\int \left(6x^2 + 2x^{-1} + \frac{1}{\sqrt{x}}\right)\ dx$

5.[R]

(a) $\int e^x\ dx$

(b) $\int e^{x/3}\ dx$

6.[R]

(a) $\int \frac{1}{1+x^2}\ dx$

(b) $\int \frac{1}{\sqrt{1-x^2}}\ dx$

7.[R]

(a) $\int \cos(x)\ dx$

(b) $\int \cos(2x)\ dx$

8.[R]

(a) $\int \sin(x)\ dx$

(b) $\int \sin(3x)\ dx$

9.[R]

(a) $\int (2\sin(x) + 3\cos(x))\ dx$

(b) $\int (\sin(2x) + \cos(3x))\ dx$

10.[R] $\int \sec(x)\tan(x)\ dx$

11.[R] $\int (\sec(x))^2\ dx$

12.[R] $\int (\csc(x))^2\ dx$

13.[R] State the mean-value theorem for definite integrals in words, using no mathematical symbols.

14.[R] Define the average value of a function over an interval, using no mathematical symbols.

15.[R] Evaluate

(a) $\int_2^5 x^2\ dx$

(b) $\int_5^2 x^2\ dx$

(c) $\int_5^5 x^2\ dx$

16.[R] Evaluate

(a) $\int_1^2 x\ dx$

(b) $\int_2^1 x\ dx$

(c) $\int_3^3 x\ dx$

17.[R] Find

(a) $\int x\ dx$

(b) $\int_3^4 x\ dx$

18.[R] Find

(a) $\int 3x^2\ dx$

(b) $\int_1^4 3x^2\ dx$

19.[R] If $2 \leq f(x) \leq 3$, what can be said about $\int_1^6 f(x)\ dx$?

20.[R] If $-1 \leq f(x) \leq 4$, what can be said about $\int_{-2}^7 f(x)\ dx$?

21.[R] Write a sentence or two, in your own words, that tells what the symbols $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$ mean. Include examples. Use as few mathematical symbols as possible.

22.[R] Let $f(x)$ be a differentiable function. In this exercise you will determine if the following equation is true or false:

$$f(x) = \int \frac{df}{dx}(x)\,dx.$$

(a) Pick several functions of your choice and test if the equation is true.

(b) Determine if the equation is always true. Write a brief justification for your answer. HINT: Read the equation out loud.

The mean-value theorem for definite integrals asserts that if $f(x)$ is continuous throughout the interval with endpoints a and b, then $\int_a^b f(x)\,dx = f(c)(b-a)$ for some number c in $[a,b]$. In each of Exercises 23 to 26 find $f(c)$ and at least one value of c in $[a,b]$.

23.[R] $f(x) = 2x$; $[a,b] = [1,5]$

24.[R] $f(x) = 5x + 2$; $[a,b] = [1,2]$

25.[R] $f(x) = x^2$; $[a,b] = [0,4]$

26.[R] $f(x) = x^2 + x$; $[a,b] = [1,4]$

27.[R] If $\int_1^2 f(x)\,dx = 3$ and $\int_1^5 f(x)\,dx = 7$, find

(a) $\int_2^1 f(x)\,dx$

(b) $\int_2^5 f(x)\,dx$

28.[R] If $\int_1^3 f(x)\,dx = 4$ and $\int_1^3 g(x)\,dx = 5$, find

(a) $\int_1^3 (2f(x) + 6g(x))\,dx$

(b) $\int_3^1 (f(x) - g(x))\,dx$

29.[R] If the maximum value of $f(x)$ on $[a,b]$ is 7 and the minimum value on $[a,b]$ is 4, what can be said about

(a) $\int_a^b f(x)\,dx$?

(b) the mean value of $f(x)$ on $[a,b]$?

30.[R] Let $f(x) = c$ (constant) for all x in $[a,b]$. Find the average value of $f(x)$ on $[a,b]$.

Exercises 31 to 34 concern the average of a function over an interval. In each case, find the minimum, maximum, and average value of the function over the given interval.

31.[R] $f(x) = x^2$, $[2,3]$

32.[R] $f(x) = x^2$, $[0,5]$

33.[R] $f(x) = 2^x$, $[0,4]$

34.[R] $f(x) = 2^x$, $[2,4]$

35.[M] Prove the mean-value theorem for definite integrals in the case when $b < a$. HINT: Use the definition of $\int_a^b f(x)\,dx$ when $b < a$.

36.[M] Is $\int f(x)g(x)\,dx$ always equal to $\int f(x)\,dx \int g(x)\,dx$? Are they ever equal? (Explain.)

37.[M]

(a) Show that $\frac{1}{3}(\sin(x))^3$ is *not* an antiderivative of $\sin(x))^2$.

(b) Use the identity $(\sin(x))^2 = \frac{1}{2}(1-\cos(2x))$ to find an antiderivative of $\sin(x))^2$.

(c) Verify your answer in (b) by differentiation.

In Exercises 38 and 39 verify the equations quoted from a table of antiderivatives (integrals). Just differentiate each of the alleged antiderivatives and see whether you obtain the quoted integrand. (The number a is a constant in each case.)

38.[M] $\int x^2 \sin(ax)\,dx = \frac{2x}{a^2}\sin(ax) + \frac{2}{a^3}\cos(ax) - \frac{x^2}{a}\cos(ax) + C$

39.[M] $\int x(\sin(ax))^2\,dx = \frac{x^2}{4} - \frac{x}{4a}\sin(2ax) - \frac{1}{8a^2}\cos(2ax) + C$

40.[M] Define $f(x) = \begin{cases} -x & 0 < x \leq 1 \\ -1 & 1 < x \leq 2 \\ 1 & 2 < x \leq 3 \\ 4 - x & 3 < x \leq 4 \end{cases}$.

(a) Sketch the graphs of $y = f(x)$ and $y = (f(x))^2$ on the interval $[0, 4]$.

(b) Find the average value of f on the interval $[0, 4]$.

(c) The **root mean square** (RMS) of a function f on $[a, b]$ is defined as $\sqrt{\frac{1}{b-a} \int_a^b f(x)^2 \, dx}$. (The voltage, e.g., 110 volts, for an alternating electric current is the root mean square of a varying voltage.) Find the "root mean square" value of f on the interval $[0, 4]$. That is, compute $\sqrt{\frac{1}{4-0} \int_0^4 (f(x))^2 \, dx}$.

(d) Why is it not surprising that your answer in (b) is zero and your answer in (c) is positive?

41.[M]

Sam: The text makes the average value of a function on $[a, b]$ too hard.

Jane: How so?

Sam: It's easy. Just average $f(a)$ and $f(b)$.

Jane: That sure is easier.

(a) Show that Sam is correct when $f(x)$ is any polynomial of degree 0 or 1.

(b) Is Sam always correct? Explain.

Exercise 42 describes the famous **Buffon neeedle** problem, now over 200 years old. Exercise 43 is related, but not nearly as famous.

42.[M] On the floor there are parallel lines a distance d from each other, such as the edges of slats. You throw a straight wire of length d on the floor at random. Sometimes it ends up crossing a line, sometimes it avoids a line.

(a) Perform the experiment at least 20 times and use the results to estimate the percentage of times the wire crosses a line.

(b) If the wire makes an angle θ with a line perpendicular to the lines, show that the probability that it crosses a line is $\cos(\theta)$.

(c) Find the average value of that probability. That average is the probability that the wire crosses a line.

(d) How close is the experimental value in (a) to the theoretical value in (c)?

43.[C] An infinite floor is composed of congruent square tiles arranged as in a checkerboard. You have a straight wire whose length is the same as the length of a side of a square. The edges of the squares form lines in perpendicular directions. What is the probability that when you throw the wire at random it crosses two lines, one in each of the two perpendicular directions? (This is related to Exercise 42, the classic Buffon needle problem.) NOTE: You can check if your answer is reasonable by carrying out the experiment.

44.[C] The average value of a certain function $f(x)$ on $[1, 3]$ is 4. On $[3, 6]$ the average value of the same function is 5. What is its average value on $[1, 6]$? (Explain your answer.)

45.[C] This exercise evaluates two definite integrals that appear often in applications.

(a) Draw the graphs of $y = (\cos(x))^2$ and $y = (\sin(x))^2$. On the basis of your picture, decide how $\int_0^{\pi/2} (\cos(x))^2 \, dx$ and $\int_0^{\pi/2} (\sin(x))^2 \, dx$ compare.

(b) Using (a) and a trigonometric identity, show that

$$\int_0^{\pi/2} (\cos(x))^2 \, dx = \frac{\pi}{4} = \int_0^{\pi/2} (\sin(x))^2 \, dx.$$

(c) Evaluate $\int_0^{\pi} (\cos(x))^2 \, dx$.

6.4 The Fundamental Theorem of Calculus

Introduction and Motivation

In this section we obtain two closely related theorems. They are called the Fundamental Theorems of Calculus I and II, or simply The Fundamental Theorem of Calculus (FTC). The first part of the FTC provides a way to evaluate a definite integral if you are lucky enough to know an antiderivative of the integrand. That means that the derivative, developed in Chapter 3, has yet another application.

This is the most important section of the entire book.

FTC I gives a shortcut to evaluating $\int_a^b f(x)\,dx$

FTC II gives a way to evaluate $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right)$

The second fundamental theorem tells how rapidly the value of a definite integral changes as you change the interval $[a, b]$ over which you are integrating. This part of the Fundamental Theorem is used to prove the first part of the FTC.

Motivation for the Fundamental Theorem of Calculus I

In Section 6.2 we found that $\int_a^b c\,dx = cb - ca$ and $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$. In the same section we found that $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$; in this case our reasoning was based, on the fact that congruent lopsided tents fill a cube. Finally, using the formula for the sum of a geometric series, we showed that $\int_a^b 2^x\,dx = \frac{2^b}{\ln(2)} - \frac{2^a}{\ln(2)}$.

Notice that all four results follow a similar pattern:

$$\begin{array}{rclcrcl} \int_a^b c\,dx &=& cb - ca & \quad & \int_a^b x\,dx &=& \frac{b^2}{2} - \frac{a^2}{2} \\ \int_a^b x^2\,dx &=& \frac{b^3}{3} - \frac{a^3}{3} & & \int_a^b 2^x\,dx &=& \frac{2^b}{\ln(2)} - \frac{2^a}{\ln(2)} \end{array}$$

To describe the similarity in detail, compute an antiderivative of each of the four integrands:

We omit "$+C$" since only one antiderivative is needed here. See Exercises 40 and 41.

$$\begin{array}{rclcrcl} \int c\,dx &=& cx & \quad & \int x\,dx &=& \frac{x^2}{2} \\ \int x^2\,dx &=& \frac{x^3}{3} & & \int 2^x\,dx &=& \frac{2^x}{\ln(2)}. \end{array}$$

In each case *the definite integral equals the difference between the values of an antiderivative of the integrand evaluated at b and at a*, the endpoints of the interval.

This suggests that *maybe* for any integrand $f(x)$, the following may be true: If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{6.4.1}$$

If this is correct, then, instead of resorting to special tricks to evaluate a definite integral, such as cutting up a cube or summing a geometric series, we should look for an antiderivative of the integrand.

We may reason using "velocity and distance" to provide further evidence for (6.4.1). Picture a particle moving upwards on the y-axis. At time t it is at position $F(t)$ on that line. The velocity at time t is $F'(t)$.

But we saw that the definite integral of the velocity from time a to time b tells the change in position, that is,

"the definite integral of the velocity = the final position − the initial position"

In symbols,

$$\int_a^b F'(t)\, dt = F(b) - F(a). \tag{6.4.2}$$

If we give $F'(t)$ the name $f(t)$, then we can restate (6.4.2) as:

$$\text{If } f(t) = F'(t), \text{ then } \int_a^b f(t)\, dt = F(b) - F(a).$$

In other words,

$$\text{If } F \text{ is an antiderivative of } f, \text{ then } \int_a^b f(t)\, dt = F(b) - F(a).$$

Formulas we found for the integrands c, x, x^2, and 2^x and reasoning about motion are all consistent with

FTC I

Theorem 6.4.1 (Fundamental Theorem of Calculus I)**.**
If f is continuous on $[a, b]$ and if F is an antiderivative of f then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

In practical terms this theorem says, "To evaluate the definite integral of f from a to b, look for an antiderivative of f. Evaluate the antiderivative at b and subtract its value at a. This difference is the value of the definite integral you are seeking". The success of this approach hinges on finding an antiderivative of the integrand f. For many functions, it is easy to find an antiderivative. For some it is hard, but they can be found. For others, the antiderivatives cannot be expressed in terms of the functions met in Chapters 2 and 3, namely polynomials, quotients of polynomials, and functions built up from trigonometric, exponential, and logarithm functions and their inverses.

Example 1 shows the power of FTC I.

Some techniques for finding antiderivatives are discussed in Chapter 7.

EXAMPLE 1 Use the Fundamental Theorem of Calculus to evaluate $\int_0^{\pi/2} \cos(x)\, dx$.

SOLUTION Since $(\sin(x))' = \cos(x)$, $\sin(x)$ is an antiderivative of $\cos(x)$. By FTC I,

$$\int_0^{\pi/2} \cos(x)\, dx \quad = \quad \sin(\frac{\pi}{2}) - \sin(0) = 1 - 0 = 1.$$

This tells us that the area under the curve $y = \cos(x)$ and above $[0, \pi/2]$, shown in Figure 6.4.1 is 1.

This result is reasonable since the area lies inside a rectangle of area $1 \times \frac{\pi}{2} = \frac{\pi}{2} \approx 1.5708$ and contains a triangle of area $\frac{1}{2}\left(\frac{\pi}{2}\right)1 = \frac{\pi}{4} \approx 0.7854$. ◇ How would the evaluation be different if we used $\sin(x) + 5$ as the antiderivative of $\cos(x)$?

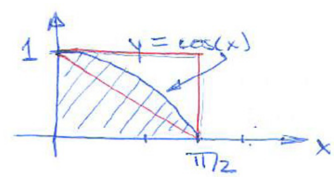
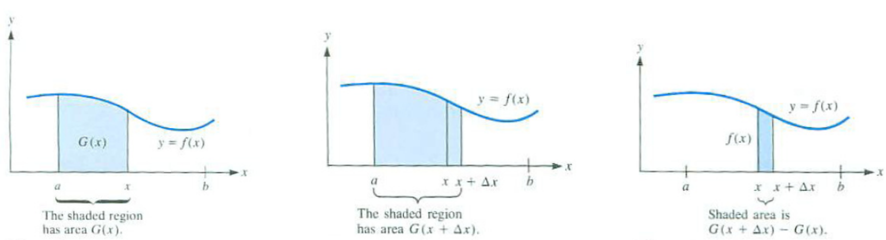


Figure 6.4.1:

Motivation for the Fundamental Theorem of Calculus II

Let f be a continuous function such that $f(x)$ is positive for x in $[a, b]$. For x in $[a, b]$, let $G(x)$ be the area of the region under the graph of f and above the interval $[a, x]$, as shown in Figure 6.4.2(a). In particular, $G(a) = 0$.



(a) (b) (c)

Figure 6.4.2:

We will compute the derivative of $G(x)$, that is,

$$G'(x) = \lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x} = \lim_{\Delta x \to 0} \frac{G(x + \Delta x) - G(x)}{\Delta x}.$$

(This is one of several occasions when we must go back to the definition of the derivative as a limit.) For simplicity, keep Δx positive. Then $G(x + \Delta x)$ is the area under the curve $y = f(x)$ above the interval $[a, x + \Delta x]$. If Δx is small, $G(x+\Delta x)$ is only slightly larger than $G(x)$, as shown in Figure 6.4.2(b). Then $\Delta G = G(x+\Delta x) - G(x)$ is the area of the thin shaded strip in Figure 6.4.2(c).

When Δx is small, the narrow shaded strip above $[x, x + \Delta x]$ resembles a rectangle of base Δx and height $f(x)$, with area $f(x)\Delta x$. Therefore, it seems reasonable that when Δx is small,

$$\frac{\Delta G}{\Delta x} \approx \frac{f(x)\Delta x}{\Delta x} = f(x).$$

In short, it seems plausible that

$$\lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x} = f(x).$$

Briefly,

$$G'(x) = f(x).$$

In words, "the derivative of the area of the region under the graph of f and above $[a, x]$ with respect to x is the value of f at x".

Now we state these observations in terms of definite integrals.

We use t in the integrand to avoid using x to denote both an end of the interval and a variable that takes values between a and x.

Let f be a continuous function. Let $G(x) = \int_a^x f(t)\, dt$. Then we expect that

$$\frac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x).$$

This equation says that "the derivative of the definite integral of f with respect to the right end of the interval is simply f evaluated at that end". This is the substance of the Fundamental Theorem of Calculus II. It tells how rapidly the definite integral changes as we change the upper limit of integration, b.

FTC II

Theorem 6.4.2 (Fundamental Theorem of Calculus II)**.**
Let f be continuous on the interval $[a, b]$. Define

$$G(x) = \int_a^x f(t)\, dt \qquad \text{for all } a \le x \le b.$$

Then G is differentiable on $[a, b]$ and its derivative is f; that is,

$$G'(x) = f(x).$$

As a consequence of FTC II, *every continuous function is the derivative of some function*.

See Exercise 63.

There is a similar theorem for $H(x) = \int_x^b f(t)\,dt$: $H'(x) = -f(x)$. A glance at Figure 6.4.3 shows why there is a minus sign: the area in this figure shrinks as x increases.

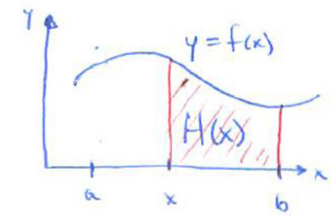
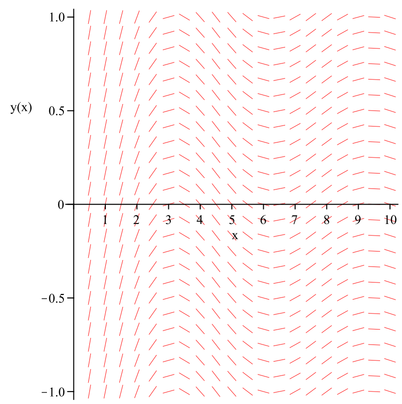


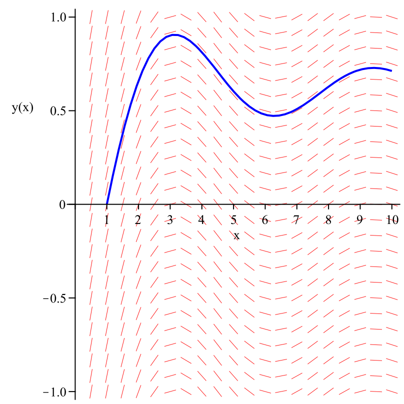
Figure 6.4.3:

EXAMPLE 2 Give an example of an antiderivative of $\frac{\sin(x)}{x}$.

SOLUTION There are many antiderivatives of $\frac{\sin(x)}{x}$. Any two antiderivatives differ by a constant. These curves can be seen in the slope field for $y' = \frac{\sin(x)}{x}$ shown in Figure 6.4.4 (a).



(a)



(b)

Figure 6.4.4: (a) slope field for $y' = \frac{\sin(x)}{x}$ and (b) same slope field with solution with $y'(1) = \sin(1)$

Let $G(x) = \int_1^x \frac{\sin(t)}{t}\,dt$. By FTC II, $G'(x) = \frac{\sin(x)}{x}$. The graph of $y = G(x)$ is shown in Figure 6.4.4 (b). Notice that $G(1) = 0$. ◇

You probably expected the answer in Example 2 to be an explicit formula for the antiderivative expressed in terms of the familiar functions discussed in Chapters 2 and 3. Recall, from Section 3.6, that the derivative of every elementary function is an elementary function. Liouville proved that there are (many) elementary functions that do not have elementary antiderivatives. Nobody will ever find an explicit formula in terms of elementary functions for an antiderivative of $\frac{\sin(x)}{x}$. (The proof is reserved for a graduate course.)

Joseph Liouville (1809–1882) `http://en.wikipedia.org/wiki/Joseph_Liouville`

EXAMPLE 3 Give an example of an antiderivative of $\frac{\sin(\sqrt{x})}{\sqrt{x}}$.

SOLUTION This integrand appears more terrifying than $\frac{\sin(x)}{x}$, yet it does have an elementary antiderivative, namely $-2\cos(\sqrt{x})$. To check, we differentiate $y = -2\cos(\sqrt{x})$ by the Chain Rule. We have $y = -2\cos(u)$ where

$u = \sqrt{x}$. Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -2(-\sin(u))\frac{1}{2\sqrt{x}} = \frac{\sin(\sqrt{x})}{\sqrt{x}}.$$

◇

Because the antiderivatives of $\frac{\sin(\sqrt{x})}{\sqrt{x}}$ are elementary functions, it would be easy to calculate $\int_1^2 \frac{\sin(\sqrt{x}}{\sqrt{x}}\,dx$.

More generally, if $H(t) = \int_a^t f(x)\,dx$, then $H'(t) = f(t)$. See also Exercise 63 .

Any antiderivative of e^x is of the form $e^x + C$, an elementary function. However, *no* antiderivative of e^{-x^2} is elementary. Statisticians define the **error function** to be $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2/2}\,dt$. Except that $\mathrm{erf}(0) = 0$, there is no easy way to evaluate $\mathrm{erf}(x)$. Since $\mathrm{erf}(x)$ is not elementary, it is customary to collect approximate values of it for various values of x in a table. Approximate values of special functions such as the error function can also be obtained from mathematical software and even a few calculators.

Net Area

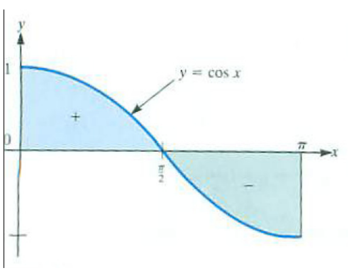


Figure 6.4.5: The area of a region below the x-axis is negative.

When we evaluate $\int_0^\pi \cos(x)\,dx$, we obtain $\sin(\pi) - \sin(0) = 0 - 0 = 0$. What does this say about areas? Inspection of Figure 6.4.5 shows what is happening.

For x in $[\pi/2, \pi]$, $\cos(x)$ is negative and the curve $y = \cos(x)$ lies *below* the x-axis. If we interpret the corresponding area as negative, then we see that it cancels with the area from 0 to $\pi/2$. Let us agree that when we say "$\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$", we mean that it represents the area between the curve and the x-axis, *with area below the x-axis taken as negative*, that is the net area.

EXAMPLE 4 Evaluate $\int_1^2 \frac{1}{x^2}\,dx$ by the Fundamental Theorem of Calculus I.

SOLUTION In order to apply FTC I we have to find an antiderivative of $\frac{1}{x^2}$. In Section 6.3 it was observed that

$$\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C \qquad a \neq -1.$$

In particular, with $a = -2$,

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{1}{(-2)+1}x^{(-2)+1} + C = \frac{1}{-1}x^{-1} + C = \frac{-1}{x} + C$$

By FTC I

$$\int\limits_1^2 \frac{1}{x^2}\,dx = \left(\frac{-1}{x} + C\right)\Big|_1^2 = \left(\frac{-1}{2} + C\right) - \left(\frac{-1}{1} + C\right) = \frac{-1}{2} - (-1) = \frac{1}{2}.$$

Note that the C's cancel. We do not need the C when applying FTC I.

◇

The First Fundamental Theorem of Calculus asserts that

$$\underbrace{\int_1^2 \frac{1}{x^2}\,dx}_{\text{The definite integral: a limit of sums}} = \underbrace{\left.\int \frac{1}{x^2}\,dx\right|_1^2}_{\text{The difference between an antiderivative evaluated at 2 and at 1}}$$

The symbols on the right and left of the equal sign are so similar that it is tempting to think that the equation is obvious or says nothing whatsoever.

> **WARNING** *(Notation)* This equation is a special instance of the First Fundamental Theorem of Calculus, FTC I.
>
> **Remark:** Often we write $\int \frac{1}{x^2}\,dx$ as $\int \frac{dx}{x^2}$, merging the 1 with the dx. More generally, $\int \frac{f(x)}{g(x)}\,dx$ may be written as $\int \frac{f(x)\,dx}{g(x)}$.

Some Terms and Notation

The related processes of computing $\int_a^b f(x)\,dx$ and of finding an antiderivative $\int f(x)\,dx$ are both called **integrating** $f(x)$. Thus integration refers to two separate but related problems: computing a number $\int_a^b f(x)\,dx$ or finding a function $\int f(x)\,dx$.

In practice, both FTC I and FTC II are called "the Fundamental Theorem of Calculus." The context always makes it clear which one is meant.

Proofs of the Two Fundamental Theorems of Calculus

We now prove both parts of the Fundamental Theorem of Calculus — without referring to motion, area, or concrete examples. The proofs use only the mathematics of functions and limits. We prove FTC II first; then we will use it to prove FTC I.

Proof of the Second Fundamental Theorem of Calculus

The Second Fundamental Theorem of Calculus asserts that the derivative of $G(x) = \int_a^x f(t)\,dt$ is $f(x)$. We gave a convincing argument using areas of regions. However, since definite integrals are defined in terms of approximating sums, not areas, we include a proof that uses only properties of definite integrals.

Proof of Fundamental Theorem of Calculus II

We wish to show that $G'(x) = f(x)$. To do this we must make use of the definition of the derivative of a function.

We have

$$
\begin{aligned}
G'(x) &= \lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} && \text{(definition of derivative)} \\
&= \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)\ dt - \int_a^x f(t)\ dt}{\Delta x} && \text{(definition of } G\text{)} \\
&= \lim_{\Delta x \to 0} \frac{\int_a^{x} f(t)\ dt + \int_x^{x+\Delta x} f(t)\ dt - \int_a^x f(t)\ dt}{\Delta x} && \text{(property 6 in Section 6.} \\
&= \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)\ dt}{\Delta x} && \text{(canceling)} \\
&= \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x} && \text{(MVT for Definite Integ} \\
& && \text{tween } x \text{ and } x+\Delta x\text{)} \\
&= \lim_{\Delta x \to 0} f(c) && \text{(canceling)} \\
&= f(x). && \text{(continuity of } f\text{; } c \to x \text{ a}
\end{aligned}
$$

Hence

$$G'(x) = f(x),$$

which is what we set out to prove. •

A similar argument shows that

$$\frac{d}{dx} \int_x^b f(t)\ dt = -f(x).$$

For integrands whose values are positive, the minus sign is to be expected. As x increases, the interval shrinks, and so the (positive) area under the curve shrinks as well.

Proof of the First Fundamental Theorem of Calculus

The First Fundamental Theorem of Calculus asserts that if $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$. We persuaded ourselves that this is true by thinking of f as "velocity" and F as "position", and also by four special cases ($f(x) = c$, $f(x) = x$, $f(x) = x^2$, and $f(x) = 2^x$). We now prove the theorem, which is an immediate consequence of the Second Fundamental Theorem of Calculus and the fact that two antiderivatives of the same function differ by a constant.

Proof of the Fundamental Theorem of Calculus I

We are assuming that $F' = f$ and wish to show that $F(b) - F(a) = \int_a^b f(x)\,dx$. Define $G(x)$ to be $\int_a^x f(t)\,dt$. By FTC II, G is an antiderivative of f. Since F and G are both antiderivatives of f, they differ by a constant, say C. That is,

$$F(x) = G(x) + C.$$

Thus,

$$\begin{aligned}
F(b) - F(a) &= (G(b) + C) - (G(a) + C) && \\
&= G(b) - G(a) && (C\text{'s cancel}) \\
&= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt && (\text{definition of } G) \\
&= \int_a^b f(t)\,dt && \left(\int_a^a f(t)\,dt = 0\right)
\end{aligned}$$

•

Summary

This section links the two basic ideas of calculus, the derivative (more precisely, the antiderivative) and the definite integral.

FTC I says that if you can find a formula for an antiderivative F of f, then you can evaluate $\int_a^b f(x)\,dx$:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

FTC II says that if f is continuous then it has an antiderivative, namely $G(x) = \int_a^x f(t)\,dt$; that is $G'(x) = f(x)$. Unfortunately, G might not be an elementary function. However, a reasonable graph of an antiderivative of f can be obtained from the slope field for $\frac{dy}{dx} = f(x)$.

EXERCISES for 6.4 *Key:* R–routine, M–moderate, C–challenging

1.[R] State (a) FTC I and (b) FTC II.

2.[R] Using only words, no mathematical symbols, state the First Fundamental Theorem of Calculus.

3.[R] Using only words, no mathematical symbols, state the Second Fundamental Theorem of Calculus.

In Exercises 4 and 5 evaluate the given expressions.
4.[R]

(a) $x^3\big|_1^2$

(b) $x^2\big|_{-1}^2$

(c) $\cos(x)\big|_0^\pi$

5.[R]

(a) $(x+\sec(x))\big|_0^{\pi/4}$

(b) $\left.\dfrac{1}{x}\right|_2^3$

(c) $\sqrt{x-1}\big|_5^{10}$

In Exercises 6 to 19 use FTC I to evaluate the given definite integrals.

6.[R] $\displaystyle\int_1^2 5x^3\,dx$

7.[R] $\displaystyle\int_{-1}^3 2x^4\,dx$

8.[R] $\displaystyle\int_1^4 (x+5x^2)\,dx$

9.[R] $\displaystyle\int_1^2 (6x-3x^2)\,dx$

10.[R] $\int_{\pi/6}^{\pi/3} 5\cos(x)\ dx$

11.[R] $\int_{\pi/4}^{3\pi/4} 3\sin(x)\ dx$

12.[R] $\int_{0}^{\pi/2} \sin(2x)\ dx$

13.[R] $\int_{0}^{\pi/6} \cos(3x)\ dx$

14.[R] $\int_{4}^{9} 5\sqrt{x}\ dx$

15.[R] $\int_{1}^{9} \frac{1}{\sqrt{x}}\ dx$

16.[R] $\int_{1}^{8} \sqrt[3]{x^2}\ dx$

17.[R] $\int_{2}^{4} \frac{4}{x^3}\ dx$

18.[R] $\int_{0}^{1} \frac{dx}{1+x^2}$

19.[R] $\int_{1/4}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

In Exercises 20 to 25 find the average value of the given function over the given interval.

20.[R] x^2; $[3,5]$

21.[R] x^4; $[1,2]$

22.[R] $\sin(x)$; $[0,\pi]$

23.[R] $\cos(x)$; $[0,\pi/2]$

24.[R] $(\sec(x))^2$; $[\pi/6,\pi/4]$

25.[R] $\sec(2x)\tan(2x)$; $[\pi/8,\pi/6]$

In Exercises 26 to 33 evaluate the given quantities.

26.[R] The area of the region under the curve $y = 3x^2$ and above $[1, 4]$.

27.[R] The area of the region under the curve $y = 1/x^2$ and above $[2, 3]$.

28.[R] The area of the region under the curve $y = 6x^4$ and above $[-1, 1]$.

29.[R] The area of the region under the curve $y = \sqrt{x}$ and above $[25, 36]$.

30.[R] The distance an object travels from time $t = 1$ second to time $t = 2$ seconds, if its velocity at time t seconds is t^5 feet per second.

31.[R] The distance an object travels from time $t = 1$ second to time $t = 8$ seconds, if its velocity at time t seconds is $7\sqrt[3]{t}$ feet per second.

32.[R] The volume of a solid located between a plane at $x = 1$ and a plane located at $x = 5$ if the cross-sectional area of the intersection of the solid with the plane perpendicular to the x-axis through the point $(x, 0)$ has area $6x^3$ square centimeters. (See Figure 6.4.6.)

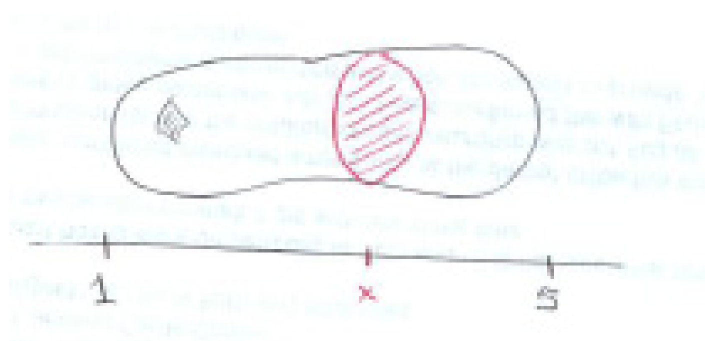


Figure 6.4.6:

33.[R] The volume of a solid located between a plane at $x = 1$ and a plane located at $x = 5$ if the cross-sectional area of the intersection of the solid with the plane perpendicular to the x-axis through the point $(x, 0)$ has area $1/x^3$ square centimeters.

34.[R] Let f be a continuous function. Estimate $f(7)$ if $\int_5^7 f(x)dx = 20.4$ and $\int_5^{7.05} f(x)dx = 20.53$.

35.[R] Determine if each of the following expressions is a function or a number.

(a) $\int x^2\ dx$

(b) $\int x^2\ dx\Big|_1^3$

(c) $\int_1^3 x^2\ dx$

36.[R]

(a) Which of these two numbers is defined as a limit of sums?

$$\int x^2\,dx\Big|_1^2 \qquad \text{and} \qquad \int_1^2 x^2\,dx$$

(b) How is the other number defined?

(c) Why are the two numbers in (a) equal?

37.[R] There is no elementary antiderivative of $\sin(x^2)$. Does $\sin(x^2)$ have an antiderivative? Explain.

38.[R] True or false:

(a) Every elementary function has an elementary derivative.

(b) Every elementary function has an elementary antiderivative.

Explain.

39.[R]

(a) Draw the slope field for $\frac{dy}{dx} = \frac{e^{-x}}{x}$ for $x > 0$.

(b) Use (a) to sketch the graph of an antiderivative of $\frac{e^{-x}}{x}$.

(c) On the slope field drawn in (a), sketch the graph of $f(x) = \int_1^x \frac{e^{-t}}{t}\,dt$. (For which one value of x is $f(x)$ easy to compute?)

Exercises 40 and 41 illustrate why FTC I can be applied using any antiderivative of the integrand.

40.[R] Evaluate the definite integral $\int_a^b x\ dx$ using each of the following antiderivatives of $f(x) = x$.

(a) $F(x) = \frac{1}{2}x^2 + 1$.

(b) $F(x) = \frac{1}{2}x^2 - 3$.

(c) $F(x) = \frac{1}{2}x^2 + C$.

41.[R] Evaluate the definite integral $\int_a^b 2^x\ dx$ using each of the following antiderivatives of $f(x) = 2^x$.

(a) $F(x) = \frac{1}{\ln(2)}2^x + 11$.

(b) $F(x) = \frac{1}{\ln(2)}2^x - 7$.

(c) $F(x) = \frac{1}{\ln(2)}2^x + C$.

42.[M] Let $F(x) = \int_0^x e^{t^2}\ dt$.

(a) Does the graph of $F(x)$ have inflection points? If so, find them.

(b) Make a rough sketch of the graph of $F(x)$.

43.[M] Area was used in Section 6.2 to develop $\int_a^b x\ dx = \frac{b^2}{2} - \frac{a^2}{2}$ when $0 < a < b$. To see that this result is true for all values of a and b (with $b > a$) we will consider these additional cases:

(a) If $a < b < 0$, work with negative area.

(b) If $a < 0 < b$, divide the interval $[a, b]$ into two pieces and work with signed areas.

44.[M] Find $\frac{dy}{dx}$ if

(a) $y = \int \sin(x^2)\ dx$

(b) $y = 3x + \int_{-2}^{3} \sin(x^2)\ dx$

(c) $y = \int_{-2}^{x} \sin(t^2)\ dt$

In Exercises 45 to 48 differentiate the given functions.

45.[M]

(a) $\int_1^x t^4\ dt$

(b) $\int_x^1 t^4\ dt$ HINT: Re-write this integral with x as the upper limit of integration.

46.[M]

(a) $\int_1^x \sqrt[3]{1+\sin(t)}\ dt$

(b) $\int_1^{x^2} \sqrt[3]{1+\sin(t)}\ dt$ HINT: Use the Chain Rule.

47.[M] $\int_{-1}^x 3^{-t}\ dt$

48.[M] $\int_{2x}^{3x} t\tan(t)\ dt$ (Assume x is in the interval $(-\pi/6, \pi/6)$.) HINT: First rewrite the integral as $\int_{2x}^0 t\tan(t)\ dt + \int_0^{3x} t\tan(t)\ dt$.

49.[M] Figure 6.4.7(a) shows the graph of a function $f(x)$ for x in $[1,3]$. Let $G(x) = \int_1^x f(t)\ dt$. Graph $y = G(x)$ for x in $[1,3]$ as well as you can. Explain your reasoning.

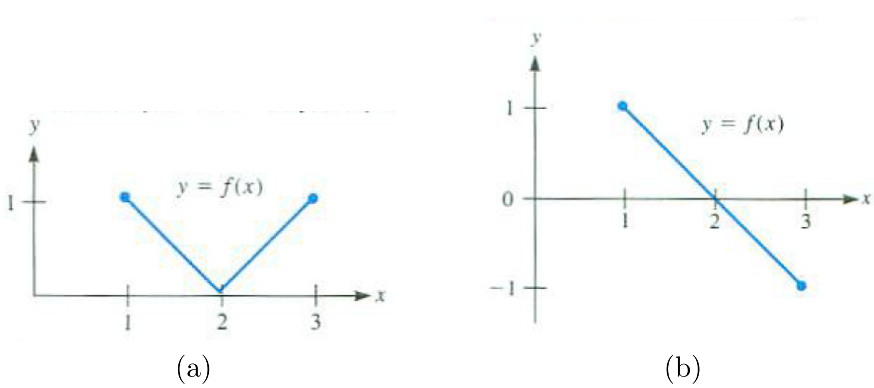


Figure 6.4.7:

50.[M] Figure 6.4.7(b) shows the graph of a function $f(x)$ for x in $[1,3]$. Let $G(x) = \int_1^x f(t)\ dt$. Graph $y = G(x)$ for x in $[1,3]$ as well as you can. Explain your reasoning.

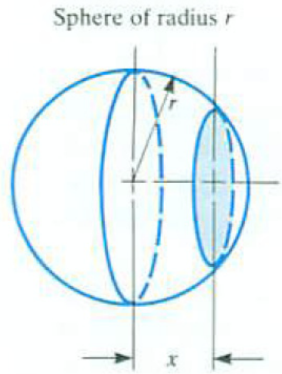


Figure 6.4.8: ARTIST: Change “Sphere” to “Ball”

51.[M] A plane at a distance x from the center of the ball of radius r, $0 \leq x \leq 4$, meets the ball in a disk. (See Figure 6.4.8.)

(a) Show that the radius of the disk is $\sqrt{r^2 - x^2}$.

(b) Show that the area of the disk is $\pi r^2 - \pi x^2$.

(c) Using the FTC, find the volume of the ball.

52.[M] Let $v(t)$ be the velocity at time t of an object moving on a straight line. The velocity may be positive or negative.

(a) What is the physical meaning of $\int_a^b v(t)\ dt$? Explain.

(b) What is the physical meaning of the slope of the graph of $y = v(t)$? Explain.

(c) What is the physical meaning of $\int_a^b |v(t)|\ dt$? Explain.

53.[M] Give an example of a function f such that $f(4) = 0$ and $f'(x) = \sqrt[3]{1 + x^2}$.

54.[M] Let f be a continuous function. Show that $\frac{d}{dx} \int_x^b f(x)\ dx = -f(x)$

(a) by using the definition of derivative as a limit

(b) by using properties of the definite integral and FTC II.

55.[M] If $f(x) = \int_{-1}^{x} \sin^3\left(e^{t^2}\right) dt$, find $f'(1)$.

56.[M] If $\int_1^x f(t) dt = \sin^3(5x)$, find $f'(3)$.

57.[M] Figure 6.4.9 shows the graph of a function f. Let $A(x)$ be the area under the graph of f and above the interval $[1, x]$.

(a) Find $A(1)$, $A(2)$, and $A(3)$.

(b) Find $A'(1)$, $A'(2)$, and $A'(3)$.

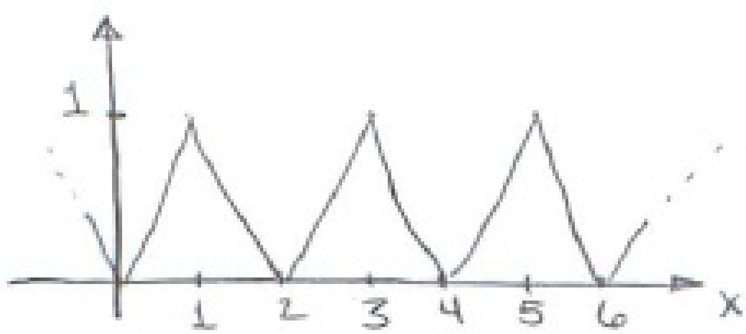


Figure 6.4.9:

58.[M]

(a) If $\int_x^{x+4} g(t)dt = 5$ for all x, what can be said about the graph of g?

(b) How would you construct such a function?

59.[M] Find $D\left(\int_{x^2}^{x^3} e^{t^2} dt\right)$.

60.[M] Find $D\left(\int_{x^2}^{5} \sin^{10}(3t)dt\right)$.

61.[M] Find the derivative of $\cos(t^2)\big|_{2x}^{3x}$.

62.[C] How often should a machine be overhauled? This depends on the rate $f(t)$ at which it depreciates and the cost A of overhaul. Denote the time between overhauls by T.

(a) Explain why you would like to minimize $g(T) = \frac{1}{T}(A + \int_0^T f(t)\ dt)$.

(b) Find $\frac{dg}{dT}$.

(c) Show that if $\frac{dg}{dT} = 0$, then $f(T) = g(T)$.

(d) Is this reasonable? Explain.

63.[C] Let $f(x)$ be a continuous function with only positive values. Define $H(x) = \int_x^b f(t)\ dt$ for all $a \le x \le b$. Let Δx be positive.

(a) Interpreting the definite integral as an area of a region, draw the regions whose areas are $H(x)$ and $H(x + \Delta x)$.

(b) Is $H(x + \Delta x) - H(x)$ positive or negative?

(c) Draw the region whose area is related to $H(x + \Delta x) - H(x)$.

(d) When Δx is small, estimate $H(x + \Delta x) - H(x)$ in terms of the integrand f.

(e) Use (d) to evaluate the derivative $H'(x)$:

$$\frac{dH}{dx} = \lim_{\Delta x \to 0} \frac{H(x + \Delta x) - H(x)}{\Delta x}.$$

64.[C] Say that you want to find the area of a certain planar cross-section of a rock. One way to find it is by sawing the rock in two and measuring the area directly. But suppose you do not want to ruin the rock. However, you do have a measuring glass, as shown in Figure 6.4.10, which gives you excellent volume measurements. How could you use the glass to get a good estimate of the cross-sectional area?

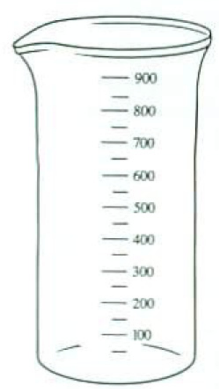


Figure 6.4.10:

65.[C] Let R be a function with continuous second derivative R''. Assume $R(1) = 2$, $R'(1) = 6$, $R(3) = 5$, and $R'(3) = 8$. Evaluate $\int_1^3 R''(x)\ dx$. NOTE: Not all of the information provided is needed.

66.[C] Two conscientious calculus students are having an argument:

Jane: $\int_a^b f(x)\,dx$ is a number.

Sam: But if I treat b as a variable, then it is a function.

Jane: How can it be both a number and a function?

Sam: It depends on what "it" means.

Jane: You can't get out of this so easily.

Which student is correct? That is, either give two interpretations of "it" or explain why "it" has only one meaning.

67.[C] The function $\frac{e^x}{x}$ does not have an elementary antiderivative. Show that its reciprocal, $\frac{x}{e^x}$, does have an elementary antiderivative. HINT: Write $\frac{x}{e^x}$ as xe^{-x} and then experiment for a few minutes.

68.[C] Show that if we knew that every continuous function has an antiderivative, then FTC I would imply FTC II.

69.[C]

(a) Show that for any constant function, $f(x) = c$, the average value of f over $[a, b]$ is the same as the value of the function at the midpoint of the interval $[a, b]$.

(b) Give an example of a non-constant function f such that for any interval $[a, b]$,

$$\frac{\int_a^b f(t)dt}{b-a} = f\left(\frac{a+b}{2}\right).$$

(c) Show that if a continuous function f on $(-\infty, \infty)$ satisfies the equation in (b), it is differentiable.

(d) Find all continuous functions that satisfy the equation in (b).

70.[C] Find all continuous functions f such that their average over $[0, t]$ always equals $f(t)$.

71.[C] Give a geometric explanation of the following properties of definite integrals:

(a) if f is an even function, then $\int_{-a}^{a} f(t)dt = 2\int_0^a f(t)dt$.

(b) if f is an odd function, then $\int_{-a}^{a} f(t)dt = 0$.

(c) if f is a periodic function with period p, then, for any integers m and n, $\int_{mp}^{np} f(t)dt = (n-m)\int_0^p f(t)dt$.

72.[C] Use FTC II to explain why, if u and v are differentiable functions,

(a) $\frac{d}{dx}\int_a^{v(x)} f(t)\ dt = f(v(x))v'(x)$

(b) $\frac{d}{dx}\int_{u(x)}^{b} f(t)\ dt = -f(u(x))u'(x)$

(c) $\frac{d}{dx}\int_{u(x)}^{v(x)}\ dt = f(v(x))v'(x) - f(u(x))u'(x)$

Hint: In (c), break the integral into two convenient integrals.

73.[C] For which continuous functions f is the average value of f on the interval $[0, b]$ a non-decreasing function of b?

6.5 Estimating a Definite Integral

It is easy to evaluate $\int_0^1 x^2\sqrt{1+x^3}\ dx$ by the Fundamental Theorem of Calculus, for the integrand has an elementary antiderivative, $\frac{2}{9}(1+x^3)^{3/2}$. (Check that $\frac{d}{dx}\frac{2}{9}(1+x^3)^{3/2}$ simplifies to $x^2\sqrt{1+x^3}$.) However, an antiderivative of $\sqrt{1+x^3}$ is not elementary, so $\int_0^1 \sqrt{1+x^3}\ dx$ cannot be evaluated so easily. In this case we have to estimate it. This section describes three ways to do this.

Approximation by Rectangles

The definite integral $\int_a^b f(x)\ dx$ is, by definition, a limit of sums of the form

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}). \qquad (6.5.1)$$

Any such sum is an estimate of $\int_a^b f(x)\ dx$.

In terms of area, the area of a rectangle gives a local estimate of the area under the graph of $y = f(x)$ above the interval $[x_{i-1}, x_i]$. See Figure 6.5.1. The sum of the areas of individual rectangles is an estimate the area under the curve.

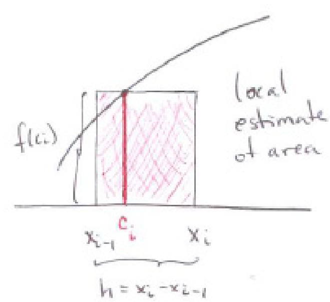


Figure 6.5.1:

To use rectangles to estimate $\int_a^b f(x)\ dx$, divide the interval $[a, b]$ into n sections of equal length by the $n+1$ numbers $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. (Choosing the sections to have the same length simplifies the arithmetic.) The width of each section is $h = (b-a)/n$. Then choose a sampling number c_i in the i^{th} section, $i = 1, 2, \ldots, n$ and form the Riemann sum $\sum_{i=1}^n f(c_i)h$. By the very definition of the definite integral, this sum is an estimate of the definite integral.

Denoting $f(x_i)$ by y_i, and using the left endpoint x_{i-1} of each interval $[x_{i-1}, x_i]$ as the sampling number, we have this **left endpoint rectangular estimate**

$$\int_a^b f(x)\ dx \approx h(y_0 + y_1 + y_2 + \cdots + y_{n-2} + y_{n-1}), \qquad (h = (b-a)/n).$$

If the right endpoints are used, we have the **right endpoint rectangular estimate**:

$$\int_a^b f(x)\,dx \approx h(y_1 + y_2 + \cdots + y_{n-1} + y_n), \qquad (h = (b-a)/n).$$

We will illustrate this and other ways to estimate a definite integral by estimating $\int_0^1 \frac{dx}{1+x^2}$. We chose this integral because it can be easily computed by the FTC:

$$\int_0^1 \frac{dx}{1+x^2} = \arctan(x)|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} \approx 0.785398.$$

That enables us to judge the accuracy of each method.

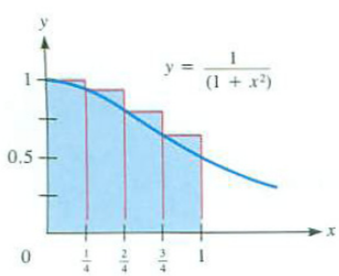


Figure 6.5.2:

EXAMPLE 1 Use four rectangles with equal widths to estimate $\int_0^1 \frac{dx}{1+x^2}$. Use the left endpoint of each section as the sampling number to determine the height of each rectangle.

SOLUTION Since the length of $[0,1]$ is 1, each of the four sections of equal length has length $\frac{1}{4}$. See Figure 6.5.2. The sum of the areas of the rectangles is

$$\frac{1}{1+0^2}\cdot\frac{1}{4} + \frac{1}{1+\left(\frac{1}{4}\right)^2}\cdot\frac{1}{4} + \frac{1}{1+\left(\frac{2}{4}\right)^2}\cdot\frac{1}{4} + \frac{1}{1+\left(\frac{3}{4}\right)^2}\cdot\frac{1}{4},$$

which equals $$\frac{1}{4}\left(1 + \frac{16}{17} + \frac{16}{20} + \frac{16}{25}\right).$$

This is approximately

$$\frac{1}{4}(1.0000 + 0.9411 + 0.8000 + 0.6400) = \frac{1}{4}(3.3811) \approx 0.845294.$$

◇

As Figure 6.5.2 shows, it is an overestimate; it exceeds the definite integral by about 0.06.

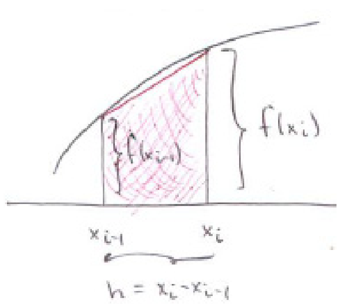


Figure 6.5.3: ARTIST: indicate height is $f(x_i) = y_i$

Approximation by Trapezoids

Trapezoids can also be used to find a local estimate of the area under the graph of $y = f(x)$ above the interval $[x_{i-1}, x_i]$. The basic idea is shown in Figure 6.5.3.

The area, A, of a trapezoid with base width h and side lengths b_1 and b_2 is the product of the base width and the average of the two side lengths: $A = \frac{1}{2}(b_1 + b_2)h$. (See Figure 6.5.4.)

The formula for the trapezoidal estimate of $\int_a^b f(x)\ dx$ follows from an argument like the one for the rectangular estimate.

Let n be a positive integer. Divide the interval $[a,b]$ into n sections of equal length $h=(b-a)/n$ with

$$x_0=a,\ x_1=a+h,\ x_2=a+2h,\ \ldots,\ x_n=a+nh=b.$$

Denote $f(x_i)$ by y_i. The local estimate of the area under $y=f(x)$ and above $[x_{i-1},x_i]$ is

$$\frac{1}{2}(y_{i-1}+y_i)h.$$

Summing the n local estimates of area gives the formula for the trapezoidal estimate of $\int_a^b f(x)\ dx$:

$$\frac{y_0+y_1}{2}\cdot h+\frac{y_1+y_2}{2}\cdot h+\cdots+\frac{y_{n-1}+y_n}{2}\cdot h$$

Factoring out $h/2$ and collecting like terms gives us the **trapezoidal estimate**:

$$\int_a^b f(x)\ dx \approx \frac{h}{2}\left(y_0+2y_1+2y_2+\cdots+2y_{n-1}+y_n\right). \qquad (6.5.2)$$

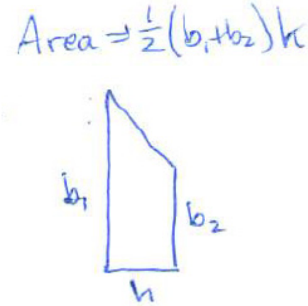


Figure 6.5.4:

There are n sections of width $h=(b-a)/n$, each corresponding to one trapezoid. However, the function is evaluated at $n+1$ points, including both ends of the interval $[a,b]$.

Note that y_0 and y_n have coefficient 1 while all other y_i's have coefficient 2. This is due to the double counting of the edges common to two trapezoids.

If $f(x)$ is a polynomial of the form $A+Bx$, its graph is a straight line. The top edge of each approximating trapezoid coincides with the graph. The approximation (6.5.2) in this special case gives the exact value of $\int_a^b f(x)\ dx$. There is no error.

Figures 6.5.5 and 6.5.6 illustrate the trapezoidal estimate for the case $n=4$. Notice that in Figure 6.5.5 the function is concave down and the trapezoidal estimate underestimates $\int_a^b f(x)\ dx$. On the other hand, when the curve is concave up the trapezoids overestimate, as shown in Figure 6.5.6. In both cases the trapezoids appear to give a better approximation of $\int_a^b f(x)\ dx$ than the same number of rectangles. For this reason we expect the trapezoidal method to provide better estimates of a definite integral than we obtain by rectangles.

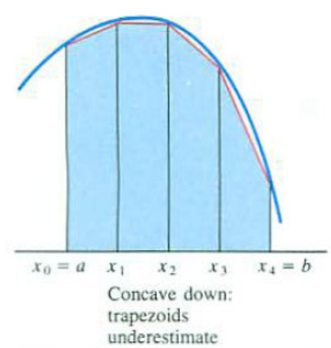


Figure 6.5.5:

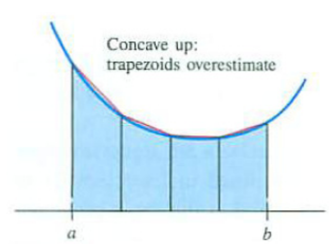


Figure 6.5.6:

EXAMPLE 2 Use the trapezoidal method with $n=4$ to estimate $\int_0^1 \frac{dx}{1+x^2}$.

SOLUTION In this case $a=0$, $b=1$, and $n=4$, so $h=(1-0)/4=\frac{1}{4}$. The four trapezoids are shown in Figure 6.5.7. The trapezoidal estimate is

$$\frac{h}{2}\left(f(0)+2f\left(\frac{1}{4}\right)+2f\left(\frac{2}{4}\right)+2f\left(\frac{3}{4}\right)+f(1)\right).$$

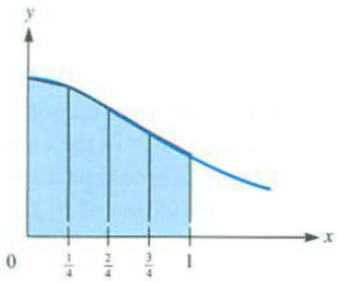


Figure 6.5.7: ARTIST: Try to make top side of trapezoids more visible.

Now, $h/2 = \frac{1}{4}/2 = 1/8$. To compute the sum of the five terms involving values of $f(x) = \frac{1}{1+x^2}$, make a list as shown in Table 6.5.1.

x_i	$f(x_i)$	coefficient	summand	decimal form
0	$\frac{1}{1+0^2}$	1	$1\cdot\frac{1}{1+0}$	1.0000
$\frac{1}{4}$	$\frac{1}{1+\left(\frac{1}{4}\right)^2}$	2	$2\cdot\frac{1}{1+\frac{1}{16}}$	1.8823
$\frac{2}{4}$	$\frac{1}{1+\left(\frac{2}{4}\right)^2}$	2	$2\cdot\frac{1}{1+\frac{4}{16}}$	1.6000
$\frac{3}{4}$	$\frac{1}{1+\left(\frac{3}{4}\right)^2}$	2	$2\cdot\frac{1}{1+\frac{9}{16}}$	1.2800
$\frac{4}{4}$	$\frac{1}{1+\left(\frac{4}{4}\right)^2}$	1	$1\cdot\frac{1}{1+\frac{16}{16}}$	0.5000

Table 6.5.1:

The trapezoidal sum is therefore, approximately,

$$\frac{1}{8}(1.0000+1.8823+1.6000+1.2800+0.5000) \approx \frac{1}{8}(6.2623)\approx 0.7827.$$

Thus

$$\int_0^1 \frac{dx}{1+x^2} \approx 0.782794.$$

This estimate differs from the definite integral by about 0.0026, which is much smaller than the error in the rectangular method, which had an error of 0.06. ◇

Comparison of Rectangular and Trapezoidal Estimates

If we divide out the 2 in the trapezoidal estimate, it takes the form

$$h\left(\frac{y_0}{2}+y_1+y_2+\cdots+y_{n-1}+\frac{y_n}{2}\right). \tag{6.5.3}$$

In this form it looks much like the rectangular estimate. It has $n+1$ summands, while the rectangular estimate has only n summands. However, if $f(a)$ happens to equal $f(b)$, that is, $y_0 = y_n$, then (6.5.3) can be written either as $h(y_0 + y_1 + y_2 + \cdots + y_{n-1})$ (the left endpoint rectangular estimate) or as $h(y_1 + y_2 + \cdots + y_{n-1} + y_n)$ (the right endpoint rectangular estimate). In this special case when $f(a) = f(b)$ the three estimates for $\int_a^b f(x)\,dx$ coincide.

Simpson's Estimate: Approximation by Parabolas

In the trapezoidal estimate a curve is approximated by chords. Simpson's estimate for $\int_a^b f(x)\ dx$ approximates the curve by parabolas. Given *three* points on a curve, there is a unique parabola of the form $y = Ax^2 + Bx + C$ that passes through them, as shown in Figure 6.5.8. (See Exercise 28.) The area under the parabola is then used to approximate the area under the curve.

Thomas Simpson, 1710–1761, http://en.wikipedia.org/wiki/Thomas_Simpson

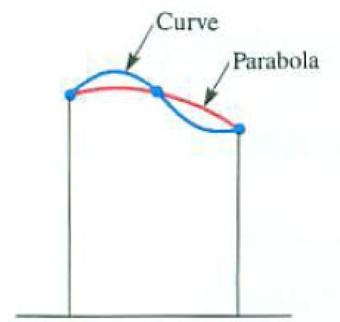
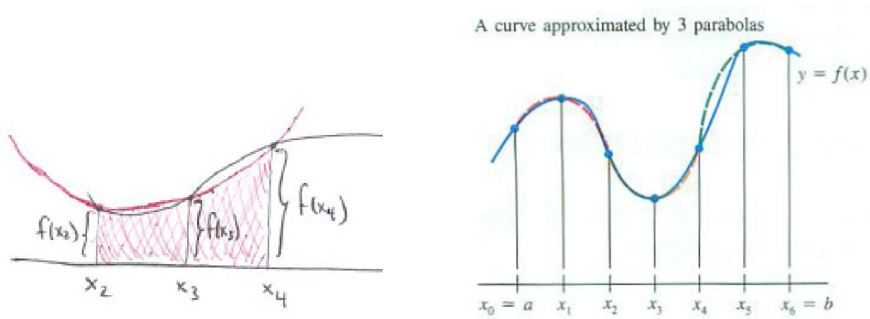


Figure 6.5.8: Curve: $y = f(x)$, Parabola: $y = Ax^2 + Bx + C$

The computations leading to the formula for the area under the parabola are more involved than those for the area of a trapezoid. (They are outlined in Exercises 28 to 29.) However, the final formula is fairly simple. Let the three points be $(x_1, f(x_1))$, $(x_2, f(x_2))$, $(x_3, f(x_3))$, with $x_1 < x_2 < x_3$, $x_2 - x_1 = h$, and $x_3 - x_2 = h$, as shown in Figure 6.5.9(a). The shaded area under the parabola turns out to be

$$\frac{h}{3}\left(f(x_1) + 4f(x_2) + f(x_3)\right). \tag{6.5.4}$$



(a) (b)

Figure 6.5.9: ARTIST: In (a), x_2, x_3, and x_4 should be labeled as x_1, x_2, and x_3.

To estimate $\int_a^b f(x)\ dx$, we pick an *even* number n and use $n/2$ parabolic arcs, each of width $2h$. As in the trapezoidal method, we start with a partition of $[a, b]$ into n sections of equal width, h: $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Denoting $f(x_i)$ by y_i, form the sum

$$\frac{h}{3}\left((y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)\right).$$

Collecting like terms gives us **Simpson's estimate** for the definite integral $\int_a^b f(x)\ dx$:

$$\frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right) \qquad (6.5.5)$$

Except for the first and last terms, the coefficients alternate 4, 2, 4, 2, ..., 2, 4. To apply (6.5.5), pick an even number n. Then $h = (b-a)/n$. The estimate uses $n+1$ points, $x_0, x_1, \ldots, x_n$, and $n/2$ parabolas. Example 3 illustrates the method, with $n = 4$.

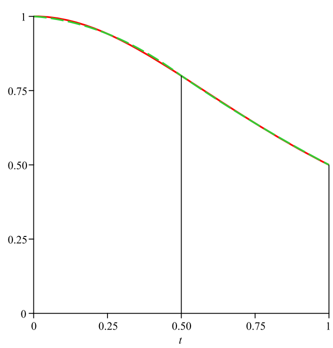


Figure 6.5.10:

EXAMPLE 3 Use Simpson's method with $n = 4$ to estimate $\int_0^1 \frac{dx}{1+x^2}$.
SOLUTION In this case, the estimate takes the form

$$\frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

with $h = (1-0)/4 = 1/4$. There are two parabolas, shown in Figure 6.5.10. Because the parabolas look almost like the curve, we expect Simpson's estimate to be even better than the trapezoidal estimate.

The computations are shown in Table 6.5.2.

x_i	$f(x_i)$	coefficient	summand	decimal form
0	$\frac{1}{1+0^2}$	1	$1 \cdot \frac{1}{1+0}$	1.0000
$\frac{1}{4}$	$\frac{1}{1+\left(\frac{1}{4}\right)^2}$	4	$4 \cdot \frac{1}{1+\frac{1}{16}}$	3.7647
$\frac{2}{4}$	$\frac{1}{1+\left(\frac{2}{4}\right)^2}$	2	$2 \cdot \frac{1}{1+\frac{4}{16}}$	1.6000
$\frac{3}{4}$	$\frac{1}{1+\left(\frac{3}{4}\right)^2}$	4	$4 \cdot \frac{1}{1+\frac{9}{16}}$	2.5600
$\frac{4}{4}$	$\frac{1}{1+\left(\frac{4}{4}\right)^2}$	1	$1 \cdot \frac{1}{1+\frac{16}{16}}$	0.5000

Table 6.5.2:

Combining the data in the table with the factor $h/3 = 1/12$ provides the estimate

$$\frac{1}{12}\left(1.0000 + 3.7647 + 1.6000 + 2.5600 + 0.5000\right) \quad = \quad \frac{1}{12}(9.4247) \approx 0.7853.$$

As the decimal form of $\int_0^1 dx/(1+x^2)$ begins 0.78539, this Simpson estimate is accurate to all four decimal places given. ◇

Method	Estimate	Error
Rectangles	0.845294	0.059896
Trapezoids	0.782794	0.002604
Simpson's (Parabolas)	0.785392	0.000006

Table 6.5.3:

Comparison of the Three Methods

We know the value of $\int_0^1 \frac{dx}{1+x^2}$ is 0.78539816, to eight decimal places. Table 6.5.3 compares the estimates made in the three examples to this value.

Error = |Exact − Estimate|

Though each method takes about the same amount of work, the table shows that Simpson's method gives the best estimate. The trapezoidal method is next best. The rectangular method has the largest error. These results should not come as a surprise. Parabolas should fit the curve better than chords do, and chords should fit better than horizontal line segments. Note that the trapezoidal and Simpson's methods in Examples 2 and 3 used the same sampling numbers to evaluate the integrand; their only difference is in the "weights" (coefficients) given the outputs of the integrand.

The size of the error is closely connected to the derivatives of the integrand. For a positive number k, let M_k be the largest value of $\left|f^{(k)}(x)\right|$ for x in $[a, b]$. Table 6.5.4 lists the general upper bounds for the error when $\int_a^b f(x)\,dx$ is estimated by sections of length $h = (b-a)/n$. These results are usually developed in a course on numerical analysis. They can also be obtained by a straightforward use of the Growth Theorem of Section 5.3 and the Fundamental Theorem of Calculus. (See Exercises 43 and 44 in this section and Exercise 73 in the Chapter 6 Summary.) They offer a good review of basic ideas.

Recall that $f^{(k)}(x)$ is the k^{th} derivative of f. For instance, $f^{(2)}(x)$ is the second derivative.

Table 6.5.4 expresses the bounds on the size of the error for each method in terms of $h = (b-a)/n$ and n.

Method	Bound on Error in Terms of h	Bound on Error in Terms of n
Rectangles	$M_1(b-a)h$	$M_1(b-a)^2/n$
Trapezoids	$\frac{1}{12}M_2(b-a)h^2$	$\frac{1}{12}M_2(b-a)^3/n^2$
Simpson's (Parabolas)	$\frac{1}{180}M_4(b-a)h^4$	$\frac{1}{180}M_4(b-a)^5/n^4$

Table 6.5.4:

The coefficients in the error bounds tell us a great deal. For instance, if $M_4 = 0$, then there is no error in Simpson's method. That is, if $f^{(4)}(x) = 0$ for all x in $[a, b]$, then Simpson's method produces an exact answer. For in this case the error is $M_4(b-a)h^4/180 = 0$. As a consequence, for polynomials of at most degree 3, Simpson's approximation is exact. (See Exercise 76 in Section 6.6.)

We know that the trapezoidal method is exact for polynomials of degree at most one, in other words, for functions whose second derivative is zero. That suggests that the error in this method is controlled by the size of the second derivative; Table 6.5.4 shows that it is.

The power of h that appears in the error bound is even more important. For instance, if you reduce the width h by a factor of 10 (using 10 times as many sections) you expect the error of the rectangular method to shrink by a factor of 10, the error in the trapezoidal method to shrink by a factor of $10^2 = 100$, and the error in Simpson's method by a factor of $10^4 = 10,000$. These observations are recorded in Table 6.5.5.

Method	Reduction Factor of h	Expected Reduction Factor of Error
Rectangles	10	10
Trapezoids	10	100
Simpson's (Parabolas)	10	10,000

Table 6.5.5:

Because the error in the rectangular method approaches 0 so slowly as $h \to 0$, we will not refer to it further.

Technology and Definite Integrals

The trapezoidal method and Simpson's method are just two examples of what is called **numerical integration**. Such techniques are studied in detail in courses on numerical analysis. While the Fundamental Theorem of Calculus is useful for evaluating definite integrals, it applies only when an antiderivative is readily available. Numerical integration is an important tool in estimating definite integrals, particularly when the FTC cannot be applied. Numerical integration can always be used to find out something about the value of a definite integral.

The design of an efficient and accurate general-purpose numerical integration algorithm is harder than it might seem. Effective algorithms typically divide the interval into unequal-length sections. The sections can be longer where the function is tame, that is, almost constant. Shorter sections are used where the function is wild, that is, changes very rapidly. Large, even unbounded, intervals can lead to another set of difficulties. Some examples of challenging definite integrals include:

$$\int_0^2 \sqrt{x(4-x)}\,dx \qquad \int_{-1}^{1} \frac{dx}{x^2+10^{-10}} \qquad \int_0^{600\pi} \frac{(\sin(x))^2}{\sqrt{x}+\sqrt{x+\pi}}\,dx$$

The HP-34C was, in 1980, the first handheld calculator to perform numerical integration. Now this is a common feature on most scientific calculators. The algorithms used vary greatly, and the details are often corporate secrets. The techniques are similar to those presented in this section and in Exercise 41.

Reference: Handheld Calculator Evaluates Integrals, William Kahan, Hewlett-Packard Journal, vol. 31, no. 8, Aug. 1980, pp. 23–32, `http://www.cs.berkeley.edu/~wkahan/Math128/INTGTkey.pdf`.

Summary

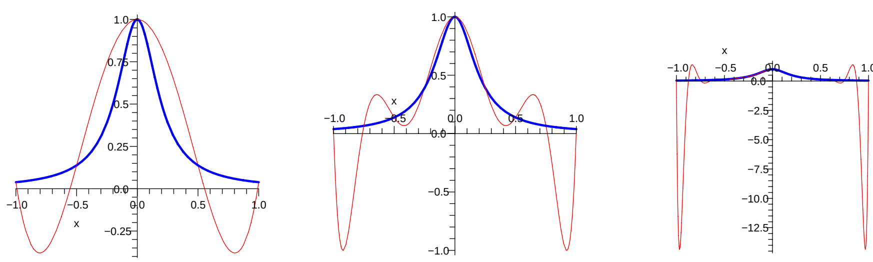
We presented three techniques for estimating definite integral: suggested by the areas of rectangles, areas of trapezoids, or areas under parabolas. We also observed that the error in each method is influenced by a derivative of the integrand and the distance, $h = (b-a)/n$, between the numbers at which we evaluate the integrand. The main difference between the methods is the coefficients used to weight the function values $y_i = f(x_i)$. In the left-hand rectangular estimate the coefficients are 1, 1, 1, ..., 1, 0 (because $y_n = f(b)$ is not used). In the right-hand rectangular estimate the coefficients are 0, 1, 1, ..., 1. In the trapezoidal estimate, 1, 2, 2, ..., 2, 1 and in Simpson's estimate 1, 4, 2, 4, 2, ..., 2, 4, 1. A course in numerical analysis presents several other ways to estimate a definite integral.

Carle Runge, 1856–1927, http://en.wikipedia.org/wiki/Carle_David_Tolm%C3%A9_Runge

Higher-Order Interpolation Methods and Runge's Counterexample

In the trapezoidal method you pass a line through two points to approximate the curve. That uses a first-degree polynomial, $Ax + B$. In Simpson's method you pass a parabola through three points, using a second-degree polynomial, $Ax^2 + Bx + C$. You would expect that as you pass higher-degree polynomials through more points on the curve you would get even better approximations. This is not always the case.

For the function $f(x) = 1/(1 + 25x^2)$, defined on $[-1, 1]$, known as Runge's Counterexample, the higher-degree polynomials passing through equally-spaced points do not resemble the function. Figure 6.5.11 shows the **interpolating polynomials** of degree 4 (a), 8 (b), and 16 (c). Notice how the approximations improve away from the endpoints and exhibit increasingly large oscillations near the endpoints. These oscillations result in poor estimates of $\int_{-1}^{1} \frac{dx}{1+25x^2}$. A Google search for "Runge's Counterexample" yields more information on this function.



(a) (b) (c)

Figure 6.5.11: In each figure the thick curve is the graph of Runge's Counterexample and the thin curve is the graph of the interpolating polynomials of degree 4 (a), 8 (b), and 12 (c). Notice the very different vertical scales in these three graphs. EDITOR: Please move these figures inside the box.

EXERCISES for 6.5

Key: R–routine, M–moderate, C–challenging

In the Exercises, T_n refers to the trapezoidal estimate with n trapezoids (partition with n sections and $n+1$ points), and S_n refers to Simpson's estimate with $n/2$ parabolas (partition with n sections and $n+1$ points)

In Exercises 1 to 8 approximate the given definite integrals by the trapezoidal estimate with the indicated T_n.

1.[R] $\int_0^2 \frac{dx}{1+x^2}$, T_2

2.[R] $\int_0^2 \frac{dx}{1+x^2}$, T_4

3.[R] $\int_0^2 \sin(\sqrt{x})\ dx$, T_2

4.[R] $\int_0^2 \sin(\sqrt{x})\ dx$, T_3

5.[R] $\int_1^3 \frac{2^x}{x}\ dx$, T_3

6.[R] $\int_1^3 \frac{2^x}{x}\ dx$, T_6

7.[R] $\int_1^3 \cos(x^2)\ dx$, T_2

8.[R] $\int_1^3 \cos(x^2)\ dx$, T_4

In Exercises 9 to 12 use Simpson's estimate to approximate each definite integral with the given S_n.

9.[R] $\int_0^1 \frac{dx}{1+x^3}$, S_2

10.[R] $\int_0^1 \frac{dx}{1+x^3}$, S_4

11.[R] $\int_0^1 \frac{dx}{1+x^4}$, S_2

12.[R] $\displaystyle\int_0^1 \frac{dx}{1+x^4}$, S_4

13.[R] Write out T_6 for $\int_1^4 5^x\, dx$ but do not carry out any of the calculations.

14.[R] Write out S_{10} for $\int_0^1 e^{x^2}\, dx$ but do not carry out any of the calculations.

15.[R] By a direct computation, show that the trapezoidal estimate is not exact for second-order polynomials. HINT: Take the simplest case, $\int_0^1 x^2\, dx$.

16.[R] By a direct computation, show that the Simpson's estimate is not exact for fourth-order polynomials. HINT: Take the simplest case, $\int_0^1 x^4\, dx$.

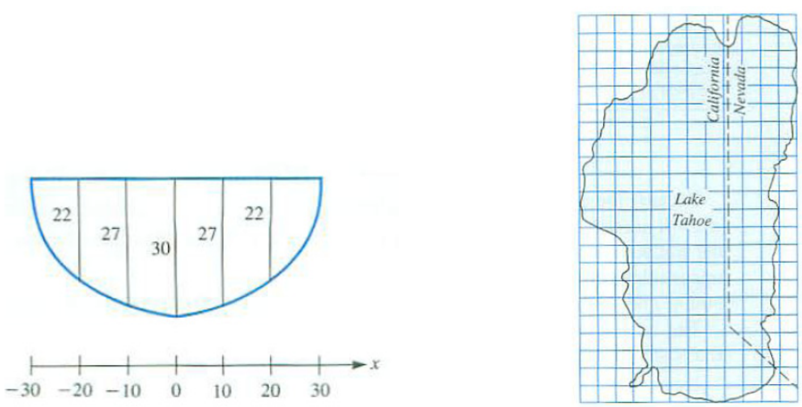
17.[R] In an interval $[a, b]$ in which $f''(x)$ is positive, do trapezoidal estimates of $\int_a^b f(x)\, dx$ underestimate or overestimate the definite integral? Explain.

18.[R] The cross-section of a ship's hull is shown in Figure 6.5.12(a). Estimate the area of this cross-section by

(a) T_6

(b) S_6

Dimensions are in feet. Give your answer to four decimal places.



(a) (b)

Figure 6.5.12:

19.[R] A ship is 120 feet long. The area of the cross-section of its hull is given at intervals in the table below:

x	0	20	40	60	80	100	120	feet
area	0	200	400	450	420	300	150	square feet

Estimate the volume of the hull in cubic feet by

(a) the trapezoidal estimate and

(b) Simpson's estimate.

Give your answer to four decimal places. HINT: What is largest n you can use in this problem?

20.[R] A map of Lake Tahoe is shown in Figure 6.5.12(b). Use Simpson's method and data from the map to estimate the surface area of the lake. Use cross-sections parallel to the side of the page. (Each little square represents a mile on each side.)

Exercises 21 and 22 present cases in which the maximum bound on the error is assumed.

21.[R] Show that the error for the trapezoidal estimate of $\int_0^1 x^2 dx$ is exactly $(b-a)M_2h^2/12$ where $a = 0$, $b = 1$, $h = 1$, and M_2 is the maximum value of $\left|D^2(x^2)\right|$ for x in $[0, 1]$.

22.[R] Show that the error for the Simpson estimate of $\int_0^1 x^4 dx$ is exactly $(b-a)M_4h^4/180$ where $a = 0$, $b = 1$, $h = 1/2$, and M_4 is the maximum value of $\left|D^4(x^4)\right|$ for x in $[0, 1]$.

23.[M] Figure 6.5.13(b) shows cross-sections of a pond in two directions. Use Simpson's method to estimate the area of the pond using

(a) vertical cross-sections, three parabolas and

(b) horizontal cross-sections, two parabolas.

24.[M] In the case of trapezoidal estimates, if you double the length of the interval $[a, b]$ and also the number of trapezoids, would you expect the error in the estimates to increase, decrease, or stay about the same? Explain.

25.[M] In the case of Simpson estimates, if you double the length of the interval $[a, b]$ and also the number of parabolas, would you expect the error in the estimates to increase, decrease, or stay about the same? Explain.

26.[M]

(a) Fill in this table concerning $\int_0^6 x^2\,dx$ and its trapezoidal estimates.

	$\int_0^6 x^2\,dx$	T_1	T_2	T_3
Value				
Error		—		

(b) Are the errors in (a) proportional to h^c for some constant c? (Recall that h is the width of the trapezoids.)

27.[M]

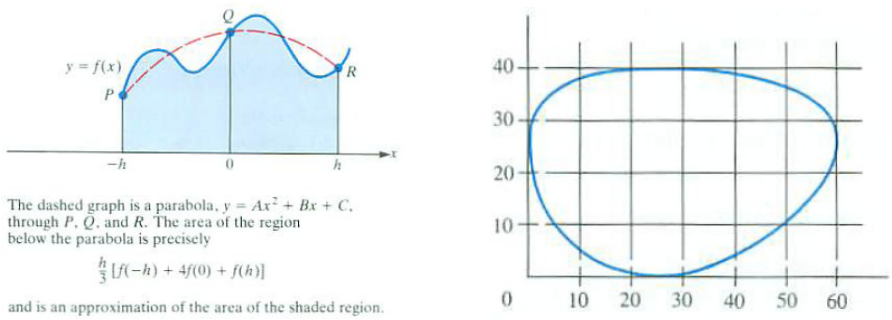
(a) Fill in this table concerning $\int_1^7 dx/(1+x)^2$ and its Simpson estimates.

	$\int_1^7 dx/(1+x)^2$	S_2	S_4	S_6
Value				
Error		—		

(b) Are the errors in (a) using S_n roughly proportional to h^k for some constant k? (Recall that h is the width of the sections.)

Exercises 28 to 30 provide the basis of Simpson estimates. For convenience we place the origin of the x-axis at the midpoint of the interval for which a single parabola will approximate the function. Because the interval has length $2h$, its ends are $-h$ and h.

28.[M] Let $f(x)$ be a function defined on at least $[-h, h]$, with $f(-h) = y_1$, $f(0) = y_2$, and $f(h) = y_3$. Show that there is exactly one parabola $P(x) = Ax^2+Bx+C$ that passes through the three points $(-h, y_1)$, $(0, y_2)$, and (h, y_3). (See Figure 6.5.13(a).)



(a) (b)

Figure 6.5.13:

29.[M] Let $p(x) = Ax^2 + Bx + C$. Show, by computing both sides of the equation, that

$$\int_{-h}^{h} p(x)\ dx = \frac{h}{3}\left(p(-h) + 4p(0) + p(h)\right).$$

This equation, expressed geometrically, was known to the ancient Greeks. In modern terms it says that Simpson's estimates are exact for polynomials of degree at most two.

30.[M] Let $f(x) = x^3$. Show that

$$\int_{-h}^{h} f(x)\ dx = \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right).$$

This information, combined with Exercise 29, shows that Simpson's method is exact for polynomials of degree at most 3.

31.[M] The table lists the values of a function f at the given points.

x	1	2	3	4	5	6	7
$f(x)$	1	2	1.5	1	1.5	3	3

(a) Plot the corresponding seven points on the graph of f.

(b) Sketch six trapezoids that can be used to estimate $\int_1^7 f(x)\,dx$.

(c) Find the trapezoidal estimate of $\int_1^7 f(x)\,dx$.

(d) Sketch, by eye, the three parabolas used in Simpson's method to estimate $\int_1^7 f(x)\,dx$.

(e) Find Simpson's estimate of $\int_1^7 f(x)\,dx$.

32.[M] A function f is defined on $[a, b]$ and $f(x)$, $f'(x)$, and $f''(x)$ are all positive for x in that interval. Arrange the following quantities in order of size, from smallest to largest. (Some may be equal.) Sketches may help.

(a) the area of the trapezoid with base $[a, b]$ and parallel sides of lengths $f(a)$ and $f(b)$

(b) the area of the "midpoint" rectangle with base $[a, b]$ and height $f((a + b)/2)$

(c) the area of the "right-endpoint" rectangle with base $[a, b]$ and height $f(b)$

(d) the area of the "left-endpoint" rectangle with base $[a, b]$ and height $f(a)$

(e) the average of (c) and (d)

(f) the trapezoid whose base is $[a, b]$ and whose top edge lies on the tangent line at $((a + b)/2, f((a + b)/2))$

(g) $\int_a^b f(x)\,dx$.

Exercises 33 to 35 describe the **midpoint estimate**, yet another way to estimate a definite integral.

33.[M] Another way to estimate a definite integral is by a Riemann sum $\sum_{i=1}^{n} f(c_i)h$, where the c_i are the midpoints of the intervals. Call such an estimate with n sections, M_n. Find M_4 for $\int_0^1 dx/(1 + x^2)$.

34.[M] With the aid of a diagram, show that the midpoint estimate is exact for functions of the form $f(x) = Ax + B$.

35.[M] Assume that $f''(x)$ is negative for x in $[a,b]$. With the aid of a diagram, show that the midpoint method overestimates $\int_a^b f(x)\ dx$. HINT: Draw a tangent at the point $((a+b)/2, f((a+b)/2))$.

36.[M] If the Simpson estimate with 4 parabolas estimate a certain definite integral with an error of 0.35, what error would you expect with (a) 8 parabolas? (b) 5 parabolas?

37.[C] The equation in Exercise 28 is called the **prismoidal formula**. Use it to compute the volume of

(a) a sphere of radius a and

(b) a right circular cone of radius a and height h.

NOTE: The prisomoidal formula was known to the Greeks. Reference: `http://www.mathpages.com/home/kmath189/kmath189.htm`

Exercise 38 provides a review of several basic ideas as it involves the Fundamental Theorem of Calculus (FTC I), the chain rule, l'Hôpital's rule, and the intermediate-value theorem. The midpoint estimate is defined in Exercise 33.

38.[C] Assume that $f''(x)$ is continuous and negative for x in $[0, 2h]$. Then the midpoint estimate, M, for $\int_{-h}^{h} f(x)\ dx$ is too large and the trapezoidal estimate, T, is too small. The error of the first is $M - \int_{-h}^{h} f(x)\ dx$ and of the second is $\int_{-h}^{h} f(x)\ dx - T$. Show that

$$\lim_{h\to 0} \frac{M - \int_{-h}^{h} f(x)\ dx}{\int_{-h}^{h} f(x)\ dx - T} = \frac{1}{2}.$$

This suggests that the error in the midpoint estimate when h is small is about half the error of the trapezoidal estimate. However, the midpoint estimate is seldom used because data at midpoints are usually not available (and because the Simpson estimate provides an even more accurate estimate using same data as the trapezoidal estimate).

39.[C] Another way to estimate a definite integral is to use Taylor polynomials (discussed in Section 5.4). If the Maclaurin polynomial $P_2(x)$ for $f(x)$ of degree 2 is used to approximate $f(x)$ for x in $[0, h]$, express the possible error in using $\int_0^h P_2(x)\ dx$ to estimate $\int_0^h f(x)\ dx$.

40.[C] Simpson's estimate is not exact for fourth-degree polynomials.

(a) Estimate $\int_0^h x^4\,dx$ by S_2.

(b) What is the ratio between that estimate and $\int_0^h x^4\,dx$?

(c) What does (b) imply about the ratio between Simpson's estimate and $\int_0^h P(x)\,dx$ for any polynomial of degree at most 4?

41.[C] There are many other methods for estimating definite integrals. Some old methods, which had been of only theoretical interest because of their messy arithmetic, have, with the advent of computers, assumed practical importance. This exercise illustrates the simplest of the so-called **Gaussian quadrature** formulas. For convenience, we consider only integrals over $[-1, 1]$.

(a) Show that

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

for $f(x) = 1$, x, x^2, and x^3.

(b) Let a and b be two numbers, $-1 \le a < b \le 1$, such that

$$\int_{-1}^{1} f(x)\,dx = f(a) + f(b)$$

for $f(x) = 1$, x, x^2, and x^3. Show that only $a = \frac{-1}{\sqrt{3}}$ and $b = \frac{1}{\sqrt{3}}$ (or $a = \frac{1}{\sqrt{3}}$ and $b = \frac{-1}{\sqrt{3}}$) satisfy this equation.

(c) Show that the Gaussian approximation

$$\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

has no error when f is a polynomial of degree at most 3.

(d) Use the formula in (a) to estimate $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$.

(e) Compare the answer in (d) to the exact value of $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$. How large is the error?

42.[C] Let f be a function such that $\left|f^{(2)}(x)\right| \leq 10$ and $\left|f^{(4)}(x)\right| \leq 50$ for all x in $[1,5]$. If $\int_1^5 f(x)\,dx$ is to be estimated with an error of at most 0.01, how small must h be in

(a) the trapezoidal approximation?

(b) Simpson's approximation?

In Section 5.4 we showed why a higher derivative controls the error in using a Taylor polynomial to approximate a function value. Exercises 43 and 44 show why a higher derivative controls the error in using the trapezoidal or Simpson estimate of a definite integral $\int_a^b f(x)dx$. (See Exercise 73 in Section 6.6 for the derivation of the corresponding error estimate for the midpoint estimate.) In each case $h = (b-a)/n$ and a function $E(t)$, $0 \leq t \leq h$, is introduced. The "local error" is $E(h)$, that is, the error in using one trapezoid of width h or one parabola of width $2h$. Once $E(h)$ is controlled by a higher derivative, we multiply by n, where $nh = b-a$, to obtain a measure of the total error in estimating $\int_a^b f(x)\,dx$. The argument involves both FTC I and FTC II and provides a review of basic concepts.

43.[C] *(The error in the trapezoid estimate.)* As usual, let $h = (b-a)/n$. We will estimate the error for a single section of width h and then multiply by n to find the error in estimating $\int_a^b f(x)\,dx$. For convenience, we move the graph so the interval (of length h) is $[0,h]$.

(a) Show that the error when using T_1 is $E(h) = \int_0^h f(x)\,dx - \frac{h}{2}(f(0)+f(h))$.

(b) For t in $[0,h]$ let $E(t) = \int_0^t f(x)\,dx - \frac{t}{2}(f(0)+f(t))$. Show that $E(0) = 0$, $E'(0) = 0$, and $E''(t) = -\frac{t}{2}f''(t)$.

(c) Let M be the maximum of $f''(x)$ on $[a,b]$ and m be the minimum. Show that $\frac{-mt}{2} \geq E''(t) \geq \frac{-Mt}{2}$.

(d) Using (b) and (c), show that $\frac{-mt^2}{4} \geq E'(t) \geq \frac{-Mt^2}{4}$.

(e) Show that $\frac{-mt^3}{12} \geq E(t) \geq \frac{-Mt^3}{12}$.

(f) Show that $\frac{-mh^3}{12} \geq E(h) \geq \frac{-Mh^3}{12}$.

(g) Show that $\frac{-m(b-a)h^2}{12} \geq \int_a^b f(x)\,dx - T_n \geq \frac{-M(b-a)h^2}{12}$.

(h) Show that $\int_a^b f(x)\,dx - T_n = \frac{-f''(c)(b-a)h^2}{12}$ for some number c in $[a,b]$.

(i) Deduce that $\left|\int_a^b f(x)\,dx - T_n\right| \leq \frac{M_2(b-a)h^2}{12}$, where M_2 is the maximum of $|f''(x)|$ for x in $[a,b]$.

44.[C] *(The error in the Simpson estimate.)* Now n is even and $[a, b]$ is divided into n sections of width $h = (b - a)/n$. The Simpson estimate is based on $n/2$ intervals of length $2h$. We will place the origin at the midpoint of an interval, so that its ends are $-h$ and h. In this case we wish to control the size of $E(h) = \int_{-h}^{h} f(x)\, dx - \frac{h}{3}(f(-h)+4f(0)+f(h))$. Introduce the function $E(t)$, for $-h \le t \le h$, defined by $E(t) = \int_{-t}^{t} f(x)\, dx - \frac{t}{3}(f(-t) + 4f(0) + f(t))$.

(a) Show that

$$E'(t) = \frac{2}{3}(f(t) + f(-t)) - \frac{4}{3}f(0) - \frac{t}{3}(f'(t) - f'(-t)).$$

(b) Show that $E''(t) = \frac{1}{3}(f'(t) - f'(-t)) - \frac{t}{3}(f''(t) + f''(-t))$.

(c) Show that $E'''(t) = -\frac{t}{3}(f'''(t) - f'''(-t))$.

(d) Show that $E'''(t) = \frac{-2t^2}{3} f^{(4)}(c)$ for some c in $[-h, h]$.

(e) Show that $E(0) = E'(0) = E''(0) = 0$.

(f) Let M_4 be the maximum of $|f^{(4)}(t)$ on $[a, b]$. Show that $|E(t)| \le \frac{2t^5}{180} M_4$.

(g) Deduce that $\left| \int_a^b f(x)\, dx - S_n \right| \le \frac{M_4(b-a)h^4}{180}$.

45.[C]

Sam: I bet I can find a better way than Simpson's estimate to approximate $\int_{-h}^{h} f(x)\,dx$ using the same three arguments ($-h$, 0, and h).

Jane: How so?

Sam: Look at his formula $\frac{h}{3}(f(-h)+4f(0)+f(h))$, which equals $2h\left(\frac{1}{6}f(-h)+\frac{4}{6}f(0)+\frac{1}{6}f(h)\right)$. The $2h$ is the width of the interval. I can't change that.

Jane: What would you change?

Sam: The weights $\frac{1}{6}$, $\frac{4}{6}$, and $\frac{1}{6}$. I'll use weights w_1, w_2, and w_3 and demand that the estimates I get be exact when the function $f(x)$ is either constant, x, or x^2.

Jane: Go ahead.

Sam: If $f(x) = c$, a constant, then, because $\int_{-h}^{h} c\,dx = 2hc$, I must have $2hc = 2h(w_1c + w_2c + w_3c)$. That tells me that $w_1 + w_2 + w_3$ must be 1.

Jane: But you need three equations for three unknowns.

Sam: When $f(x) = x$, I get $\int_{-h}^{h} f(x)\,dx = 0$, so $0 = 2h(-w_1h + w_20 + w_3h)$. Now I know that w_1 equals w_3.

Jane: And the third equation?

Sam: With $f(x) = x^2$, I find that $\frac{2}{3}h^3 = 2h^3(w_1 + w_3)$.

Jane: So what are your three w's?

Sam: A little high school algebra shows they are $\frac{1}{6}$, $\frac{4}{6}$, and $\frac{1}{6}$. What a disappointment. But at least I avoided all the geometry of parabolas. It's really all about assigning proper weights.

Check the missing details and show that Sam is right.

6.6 Chapter Summary

Chapter 6 introduced the second major concept in calculus, the definite integral, defined as a limit:

$$\int_a^b f(x)\ dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

For a continuous function this limit always exists and $\int_a^b f(x)\ dx$ can be viewed as the (net) area under the graph of $y = f(x)$ on the interval $[a, b]$. Both the definite integral and an antiderivative of a function f are called "integrals." Context tells which is meant. An antiderivative is also called an "indefinite integral."

The definite integral, in contrast to the derivative, gives global information.

Integrand: $f(x)$	Integral: $\int_a^b f(x)\ dx$
velocity	change in position
speed (|velocity|)	distance traveled
cross-sectional length of plane region	area of a plane region
cross-sectional area of solid	volume of solid
rate bacterial colony grows	total growth

As the first and last of these applications show, if you compute the definite integral of the rate at which some quantity is changing, you get the total change. To put this in mathematical symbols, let $F(x)$ be the quantity present at time x. Then $F'(x)$ is the rate at which the quantity changes. Thus $\int_a^b F'(x)$ equals the change in $F(x)$ as x goes from a to b, which is $F(b) - F(a)$. In short, $\int_a^b F'(x)\ dx = F(b) - F(a)$. This is another way of stating the Fundamental Theorem of Calculus, because F is an antiderivative of F'.

That equation gives a shortcut for evaluating many common definite integrals. However, finding an antiderivative can be tedious or impossible. For instance, $\exp(x^2)$ does not have an elementary antiderivative. However, continuous functions do have antiderivatives, as slope fields suggest. Indeed $G(x) = \int_a^x f(t)\ dt$ is an antiderivative of the integrand.

One way to estimate a definite integral is to employ one of the sums $\sum_{i=1}^{n} f(c_i)\ \Delta x_i$ that appear in its definition.

A more accurate method, which involves the same amount of arithmetic, uses trapezoids. Then the estimate takes the form

$$\int_a^b f(x)\ dx \approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right),$$

where consecutive x_i's are a fixed distance $h = (b-a)/n$ apart. In Simpson's method the graph is approximated by parts of parabolas, n is even, and the estimate is

$$\frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right).$$

The remaining chapters are simply elaborations of the derivative and the definite integral or further applications of them. For instance, instead of integrals over intervals, Chapter 17 deals with integrals over sets in the plane or in space. Chapter 15 treats derivatives of functions of several variables. In both cases the definitions involve limits similar to those that appear in the definitions of the derivative and the definite integral. That is one reason not to lose sight of those two definitions in the many applications.

EXERCISES for 6.6 *Key:* R–routine, M–moderate, C–challenging

1.[R] State FTC II in words, using no mathematical symbols. (It refers to $F(b) - F(a)$.)

2.[R] State FTC I in words, using no mathematical symbols. (It refers to the derivative of $\int_a^x f(t)\ dt$.)

Evaluate the definite integrals in Exercises 3 to 16.

3.[R] $\int_1^2 (2x^3 + 3x - 5)\ dx$

4.[R] $\int_5^7 \frac{3}{x}\ dx$

5.[R] $\int_1^4 \frac{dx}{\sqrt{x}}$

6.[R] $\int_1^4 \frac{x+2x^3}{\sqrt{x}}\ dx$

7.[R] $\int_0^1 x(3+x)\ dx$

8.[R] $\int_0^2 (2+3x)^2\ dx$

9.[R] $\int_1^2 \frac{(2+3x)^2}{x^2}\ dx$

10.[R] $\int_1^2 e^{2x}\ dx$

11.[R] $\int_0^\pi \sin(3x)\ dx$

12.[R] $\int_0^{\pi/4} \sec^2(x)\ dx$

13.[R] $\int_0^{\sqrt{2}/2} \frac{3\ dx}{\sqrt{1-x^2}}\ dx$

14.[R] $\int_0^{\pi/4} \cos(x)\ dx$

15.[R] $\int_0^{\pi/4} \sec(x)\tan(x)\ dx$

16.[R] $\int_{1/2}^{\sqrt{2}/2} \frac{dx}{x\sqrt{x^2-1}}$

In Exercises 17 to 24 find an antiderivative of the given function by guess and experiment. Check your answer by differentiating it.

17.[R] $(2x+1)^5$

18.[R] $\frac{1}{(2x+1)^5}$

19.[R] $\frac{1}{x+1}$

20.[R] $\frac{1}{2x+1}$

21.[R] $\ln(x)$

22.[R] $x\sin(x)$

23.[R] $\sin(2x)$

24.[R] xe^{x^2}

Use Simpson's estimate with three parabolas ($n = 6$) to approximate the definite integrals in Exercises 25 and 26.

25.[R] $\int_0^{\pi/2} \sin(x^2)\ dx$

26.[R] $\int_1 2\sqrt{1+x^2}\ dx$

27.[R] Use the trapezoidal estimate with $n = 6$ to estimate the integral in Exercise 25.

28.[R] Use the trapezoidal estimate with $n = 6$ to estimate the integral in Exercise 26.

29.[R]

(a) What is the area under $y = 1/x$ and above $[1, b]$, $b > 1$?

(b) Is the area under $y = 1/x$ and above $[1, \infty)$ finite or infinite?

(c) The region under $y = 1/x$ and above $[1, b]$ is rotated around the x-axis. What is the volume of the solid produced?

30.[R] The basis for this chapter is that if f is continuous and $x > a$, then $\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$.

(a) Review how this equation was obtained.

(b) Use a similar method to show that, if $x < b$, then $\frac{d}{dx}\int_x^b f(t)\ dt = -f(x)$.

31.[R] Let $f(x)$ and $g(x)$ be differentiable functions with $f(x) \geq g(x)$ for all x in $[a, b]$, $a < b$.

(a) Is $f'(x) \geq g'(x)$ for all x in $[a, b]$? Explain.

(b) Is $\int_a^b f(x)dx \geq \int_a^b g(x)dx$? Explain.

32.[R] Find $D\left(\int_{x^2}^{x^3} e^{-t^2} dt\right)$.

33.[R]

Jane: I'm not happy. The text says that a definite integral measures area. But they never defined "area under a curve." I know what the area of a rectangle is: width times length. But what is meant by the area under a curve? If they say, "Well, its the definite integral of the cross-sections," that won't do. What if I integrate cross-sections that are parallel to the x-axis instead of the y-axis? How do I know I'll get the same answer? Once again, the authors are hoping no one will notice a big gap in their logic.

Is Jane right? Have the authors tried to slip something past the reader?

34.[M] Let T_n be the trapezoidal estimate of $\int_a^b f(x)\, dx$ with n trapezoids and M_n be the midpoint estimate with n sections. Show that $\frac{1}{3}T_n + \frac{2}{3}M_n$ equals the Simpson estimate S_{2n} with n parabolas. Hint: Consider a typical interval of length h.

35.[M] A river flows at the (varying) rate of $r(t)$ cubic feet per second.

(a) Approximate how many cubic feet passes during the short time interval from time t to time $t + \Delta t$ seconds.

(b) How much passes from time t_1 to time t_2 seconds?

36.[M] Let $f(x) = xe^{-x}$ for $x \geq 0$. For which interval of length 1 is the area below the graph of f and above that interval a maximum?

37.[M] Let $f(x) = x/(x + 1)^2$ for $x \geq 0$.

(a) Graph f, showing any extrema.

(b) Looking at your graph, estimate for which interval of length one, the area below the graph of f and above the interval is a maximum.

(c) Using calculus, find the interval in (b) that yields the maximum area.

38.[M]

(a) Estimate $\int_0^1 \frac{\sin(x)}{x} dx$ by approximating $\sin(x)$ by the Taylor polynomial $P_6(x;0)$.

(b) Use the Lagrange bound on the error to bound the error in (a).

39.[M]

(a) Estimate $\int_1^3 \frac{e^x}{x} dx$ by using the Taylor polynomial $P_3(x;2)$ to approximate e^x. (To avoid computing e^2, approximate e by 2.71828.)

(b) Use the Lagrange bound on the error to bound the error in (a).

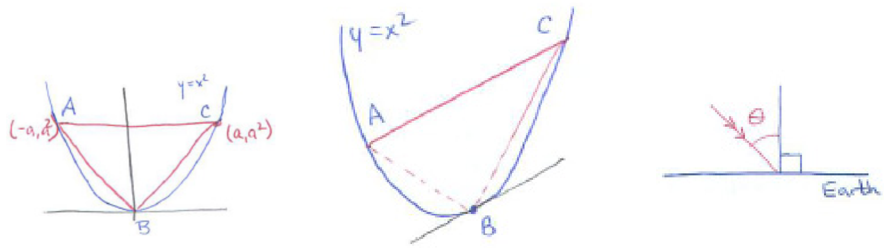
40.[M] Assume $f(2) = 0$ and $f'(2) = 0$ and $f''(x) \leq 5$ for all x in $[0, 7]$. Show that $\int_2^3 f(x)dx \leq 5/6$.

41.[M] Find $\lim_{t \to 0} \dfrac{\int_0^t \left(e^{x^2} - 1\right)\ dx}{\int_0^t \sin(2x^2)\ dx}$.

42.[M] Let $G(t) = \int_0^t \cos^5(\theta)\ d\theta$ for t in $[0, 2\pi]$.

(a) Sketch a rough graph of $y = G'(t)$.

(b) Sketch a rough graph of $y = G(t)$.



(a) (b) (c)

Figure 6.6.1:

43.[M] Figure 6.6.1(a) shows a triangle ABC inscribed in the parabola $y = x^2$ $A = (-a, a^2)$, $B = (0, 0)$, and $C = (a, a^2)$. Let $T(a)$ beits area and $P(a)$ the area bounded by AC and the parabola above the interval $[-a, a]$. Find $\lim_{a\to 0} \frac{T(a)}{P(a)}$. Note: Archimedes established a much more general result. In Figure 6.6.1(b) the tangent line at B is parallel to AC. He determined for any chord AC the ratio between the area of triangle ABC and the area of the parabolic section .

Usually we use a sum to estimate a definite integral. We can also use a definite integral to estimate a sum. In Exercises 44 and 45, rewrite each sum so that it becomes the sum estimating a definite integral. Then use the definite integral to estimate the sum.

44.[M] $\frac{1}{100}\sum_{i=1}^{100} \frac{1}{i^2}$

45.[M] $\sum_{n=51}^{100} \frac{1}{n}$

46.[M]

(a) Show that the average value of $\cos(\theta)$ for θ in $[0, \pi/2]$ is about 0.637.

(b) The average in (a) is fairly large, being much more than half of the maximum value of $\cos(\theta)$. Why is that good news for a farmer or solar engineer on Earth who depends on heat from the sun? Hint: See Figure 6.6.1(c).

47.[M] Assume f' is continuous on $[0, t]$.

(a) Find the derivative of $F(t) = 2\int_0^t f(x)f'(x)\, dx - f(t)^2$.

(b) Give a shorter formula for $F(t)$.

48.[M] Find a simple expression for the function $F(t) = \int_1^t \cos(x^2)\, dx - \int_1^{t^2} \frac{\cos(u)}{2\sqrt{u}}\, du$.

49.[M] A tent has a square base of side b and a pole of length $b/2$ above the center of the base.

(a) Set up a definite integral for the volume of the tent.

(b) Evaluate the integral in (a) by the Fundamental Theorem of Calculus.

(c) Find the volume of the tent by showing that six copies of it fill up a cube of side b.

50.[M]

Sam: I can get the second FTC, the one about $F(b) - F(a)$, without all that stuff in the first FTC.

Jane: That would be nice.

Sam: As usual, I assume F' is continuous and $\int_a^b F'(x)dx$ exists. Now, $F(b) - F(a)$ is the total change in F. Well, bust up $[a, b]$ by t_0, t_1, ..., t_n in the usual way. Then the total change is just the sum of the changes in F over each of the n intervals, $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

Jane: That's a no-brainer, but then what?

Sam: The change in F over the typical interval is $F(t_i) - F(t_{i-1})$. By the Mean Value Theorem for F, that equals $F'(t_i^*)(t_i - t_{i-1})$ for some t_i^* in the i^{th} interval. The rest is automatic.

Jane: I see. You let all the intervals get shorter and shorter and the sums of the $F'(t_i^*)(t_i - t_{i-1})$ approach $\int_a^b F'(x)\ dx$. But they are all already equal to $F(b) - F(a)$.

Sam: Pretty neat, yes?

Jane: Something must be wrong.

Is anything wrong?

51.[M]

Sam: There are two authors and they are both wrong.

Jane: How so?

Sam: Light can be both a wave and a particle, right?

Jane: Yes.

Sam: Well the definite integral is both a number and a function.

Jane: Did you get enough sleep?

Sam: This is serious. Take $\int_0^b x^2$. That equals $b^3/3$. Right?

Jane: So far, right.

Sam: Well, as b varies, so does $b^3/3$. So it's a function.

Jane: ...

What is Jane's reply?

52.[M]

(a) Graph $y = e^x$ for x in $[0, 1]$.

(b) Let c be the number such that the area under the graph of $y = e^x$ above $[0, c]$ equals the area under the graph above $[c, 1]$. Looking at the graph in (a), decide whether c is bigger or smaller than $1/2$.

(c) Find c.

53.[M] Find $\lim_{\Delta x \to 0} \left(\frac{1}{\Delta x} \int_5^{7+\Delta x} e^{x^3}\ dx - \frac{1}{\Delta x} \int_5^7 e^{x^3}\ dx \right)$.

54.[M] Find $\lim_{\Delta x \to 0} \left(\frac{1}{\Delta x} \int_{5+\Delta x}^{7} e^{x^3}\ dx - \frac{1}{\Delta x} \int_5^7 e^{x^3}\ dx \right)$.

55.[M] A company is founded with capital investment A. It plans to have its rate of investment proportional to its total investment at any time. Let $f(t)$ denote the rate of investment at time t.

(a) Show that there is a constant k such that $f(t) = k(A + \int_0^t f(x)dx)$ for any $t \geq 0$.

(b) Find a formula for f.

There are two definite integrals in each of Exercises 56 to 59. One can be evaluated by the FTC, the other not. Evaluate the one that can be evaluated by the FTC and approximate the other by Simpson's estimate with $n = 4$ (2 parabolas).

56.[M] $\int_0^1 (e^x)^2\ dx$; $\int_0^1 e^{x^2}\ dx$.

57.[M] $\int_0^{\pi/4} \sec(x^2)\ dx$; $\int_0^{\pi/4} (\sec(x))^2\ dx$.

58.[M] $\int_1^3 e^{x^2} x\ dx$; $\int_1^3 \frac{e^{x^2}}{x}\ dx$.

59.[M] $\int_{0.2}^{0.4} \frac{dx}{\sqrt{1-x^2}}$; $\int_{0.2}^{0.4} \frac{dx}{\sqrt{1-x^3}}$.

60.[M] If $F'(x) = f(x)$, find an antiderivative for (a) $g(x) = x + f(x)$, (b) $g(x) = 2f(x)$, and (c) $g(x) = f(2x)$.

61.[M] John M. Robson in The Physics of Fly Casting, American J. Physics 58(1990), pp. 234–240, lets the reader fill in the calculus steps. For instance, he has the equation

$$\mu(4z + h)\dot{z}^2 = 2\int_0^t crh\rho\dot{z}^3 dt + T(0)$$

where z is a function of time t, $\dot{z} = dz/dt$, and $\ddot{z} = d^z/dt^2$. He then states, "differentiating this gives

$$(2\mu - crh\rho)\dot{z}^2 + (4z + h)\mu\ddot{z} = 0."$$

Check that he is correct.

62.[C] Jane is running from a to b, on the x-axis. When she is at x, her speed is $v(x)$. How long does it take her to go from a to b?

63.[C]

(a) Find all continuous functions $f(t)$, $t \geq 0$, such that $\int_0^{x^2} f(t)\ dt = 3x^3$. $x \geq 0$.

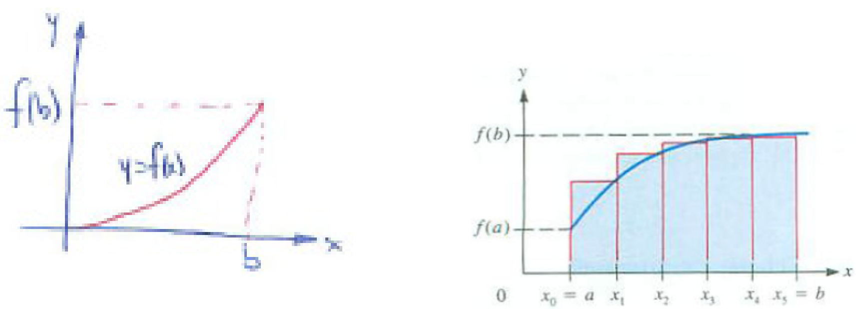
(b) Check that they satisfy the equation in (a).

64.[C] Let $f(x)$ be defined for x in $[0, b]$, $b > 0$. Assume that $f(0) = 0$ and $f'(x)$ is positive.

(a) Use Figure 6.6.2(a) to show that $\int_0^b f(x)\,dx + \int_0^{f(b)} (\mathsf{inv} f)(x)\,dx = bf(b)$.

(b) As a check on the equation in (a), differentiate both sides of it with respect to b. You should get a valid equation.

(c) Use (a) to evaluate $\int_0^1 \arcsin(x)\,dx$.



(a) (b)

Figure 6.6.2:

65.[C]

(a) Verify, without using the FTC, that $\int_0^2 \sqrt{x(4-x)}\,dx = \pi$. HINT: What region has an area give by that integral?

(b) Approximate the definite integral in (a) by the trapezoidal estimate with 4 trapezoids and also with 8 trapezoids.

(c) Compute the error in each case.

(d) By trial-and-error, estimate how many trapezoids are needed to have an approximation that is accurate to three decimal places?

(e) Why is the error bound for the trapezoidal estimate of no use in (d)?

66.[C]

(a) Approximate the definite integral in Exercise 65 by Simpson's estimate with 2 parabolas and again with 4 parabolas. (These use the same number of arguments as in Exercise 65.)

(b) Compute the error in each case.

(c) By trial-and-error, estimate how many parabolas are needed to have an estimate accurate to 3 decimal places. HINT: Use your calculator or computer to automate the calculations.

(d) Why is the error bound for the Simpson's estimate of no use in (c)?

67.[C] In his *Principia*, published in 1607, Newton examined the error in approximating an area by rectangles. He considered an increasing, differentiable function f defined on the interval $[a, b]$ and drew a figure similar to Figure 6.6.2(b). All rectangles have the same width h. Let R equal the sum of the areas of the rectangles using right endpoints and let L equal the sum of the areas of the rectangles using left endpoints. Let A be the area under the curve $y = f(x)$ and above $[a, b]$.

(a) Why is $R - L = (f(b) - f(a))h$?

(b) Show that any approximating sum for A, formed with rectangles of equal width h and any sampling points, differs from A by at most $(f(b) - f(a))h$.

(c) Let M_1 be the maximum value of $|f'(x)|$ for x in $[a, b]$. Show that any approximating sum for A formed with equal widths h differs from A by at most $M_1(b - a)h$.

(d) Newton also considered the case where the rectangles do not necessarily have the same widths. Let h be the largest of their widths. What can be said about the error in this case?

68.[C] Let f be a continuous function such that $f(x) > 0$ for $x > 0$ and $\int_0^x f(t)\,dt = (f(x))^2$ for $x \geq 0$.

(a) Find $f(0)$.

(b) Find $f(x)$ for $x > 0$.

69.[C] A particle moves on a line in such a way that its average velocity over any interval of time $[a, b]$ is the same as its velocity at $(a + b)/2$. Prove that the velocity $v(t)$ must be of the form $ct + d$ for some constants c and d. HINT: Differentiate the relationship $\int_a^b v(t)\,dt = v\left(\frac{a+b}{2}\right)(b-a)$ with respect to b and with respect to a.

70.[C] A particle moves on a line in such a way that the average velocity over any interval of the form $[a, b]$ is equal to the average of the velocities at the beginning and the end of the interval of time. Prove that the velocity $v(t)$ must be of the form $ct + d$ for some constants c and d.

Exercises 71 and 72 present Archimedes' derivations for the area of a disk and the volume of a ball. He viewed these explanations as informal, and also presented rigorous proofs for them.

71.[C] Archimedes pictured a disk as made up of "almost" isosceles triangles, with one vertex of each triangle at the center of the disk and the base of the triangle part of the boundary of the disk. On the basis of this he conjectured that the area of a disk is one-half the product of the radius and its circumference. Explain why Archimedes' reasoning is plausible.

72.[C] Archimedes pictured a ball as made up of "almost" pyramids, with the vertex of each pyramid at the center of the ball and the base of the pyramid as part of the surface of the ball. On the basis of this he conjectured that the volume of a ball is one-third the product of the radius and its surface area. Explain why Archimedes' reasoning is plausible.

73.[C] (The midpoint estimates for a definite integral is described in Exercises 33 to 35 in Section 6.5.) Let M_n be the midpoint estimate of $\int_a^b f(x)\,dx$ based on n sections of width $h = (b-a)/n$. This exercise shows that the bound on the error, $\left|\int_a^b f(x)\,dx - M_n\right|$ is half of the bound on the trapezoidal estimate. The argument is like that in Exercises 43 and 44 of Section 6.5, a direct application of the Growth Theorem of Section 5.3.
Let $E(t) = \int_{-t/2}^{t/2} f(x)\,dx - f(0)t$.

(a) Show that $E(0) = E'(0) = 0$, and that $E''(t) = \frac{1}{4}\left(f'\left(\frac{t}{2}\right) - f'\left(\frac{-t}{2}\right)\right)$.

(b) Show that $\left|\int_a^b f(x)\,dx - M_n\right| \leq \frac{1}{24}M(b-a)h^2$, where M is the maximum of $|f''(x)|$ for x in $[a,b]$.

74.[C] Let $y = f(x)$ be a function such that $f(x) \geq 0$, $f'(x) \geq 0$, and $f''(x) \geq 0$ for all x in $[1,4]$. An estimate of the area under $y = f(x)$ is made by dividing the interval into sections and forming rectangles. The height of each rectangle is the value of $f(x)$ at the midpoint of the corresponding section.

(a) Show that the estimate is less than or equal to the area under the curve. Hint: Draw a tangent to the curve at each of the midpoints.

(b) How does the estimate compare to the area under the curve if, instead, $f''(x) \leq 0$ for all x in $[1,4]$?

75.[C] The definite integral $\int_0^1 \sqrt{x}\, dx$ gives numerical analysts a pain. The integrand is not differentiable at 0. What is worse, the derivatives (first, second, etc.) of $\sqrt{x}$ become arbitrarily large for x near 0. It is instructive, therefore, to see how the error in Simpson's estimate behaves as h is made small.

(a) Use the FTC to show that $\int_0^1 \sqrt{x}\, dx = \frac{2}{3}$.

(b) Fill in the table. (Keep at least 7 decimal places in each answer.)

h	Simpson's Estimate	Error
$\frac{1}{2}$		
$\frac{1}{4}$		
$\frac{1}{8}$		
$\frac{1}{16}$		
$\frac{1}{32}$		
$\frac{1}{64}$		

(c) In the typical application of Simpson's method, when you cut h by a factor of 2, you find that the error is cut by a factor of $2^4 = 16$. (That is, the ratio of the two errors would be $\frac{1}{16} = 0.0625$.) Examine the five ratios of consecutive errors in the table.

(d) Let $E(h)$ be the error in using Simpson's method to estimate $\int_0^1 \sqrt{x}\, dx$ with sections of length h. Assume that $E(h) = Ah^k$ for some constants k and A. Estimate k and A.

76.[C] Since Simpson's method was designed to be exact when $f(x) = Ax^2+Bx+C$, one would expect the error associated with it to involve $f^{(3)}(x)$. By a quirk of good fortune, Simpson's method happens to be exact even when $f(x)$ is a cubic, $Ax^3 + Bx^2 + Cx + D$. This suggests that the error involves $f^{(4)}(x)$, not $f^{(3)}(x)$. Confirm that this is the case. Note: Exercise 44 in Section 6.5 did this using the Growth Theorem.

(a) Show that $\displaystyle\int_c^d x^3\, dx = \frac{d-c}{6}\left(f(c) + 4f\left(\frac{c+d}{2}\right) + f(d)\right).$

(b) Why is Simpson's estimate exact for cubic polynomials?

77.[C] A producer of wine can choose to store it and sell it at a higher price after it has aged. However, he also must consider storage costs, which should not exceed the revenue.
Assume the revenue he would receive when selling the wine at time t is $V(t)$. If the interest rate on bank balances is r, which we will assume is constant, the present value of that sale is $V(t)e^{-rt}$.
The cost of storing the wine varies with time. Assume $c(t)$ represents that cost, that is, the cost of storing the wine during the short interval $[t, t + \Delta t]$ is approximately $c(t)\Delta t$.

(a) What is the present value of storing the wine for the period $[0, x]$?

(b) What is the present value, $P(x)$, of the profit (or loss) selling all the wine at time x? That is, the present value of the revenue minus the present value of the storage cost if sold at time x?

(c) Show that $P'(x) = V'(x)e^{-rx} - rV(x)e^{-rx} - c(x)e^{-rx}$.

(d) Show that if $V'(x)e^{-rx} > rV(x)e^{-rx} + c(x)e^{-rx}$, then $P'(x)$ is positive, and he should continue to store the wine.

(e) What is the meaning of each of the three terms in the inequality in (d)? Why does that inequality make economic sense?

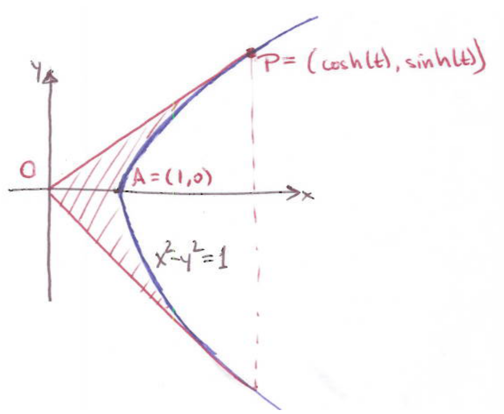


Figure 6.6.3:

78.[M] This exercise verifies the claims made in the last paragraph of Section 5.7.

(a) Explain why, for each angle θ in $[0, \pi]$, a sector of the unit circle with angle 2θ has area θ.

(b) In Figure 6.6.3, the area of the shaded region is twice the area of region OAP. The area of OAP is the area of a triangle less the area under the hyperbola.

Express this area in terms of the parameter t. HINT: This will include a definite integral with integrand $\sqrt{x^2-1}$.

(c) Verify that $\frac{1}{2}\left(x\sqrt{x^2-1}-\ln(x+\sqrt{x^2-1})\right)$ is an antiderivative of $\sqrt{x^2-1}$ for $x>1$.

(d) Show that the area of the shaded region in Figure 6.6.3 is t.

NOTE: Alternate ways to compute the area of the shaded region are found in Exercises 74 on page 764 and 7 on page 1226.

SKILL DRILL: DERIVATIVES

Exercises 79 to 84 offer an opportunity to practice differentiation skills. In each case, verify that the derivative of the first function is the second function.

79.[R] $\ln\left(\frac{e^x}{1+e^x}\right)$; $\frac{1}{1+e^x}$ HINT: To simplify, first take logs.

80.[R] $\frac{1}{m}\arctan(e^{mx})$; $\frac{1}{e^{mx}+e^{-mx}}$ (m is a constant).

81.[R] $\ln(\tan(x))$; $\frac{1}{\sin(x)\cos(x)}$

82.[R] $\tan\left(\frac{x}{2}\right)$; $\frac{1}{1+\cos(x)}$

83.[R] $\frac{1}{2}\ln\left(\frac{1+\sin(x)}{1-\sin(x)}\right)$; $\sec(x)=\frac{1}{\cos(x)}$

84.[R] $\arcsin(x)-\sqrt{1-x^2}$; $\sqrt{\frac{1+x}{1-x}}$

In Exercises 85 to 87 differentiate the given functions.

85.[R] $\frac{\sin(2x)\tan(3x)}{x^3}$

86.[R] $2^{x^2}x^3\cos(4x)$

87.[R] $\frac{x^2e^{3x}}{\sqrt{1+x^2}}$

Calculus is Everywhere
Peak Oil Production

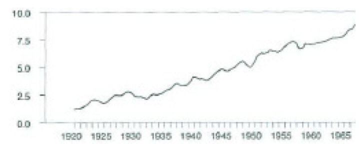


Figure 6.6.1:

The United States in 1956 produced most of the oil it consumed, and the rate of production was increasing. Even so, M. King Hubbert, a geologist at Shell Oil, predicted that production would peak near 1970 and then gradually decline. His prediction did not convince geologists, who were reassured by the rising curve in Figure 6.6.1.

Hubbert was right and the moment of maximum production is known today as Hubbert's Peak.

We present below Hubbert's reasoning in his own words, drawn from "Nuclear Energy and the Fossil Fuels," available at `http://www.hubbertpeak.com/hubbert/1956/1956.pdf`. In it he uses an integral over the entire positive x-axis, a concept we will define in Section 7.8. However, since a finite resource is exhausted in a finite time, his integral is an ordinary definite integral, whose upper bound is not known.

First he stated two principles when analyzing curves that describe the rate of exploitation of a finite resource:

1. For any production curve of a finite resource of fixed amount, two points on the curve are known at the outset, namely that at $t = 0$ and again at $t = \infty$. The production rate will be zero when the reference time is zero, and the rate will again be zero when the resource is exhausted; that is to say, in the production of any resource of fixed magnitude, the production rate must begin at zero, and then after passing through one or several maxima, it must decline again to zero.

2. The second consideration arises from the fundamental theorem of integral calculus; namely, if there exists a single-valued function $y = f(x)$, then

$$\int_0^{x_1} y\,dx = A, \tag{6.6.1}$$

where A is the area between the curve $y = f(x)$ and the x-axis from the origin out to the distance x_1.

In the case of the production curve plotted against time on an arithmetical scale, we have as the ordinate

$$P = \frac{dQ}{dt}, \tag{6.6.2}$$

where dQ is the quantity of the resource produced in time dt. Likewise, from equation (6.6.1) the area under the curve up to any time t is given by

$$A = \int_0^t P\ dt = \int_0^t \left(\frac{dQ}{dt}\right)\ dt = Q, \tag{6.6.3}$$

where Q is the cumulative production up to the time t. Likewise, the ultimate production will be given by

$$Q_{max} = \int_0^\infty P\ dt, \tag{6.6.4}$$

and will be represented on the graph of production-versus-time as the total area beneath the curve.

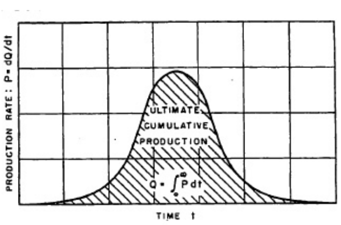


Figure 6.6.2:

These basic relationships are indicated in Figure 6.6.2. The only a priori information concerning the magnitude of the ultimate cumulative production of which we may be certain is that it will be less than, or at most equal to, the quantity of the resource initially present. Consequently, if we knew the production curves, all of which would exhibit the common property of beginning and ending at zero, and encompassing an area equal to or less than the initial quantity.

That the production of exhaustible resources does behave this way can be seen by examining the production curves of some of the older producing areas.

He then examines those curves for Ohio and Illinois. They resembled the curves below, which describe more recent data on production in Alaska, the United States, the North Sea, and Mexico.

Hubbert did not use a particular formula. Instead he employed the key idea in calculus, expressed in terms of production of oil, "The definte integral of the rate of production equals the total production."

He looked at the data up to 1956 and extrapolated the curve by eye, and by logic. This is his reasoning:

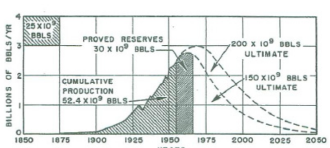


Figure 6.6.4: Ultimate United States crude-oil production based on assumed initial reserves of 150 and 200 billion barrels.

> Figure 6.6.4 shows "a graph of the production up to the present, and two extrapolations into the future. The unit rectangle in this case represents 25 billion barrels so that if the ultimate potential production is 150 billion barrels, then the graph can encompass but six rectangles before returning to zero. Since the cumulative production is already a little more than 50 billion barrels, then only four more rectangles are available for future production. Also, since the production rate is still increasing, the ultimate production peak must be greater than the present rate of production and must occur sometime in the future. At the same time it is possible to delay

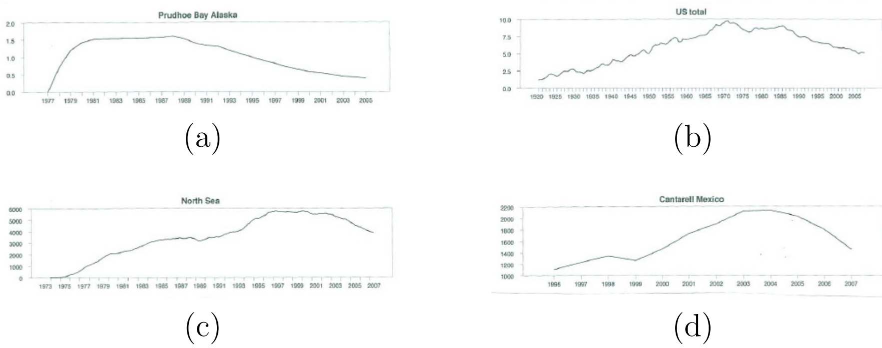


Figure 6.6.3: Annual production of oil in millions of barrels per day for (a) Annual oil production for Prudhoe Bay in Alaska, 1977–2005 [Alaska Department of Revenue], (b) moving average of preceding 12 months of monthly oil production for the United States, 1920–2008 [EIA, "Crude Oil Production"], (c) moving average of preceding 12 months of sum of U.K. and Norway crude oil production, 1973–2007 [EIA, Table 11.1b], and (d) annual production from Cantarell complex in Mexico, 1996–2007 [Pemex 2007 Statistical Yearbook and Green Car Congress (`http://www.greencarcongress.com/2008/01/mexicos-cantare.html`).

> the peak for more than a few years and still allow time for the unavoidable prolonged period of decline due to teh slowing rates of extraction from depleting reservoirs.
>
> With due regard for these considerations, it is almost impossible to draw the production curve based upon an assumed ultimate production of 150 billion barrels in any manner differing significantly from that shown in Figure 6.6.4, according to which the curve must culminate in about 1965 and then must decline at a rate comparable to its earlier rate of growth.
>
> If we suppose the figure of 150 billion barrels to be 50 billion barrels too low — an amount equal to eight East Texas oil fields — then the ultimate potential reserve would be 200 billion barrels. The second of the two extrapolations shown in Figure 6.6.4 is based upon this assumption; but it is interesting to note that even then the date of culmination is retarded only until about 1970."

Geologists are now trying to predict when world production of oil will peak. (Hubbert predicted the peak to occur in the year 2000.) In 2009 oil was being extracted at the rate of 85 million barrels per day. Some say the peak occurred as early as 2005, but others believe it may not occur until after 2020.

To see some of the latest estimates, do a web search for "Hubbert peak oil estimate".

What is just as alarming is that the world is burning oil faster than we are discovering new deposits.

In the CIE on Hubbert's Peak in Chapter 10 (see page 901) we present a later work of Hubbert, in which he uses a specific formula to analyze oil use and depletion.

Summary of Calculus I

The limit is the fundamental concept that forms the foundation for all of calculus. Limits are introduced in Chapter 2.

Chapters 3 through 5 were devoted to one of the two basic concepts in calculus, the derivative, defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

It tells how rapidly a function changes for inputs near x. That is local information.

Chapter 6 introduced the other major concept in calculus, the definite integral, also defined as a limit

$$\int_a^b f(x)\ dx = \lim_{\max \Delta x_i} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

For a continuous function this limit exists. $\int_a^b f(x)\ dx$ can be viewed as the (net) area under the graph of $y = f(x)$ above the interval $[a, b]$. Both the definite integral and an antiderivative of a function are called “integrals.” Context tells which is meant. An antiderivative is also called an “indefinite integral.”

The definite integral, in contrast to the derivative, gives global or overall information.

Integrand: $f(x)$	Integral: $\int_a^b f(x)\ dx$
velocity	change in position
speed (= |velocity|)	distance traveled
length of cross-section of plane region	area of region
area of cross-section of solid	volume of solid
rate bacterial colony grows	total growth

As the first and last of these applications show, if you compute the definite integral of the rate at which some quantity is changing, you get the total change. To put this in mathematical symbols, let $F(x)$ be the quantity present at time x. Then $F'(x)$ is the rate at which it changes.

That equation gives a shortcut for evaluating many common definite integrals. However, finding an antiderivative can be tedious or impossible.

For instance, e^{x^2} does not have an elementary antiderivative. However, continuous functions do have antiderivatives, as slope fields suggest. Indeed $G(x) = \int_a^x f(t)\ dt$ is an antiderivative of the integrand.

One way to estimate a definite integral is to employ one of the sums $\sum_{i=1}^{n} f(c_i)\Delta x_i$ that appear in its definition. A more accurate method, which uses the same amount of arithmetic, uses trapezoids. The trapezoidal estimate takes the form

$$\int_a^b f(x)\ dx \approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right),$$

where consecutive x_is are a fixed distance $hx = (b-a)/n$ apart.

In the even more accurate Simpson's estimate the graph is approximated by parts of parabolas, n is even, and the estimate is

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{x-1}) + 4f(x_{n-1}) + f(x_n)\right).$$

Long Road to Calculus

It is often stated that Newton and Leibniz invented calculus in order to solve problems in the physical world. There is no evidence for this claim. Rather, as with their predecessors, Newton and Leibniz were driven by curiosity to solve the "tangent" and "area" problems, that is, to construct a general procedure for finding tangents and areas. Once calculus was available, it was then applied to a variety of fields, notably physics, with spectacular success.

The first five chapters have presented the foundations of calculus in this order: functions, limits and continuity, the derivative, the definite integral, and the fundamental theorem that joins the last two. This bears little relation to the order in which these concepts were actually developed. Nor can we sense in this approach, which follows the standard calculations syllabus, the long struggle that culminated in the creation of calculus.

The origins of calculus go back over 2000 years to the work of the Greeks on areas and tangents. Archimedes (287–212 B.C.) found the area of a section of a parabola, an accomplishment that amounts in our terms to evaluating $\int_0^b x^2 dx$. He also found the area of an ellipse and both the surface area and the volume of a sphere. Apollonius (around 260–200 B.C.) wrote about tangents to ellipses, parabolas, and hyperbolas, and Archimedes discussed the tangents to a certain spiral-shaped curve. Little did they suspect that the "area" and "tangent" problems were to converge many centuries later.

With the collapse of the Greek world, symbolized by the Emperor Justinian's closing in A.D. 529 of Plato's Academy, which had survived for a thousand years, it was the Arab world that preserved the works of Greek mathematicians. In its liberal atmosphere, Arab, Christian, and Jewish scholars worked together, translating and commenting on the old writings, occasionally adding their own embellishments. For instance, Alhazen (A.D. 965–1039) computed volumes of certain solids, in essence evaluating $\int_0^b x^3 dx$ and $\int_0^b x^4 dx$.

It was not until the seventeenth century that several ideas came together to form calculus. In 1637, both Descartes (1596–1650) and Fermat (1601–165) introduced analytic geometry. Descartes examined a given curve with the aid of algebra, while Fermat took the opposite tack, exploring the geometry hidden in a given equation. For instance, Fermat showed that the graph of $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is always an ellipse, hyperbola, parabola, or one of their degenerate forms.

In this same period, Cavalieri (1598–1647) found the area under the curve $y = x^n$ for $n = 1, 2, 3, \dots, 9$ by a method the length of whose computations grew rapidly as the exponent increased. Stopping at $n = 0$, he conjectured that the pattern would continue for larger exponents. In the next 20 years, several mathematicians justified his guess. So, even the calculation of the area under $y = x^n$ for a positive integer n, which we take for granted, represented a hard-won triumph.

"What about the other exponents?" we may wonder. Before 1665 there were no other exponents. Nevertheless, it was possible to work with the function which we denote $y = x^{p/q}$ for positive integers p and q by describing it as the function y such that $y^q = x^p$. (For instance, $y = x^{2/3}$ would be the function y that satisfies $y^3 = x^2$.) Wallis (1616–1703) found the area by a method that smacks more of magic than of mathematics. However, Fermat obtained the same result with the aid of an infinite geometric series.

The problem of determining tangents to curves was also in vogue in the first half of the seventeenth century. Descartes showed how to find a line perpendicular to a curve at a point P (by constructing a circle that meets the curve only at P); the tangent was then the line through P perpendicular to that line. Fermat found tangents in a way similar to ours and applied it to maximum-minimum problems.

Newton (1642-1727) arrived in Cambridge in 1661, and during the two years 1665–1666, which he spent at his family's farm to avoid the plague, he developed the essentials of calculus — recognizing that finding tangents and calculating areas are inverse problems. The first integral table ever compiled is to be found in one of his manuscripts of this period. But Newton did not publish his results at that time, perhaps because of the depression in the book trade after the Great Fire of London in 1665. During those two remarkable years he also introduced negative and fractional exponents, thus demonstrating that such diverse operations as multiplying a number by itself several times, taking its reciprocal, and finding a root of some power of that number are just special cases of a single general exponential function a^x, where x is a positive integer, -1, or a fraction, respectively.

Independently, however, Leibniz (1646–1716) also invented calculus. A lawyer, diplomat, and philosopher, for whom mathematics was a serious avocation, Leibniz established his version in the years 1673–1676, publishing his researches in 1684 and 1686, well before Newton's first publication in 1711. To Leibniz we owe the notations dx and dy, the terms "differential calculus" and "integral calculus," the integral sign, and the work "function." Newton's notation survives only in the symbol $\dot{x}$ for differentiation with respect to time, which is still used in physics.

It was to take two more centuries before calculus reached its present state of precision and rigor. The notion of a function gradually evolved from "curve" to "formula" to any rule that assigns one quantity to another. The great calculus text of Euler, published in 1748, emphasized the function concept by including not even one graph.

In several texts of the 1820s, Cauchy (1789–1857) defined "limit" and "continuous function" much as we do today. He also gave a definition of the definite integral, which with a slight change by Riemann (1826–1866) in 1854 became the definition standard today. So by the mid-nineteenth century the discoveries of Newton and Leibniz were put on a solid foundation.

In 1833, Liouville (1808–1882) demonstrated that the fundamental theorem could not be used to evaluate integrals of all elementary functions. In fact, he showed that the only values of the constant k for which $\int \sqrt{1-x^2}\sqrt{1-kx^2}dx$ is elementary are 0 and 1.

Still some basic questions remained, such as "What do we mean by area?" (For instance, does the set of points situated within some square and having both coordinates rational have an area? If so, what is this area?) It was as recently as 1887 that Peano (1858–1932) gave a precise definition of area — that quantity which earlier mathematicians had treated as intuitively given.

The history of calculus therefore consists of three periods. First, there was the long stretch when there was no hint that the tangent and area problems were related. Then came the discovery of their intimate connection and the exploitation of this relation from the end of the seventeenth century through the eighteenth century. This was followed by a century in which the loose ends were tied up.

The twentieth century saw calculus applied in many new areas, for it is the natural language for dealing with continuous processes, such as change with time. In that century mathematicians also obtained some of the deepest theoretical results about its foundations.

References

E. T. Bell, *Men of Mathematics*, Simon and Schuster, New York, 1937. (In particular, Chapter 6 on Newton and Chapter 7 on Leibniz.)

Carl B. Boyer, *The History of the Calculus and Its Conceptual Development*, Dover, New York, 1959.

Carl B. Boyer, *A History of Mathematics*, Wiley, New York, 1968.

M. J. Crowe, *A History of Vector Analysis*, Dover, New York, 1985. (In particular, see Chapter 5.)

Philip J. Davis and Reuben Hersh, *The Mathematical Experience*, Birkhäuser Boston, Cambridge, 1981.

C. H. Edwards, Jr. *The Historical Development of the Calculus*, Springer-Verlag, New York, 1879. (In particular, Chapter 8, "The Calculus According to Newton," and Chapter 9, "The Calculus According to Leibniz.")

Morris Kline, *Mathematical Thoughts from Ancient to Modern Times*, Oxford, New York, 1972. (In particular, Chapter 17.)

Pronunciation

Descartes	“Day-CART”
Fermat	“Fair-MA”
Leibniz	“LIBE-nits”
Euler	“OIL-er”
Cauchy	“KOH-shee”
Riemann	“REE-mahn”
Liouville	“LYU-veel”
Peano	“Pay-AHN-oh”