Chapter 6

The Definite Integral

Up to this point we have been concerned with the derivative, which provides local information, such as the slope at a particular point on a curve or the velocity at a particular time. Now we introduce the second major concept of calculus, the definite integral. In contrast to the derivative, the definite integral provides global information, such as the area under a curve.

Section 6.1 motivates the definite integral through three of its applications. Section 6.2 defines the definite integral and Section 6.3 presents ways to estimate it. Sections 6.4 and 6.5 develop the connection between the derivative and the definite integral, which culminates in the Fundamental Theorems of Calculus. The derivative turns out to be essential for evaluating many definite integrals.

Chapters 2 to 6 form the core of calculus. Later chapters are mostly variations or applications of the key ideas in those chapters.

6.1 Three Problems That Are One Problem

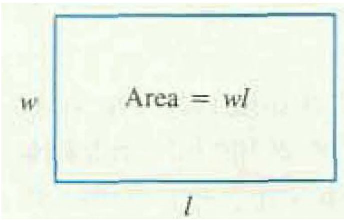


Figure 6.1.1:

The definite integral is introduced with three problems. At first glance these problems may seem unrelated, but by the end of the section it will be clear that they represent one basic problem in various guises. They lead up to the concept of the definite integral, defined in the next section.

Estimating an Area

It is easy to find the exact area of a rectangle: multiply its length by its width (see Figure 6.1.1). But how do you find the area of the region in Figure 6.1.2? In this section we will show how to make accurate *estimates* of that area. The technique we use will lead up in the next section to the definition of the definite integral of a function.

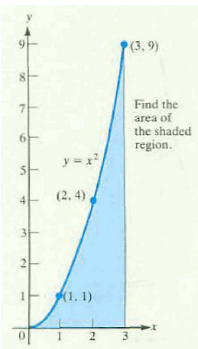


Figure 6.1.2:

PROBLEM 1 Estimate the area of the region bounded by the curve $y = x^2$, the x-axis, and the vertical line $x = 3$, as shown in Figure 6.1.2.

Since we know how to find the area of a rectangle, we will use rectangles to approximate the region. Figure 6.1.3(a) shows an approximation by six rectangles whose total area is more than the area under the parabola. Figure 6.1.3(b) shows a similar approximation whose area is less than the area under the parabola.

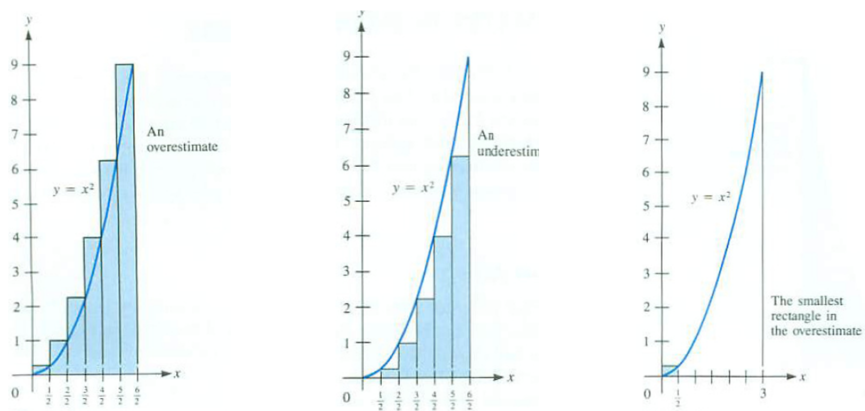


Figure 6.1.3:

In each case we break the interval $[0, 3]$ into six short intervals, all of width $\frac{1}{2}$. In order to find the areas of the overestimate and of the underestimate, we must find the height of each rectangle. That height is determined by the curve $y = x^2$. Let us examine only the overestimate, leaving the underestimate for the Exercises.

There are six rectangles in the overestimate shown in Figure 6.1.3(a). The smallest rectangle is shown in Figure 6.1.3(c). The height of this rectangle is equal to the value of x^2 when $x = \frac{1}{2}$. Its height is therefore $\left(\frac{1}{2}\right)^2$ and its area is $\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)$, the product of its height and its width. The areas of the other five rectangles can be found similarly. In each case evaluate x^2 at the right end of the rectangle's base in order to find the height. The total area of the six rectangles is

$$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{2}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{6}{2}\right)^2\left(\frac{1}{2}\right).$$

This equals

$$\frac{1}{8}\left(1^2+2^2+3^2+4^2+5^2+6^2\right)=\frac{91}{8}=11.375. \tag{6.1.1}$$

The area under the parabola is therefore less than 11.375.

To get a closer estimate we should use more rectangles. Figure 6.1.4 shows an overestimate in which there are 12 rectangles. Each has width $\frac{3}{12}=\frac{1}{4}$. The total area of the overestimate is

$$\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{2}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)+\cdots+\left(\frac{12}{4}\right)^2\left(\frac{1}{4}\right).$$

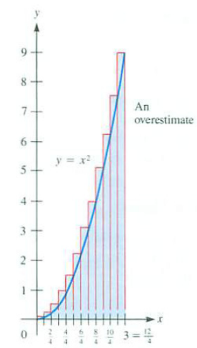


Figure 6.1.4:

This equals

$$\frac{1}{4^3}\left(1^2+2^2+3^2+\cdots+12^2\right)=\frac{650}{64}=10.15625. \tag{6.1.2}$$

Now we know the area under the parabola is less than 10.15625.

To get closer estimates we would cut the interval $[0,3]$ into more sections, maybe 100 or 10,000 or more, and calculate the total area of the corresponding rectangles. (This is an easy computation on a computer.)

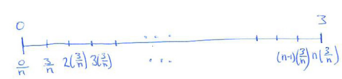


Figure 6.1.5:

In general, we would divide $[0,3]$ into n sections of equal length. The length of each section is then $\frac{3}{n}$. Their endpoints are shown in Figure 6.1.5.

Then, for each integer $i = 1,\ 2,\ \ldots, n$, the i^{th} section from the left has endpoints $(i-1)\left(\frac{3}{n}\right)$ and $i\left(\frac{3}{n}\right)$, as shown in Figure 6.1.6.

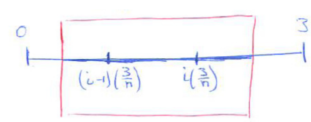


Figure 6.1.6: [ARTIST: Redraw Figure 6.1.6 to give effect of zooming in on ith interval]

To make an overestimate, observe that x^2 is increasing for $x > 0$ and evaluate x^2 at the right endpoint of each interval. Then multiply the result by the width of the interval, getting

$$\left(i\left(\frac{3}{n}\right)\right)^2\frac{3}{n}=3^3\frac{i^2}{n^3}.$$

Then, sum these overestimates for all n intervals:

$$3^3\frac{1^2}{n^3}+3^3\frac{2^2}{n^3}+3^3\frac{3^2}{n^3}+\cdots+3^3\frac{(n-1)^2}{n^3}+3^3\frac{n^2}{n^3}$$

which simplifies to

$$3^3 \left(\frac{1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2}{n^3} \right). \tag{6.1.3}$$

In the summation notation described in Appendix C, this equals

$$\frac{3^3}{n^3} \sum_{i=1}^{n} i^2.$$

We have already seen that these overestimates become more and more accurate as the number of intervals increases. We would like to know what happens to the overestimate as n gets larger and larger. More specifically, does

$$\lim_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2}{n^3} \tag{6.1.4}$$

exist? If it does exist, call it L. (Then the area would be 3^3L.)

The numerator gets large, tending to make the fraction large. But the denominator also gets large, which tends to make the fraction small. Once again we encounter one of the "limit battles" that occurs in the foundation of calculus.

Archimedes, some 2200 years ago, found a short formula for the numerator in (6.1.3), enabling him to find the limit in (6.1.4). See, for instance, S. Stein, "Archimedes: What did he do besides cry Eureka?".

To estimate L, use, say, $n = 6$. Then we have

$$\frac{1}{6^3}\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right) = \frac{91}{216} \approx 0.42130.$$

Try a larger value of n to get a closer estimate of L.

If we knew L we would know the area under the parabola and above the interval $[0, 3]$, for the area is 3^3L. Since we do not know L, we don't know the area. Be patient. We will find L indirectly in this section. You may want to compute the quotient in (6.1.4) for some n and guess what L is. For example, with $n = 12$, the estimate is $\frac{650}{12^3} = \frac{650}{1728} \approx 0.37616$.

Estimating a Distance Traveled

The units simplify: $\frac{\text{mi}}{\text{hr}} \times \text{hr} = \text{mi}$.

If you drive at a constant speed of v miles per hour for a period of t hours, you travel vt miles:

$$\text{Distance} = \text{Speed} \times \text{Time} = vt \text{ miles}.$$

But how would you compute the total distance traveled if your speed were not constant? (Imagine that your odometer, which records distance traveled, was broken. However, your speedometer is still working fine, so you know your speed at any instant.) The next problem illustrates how you could make accurate estimates of the total distance traveled.

PROBLEM 2 A snail is crawling about for three minutes. This remarkable snail knows that she is traveling at the rate of t^2 feet per minute at time t minutes. For instance, after half a minute, she is slowly moving at the rate of $\left(\frac{1}{2}\right)^2$ feet per minute. At the end of her journey she is moving along at 3^2 feet per minute. Estimate how far she travels during the three minutes.

Figure 6.1.7:

The speed during the three-minute trip increases from 0 to 9 feet per minute. During shorter time intervals, such a wide fluctuation does not occur. As in Problem 1, cut the three minutes of the trip into six equal intervals each 1/2 minute long, and use them to estimate the total distance covered. Represent time by a line segment cut into six parts of equal length, as in Figure 6.1.7.

Speed increases as t increases.

Consider the distance she travels during one of the six half-minute intervals, say during the interval $[\frac{3}{2}, \frac{4}{2}]$. At the beginning of this time interval her speed was $\left(\frac{3}{2}\right)^2$ feet per minute; at the end she was going $\left(\frac{4}{2}\right)^2$ feet per minute. The highest speed during this half hour was $\left(\frac{4}{2}\right)^2$ feet per minute. Therefore, she traveled at most $\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)$ feet during the time interval $[3/2, 4/2]$. Similar reasoning applies to the other five half-minute periods. Adding up these upper estimates for the distance traveled during each interval of time, we get the total distance traveled is less than

$$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{2}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{6}{2}\right)^2\left(\frac{1}{2}\right).$$

If we divide the time interval into n equal sections of duration $\frac{3}{n}$, the right endpoint of the i^{th} interval is $i\left(\frac{3}{n}\right)$. At that time the speed is $(3i/n)^2$ feet per minute. So the distance covered during the i^{th} interval of time is less than

$$\underbrace{\left(\frac{3i}{n}\right)^2}_{\text{max speed}}\underbrace{\frac{3}{n}}_{\text{time}} = \frac{3^3 i^2}{n^3}.$$

The total overestimate is then

$$3^3\frac{1^2}{n^3} + 3^3\frac{2^2}{n^3} + 3^3\frac{3^2}{n^3} + \cdots + 3^3\frac{(n-1)^2}{n^3} + 3^3\frac{n^2}{n^3}$$

or

$$3^3\left(\frac{1^2+2^2+3^2+\cdots+(n-1)^2+n^2}{n^3}\right). \tag{6.1.5}$$

The calculations in the area problem, (6.1.3), and in the distance problem, (6.1.5), are the same. Thus, the area and distance have the same upper estimates. Their lower estimates are also the same, as you may check. The limit of (6.1.5) is 3^3L. The two problems are really the same problem.

Estimating a Volume

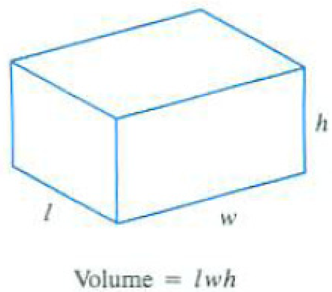
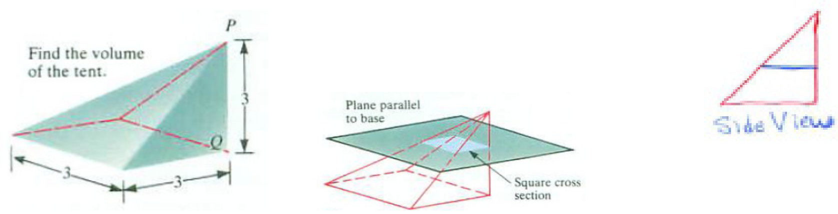


Figure 6.1.8:

The volume of a rectangular box is easy to compute; it is the product of its length, width, and height. See Figure 6.1.8. But finding the volume of a pyramid or ball requires more work. The next example illustrates how we can estimate the volume inside a certain tent.

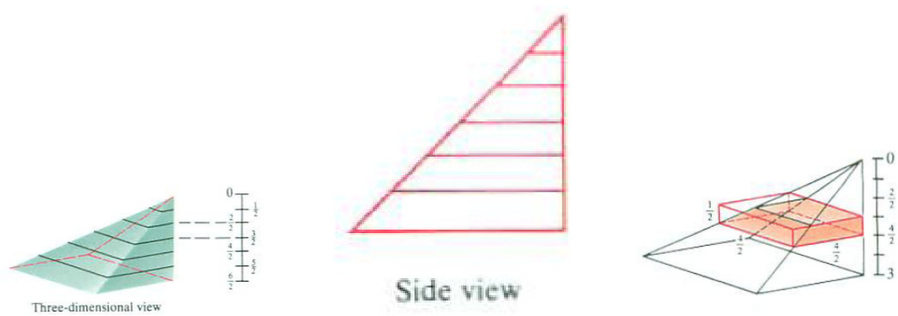
PROBLEM 3 Estimate the volume inside a tent with a square floor of side 3 feet, whose vertical pole, 3 feet long, is located *above one corner* of the floor. The tent is shown in Figure 6.1.9(a).



(a) (b) (c)

Figure 6.1.9:

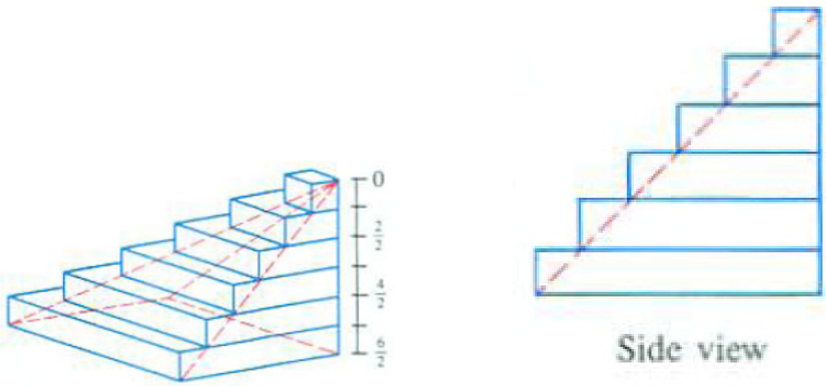
The cross-section of the tent made by any plane parallel to the base is a square, as shown in Figure 6.1.9(b). The width of the square equals its distance from the top of the pole, as shown in Figure 6.1.9(c). Using this fact, we can approximate the volume inside the tent with rectangular boxes with square cross-sections. Begin by cutting a vertical line, representing the pole, into six



(a) (b) (c)

Figure 6.1.10:

sections of equal length, each $\frac{1}{2}$ foot long. Draw the corresponding square cross section of the tent, as in Figure 6.1.10(a). Use these square cross-sections to



(a) (b)

Figure 6.1.11:

form rectangular boxes. Consider the part of the tent corresponding to the interval $[\frac{3}{2}, \frac{4}{2}]$ on the pole. The base of this section is a square with sides $\frac{4}{2}$ feet. The box with this square as a base and height $\frac{1}{2}$ foot encloses completely the part of the tent corresponding to $[\frac{3}{2}, \frac{4}{2}]$. (See Figure 6.1.10(c).) The volume of this box is $\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)$ cubic feet. Figure 6.1.11(a) shows six such boxes, whose total volume is greater than the volume of the tent.

Since the volume of each box is the area of its base times its height, the total volume of the six boxes is

$$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{2}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{6}{2}\right)^2\left(\frac{1}{2}\right) \quad \text{cubic feet.}$$

This sum, which we have encountered twice before, equals 11.375. It is an *overestimate* of the volume of the tent. Better (over)estimates can be obtained by cutting the pole into shorter pieces. Evidently, the arithmetic for the tent volume is the same as for the previous two problems.

We now know that the number describing the volume of the tent is the same as the number describing the area under the parabola and also the length of the snail's journey. That number is 3^3L. The arithmetic of the estimates is the same in all three cases.

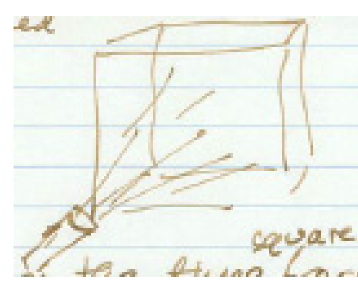
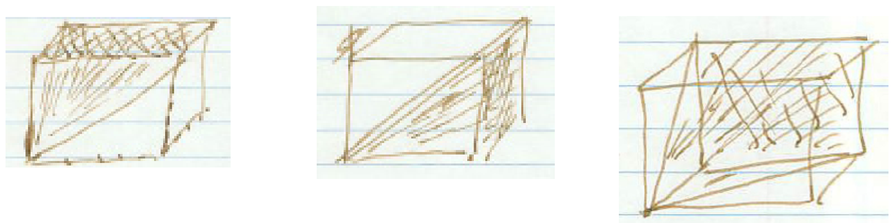


Figure 6.1.12:

A Neat Bit of Geometry

If we knew the limit L in (6.1.3), we would then find the answers to all three problems. But we haven't found L. Luckily, there is a way to find the volume of the tent without knowing L.

The key is that *three identical copies of the tent* fill up a cube of side 3 feet. To see why, imagine a flashlight at one corner of the cube, aimed into the cube, as in Figure 6.1.12.



(a) (b) (c)

Figure 6.1.13:

This trick is like the way the area of a right triangle is found by arranging two copies to form a rectangle.

The flashlight illuminates the three square faces not meeting the corner at the flashlight. The rays from the flashlight to each of the faces fill out a copy of the tent, as shown in Figure 6.1.13.

Since three copies of the tent fill a cube of volume $3^3 = 27$ cubic feet, the tent has volume 9 cubic feet. From this, we see that the area under the parabola above $[0, 3]$ is 9 and the snail travels 9 feet. Incidentally, the limit L must be $\frac{1}{3}$, since the area under the parabola is both 9 and 3^3L. In short,

$$\lim_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2. = \frac{1}{3}$$

Summary

Using upper estimates, we showed that problems concerning area, distance traveled, and volume were the same problem in various disguises. We were really studying a problem concerning a particular function, x^2, over a particular interval $[0, 3]$. We solved this problem by cutting a cube into three congruent pieces. By the end of this chapter you will learn general techniques that will make such a special device unnecessary.

EXERCISES for Section 6.1 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 21 concern estimates of areas under curves.

1.[R] In Problem 1 we broke the interval $[0,3]$ into six sections. Instead, break $[0,3]$ into four sections of equal lengths and estimate the area under $y = x^2$ and above $[0,3]$ as follows.

(a) Draw the four rectangles whose total area is larger than the area under the curve. The value of x^2 at the right endpoint of each section determines the height of each rectangle.

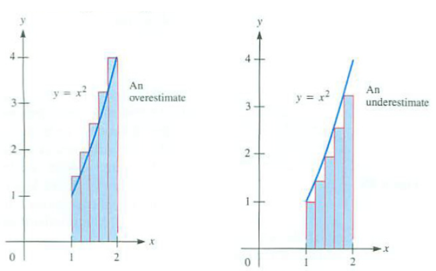
(b) On the diagram in (a), show the height and width of each rectangle.

(c) Find the total area of the four rectangles.

2.[R] Like Exercise 1, but this time obtain an underestimate of the area by using the value of x^2 at the left endpoint of each section to determine the height of the rectangles.

3.[R] Estimate the area under $y = x^2$ and above $[1,2]$ using the five rectangles with equal widths shown in Figure 6.1.14(a).

4.[R] Repeat Exercise 3 with the five rectangles in Figure 6.1.14(b).



(a) (b)

Figure 6.1.14:

5.[R] Evaluate

(a) $\sum_{i=1}^{4} i^2$

(b) $\sum_{i=1}^{4} 2^i$

(c) $\sum_{n=3}^{4} (n-3)$

6.[R] Evaluate

(a) $\sum_{i=1}^{4} i^3$

(b) $\sum_{i=2}^{5} 2^i$

(c) $\sum_{k=1}^{4} (k^3 - k^2)$

7.[R] Figure 6.1.15(a) shows the curve $y = \frac{1}{x}$ above the interval $[1,2]$ and an approximation to the area under the curve by five rectangles of equal width.

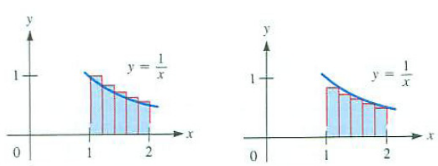
(a) Make a large copy of Figure 6.1.15(a).

(b) On your diagram show the height and width of each rectangle.

(c) Find the total area of the five rectangles.

(d) Find the total area of the five rectangles in Figure 6.1.15(b).

(e) On the basis of (c) and (d), what can you say about the area under the curve $y = 1/x$ and above $[1,2]$?



(a) (b)

Figure 6.1.15:

Exercises 8 and 9 develop underestimates for each of the problems considered in this section.

8.[R] In Problem 1 we found overestimates for the area under the parabola x^2 over the interval $[0, 3]$. Here we obtain underestimates for this area as follows.

(a) Break $[0, 3]$ into six sections of equal lengths and draw the six rectangles whose total area is smaller than the area under the curve.

(b) Because x^2 is increasing on $[0, 3]$, the left endpoint of each section determines the height of each rectangle. Show the height and width of each rectangle you drew (a).

(c) Find the total area of the six rectangles.

9.[R] Repeat Exercise 8 with 12 sections of equal lengths.

10.[R] Consider the area under $y = 2^x$ and above $[-1, 1]$.

(a) Graph the curve and estimate the area by eye.

(b) Make an overestimate of the area, using four sections of equal width.

(c) Make an underestimate of the area, using four sections of equal width.

11.[R] Use the information found in Exercises 3 and 4 to complete this sentence:
The area in Problem 1 is certainly less than ______ but larger than ______.

12.[R] Estimate the area in Problem 1, using the division of $[0, 3]$ into four sections with endpoints 0, 1, $\frac{5}{3}$, $\frac{11}{4}$, and 3 (see Figure 6.1.16(a)).

(a) Estimate the area when the right-hand endpoints of each section are used to find the heights of the rectangles.

(b) Repeat (a), using the left-hand endpoints of each section to find the heights of the rectangles.

(c) Repeat (a) computing the heights of the rectangles at the points $\frac{1}{2}$, $\frac{3}{2}$, 2, and $\frac{14}{5}$.

Figure 6.1.16:

In each of Exercises 13 to 18

(a) Draw the region.

(b) Draw six rectangles of equal widths whose total area overestimates the area of the region.

(c) On your diagram indicate the height and width of each rectangle.

(d) Find the total area of the six rectangles. (Give this answer accurate to two decimal places.)

13.[R] Under $y = x^2$, above $[2, 3]$.

14.[R] Under $y = \frac{1}{x}$, above $[2, 3]$.

15.[R] Under $y = x^3$, above $[0, 1]$.

16.[R] Under $y = \sqrt{x}$, above $[1, 4]$.

17.[M] Under $y = \sin(x)$, above $[0, \pi/2]$.

18.[M] Under $y = \ln(x)$, above $[1, e]$.

19.[M] Estimate the area under $y = x^2$ and above $[-1, 2]$ by dividing the interval into six sections of equal lengths.

(a) Draw the six rectangles that form an overestimate for the area under the curve. Note that you cannot do this using only left-endpoints or only right-endpoints.

(b) Find the total area of all six rectangles.

(c) Repeat (a) and (b) to find an underestimate for this area.

20.[M] Estimate the area between the curve $y = x^3$, the x-axis, and the vertical line $x = 6$ using a division into

(a) six sections of equal lengths with left endpoints;

(b) six sections of equal lengths with right endpoints;

(c) three sections of equal lengths with midpoints;

(d) six sections of equal lengths with midpoints.

21.[M] Estimate the area below the curve $y = \frac{1}{x^2}$ and above $[1, 7]$ following the directions in Exercise 20.

22.[M] To estimate the area in Problem 1 you divide the interval $[0,3]$ into n sections of equal lengths. Using the right-hand endpoint of each of the n sections you then obtain an overestimate. Using the left-hand endpoint, you obtain an underestimate.

(a) Show that these two estimates differ by $\frac{27}{n}$.

(b) How large should n be chosen in order to be sure the difference between the upper estimate and the area under the parabola is less than 0.01?

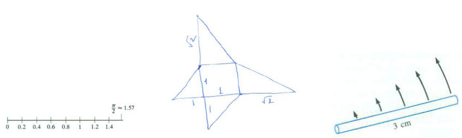
23.[M] Estimate the area of the region under the curve $y = \sin(x)$ and above the interval $[0, \frac{\pi}{2}]$, cutting the interval as shown in Figure 6.1.17(a) and using

(a) left endpoints

(b) right endpoints

(c) midpoints.

(All but the last section are of the same length.)



(a) (b) (c)

Figure 6.1.17:

24.[M] Make three copies of the tent in Problem 3 by folding a pattern as shown in Figure 6.1.17(b). Check that they fill up a cube.

25.[M] An electron is being accelerated in such a way that its velocity is t^3 kilometers per second after t seconds. Estimate how far it travels in the first 4 seconds, as follows:

(a) Draw the interval $[0,4]$ as the time axis and cut it into eight sections of equal length.

(b) Using the sections in (a), make an estimate that is too large.

(c) Using the sections in (a), make an estimate that is too small.

26.[M] A business which now shows no profit is to increase its profit flow gradually in the next 3 years until it reaches a rate of 9 million dollars per year. At the end of the first half year the rate is to be $\frac{1}{4}$ million dollars per year; at the end of 2 years, 4 million dollars per year. In general, at the end of t years, where t is any number between 0 and 3, the rate of profit is to be t^2 million dollars per year. Estimate the total profit during its first 3 years if the plan is successful using

(a) using six intervals and left endpoints;

(b) using six intervals and right endpoints;

(c) using six intervals and midpoints.

27.[M] Oil is leaking out of a tank at the rate of 2^{-t} gallons per minute after t minutes. Describe how you would estimate how much oil leaks out during the first 10 minutes. Illustrate your procedure by computing one estimate.

28.[C] Archimedes showed that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. You can prove this as follows:

(a) Check that the formula is correct for $n = 1$.

(b) Show that if the formula is correct for the integer n, it is also correct for the next integer, $n+1$.

(c) Why do (a) and (b) together show that Archimedes' formula holds for all positive integers n?

NOTE: This type of proof is known as **mathematical induction**.

29.[C]

(a) Explain why the area of the region under the curve $y = x^2$ and above the interval $[0, b]$ is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{bi}{n}\right)^2 \frac{b}{n}.$$

(b) Use Exercise 28 to find this limit.

(c) Give an explicit formula for the area of the region under $y = x^2$ and above $[0, b]$.

(d) For $0 < a < b$, what is the area under the curve $y = x^2$ and above the interval $[a, b]$?

30.[C] The function $f(x)$ is increasing for x in the interval $[a,b]$ and is positive. To estimate the area under the graph of $y = f(x)$ and above $[a,b]$ you divide the interval $[a,b]$ into n sections of equal lengths. You then form an overestimate B (for "big") using right-hand endpoints of the sections and an underestimate S (for "small") using left-hand endpoints. Express the difference between the two estimates, $B - S$, as simply as possible.

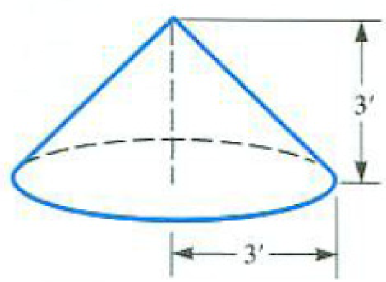
31.[C] A right circular cone has a height of 3 feet and a radius of 3 feet, as shown in Figure 6.1.18. Estimate its volume by the sum of the volumes of six cylindrical slabs, just as we estimated the volume of the tent with the aid of six rectangular slabs.

(a) Make a large and neat diagram that shows the six cylinders used in making an overestimate.

(b) Compute the total volume of the six cylinders in (a).

(c) Make a separate diagram showing a corresponding underestimate.

(d) Compute the total volume of the six cylinders in (c). (Note: One of the cylinders has radius 0.)



Right circular cone
of height 3 feet
and radius 3 feet

Figure 6.1.18:

32.[C] The kinetic energy of an object, for example, a baseball or car, of mass m gram
timeters per second is defined as $\frac{1}{2}$
certain machine a uniform rod 3 c
weighing 32 grams rotates once pe
of its ends as shown in Figure 6.1.
kinetic energy of this rod by cuttin
each $\frac{1}{2}$ centimeter long, and takin
section" the speed of its midpoint.

33.[C] Express the sum $\sum_{i=1}^{n} \ln \Big($
possible. (So that you could comp
fewest steps.)

Skill Drill

In Exercises 34 to 39 differentiate

34.[R] $(1+x^2)^{4/3}$

35.[R] $\frac{(1+x^3)\sin(3x)}{\sqrt[3]{5x}}$ **38.**[R]

36.[R] $\frac{3x}{8} + \frac{3x\sin(4x)}{32} + \frac{\cos^3(2x)\sin(2x)}{8}$ **39.**[R]

37.[R] $\frac{3}{8(2x+3)^2} - \frac{1}{4(2x+3)}$

In Exercises 40 to 50 give an antic
pression.

40.[R] $(x+2)^3$ **46.**[R]

41.[R] $(x^2+1)^2$ **47.**[R]

42.[R] $x\sin(x^2)$ **48.**[R]

43.[R] $x^3 + \frac{1}{x^3}$ **49.**[R]

44.[R] $\frac{1}{\sqrt{x}}$ **50.**[R]

45.[R] $\frac{3}{x}$

6.2 The Definite Integral

We now introduce the other main concept in calculus, the "definite integral of a function over an interval."

The preceding section was not really about area under a parabola, distance a snail traveled, or volume of a tent. The common theme of all three was a procedure we carried out with the function x^2 and the interval $[0, 3]$: Cut the interval into small pieces, evaluate the function somewhere in each section, form certain sums, and then see how those sums behave as we choose the sections smaller and smaller.

Here is the general procedure. We have a function f defined at least on an interval $[a, b]$. We cut, or "partition," the interval into n sections by the numbers $x_0 = a$, x_1, x_2, …, x_{n-1}, $x_n = b$, as in Figure 6.2.1. They need not all be of the same length, though usually, for convenience, they will be.

The sections $[a, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, b]$ form a **partition** of $[a, b]$.

Figure 6.2.1:

Then we pick a **sampling number** in each interval, c_1 in $[x_0, x_1]$, c_2 in $[x_1, x_2]$, …, c_i in $[x_{i-1}, x_i]$, …, c_n in $[x_{n-1}, x_n]$ (as in Figure 6.2.1). In Section 6.1, the c_i's were mostly either right-hand or left-hand endpoints or midpoints. However, they can be anywhere in each section.

Next we bring in the particular function f. (In Section 6.1 the function was x^2.) We evaluate that function at each c_i and form the sum

$$f(c_1)(x_1 - x_0) + f(c_2)(x_2 - x_1) + \cdots + f(c_i)(x_i - x_{i-1}) \qquad (6.2.1)$$
$$+ \cdots + f(c_{n-1})(x_{n-1} - x_{n-2}) + f(c_n)(x_n - x_{n-1}). \qquad (6.2.2)$$

Rather than continue to write out such a long expression, we choose to take advantage of the fact that each term in (6.2.1) follows the same general pattern: for each of the n sections, multiply the function value at the sampling number by the length of the section. This pattern is easily expressed in the shorthand Σ-notation as:

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}). \qquad (6.2.3)$$

If the length of section i is written as $\Delta x_i = x_i - x_{i-1}$, the expression for the sum becomes even shorter:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i. \tag{6.2.4}$$

If all the sections have the same length, each Δx_i equals $(b-a)/n$, since the length of $[a,b]$ is $b-a$. Let Δx denote $\frac{b-a}{n}$. We can write (6.2.3) and (6.2.4) also as

$$\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) \qquad \text{or as} \qquad \sum_{i=1}^{n} f(c_i)\Delta x \tag{6.2.5}$$

where $\Delta x = \frac{b-a}{n}$.

The final step is to investigate what happens to the sums of the form (6.2.4) (or (6.2.5)) as the lengths of all the sections approach 0. That is, we try to find

$$\lim_{\text{all } \Delta x_i \text{ approach } 0} \sum_{i=1}^{n} f(c_i)\Delta x_i. \tag{6.2.6}$$

The sums in (6.2.1)–(6.2.5) are called **Riemann sums** in honor of the nineteenth century mathematician, Bernhard Riemann.

Bernhard Riemann, 1826–1866, http://en.wikipedia.org/wiki/Bernhard_Riemann.

In advanced mathematics it is proved that if f is continuous on $[a,b]$ then the sums in (6.2.6) do approach a single number. This brings us to the definition of the definite integral.

The Definite Integral

DEFINITION *(Definite Integral of a function f over an interval $[a,b]$)* Let f be a continuous function defined at least on the interval $[a,b]$. The limit of sums of the form $\sum_{i=1}^{n} f(c_i)\Delta x_i$, for partitions of $[a,b]$ where every Δx_i approaches 0, exists (no matter how the sampling numbers c_i are chosen). The limiting value is called the **definite integral of f over the interval** $[a,b]$ and is denoted

$$\int_a^b f(x)\,dx.$$

Gottfried Liebniz, 1646–1716, http://en.wikipedia.org/wiki/Gottfried_Leibniz.

NOTE: The symbol $\int$ comes from "S," for "sum". The "dx," strictly speaking, is not needed. Both symbols were introduced by Liebniz.

The limit in this definition is a little unusual. It requires the length of every segment within the partition to approach 0. It is not sufficient to simply consider partitions of $[a,b]$ with more and more segments as this does not prevent segments with lengths that do not approach 0. Another way of stating this requirement is that the length of the largest segment in the partion must approach zero. This

EXAMPLE 1 Express the area under $y = x^2$ and above $[0,3]$ as a definite integral.
SOLUTION Here the function is $f(x) = x^2$ and the interval is $[0,3]$. As we saw in the previous section, the area equals the limit of Riemann sums

$$\lim_{\Delta x \to 0^+} \sum_{i=1}^{n} c_i^2 \Delta x = \int_0^3 x^2\ dx. \tag{6.2.7}$$

◇

The dx traditionally suggests the length of a small section of the x-axis and denotes the **variable of integration** (usually x, as in this case). The function $f(x)$ is called the **integrand**, while the numbers a and b are called the **limits of integration**; a is the **lower limit of integration** and b is the **upper limit of integration**.

The symbol $\int_a^b x^2\ dx$ is read as "the integral from a to b of x^2". Freeing ourselves from the variable x, we could say, "the integral from a to b of the squaring function". There is nothing special about the symbol x in "x^2." We could just as well have used the letter t — or any other letter. (We would typically pick a letter near the end of the alphabet, since letters near the beginning are customarily used to denote constants.) The notations

$$\int_a^b x^2\ dx, \qquad \int_a^b t^2\ dt, \qquad \int_a^b z^2\ dz, \qquad \int_a^b u^2\ du, \qquad \int_a^b \theta^2\ d\theta$$

all denote the same number, that is, "the definite integral of the squaring function from a to b". Taken to the extreme, we could express (6.2.7) as

$$\int_a^b (\ )^2\ d(\ ).$$

Usually, however, we find it more convenient to use some letter to name the independent variable. Since the letter chosen to represent the variable has no significance of its own, it is called a **dummy variable**. Later in this chapter there will be cases where the interval of integration is $[a,x]$ instead of $[a,b]$. Were we to write $\int_a^x x^2\ dx$, it would be easy to think there is some relation between the x in x^2 and the x in the upper limit of integration. To avoid possible confusion, we prefer to use a different dummy variable and write, for example, $\int_a^x t^2 \, dt$ in such cases.

It is important to realize that area, distance traveled, and volume are merely applications of the definite integral. (It is a mistake to link the definite integral too closely with one of its applications, just as it narrows our understanding of the number 2 to link it always with the idea of two fingers.) The definite integral $\int_a^b f(x) \, dx$ is also call the **Riemann integral**.

derivatives are limits

Slope and velocity are particular interpretations or applications of the derivative, which is a purely mathematical concept defined as a limit:

$$\text{derivative of } f \text{ at } x = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

definite integrals are also limits

Similarly, area, total distance, and volume are just particular interpretations of the definite integral, which is also defined as a limit:

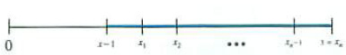
$$\text{definite integral of } f \text{ over } [a, b] = \lim_{\text{as all } \Delta x_i \to 0^+} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}).$$

The Definite Integral of a Constant Function

To bring the definition down to earth, let us use it to evaluate the definite integral of a constant function.

EXAMPLE 2 Let f be the function whose value at any number x is 4; that is, f is the constant function given by the formula $f(x) = 4$. *Use only the definition* of the definite integral to compute

$$\int_1^3 f(x) \, dx.$$



A typical partition of [1,3]

Figure 6.2.2:

SOLUTION In this case, every partition of the interval $[1, 3]$ has $x_0 = 1$ and $x_n = 3$. See Figure 6.2.2. Since, no matter how the sampling number c_i is chosen, $f(c_i) = 4$, the approximating sum equals

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} 4(x_i - x_{i-1})$$

Now

$$\sum_{i=1}^{n} 4(x_i - x_{i-1}) = 4\sum_{i=1}^{n}(x_i - x_{i-1}) = 4 \cdot (x_n - x_0) = 4 \cdot 2 = 8.$$

This is true because the sum of the widths of the sections is the width of the interval $[1, 3]$, namely 2. All approximating sums have the same value, namely, 8. For every partition,

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = 8.$$

Thus, as all sections are chosen smaller, the values of the sums are always 8. This number must be the limit:

$$\int_1^3 4\ dx = 8.$$

◇

We could have guessed the value of $\int_1^3 4\ dx$ by interpreting the definite integral as as area. To do so, draw a rectangle of height 4 and base coinciding with the interval $[1, 3]$. (See Figure 6.2.3.) Since the area of a rectangle is its base times its height, it follows again that $\int_1^3 4\ dx = 8$.

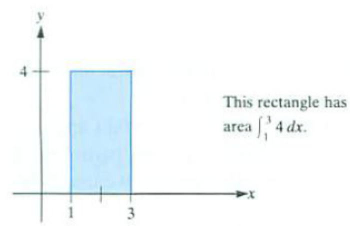


Figure 6.2.3:

Similar reasoning shows that for any constant function that has the fixed value c,

$$\int_a^b c\ dx = c(b-a) \qquad (c \text{ is a constant function})$$

The Definite Integral of x

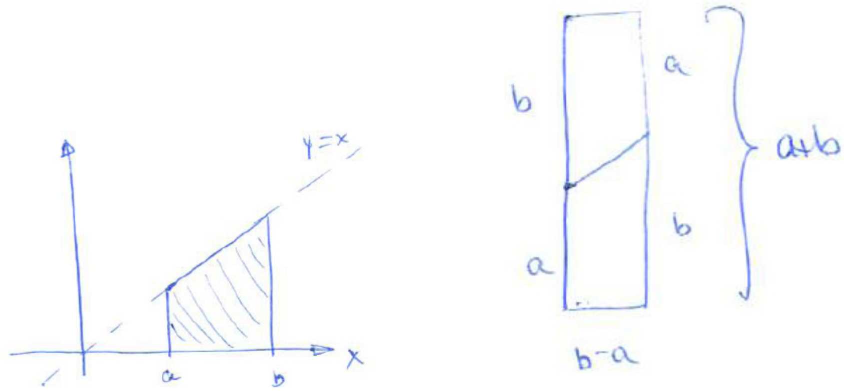
Exercise 34 shows us how to find $\int_a^b x\ dx$ directly from the definition. Alternatively, let us use the "area" interpretation of the definite integral to predict the value of $\int_a^b x\ dx$.

When the integrand is positive, that is, $0 < a < b$, the area in question then lies above the x-axis, as shown in Figure 6.2.4(a). Two copies of this region form a rectangle of width $b - a$ and height $a + b$, as shown in Figure 6.2.4(b). Thus, the area shown in Figure 6.2.4(a) is half of $(b - a)(b + a) = b^2 - a^2$. Hence,

$$\int_a^b x\ dx = \frac{b^2}{2} - \frac{a^2}{2}.$$

The Definite Integral of x^2

We will find $\int_0^b x^2\ dx$ by examining the approximating sums when all the sections have the same length, as they did in Section 6.1.



(a) (b)

Figure 6.2.4:

Pick a positive integer n and cut the interval $[0, b]$ into n sections of length $\Delta x = b/n$ as in Figure 6.2.5. Then the points of subdivision are 0, Δx, $2\Delta x$, $\dots$, $(n-1)\Delta x$, and $n\Delta x = b$.

Figure 6.2.5:

In the typical section $[(i-1)\Delta x, i\Delta x]$ we pick the right-hand endpoint as the sampling number. Thus the approximating sum is

$$\sum_{i=1}^{n} (i\Delta x)^2 \ (\Delta x) = (\Delta x)^3 \sum_{i=1}^{n} i^2.$$

Since $\Delta x = b/n$, these overestimates can be written as

$$\frac{b^3}{n^3} \sum_{i=1}^{n} i^2. \tag{6.2.8}$$

Or, see Exercise 29 in Section 6.1.

In Section 6.1 we used geometry to find that

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{3}.$$

Thus, (6.2.8) approaches $b^3/3$ as n increases, and we conclude that

$$\int_0^b x^2 \ dx = \frac{b^3}{3}.$$

Note that when $b = 3$, we have $b^3/3 = 9$, agreeing with the three problems in Section 6.1.

A little geometry suggests the value of $\int_a^b x^2\,dx$, for $0 \le a < b$. Interpret $\int_a^b x^2\,dx$ as the area under $y = x^2$ and above $[a,b]$. This area is equal to the area under $y = x^2$ and above $[0,b]$ minus the area under $y = x^2$ and above $[0,a]$, as shown in Figure 6.2.6. Then

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

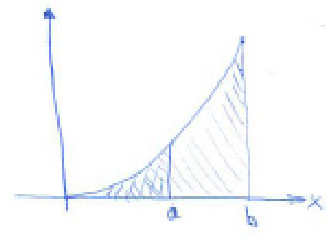


Figure 6.2.6:

The Definite Integral of 2^x

EXAMPLE 3 Use the definition of the definite integral to evaluate $\int_0^b 2^x\,dx$. (Assume $b > 0$.)

SOLUTION Divide the interval $[0,b]$ into n sections of equal length, $d = b/n$. This time let's evaluate the integrand at the left-hand endpoint of each section. Call this number c_i, $c_i = (i-1)d$. The approximating sum has one term for each section. The contribution from the i^{th} section is

d = width of section

$$2^{c_i}d = 2^{(i-1)d}d.$$

The total estimate is the sum

$$2^0d + 2^dd + 2^{2d}d + \cdots + 2^{(i-1)d}d + \cdots + 2^{(n-1)d}d.$$

This equals

$$d\left(1 + 2^d + (2^d)^2 + \cdots + (2^d)^i + \cdots + (2^d)^{n-1}\right). \tag{6.2.9}$$

The terms inside the large parentheses in (6.2.9) form a geometric series with n terms, whose first term is 1 and whose ratio is 2^d. Thus, its sum is

Sum of geometric series: $a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}$.

$$\frac{1-(2^d)^n}{1-2^d}.$$

Therefore this typical underestimate is

$$\frac{d(1-(2^d)^n)}{1-2^d} = \frac{d(1-2^{dn})}{1-2^d} = \frac{d(1-2^b)}{1-2^d}. \tag{6.2.10}$$

In the last step we used the fact that $dn = b$. We can rewrite (6.2.10) as

$$\frac{d}{2^d-1}\left(2^b-1\right). \tag{6.2.11}$$

It still remains to take the limit as n increases without bound. To find what happens to (6.2.11) as $n \to \infty$, we must investigate how $\frac{d}{2^d-1}$ behaves as

d approaches 0 (from the right). Though we haven't met this quotient before, we have met its reciprocal, $\frac{2^d-1}{d}$. This quotient occurs in the definition of the derivative of 2^x at $x = 0$:

$$\lim_{x\to 0} \frac{2^x - 2^0}{x} = \lim_{x\to 0} \frac{2^x - 1}{x}.$$

As we saw in Section 3.5, the derivative of 2^x is $2^x \ln(2)$. Thus $D(2^x)$ at $x = 0$ is $\ln(2)$. Hence

$$\lim_{d\to 0^+} \frac{d}{2^d - 1}(2^b - 1) = \lim_{d\to 0^+} \frac{1}{\left(\frac{2^d-1}{d}\right)} \left(2^b - 1\right) = \frac{2^b - 1}{\ln(2)}.$$

Incidentally, $\frac{1}{\ln(2)} \approx 1.443$.

We conclude that

$$\int_0^b 2^x \, dx = \frac{1}{\ln(2)} \left(2^b - 1\right).$$

$\diamond$

To evaluate $\int_a^b 2^x \, dx$ with $b > a \geq 0$, we reason as we did when we generalized $\int_0^b x^2 \, dx$ to $\int_a^b x^2 \, dx$. Namely,

$$\int_a^b 2^x \, dx = \int_0^b 2^x \, dx - \int_0^a 2^x \, dx = \frac{2^b - 1}{\ln(2)} - \frac{2^a - 1}{\ln(2)} = \frac{2^b}{\ln(2)} - \frac{2^a}{\ln(2)}.$$

Summary

We defined the definite integral of a function $f(x)$ over an interval $[a, b]$. It is the limit of sums of the form $\sum_{i=1}^{n} f(c_i)\Delta x_i$ created from partitions of $[a, b]$. It is a purely mathematical idea. You could estimate $\int_a^b f(x) \, dx$ with your calculator – even without having any application in mind. However, the definite integral has many applications: three of them are "area under a curve," "distance traveled" and "volume."

The following table contains a great deal of information. Compare the first three cases with the fourth, which describes the fundamental definition of integral calculus. In this table, all the functions, whether cross-sectional length, velocity, or cross-sectional area, are denoted by the same symbol $f(x)$.

Underlying these three applications is one purely mathematical concept, the definite integral, $\int_a^b f(x) \, dx$. The definite integral is defined as a certain limit; it is a number. It is essential to keep the definition of the number $\int_a^b f(x) \, dx$ clear. *It is a limit of certain sums.*

Spend some time studying this table. The concepts it summarizes will be used often.

$f(x)$	$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$	$\int_a^b f(x)\ dx$
Variable length of cross section of set in plane	Approximate area of set in the plane	The area of set in the plane
Variable velocity	Approximation to the distance traveled	The distance traveled
Variable cross section of a solid	Approximate of volume	The volume of a solid
A function	Just a certain sum	The limit of the sums as the $\Delta x_i \to 0$

EXERCISES for Section 6.2 *Key:* R–routine, M–moderate, C–challenging

1.[R] Using the formula for $\int_a^b x^2\,dx$, find the area under the curve $y = x^2$ and above the interval

(a) $[0, 5]$

(b) $[0, 4]$

(c) $[4, 5]$

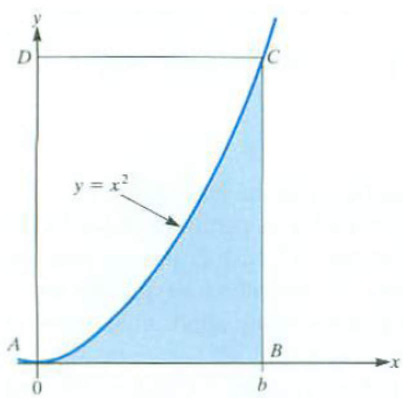


Figure 6.2.7:

2.[R] Figure 6.2.7 shows the curve $y = x^2$. What is the ratio between the shaded area under the curve and the area of the rectangle $ABCD$?

3.[R]

(a) Define "the definite integral of $f(x)$ from a to b, $\int_a^b f(x)\,dx$."

(b) Define the definite integral, using as few mathematical symbols as you can.

(c) Give three applications of the definite integral.

4.[R] Assume $f(x)$ is decreasing for x in $[a, b]$. When you form an approximating sum for $\int_a^b f(x)\,dx$ with left-hand endpoints as sampling points, is your estimate too large or too small? Explain (in one or more complete sentneces).

In Exercises 5 to 8 evaluate the sum

5.[R]

(a) $\sum_{i=1}^{3} i$

(b) $\sum_{i=3}^{7}(2i+3)$

(c) $\sum_{d=1}^{3} d^2$

6.[R]

(a) $\sum_{i=2}^{4} i^2$

(b) $\sum_{j=2}^{4} j^2$

(c) $\sum_{i=1}^{3}(i^2+i)$

7.[R]

(a) $\sum_{i=1}^{4} 1^i$

(b) $\sum_{k=2}^{6}(-1)^k$

(c) $\sum_{j=1}^{150} 3$

8.[R]

(a) $\sum_{i=3}^{5} \frac{1}{i}$

(b) $\sum_{i=0}^{4} \cos(2\pi i)$

(c) $\sum_{i=1}^{3} 2^{-i}$

In Exercises 9 to 12 write each sum in Σ-notation. (Do not evaluate the sum.)

9.[R]

(a) $1+2+2^2+2^3+\cdots+2^{100}$

(b) $x^3+x^4+x^5+x^6+x^7$

(c) $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{102}+\frac{1}{103}$

10.[R]

(a) $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}$

(b) $\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\frac{1}{14}$

(c) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\frac{1}{101^2}$

11.[R]

(a) $x_0^2(x_1-x_0)+x_1^2(x_2-x_1)+x_2^2(x_3-x_2)$

(b) $x_1^2(x_1-x_0)+x_2^2(x_2-x_1)+x_3^2(x_3-x_2)$

12.[R]

(a) $8t_0^2(t_1-t_0)+8t_1^2(t_2-t_1)+\cdots+8t_{99}^2(t_{100}-t_{99})$

(b) $8t_1^2(t_1-t_0)+8t_2^2(t_2-t_1)+\cdots+8t_n^2(t_n-t_{n-1})$

13.[R]

(a) Use the definition of definite integral to evaluate $\int_0^b e^x\,dx$. (See Example 3.)

(b) From (a), deduce that, for $0 \le a < b$, $\int_a^b e^x \, dx = e^b - e^a$.

14.[R]

(a) Use the definition of definite integral to evaluate $\int_0^b 3^x \, dx$.

(b) From (a), deduce that, for $0 \le a < b$, $\int_a^b 3^x \, dx = (3^b - 3^a)/\ln(3)$.

15.[R] The fact that $\int_a^b f(x) \, dx = \lim_{n\to\infty} \sum_{i=1}^n f(c_i)\Delta x$ provides another way to evaluate some limits of sums that would otherwise be very challenging to evaluate. Use this idea to write each of the following limits as a definite integral. (Do not evaluate the definite integrals.)

(a) $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} e^{i/n} \frac{1}{n}$

(b) $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1 + \left(1 + \frac{2i}{n}\right)^2} \frac{2}{n}$

(c) $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right) \frac{\pi}{n}$

(d) $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{3i}{n}\right)^4 \frac{3}{n}$

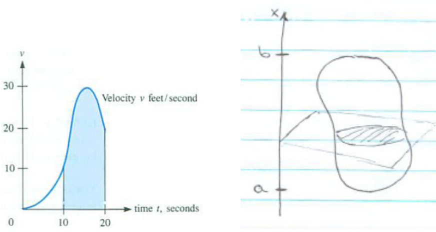
In Exercises 16 to 18 evaluate $\sum_{i=1}^n f(c_i)(x_i - x_{i-1})$ for the given function, partition, and sampling numbers.

16.[R] $f(x) = \sqrt{x}$, $x_0 = 1$, $x_1 = 3$, $x_2 = 5$, $c_1 = 1$, $c_2 = 4$ $(n = 2)$

17.[R] $f(x) = \sqrt[3]{x}$, $x_0 = 0$, $x_1 = 1$, $x_2 = 4$, $x_3 = 10$, $c_1 = 0$, $c_2 = 1$, $c_3 = 8$ $(n = 3)$

18.[R] $f(x) = 1/x$, $x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$, $c_1 = 1$, $c_2 = 1.25$, $c_3 = 1.6$, $c_4 = 2$ $(n = 4)$

19.[M] The velocity of an automobile at time t is $v(t)$ feet per second. [Assume $v(t) \ge 0$.] The graph of v for t in $[0, 20]$ is shown in Figure 6.2.8(a). Explain, in complete sentences, why the shaded area under the curve equals the change in position.



(a) (b)

Figure 6.2.8:

In Exercises 20 to 23 partition the interval into 4 sections of equal lengths. Estimate the definite integral using sampling numbers chosen to be (a) the left endpoints and (b) the right endpoints.

20.[M] $\int_1^2 (1/x^2) \, dx$.

21.[M] $\int_1^5 \ln(x) \, dx$.

22.[M] $\int_1^5 \frac{2^x}{x} \, dx$.

23.[M] $\int_0^1 \sqrt{1 + x^3} \, dx$.

24.[M] Write the following expression using summation notation.

$$c^{n-1} + c^{n-2}d + c^{n-3}d^2 + \cdots + cd^{n-2} + d^{n-1}.$$

25.[M] Assume that $f(x) \le -3$ for all x in $[1, 5]$. What can be said about the value of $\int_1^5 f(x) \, dx$? Explain, in detail, using the definition of the definite integral.

26.[M] A rocket moving with a varying speed travels $f(t)$ miles per second at time t seconds. Let $t_0, \ldots, t_n$ be a partition of $[a, b]$, and let $T_1, \ldots, T_n$ be sampling numbers. What is the physical interpretation of each of the following quantities?

(a) $t_i - t_{i-1}$

(b) $f(T_i)$

(c) $f(T_i)(t_i - t_{i-1})$

(d) $\sum_{i=1}^{n} f(T_i)(t_i - t_{i-1})$

(e) $\int_a^b f(t)\,dt$

27.[M]

(a) Sketch $y = \cos(x)$, for x in $[0, \pi/2]$.

(b) Estimate, by eye, the area under the curve and above $[0, \pi/2]$.

(c) Partition $[0, \pi/2]$ into three equal sections and use them to provide an overestimate of the area under the curve.

(d) Use the same partition to provide an underestimate of the area under the curve.

28.[M] Repeat Exercise 27 for the area under the curve $y = e^{-x}$ above $[0, 3]$.

29.[M] For x in $[a, b]$, let $A(x)$ be the area of the cross section of a solid perpendicular to the x-axis at x (think of slicing a potato). Let x_0, x_1, $\ldots$, x_n be a partition of $[a, b]$. Let c_1, $\ldots$, c_n be the corresponding sampling numbers. What is the geometric interpretation of each of the following quantities? HINT: Refer to Figure 6.2.8(b).

(a) $x_i - x_{i-1}$

(b) $A(c_i)$

(c) $A(c_i)(x_i - x_{i-1})$

(d) $\displaystyle\sum_{i=1}^{n} A(c_i)(x_i - x_{i-1})$

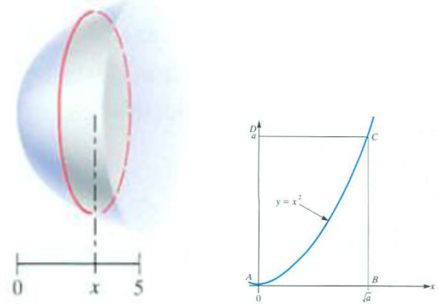
(e) $\int_a^b A(x)\ dx$

30.[M] Show that the volume of a right circular cone of radius a and height h is $\frac{\pi a^3 h}{3}$. HINT: First show that a cross section by a plane perpendicular to the axis of the cone and a distance x from the vertex is a circle of radius ax/h. NOTE: See Exercise 29.

31.[M]

(a) Set up an appropriate definite integral $\int_a^b f(x)\ dx$ which equals the volume of the headlight in Figure 6.2.9(a) whose cross section by a typical plane perpendicular to the x-axis at x is a disk whose radius is $\sqrt{x/\pi}$. NOTE: A circle is a curve and a disk is the flat region inside a circle.

(b) Evaluate the definite integral found in (a).



(a) (b)

Figure 6.2.9:

32.[M]

(a) By considering Figure 6.2.9(b), in particular the area of region ACD, show that $\int_0^a \sqrt{x}\ dx = \frac{2}{3}a^{3/2}$.

(b) Use (a) to evaluate $\int_a^b \sqrt{x}\ dx$ when $0 < a < b$.

Exercises 33 to 36 involve "telescoping sums". Let f be a function defined at least for positive integers. A sum of the form $\sum_{i=1}^{n}(f(i+1) - f(i))$ is called telescoping. To show why, write the sum out in longhand:

$$(f(2)-f(1))+(f(3)-f(2))+(f(4)-f(3))+\cdots+(f(n)-f(n-1))+($$

Everything cancels except $-f(1)$ and $f(n+1)$. The whole sum shrinks like a collapsible telescope, with value $f(n+1) - f(1)$.

33.[C]

(a) Show that $\sum_{i=1}^{n}\left((i+1)^2 - i^2\right) = (n+1)^2 - 1$.
HINT: This is a telescoping sum.

(b) From (a), show that $\sum_{i=1}^{n}(2i+1) = (n+1)^2 - 1$.

(c) From (b), show that $n + 2\sum_{i=1}^{n} i = (n+1)^2 - 1$.

(d) From (c), show that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

34.[C] Exercise 33 showed that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Use this information to find $\int_0^b x\ dx$ directly from the definition of the definite integral (not by interpreting it as an area). No picture is needed.

35.[C]

(a) Starting with the telescoping sum $\sum_{i=1}^{n}\left((i+1)^3 - i^3\right)$ show that

$$n+3\sum_{i=1}^{n} i^2+3\sum_{i=1}^{n}$$

(b) Use (a) to show th $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n$ $1)(2n+1)$.

(c) Use (b) to show th $\int_0^b x^2\ dx = \frac{b^3}{3}$.

NOTE: See Exercise 34.

36.[C]

(a) Using the techniques of Exercises 33 to 35, find a short formula for the sum $\sum_{i=1}^{n} i^3$.

(b) Use the formula found in (a) to show that $\int_0^b x^3\ dx = \frac{b^4}{4}$.

37.[C] The function $f(x) = 1/x$ has a remarkable property, namely, for a and b greater than 1,

$$\int_1^a \frac{1}{x}\ dx = \int_b^{ab} \frac{1}{x}\ dx.$$

In other words, "magnifying the interval $[1, a]$ by a positive number b does not change the value of the definite integral." The following steps show why this is so.

(a) Let $x_0 = 1, x_1, x_2, \ldots, x_n = a$ divide the interval $[1, a]$ into n sections. Using left endpoints write out an approximating sum for $\int_1^a \frac{1}{x}\ dx$.

(b) Let $bx_0 = b, bx_1, bx_2, \ldots, bx_n = ab$ divide the interval $[b, ab]$ into n sections. Using left endpoints write out an approximating sum for $\int_b^{ab} \frac{1}{x}\ dx$.

(c) Explain why $\int_1^a \frac{1}{x}\ dx = \int_b^{ab} \frac{1}{x}\ dx$.

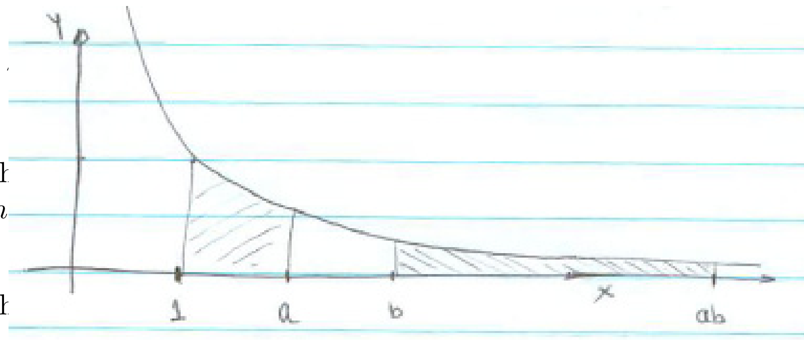


Figure 6.2.10:

38.[C] Let $L(t) = \int_1^t \frac{1}{x}\ dx$, $t > 1$.

(a) Show that $L(a) = L(ab) - L(b)$.

(b) By (a), conclude that $L(ab) = L(a) + L(b)$.

(c) What familiar function has the property listed in (b)?

Gregory St. Vincent noticed the property (a) in 1647, and his friend A.A. de Sarasa saw that (b) followed. Euler, in the 18th century, recognized that $L(x)$ is the logarithm of x to the base e. In short, the area under the hyperbola $y = 1/x$ and above $[1, a]$, $a > 1$, is $\ln(a)$. It can be shown that for a in $(0, 1)$, the negative of the area below that curve and above $[a, 1]$ is $\ln(a)$. (See C. H. Edwards Jr., The Historical development of the Calculus, pp. 154–158.)

39.[C] In Exercise 13 it was shown that for $0 \le a \le b$, $\int_a^b e^x\ dx = e^b - e^a$.

(a) Use this information and a diagram to show that $\int_{e^a}^{e^b} \ln(x)\ dx = e^b(b-1) - e^a(a-1)$.

(b) From (a), deduce that for $1 \le c \le d$, $\int_c^d \ln(x)\ dx = (d\ln(d) - d) - (c\ln(c) - c)$.

(c) By differentiating $x\ln(x) - x$, show that it is an antiderivative of $\ln(x)$.

40.[C]

(a) To estimate $\int_1^2 \frac{1}{x}\ dx$ divide $[1, 2]$ into n sections of equal lengths and use right endpoints as the sampling points.

(b) Deduce from (a) that

$$\lim_{n\to\infty} \sum_{i=n+1}^{2n} \frac{1}{i} = \lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \text{area under } y = 1/x \text{ and above } [1, 2].$$

(c) Let $g(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$. Show that $\frac{1}{2} \le g(n) < 1$ and $g(n+1) < g(n)$.

41.[C] (This Exercise is used in Exercise 42.) Consider $b > 1$ and n a positive integer. Define $r(n)$ by the equation $(r(n))^n = b$.

(a) In the case $b = 5$, find $r(n)$ for $n = 1$, 2, 3, and 10. (Note that $r = b^{1/n}$, so you could use the x^y key on a calculator.)

(b) The calculations in (a) suggest that $\lim_{n\to\infty} r(n) = 1$. Show that this conjecture is correct. HINT: Start by taking ln of both sides of the equation $(r(n))^n = b$.

42.[C] For $b > 1$ and k and number, Pierre Fermat (1601–1665) found the area under $y = x^k$ and above $[1, b]$ by using approximating sums. However, he did not cut the interval $[1, b]$ into n sections of equal widths. Instead, for a given positive integer n, he introduced the number r such that $r^n = b$. As n increases, r approaches 1, as Exercise 41 shows. Then he divided the interval $[0, b]$ into sections using the number r, r^2, r^3, …, r^{n-1}, as shown in Figure 6.2.11. The n sections are $[1, r]$, $[r, r^2]$, …, $[r^{n-1}, r^n] = [r^{n-1}, b]$.

(a) Show that the width of the i^{th} section, $[r^{i-1}, r^i]$, is $r^{i-1}(r-1)$.

(b) Using the left endpoints of each section, obtain an underestimate of $\int_1^b x^2\ dx$.

(c) Show that the estimate in (b) is equal to

$$\frac{b^3 - 1}{1 + r + r^2}.$$

(d) Find $\lim_{n\to\infty} \frac{b^3-1}{1+r+r^2}$. HINT: Remember that r depends on n.

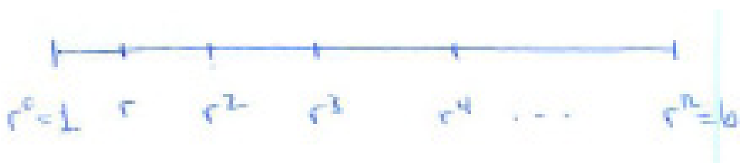


Figure 6.2.11:

43.[C] Use Fermat's approach outlined in Exercise 42, but with right endpoints as the sampling points, to obtain an overestimate of the area under x^2, above $[1, b]$, and then find its limit as $n \to \infty$.

44.[C]

(a) Obtain an underestimate and an overestimate of $\int_0^{\pi/2} \cos(x)\ dx$ that differ by at most 0.1. NOTE: Remember that the angles are measured in radians.

(b) Average the two estimates in (a).

(c) If $\int_0^{\pi/2} \cos(x)\ dx$ is a famous number, what do you think it is?

45.[C] Is $\int_1^2 \frac{1}{x^2}\ dx$ equal to $1/\int_1^2 x^2\ dx$? HINT: Use Fermat's formula from Exercise 42.

46.[C] By considering the approximating sums in the definition of a definite integral, show that $\int_3^4 \frac{dx}{(x+5)^3}$ equals $\int_2^3 \frac{dx}{(x+6)^3}$.

47.[C] For a continuous function f defined for all x, is $\int_a^b f(x+1)\ dx$ equal to $\int_{a+1}^{b+1} f(x)\ dx$?

48.[C] For continuous functions f and g defined for all x, is $\int_a^b f(x)g(x)\ dx$ equal to $\int_a^b f(x)\ dx \int_a^b g(x)\ dx$?

49.[C] If f is an increasing function such that $f(1) = 3$ and $f(6) = 7$, what can be said about $\int_2^4 f(x)\ dx$? Explain.

50.[C]

(a) Using formulas already developed, evaluate $G(x) = \int_1^x t^2\ dt$.

(b) Find $G'(x)$.

(c) Repeat (a) and (b) for $G(x) = \int_1^x 2^t\ dt$.

(d) Do you noice what appears in (b) and (c)?

Skill Drill

In Exercises 51 to 58 give two ant given functions.

51.[R] x^2

52.[R] $1/x^3$

53.[R] e^{-4x}

54.[R] $1/(2x+1)$

55.[R] 2^x

56.[R]

57.[R]

58.[R]

6.3 Properties of the Antiderivative and the Definite Integral

In Section 3.6 we defined an antiderivative of a function $f(x)$. It is any function $F(x)$ whose derivative is $f(x)$. For instance, x^3 is an antiderivative of $3x^2$. So is $x^3 + 2011$. Keep in mind that *an antiderivative is a function.*

F is an antiderivative of f when $F'(x) = f(x)$

In this section we discuss various properties of antiderivatives and definite integrals. These properties will be needed in Section 6.4 where we obtain a relation between antiderivatives and definite integrals. That relation will be a great time-saver in evaluating many (but not all) definite integrals.

We have not yet introduced a symbol for an antiderivative of a function. We will adopt the following standard notation:

Warning: If a function has an antiderivative, then it has lots of antiderivatives.

Notation: Any antiderivative of f is denoted $\int f(x)\ dx$.

For instance, $x^3 = \int 3x^2\ dx$. This equation is read "x^3 is an antiderivative of $3x^2$". That means simply that "the derivative of x^3 is $3x^2$". It is true that $x^3 + 2011 = \int 3x^2\ dx$, since $x^3 + 2011$ is also an antiderivative of $3x^2$. That does *not* mean that the functions x^3 and $x^3 + 2011$ are equal. All it means is that these two functions both have the same derivative, $3x^2$. The symbol $\int 3x^2\ dx$ refers to *any function* whose derivative is $3x^2$.

$\int f(x)\ dx$ is a function

$\int_a^b f(x)\ dx$ is a number.

If $F'(x) = f(x)$ we write $F(x) = \int f(x)\ dx$. The function $f(x)$ is called the **integrand**. The function $F(x)$ is called an antiderivative of $f(x)$. The symbol for an antiderivative, $\int f(x)\ dx$, is similar to the symbol for a definite integral, $\int_a^b f(x)\ dx$, but they denote vastly different concepts. An antiderivative is often called an "integral" or "indefinite integral," but should not be confused with a definite integral. The symbol $\int f(x)dx$ denotes a *function* — any function whose derivative is $f(x)$. The symbol $\int_a^b f(x)\ dx$ denotes a *number* — one that is defined by a limit of certain sums. The value of the definite integral may vary as the interval $[a, b]$ changes.

We apologize for the use of such similar notations, $\int f(x)\ dx$ and $\int_a^b f(x)\ dx$, for such distinct concepts. However, it is not for us to undo over three centuries of custom. Rather, it is up to you to read the symbols $\int f(x)\ dx$ and $\int_a^b f(x)\ dx$ carefully. You distinguish between such similar-looking words as "density" and "destiny" or "nuclear" and "unclear". Be just as careful when reading mathematics.

Properties of Antiderivatives

The tables inside the covers of this book list many antiderivatives. One example is $\int \sin(x)\ dx = -\cos(x)$. Of course, $-\cos(x) + 17$ also is an antiderivative of $\sin(x)$. In Section 4.1 it was shown that if F and G have the same derivative on an interval, they differ by a constant, C. So $F(x) - G(x) = C$ or $F(x) = G(x) + C$. For emphasis, we state this as a theorem.

The following theorem asserts that if you find an antiderivative $F(x)$ for a function $f(x)$, then any other antiderivative of $f(x)$ is of the form $F(x) + C$ for some constant C.

This result was anticipated back in Section 3.6.

Theorem. *If F and G are both antiderivatives of f on some interval, then there is a constant C such that*

$$F(x) = G(x) + C.$$

Many tables of integrals, including the ones in the cover of this book, omit the $+C$.

When using an antiderivative, it is best to include the constant C. (It was needed in the study of differential equations in Section 5.2.) For example,

$$\int 5\ dx = 5x + C$$

$$\int e^x\ dx = e^x + C$$

and

$$\int \sin(2x)\ dx = \frac{-1}{2}\cos(2x) + C.$$

Observe that

$$\frac{d}{dx}\left(\int x^3\ dx\right) = x^3 \qquad \text{and} \qquad \frac{d}{dx}\left(\int \sin(2x)\ dx\right) = \sin(2x). \tag{6.3.1}$$

Are these two equations profound or trivial? Read them aloud and decide.

The first says, "The derivative of an antiderivative of x^3 is x^3." *It is true simply because that is how we defined the antiderivative.* We know that

$$\frac{d}{dx}\left(\int \frac{\ln(1+x^2)}{(\sin(x))^2}\ dx\right) = \frac{\ln(1+x^2)}{(\sin(x))^2}$$

We know that the square of the square root of 7 is 7 and that $e^{\ln(3)} = 3$, both by the definition of inverse functions.

even though we cannot write out a formula for an antiderivative of $\frac{\ln(1+x^2)}{(\sin(x))^2}$. In other words, by the very definition of the antiderivative,

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Any property of derivatives gives us a corresponding property of antiderivatives. Three of the most important properties of antiderivatives are recorded in the next theorem.

Properties of antiderivatives

Theorem 6.3.1 (Properties of Antiderivatives)**.** *Assume that f and g are functions with antiderivatives $\int f(x)\ dx$ and $\int g(x)\ dx$. Then the following hold:*

A. $\int cf(x)\ dx = c\int f(x)\ dx$ *for any constant c.*

B. $\int (f(x)+g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx$.

C. $\int (f(x)-g(x))\ dx = \int f(x)\ dx - \int g(x)\ dx$.

Proof

(A) Before we prove that $\int cf(x)\ dx = c\int f(x)\ dx$, we stop to see what it means. This equation says that "c times an antiderivative of $f(x)$ is an antiderivative of $cf(x)$". Let $F(x)$ be an antiderivative of $f(x)$. Then the equation says "c times $F(x)$ is an antiderivative of $cf(x)$". To determine if this statement is true we must differentiate $cF(x)$ and check that we get $cf(x)$. So, we compute $(cF(x))'$:

$$\begin{aligned}(cF(x))' &= cF'(x) && [c \text{ is a constant}]\\ &= cf(x). && [F \text{ is antiderivative of } f]\end{aligned}$$

Thus $cF(x)$ is indeed an antiderivative of $cf(x)$. Therefore, we may write

$$cF(x) = \int cf(x)\ dx.$$

Since $F(x) = \int f(x)\ dx$, we conclude that

$$c\int f(x)\ dx = \int cf(x)\ dx.$$

(B) The proof is similar. We show that $\int f(x)\ dx + \int g(x)\ dx$ is an antiderivative of $f(x)+g(x)$. To do this we compute the derivative of the sum $\int f(x)\ dx + \int g(x)\ dx$:

$$\begin{aligned}\tfrac{d}{dx}\left(\int f(x)\ dx + \int g(x)\ dx\right) &= \tfrac{d}{dx}\left(\int f(x)\ dx\right) + \tfrac{d}{dx}\left(\int g(x)\ dx\right) && [\text{derivative of a sum}]\\ &= f(x)+g(x). && [\text{definition of antiderivatives}]\end{aligned}$$

(C) The proof is similar to the one for (b). •

EXAMPLE 1 Find (a) $\int 6\cos(x)\,dx$, (b) $\int(6\cos(x)+3x^2)\,dx$, and (c) $\int(6\cos(x)-\frac{5}{1+x^2}\,dx$.

SOLUTION (a) Part (a) of the theorem is used to move the "6" (a constant) past the integral sign, "$\int$". We then have:

$$\int 6\cos(x)\,dx = 6\int \cos(x)\,dx = 6\sin(x)+C.$$

Notice that the "C" is added as the last step in finding an antiderivative.

(b)

$$\begin{aligned}\int(6\cos(x)+3x^2)\,dx &= \int 6\cos(x)\,dx+\int 3x^2\,dx \qquad \text{[part (b) of the theorem]}\\ &= 6\sin(x)+x^3+C.\end{aligned}$$

Here, notice that separate constants are not needed for each antiderivative; again only one "C" is needed for the overall antiderivative.

(c)

$$\begin{aligned}\int\left(6\cos(x)-\tfrac{5}{1+x^2}\right)\,dx &= \int 6\cos(x)\,dx-\int\tfrac{5}{1+x^2}\,dx \qquad \text{[part (c) of the theorem]}\\ &= 6\sin(x)-5\int\tfrac{1}{1+x^2}\,dx \qquad \text{[part (a) of the theorem]}\\ &= 6\sin(x)-5\arctan(x)+C \qquad [(\arctan(x))'=\tfrac{1}{1+x^2}]\end{aligned}$$

◇

The last two parts of Theorem 6.3.1 extend to any finite number of functions. For instance,

$$\int(f(x)-g(x)+h(x))\,dx = \int f(x)\,dx-\int g(x)\,dx+\int h(x)\,dx.$$

Theorem. *Let a be a number other than -1. Then*

$$\int x^a\,dx = \frac{x^{a+1}}{a+1}+C.$$

Proof

$$\left(\frac{x^{a+1}}{a+1}\right)' = \frac{(a+1)x^{(a+1)-1}}{a+1} = x^a.$$

•

EXAMPLE 2 Find $\int \left(\frac{3}{\sqrt{1-x^2}} - \frac{2}{x} + \frac{1}{x^3} \right) \, dx$, $0 < x < 1$.

SOLUTION

$$\begin{aligned} \int \left(\frac{3}{\sqrt{1-x^2}} - \frac{2}{x} + \frac{1}{x^3} \right) \, dx &= 3 \int \frac{1}{\sqrt{1-x^2}} \, dx - 2 \int \frac{1}{x} \, dx + \int x^{-3} \, dx \\ &= 3 \arcsin(x) - 2 \ln(x) + \frac{x^{-2}}{-2} + C \\ &= 3 \arcsin(x) - 2 \ln(x) - \frac{1}{2x^2} + C. \end{aligned}$$

◇

If $-1 < x < 0$, we would write the antiderivative of $1/x$ as $\ln |x|$.

Properties of Definite Integrals

Some of the properties of definite integrals look like properties of antiderivatives. However, they are assertions about numbers, not about functions. In the notation for the definite integral, $\int_a^b f(x) \, dx$, b is larger than a. It will be useful to be able to speak about "the definite integral from a to b" even if b is less than or equal to a. The following two definitions meet this need and we will use them in the proofs of the two fundamental theorems of calculus in the next section.

DEFINITION *(Integral from a to b, where $b < a$.)* If b is less than a, then

$$\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx.$$

EXAMPLE 3 Compute $\int_3^0 x^2 \, dx$, the integral from 3 to 0 of x^2.

SOLUTION The symbol $\int_3^0 x^2 \, dx$ is defined as $-\int_0^3 x^2 \, dx$. As was shown in Section 6.2, $\int_0^3 x^2 \, dx = 9$. Thus

$$\int_3^0 x^2 \, dx = -9.$$

◇

DEFINITION *(Integral from a to a.)*

$$\int_a^a f(x)\,dx = 0$$

Remark: The definite integral is defined with the aid of partitions of an interval. Rather than permit partitions to have sections of length 0, it is simpler just to make this definition.

The point of making these two definitions is that now the symbol $\int_a^b f(x)\,dx$ is defined for any numbers a and b and any continuous function f, assuming $f(x)$ is defined for x in $[a,b]$. It is no longer necessary that a be less than b.

The definite integral has several properties, some of which we will be using in this section and some in later chapters. Justifications of these properties are provided immediately after the following table.

Properties of antiderivatives

Theorem (Properties of the Definite Integral)**.** *Let f and g be continuous functions, and let c be a constant. Then*

1. ***Moving a Constant Past $\int_a^b$***
   $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$
2. ***Definite Integral of a Sum***
   $\int_a^b (f(x)+g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
3. ***Definite Integral of a Difference***
   $\int_a^b (f(x)-g(x))\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$
4. ***Definite Integral of a Non-Negative Function***
   If $f(x) \geq 0$ for all x in $[a,b]$, $a<b$, then $\displaystyle\int_a^b f(x)\ dx \geq 0.$
5. ***Definite Integrals Preserve Order***
   If $f(x) \geq g(x)$ for all x in $[a,b]$, $a<b$, then
   $$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx.$$
6. ***Sum of Definite Integrals Over Adjoining Intervals***
   If a, b, and c are numbers, then
   $$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx.$$
7. ***Bounds on Definite Integrals***
   If m and M are numbers and $m \leq f(x) \leq M$ for all x between a and b, then
   $$m(b-a) \ \leq\ \int_a^b f(x)\ dx \leq M(b-a) \qquad \text{if } a<b$$
   and
   $$m(b-a) \ \geq\ \int_a^b f(x)\ dx \geq M(b-a) \qquad \text{if } a>b$$

Proof of Property 1

Take the case $a < b$. The equation $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$ resembles part (a) of Theorem 6.3.1 about antiderivatives: $\int cf(x)\ dx = c\int f(x)\ dx$. However, its proof is quite different, since $\int_a^b cf(x)\ dx$ is defined as a limit of sums.

We have

$$\begin{aligned}
\int_a^b cf(x)\ dx &= \lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^n cf(c_i)\Delta x_i && \text{definition of definite integral}\\
&= \lim_{\text{all } \Delta x_i \to 0} c\sum_{i=1}^n f(c_i)\Delta x_i && \text{algebra (distributive law)}\\
&= c\lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^n f(c_i)\Delta x_i && \text{property of limits}\\
&= c\int_a^b f(x)\ dx. && \text{definition of definite integral}
\end{aligned}$$

•

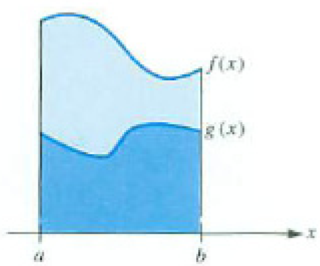


Figure 6.3.1:

Similar approaches can be used to justify each of the other properties. However, we pause only to make them plausible by giving an intuitive interpretation of each property in terms of area.

Plausibility of Argument for Property 5

This amounts to the assertion that when the graph of $y = f(x)$ is always at least as high as the graph of $y = g(x)$, then the area of a region under the curve $y = f(x)$ is greater than or equal to the area under the curve $y = g(x)$ above a given interval. (See Figure 6.3.) •

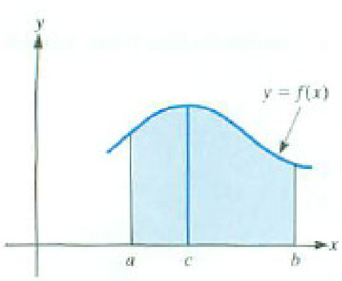


Figure 6.3.2:

Plausibility of Argument for Property 6

In the case that $a < c < b$ and $f(x)$ assumes only positive values, this property asserts that the area of the region below the graph of $y = f(x)$ and above the interval $[a, b]$ is the sum of the areas of the regions below the graph and above the smaller intervals $[a, c]$ and $[c, b]$. Figure 6.3.2 shows that this is certainly plausible. •

Plausibility of Argument for Property 7

The inequalities in this property compare the area under the graph of $y = f(x)$ with the areas of two rectangles, one of height M and one of height m. (See Figure 6.3.3.) In the case $a < b$, the area of the larger rectangle is $M(b - a)$ and the area of the smaller rectangle is $m(b - a)$. •

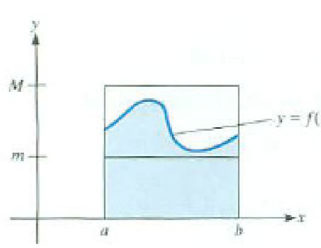
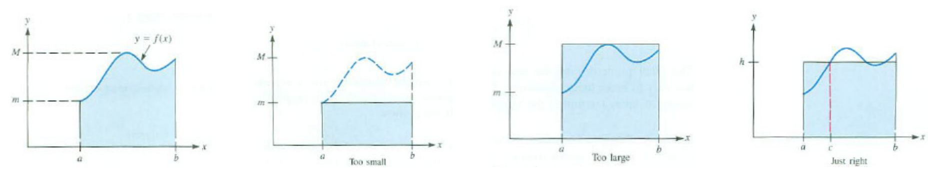


Figure 6.3.3:

The Mean-Value Theorem for Definite Integrals

The mean-value theorem for *derivatives* says that (under suitable hypotheses) $f(b)-f(a) = f'(c)(b-a)$ for some number c in $[a,b]$. The mean-value theorem for *definite integrals* has a similar flavor. First, we state it geometrically.

If $f(x)$ is positive and $a < b$, then $\int_a^b f(x)\ dx$ can be interpreted as the area of the shaded region in Figure 6.3.4(a).



(a) (b) (c) (d)

Figure 6.3.4:

What can you say about the case when $m = M$?

Let m be the minimum and M the maximum values of $f(x)$ for x in $[a,b]$. We assume that $m < M$. The area of the rectangle of height M is larger than the shaded area; the area of the rectangle of height m is smaller than the shaded area. (See Figures 6.3.4(b) and (c).) Therefore, there is a rectangle whose height h is somewhere between m and M, whose area is the same as the shaded area under the curve $y = f(x)$. (See Figure 6.3.4(d).) Hence $\int_a^b f(x)\ dx = (b-a)h$.

Now, h is a number between m and M. By the Intermediate-Value Property for continuous functions, in Section 2.5 there is a number c in $[a,b]$ such that $f(c) = h$. (See Figure 6.3.4(d).) Hence,

$$\text{Area of shaded region under curve} = f(c)(b-a).$$

This suggests the mean-value theorem for definite integrals.

Mean-Value Theorem for Definite Integrals

Theorem (Mean-Value Theorem for Definite Integrals)**.** *Let a and b be numbers, and let f be a continuous function defined at least on the interval $[a,b]$. Then there is a number c in $[a,b]$ such that*

$$\int_a^b f(x)\ dx = f(c)(b-a).$$

Proof of the Mean-Value Theorem for Definite Integrals, using only properties of the definite integral
Consider the case when $a < b$. Let M be the maximum and m the minimum of $f(x)$ on $[a, b]$. Property 7, combined with division by $b - a$, gives

$$m \leq \frac{\int_a^b f(x)\ dx}{b-a} \leq M,$$

Because f is continuous on $[a, b]$, by the Intermediate-Value Property of Section 2.5 there is a number c in $[a, b]$ such that

$$f(c) = \frac{\int_a^b f(x)\ dx}{b-a},$$

The case $b < a$ can be obtained from the case $a < b$. (see Exercise 37).

and the theorem is proved (without depending on a picture). •

EXAMPLE 4 Verify the mean-value theorem for definite integrals when $f(x) = x^2$ and $[a, b] = [0, 3]$.
SOLUTION In Section 6.2 it was shown that $\int_0^3 x^2\ dx = 9$. Since $f(x) = x^2$, we are looking for c in $[0, 3]$ such that

$$\int_0^3 x^2\ dx = 9 = c^2(3-0)$$

$-\sqrt{3}$ is not in $[0, 3]$.

That is, $9 = 3c^2$, so $c^2 = \frac{9}{3} = 3$, $c = \sqrt{3}$. (See Figure 6.3.5.) The rectangle with height $f(\sqrt{3}) = (\sqrt{3})^2 = 3$ and base $[0, 3]$ has the same area as the region under the curve $y = x^2$ and above $[0, 3]$. ◇

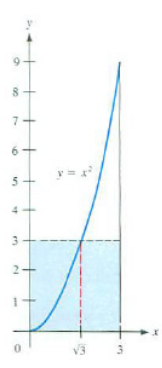


Figure 6.3.5:

The Average Value of a Function

Let $f(x)$ be a continuous function defined on $[a, b]$. What shall we mean by the "average value of $f(x)$ over $[a, b]$"? We cannot add up all the values of $f(x)$ for all x's in $[a, b]$ and divide by the number of x's, since there are an infinite number of such x's. However, we can work with the average (or mean) of n numbers $a_1, a_2, \ldots, a_n$, which is their sum divided by n: $\frac{1}{n}\sum_{i=1}^n a_i$. For example, the average of 1, 2, and 6 is $\frac{1}{3}(1+2+6) = \frac{9}{3} = 3$.

This suggests how to define the "average value of $f(x)$ over $[a, b]$". Choose a large integer n and partition $[a, b]$ into n sections of equal length, $\Delta x = (b-a)/n$. Let the sampling points c_i be the left endpoint of each section,

$c_1 = a$, $c_2 = a + \Delta x$, ..., $c_n = a + (n-1)\Delta x = b - \Delta x$. Then an estimate of the "average" would be

$$\frac{1}{n}(f(c_1) + f(c_2) + \cdots + f(c_n)). \tag{6.3.2}$$

Since $\Delta x = (b-a)/n$, it follows that $\frac{1}{n} = \frac{\Delta x}{b-a}$. Therefore, (6.3.2) can be rewritten as

$$\frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x.$$

But, $\sum_{i=1}^{n} f(c_i)\Delta x$ is an estimate of $\int_a^b f(x)\,dx$. It follows that, as $n \to \infty$, this average of the n function values approaches $\frac{1}{b-a}\int_a^b f(x)\,dx$. This motivates the following definition:

DEFINITION *(Average Value of a Function over an Interval)* Let $f(x)$ be defined on the interval $[a, b]$. Assume that $\int_a^b f(x)\,dx$ exists. The **average value** or **mean value** of f on $[a, b]$ is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Geometrically speaking (if $f(x)$ is positive), this average value is the height of the rectangle that has the base $[a, b]$ and the same area as the area of the region under the curve $y = f(x)$, above $[a, b]$. (See Figure 6.3.6.) Observe that the average value of $f(x)$ over $[a, b]$ is between its maximum and minimum values for x in $[a, b]$. However, it is not necessarily the average of these two numbers.

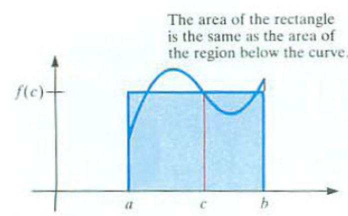


Figure 6.3.6:

EXAMPLE 5 Find the average value of 2^x over the interval $[1, 3]$.
SOLUTION The average value of 2^x over $[1, 3]$ by definition equals

$$\frac{1}{3-1}\int_1^3 2^x\,dx.$$

First, by Example 3 in Section 6.2,

$$\int_1^3 2^x\,dx = \frac{1}{\ln(2)}(2^3 - 2^1) = \frac{6}{\ln(2)}.$$

The average of the maximum and minimum values of 2^x on $[1, 3]$ is $\frac{1}{2}(2^3 + 2^1) = 5$. It's not the same as the average value.

Hence,

$$\text{average value of } 2^x \text{ over } [1, 3] = \frac{1}{3-1}\frac{6}{\ln(2)} = \frac{3}{\ln(2)} \approx 4.2381.$$

$\diamond$

The Zero-Integral Principle

Let f be a continuous function on the interval $[a, b]$. Suppose *for every subinterval* $[c, d]$ *of* $[a, b]$ that $\int_c^d f(x)\ dx$ is zero. For example, the constant function $f(x) = 0$ has this property. We now show that this is the only such function with this property.

Let $f(x)$ be any continuous function on $[a, b]$ that is not the constant function 0. Then there is a number q in $[a, b]$ such that $f(q) = p$ is not zero. We consider the case when p is positive. (The case when p is negative can be treated the same way. See Exercise 46.)

By the Permanence Property (see Theorem 2.5.1 in Section 2.5), there is a subinterval $[c, d]$ of $[a, b]$, where the function values remain larger than $p/2$. The integral of f over $[c, d]$ is at least $p/2$ times the length of the interval $[c, d]$, hence not 0. This contradicts the assumption that $\int_c^d f(x)\ dx = 0$ for all subintervals $[c, d]$ of the domain of f. As a result, the hypothesis must also be false and so f is zero on $[a, b]$.

This interesting result will be applied in Section 18.5 and 18.8.

Zero-Integral Principle

Let f be a continuous function on an interval $[a, b]$. If f has the property that $\int_c^d f(x)\ dx = 0$ for every subinterval $[c, d]$ of $[a, b]$, then $f(x) = 0$ on $[a, b]$.

WARNING *(Antiderivative Terminology)* As mentioned earlier, in the real world an antiderivative is most often called an "integral" or "indefinite integral". If you stay alert, the context will always reveal whether the word "integral" refers to an antiderivative (a function) or to a definite integral (a number). They are two wildly different beasts. Even so, the next section will show that there is a very close connection between them. This connection ties the two halves of calculus — differential calculus and integral calculus — into one neat package.

Summary

We introduced the notation $\int f(x)\ dx$ for an **antiderivative** of $f(x)$. Using this notation we stated several properties of antiderivatives.

We defined the symbol $\int_a^b f(x)\ dx$ in the special case when $b \leq a$, and stated various properties of definite integrals.

The mean-value theorem for definite integrals asserts that for a continuous function $f(x)$, $\int_a^b f(x)\ dx$ equals $f(c)$ times $(b-a)$ for at least one value of c in $[a, b]$.

The quantity $\frac{1}{b-a}\int_a^b f(x)\ dx$ is called the **average value** (or **mean value**) of $f(x)$ over $[a, b]$. It can be thought of as the height of the rectangle whose area is the same as the area of the region under the curve $y = f(x)$.

EXERCISES for Section 6.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 12 evaluate each antiderivative. Remember to add a constant to each answer. Check each answer by differentiating it.

1.[R] $\int 5x^2\ dx$

2.[R] $\int \left(7/x^2\right)\ dx$

3.[R] $\int (2x-x^3+x^5)\ dx$

4.[R] $\int \left(6x^2+2x^{-1}+\frac{1}{\sqrt{x}}\right)\ dx$

5.[R]

(a) $\int e^x\ dx$

(b) $\int e^{x/3}\ dx$

6.[R]

(a) $\int \frac{1}{1+x^2}\ dx$

(b) $\int \frac{1}{\sqrt{1-x^2}}\ dx$

7.[R]

(a) $\int \cos(x)\ dx$

(b) $\int \cos(2x)\ dx$

8.[R]

(a) $\int \sin(x)\ dx$

(b) $\int \sin(3x)\ dx$

9.[R]

(a) $\int (2\sin(x) + 3\cos(x))\ dx$

(b) $\int (\sin(2x) + \cos(3x))\ dx$

10.[R] $\int \sec(x)\tan(x)\ dx$

11.[R] $\int (\sec(x))^2\ dx$

12.[R] $\int (\csc(x))^2\ dx$

13.[R] State the mean-value theorem for definite integrals in words, using no mathematical symbols.

14.[R] Define the average value of a function over an interval, using no mathematical symbols.

15.[R] Evaluate

(a) $\int_2^5 x^2\ dx$

(b) $\int_5^2 x^2\ dx$

(c) $\int_5^5 x^2\ dx$

16.[R] Evaluate

(a) $\int_1^2 x\ dx$

(b) $\int_2^1 x\ dx$

(c) $\int_3^3 x\ dx$

17.[R] Find

(a) $\int x\ dx$

(b) $\int_3^4 x\ dx$

18.[R] Find

(a) $\int 3x^2\ dx$

(b) $\int_1^4 3x^2\ dx$

19.[R] If $2 \leq f(x) \leq 3$, what can be said about $\int_1^6 f(x)\ dx$?

20.[R] If $-1 \leq f(x) \leq 4$, what can be said about $\int_{-2}^7 f(x)\ dx$?

21.[R] Write a sentence or two, in your own words, that tells what the symbols $\int f(x)\ dx$ and $\int_a^b f(x)\ dx$ mean. Include examples. Use as few mathematical symbols as possible.

22.[R] Let $f(x)$ be a differentiable function. In this exercise you will determine if the following equation is true or false:

$$f(x) = \int \frac{df}{dx}(x)\ dx.$$

(a) Pick several functions of your choice and test if the equation is true.

(b) Determine if the equation is always true. Write a brief justification for your answer. HINT: Read the equation out loud.

The mean-value theorem for definite integrals asserts that if $f(x)$ is continuous throughout the interval with endpoints a and b, then $\int_a^b f(x)\ dx = f(c)(b-a)$ for some number c in $[a,b]$. In each of Exercises 23 to 26 find $f(c)$ and at least one value of c in $[a,b]$.

23.[R] $f(x) = 2x$; $[a,b] = [1,5]$

24.[R] $f(x) = 5x + 2$; $[a,b] = [1,2]$

25.[R] $f(x) = x^2$; $[a,b] = [0,4]$

26.[R] $f(x) = x^2 + x$; $[a,b] = [1,4]$

27.[R] If $\int_1^2 f(x)\ dx = 3$ and $\int_1^5 f(x)\ dx = 7$, find

(a) $\int_2^1 f(x)\ dx$

(b) $\int_2^5 f(x)\ dx$

28.[R] If $\int_1^3 f(x)\ dx = 4$ and $\int_1^3 g(x)\ dx = 5$, find

(a) $\int_1^3 (2f(x) + 6g(x))\ dx$

(b) $\int_3^1 (f(x) - g(x))\ dx$

29.[R] If the maximum value of $f(x)$ on $[a,b]$ is 7 and the minimum value on $[a,b]$ is 4, what can be said about

(a) $\int_a^b f(x)\ dx$?

(b) the mean value of $f(x)$ on $[a,b]$?

30.[R] Let $f(x) = c$ (constant) for all x in $[a,b]$. Find the average value of $f(x)$ on $[a,b]$.

Exercises 31 to 34 concern the average of a function over an interval. In each case, find the minimum, maximum, and average value of the function over the given interval.

31.[R] $f(x) = x^2$, $[2,3]$

32.[R] $f(x) = x^2$, $[0,5]$

33.[R] $f(x) = 2^x$, $[0,4]$

34.[R] $f(x) = 2^x$, $[2,4]$

35.[R] Let a, b, and c be constants. Assume that the integral of $\left(ax^2 + bx + c\right)^2$ over any interval is zero. Find a, b, and c.

36.[R] Let a and b be constants. Assume that the integral of $ae^{x^3} + b\cos^{10}(x)$ over every interval is zero. Find a and b.

37.[M] Prove the mean-value theorem for definite integrals in the case when $b < a$. Hint: Use the definition of $\int_a^b f(x)\ dx$ when $b < a$.

38.[M] Is $\int f(x)g(x)\ dx$ always equal to $\int f(x)\ dx \int g(x)\ dx$? Are they ever equal? (Explain.)

39.[M]

(a) Show that $\frac{1}{3}(\sin(x))^3$ is *not* an antiderivative of $\sin(x))^2$.

(b) Use the identity $(\sin(x))^2 = \frac{1}{2}(1 - \cos(2x))$ to find an antiderivative of $\sin(x))^2$.

(c) Verify your answer in (b) by differentiation.

In Exercises 40 and 41 verify the equations quoted from a table of antiderivatives (integrals). Just differentiate each of the alleged antiderivatives and see whether you obtain the quoted integrand. (The number a is a constant in each case.)

40.[M] $\int x^2 \sin(ax)\ dx = \frac{2x}{a^2}\sin(ax) + \frac{2}{a^3}\cos(ax) - \frac{x^2}{a}\cos(ax) + C$

41.[M] $\int x(\sin(ax))^2\ dx = \frac{x^2}{4} - \frac{x}{4a}\sin(2ax) - \frac{1}{8a^2}\cos(2ax) + C$

42.[M] Define $f(x) = \begin{cases} -x & 0 < x \le 1 \\ -1 & 1 < x \le 2 \\ 1 & 2 < x \le 3 \\ 4 - x & 3 < x \le 4 \end{cases}$.

(a) Sketch the graphs of $y = f(x)$ and $y = (f(x))^2$ on the interval $[0,4]$.

(b) Find the average value of f on the interval $[0,4]$.

(c) The **root mean square** (RMS) of a function f on $[a,b]$ is defined as $\sqrt{\frac{1}{b-a}\int_a^b f(x)^2\ dx}$. (The voltage, e.g., 110 volts, for an alternating electric current is the root mean square of a varying voltage.) Find the "root mean square" value of f on the interval $[0,4]$. That is, compute $\sqrt{\frac{1}{4-0}\int_0^4 \left(f(x)\right)^2\ dx}$.

(d) Why is it not surprising that your answer in (b) is zero and your answer in (c) is positive?

43.[M]

Sam: The text makes the average value of a function on $[a,b]$ too hard.

Jane: How so?

Sam: It's easy. Just average $f(a)$ and $f(b)$.

Jane: That sure is easier.

(a) Show that Sam is correct when $f(x)$ is any polynomial of degree 0 or 1.

(b) Is Sam always correct? Explain.

Exercise 44 describes the famous **Buffon neeedle** problem, now over 200 years old. Exercise 47 is related, but not nearly as famous.

44.[M] On the floor there are parallel lines a distance d from each other, such as the edges of slats. You throw a straight wire of length d on the floor at random. Sometimes it ends up crossing a line, sometimes it avoids a line.

(a) Perform the experiment at least 20 times and use the results to estimate the percentage of times the wire crosses a line.

(b) If the wire makes an angle θ with a line perpendicular to the lines, show that the probability that it crosses a line is $\cos(\theta)$.

(c) Find the average value of that probability. That average is the probability that the wire crosses a line.

(d) How close is the experimental value in (a) to the theoretical value in (c)?

45.[M] Assume that f and g are continuous functions and that $\int_a^b f(x)\, dx$ equals $\int_a^b g(x)\, dx$ for every interval $[a, b]$. Show that $f(x)$ equals $g(x)$ for all x.

46.[M] Provide the details for the proof of the Zero-Integral Principle in the case when p is negative.

47.[C] An infinite floor is composed of congruent square tiles arranged as in a checke
straight wire whose length is the sa
a side of a square. The edges of th
in perpendicular directions. Wha
that when you throw the wire at ra
lines, one in each of the two perpe
(This is related to Exercise 44, the
dle problem.) NOTE: You can che
reasonable by carrying out the exp

48.[C] The average value of a ce
on $[1, 3]$ is 4. On $[3, 6]$ the averag
function is 5. What is its average
plain your answer.)

49.[C] This exercise evaluates tw
that appear often in applications.

(a) Draw the graphs of y =
$(\sin(x))^2$. On the basis of y
how $\int_0^{\pi/2}(\cos(x))^2\, dx$ and $\int_0^{\pi}$
pare.

(b) Using (a) and a trigonometric

$$\int_0^{\pi/2} (\cos(x))^2\, dx = \frac{\pi}{4} =$$

(c) Evaluate $\int_0^{\pi}(\cos(x))^2\, dx$.

6.4 The Fundamental Theorem of Calculus

Introduction and Motivation

In this section we obtain two closely related theorems. They are called the Fundamental Theorems of Calculus I and II, or simply The Fundamental Theorem of Calculus (FTC). The first part of the FTC provides a way to evaluate a definite integral if you are lucky enough to know an antiderivative of the integrand. That means that the derivative, developed in Chapter 3, has yet another application.

This is the most important section of the entire book.

FTC I gives a shortcut to evaluating $\int_a^b f(x)\ dx$

FTC II gives a way to evaluate $\frac{d}{dx}\left(\int_a^x f(t)\ dt\right)$

The second fundamental theorem tells how rapidly the value of a definite integral changes as you change the interval $[a, b]$ over which you are integrating. This part of the Fundamental Theorem is used to prove the first part of the FTC.

Motivation for the Fundamental Theorem of Calculus I

In Section 6.2 we found that $\int_a^b c\ dx = cb - ca$ and $\int_a^b x\ dx = \frac{b^2}{2} - \frac{a^2}{2}$. In the same section we found that $\int_a^b x^2\ dx = \frac{b^3}{3} - \frac{a^3}{3}$; in this case our reasoning was based, on the fact that congruent lopsided tents fill a cube. Finally, using the formula for the sum of a geometric series, we showed that $\int_a^b 2^x\ dx = \frac{2^b}{\ln(2)} - \frac{2^a}{\ln(2)}$.

Notice that all four results follow a similar pattern:

$$\begin{array}{rclcrcl} \int_a^b c\ dx &=& cb - ca & \qquad & \int_a^b x\ dx &=& \frac{b^2}{2} - \frac{a^2}{2} \\ \int_a^b x^2\ dx &=& \frac{b^3}{3} - \frac{a^3}{3} & & \int_a^b 2^x\ dx &=& \frac{2^b}{\ln(2)} - \frac{2^a}{\ln(2)} \end{array}$$

To describe the similarity in detail, compute an antiderivative of each of the four integrands:

We omit "$+C$" since only one antiderivative is needed here. See Exercises 40 and 41.

$$\begin{array}{rclcrcl} \int c\ dx &=& cx & \qquad & \int x\ dx &=& \frac{x^2}{2} \\ \int x^2\ dx &=& \frac{x^3}{3} & & \int 2^x\ dx &=& \frac{2^x}{\ln(2)}. \end{array}$$

In each case *the definite integral equals the difference between the values of an antiderivative of the integrand evaluated at b and at a*, the endpoints of the interval.

This suggests that *maybe* for any integrand $f(x)$, the following may be true: If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\ dx = F(b) - F(a). \tag{6.4.1}$$

If this is correct, then, instead of resorting to special tricks to evaluate a definite integral, such as cutting up a cube or summing a geometric series, we should look for an antiderivative of the integrand.

We may reason using "velocity and distance" to provide further evidence for (6.4.1). Picture a particle moving upwards on the y-axis. At time t it is at position $F(t)$ on that line. The velocity at time t is $F'(t)$.

But we saw that the definite integral of the velocity from time a to time b tells the change in position, that is,

"the definite integral of the velocity = the final position − the initial position"

In symbols,

$$\int_a^b F'(t)\ dt = F(b) - F(a). \tag{6.4.2}$$

If we give $F'(t)$ the name $f(t)$, then we can restate (6.4.2) as:

$$\text{If } f(t) = F'(t), \text{ then } \int_a^b f(t)\ dt = F(b) - F(a).$$

In other words,

$$\text{If } F \text{ is an antiderivative of } f, \text{ then } \int_a^b f(t)\ dt = F(b) - F(a).$$

Formulas we found for the integrands c, x, x^2, and 2^x and reasoning about motion are all consistent with

FTC I

Theorem 6.4.1 (Fundamental Theorem of Calculus I)**.**
If f is continuous on $[a, b]$ and if F is an antiderivative of f then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

In practical terms this theorem says, "To evaluate the definite integral of f from a to b, look for an antiderivative of f. Evaluate the antiderivative at b and subtract its value at a. This difference is the value of the definite integral you are seeking". The success of this approach hinges on finding an antiderivative of the integrand f. For many functions, it is easy to find an antiderivative. For some it is hard, but they can be found. For others, the antiderivatives cannot be expressed in terms of the functions met in Chapters 2 and 3, such as polynomials, quotients of polynomials, and functions built up from trigonometric, exponential, and logarithm functions and their inverses.

Some techniques for finding antiderivatives are discussed in Chapter 7.

Example 1 shows the power of FTC I.

EXAMPLE 1 Use the Fundamental Theorem of Calculus to evaluate $\int_0^{\pi/2} \cos(x)\, dx$.

SOLUTION Since $(\sin(x))' = \cos(x)$, $\sin(x)$ is an antiderivative of $\cos(x)$. By FTC I,

$$\int_0^{\pi/2} \cos(x)\, dx \;=\; \sin(\frac{\pi}{2}) - \sin(0) = 1 - 0 = 1.$$

This tells us that the area under the curve $y = \cos(x)$ and above $[0, \pi/2]$, shown in Figure 6.4.1 is 1.

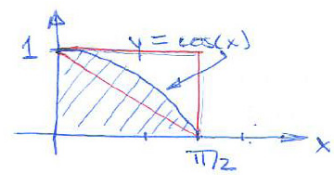


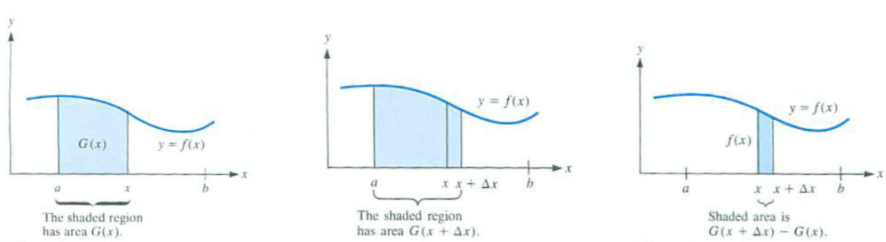
Figure 6.4.1:

This result is reasonable since the area lies inside a rectangle of area $1 \times \frac{\pi}{2} = \frac{\pi}{2} \approx 1.5708$ and contains a triangle of area $\frac{1}{2}\left(\frac{\pi}{2}\right) 1 = \frac{\pi}{4} \approx 0.7854$. ◇

How would the evaluation be different if we used $\sin(x) + 5$ as the antiderivative of $\cos(x)$?

Motivation for the Fundamental Theorem of Calculus II

Let f be a continuous function such that $f(x)$ is positive for x in $[a, b]$. For x in $[a, b]$, let $G(x)$ be the area of the region under the graph of f and above the interval $[a, x]$, as shown in Figure 6.4.2(a). In particular, $G(a) = 0$.



(a) (b) (c)

Figure 6.4.2:

We will compute the derivative of $G(x)$, that is,

$$G'(x) = \lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x} = \lim_{\Delta x \to 0} \frac{G(x + \Delta x) - G(x)}{\Delta x}.$$

(This is one of several occasions when we must go back to the definition of the derivative as a limit.) For simplicity, keep Δx positive. Then $G(x + \Delta x)$ is the area under the curve $y = f(x)$ above the interval $[a, x + \Delta x]$. If Δx is small, $G(x+\Delta x)$ is only slightly larger than $G(x)$, as shown in Figure 6.4.2(b). Then $\Delta G = G(x+\Delta x) - G(x)$ is the area of the thin shaded strip in Figure 6.4.2(c).

When Δx is small, the narrow shaded strip above $[x, x + \Delta x]$ resembles a rectangle of base Δx and height $f(x)$, with area $f(x)\Delta x$. Therefore, it seems reasonable that when Δx is small,

$$\frac{\Delta G}{\Delta x} \approx \frac{f(x)\Delta x}{\Delta x} = f(x).$$

In short, it seems plausible that

$$\lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x} = f(x).$$

Briefly,

$$G'(x) = f(x).$$

In words, "the derivative of the area of the region under the graph of f and above $[a, x]$ with respect to x is the value of f at x".

We use t in the integrand to avoid using x to denote both an end of the interval and a variable that takes values between a and x.

Now we state these observations in terms of definite integrals.

Let f be a continuous function. Let $G(x) = \int_a^x f(t)\ dt$. Then we expect that

$$\frac{d}{dx}\left(\int\limits_a^x f(t)\ dt\right) = f(x).$$

This equation says that "the derivative of the definite integral of f with respect to the right end of the interval is simply f evaluated at that end". This is the substance of the Fundamental Theorem of Calculus II. It tells how rapidly the definite integral changes as we change the upper limit of integration.

FTC II

Theorem 6.4.2 (Fundamental Theorem of Calculus II)**.**
Let f be continuous on the interval $[a, b]$. Define

$$G(x) = \int\limits_a^x f(t)\ dt \qquad \textit{for all } a \le x \le b.$$

Then G is differentiable on $[a, b]$ and its derivative is f; that is,

$$G'(x) = f(x).$$

As a consequence of FTC II, *every continuous function is the derivative of some function*.

See Exercise 63.

There is a similar theorem for $H(x) = \int_x^b f(t)\,dt$: $H'(x) = -f(x)$. A glance at Figure 6.4.3 shows why there is a minus sign: the area in this figure shrinks as x increases.

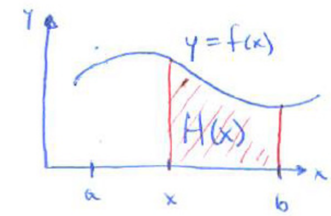
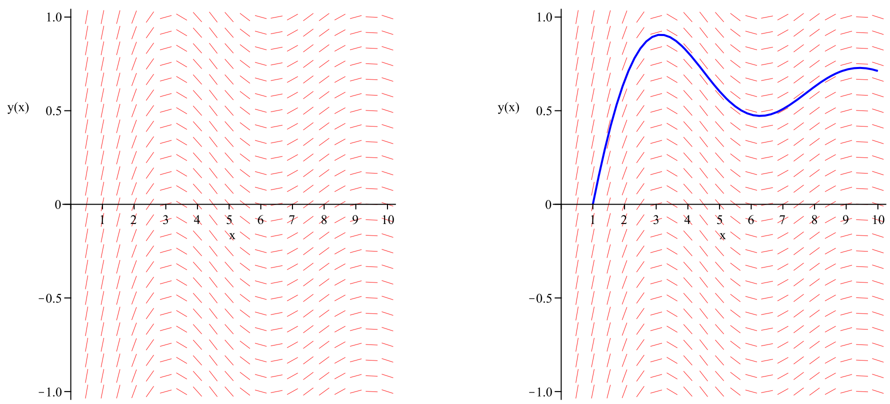


Figure 6.4.3:

EXAMPLE 2 Give an example of an antiderivative of $\frac{\sin(x)}{x}$.
SOLUTION There are many antiderivatives of $\frac{\sin(x)}{x}$. Any two antiderivatives differ by a constant. These curves can be seen in the slope field for $y' = \frac{\sin(x)}{x}$ shown in Figure 6.4.4 (a).



(a) (b)

Figure 6.4.4: (a) slope field for $y' = \frac{\sin(x)}{x}$ and (b) same slope field with solution with $y'(1) = \sin(1)$

Let $G(x) = \int_1^x \frac{\sin(t)}{t}\,dt$. By FTC II, $G'(x) = \frac{\sin(x)}{x}$. The graph of $y = G(x)$ is shown in Figure 6.4.4 (b). Notice that $G(1) = 0$. ◇

You probably expected the answer in Example 2 to be an explicit formula for the antiderivative expressed in terms of the familiar functions discussed in Chapters 2 and 3. Recall, from Section 3.6, that the derivative of every elementary function is an elementary function. Liouville proved that there are (many) elementary functions that do not have elementary antiderivatives. Nobody will ever find an explicit formula in terms of elementary functions for an antiderivative of $\frac{\sin(x)}{x}$. (The proof is reserved for a graduate course.)

Joseph Liouville (1809–1882) `http://en.wikipedia.org/wiki/Joseph_Liouville`

EXAMPLE 3 Give an example of an antiderivative of $\frac{\sin(\sqrt{x})}{\sqrt{x}}$.
SOLUTION This integrand appears more terrifying than $\frac{\sin(x)}{x}$, yet it does have an elementary antiderivative, namely $-2\cos(\sqrt{x})$. To check, we differentiate $y = -2\cos(\sqrt{x})$ by the Chain Rule. We have $y = -2\cos(u)$ where

$u = \sqrt{x}$. Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -2(-\sin(u))\frac{1}{2\sqrt{x}} = \frac{\sin(\sqrt{x})}{\sqrt{x}}.$$

◇

Because the antiderivatives of $\frac{\sin(\sqrt{x})}{\sqrt{x}}$ are elementary functions, it would be easy to calculate $\int_1^2 \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$.

More generally, if $H(t) = \int_a^t f(x)\,dx$, then $H'(t) = f(t)$. See also Exercise 63 .

Any antiderivative of e^x is of the form $e^x + C$, an elementary function. However, *no* antiderivative of e^{-x^2} is elementary. Statisticians define the **error function** to be $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2/2}\,dt$. Except that $\operatorname{erf}(0) = 0$, there is no easy way to evaluate $\operatorname{erf}(x)$. Since $\operatorname{erf}(x)$ is not elementary, it is customary to collect approximate values of it for various values of x in a table. Approximate values of special functions such as the error function can also be obtained from mathematical software and even a few calculators.

Net Area

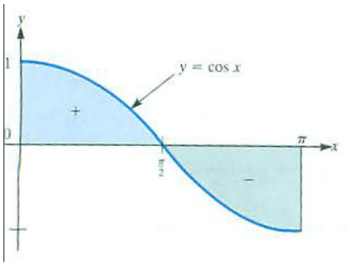


Figure 6.4.5: The area of a region below the x-axis is negative.

When we evaluate $\int_0^\pi \cos(x)\,dx$, we obtain $\sin(\pi) - \sin(0) = 0 - 0 = 0$. What does this say about areas? Inspection of Figure 6.4.5 shows what is happening.

For x in $[\pi/2, \pi]$, $\cos(x)$ is negative and the curve $y = \cos(x)$ lies *below* the x-axis. If we interpret the corresponding area as negative, then we see that it cancels with the area from 0 to $\pi/2$. Let us agree that when we say "$\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$", we mean that it represents the area between the curve and the x-axis, *with area below the x-axis taken as negative*. This is the **net area** under $y = f(x)$ on the interval $[a, b]$. Note that the net area can be positive, zero, or negative.

EXAMPLE 4 Evaluate $\int_1^2 \frac{1}{x^2}\,dx$ by the Fundamental Theorem of Calculus I.

SOLUTION In order to apply FTC I we have to find an antiderivative of $\frac{1}{x^2}$. In Section 6.3 it was observed that

$$\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C \qquad a \neq -1.$$

In particular, with $a = -2$,

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{1}{(-2)+1}x^{(-2)+1} + C = \frac{1}{-1}x^{-1} + C = \frac{-1}{x} + C$$

By FTC I

$$\int_1^2 \frac{1}{x^2}\,dx = \left(\frac{-1}{x} + C\right)\Big|_1^2 = \left(\frac{-1}{2} + C\right) - \left(\frac{-1}{1} + C\right) = \frac{-1}{2} - (-1) = \frac{1}{2}.$$

Note that the C's cancel. We do not need the C when applying FTC I.

$\diamond$

The First Fundamental Theorem of Calculus asserts that

$$\underbrace{\int_1^2 \frac{1}{x^2}\,dx}_{\text{The definite integral: a limit of sums}} = \underbrace{\int \frac{1}{x^2}\,dx\Big|_1^2}_{\text{The difference between an antiderivative evaluated at 2 and at 1}}$$

The symbols on the right and left of the equal sign are so similar that it is tempting to think that the equation is obvious or says nothing whatsoever.

WARNING *(Notation)* This equation is a special instance of the First Fundamental Theorem of Calculus, FTC I.

Remark: Often we write $\int \frac{1}{x^2}\,dx$ as $\int \frac{dx}{x^2}$, merging the 1 with the dx. More generally, $\int \frac{f(x)}{g(x)}\,dx$ may be written as $\int \frac{f(x)\,dx}{g(x)}$.

Some Terms and Notation

The related processes of computing $\int_a^b f(x)\,dx$ and of finding an antiderivative $\int f(x)\,dx$ are both called **integrating** $f(x)$. Thus integration refers to two separate but related problems: computing a number $\int_a^b f(x)\,dx$ or finding a function $\int f(x)\,dx$.

In practice, both FTC I and FTC II are called "the Fundamental Theorem of Calculus." The context always makes it clear which one is meant.

Proofs of the Two Fundamental Theorems of Calculus

We now prove both parts of the Fundamental Theorem of Calculus — without referring to motion, area, or concrete examples. The proofs use only the mathematics of functions and limits. We prove FTC II first; then we will use it to prove FTC I.

Proof of the Second Fundamental Theorem of Calculus

The Second Fundamental Theorem of Calculus asserts that the derivative of $G(x) = \int_a^x f(t)\,dt$ is $f(x)$. We gave a convincing argument using areas of regions. However, since definite integrals are defined in terms of approximating sums, not areas, we include a proof that uses only properties of definite integrals.

Proof of Fundamental Theorem of Calculus II

We wish to show that $G'(x) = f(x)$. To do this we must make use of the definition of the derivative of a function.

We have

$$
\begin{aligned}
G'(x) &= \lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} && \text{(definition of derivative)} \\
&= \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)\ dt - \int_a^x f(t)\ dt}{\Delta x} && \text{(definition of } G\text{)} \\
&= \lim_{\Delta x \to 0} \frac{\int_a^{x} f(t)\ dt + \int_x^{x+\Delta x} f(t)\ dt - \int_a^x f(t)\ dt}{\Delta x} && \text{(property 6 in Section 6.} \\
&= \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)\ dt}{\Delta x} && \text{(canceling)} \\
&= \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x} && \text{(MVT for Definite Integ} \\
& && \text{tween } x \text{ and } x+\Delta x\text{)} \\
&= \lim_{\Delta x \to 0} f(c) && \text{(canceling)} \\
&= f(x). && \text{(continuity of } f\text{; } c \to x \text{ a}
\end{aligned}
$$

Hence

$$G'(x) = f(x),$$

which is what we set out to prove. •

A similar argument shows that

$$\frac{d}{dx} \int_x^b f(t)\ dt = -f(x).$$

For integrands whose values are positive, the minus sign is to be expected. As x increases, the interval shrinks, and so the (positive) area under the curve shrinks as well.

Proof of the First Fundamental Theorem of Calculus

The First Fundamental Theorem of Calculus asserts that if $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$. We persuaded ourselves that this is true by thinking of f as "velocity" and F as "position", and also by four special cases ($f(x) = c$, $f(x) = x$, $f(x) = x^2$, and $f(x) = 2^x$). We now prove the theorem, which is an immediate consequence of the Second Fundamental Theorem of Calculus and the fact that two antiderivatives of the same function differ by a constant.

Proof of the Fundamental Theorem of Calculus I

We are assuming that $F' = f$ and wish to show that $F(b) - F(a) = \int_a^b f(x)\,dx$. Define $G(x)$ to be $\int_a^x f(t)\,dt$. By FTC II, G is an antiderivative of f. Since F and G are both antiderivatives of f, they differ by a constant, say C. That is,

$$F(x) = G(x) + C.$$

Thus,

$$\begin{array}{rcll} F(b) - F(a) & = & (G(b) + C) - (G(a) + C) & \\ & = & G(b) - G(a) & (C\text{'s cancel}) \\ & = & \int_a^b f(t)\,dt - \int_a^a f(t)\,dt & (\text{definition of } G) \\ & = & \int_a^b f(t)\,dt & (\int_a^a f(t)\,dt = 0) \end{array}$$

•

Summary

This section links the two basic ideas of calculus, the derivative (more precisely, the antiderivative) and the definite integral.

FTC I says that if you can find a formula for an antiderivative F of f, then you can evaluate $\int_a^b f(x)\,dx$:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a).}$$

FTC II says that if f is continuous then it has an antiderivative, namely $G(x) = \int_a^x f(t)\,dt$; that is, $G'(x) = f(x)$. Unfortunately, G might not be an elementary function. However, a reasonable graph of an antiderivative of f can be obtained from the slope field for $\frac{dy}{dx} = f(x)$.

EXERCISES for Section 6.4 *Key:* R–routine, M–moderate, C–challenging

1.[R] State (a) FTC I and (b) FTC II.

2.[R] Using only words, no mathematical symbols, state the First Fundamental Theorem of Calculus.

3.[R] Using only words, no mathematical symbols, state the Second Fundamental Theorem of Calculus.

In Exercises 4 and 5 evaluate the given expressions.

4.[R]

(a) $x^3\big|_1^2$

(b) $x^2\big|_{-1}^2$

(c) $\cos(x)\big|_0^\pi$

5.[R]

(a) $(x+\sec(x))\big|_0^{\pi/4}$

(b) $\left.\dfrac{1}{x}\right|_2^3$

(c) $\sqrt{x-1}\big|_5^{10}$

In Exercises 6 to 19 use FTC I to evaluate the given definite integrals.

6.[R] $\displaystyle\int_1^2 5x^3\ dx$

7.[R] $\displaystyle\int_{-1}^3 2x^4\ dx$

8.[R] $\displaystyle\int_1^4 (x+5x^2)\ dx$

9.[R] $\displaystyle\int_1^2 (6x-3x^2)\ dx$

10.[R] $\displaystyle\int_{\pi/6}^{\pi/3} 5\cos(x)\ dx$

11.[R] $\displaystyle\int_{\pi/4}^{3\pi/4} 3\sin(x)\ dx$

12.[R] $\displaystyle\int_0^{\pi/2} \sin(2x)\ dx$

13.[R] $\displaystyle\int_0^{\pi/6} \cos(3x)\ dx$

14.[R] $\displaystyle\int_4^9 5\sqrt{x}\ dx$

15.[R] $\displaystyle\int_1^9 \frac{1}{\sqrt{x}}\ dx$

16.[R] $\displaystyle\int_1^8 \sqrt[3]{x^2}\ dx$

17.[R] $\displaystyle\int_2^4 \frac{4}{x^3}\ dx$

18.[R] $\displaystyle\int_0^1 \frac{dx}{1+x^2}$

19.[R] $\displaystyle\int_{1/4}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

In Exercises 20 to 25 find the average value of the given function over the given interval.

20.[R] x^2; $[3,5]$

21.[R] x^4; $[1,2]$

22.[R] $\sin(x)$; $[0,\pi]$

23.[R] $\cos(x)$; $[0,\pi/2]$

24.[R] $(\sec(x))^2$; $[\pi/6,\pi/4]$

25.[R] $\sec(2x)\tan(2x)$; $[\pi/8,\pi/6]$

In Exercises 26 to 33 evaluate the given quantities.

26.[R] The area of the region under the curve $y = 3x^2$ and above $[1,4]$.

27.[R] The area of the region under the curve $y = 1/x^2$ and above $[2,3]$.

28.[R] The area of the region under the curve $y = 6x^4$ and above $[-1,1]$.

29.[R] The area of the region under the curve $y = \sqrt{x}$ and above $[25,36]$.

30.[R] The distance an object travels from time $t = 1$ second to time $t = 2$ seconds, if its velocity at time t seconds is t^5 feet per second.

31.[R] The distance an object travels from time $t = 1$ second to time $t = 8$ seconds, if its velocity at time t seconds is $7\sqrt[3]{t}$ feet per second.

32.[R] The volume of a solid located between a plane at $x = 1$ and a plane located at $x = 5$ if the cross-sectional area of the intersection of the solid with the plane perpendicular to the x-axis through the point $(x,0)$ has area $6x^3$ square centimeters. (See Figure 6.4.6.)

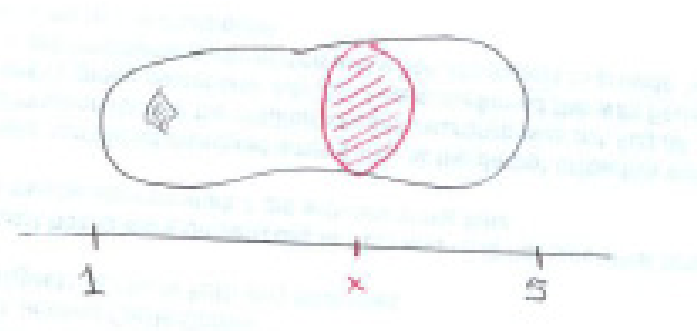


Figure 6.4.6:

33.[R] The volume of a solid located between a plane at $x = 1$ and a plane located at $x = 5$ if the cross-sectional area of the intersection of the solid with the plane perpendicular to the x-axis through the point $(x,0)$ has area $1/x^3$ square centimeters.

34.[R] Let f be a continuous function. Estimate $f(7)$ if $\int_5^7 f(x)dx = 20.4$ and $\int_5^{7.05} f(x)dx = 20.53$.

35.[R] Determine if each of the following expressions is a function or a number.

(a) $\int x^2 \, dx$

(b) $\int x^2 \, dx\Big|_1^3$

(c) $\int_1^3 x^2 \, dx$

36.[R]

(a) Which of these two numbers is defined as a limit of sums?

$$\int x^2\,dx\Big|_1^2 \qquad \text{and} \qquad \int_1^2 x^2\,dx$$

(b) How is the other number defined?

(c) Why are the two numbers in (a) equal?

37.[R] There is no elementary antiderivative of $\sin(x^2)$. Does $\sin(x^2)$ have an antiderivative? Explain.

38.[R] True or false:

(a) Every elementary function has an elementary derivative.

(b) Every elementary function has an elementary antiderivative.

Explain.

39.[R]

(a) Draw the slope field for $\frac{dy}{dx} = \frac{e^{-x}}{x}$ for $x > 0$.

(b) Use (a) to sketch the graph of an antiderivative of $\frac{e^{-x}}{x}$.

(c) On the slope field drawn in (a), sketch the graph of $f(x) = \int_1^x \frac{e^{-t}}{t}\,dt$. (For which one value of x is $f(x)$ easy to compute?)

Exercises 40 and 41 illustrate why FTC I can be applied using any antiderivative of the integrand.

40.[R] Evaluate the definite integral $\int_a^b x\ dx$ using each of the following antiderivatives of $f(x) = x$.

(a) $F(x) = \frac{1}{2}x^2 + 1$.

(b) $F(x) = \frac{1}{2}x^2 - 3$.

(c) $F(x) = \frac{1}{2}x^2 + C$.

41.[R] Evaluate the definite integral $\int_a^b 2^x\ dx$ using each of the following antiderivatives of $f(x) = 2^x$.

(a) $F(x) = \frac{1}{\ln(2)}2^x + 11$.

(b) $F(x) = \frac{1}{\ln(2)}2^x - 7$.

(c) $F(x) = \frac{1}{\ln(2)}2^x + C$.

42.[M] Let $F(x) = \int_0^x e^{t^2}\ dt$.

(a) Does the graph of $F(x)$ have inflection points? If so, find them.

(b) Make a rough sketch of the graph of $F(x)$.

43.[M] Area was used in Section 6.2 to develop $\int_a^b x\ dx = \frac{b^2}{2} - \frac{a^2}{2}$ when $0 < a < b$. To see that this result is true for all values of a and b (with $b > a$) we will consider these additional cases:

(a) If $a < b < 0$, work with negative area.

(b) If $a < 0 < b$, divide the interval $[a, b]$ into two pieces and work with signed areas.

44.[M] Find $\frac{dy}{dx}$ if

(a) $y = \int \sin(x^2)\ dx$

(b) $y = 3x + \int_{-2}^{3} \sin(x^2)\ dx$

(c) $y = \int_{-2}^{x} \sin(t^2)\ dt$

In Exercises 45 to 48 differentiate the given functions.

45.[M]

(a) $\int_1^x t^4\ dt$

(b) $\int_x^1 t^4\ dt$ Hint: Rewrite this integral with x as the upper limit of integration.

Hint: Use the Chain Rule.

46.[M]

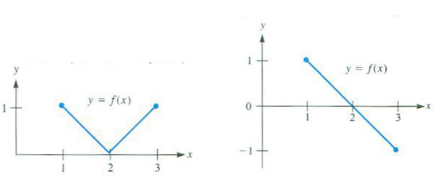
(a) $\int_1^x \sqrt[3]{1+\sin(t)}\ dt$

(b) $\int_1^{x^2} \sqrt[3]{1+\sin(t)}\ \ dt$

47.[M] $\int_{-1}^{x} 3^{-t}\ dt$

48.[M] $\int_{2x}^{3x} t\tan(t)\ dt$ (Assume x is in the interval $(-\pi/6, \pi/6)$.) Hint: First rewrite the integral as $\int_{2x}^{0} t\tan(t)\ dt + \int_0^{3x} t\tan(t)\ dt$.

49.[M] Figure 6.4.7(a) shows the graph of a function $f(x)$ for x in $[1, 3]$. Let $G(x) = \int_1^x f(t)\ dt$. Graph $y = G(x)$ for x in $[1, 3]$ as well as you can. Explain your reasoning.



(a) (b)

Figure 6.4.7:

50.[M] Figure 6.4.7(b) shows the graph of a function $f(x)$ for x in $[1, 3]$. Let $G(x) = \int_1^x f(t)\ dt$. Graph $y = G(x)$ for x in $[1, 3]$ as well as you can. Explain your reasoning.

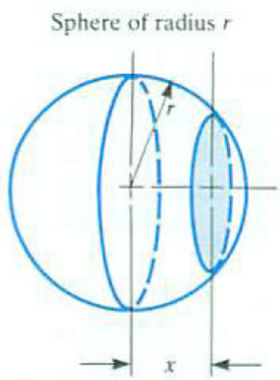


Figure 6.4.8: ARTIST: Change "Sphere" to "Ball"

51.[M] A plane at a distance x from the center of the ball of radius r, $0 \leq x \leq 4$, meets the ball in a disk. (See Figure 6.4.8.)

(a) Show that the radius of the disk is $\sqrt{r^2 - x^2}$.

(b) Show that the area of the disk is $\pi r^2 - \pi x^2$.

(c) Using the FTC, find the volume of the ball.

52.[M] Let $v(t)$ be the velocity at time t of an obj moving on a straight line. The velocity may be posit or negative.

(a) What is the physical meaning of $\int_a^b v(t)\,dt$? Explain.

(b) What is the physical meaning of the slope of the graph of $y = v(t)$? Explain.

(c) What is the physical meaning of $\int_a^b |v(t)|\,dt$? Explain.

53.[M] Give an example of a function f such that $f(4) = 0$ and $f'(x) = \sqrt[3]{1 + x^2}$.

54.[M] Let f be a continuous function. Show that $\frac{d}{dx}\int_x^b f(x)\,dx = -f(x)$

(a) by using the definition of derivative as a limit

(b) by using properties of the definite integral and FTC II.

55.[M] If $f(x) = \int_{-1}^{x} \sin^3\left(e^{t^2}\right) dt$, find $f'(1)$.

56.[M] If $\int_1^x f(t)dt = \sin^3(5x)$, find $f'(3)$.

57.[M] Figure 6.4.9 shows the graph of a function f. Let $A(x)$ be the area under the graph of f and above the interval $[1, x]$.

(a) Find $A(1)$, $A(2)$, and $A(3)$.

(b) Find $A'(1)$, $A'(2)$, and $A'(3)$.

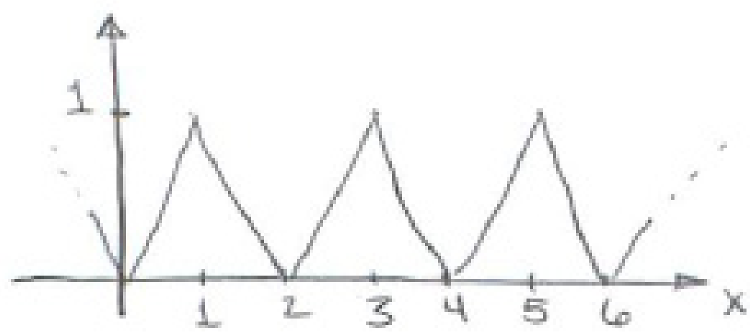


Figure 6.4.9:

58.[M]

(a) If $\int_x^{x+4} g(t)dt = 5$ for all x, what can be said about the graph of g?

(b) How would you construct such a function?

59.[M] Find $D\left(\int_{x^2}^{x^3} e^{t^2}\,dt\right)$.

60.[M] Find $D\left(\int_{x^2}^{5} \sin^{10}(3t)dt\right)$.

61.[M] Find the derivative of $\cos(t^2)\big|_{2x}^{3x}$.

62.[C] How often should a machine be overhauled? This depends on the rate $f(t)$ at which it depreciates and the cost A of overhaul. Denote the time between overhauls by T.

(a) Explain why you would like to minimize $g(T) = \frac{1}{T}(A + \int_0^T f(t)\ dt)$.

(b) Find $\frac{dg}{dT}$.

(c) Show that if $\frac{dg}{dT} = 0$, then $f(T) = g(T)$.

(d) Is this reasonable? Explain.

63.[C] Let $f(x)$ be a continuous function with only positive values. Define $H(x) = \int_x^b f(t)\ dt$ for all $a \le x \le b$. Let Δx be positive.

(a) Interpreting the definite integral as an area of a region, draw the regions whose areas are $H(x)$ and $H(x + \Delta x)$.

(b) Is $H(x + \Delta x) - H(x)$ positive or negative?

(c) Draw the region whose area is related to $H(x + \Delta x) - H(x)$.

(d) When Δx is small, estimate $H(x + \Delta x) - H(x)$ in terms of the integrand f.

(e) Use (d) to evaluate the derivative $H'(x)$:

$$\frac{dH}{dx} = \lim_{\Delta x \to 0} \frac{H(x + \Delta x) - H(x)}{\Delta x}.$$

64.[C] Say that you want to find the area of a certain planar cross-section of a rock. One way to find it is by sawing the rock in two and measuring the area directly. But suppose you do not want to ruin the rock. However, you do have a measuring glass, as shown in Figure 6.4.10, which gives you excellent volume measurements. How could you use the glass to get a good estimate of the cross-sectional area?

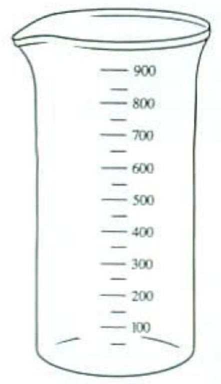


Figure 6.4.10:

65.[C] Let R be a function with continuous second derivative R''. Assume $R(1) = 2$, $R'(1) = 6$, $R(3) = 5$, and $R'(3) = 8$. Evaluate $\int_1^3 R''(x)\ dx$. NOTE: Not all of the information provided is needed.

66.[C] Two conscientious calculus students are having an argument:

Jane: $\int_a^b f(x)\,dx$ is a number.

Sam: But if I treat b as a variable, then it is a function.

Jane: How can it be both a number and a function?

Sam: It depends on what "it" means.

Jane: You can't get out of this so easily.

Which student is correct? That is, either give two interpretations of "it" or explain why "it" has only one meaning.

67.[C] The function $\frac{e^x}{x}$ does not have an elementary antiderivative. Show that its reciprocal, $\frac{x}{e^x}$, does have an elementary antiderivative. HINT: Write $\frac{x}{e^x}$ as xe^{-x} and then experiment for a few minutes.

68.[C] Show that if we knew that every continuous function has an antiderivative, then FTC I would imply FTC II.

69.[C]

(a) Show that for any constant function, $f(x) = c$, the average value of f over $[a, b]$ is the same as the value of the function at the midpoint of the interval $[a, b]$.

(b) Give an example of a non-constant function f such that for any interval $[a, b]$,

$$\frac{\int_a^b f(t)dt}{b-a} = f\left(\frac{a+b}{2}\right).$$

(c) Show that if a continuous function f on $(-\infty, \infty)$ satisfies the equation in (b), it is differentiable.

(d) Find all continuous functions that satisfy the equation in (b).

70.[C] Find all continuous functions f such that their average over $[0, t]$ always equals $f(t)$.

71.[C] Give a geometric explanation of the following properties of definite integrals:

(a) if f is an even function, then $\int_{-a}^{a} f(t)dt = 2\int_0^a f(t)dt$.

(b) if f is an odd function, then $\int_{-a}^{a} f(t)dt = 0$.

(c) if f is a periodic function with period p, then, for any integers m and n, $\int_{mp}^{np} f(t)dt = (n-m)\int_0^p f(t)dt$.

72.[C] Use FTC II to explain why, if u and v are differentiable functions,

(a) $\frac{d}{dx}\int_a^{v(x)} f(t)\ dt = f(v(x))v'(x)$

(b) $\frac{d}{dx}\int_{u(x)}^{b} f(t)\ dt = -f(u(x))u'(x)$

(c) $\frac{d}{dx}\int_{u(x)}^{v(x)}\ dt = f(v(x))v'(x) - f(u(x))u'(x)$

Hint: In (c), break the integral into two convenient integrals.

73.[C] For which continuous functions f is the average value of f on the interval $[0, b]$ a non-decreasing function of b?

6.5 Estimating a Definite Integral

It is easy to evaluate $\int_0^1 x^2\sqrt{1+x^3}\,dx$ by the Fundamental Theorem of Calculus, for the integrand has an elementary antiderivative, $\frac{2}{9}(1+x^3)^{3/2}$. (Check that $\frac{d}{dx}\frac{2}{9}(1+x^3)^{3/2}$ simplifies to $x^2\sqrt{1+x^3}$.) However, an antiderivative of $\sqrt{1+x^3}$ is not elementary, so $\int_0^1 \sqrt{1+x^3}\,dx$ cannot be evaluated so easily. In this case we have to estimate it. This section describes three ways to do this.

Approximation by Rectangles

The definite integral $\int_a^b f(x)\,dx$ is, by definition, a limit of sums of the form

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}). \tag{6.5.1}$$

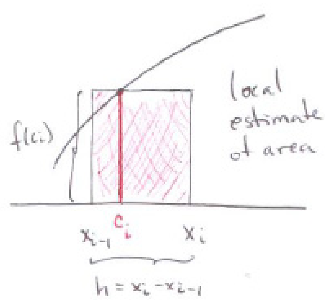


Figure 6.5.1:

Any such sum is an estimate of $\int_a^b f(x)\,dx$.

In terms of area, the area of a rectangle gives a local estimate of the area under the graph of $y = f(x)$ above the interval $[x_{i-1}, x_i]$. See Figure 6.5.1. The sum of the areas of individual rectangles is an estimate the area under the curve.

To use rectangles to estimate $\int_a^b f(x)\,dx$, divide the interval $[a, b]$ into n sections of equal length by the $n+1$ numbers $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. (Choosing the sections to have the same length simplifies the arithmetic.) The width of each section is $h = (b-a)/n$. Then choose a sampling number c_i in the i^{th} section, $i = 1, 2, \ldots, n$ and form the Riemann sum $\sum_{i=1}^{n} f(c_i)h$. By the very definition of the definite integral, this sum is an estimate of the definite integral.

Denoting $f(x_i)$ by y_i, and using the left endpoint x_{i-1} of each interval $[x_{i-1}, x_i]$ as the sampling number, we have this **left endpoint rectangular estimate**

$$\boxed{\int_a^b f(x)\,dx \approx h(y_0 + y_1 + y_2 + \cdots + y_{n-2} + y_{n-1}), \qquad (h = (b-a)/n).}$$

If the right endpoints are used, we have the **right endpoint rectangular estimate**:

$$\int_a^b f(x)\,dx \approx h(y_1 + y_2 + \cdots + y_{n-1} + y_n), \qquad (h = (b-a)/n).$$

We will illustrate this and other ways to estimate a definite integral by estimating $\int_0^1 \frac{dx}{1+x^2}$. We chose this integral because it can be easily computed by the FTC:

$$\int_0^1 \frac{dx}{1+x^2} = \arctan(x)|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} \approx 0.785398.$$

That enables us to judge the accuracy of each method.

EXAMPLE 1 Use four rectangles with equal widths to estimate $\int_0^1 \frac{dx}{1+x^2}$. Use the left endpoint of each section as the sampling number to determine the height of each rectangle.

SOLUTION Since the length of $[0, 1]$ is 1, each of the four sections of equal length has length $\frac{1}{4}$. See Figure 6.5.2. The sum of the areas of the rectangles is

$$\frac{1}{1+0^2}\cdot\frac{1}{4} + \frac{1}{1+\left(\frac{1}{4}\right)^2}\cdot\frac{1}{4} + \frac{1}{1+\left(\frac{2}{4}\right)^2}\cdot\frac{1}{4} + \frac{1}{1+\left(\frac{3}{4}\right)^2}\cdot\frac{1}{4},$$

which equals

$$\frac{1}{4}\left(1 + \frac{16}{17} + \frac{16}{20} + \frac{16}{25}\right).$$

This is approximately

$$\frac{1}{4}(1.0000 + 0.9411 + 0.8000 + 0.6400) = \frac{1}{4}(3.3811) \approx 0.845294.$$

◇

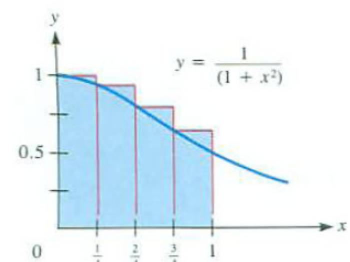


Figure 6.5.2:

As Figure 6.5.2 shows, it is an overestimate; it exceeds the definite integral by about 0.06.

Approximation by Trapezoids

Trapezoids can also be used to find a local estimate of the area under the graph of $y = f(x)$ above the interval $[x_{i-1}, x_i]$. The basic idea is shown in Figure 6.5.3.

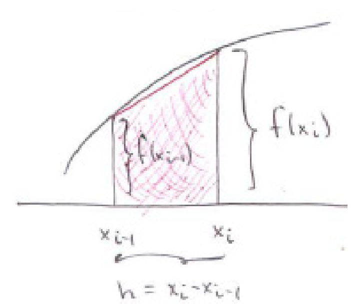


Figure 6.5.3: ARTIST: indicate height is $f(x_i) = y_i$

The area, A, of a trapezoid with base width h and side lengths b_1 and b_2 is the product of the base width and the average of the two side lengths: $A = \frac{1}{2}(b_1 + b_2)h$. (See Figure 6.5.4.)

The formula for the trapezoidal estimate of $\int_a^b f(x)\ dx$ follows from an argument like the one for the rectangular estimate.

Let n be a positive integer. Divide the interval $[a, b]$ into n sections of equal length $h = (b-a)/n$ with

$$x_0 = a,\ x_1 = a+h,\ x_2 = a+2h,\ \ldots,\ x_n = a+nh = b.$$

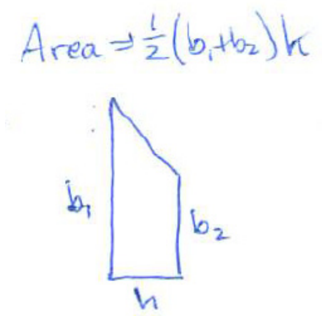


Figure 6.5.4:

Denote $f(x_i)$ by y_i. The local estimate of the area under $y = f(x)$ and above $[x_{i-1}, x_i]$ is

$$\frac{1}{2}(y_{i-1}+y_i)h.$$

Summing the n local estimates of area gives the formula for the trapezoidal estimate of $\int_a^b f(x)\ dx$:

$$\frac{y_0+y_1}{2}\cdot h + \frac{y_1+y_2}{2}\cdot h + \cdots + \frac{y_{n-1}+y_n}{2}\cdot h$$

Factoring out $h/2$ and collecting like terms gives us the **trapezoidal estimate**:

$$\int_a^b f(x)\ dx \approx \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right). \qquad (6.5.2)$$

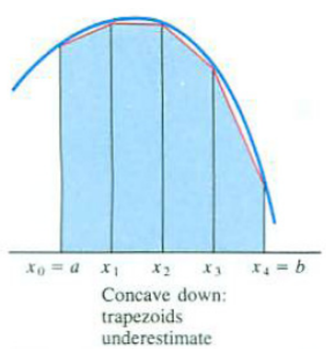


Figure 6.5.5:

There are n sections of width $h = (b-a)/n$, each corresponding to one trapezoid. However, the function is evaluated at $n+1$ points, including both ends of the interval $[a, b]$.

Note that y_0 and y_n have coefficient 1 while all other y_i's have coefficient 2. This is due to the double counting of the edges common to two trapezoids.

If $f(x)$ is a polynomial of the form $A + Bx$, its graph is a straight line. The top edge of each approximating trapezoid coincides with the graph. The approximation (6.5.2) in this special case gives the exact value of $\int_a^b f(x)\ dx$. There is no error.

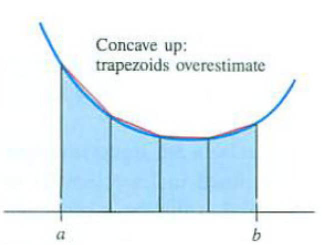


Figure 6.5.6:

Figures 6.5.5 and 6.5.6 illustrate the trapezoidal estimate for the case $n = 4$. Notice that in Figure 6.5.5 the function is concave down and the trapezoidal estimate underestimates $\int_a^b f(x)\ dx$. On the other hand, when the curve is concave up the trapezoids overestimate, as shown in Figure 6.5.6. In both cases the trapezoids appear to give a better approximation of $\int_a^b f(x)\ dx$ than the same number of rectangles. For this reason we expect the trapezoidal method to provide better estimates of a definite integral than we obtain by rectangles.

EXAMPLE 2 Use the trapezoidal method with $n = 4$ to estimate $\int_0^1 \frac{dx}{1+x^2}$.
SOLUTION In this case $a = 0$, $b = 1$, and $n = 4$, so $h = (1-0)/4 = \frac{1}{4}$. The four trapezoids are shown in Figure 6.5.7. The trapezoidal estimate is

$$\frac{h}{2}\left(f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right).$$

Now, $h/2 = \frac{1}{4}/2 = 1/8$. To compute the sum of the five terms involving values of $f(x) = \frac{1}{1+x^2}$, make a list as shown in Table 6.5.1.

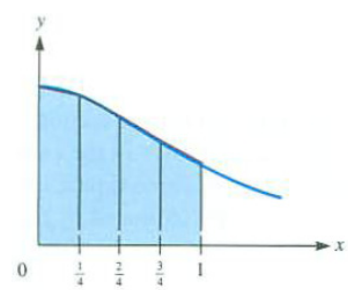


Figure 6.5.7: ARTIST: Try to make top side of trapezoids more visible.

x_i	$f(x_i)$	coefficient	summand	decimal form
0	$\frac{1}{1+0^2}$	1	$1 \cdot \frac{1}{1+0}$	1.0000
$\frac{1}{4}$	$\frac{1}{1+\left(\frac{1}{4}\right)^2}$	2	$2 \cdot \frac{1}{1+\frac{1}{16}}$	1.8823
$\frac{2}{4}$	$\frac{1}{1+\left(\frac{2}{4}\right)^2}$	2	$2 \cdot \frac{1}{1+\frac{4}{16}}$	1.6000
$\frac{3}{4}$	$\frac{1}{1+\left(\frac{3}{4}\right)^2}$	2	$2 \cdot \frac{1}{1+\frac{9}{16}}$	1.2800
$\frac{4}{4}$	$\frac{1}{1+\left(\frac{4}{4}\right)^2}$	1	$1 \cdot \frac{1}{1+\frac{16}{16}}$	0.5000

Table 6.5.1:

The trapezoidal sum is therefore, approximately,

$$\frac{1}{8}(1.0000 + 1.8823 + 1.6000 + 1.2800 + 0.5000) \quad \approx \quad \frac{1}{8}(6.2623) \approx 0.7827.$$

Thus

$$\int_0^1 \frac{dx}{1+x^2} \approx 0.782794.$$

This estimate differs from the definite integral by about 0.0026, which is much smaller than the error in the rectangular method, which had an error of 0.06. ◇

Comparison of Rectangular and Trapezoidal Estimates

If we divide out the 2 in the trapezoidal estimate, it takes the form

$$h\left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right). \qquad (6.5.3)$$

In this form it looks much like the rectangular estimate. It has $n+1$ summands, while the rectangular estimate has only n summands. However, if $f(a)$ happens to equal $f(b)$, that is, $y_0 = y_n$, then (6.5.3) can be written either as $h(y_0 + y_1 + y_2 + \cdots + y_{n-1})$ (the left endpoint rectangular estimate) or as $h(y_1 + y_2 + \cdots + y_{n-1} + y_n)$ (the right endpoint rectangular estimate). In this special case when $f(a) = f(b)$ the three estimates for $\int_a^b f(x)\,dx$ coincide.

Simpson's Estimate: Approximation by Parabolas

Thomas Simpson, 1710–1761, http://en.wikipedia.org/wiki/Thomas_Simpson

In the trapezoidal estimate a curve is approximated by chords. Simpson's estimate for $\int_a^b f(x)\,dx$ approximates the curve by parabolas. Given *three* points on a curve, there is a unique parabola of the form $y = Ax^2 + Bx + C$ that passes through them, as shown in Figure 6.5.8. (See Exercise 28.) The area under the parabola is then used to approximate the area under the curve.

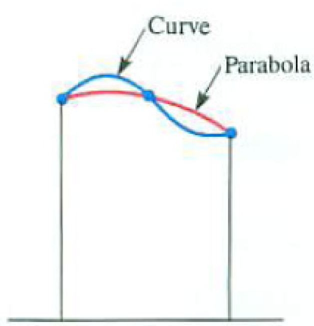
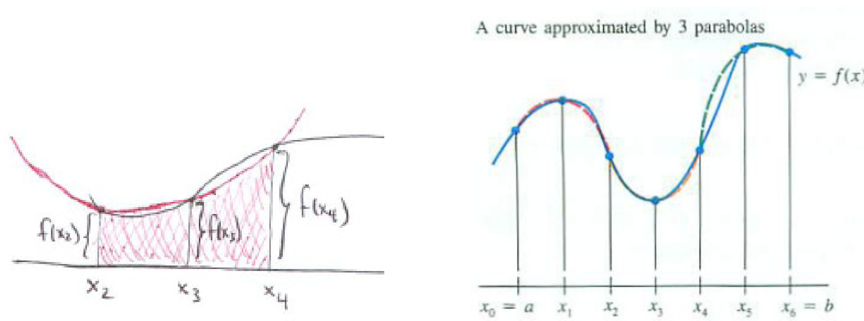


Figure 6.5.8: Curve: $y = f(x)$, Parabola: $y = Ax^2+Bx+C$

The computations leading to the formula for the area under the parabola are more involved than those for the area of a trapezoid. (They are outlined in Exercises 28 to 29.) However, the final formula is fairly simple. Let the three points be $(x_1, f(x_1))$, $(x_2, f(x_2))$, $(x_3, f(x_3))$, with $x_1 < x_2 < x_3$, $x_2 - x_1 = h$, and $x_3 - x_2 = h$, as shown in Figure 6.5.9(a). The shaded area under the parabola turns out to be

$$\frac{h}{3}\left(f(x_1) + 4f(x_2) + f(x_3)\right). \tag{6.5.4}$$



(a) (b)

Figure 6.5.9: ARTIST: In (a), x_2, x_3, and x_4 should be labeled as x_1, x_2, and x_3.

To estimate $\int_a^b f(x)\,dx$, we pick an *even* number n and use $n/2$ parabolic arcs, each of width $2h$. As in the trapezoidal method, we start with a partition of $[a, b]$ into n sections of equal width, h: $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} <$

$x_n = b$. Denoting $f(x_i)$ by y_i, form the sum

$$\frac{h}{3}\left((y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)\right).$$

Collecting like terms gives us **Simpson's estimate** for the definite integral $\int_a^b f(x)\,dx$:

$$\boxed{\frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right) \qquad (6.5.5)}$$

Except for the first and last terms, the coefficients alternate 4, 2, 4, 2, ..., 2, 4. To apply (6.5.5), pick an even number n. Then $h = (b-a)/n$. The estimate uses $n+1$ points, x_0, x_1, ..., x_n, and $n/2$ parabolas. Example 3 illustrates the method, with $n = 4$.

EXAMPLE 3 Use Simpson's method with $n = 4$ to estimate $\int_0^1 \frac{dx}{1+x^2}$.

SOLUTION In this case, the estimate takes the form

$$\frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

with $h = (1-0)/4 = 1/4$. There are two parabolas, shown in Figure 6.5.10. Because the parabolas look almost like the curve, we expect Simpson's estimate to be even better than the trapezoidal estimate.

The computations are shown in Table 6.5.2.

x_i	$f(x_i)$	coefficient	summand	decimal form
0	$\frac{1}{1+0^2}$	1	$1 \cdot \frac{1}{1+0}$	1.0000
$\frac{1}{4}$	$\frac{1}{1+\left(\frac{1}{4}\right)^2}$	4	$4 \cdot \frac{1}{1+\frac{1}{16}}$	3.7647
$\frac{2}{4}$	$\frac{1}{1+\left(\frac{2}{4}\right)^2}$	2	$2 \cdot \frac{1}{1+\frac{4}{16}}$	1.6000
$\frac{3}{4}$	$\frac{1}{1+\left(\frac{3}{4}\right)^2}$	4	$4 \cdot \frac{1}{1+\frac{9}{16}}$	2.5600
$\frac{4}{4}$	$\frac{1}{1+\left(\frac{4}{4}\right)^2}$	1	$1 \cdot \frac{1}{1+\frac{16}{16}}$	0.5000

Table 6.5.2:

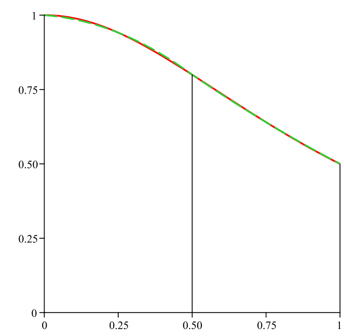


Figure 6.5.10:

Combining the data in the table with the factor $h/3 = 1/12$ provides the estimate

$$\frac{1}{12}(1.0000 + 3.7647 + 1.6000 + 2.5600 + 0.5000) \;=\; \frac{1}{12}(9.4247) \approx 0.7853.$$

As the decimal form of $\int_0^1 dx/(1+x^2)$ begins 0.78539, this Simpson estimate is accurate to all four decimal places given. ◇

Comparison of the Three Methods

Error = |Exact − Estimate|

We know the value of $\int_0^1 \frac{dx}{1+x^2}$ is 0.78539816, to eight decimal places. Table 6.5.3 compares the estimates made in the three examples to this value.

Method	Estimate	Error
Rectangles	0.845294	0.059896
Trapezoids	0.782794	0.002604
Simpson's (Parabolas)	0.785392	0.000006

Table 6.5.3:

Though each method takes about the same amount of work, the table shows that Simpson's method gives the best estimate. The trapezoidal method is next best. The rectangular method has the largest error. These results should not come as a surprise. Parabolas should fit the curve better than chords do, and chords should fit better than horizontal line segments. Note that the trapezoidal and Simpson's methods in Examples 2 and 3 used the same sampling numbers to evaluate the integrand; their only difference is in the "weights" (coefficients) given the outputs of the integrand.

The size of the error is closely connected to the derivatives of the integrand. For a positive number k, let M_k be the largest value of $\left|f^{(k)}(x)\right|$ for x in $[a, b]$. Table 6.5.4 lists the general upper bounds for the error when $\int_a^b f(x)\,dx$ is estimated by sections of length $h = (b-a)/n$. These results are usually developed in a course on numerical analysis. They can also be obtained by a straightforward use of the Growth Theorem of Section 5.3 and the Fundamental Theorem of Calculus. (See Exercises 44 and 45 in this section and Exercise 73 in the Chapter 6 Summary.) They offer a good review of basic ideas.

Recall that $f^{(k)}(x)$ is the k^{th} derivative of f. For instance, $f^{(2)}(x)$ is the second derivative.

Table 6.5.4 expresses the bounds on the size of the error for each method in terms of $h = (b-a)/n$ and n.

Method	Bound on Error in Terms of h	Bound on Error in Terms of n
Rectangles	$M_1(b-a)h$	$M_1(b-a)^2/n$
Trapezoids	$\frac{1}{12}M_2(b-a)h^2$	$\frac{1}{12}M_2(b-a)^3/n^2$
Simpson's (Parabolas)	$\frac{1}{180}M_4(b-a)h^4$	$\frac{1}{180}M_4(b-a)^5/n^4$

Table 6.5.4:

The coefficients in the error bounds tell us a great deal. For instance, if $M_4 = 0$, then there is no error in Simpson's method. That is, if $f^{(4)}(x) = 0$ for all x in $[a, b]$, then Simpson's method produces an exact answer. For in this case the error is $M_4(b-a)h^4/180 = 0$. As a consequence, for polynomials of at most degree 3, Simpson's approximation is exact. (See Exercise 76 in Section 6.5.)

We know that the trapezoidal method is exact for polynomials of degree at most one, in other words, for functions whose second derivative is zero. That suggests that the error in this method is controlled by the size of the second derivative; Table 6.5.4 shows that it is.

The power of h that appears in the error bound is even more important. For instance, if you reduce the width h by a factor of 10 (using 10 times as many sections) you expect the error of the rectangular method to shrink by a factor of 10, the error in the trapezoidal method to shrink by a factor of $10^2 = 100$, and the error in Simpson's method by a factor of $10^4 = 10,000$. These observations are recorded in Table 6.5.5.

Method	Reduction Factor of h	Expected Reduction Factor of Error
Rectangles	10	10
Trapezoids	10	100
Simpson's (Parabolas)	10	10,000

Table 6.5.5:

Because the error in the rectangular method approaches 0 so slowly as $h \to 0$, we will not refer to it further.

Reference: Handheld Calculator Evaluates Integrals, William Kahan, Hewlett-Packard Journal, vol. 31, no. 8, Aug. 1980, pp. 23–32, `http://www.cs.berkeley.edu/~wkahan/Math128/INTGTkey.pdf`.

Technology and Definite Integrals

The trapezoidal method and Simpson's method are just two examples of what is called **numerical integration**. Such techniques are studied in detail in courses on numerical analysis. While the Fundamental Theorem of Calculus is useful for evaluating definite integrals, it applies only when an antiderivative is readily available. Numerical integration is an important tool in estimating definite integrals, particularly when the FTC cannot be applied. Numerical integration can always be used to find out something about the value of a definite integral.

The design of an efficient and accurate general-purpose numerical integration algorithm is harder than it might seem. Effective algorithms typically divide the interval into unequal-length sections. The sections can be longer where the function is tame, that is, almost constant. Shorter sections are used where the function is wild, that is, changes very rapidly. Large, even unbounded, intervals can lead to another set of difficulties. Some examples of challenging definite integrals include:

$$\int_0^2 \sqrt{x(4-x)}\, dx \qquad \int_{-1}^1 \frac{dx}{x^2+10^{-10}} \qquad \int_0^{600\pi} \frac{(\sin(x))^2}{\sqrt{x}+\sqrt{x+\pi}}\, dx$$

The HP-34C was, in 1980, the first handheld calculator to perform numerical integration. Now this is a common feature on most scientific calculators. The algorithms used vary greatly, and the details are often corporate secrets. The techniques are similar to those presented in this section and in Exercise 40.

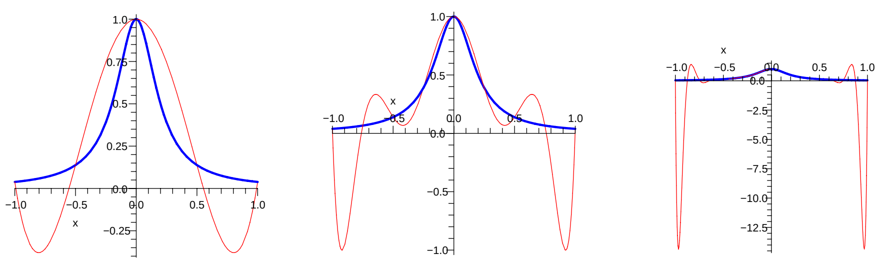
Summary

Three techniques for estimating definite integral are suggested by the areas of rectangles, the areas of trapezoids, and the areas under parabolas. We observed that the error in each method is influenced by a derivative of the integrand and the distance, $h = (b-a)/n$, between the numbers at which we evaluate the integrand. The main difference between the methods is the coefficients used to weight the function values $y_i = f(x_i)$. In the left-hand rectangular estimate the coefficients are 1, 1, 1, . . . , 1, 0 (because $y_n = f(b)$ is not used). In the right-hand rectangular estimate the coefficients are 0, 1, 1, . . . , 1. In the trapezoidal estimate, they are 1, 2, 2, . . . , 2, 1 and in Simpson's estimate they are 1, 4, 2, 4, 2, . . . , 2, 4, 1. A course in numerical analysis presents several other ways to estimate a definite integral.

Higher-Order Interpolation Methods and Runge's Counterexample
In the trapezoidal method you pass a line through two points to approximate the curve. That uses a first-degree polynomial, $Ax + B$. In Simpson's method you pass a parabola through three points, using a second-degree polynomial, $Ax^2 + Bx + C$. You would expect that as you pass higher-degree polynomials through more points on the curve you would get even better approximations. This is not always the case.

For the function $f(x) = 1/(1 + 25x^2)$, defined on $[-1, 1]$, known as Runge's Counterexample, the higher-degree polynomials passing through equally-spaced points do not resemble the function. Figure 6.5.11 shows the **interpolating polynomials** of degree 4 (a), 8 (b), and 16 (c). Notice how the approximations improve away from the endpoints and exhibit increasingly large oscillations near the endpoints. These oscillations result in poor estimates of $\int_{-1}^{1} \frac{dx}{1+25x^2}$. A Google search for "Runge's Counterexample" yields more information on this function.

Carle Runge, 1856–1927, http://en.wikipedia.org/wiki/Carle_David_Tolm%C3%A9_Runge



(a) (b) (c)

Figure 6.5.11: In each figure the thick curve is the graph of Runge's Counterexample and the thin curve is the graph of the interpolating polynomials of degree 4 (a), 8 (b), and 12 (c). Notice the very different vertical scales in these three graphs. EDITOR: Please move these figures inside the box.

EXERCISES for Section 6.5 *Key:* R–routine, M–moderate, C–challenging

In the Exercises, T_n refers to the trapezoidal estimate with n trapezoids (partition with n sections and $n+1$ points), and S_n refers to Simpson's estimate with $n/2$ parabolas (partition with n sections and $n+1$ points) In Exercises 1 to 8 approximate the given definite integrals by the trapezoidal estimate with the indicated T_n.

1.[R] $\int_0^2 \frac{dx}{1+x^2}$, T_2

2.[R] $\int_0^2 \frac{dx}{1+x^2}$, T_4

3.[R] $\int_0^2 \sin(\sqrt{x})\ dx,\ T_2$

4.[R] $\int_0^2 \sin(\sqrt{x})\ dx,\ T_3$

5.[R] $\int_1^3 \frac{2^x}{x}\ dx$, T_3

6.[R] $\int_1^3 \frac{2^x}{x}\ dx$, T_6

7.[R] $\int_1^3 \cos(x^2)\ dx$, T_2

8.[R] $\int_1^3 \cos(x^2)\ dx$, T_4

In Exercises 9 to 12 use Simpson's estimate to approximate each definite integral with the given S_n.

9.[R] $\int_0^1 \frac{dx}{1+x^3}$, S_2

10.[R] $\int_0^1 \frac{dx}{1+x^3}$, S_4

11.[R] $\int_0^1 \frac{dx}{1+x^4}$, S_2

12.[R] $\int_0^1 \frac{dx}{1+x^4}$, S_4

13.[R] Write out T_6 for $\int_1^4 5^x\ dx$ but do not carry out any of the calculations.

14.[R] Write out S_{10} for $\int_0^1 e^{x^2}\ dx$ but do not carry out any of the calculations.

15.[R] By a direct computation, show that the trapezoidal estimate is not exact for second-order polynomials. HINT: Take the simplest case, $\int_0^1 x^2\ dx$.

16.[R] By a direct computation, show that the Simpson's estimate is not exact for fourth-order polynomials. HINT: Take the simplest case, $\int_0^1 x^4\ dx$.

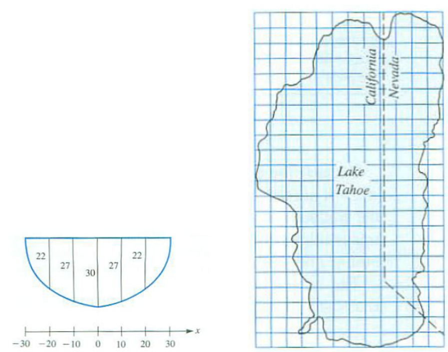
17.[R] In an interval $[a,b]$ in which $f''(x)$ is positive, do trapezoidal estimates of $\int_a^b f(x)\ dx$ underestimate or overestimate the definite integral? Explain.

18.[R] The cross-section of a ship's hull is shown in Figure 6.5.12(a). Estimate the area of this cross-section by

(a) T_6

(b) S_6

Dimensions are in feet. Give your answer to four decimal places.



(a) (b)

Figure 6.5.12:

19.[R] A ship is 120 feet long. The area of the cross-section of its hull is given at intervals in the table below:

x	0	20	40	60	80	100	120	feet
area	0	200	400	450	420	300	150	square feet

Estimate the volume of the hull in cubic feet by

(a) the trapezoidal estimate and

(b) Simpson's estimate.

Give your answer to four decimal places. HINT: What is largest n you can use in this problem?

20.[R] A map of Lake Tahoe is shown in Figure 6.5.12(b). Use Simpson's method and data from the map to estimate the surface area of the lake. Use cross-sections parallel to the side of the page. (Each little square represents a square mile.)

Exercises 21 and 22 present cases in which the maximum bound on the error is assumed.

21.[R] Show that the error for the trapezoidal estimate of $\int_0^1 x^2 dx$ is exactly $(b-a)M_2h^2/12$ where $a = 0$, $b = 1$, $h = 1$, and M_2 is the maximum value of $\left|D^2(x^2)\right|$ for x in $[0, 1]$.

22.[R] Show that the error for the Simpson estimate of $\int_0^1 x^4 dx$ is exactly $(b-a)M_4h^4/180$ where $a = 0$, $b = 1$, $h = 1/2$, and M_4 is the maximum value of $\left|D^4(x^4)\right|$ for x in $[0, 1]$.

23.[M] Figure 6.5.13(b) shows cross-sections of a pond in two directions. Use Simpson's method to estimate the area of the pond using

(a) vertical cross-sections, three parabolas and

(b) horizontal cross-sections, two parabolas.

24.[M] In the case of trapezoidal estimates, if you double the length of the interval $[a, b]$ and also the number of trapezoids, would you expect the error in the estimates to increase, decrease, or stay about the same? Explain.

25.[M] In the case of Simpson estimates, if you double the length of the interval $[a, b]$ and also the number of parabolas, would you expect the error in the estimates to increase, decrease, or stay about the same? Explain.

26.[M]

(a) Fill in this table concerning $\int_0^6 x^2\ dx$ and its trapezoidal estimates.

	$\int_0^6 x^2\ dx$	T_1	T_2	T_3
Value				
Error		—		

(b) Are the errors in (a) proportional to h^c for some constant c? (Recall that h is the width of the trapezoids.)

27.[M]

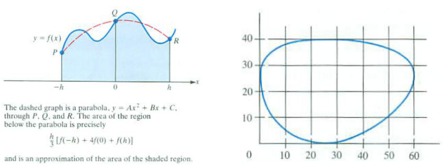
(a) Fill in this table concerning $\int_1^7 dx/(1+x)^2$ and its Simpson estimates.

	$\int_1^7 dx/(1+x)^2$	S_2	S_4	S_6
Value				
Error		—		

(b) Are the errors in (a) using S_n roughly proportional to h^k for some constant k? (Recall that h is the width of the sections.)

Exercises 28 to 30 provide the basis of Simpson estimates. For convenience we place the origin of the x-axis at the midpoint of the interval for which a single parabola will approximate the function. Because the interval has length $2h$, its ends are $-h$ and h.

28.[M] Let $f(x)$ be a function defined on at least $[-h, h]$, with $f(-h) = y_1$, $f(0) = y_2$, and $f(h) = y_3$. Show that there is exactly one parabola $P(x) = Ax^2 + Bx + C$ that passes through the three points $(-h, y_1)$, $(0, y_2)$, and (h, y_3). (See Figure 6.5.13(a).)



(a) (b)

Figure 6.5.13:

29.[M] Let $p(x) = Ax^2+Bx+C$. Show, by computing both sides of the equation, that

$$\int_{-h}^{h} p(x)\ dx = \frac{h}{3}\left(p(-h) + 4p(0) + p(h)\right).$$

This equation, expressed geometrically, was known to the ancient Greeks. In modern terms it says that Simpson's estimates are exact for polynomials of degree at most two.

30.[M] Let $f(x) = x^3$. Show that

$$\int_{-h}^{h} f(x)\ dx = \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right).$$

This information, combined with Exercise 29, implies that Simpson's method is exact for polynomials of degree at most 3.

31.[M] The table lists the values of a function f at the given points.

x	1	2	3	4	5	6	7
$f(x)$	1	2	1.5	1	1.5	3	3

(a) Plot the corresponding seven points on the graph of f.

(b) Sketch six trapezoids that can be used to estimate $\int_1^7 f(x)\,dx$.

(c) Find the trapezoidal estimate of $\int_1^7 f(x)\,dx$.

(d) Sketch, by eye, the three parabolas used in Simpson's method to estimate $\int_1^7 f(x)\,dx$.

(e) Find Simpson's estimate of $\int_1^7 f(x)\,dx$.

32.[M] A function f is defined on $[a,b]$ and $f(x)$, $f'(x)$, and $f''(x)$ are all positive for x in that interval. Arrange the following quantities in order of size, from smallest to largest. (Some may be equal.) Sketches may help.

(a) the area of the trapezoid with base $[a,b]$ and parallel sides of lengths $f(a)$ and $f(b)$

(b) the area of the "midpoint" rectangle with base $[a,b]$ and height $f((a+b)/2)$

(c) the area of the "right-endpoint" rectangle with base $[a,b]$ and height $f(b)$

(d) the area of the "left-endpoint" rectangle with base $[a,b]$ and height $f(a)$

(e) the average of (c) and (d)

(f) the trapezoid whose base is $[a,b]$ and whose top edge lies on the tangent line at $((a+b)/2, f((a+b)/2))$

(g) $\int_a^b f(x)\,dx$.

Exercises 33 to 35 describe the **midpoint estimate**, yet another way to estimate a definite integral.

33.[M] Another way to estimate a definite integral is by a Riemann sum $\sum_{i=1}^n f(c_i)h$, where the c_i are the midpoints of the intervals. Call such an estimate with n sections, M_n. Find M_4 for $\int_0^1 dx/(1+x^2)$.

34.[M] With the aid of a diagram, show that the midpoint estimate is exact for functions of the form $f(x) = Ax + B$.

35.[M] Assume that $f''(x)$ is negative for x in $[a,b]$. With the aid of a diagram, show that the midpoint method overestimates $\int_a^b f(x)\,dx$. Hint: Draw a tangent at the point $((a+b)/2, f((a+b)/2))$.

36.[M] If the Simpson estimate with 4 parabolas estimate a certain definite integral with an error of 0.35, what error would you expect with (a) 8 parabolas? (b) 5 parabolas?

37.[C] The equation in Exercise 28 is called the **prismoidal formula**. Use it to compute the volume of

(a) a sphere of radius a and

(b) a right circular cone of radius a and height h.

Note: The prisomoidal formula was known to the Greeks. Reference: `http://www.mathpages.com/home/kmath189/kmath189.htm`

Exercise 38 provides a review of several basic ideas as it involves the Fundamental Theorem of Calculus (FTC I), the chain rule, l'Hôpital's rule, and the intermediate-value theorem. The midpoint estimate is defined in Exercise 33.

38.[C] Assume that $f''(x)$ is continuous and negative for x in $[0,2h]$. Then the midpoint estimate, M, for $\int_{-h}^h f(x)\,dx$ is too large and the trapezoidal estimate, T, is too small. The error of the first is $M - \int_{-h}^h f(x)\,dx$

and of the second is $\int_{-h}^{h} f(x)\,dx - T$. Show that

$$\lim_{h\to 0} \frac{M - \int_{-h}^{h} f(x)\,dx}{\int_{-h}^{h} f(x)\,dx - T} = \frac{1}{2}.$$

This suggests that the error in the midpoint estimate when h is small is about half the error of the trapezoidal estimate. However, the midpoint estimate is seldom used because data at midpoints are usually not available (and because the Simpson estimate provides an even more accurate estimate using same data as the trapezoidal estimate).

39.[C] Simpson's estimate is not exact for fourth-degree polynomials.

(a) Estimate $\int_0^h x^4\,dx$ by S_2.

(b) What is the ratio between that estimate and $\int_0^h x^4\,dx$?

(c) What does (b) imply about the ratio between Simpson's estimate and $\int_0^h P(x)\,dx$ for any polynomial of degree at most 4?

40.[C] There are many other methods for estimating definite integrals. Some old methods, which had been of only theoretical interest because of their messy arithmetic, have, with the advent of computers, assumed practical importance. This exercise illustrates the simplest of the so-called **Gaussian quadrature** formulas. For convenience, we consider only integrals over $[-1,1]$.

(a) Show that

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

for $f(x) = 1$, x, x^2, and x^3.

(b) Let a and b be two numbers, $-1 \le a < b \le 1$, such that

$$\int_{-1}^{1} f(x)\,dx = f(a) + f(b)$$

for $f(x) = 1$, x, x^2, and x^3. Show that only $a = \frac{-1}{\sqrt{3}}$ and $b = \frac{1}{\sqrt{3}}$ (or $a = \frac{1}{\sqrt{3}}$ and $b = \frac{-1}{\sqrt{3}}$) satisfy this equation.

(c) Show that the Gaussian approximation

$$\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

has no error when f is a polynomial of degree at most 3.

(d) Use the formula in (a) to estimate $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$.

(e) Compare the answer in (d) to the exact value of $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$. How large is the error?

41.[C] Let f be a function such that $\left|f^{(2)}(x)\right| \le 10$ and $\left|f^{(4)}(x)\right| \le 50$ for all x in $[1,5]$. If $\int_1^5 f(x)\,dx$ is to be estimated with an error of at most 0.01, how small must h be in

(a) the trapezoidal approximation?

(b) Simpson's approximation?

42.[C]

Sam: I bet I can find a better way than Simpson's estimate to approximate $\int_{-h}^{h} f(x)\,dx$ using the same three arguments ($-h$, 0, and h).

Jane: How so?

Sam: Look at his formula $\frac{h}{3}(f(-h) + 4f(0) + f(h))$, which equals $2h\left(\frac{1}{6}f(-h) + \frac{4}{6}f(0) + \frac{1}{6}f(h)\right)$. The $2h$ is the width of the interval. I can't change that.

Jane: What would you change?

Sam: The weights $\frac{1}{6}$, $\frac{4}{6}$, and $\frac{1}{6}$. I'll use weights w_1, w_2, and w_3 and demand that the estimates I get be exact when the function $f(x)$ is either constant, x, or x^2.

Jane: Go ahead.

Sam: If $f(x) = c$, a constant, then, because $\int_{-h}^{h} c\ dx = 2hc$, I must have $2hc = 2h(w_1 c + w_2 c + w_3 c)$. That tells me that $w_1 + w_2 + w_3$ must be 1.

Jane: But you need three equations for three unknowns.

Sam: When $f(x) = x$, I get $\int_{-h}^{h} f(x)\ dx = 0$, so $0 = 2h(-w_1 h + w_2 0 + w_3 h)$. Now I know that w_1 equals w_3.

Jane: And the third equation?

Sam: With $f(x) = x^2$, I find that $\frac{2}{3}h^3 = 2h^3(w_1 + w_3)$.

Jane: So what are your three w's?

Sam: A little high school algebra shows they are $\frac{1}{6}$, $\frac{4}{6}$, and $\frac{1}{6}$. What a disappointment. But at least I avoided all the geometry of parabolas. It's really all about assigning proper weights.

Check the missing details and show that Sam is right.

43.[C] Another way to estimate a definite integral is to use Taylor polynomials (discussed in Section 5.4). If the Maclaurin polynomial $P_2(x)$ for $f(x)$ of degree 2 is used to approximate $f(x)$ for x in $[0, h]$, express the possible error in using $\int_0^h P_2(x)\ dx$ to estimate $\int_0^h f(x)\ dx$.

In Section 5.4 we showed why a higher derivative controls the error in using a Taylor polynomial to approximate a function value. Exercises 44 and 45 show why a higher derivative controls the error in using the trapezoidal or Simpson estimate of a definite integral $\int_a^b f(x)dx$. (See Exercise 73 in Section 6.5 for the derivation of the corresponding error estimate for the midpoint estimate.) In each case $h = (b-a)/n$ and a function $E(t)$, $0 \le t \le h$, is introduced. The "local error" is $E(h)$, that is, the error in using one trapezoid of width h or one parabola of width $2h$. Once $E(h)$ is controlled by a higher derivative, we multiply by n, where $nh = b - a$, to obtain a measure of the total error in estimating $\int_a^b f(x)\ dx$. The argument involves both FTC I and FTC II and provides a review of basic concepts.

44.[C] *(The error in the trapezoid estimate.)* As usual, let $h = (b - a)/n$. We will estimate the error for a single section of width h and then multiply by n to find the error in estimating $\int_a^b f(x)\ dx$. For convenience, we move the graph so the interval (of length h) is $[0, h]$.

(a) Show that the error when using T_1 is $E(h) = \int_0^h f(x)\ dx - \frac{h}{2}(f(0) + f(h))$.

(b) For t in $[0, h]$ let $E(t) = \int_0^t f(x)\ dx - \frac{t}{2}(f(0) + f(t))$. Show that $E(0) = 0$, $E'(0) = 0$, and $E''(t) = -\frac{t}{2}f''(t)$.

(c) Let M be the maximum of $f''(x)$ on $[a, b]$ and m be the minimum. Show that $\frac{-mt}{2} \ge E''(t) \ge \frac{-Mt}{2}$.

(d) Using (b) and (c), show that $\frac{-mt^2}{4} \ge E'(t) \ge \frac{-Mt^2}{4}$.

(e) Show that $\frac{-mt^3}{12} \ge E(t) \ge \frac{-Mt^3}{12}$.

(f) Show that $\frac{-mh^3}{12} \ge E(h) \ge \frac{-Mh^3}{12}$.

(g) Show that $\frac{-m(b-a)h^2}{12} \ge \int_a^b f(x)\ dx - T_n \ge \frac{-M(b-a)h^2}{12}$.

(h) Show that $\int_a^b f(x)\ dx - T_n = \frac{-f''(c)(b-a)h^2}{12}$ for some number c in $[a, b]$.

(i) Deduce that $\left|\int_a^b f(x)\ dx - T_n\right| \le \frac{M_2(b-a)h^2}{12}$, where M_2 is the maximum of $|f''(x)|$ for x in $[a, b]$.

45.[C] *(The error in the Simpson estimate.)* Now n is even and $[a, b]$ is divided into n sections of width $h = (b-a)/n$. The Simpson estimate is based on $n/2$ intervals of length $2h$. We will place the origin at the midpoint of an interval, so that its ends are $-h$ and h. In this case we wish to control the size of $E(h) = \int_{-h}^{h} f(x)\ dx - \frac{h}{3}(f(-h) + 4f(0) + f(h))$. Introduce the function $E(t)$, for $-h \le t \le h$, defined by $E(t) = \int_{-t}^{t} f(x)\ dx - \frac{t}{3}(f(-t) + 4f(0) + f(t))$.

(a) Show that
$$E'(t) = \frac{2}{3}(f(t)+f(-t)) - \frac{4}{3}f(0) - \frac{t}{3}(f'(t)-f'(-t)).$$

(b) Show that $E''(t) = \frac{1}{3}(f'(t) - f'(-t)) - \frac{t}{3}(f''(t) + f''(-t))$.

(c) Show that $E'''(t) = -\frac{t}{3}(f'''(t) - f'''(-t))$.

(d) Show that $E'''(t) = \frac{-2t^2}{3} f^{(4)}(c)$ for some c in $[-h, h]$.

(e) Show that $E(0) = E'(0) = E''(0) = 0$.

(f) Let M_4 be the maximum of $|f^{(4)}(t)$ on $[a, b]$. Show that $|E(t)| \le \frac{2t^5}{180} M_4$.

(g) Deduce that $\left| \int_a^b f(x)\ dx - S_n \right| \le \frac{M_4(b-a)h^4}{180}$.

6.S Chapter Summary

Chapter 6 introduced the second major concept in calculus, the definite integral, defined as a limit:

$$\int_a^b f(x)\ dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

For a continuous function this limit always exists and $\int_a^b f(x)\ dx$ can be viewed as the (net) area under the graph of $y = f(x)$ on the interval $[a, b]$. Both the definite integral and an antiderivative of a function f are called "integrals." Context tells which is meant. An antiderivative is also called an "indefinite integral."

The definite integral, in contrast to the derivative, gives global information.

Integrand: $f(x)$	Integral: $\int_a^b f(x)\ dx$
velocity	change in position
speed (|velocity|)	distance traveled
cross-sectional length of plane region	area of a plane region
cross-sectional area of solid	volume of solid
rate bacterial colony grows	total growth

As the first and last of these applications show, if you compute the definite integral of the rate at which some quantity is changing, you get the total change. To put this in mathematical symbols, let $F(x)$ be the quantity present at time x. Then $F'(x)$ is the rate at which the quantity changes. Thus $\int_a^b F'(x)\ dx$ equals the change in $F(x)$ as x goes from a to b, which is $F(b) - F(a)$. In short, $\int_a^b F'(x)\ dx = F(b) - F(a)$. This is another way of stating the Fundamental Theorem of Calculus, because F is an antiderivative of F'.

That equation gives a shortcut for evaluating many common definite integrals. However, finding an antiderivative can be tedious or impossible. For instance, $\exp(x^2)$ does not have an elementary antiderivative. However, continuous functions do have antiderivatives, as slope fields suggest. Indeed $G(x) = \int_a^x f(t)\ dt$ is an antiderivative of the integrand.

One way to estimate a definite integral is to employ one of the sums $\sum_{i=1}^{n} f(c_i)\ \Delta x_i$ that appear in its definition.

A more accurate method, which involves the same amount of arithmetic, uses trapezoids. Then the estimate takes the form

$$\int_a^b f(x)\, dx \approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right),$$

where consecutive x_i's are a fixed distance $h = (b-a)/n$ apart. In Simpson's method the graph is approximated by parts of parabolas, n is even, and the estimate is

$$\frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right).$$

The remaining chapters are simply elaborations of the derivative and the definite integral or further applications of them. For instance, instead of integrals over intervals, Chapter 17 deals with integrals over sets in the plane or in space. Chapter 15 treats derivatives of functions of several variables. In both cases the definitions involve limits similar to those that appear in the definitions of the derivative and the definite integral. That is one reason not to lose sight of those two definitions in their many applications.

EXERCISES for 6.S *Key:* R–routine, M–moderate, C–challenging

1.[R] State FTC II in words, using no mathematical symbols. (It refers to $F(b) - F(a)$.)

2.[R] State FTC I in words, using no mathematical symbols. (It refers to the derivative of $\int_a^x f(t)\, dt$.)

Evaluate the definite integrals in Exercises 3 to 16.

3.[R] $\int_1^2 (2x^3 + 3x - 5)\, dx$

4.[R] $\int_5^7 \frac{3}{x}\, dx$

5.[R] $\int_1^4 \frac{dx}{\sqrt{x}}$

6.[R] $\int_1^4 \frac{x+2x^3}{\sqrt{x}}\, dx$

7.[R] $\int_0^1 x(3+x)\, dx$

8.[R] $\int_0^2 (2+3x)^2\, dx$

9.[R] $\int_1^2 \frac{(2+3x)^2}{x^2}\, dx$

10.[R] $\int_1^2 e^{2x}\, dx$

11.[R] $\int_0^\pi \sin(3x)\, dx$

12.[R] $\int_0^{\pi/4} \sec^2(x)\, dx$

13.[R] $\int_0^{\sqrt{2}/2} \frac{3\, dx}{\sqrt{1-x^2}}\, dx$

14.[R] $\int_0^{\pi/4} \cos(x)\, dx$

15.[R] $\int_0^{\pi/4} \sec(x)\tan(x)\, dx$

16.[R] $\int_{1/2}^{\sqrt{2}/2} \frac{dx}{x\sqrt{x^2-1}}$

In Exercises 17 to 24 find an antiderivative of the given function by guess and experiment. Check your answer by differentiating it.

17.[R] $(2x+1)^5$

18.[R] $\frac{1}{(2x+1)^5}$

19.[R] $\frac{1}{x+1}$

20.[R] $\frac{1}{2x+1}$

21.[R] $\ln(x)$

22.[R] $x\sin(x)$

23.[R] $\sin(2x)$

24.[R] xe^{x^2}

Use Simpson's estimate with three parabolas ($n = 6$) to approximate the definite integrals in Exercises 25 and 26.

25.[R] $\int_0^{\pi/2} \sin(x^2)\, dx$

26.[R] $\int_1 2\sqrt{1+x^2}\, dx$

27.[R] Use the trapezoidal estimate with $n = 6$ to estimate the integral in Exercise 25.

28.[R] Use the trapezoidal estimate with $n = 6$ to estimate the integral in Exercise 26.

29.[R]

(a) What is the area under $y = 1/x$ and above $[1, b]$, $b > 1$?

(b) Is the area under $y = 1/x$ and above $[1, \infty)$ finite or infinite?

(c) The region under $y = 1/x$ and above $[1, b]$ is rotated around the x-axis. What is the volume of the solid produced?

30.[R] The basis for this chapter is that if f is continuous and $x > a$, then $\frac{d}{dx} \int_a^x f(t)\ dt = f(x)$.

(a) Review how this equation was obtained.

(b) Use a similar method to show that, if $x < b$, then $\frac{d}{dx} \int_x^b f(t)\ dt = -f(x)$.

31.[R] Let $f(x)$ and $g(x)$ be differentiable functions with $f(x) \geq g(x)$ for all x in $[a, b]$, $a < b$.

(a) Is $f'(x) \geq g'(x)$ for all x in $[a, b]$? Explain.

(b) Is $\int_a^b f(x)dx \geq \int_a^b g(x)dx$? Explain.

32.[R] Find $D\left(\int_{x^2}^{x^3} e^{-t^2} dt\right)$.

33.[R]

Jane: I'm not happy. The text says that a definite integral measures area. But they never defined "area under a curve." I know what the area of a rectangle is: width times length. But what is meant by the area under a curve? If they say, "Well, its the definite integral of the cross-sections," that won't do. What if I integrate cross-sections that are parallel to the x-axis instead of the y-axis? How do I know I'll get the same answer? Once again, the authors are hoping no one will notice a big gap in their logic.

Is Jane right? Have the authors tried to slip something past the reader?

34.[M] Let T_n be the trapezoidal estimate of $\int_a^b f(x)\ dx$ with n trapezoids and M_n be the midpoint estimate with n sections. Show that $\frac{1}{3}T_n + \frac{2}{3}M_n$ equals the Simpson estimate S_{2n} with n parabolas. HINT: Consider a typical interval of length h.

35.[M] A river flows at the (varying) rate of $r(t)$ cubic feet per second.

(a) Approximate how many cubic feet passes during the short time interval from time t to time $t+\Delta t$ seconds.

(b) How much passes from time t_1 to time t_2 seconds?

36.[M] Let $f(x) = xe^{-x}$ for $x \geq 0$. For which interval of length 1 is the area below the graph of f and above that interval a maximum?

37.[M] Let $f(x) = x/(x+1)^2$ for $x \geq 0$.

(a) Graph f, showing any extrema.

(b) Looking at your graph, estimate for which interval of length one, the area below the graph of f and above the interval is a maximum.

(c) Using calculus, find the interval in (b) that yields the maximum area.

38.[M]

(a) Estimate $\int_0^1 \frac{\sin(x)}{x} dx$ by approximating $\sin(x)$ by the Taylor polynomial $P_6(x; 0)$.

(b) Use the Lagrange bound on the error to bound the error in (a).

39.[M]

(a) Estimate $\int_1^3 \frac{e^x}{x} dx$ by using the Taylor polynomial $P_3(x; 2)$ to approximate e^x. (To avoid computing e^2, approximate e by 2.71828.)

(b) Use the Lagrange bound on the error to bound the error in (a).

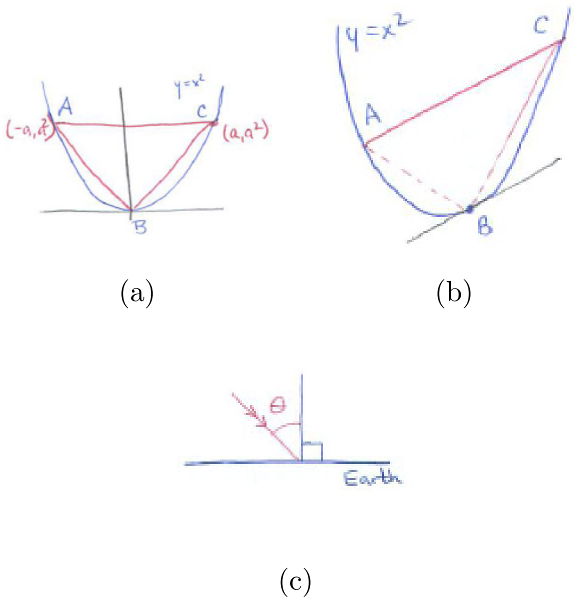
40.[M] Assume $f(2) = 0$ and $f'(2) = 0$ and $f''(x) \leq 5$ for all x in $[0, 7]$. Show that $\int_2^3 f(x)dx \leq 5/6$.

41.[M] Find $\displaystyle\lim_{t\to 0} \frac{\int_0^t \left(e^{x^2} - 1\right)\ dx}{\int_0^t \sin(2x^2)\ dx}$.

42.[M] Let $G(t) = \int_0^t \cos^5(\theta)\ d\theta$ for t in $[0, 2\pi]$.

(a) Sketch a rough graph of $y = G'(t)$.

(b) Sketch a rough graph of $y = G(t)$.



(a) (b)

(c)

Figure 6.S.1:

43.[M] Figure 6.S.1(a) shows a triangle ABC inscribed in the parabola $y = x^2$ $A = (-a, a^2)$, $B = (0,0)$, and $C = (a, a^2)$. Let $T(a)$ be its area and $P(a)$ the area bounded by AC and the parabola above the interval $[-a, a]$. Find $\lim_{a\to 0} \frac{T(a)}{P(a)}$. NOTE: Archimedes established a much more general result. In Figure 6.S.1(b) the tangent line at B is parallel to AC. He determined for any chord AC the ratio between the area of triangle ABC and the area of the parabolic section .

Usually we use a sum to estimate a definite integral. We can also use a definite integral to estimate a sum. In Exercises 44 and 45, rewrite each sum so that it becomes the sum estimating a definite integral. Then use the definite integral to estimate the sum.

44.[M] $\frac{1}{100}\sum_{i=1}^{100}\frac{1}{i^2}$

45.[M] $\sum_{n=51}^{100}\frac{1}{n}$

46.[M]

(a) Show that the average value of $\cos(\theta)$ for θ in $[0, \pi/2]$ is about 0.637.

(b) The average in (a) is fairly large, being much more than half of the maximum value of $\cos(\theta)$. Why is that good news for a farmer or solar engineer on Earth who depends on heat from the sun? HINT: See Figure 6.S.1(c).

47.[M] Assume f' is continuous on $[0, t]$.

(a) Find the derivative of $F(t) = 2\int_0^t f(x)f'(x)\,dx - f(t)^2$.

(b) Give a shorter formula for $F(t)$.

48.[M] Find a simple expression for the function $F(t) = \int_1^t \cos(x^2)\,dx - \int_1^{t^2} \frac{\cos(u)}{2\sqrt{u}}\,du$.

49.[M] A tent has a square base of side b and a pole of length $b/2$ above the center of the base.

(a) Set up a definite integral for the volume of the tent.

(b) Evaluate the integral in (a) by the Fundamental Theorem of Calculus.

(c) Find the volume of the tent by showing that six copies of it fill up a cube of side b.

50.[M]

Sam: I can get the second FTC, the one about $F(b) - F(a)$, without all that stuff in the first FTC.

Jane: That would be nice.

Sam: As usual, I assume F' is continuous and $\int_a^b F'(x)dx$ exists. Now, $F(b) - F(a)$ is the total change in F. Well, bust up $[a, b]$ by $t_0, t_1, \ldots, t_n$ in the usual way. Then the total change is just the sum of the changes in F over each of the n intervals, $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

Jane: That's a no-brainer, but then what?

Sam: The change in F over the typical interval is $F(t_i) - F(t_{i-1})$. By the Mean Value Theorem for F, that equals $F'(t_i^*)(t_i - t_{i-1})$ for some t_i^* in the i^{th} interval. The rest is automatic.

Jane: I see. You let all the intervals get shorter and shorter and the sums of the $F'(t_i^*)(t_i - t_{i-1})$ approach $\int_a^b F'(x)\ dx$. But they are all already equal to $F(b) - F(a)$.

Sam: Pretty neat, yes?

Jane: Something must be wrong.

Is anything wrong?

51.[M]

Sam: There are two authors and they are both wrong.

Jane: How so?

Sam: Light can be both a wave and a particle, right?

Jane: Yes.

Sam: Well the definite integral is both a number and a function.

Jane: Did you get enough sleep?

Sam: This is serious. Take $\int_0^b x^2$. That equals $b^3/3$. Right?

Jane: So far, right.

Sam: Well, as b varies, so does $b^3/3$. So it's a function.

Jane: ...

What is Jane's reply?

52.[M]

(a) Graph $y = e^x$ for x in $[0, 1]$.

(b) Let c be the number such that the area under the graph of $y = e^x$ above $[0, c]$ equals the area under the graph above $[c, 1]$. Looking at the graph in (a), decide whether c is bigger or smaller than $1/2$.

(c) Find c.

53.[M] Find $\lim_{\Delta x \to 0} \left(\frac{1}{\Delta x} \int_5^{7+\Delta x} e^{x^3}\ dx - \frac{1}{\Delta x} \int_5^7 e^{x^3}\ dx \right)$.

54.[M] Find $\lim_{\Delta x \to 0} \left(\frac{1}{\Delta x} \int_{5+\Delta x}^{7} e^{x^3}\ dx - \frac{1}{\Delta x} \int_5^7 e^{x^3}\ dx \right)$.

55.[M] A company is founded with capital investment A. It plans to have its rate of investment proportional to its total investment at any time. Let $f(t)$ denote the rate of investment at time t.

(a) Show that there is a constant k such that $f(t) = k(A + \int_0^t f(x)dx)$ for any $t \geq 0$.

(b) Find a formula for f.

There are two definite integrals in each of Exercises 56 to 59. One can be evaluated by the FTC, the other not. Evaluate the one that can be evaluated by the FTC and approximate the other by Simpson's estimate with $n = 4$ (2 parabolas).

56.[M] $\int_0^1 (e^x)^2\ dx$; $\int_0^1 e^{x^2}\ dx$.

57.[M] $\int_0^{\pi/4} \sec(x^2)\ dx$; $\int_0^{\pi/4} (\sec(x))^2\ dx$.

58.[M] $\int_1^3 e^{x^2} x\ dx$; $\int_1^3 \frac{e^{x^2}}{x}\ dx$.

59.[M] $\int_{0.2}^{0.4} \frac{dx}{\sqrt{1-x^2}}$; $\int_{0.2}^{0.4} \frac{dx}{\sqrt{1-x^3}}$.

60.[M] If $F'(x) = f(x)$, find an antiderivative for (a) $g(x) = x + f(x)$, (b) $g(x) = 2f(x)$, and (c) $g(x) = f(2x)$.

61.[M] John M. Robson in The Physics of Fly Casting, American J. Physics 58(1990), pp. 234–240, lets the reader fill in the calculus steps. For instance, he has the equation

$$\mu(4z + h)\dot{z}^2 = 2 \int_0^t crh\rho\dot{z}^3 dt + T(0)$$

where z is a function of time t, $\dot{z} = dz/dt$, and $\ddot{z} = d^z/dt^2$. He then states, "differentiating this gives

$$(2\mu - crh\rho)\dot{z}^2 + (4z + h)\mu\ddot{z} = 0."$$

Check that he is correct.

62.[C] Jane is running from a to b, on the x-axis. When she is at x, her speed is $v(x)$. How long does it take her to go from a to b?

63.[C]

(a) Find all continuous functions $f(t)$, $t \geq 0$, such that $\int_0^{x^2} f(t)\, dt = 3x^3$. $x \geq 0$.

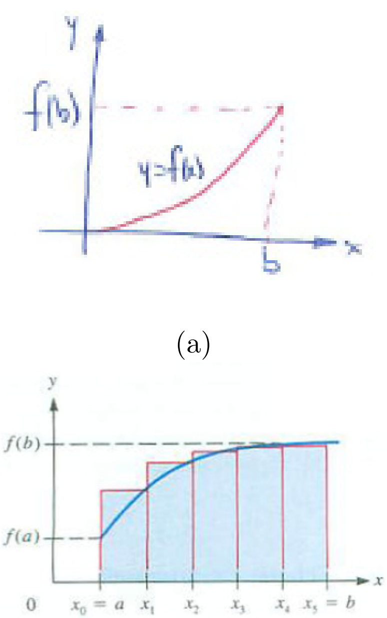
(b) Check that they satisfy the equation in (a).

64.[C] Let $f(x)$ be defined for x in $[0, b]$, $b > 0$. Assume that $f(0) = 0$ and $f'(x)$ is positive.

(a) Use Figure 6.S.2(a) to show that $\int_0^b f(x)\ dx + \int_0^{f(b)} (\mathsf{inv}f)(x)\ dx = bf(b)$.

(b) As a check on the equation in (a), differentiate both sides of it with respect to b. You should get a valid equation.

(c) Use (a) to evaluate $\int_0^1 \arcsin(x)\ dx$.



(a)

(b)

Figure 6.S.2:

65.[C]

(a) Verify, without using the FTC, that $\int_0^2 \sqrt{x(4-x)}\ dx = \pi$. HINT: What region has an area give by that integral?

(b) Approximate the definite integral in (a) by the trapezoidal estimate with 4 trapezoids and also with 8 trapezoids.

(c) Compute the error in each case.

(d) By trial-and-error, estimate how many trapezoids are needed to have an approximation that is accurate to three decimal places?

(e) Why is the error bound for the trapezoidal estimate of no use in (d)?

66.[C]

(a) Approximate the definite integral in Exercise 65 by Simpson's estimate with 2 parabolas and again with 4 parabolas. (These use the same number of arguments as in Exercise 65.)

(b) Compute the error in each case.

(c) By trial-and-error, estimate how many parabolas are needed to have an estimate accurate to 3 decimal places. HINT: Use your calculator or computer to automate the calculations.

(d) Why is the error bound for the Simpson's estimate of no use in (c)?

67.[C] In his *Principia*, published in 1607, Newton examined the error in approximating an area by rectangles. He considered an increasing, differentiable function f defined on the interval $[a, b]$ and drew a figure similar to Figure 6.S.2(b). All rectangles have the same width h. Let R equal the sum of the areas of the rectangles using right endpoints and let L equal the sum of the areas of the rectangles using left endpoints. Let A be the area under the curve $y = f(x)$ and above $[a, b]$.

(a) Why is $R - L = (f(b) - f(a))h$?

(b) Show that any approximating sum for A, formed with rectangles of equal width h and any sampling points, differs from A by at most $(f(b) - f(a))h$.

(c) Let M_1 be the maximum value of $|f'(x)|$ for x in $[a, b]$. Show that any approximating sum for A formed with equal widths h differs from A by at most $M_1(b - a)h$.

(d) Newton also considered the case where the rectangles do not necessarily have the same widths. Let h be the largest of their widths. What can be said about the error in this case?

68.[C] Let f be a continuous function such that $f(x) > 0$ for $x > 0$ and $\int_0^x f(t)\,dt = (f(x))^2$ for $x \geq 0$.

(a) Find $f(0)$.

(b) Find $f(x)$ for $x > 0$.

69.[C] A particle moves on a line in such a way that its average velocity over any interval of time $[a, b]$ is the same as its velocity at $(a + b)/2$. Prove that the velocity $v(t)$ must be of the form $ct + d$ for some constants c and d. HINT: Differentiate the relationship $\int_a^b v(t)\,dt = v\left(\frac{a+b}{2}\right)(b - a)$ with respect to b and with respect to a.

70.[C] A particle moves on a line in such a way that the average velocity over any interval of the form $[a, b]$ is equal to the average of the velocities at the beginning and the end of the interval of time. Prove that the velocity $v(t)$ must be of the form $ct + d$ for some constants c and d.

Exercises 71 and 72 present Archimedes' derivations for the area of a disk and the volume of a ball. He viewed these explanations as informal, and also presented rigorous proofs for them.

71.[C] Archimedes pictured a disk as made up of "almost" isosceles triangles, with one vertex of each triangle at the center of the disk and the base of the triangle part of the boundary of the disk. On the basis of this he conjectured that the area of a disk is one-half the product of the radius and its circumference. Explain why Archimedes' reasoning is plausible.

72.[C] Archimedes pictured a ball as made up of "almost" pyramids, with the vertex of each pyramid at the center of the ball and the base of the pyramid as part of the surface of the ball. On the basis of this he conjectured that the volume of a ball is one-third the product of the radius and its surface area. Explain why Archimedes' reasoning is plausible.

73.[C] (The midpoint estimates for a definite integral is described in Exercises 33 to 35 in Section 6.5.) Let M_n be the midpoint estimate of $\int_a^b f(x)\ dx$ based on n sections of width $h = (b-a)/n$. This exercise shows that the bound on the error, $\left|\int_a^b f(x)\ dx - M_n\right|$ is half of the bound on the trapezoidal estimate. The argument is like that in Exercises 44 and 45 of Section 6.5, a direct application of the Growth Theorem of Section 5.3.
Let $E(t) = \int_{-t/2}^{t/2} f(x)\ dx - f(0)t$.

(a) Show that $E(0) = E'(0) = 0$, and that $E''(t) = \frac{1}{4}\left(f'\left(\frac{t}{2}\right) - f'\left(\frac{-t}{2}\right)\right)$.

(b) Show that $\left|\int_a^b f(x)\ dx - M_n\right| \leq \frac{1}{24}M(b-a)h^2$, where M is the maximum of $|f''(x)|$ for x in $[a,b]$.

74.[C] Let $y = f(x)$ be a function such that $f(x) \geq 0$, $f'(x) \geq 0$, and $f''(x) \geq 0$ for all x in $[1,4]$. An estimate of the area under $y = f(x)$ is made by dividing the interval into sections and forming rectangles. The height of each rectangle is the value of $f(x)$ at the midpoint of the corresponding section.

(a) Show that the estimate is less than or equal to the area under the curve. HINT: Draw a tangent to the curve at each of the midpoints.

(b) How does the estimate compare to the area under the curve if, instead, $f''(x) \leq 0$ for all x in $[1,4]$?

75.[C] The definite integral $\int_0^1 \sqrt{x}\, dx$ gives numerical analysts a pain. The integrand is not differentiable at 0. What is worse, the derivatives (first, second, etc.) of $\sqrt{x}$ become arbitrarily large for x near 0. It is instructive, therefore, to see how the error in Simpson's estimate behaves as h is made small.

(a) Use the FTC to show that $\int_0^1 \sqrt{x}\, dx = \frac{2}{3}$.

(b) Fill in the table. (Keep at least 7 decimal places in each answer.)

h	Simpson's Estimate	Error
$\frac{1}{2}$		
$\frac{1}{4}$		
$\frac{1}{8}$		
$\frac{1}{16}$		
$\frac{1}{32}$		
$\frac{1}{64}$		

(c) In the typical application of Simpson's method, when you cut h by a factor of 2, you find that the error is cut by a factor of $2^4 = 16$. (That is, the ratio of the two errors would be $\frac{1}{16} = 0.0625$.) Examine the five ratios of consecutive errors in the table.

(d) Let $E(h)$ be the error in using Simpson's method to estimate $\int_0^1 \sqrt{x}\, dx$ with sections of length h. Assume that $E(h) = Ah^k$ for some constants k and A. Estimate k and A.

76.[C] Since Simpson's method was designed to be exact when $f(x) = Ax^2 + Bx + C$, one would expect the error associated with it to involve $f^{(3)}(x)$. By a quirk of good fortune, Simpson's method happens to be exact even when $f(x)$ is a cubic, $Ax^3 + Bx^2 + Cx + D$. This suggests that the error involves $f^{(4)}(x)$, not $f^{(3)}(x)$. Confirm that this is the case. NOTE: Exercise 45 in Section 6.5 does this using the Growth Theorem.

(a) Show that $\displaystyle\int_c^d x^3\, dx = \frac{d-c}{6}\left(f(c) + 4f\left(\frac{c+d}{2}\right) + f(d)\right)$.

(b) Why is Simpson's estimate exact for cubic polynomials?

77.[C] A producer of wine can choose to store it and sell it at a higher price after it has aged. However, he also must consider storage costs, which should not exceed the revenue.

Assume the revenue he would receive when selling the wine at time t is $V(t)$. If the interest rate on bank balances is r, which we will assume is constant, the present value of that sale is $V(t)e^{-rt}$.

The cost of storing the wine varies with time. Assume $c(t)$ represents that cost, that is, the cost of storing the wine during the short interval $[t, t + \Delta t]$ is approximately $c(t)\Delta t$.

(a) What is the present value of storing the wine for the period $[0, x]$?

(b) What is the present value, $P(x)$, of the profit (or loss) selling all the wine at time x? That is, the present value of the revenue minus the present value of the storage cost if sold at time x?

(c) Show that $P'(x) = V'(x)e^{-rx} - rV(x)e^{-rx} - c(x)e^{-rx}$.

(d) Show that if $V'(x)e^{-rx} > rV(x)e^{-rx} + c(x)e^{-rx}$, then $P'(x)$ is positive, and he should continue to store the wine.

(e) What is the meaning of each of the three terms in the inequality in (d)? Why does that inequality make economic sense?

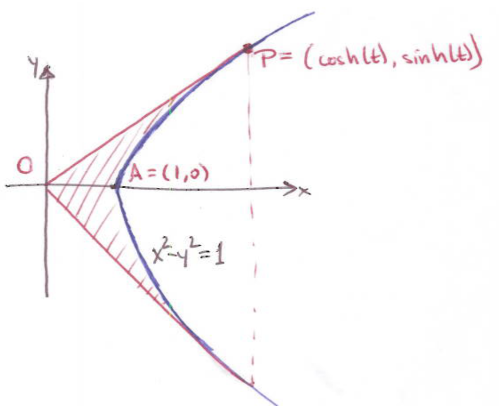


Figure 6.S.3:

78.[M] This exercise verifies the claims made in the last paragraph of Section 5.7.

(a) Explain why, for each angle θ in $[0, \pi]$, a sector of the unit circle with angle 2θ has area θ.

(b) In Figure 6.S.3, the area of the shaded region is twice the area of region OAP. The area of OAP is the area of a triangle less the area under the hyperbola. Express this area in terms of the parameter t. Hint: This will include a definite integral with integrand $\sqrt{x^2-1}$.

(c) Verify that $\frac{1}{2}\left(x\sqrt{x^2-1} - \ln(x+\sqrt{x^2-1})\right)$ is an antiderivative of $\sqrt{x^2-1}$ for $x > 1$.

(d) Show that the area of the shaded region in Figure 6.S.3 is t.

Note: Alternate ways to compu
shaded region are found in Exerci
and 8 on page 1059.

Skill Drill: Deriv

Exercises 79 to 84 offer an opportu
ferentiation skills. In each case, ve
tive of the first function is the seco

79.[R] $\ln\left(\frac{e^x}{1+e^x}\right)$; $\frac{1}{1+e^x}$
Hint: To simplify, first take logs.

80.[R] $\frac{1}{m}\arctan\left(e^{mx}\right)$; $\frac{1}{e^{mx}+e^{-mx}}$ (m is a constant).

81.[R] $\ln(\tan(x))$; $\frac{1}{\sin(x)\cos(x)}$

82.[R]

83.[R]
$\sec(x)$

84.[R]
$\sqrt{1-}$

In Exercises 85 to 87 differentiate t

85.[R] $\frac{\sin(2x)\tan(3x)}{x^3}$

86.[R] $2^{x^2}x^3\cos(4x)$

87.[R]

Calculus is Everywhere # 7
Peak Oil Production

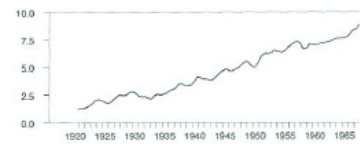


Figure C.7.1:

The United States in 1956 produced most of the oil it consumed, and the rate of production was increasing. Even so, M. King Hubbert, a geologist at Shell Oil, predicted that production would peak near 1970 and then gradually decline. His prediction did not convince geologists, who were reassured by the rising curve in Figure C.7.1.

Hubbert was right and the moment of maximum production is known today as Hubbert's Peak.

We present below Hubbert's reasoning in his own words, drawn from "Nuclear Energy and the Fossil Fuels," available at `http://www.hubbertpeak.com/hubbert/1956/1956.pdf`. In it he uses an integral over the entire positive x-axis, a concept we will define in Section 7.8. However, since a finite resource is exhausted in a finite time, his integral is an ordinary definite integral, whose upper bound is not known.

First he stated two principles when analyzing curves that describe the rate of exploitation of a finite resource:

1. For any production curve of a finite resource of fixed amount, two points on the curve are known at the outset, namely that at $t = 0$ and again at $t = \infty$. The production rate will be zero when the reference time is zero, and the rate will again be zero when the resource is exhausted; that is to say, in the production of any resource of fixed magnitude, the production rate must begin at zero, and then after passing through one or several maxima, it must decline again to zero.

2. The second consideration arises from the fundamental theorem of integral calculus; namely, if there exists a single-valued function $y = f(x)$, then

$$\int_0^{x_1} y \, dx = A, \qquad \text{(C.7.1)}$$

where A is the area between the curve $y = f(x)$ and the x-axis from the origin out to the distance x_1.

In the case of the production curve plotted against time on an arithmetical scale, we have as the ordinate

$$P = \frac{dQ}{dt}, \qquad \text{(C.7.2)}$$

where dQ is the quantity of the resource produced in time dt. Likewise, from equation (C.7.1) the area under the curve up to any time t is given by

$$A = \int_0^t P\,dt = \int_0^t \left(\frac{dQ}{dt}\right)\,dt = Q, \tag{C.7.3}$$

where Q is the cumulative production up to the time t. Likewise, the ultimate production will be given by

$$Q_{max} = \int_0^\infty P\,dt, \tag{C.7.4}$$

and will be represented on the graph of production-versus-time as the total area beneath the curve.

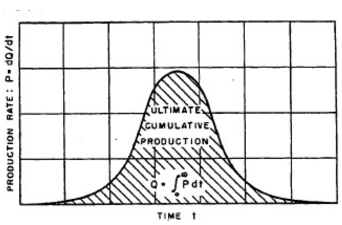


Figure C.7.2:

These basic relationships are indicated in Figure C.7.2. The only a priori information concerning the magnitude of the ultimate cumulative production of which we may be certain is that it will be less than, or at most equal to, the quantity of the resource initially present. Consequently, if we knew the production curves, all of which would exhibit the common property of beginning and ending at zero, and encompassing an area equal to or less than the initial quantity.

That the production of exhaustible resources does behave this way can be seen by examining the production curves of some of the older producing areas.

He then examines those curves for Ohio and Illinois. They resembled the curves below, which describe more recent data on production in Alaska, the United States, the North Sea, and Mexico.

Hubbert did not use a particular formula. Instead he employed the key idea in calculus, expressed in terms of production of oil, "The definte integral of the rate of production equals the total production."

He looked at the data up to 1956 and extrapolated the curve by eye, and by logic. This is his reasoning:

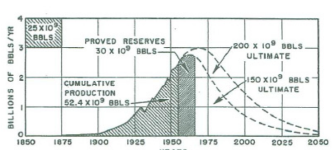


Figure C.7.4: Ultimate United States crude-oil production based on assumed initial reserves of 150 and 200 billion barrels.

> Figure C.7.4 shows "a graph of the production up to the present, and two extrapolations into the future. The unit rectangle in this case represents 25 billion barrels so that if the ultimate potential production is 150 billion barrels, then the graph can encompass but six rectangles before returning to zero. Since the cumulative production is already a little more than 50 billion barrels, then only four more rectangles are available for future production. Also, since the production rate is still increasing, the ultimate production peak must be greater than the present rate of production and must occur sometime in the future. At the same time it is possible to delay

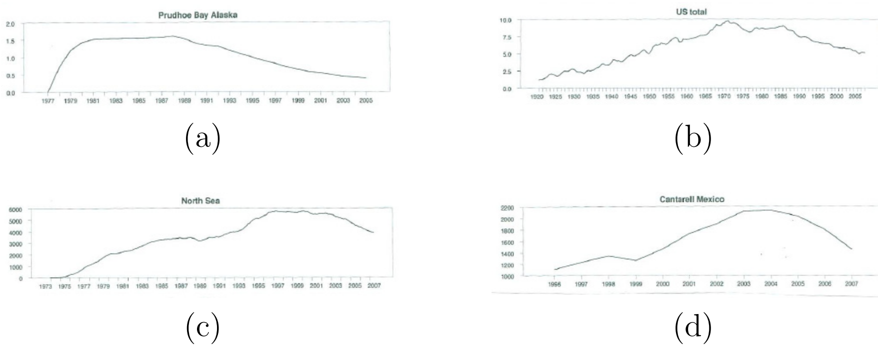


Figure C.7.3: Annual production of oil in millions of barrels per day for (a) Annual oil production for Prudhoe Bay in Alaska, 1977–2005 [Alaska Department of Revenue], (b) moving average of preceding 12 months of monthly oil production for the United States, 1920–2008 [EIA, "Crude Oil Production"], (c) moving average of preceding 12 months of sum of U.K. and Norway crude oil production, 1973–2007 [EIA, Table 11.1b], and (d) annual production from Cantarell complex in Mexico, 1996–2007 [Pemex 2007 Statistical Yearbook and Green Car Congress (`http://www.greencarcongress.com/2008/01/mexicos-cantare.html`).

> the peak for more than a few years and still allow time for the unavoidable prolonged period of decline due to the slowing rates of extraction from depleting reservoirs.
>
> With due regard for these considerations, it is almost impossible to draw the production curve based upon an assumed ultimate production of 150 billion barrels in any manner differing significantly from that shown in Figure C.7.4, according to which the curve must culminate in about 1965 and then must decline at a rate comparable to its earlier rate of growth.
>
> If we suppose the figure of 150 billion barrels to be 50 billion barrels too low — an amount equal to eight East Texas oil fields — then the ultimate potential reserve would be 200 billion barrels. The second of the two extrapolations shown in Figure C.7.4 is based upon this assumption; but it is interesting to note that even then the date of culmination is retarded only until about 1970."

Geologists are now trying to predict when world production of oil will peak. (Hubbert predicted the peak to occur in the year 2000.) In 2009 oil was being extracted at the rate of 85 million barrels per day. Some say the peak occurred as early as 2005, but others believe it may not occur until after 2020.

To see some of the latest estimates, do a web search for "Hubbert peak oil estimate".

What is just as alarming is that the world is burning oil faster than we are discovering new deposits.

In the CIE on Hubbert's Peak in Chapter 10 (see page 786) we present a later work of Hubbert, in which he uses a specific formula to analyze oil use and depletion.

Summary of Calculus I

The limit is the fundamental concept that forms the foundation for all of calculus. Limits are introduced in Chapter 2.

Chapters 3 through 5 were devoted to one of the two basic concepts in calculus, the derivative, defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

It tells how rapidly a function changes for inputs near x. That is local information.

Chapter 6 introduced the other major concept in calculus, the definite integral, also defined as a limit

$$\int_a^b f(x)\ dx = \lim_{\max \Delta x_i} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

For a continuous function this limit exists. $\int_a^b f(x)\ dx$ can be viewed as the (net) area under the graph of $y = f(x)$ above the interval $[a, b]$. Both the definite integral and an antiderivative of a function are called "integrals." Context tells which is meant. An antiderivative is also called an "indefinite integral."

The definite integral, in contrast to the derivative, gives global or overall information.

Integrand: $f(x)$	Integral: $\int_a^b f(x)\ dx$
velocity	change in position
speed (= |velocity|)	distance traveled
length of cross-section of plane region	area of region
area of cross-section of solid	volume of solid
rate bacterial colony grows	total growth

As the first and last of these applications show, if you compute the definite integral of the rate at which some quantity is changing, you get the total change. To put this in mathematical symbols, let $F(x)$ be the quantity present at time x. Then $F'(x)$ is the rate at which it changes.

That equation gives a shortcut for evaluating many common definite integrals. However, finding an antiderivative can be tedious or impossible.

For instance, e^{x^2} does not have an elementary antiderivative. However, continuous functions do have antiderivatives, as slope fields suggest. Indeed $G(x) = \int_a^x f(t)\ dt$ is an antiderivative of the integrand.

One way to estimate a definite integral is to employ one of the sums $\sum_{i=1}^{n} f(c_i)\Delta x_i$ that appear in its definition. A more accurate method, which uses the same amount of arithmetic, uses trapezoids. The trapezoidal estimate takes the form

$$\int_a^b f(x)\ dx \approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right),$$

where consecutive x_is are a fixed distance $hx = (b-a)/n$ apart.

In the even more accurate Simpson's estimate the graph is approximated by parts of parabolas, n is even, and the estimate is

$$\int_a^b f(x)\, dx \approx \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{x-1}) + 4f(x_{n-1}) + f(x_n)\right).$$

Long Road to Calculus

It is often stated that Newton and Leibniz invented calculus in order to solve problems in the physical world. There is no evidence for this claim. Rather, as with their predecessors, Newton and Leibniz were driven by curiosity to solve the "tangent" and "area" problems, that is, to construct a general procedure for finding tangents and areas. Once calculus was available, it was then applied to a variety of fields, notably physics, with spectacular success.

The first five chapters have presented the foundations of calculus in this order: functions, limits and continuity, the derivative, the definite integral, and the fundamental theorem that joins the last two. This bears little relation to the order in which these concepts were actually developed. Nor can we sense in this approach, which follows the standard calculations syllabus, the long struggle that culminated in the creation of calculus.

The origins of calculus go back over 2000 years to the work of the Greeks on areas and tangents. Archimedes (287–212 B.C.) found the area of a section of a parabola, an accomplishment that amounts in our terms to evaluating $\int_0^b x^2 dx$. He also found the area of an ellipse and both the surface area and the volume of a sphere. Apollonius (around 260–200 B.C.) wrote about tangents to ellipses, parabolas, and hyperbolas, and Archimedes discussed the tangents to a certain spiral-shaped curve. Little did they suspect that the "area" and "tangent" problems were to converge many centuries later.

With the collapse of the Greek world, symbolized by the Emperor Justinian's closing in A.D. 529 of Plato's Academy, which had survived for a thousand years, it was the Arab world that preserved the works of Greek mathematicians. In its liberal atmosphere, Arab, Christian, and Jewish scholars worked together, translating and commenting on the old writings, occasionally adding their own embellishments. For instance, Alhazen (A.D. 965–1039) computed volumes of certain solids, in essence evaluating $\int_0^b x^3 dx$ and $\int_0^b x^4 dx$.

It was not until the seventeenth century that several ideas came together to form calculus. In 1637, both Descartes (1596–1650) and Fermat (1601–165) introduced analytic geometry. Descartes examined a given curve with the aid of algebra, while Fermat took the opposite tack, exploring the geometry hidden in a given equation. For instance, Fermat showed that the graph of $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is always an ellipse, hyperbola, parabola, or one of their degenerate forms.

In this same period, Cavalieri (1598–1647) found the area under the curve $y = x^n$ for $n = 1, 2, 3, \ldots, 9$ by a method the length of whose computations grew rapidly as the exponent increased. Stopping at $n = 0$, he conjectured that the pattern would continue for larger exponents. In the next 20 years, several mathematicians justified his guess. So, even the calculation of the area under $y = x^n$ for a positive integer n, which we take for granted, represented a hard-won triumph.

"What about the other exponents?" we may wonder. Before 1665 there were no other exponents. Nevertheless, it was possible to work with the function which we denote $y = x^{p/q}$ for positive integers p and q by describing it as the function y such that $y^q = x^p$. (For instance, $y = x^{2/3}$ would be the function y that satisfies $y^3 = x^2$.) Wallis (1616–1703) found the area by a method that smacks more of magic than of mathematics. However, Fermat obtained the same result with the aid of an infinite geometric series.

The problem of determining tangents to curves was also in vogue in the first half of the seventeenth century. Descartes showed how to find a line perpendicular to a curve at a point P (by constructing a circle that meets the curve only at P); the tangent was then the line through P perpendicular to that line. Fermat found tangents in a way similar to ours and applied it to maximum-minimum problems.

Newton (1642-1727) arrived in Cambridge in 1661, and during the two years 1665–1666, which he spent at his family's farm to avoid the plague, he developed the essentials of calculus — recognizing that finding tangents and calculating areas are inverse problems. The first integral table ever compiled is to be found in one of his manuscripts of this period. But Newton did not publish his results at that time, perhaps because of the depression in the book trade after the Great Fire of London in 1665. During those two remarkable years he also introduced negative and fractional exponents, thus demonstrating that such diverse operations as multiplying a number by itself several times, taking its reciprocal, and finding a root of some power of that number are just special cases of a single general exponential function a^x, where x is a positive integer, -1, or a fraction, respectively.

Independently, however, Leibniz (1646–1716) also invented calculus. A lawyer, diplomat, and philosopher, for whom mathematics was a serious avocation, Leibniz established his version in the years 1673–1676, publishing his researches in 1684 and 1686, well before Newton's first publication in 1711. To Leibniz we owe the notations dx and dy, the terms "differential calculus" and "integral calculus," the integral sign, and the work "function." Newton's notation survives only in the symbol $\dot{x}$ for differentiation with respect to time, which is still used in physics.

It was to take two more centuries before calculus reached its present state of precision and rigor. The notion of a function gradually evolved from "curve" to "formula" to any rule that assigns one quantity to another. The great calculus text of Euler, published in 1748, emphasized the function concept by including not even one graph.

In several texts of the 1820s, Cauchy (1789–1857) defined "limit" and "continuous function" much as we do today. He also gave a definition of the definite integral, which with a slight change by Riemann (1826–1866) in 1854 became the definition standard today. So by the mid-nineteenth century the discoveries of Newton and Leibniz were put on a solid foundation.

In 1833, Liouville (1808–1882) demonstrated that the fundamental theorem could not be used to evaluate integrals of all elementary functions. In fact, he showed that the only values of the constant k for which $\int \sqrt{1-x^2}\sqrt{1-kx^2}dx$ is elementary are 0 and 1.

Still some basic questions remained, such as "What do we mean by area?" (For instance, does the set of points situated within some square and having both coordinates rational have an area? If so, what is this area?) It was as recently as 1887 that Peano (1858–1932) gave a precise definition of area — that quantity which earlier mathematicians had treated as intuitively given.

The history of calculus therefore consists of three periods. First, there was the long stretch when there was no hint that the tangent and area problems were related. Then came the discovery of their intimate connection and the exploitation of this relation from the end of the seventeenth century through the eighteenth century. This was followed by a century in which the loose ends were tied up.

The twentieth century saw calculus applied in many new areas, for it is the natural language for dealing with continuous processes, such as change with time. In that century mathematicians also obtained some of the deepest theoretical results about its foundations.

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Pronunciation

Descartes	"Day-CART"
Fermat	"Fair-MA"
Leibniz	"LIBE-nits"
Euler	"OIL-er"
Cauchy	"KOH-shee"
Riemann	"REE-mahn"
Liouville	"LYU-veel"
Peano	"Pay-AHN-oh"

Overview of Calculus II

The first part of this book was mainly about the derivative and the definite integral. The derivative measures a rate of change. The integral measures total change of a quantity that has a varying rate of change. The derivative and definite integral are linked by the Fundamental Theorem of Calculus. Both concepts are defined with the aid of limits, the basis of calculus.

The next six chapters apply the derivative and integral in a variety of contexts. Chapters 7 and 8 apply the definite integral and describe a few ways to find antiderivatives. Chapter 9, which stands by itself, concerns the geometry of curves and the physics of objects moving in a curved path. The next three chapters emphasize power series, which you may think of as "polynomials of infinite degree." That functions such as e^x and $\sin(x)$ can be represented by power series gives a way to compute them. With the aid of power series and complex numbers we show that the trigonometric functions can be expressed in terms of exponential functions (a relation applied, for instance, in the theory of alternating currents). Chapter 13, which discusses equations involving derivatives, could be studied any time after Chapter 8.