Chapter 5

More Applications of Derivatives

Chapter 2 constructed the foundation for derivatives, namely the concept of a limit. Chapters 3 and 4 developed the derivative and applied it to graphs of functions. The present chapter will apply the derivative in a variety of ways, such as: finding the most efficient way to accomplish a task (Section 5.1), connecting the rate one variable changes to the rate another changes (Section 5.2), the approximation of functions such as e^x by polynomials (Sections 5.3 and 5.4), the evaluation of certain limits (Section 5.5), natural growth and decay (Section 5.6), and to certain special functions (Section 5.7).

5.1 Applied Maximum and Minimum Problems

In Chapter 4, we saw how the derivative and second derivative are of use in finding the maxima and minima of a given function – the locally high and low points on its graph. Now we will use these same techniques to find extrema in applied problems. Though the examples will be drawn mainly from geometry they illustrate the general procedure. The main challenge in these situations is figuring out the formula for the function that describes the quantity to be maximized (or minimized).

The General Procedure

The general procedure runs something along these lines.

1. Get a feel for the problem (experiment with particular cases.)
2. Devise a formula for the function whose maximum or minimum you want to find.
3. Determine the domain of the function – that is, the inputs *that make sense in the application.*
4. Find the maximum or minimum of the function found in Step 2 for inputs that are in the domain identified in Step 3.

Additional worked examples can be found on the website for this book.

The most important step is finding a formula for the function. To become skillful at doing this takes practice. First, carefully read and study the three examples that comprise the remainder of this section.

A Large Garden

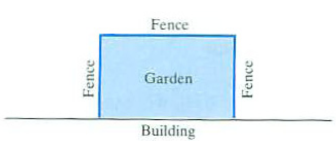


Figure 5.1.1:

EXAMPLE 1 A couple have enough wire to construct 100 feet of fence. They wish to use it to form three sides of a rectangular garden, one side of which is along a building, as shown in Figure 5.1.1. What shape garden should they choose in order to enclose the largest possible area?

SOLUTION *Step* 1. First make a few experiments. Figures 5.1.2 show some possible ways of laying out the 100 feet of fence. In the first case the side parallel to the building is very long, in an attempt to make a large area. However, doing this forces the other sides of the garden to be small. The area is $90 \times 5 = 450$ square feet. In the second case, the garden has a larger area, $60 \times 20 = 1200$ square feet. In the third case, the side parallel to the building

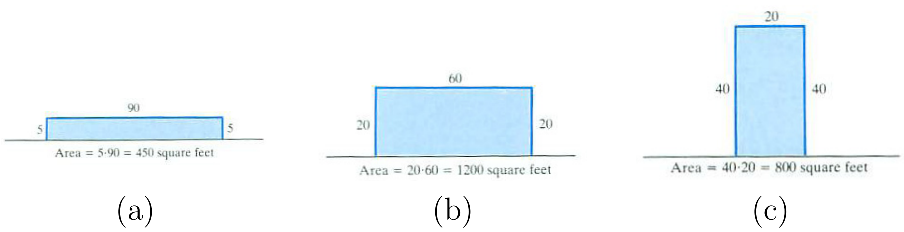


Figure 5.1.2:

is only 20 feet long, but the other sides are longer. The area is $20 \times 40 = 800$ square feet.

In all three cases, once the length of the side parallel to the building is set, the other side lengths are known and the area can be computed.

Clearly, we may think of the *area of the garden as a function of the length of the side parallel to the building.*

Step 2. Let $A(x)$ be the area of the garden when the length of the side parallel to the building is x feet, as in Figure 5.1.3. The other sides of the garden have length y. But y is completely determined by x since the total length of the fence is 100 feet:

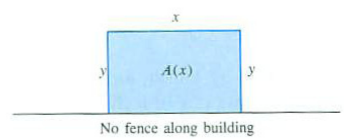


Figure 5.1.3:

$$x + 2y = 100.$$

Thus $y = (100 - x)/2$.

Since the area of a rectangle is its length times its width,

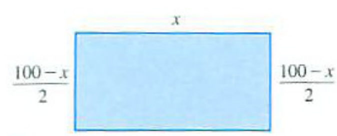


Figure 5.1.4:

$$A(x) \quad = \quad xy = x\left(\frac{100-x}{2}\right) = 50x - \frac{x^2}{2}.$$

(See Figure 5.1.4.) We now have the function.

Step 3. Which values of x in (5.1.1) correspond to possible gardens?

Since there is only 100 feet of fence, $x \leq 100$. Furthermore, it makes no sense to have a negative amount of fence; hence $x \geq 0$. Therefore the domain on which we wish to consider the function (5.1.1) is the closed interval $[0, 100]$.

Step 4. To maximize $A(x) = 50x - x^2/2$ on $[0, 100]$ we examine $A(0)$, $A(100)$, and the value of $A(x)$ at any critical numbers.

To find critical numbers, differentiate $A(x)$:

$$A(x) = 50x - \frac{x^2}{2} \qquad \text{so} \qquad A'(x) = 50 - x$$

and solve $A'(x) = 0$ to find:

$$0 = 50 - x \qquad \text{or} \qquad x = 50.$$

There is one critical number, 50.

All that is left is to find the largest of $A(0)$, $A(100)$, and $A(50)$. We have

$$\begin{aligned} A(0) &= 50 \cdot 0 - \frac{0^2}{2} = 0, \\ A(100) &= 50 \cdot 100 - \frac{100^2}{2} = 0, \\ \text{and} \quad A(50) &= 50 \cdot 50 - \frac{50^2}{2} = 1250. \end{aligned}$$

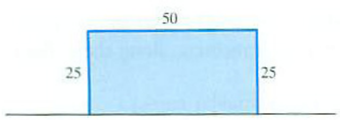
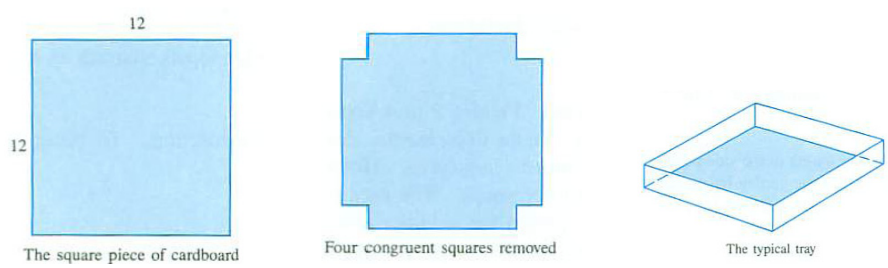


Figure 5.1.5:

The maximum possible area is 1250 square feet, and the fence should be laid out as shown in Figure 5.1.5. ◇

A Large Tray

EXAMPLE 2 Four congruent squares are cut out of the corners of a square piece of cardboard 12 inches on each side and the four remaining flaps can be folded up to obtain a tray without a top. (See Figure 5.1.6.) What size squares should be cut in order to maximize the volume of the tray?



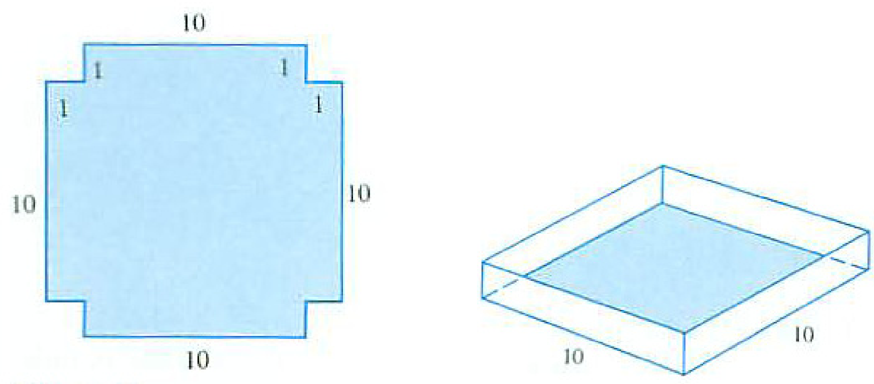
(a) (b) (c)

Figure 5.1.6:

Step 1. *SOLUTION* First we get a feel for the problem. Let us make a couple of experiments.

Say that we remove small squares that are 1 inch by 1 inch, as in Figure 5.1.7(a). When we fold up the flaps we obtain a tray whose base is a 10-inch by 10-inch square and whose height is 1 inch, as in Figure 5.1.7(b). The volume of the tray is

$$\text{Area of base} \times \text{height} = \underbrace{10 \times 10}_{\text{base area}} \times \underbrace{1}_{\text{height}} = 100 \text{ cubic inches.}$$



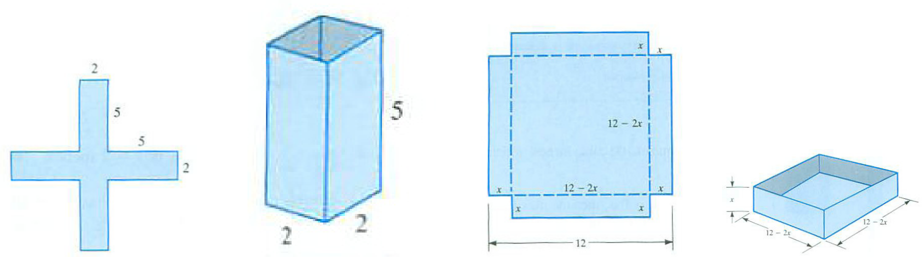
(a) (b)

Figure 5.1.7:

For our second experiment, let's try cutting out a large square, say 5 inches by 5 inches, as in Figure 5.1.8(a). When we fold up the flaps, we get a very tall tray with a very small base, as in Figure 5.1.8(b). It volume is

$$\text{Area of base} \times \text{height} = 2 \times 2 \times 5 = 20 \text{ cubic inches.}$$

Clearly *volume depends on the size of the cut-out squares*. The function



(a) (b) (c) (d)

Figure 5.1.8:

we will investigate is $V(x)$, the volume of the tray formed by removing four squares whose sides all have length x.

To find the formula for $V(x)$ we make a *large*, clear diagram of the typical case, as in Figure 5.1.8(c) and Figure 5.1.8(d). Now Step 2.

$$\text{Volume of tray} = \underbrace{(12-2x)}_{\text{length}}\underbrace{(12-2x)}_{\text{width}}\underbrace{\ x\ }\text{height} = (12-2x)^2x,$$

hence

$$V(x) = (12 - 2x)^2 x = 4x^3 - 48x^2 + 144x. \qquad (5.1.1)$$

We have obtained a formula for volume as a function of the length of the sides of the cut-out squares.

Step 3. Next determine the domain of the function $V(x)$ that is *meaningful* in the problem.

The smallest that x can be is 0. In this case the tray has height 0 and is just a flat piece of cardboard. (Its volume is 0.) The size of the cut is not more than 6 inches, since the cardboard has sides of length 12 inches. The cut can be as near 6 inches as we please, and the nearer it is to 6 inches, the smaller is the base of the tray. For convenience of our calculations, we allow cuts with $x = 6$, when the area of the base is 0 square inches and the height is 6 inches. (The volume in each of these cases is 0 cubic inches.) Therefore the domain of the volume function $V(x)$ is the closed interval $[0, 6]$.

Step 4. To maximize $V(x) = 4x^3 - 48x^2 + 144x$ on $[0, 6]$, evaluate $V(x)$ at critical numbers in $[0, 6]$ and at the endpoints of $[0, 6]$.

We have

$$V'(x) \quad = \quad 12x^2 - 96 + 144 = 12(x^2 - 8x + 12) = 12(x - 2)(x - 6).$$

A critical number is a solution to the equation

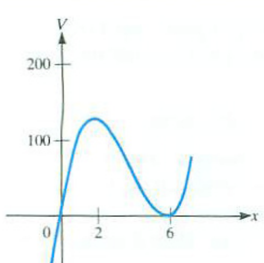
$$0 = 12(x - 2)(x - 6).$$

Hence $x - 2 = 0$ or $x - 6 = 0$. The critical numbers are 2 and 6.

The endpoints of the interval $[0, 6]$ are 0 and 6. Therefore the maximum value of $V(x)$ for x in $[0, 6]$ is the largest of $V(0)$, $V(2)$, and $V(6)$. Since $V(0) = 0$ and $V(6) = 0$, the largest value is

$$V(2) = 4(2^3) - 48(2^2) - 144 \cdot 2 = 128 \text{ cubic inches.}$$

The cut that produces the tray with the largest volume is $x = 2$ inches. ⋄



Only values of x in the portion above [0, 6] correspond to physically realizable trays.

Figure 5.1.9:

As a matter of interest, let us graph the function V, showing its behavior for all x, not just for values of x significant in the problem. Note in Figure 5.1.9 that at $x = 2$ and $x = 6$ the tangent is horizontal.

Remark: In Example 2 you might say $x = 0$ and $x = 6$ don't really correspond to what you would call a tray. If so, you would restrict the domain of $V(x)$ to the open interval $(0, 6)$. You would then have to examine the behavior of $V(x)$ for x *near* 0 and for x *near* 6. By making the domain $[0, 6]$ from the start, you avoid the extra work of examining $V(x)$ for x *near* the ends of the interval.

The key step in these two examples, and in any applied problem, is Step 2: findng a formula for the quantity whose extremum you are seeking. In case the problem is geometrical, the following chart may be of aid.

Setting Up the Function

1. Draw and label the appropriate diagrams.
   (Make them large enough so that there is room for labels.)
2. Label the various quantities by letters, such as x, y, A, V.
3. Identify the quantity to be maximized (or minimized).
4. Express the quantity to be maximized (or minimized) in terms of one or more of the other variables.
5. Finally, express that quantity in terms of only one variable.

An Economical Can

EXAMPLE 3 Of all the tin cans that enclose a volume of 100π cubic centimeters, which requires the least metal?

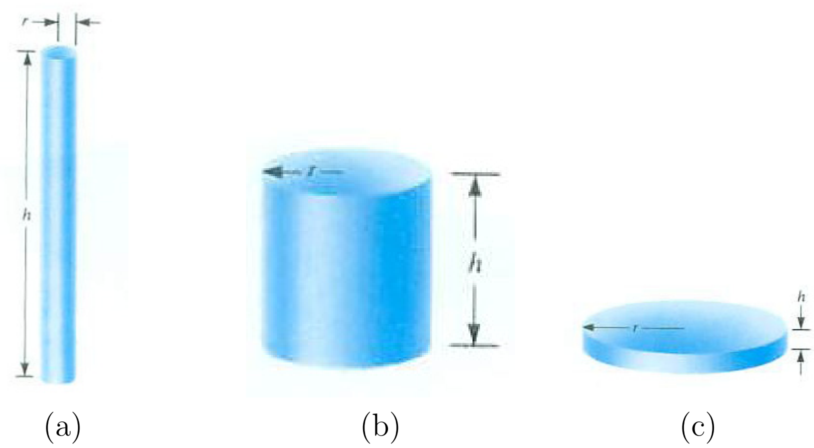
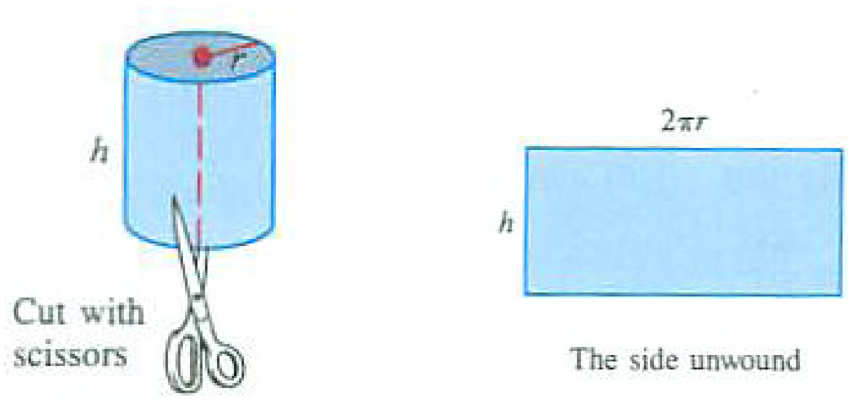


Figure 5.1.10:

SOLUTION The can may be flat or tall. If the can is flat, the side uses little metal, but then the top and bottom bases are large. If the can is shaped like a mailing tube, then the two bases require little metal, but the curved side requires a great deal of metal. (See Figure 5.1.10, where r denotes the radius and h the height of the can.) What is the ideal compromise between these two extremes? Step 1



(a) (b)

Figure 5.1.11:

Step 2 The surface area S of the can is the sum of the area of the top, side, and bottom. The top and bottom are circles with radius r so their total area is $2\pi r^2$. Figure 5.1.11 shows why the area of the side is $2\pi rh$. The total surface area of the can is given by

$$S = 2\pi r^2 + 2\pi rh. \tag{5.1.2}$$

Since the amount of metal in the can is proportional to S, it suffices to minimize S.

Equation (5.1.2) gives S as a function of two variables, but we can express one of the variables in terms of the other. The radius and height are related by the equation

$$V = \pi r^2 h = 100\pi, \tag{5.1.3}$$

since their volume is 100π cubic centimeters. In order to express S as a function of one variable, use (5.1.3) to eliminate either r or h. Choosing to eliminate h, we solve (5.1.3) for h,

$$h = \frac{100}{r^2}.$$

Substitution into (5.1.2) yields

$$S = 2\pi r^2 + 2\pi r \frac{100}{r^2} \qquad \text{or} \qquad S = 2\pi r^2 + \frac{200}{r}\pi. \tag{5.1.4}$$

Equation (5.1.4) expresses S as a function of just one variable, r.

Step 3 The cans have a positive radius as large as you please. The function $S(r)$ is continuous and differentiable on $(0, \infty)$.

Step 4 Compute dS/dr:

$$\frac{dS}{dr} = 4\pi r - \frac{200\pi}{r^2}\pi = \frac{4\pi r^3 - 200\pi}{r^2}. \tag{5.1.5}$$

Set the derivative equal to 0 to find any critical numbers. We have

$r = 0$ is *not* a critical number because it is not in the domain of V.

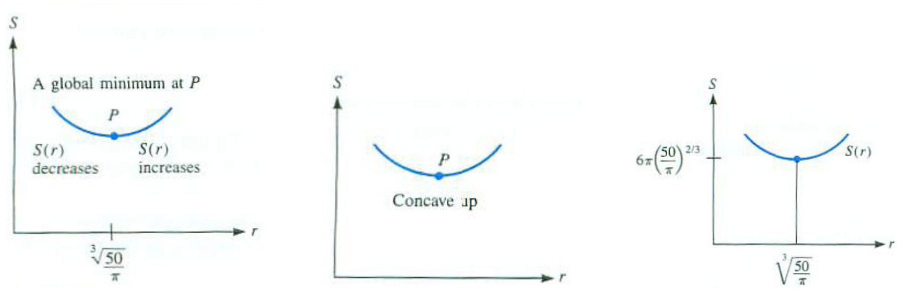
$$\begin{aligned} 0 &= \frac{4\pi r^3 - 200}{r^2}, \\ \text{hence} \quad 0 &= 4\pi r^3 - 200\pi \\ \text{or} \quad 4\pi r^3 &= 200\pi \\ r^3 &= \frac{200}{4} \\ r &= \sqrt[3]{50} \approx 0.7071. \end{aligned}$$

There is only one critical number. Does it provide a minimum? Let's check it two ways, first by the first-derivative test, then by the second-derivative test.

The first derivative is

$$\frac{dS}{dr} = \frac{4\pi r^3 - 200\pi}{r^2}. \tag{5.1.6}$$

When $r = \sqrt[3]{50}$, the numerator in (5.1.6) is 0. When $r < \sqrt[3]{50}$ the numerator is negative and when $r > \sqrt[3]{50}$ the numerator is positive. (The denominator is always positive.) Since $dS/dr < 0$ for $r < \sqrt[3]{50}$, and $dS/dr > 0$ for $r > \sqrt[3]{50}$, the function $S(r)$ decreases for $r < \sqrt[3]{50}$ and increases for $r > \sqrt[3]{50}$. That shows that a global minimum occurs at $\sqrt[3]{50}$. (See Figure 5.1.12(a).)



(a) (b) (c)

Figure 5.1.12:

Let us instead use the second-derivative test. Differentiation of (5.1.5) gives

$$\frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}\pi. \tag{5.1.7}$$

Inspection of (5.1.7) shows that for all meaningful values of r, that is r in $(0, \infty)$, d^2S/dr^2 is positive. (The function is concave up as shown in Figure 5.1.12(b).) Not only is P a relative minimum, it is a global minimum, since the graph lies above its tangents, in particular, the tangent at P.

The minimum of $S(r)$ is shown in Figure 5.1.12(c).

To find the height of the most economical can, solve (5.1.7) for h:

$$\begin{aligned} h = \tfrac{100}{r^2} &= \tfrac{100}{\pi(\sqrt[3]{50})^2} \\ &= \tfrac{100}{\pi(\sqrt[3]{50})^2}\tfrac{\sqrt[3]{50}}{\sqrt[3]{50}} && \text{rationalize the denominator} \\ &= \tfrac{100}{\pi(50)}\sqrt[3]{50} = 2\sqrt[3]{50}. \end{aligned}$$

The height of the can is equal to twice its radius, that is, its diameter. The total surface area of the can is

$$S = 2\pi r^3 + \frac{200\pi}{r}\bigg|_{r=50^{1/3}} = (100 + 4 \cdot 50^{2/3}) \approx 154.288 \text{ square centimeters.}$$

◇

Summary

We showed how to use calculus to solve applied problems: experiment, set up a function, find its domain, and its critical points. Then test the critical points and endpoints of the domain to determine the extrema.

1. Draw and label appropriate diagrams.
2. Express the quantity to be optimized in terms of one other variable.
3. Determine the domain of the function.
4. Use the first or second derivative test to determine the maximum or minimum of the function in its domain.

If the interval is closed, the maximum or minimum will occur at a critical point or an endpoint. If the interval is not closed, a little more care is needed to confirm that a critical number provides an extremum.

With practice this process becomes second nature.

EXERCISES for Section 5.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] A gardener wants to make a rectangular garden with 100 feet of fence. What is the largest area the fence can enclose?

2.[R] Of all rectangles with area 100 square feet, find the one with the shortest perimeter.

3.[R] Solve Example 1, expressing A in terms of y instead of x.

4.[R] A gardener is going to put a rectangular garden inside one arch of the cosine curve, as shown in Figure 5.1.13. What is the garden with the largest area.

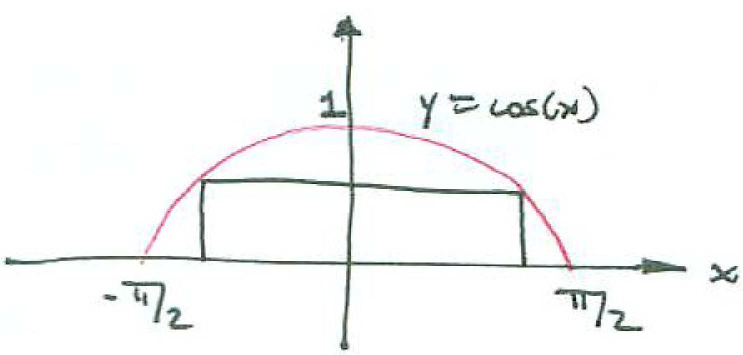


Figure 5.1.13:

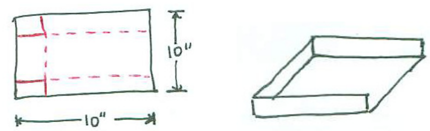
Exercises 5 to 8 are related to Example 2. In each case find the length of the cut that maximizes the volume of the tray. The dimensions of the cardboard are given.

5.[R] 5 inches by 5 inches

6.[R] 5 inches by 7 inches

7.[R] 4 inches by 8 inches,

8.[R] 6 inches by 10 inches,



(a) (b)

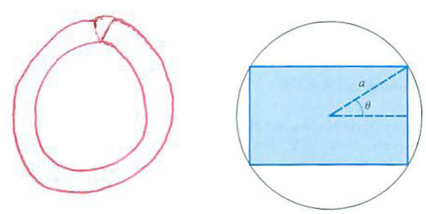
Figure 5.1.14:

9.[R] Starting with a square piece of paper 10″ on a side, Sam wants to make a paper holder with three sides. The pattern he will use is shown in Figure 5.1.14 along with the tray. He will remove two squares and fold up three flaps.

(a) What size square maximizes the volume of the tray?

(b) What is that volume?

10.[C] A chef wants to make a cake pan out of a circular piece of aluminum of radius 12 inches. To do this he plans to cut the circular base from the center of the piece and then cut the side from the remainder. What should the radius and height be to maximize the volume of the pan? (See Figure 5.1.15(a).)



(a) (b)

Figure 5.1.15:

11.[R] Solve Example 3, expressing S in terms of h instead of r.

12.[R] Of all cylindrical tin cans *without a top* that contains 100 cubic inches, which requires the least material?

13.[R] Of all enclosed rectangular boxes with square bases that have a volume of 1000 cubic inches, which uses the least material?

14.[R] Of all topless rectangular boxes with square bases that have a volume of 1000 cubic inches, which uses the least material?

15.[M] Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius a. The typical rectangle is shown in Figure 5.1.15(b). HINT: Express the area in terms of the angle θ shown.

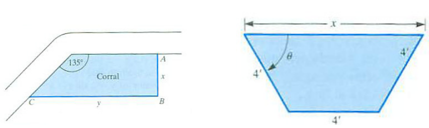
16.[M] Solve Exercise 15, expressing the area in terms of half the width of the rectangle, x. HINT: Square the area to avoid square roots.

17.[M] Find the dimensions of the rectangle of largest perimeter that can be inscribed in a circle of radius a.

18.[M] Show that of all rectangles of a given area, the square has the shortest perimeter. *Suggestion:* Call the fixed area A and keep in mind that it is a constant.

19.[M] A rancher wants to construct a rectangular corral. He also wants to divide the corral by a fence parallel to one of the sides. He has 240 feet of fence. What are the dimensions of the corral of largest area he can enclose?

20.[M] A river has a 45° turn, as indicated in Figure 5.1.16(a). A rancher wants to construct a corral bounded on two sides by the river and on two sides by 1 mile of fence ABC, as shown. Find the dimensions of the corral of largest area.



(a) (b)

Figure 5.1.16:

21.[M]

(a) How should one choose two nonnegative numbers whose sum is 1 in order to maximize the sum of their squares?

(b) To minimize the sum of their squares?

22.[M] How should one choose two nonnegative numbers whose sum is 1 in order to maximize the product of the square of one of them and the cube of the other?

23.[M] An irrigation channel made of concrete is to have a cross section in the form of an isosceles trapezoid, three of whose sides are 4 feet long. See Figure 5.1.16(b). How should the trapezoid be shaped if it is to have the maximum possible area? HINT: Consider the area as a function of x and solve.

24.[R]

(a) Solve Exercise 23 expressing the area as a function of θ instead of x.

(b) Do the answers in (a) and Exercise 23 agree? Explain.

In Exercises 25 to 28 use the fact that the combined length and girth (distance around) of a package to be sent through the mail by the United States Postal Service (USPS) cannot exceed 108 inches. NOTE: The combined length and girth of a packages sent as "parcel post" is 130 inches. The United Parcel Service (UPS) limit is 165 inches for combined length and girth with the length not exceeding 108 inches. Why do you think they have this restriction?

25.[R] Find the dimensions of the right circular cylinder of largest volume that can be sent through the mail.

26.[R] Find the dimensions of the right circular cylinder of largest surface area that can be sent through the USPS.

27.[R] Find the dimensions of the rectangular box with square base of largest volume that can be sent through the USPS.

28.[R] Find the dimensions of the rectangular box with square base of largest surface area that can be sent through the USPS.

29.[M]

(a) Repeat Exercise 25 with for a package sent by UPS.

(b) Generalize your solutions to Exercise 25 for a packages subject to a combined length and girth that does not exceed M inches.

30.[M]

(a) Repeat Exercise 26 with for a package sent by UPS.

(b) Generalize your solutions to Exercise 26 for a packages subject to a combined length and girth that does not exceed M inches.

Exercises 31 to 38 concern "minimal cost" problems.

31.[MR] A cylindrical can is to be made to hold 100 cubic inches. The material for its top and bottom costs twice as much per square inch as the material for its side. Find the radius and height of the most economical can. *Warning:* This is not the same as Example 3.

(a) Would you expect the most economical can in this problem to be taller or shorter than the solution to Example 3? (Use common sense, not calculus.)

(b) For convenience, call the cost of 1 square inch of the material for the side k cents. Thus the cost of 1 square inch of the material for the top and bottom is $2k$ cents. (The precise value of k will not affect the answer.) Show that a can of radius r and height h costs

$$C = 4k\pi r^2 + 2k\pi rh \text{ cents.}$$

(c) Find r that minimizes the functions C in (b). Keep in mind during any differentiation that k is constant.

(d) Find the corresponding h.

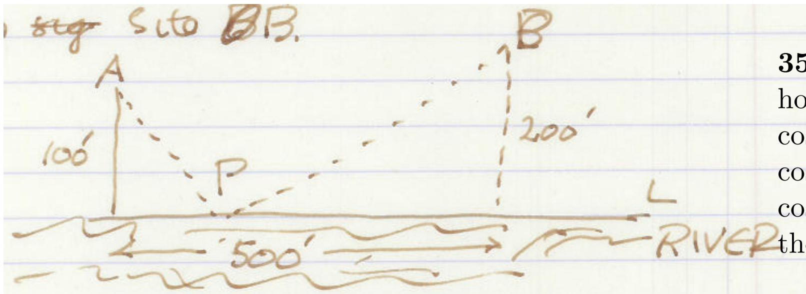


Figure 5.1.17: Sketch of situation in Exercise 32.

32.[M] A camper at A will walk to the river, put some water in a pail at P, and take it to the campsite at B.

(a) Express $\bar{AP} + \bar{PB}$ as a function of x.

(b) Where should P be located to minimize the length of the walk, $AP + PB$? (See Figure 5.1.17.) Hint: Reflect B across the line L.

Note: This exercise was first encountered as Exercise 34 in Section 1.1.

33.[M] Sam is at the edge of a circular lake of radius one mile and Jane is at the edge, directly opposite. Sam wants to visit Jane. He can walk 3 miles per hour and he has a canoe. What mix of paddling and walking should Sam use to minimize the time needed to reach Jane if

(a) he paddles at least three miles an hour?

(b) he paddles at 1.5 miles per hour?

(c) he paddles at 2 miles per hour?

34.[M] Consider a right triangle ABC, with C being at the right angle. There are two routes from A to B. One is direct, along the hypotenuse. The other is along the two legs, from A to C and then to B. Now, the shortest path between two points is the straight one. That raises this question: What is the largest percentage saving possible by walking along the hypotenuse instead of along the two legs? For which shape right triangle does this savings occur?

35.[M] A rectangular box with a square base is to hold 100 cubic inches. Material for the top of the box costs 2 cents per square inch; material for the sides costs 3 cents per square inch; material for the bottom costs 5 cents per square inch. Find the dimensions of the most economical box.

36.[M] The cost of operating a certain truck (for gasoline, oil, and depreciation) is $(20 + s/2)$ cents per mile when it travels at a speed of s miles per hour. A truck driver earns \$18 per hour. What is the most economical speed at which to operate the truck during a 600-mile trip?

(a) If you considered only the truck, would you want s to be small or large?

(b) If you, the employer, considered only the expense of the driver's wages, would you want s to be small or large?

(c) Express cost as a function of s and solve. (Be sure to put the costs all in terms of cents or all in terms of dollars.)

(d) Would the answer be different for a 1000-mile trip?

37.[R] A government contractor who is removing earth from a large excavation can route trucks over either of two roads. There are $10,000$ cubic yards of earth to move. Each truck holds 10 cubic yards. On one road the cost per truckload is $1 + 2x^2$ cents, when x trucks use that raod; the function records the cost of congestion. On the other road the cost is $2 + x^2$ cents per truckload when x trucks use that road. How many trucks should be dispatched to each of the two roads?

38.[R] On one side of a river 1 mile wide is an electric power station; on the other side, s miles upstream, is a factory. (See Figure 5.1.18.) It costs 3 dollars per foot to run cable over land and 5 dollars per foot under water. What is the most economical way to run cable from the station to the factory?

(a) Using no calculus, what do you think would be (approximately) the best route if s were very small? if s were very large?

(b) Solve with the aid of calculus, and draw the routes for $s = \frac{1}{2}$, $\frac{3}{4}$, 1, and 2.

(c) Solve for arbitrary s.

Warning: Minimizing the length of cable is *not* the same as minimizing its cost.

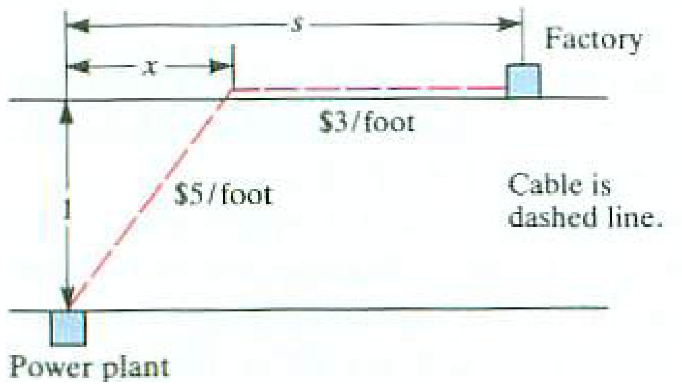


Figure 5.1.18:

39.[R] (From a text on the dynamics of airplanes.) "Recalling that

$$I = A\cos^2\theta + C\sin^2\theta - 2E\cos\theta\sin\theta,$$

we wish to find θ when I is a maximum or a minimum." Show that at an extremum of I,

$$\tan 2\theta = \frac{2\ E}{C-A}.\ (\text{assume that} A \neq C)$$

40.[R] (From a physics text.) "By differentiating the equation for the horizontal range,

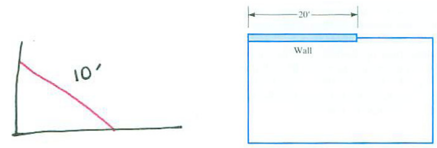
$$R = \frac{v_0^2\sin(2\theta)}{g},$$

show that the initial elevation angle θ for maximum range is 45°." In the formula for R, v_0 and g are constants. (R is the horizontal distance a baseball covers if you throw it at an angle θ with speed v_0. Air resistance is disregarded.)

(a) Using calculus, show that the maximum range occurs when $\theta = 45°$.

(b) Solve the same problem without calculus.

41.[R] A gardener has 10 feet of fence and wishes to make a triangular garden next to two buildings, as in Figure 5.1.19(a). How should he place the fence to enclose the maximum area?



(a) (b)

Figure 5.1.19:

42.[R] Fencing is to be added to an existing wall of length 20 feet, as shown in Figure 5.1.19(b). How should the extra fence be added to maximum the area of the enclosed rectangle if the additional fence is

(a) 40 feet long?

(b) 80 feet long?

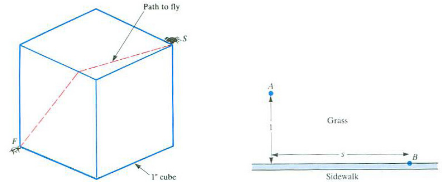
(c) 60 feet long?

43.[R] Let A and B be constants. Find the maximum and mimimum values of $A\cos t + B\sin t$.

44.[R] A spider at corner S of a cube of side 1 inch wishes to capture a fly at the opposite corner F. (See Figure 5.1.20(a).) The spider, who must walk on the surface of the solid cube, wishes to find the shortest path.

(a) Find a shortest path without the aid of calculus.

(b) Find a shortest path with calculus.



(a) (b)

Figure 5.1.20:

45.[R] A ladder of length b leans against a wall of height a, $a < b$. What is the maximal horizontal distance that the ladder can extend beyond the wall if its base rests on the horizontal ground?

46.[R] A woman can walk 3 miles per hour on grass and 5 miles per hour on sidewalk. She wishes to walk from point A to point B, shown in Figure 5.1.20(b), in the least time. What route should she follow if s is

(a) $\frac{1}{2}$?

(b) $\frac{3}{4}$?

(c) 1?

47.[R] The potential energy in a diatomic molecule is given by the formula

$$U(r) = u_0 \left(\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right),$$

where U_0 and r_0 are constants and r is the distance between the atoms. For which value of r is $U(r)$ a minimum?

48.[R] What are the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius a?

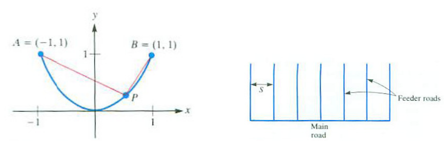
49.[R] The stiffness of a rectangular beam is proportional to the product of the width and the cube of the height of its cross section. What shape beam should be cut from a log in the form of a right circular cylinder of radius r in order to maximize its stiffness.

50.[R] A rectangular box-shaped house is to have a square floor. Three times as much heat per square foot enters through the roof as through the walls. What shape should the house be if it is to enclose a volume of $12,000$ cubic feet and minimize heat entry. (Assume no heat enters through the floor.)

51.[R] (See Figure 5.1.21(a).) Find the coordinates of the points $P = (x, y)$, with $y \leq 1$, on the parabola $y = x^2$, that

(a) minimize $\overline{PA}^2 + \overline{PB}^2$,

(b) maximize $\overline{PA}^2 + \overline{PB}^2$.



(a) (b)

Figure 5.1.21:

52.[R] The speed of traffic through the Lincoln Tunnel in New York City depends on the amount of traffic. Let S be the speed in miles per hour and let D be the amount of traffic measured in vehicles per mile. The relation between S and D was seen to be approximated closely, for $D \leq 100$, by the formula

$$S = 42 - \frac{D}{3}.$$

(a) Express in terms of S and D the total number of vehicles that enter the tunnel in an hour.

(b) What value of D will maximize the flow in (a)?

53.[R] When a tract of timber is to be logged, a main logging road is built from which small roads branch off as feeders. The question of how many feeders to build arises in practice. If too many are built, the cost of construction would be prohibitive. If too few are built, the time spent moving the logs to the roads would be prohibitive. The formula for total cost,

$$y = \frac{CS}{4} + \frac{R}{VS},$$

is used in a logger's manual to find how many feeder roads are to be built. R, C, and V are known constants: R is the cost of road at "unit spacing"; C is the cost of moving a log a unit distance; V is the value of timber per acre. S denotes the distance between the regularly spaced feeder roads. (See Figure 5.1.21(b).) Thus the cost y is a function of S, and the object is to find that value of S that minimizes y. The manual says, "To find the desired S set the two summands equal to each other and solve

$$\frac{CS}{4} = \frac{r}{VS}.''$$

Show that the method if valid.

54.[R] A delivery service is deciding how many warehouses to set up in a large city. The warehouses will serve similarly shaped regions of equal area A and, let us assume, an equal number of people.

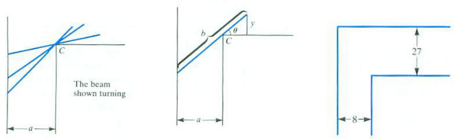
(a) Why would transportation costs per item presumably be proportional to $\sqrt{A}$?

(b) Assuming that the warehouse cost per item is inversely proportional to A, show that C, the cost of transportation and storage per item, is of the form $t\sqrt{A} + w/A$, where t and w are appropraite constants.

(c) Show that C is a minimum when $A = (2w/t)^{2/3}$.

Exercises 55 and 56 are related.

55.[R] A pipe of length b is carried down a long corridor of width $a < b$ and then around corner C. (See Figure 5.1.22.) During the turn y starts out at 0, reaches a maximum, and then returns to 0. (Try this with a short stick.) Find that maximum in terms of a and b. *Suggestion:* Express y in terms of a, b, and θ; θ is a variable, while a and b are constants.



(a) (b) (c)

Figure 5.1.22:

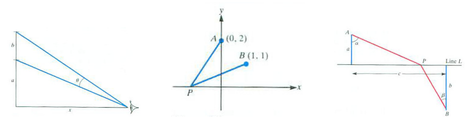
56.[M] Figure 5.1.22(c) shows two corridors meeting at right angle. One has width 8; the other, width 27. Find the length of the longest pipe that can be carried horizontally from one hall, around the corner and into the other hall. *Suggestion:* Do Exercise 55 first.

57.[M] Two houses, A and B, are a distance p apart. They are distances q and r, respectively, from a straight road, and on the same side of the road. Find the length of the shortest path that goes from A to the road, and then on to the other house B.

(a) Use calculus.

(b) Use only elementary geometry. *Hint:* Introduce an imaginary house C such that the midpoint of B and C is on the road and the segment BC is perpendicular to the road; that is, "reflect" B across the road to become C.

58.[R] The base of a painting on a wall is a feet above the eye of an observer, as shown in Figure 5.1.23(a). The vertical side of the painting is b feet long. How far from the wall should the ovserver stand to maximize the angle that the painting subtends? *Hint:* It is more convenient to maximize $\tan\theta$ than θ itself. HINT: Recall that $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$.



(a) (b) (c)

Figure 5.1.23:

59.[R] Find the point P on the x-axis such that the angle APB in Figure 5.1.23(b) is maximal. HINT: See the hint in Exercise 58.

60.[R] *(Economics)* Let p denote the price of some commodity and y the number sold at that price. To be concrete, assume that $y = 250 - p$ for $0 \le p \le 250$. Assume that it costs the producer $100 + 10y$ dollars to manufacture y units. What price p should the producer choose in order to maximize total profit, that is, "revenue minus cost"?

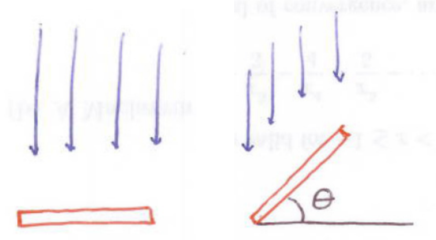
61.[R] *(Leibniz on light)* A ray of light travels from point A to point B in Figure 5.1.23(c) in minimal time. The point A is in one medium, such as air or a vacuum. The point B is in another medium, such as water or glass. In the first medium, light travels at velocity v_1 and in the second at velocity v_2. The media are separated by line L. Show that for the path APB of minimal time,

$$\frac{\sin\alpha}{v_1} = \frac{\sin(\beta)}{v_2}.$$

Leibniz solved this problem with calculus in a paper published in 1684. (The result is called **Snell's law of refraction**.)

Leibniz then wrote, "other very learned men have sought in many devious ways what someone versed in this calculus can accomplish in these lines as by magic." (See C. H. Edwards Jr., *The Historical Development of the Calculus*, p. 259, Springer-Verlag, New York, 1979.)

Exercises 62 to 65 concern the intensity of light.



(a) (b)

Figure 5.1.24:

62.[R] Why is it reasonable to assume that the intensity of light from a lamp is inversely proportional to the square of the distance from the lamp? Hint: Imagine the light spreading out in all directions.

63.[R] A solar panel perpendicular to the sun's rays catches more light than when it is tilted at any other angle, as shown in Figure 5.1.24(a). Let θ be the angle the panel is tilted, as in Figure 5.1.24(b). Show that it then receives $\cos(\theta)$ times the light the panel would receive when perpendicular to the sun's rays.

64.[M] In view of the preceding introduction and exercises, the intensity of light on a small (flat) surface is inversely proportional to the square of the distance from the source and proportional to the angle between the surface and a surface perpendicular to the source.

(a) A person wants to put a light at a horizontal distance of ten feet from his address, which is on a wall (a vertical surface). At what height should the lamp be placed to maximize the intensity of light at the address? Hint: No calculus is needed for this.

(b) Now the person paints the address on the horizontal surface of the curb. Again the lamp will be placed at a horizontal distance of ten feet from the address. Without doing any calculations sketch what the graph of "intensity of light on the address versus height of lamp" might look like.

(c) Find the height the lamp should have to maximize the light on the address. Hint: Use height as the independent variable.

65.[M] Solve Exercise 64(c) using an angle as the independent variable.

66.[M] The following calculation occurs in an article concerning the optimum size of new cities: "The net utility to the total client-centered system is

$$U = \frac{RLv}{A}n^{1/2} - nK - \frac{ALc}{v}n^{-1/2}.$$

All symbols except U and n are constant; n is a measure of decentralization. Regarding U as a differentiable function of n, we can determine when $dU/dn = 0$. This occurs when

$$\frac{RLv}{2A}n^{-1/2} - K + \frac{ALc}{2v}n^{-3/2} = 0.$$

This is a cubic equation for $n^{-1/2}$."

(a) Check that the differentiation is correct.

(b) Of what cubic polynomial is $n^{-1/2}$ a root?

67.[C] Consider the curve $y = x^{-2}$ in the first quadrant. A tangent to this curve, together with axes, determine a triangle.

(a) What is the largest area of such a triangle?

(b) The smallest area?

68.[C] Let f be a differentiable function that is never zero on its domain. Let $g(x) = (f(x))^2$. Show that the functions f and g have the same critical numbers. Note: This is useful for getting rid of square roots.

69.[C] Let f be a differentiable function. Define the function g by $g(x) = \tan(f(x))$. Show that the functions f and g have the same critical numbers.

5.2 Implicit Differentiation and Related Rates

Sometimes a function $y = f(x)$ is given indirectly by an equation that links y and x. This section shows how to differentiate y without solving for y explicitly in terms of x.

We will apply this technique to determine how the rate at which one quantity changes influences the rate at which another changes.

A Function Given Implicitly

The equation

$$x^2 + y^2 = 25 \tag{5.2.1}$$

describes a circle of radius 5 and center at the origin, as in Figure 5.2.1(a). This circle is not the graph of a function, since some vertical lines meet the

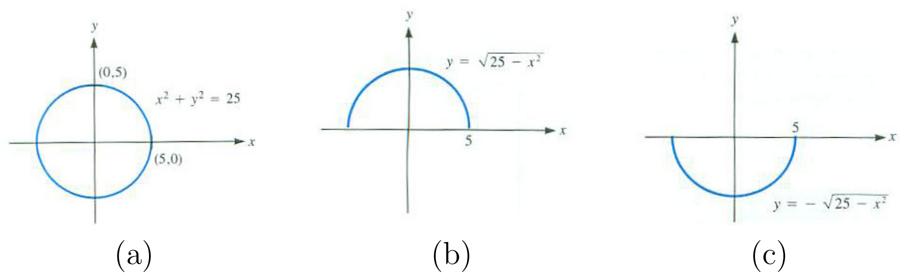


Figure 5.2.1:

circle in two points. However, the top half is the graph of a function and so is the bottom half. To find these functions explicitly, solve (5.2.1) for y:

$$\begin{aligned} y^2 &= 25 - x^2 \\ y &= \pm\sqrt{25 - x^2}. \end{aligned}$$

So either $y = \sqrt{25 - x^2}$ or $y = -\sqrt{25 - x^2}$. The graph of $y = \sqrt{25 - x^2}$ is the top semicircle (see Figure 5.2.1(b)); the graph of $y = -\sqrt{25 - x^2}$ is the bottom semicircle (see Figure 5.2.1(c)). There are two continuous functions that satisfy (5.2.1).

The equation $x^2 + y^2 = 25$ is said to describe the function $y = f(x)$ **implicitly**. The equations

$$y = \sqrt{25 - x^2} \qquad \text{and} \qquad y = -\sqrt{25 - x^2}$$

describe the function $y = f(x)$ **explicitly**.

Differentiating an Implicit Function

It is possible to differentiate a function given implicitly without having to solve for it and express it explicitly. An example will illustrate the method, which is to differentiate both sides of the equation that defines the function implicitly. This procedure is called **implicit differentiation**.

EXAMPLE 1 Let $y = f(x)$ be the continuous function that satisfies the equation

$$x^2 + y^2 = 25$$

such that $y = 4$ when $x = 3$. Find dy/dx when $x = 3$ and $y = 4$.
SOLUTION (We could, of course, solve for y, $y = \sqrt{25 - x^2}$, and differentiate directly. However, the algebra would be more involved since square roots would appear.) Differentiating both sides of the equation

$$x^2 + y^2 = 25$$

with respect to x yields

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(25), \\ 2x + \frac{d(y^2)}{dx} &= 0. \end{aligned}$$

To differentiate y^2 with respect to x, write $w = y^2$, where y is a function of x.

By the chain rule
$$\frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx},$$
which gives us
$$\frac{d(y^2)}{dx} = 2y\frac{dy}{dx}.$$
Thus
$$2x + 2y\frac{dy}{dx} = 0,$$
or
$$x + y\frac{dy}{dx} = 0.$$
In particular, when $x = 3$ and $y = 4$,
$$3 + 4\frac{dy}{dx} = 0,$$
and therefore,
$$\frac{dy}{dx} = -\frac{3}{4}.$$

◇

Observe that the algebra involves no square roots.

If you look back at Section 3.5, you will see that we already used implicit differentiation to find derivatives of inverse functions. For instance, we differentiated both sides of $y = e^x$ with respect to y, obtaining $1 = e^x(dx/dy)$. Then $dx/dy = 1/e^x = 1/y$. In short, $D(\ln(y)) = 1/y$.

In the next example implicit differentiation is the only way to find the derivative, for in this case there is no formula expressible in terms of trigonometric and algebraic functions giving y explicitly in terms of x.

EXAMPLE 2 Assume that the equation

$$2xy + \pi \sin(y) = 2\pi$$

defines a function $y = f(x)$. Find dy/dx when $x = 1$ and $y = \pi/2$.

Verify that the point $(1, \pi/2)$ is on the graph of $y = f(x)$ by checking that the equation is satisfied when $x = 1$ and $y = \pi/2$.

SOLUTION Implicit differentiation yields

$$\begin{aligned} \frac{d}{dx}(2xy + \pi \sin y) &= \frac{d(2\pi)}{dx}, \\ \left(2\frac{dx}{dx}y + 2x\frac{dy}{dx}\right) + \pi(\cos y)\frac{dy}{dx} &= 0, \end{aligned}$$

by the formula for the derivative of a product and the chain rule. Hence

$$2y + 2x\frac{dy}{dx} + \pi(\cos y)\frac{dy}{dx} = 0.$$

Solving for the derivative, dy/dx, we get

$$\frac{dy}{dx} = \frac{-2y}{2x + \pi \cos y}.$$

In particular, when $x = 1$ and $y = \pi/2$,

$$\frac{dy}{dx} = -\frac{2 \cdot \frac{\pi}{2}}{2 \cdot 1 + \pi \cos \frac{\pi}{2}} = -\frac{\pi}{2 + \pi \cdot 0} = -\frac{\pi}{2}.$$

◇

Implicit Differentiation and Extrema

Example 3 of Section 5.1 answered the question, "Of all the tin cans that enclose a volume of 100 cubic inches, which requires the least metal?" The radius of the most economical can is $\sqrt[3]{50/\pi}$. From this and the fact that its volume is 100 cubic inches, its height was found to be $2\sqrt[3]{50/\pi}$, exactly twice the radius. In the next example implicit differentiation is used to answer the same question. Not only will the algebra be simpler but it will provide the shape – the proportion between height and radius – easily.

EXAMPLE 3 Of all the tin cans that enclose a volume of 100 cubic inches, which requires the least metal?

SOLUTION The height h and radius r of any can of volume 100 cubic inches are related by the equation

$$\pi r^2 h = 100. \tag{5.2.2}$$

The surface area S of the can is

$$S = 2\pi r^2 + 2\pi r h \tag{5.2.3}$$

Consider h, and hence S, as functions of r. It is *not* necessary to find h and S explicitly in terms of r. Differentiation of (5.2.2) and (5.2.3) with respect to r yields

$$\pi r^2 \frac{dh}{dr} + 2\pi r h = \frac{d(100)}{dr} = 0 \tag{5.2.4}$$

and

$$\frac{dS}{dr} = 4\pi r + 2\pi r \frac{dh}{dr} + 2\pi h. \tag{5.2.5}$$

When S is a minimum, $dS/dr = 0$, so we have

$$0 = 4\pi r + 2\pi r \frac{dh}{dr} + 2\pi h. \tag{5.2.6}$$

Equations (5.2.4) and (5.2.6) yield, with a little algebra, a relation between h and r, as follows:

Factoring πr out of (5.2.4) and 2π out of (5.2.6) shows that

$$r\frac{dh}{dr} + 2h = 0 \qquad \text{and} \qquad 2r + r\frac{dh}{dr} + h = 0. \tag{5.2.7}$$

Elimination of dh/dr from (5.2.7) yields

$$2r + r\left(\frac{-2h}{r}\right) + h = 0,$$

which simplifies to

$$2r = h. \tag{5.2.8}$$

We have obtained the shape before the specific dimensions. Equation (5.2.8) asserts that the height of the most economical can is the same as its diameter. Moreover, this is the ideal shape, no matter what the prescribed volume happens to be.

The specific dimensions of the most economical can are found by eliminating h from equations (5.2.2) and (5.2.4). This shows that

$$\pi r^2(2r) = 100 \qquad \text{or} \qquad r^3 = \frac{50}{\pi}.$$

Hence

$$r = \sqrt[3]{\frac{50}{\pi}} \qquad \text{and} \qquad h = 2r = 2\sqrt[3]{\frac{50}{\pi}}$$

◇

The procedure illustrated in Example 3 is quite general. It may be of use when maximizing (or minimizing) a quantity that at first is expressed as a function of two variable which are linked by an equation. The equation that links them is called the **constraint**. In Example 3, the constraint is $\pi r^2 h = 100$.

Using Implicit Differentiation in an Extremum Problem

1. Name the various quantities in the problem by letters, such as x, y, h, r, A, V.
2. Identify the quantity to be maximized (or minimized).
3. Express that quantity in terms of other quantities, such as x and y.
4. Obtain an equation relating x and y.
   (This equation is called a constraint.)
5. Differentiate implicitly both the constraint and the quantity to be maximized (or minimized), interpreting all quantities to be functions of a single variable (which you choose).
6. Set the derivative of the quantity to be maximized (or minimized) equal to 0 and combine with the derivative of the constraint to obtain an equation relating x and y at a maximum (or minimum).
7. Step 6 gives only a relation between x and y at an extremum. If the explicit values of x and y are desired, find them by using the fact that x and y also satisfy the constraint.

Exercise 22 illustrates this possibility.

Warning: Sometimes an extremum occurs where a derivative, such as dy/dx, is not defined.

Related Rates

Implicit differentiation also comes in handy when showing how the rate of change of one quantity affects the rate of change of another.

EXAMPLE 4 An angler has a fish at the end of his line, which is reeled in at 2 feet per second from a bridge 30 feet above the water. At what speed

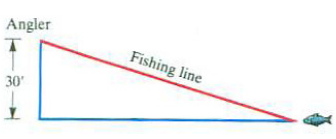


Figure 5.2.2:

is the fish moving through the water when the amount of line out is 50 feet? 31 feet? Assume the fish is at the surface of the water. (See Figure 5.2.2.)

SOLUTION Our first impression might be that since the line is reeled in at a constant speed, the fish at the end of the line moves through the water at a constant speed. As we will see, this is not the case.

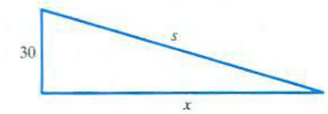


Figure 5.2.3:

Let s be the length of the line and x the horizontal distance of the fish from the bridge. (See Figure 5.2.3.)

Since the line is reeled in at the rate of 2 feet per second, s is shrinking, and

$$\frac{ds}{dt} = -2.$$

The rate at which the fish moves through the water is given by the derivative, dx/dt. The problem is to find dx/dt when $s = 50$ and also when $s = 31$.

We need an equation that relates s and x at *any* time, not just when $x = 50$ or $x = 31$. If we consider only $x = 50$ or $x = 31$, there would be no motion, and no chance to use derivatives.

The quantities x and s are related by the Pythagorean Theorem:

This equation is the heart of the example.

$$x^2 + 30^2 = s^2.$$

Both x and s are functions of time t. Thus both sides of the equation may be differentiated with respect to t, yielding

$$\begin{aligned} \frac{d(x^2)}{dt} + \frac{d(30^2)}{dt} &= \frac{d(s^2)}{dt} \\ \text{or} \qquad 2x\frac{dx}{dt} + 0 &= 2s\frac{ds}{dt}. \\ \text{Hence} \qquad x\frac{dx}{dt} &= s\frac{ds}{dt}. \end{aligned}$$

This last equation provides the tool for answering the questions.

Since $ds/dt = -2$,

$$\begin{aligned} x\frac{dx}{dt} &= (s)(-2). \\ \text{Hence} \qquad \frac{dx}{dt} &= \frac{-2s}{x}. \\ \text{When } s = 50, \qquad x^2 + 30^2 &= 50^2, \end{aligned}$$

so $x = 40$. Thus when 50 feet of line is out, the speed is

$$\left|\frac{dx}{dt}\right| = \frac{2s}{x} = \frac{2 \cdot 50}{40} = 2.5 \text{ feet per second.}$$

When $s = 31$, $x^2 + 30^2 = 31^2$.

Hence $x = \sqrt{31^2 - 30^2} = \sqrt{961 - 900} = \sqrt{61}$.

Thus when 31 feet of line is out, the fish is moving at the speed of

$$\frac{dx}{dt} = \frac{2s}{x} = \frac{2 \cdot 31}{\sqrt{61}} = \frac{62}{\sqrt{61}} \approx 7.9 \text{ feet per second.}$$

Let us look at the situation from the fish's point of view. When it is x feet from the point in the water directly below the bridge, its speed is $2s/x$ feet per second. Since s is larger than x, its speed is always greater than 2 feet per second. When x is very large, s/x is near 1 so the fish is moving through the water only a little faster than the line is reeled in. However, when the fish is almost at the point under the bridge, x is very small; then $2s/x$ is huge, and the fish finds itself moving at huge speeds, but according to Einstein, not faster than the speed of light. ◇

In Example 4 it would be a tactical mistake to indicate in Figure 5.2.3 that the hypotenuse of the triangle is 50 feet long, for if one leg is 30 feet and the hypotenuse is 50 feet, the triangle is determined; there is nothing left free to vary with time.

In general, label all the lengths or quantities that can change with letters x, y, s, and so on, even if not all are needed in the solution. Only after you finish differentiating do you determine what the rates are at a specified value of the variable.

The General Procedure

The method used in Example 4 applies to many related rate problems. This is the general procedure, broken into steps:

Procedure for Finding a Related Rate

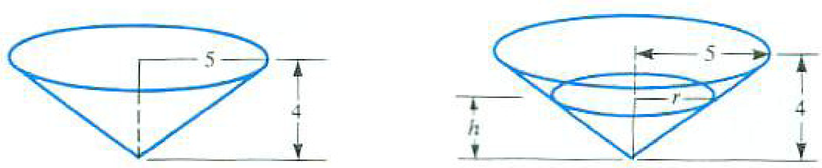
1. Find an equation that relates the varying quantities.
   (If the quantities are geometric, draw a picture and label the varying quantities with letters.)
2. Differentiate both sides of the equation with respect to time, obtaining an equation that relates the various rates of change.
3. Solve the equation obtained in Step 2 for the unknown rate.
   (Only at this step do you substitute constants for variable.)

WARNING Differentiate, then substitute the specific numbers for the variables. If you reversed the order, you would just be differentiating constants.

Finding an Acceleration

The method described in Example 4 for determining unknown rates from known ones extends to finding an unknown acceleration. Just differentiate another time. Example 5 illustrates the procedure.

EXAMPLE 5 Water flows into a conical tank at the constant rate of 3 cubic meters per second. The radius of the cone is 5 meters and its height is 4 meters. Let $h(t)$ represent the height of the water above the bottom of the cone at time t. Find dh/dt (the rate at which the water is rising in the tank) and d^2h/dt^2 (the rate at which that rate changes) when the tank is filled to a height of 2 meters. (See Figure 5.2.4.)



(a) (b)

Figure 5.2.4:

SOLUTION Let $V(t)$ be the volume of water in the tank at time t. The fact that water flows into the tank at 3 cubic meters per second is expressed as

$$\frac{dV}{dt} = 3,$$

and, since this rate is constant,

$$\frac{d^2V}{dt^2} = 0.$$

To find dh/dt and d^2h/dt^2, first obtain an equation relating V and h.

When the tank is filled to the height h, the water forms a cone of height h and radius r. (See Figure 5.2.4(b).) By similar triangles,

$$\frac{r}{h} = \frac{5}{4} \qquad \text{or} \qquad r = \frac{5h}{4}.$$

Thus

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{4}h\right)^2 h = \frac{25}{48}\pi h^3.$$

So the equation relating V and h is

$$V = \frac{25\pi}{48}h^3. \tag{5.2.9}$$

From here on, just differentiate as often as needed.

Differentiating both sides of (5.2.9) once (using the chain rule) yields

$$\frac{dV}{dt} = \frac{25\pi}{48}\frac{d(h^3)}{dh}\frac{dh}{dt}$$

or

$$\frac{dV}{dt} = \frac{25\pi}{16}h^2\frac{dh}{dt}.$$

Since $dV/dt = 3$ all the time,

$$3 = \frac{25\pi h^2}{16}\frac{dh}{dt},$$

from which it follows that

$$\frac{dh}{dt} = \frac{48}{25\pi h^2} \text{ meters per second.} \tag{5.2.10}$$

Even though the water enters the tank at a constant rate, it does not rise at a constant rate.

As (5.2.10) shows, the larger h is, the slower the water rises. (Why is this to be expected?)

To find dh/dt when $h = 2$ meters, substitute 2 for h in (5.2.10), obtaining

$$\frac{dh}{dt} = \frac{48}{25\pi 2^2} = \frac{12}{25\pi} \approx 0.15279 \text{ meters per second.}$$

Now we turn to the acceleration, d^2h/dt^2. *We do not* differentiate the equation $dh/dt = 12/(25\pi)$ since this equation holds only when $h = 2$. We must go back to (5.2.10), which holds at any time.

Differentiating (5.2.10) with respect to t yields

$$\frac{d^2h}{dt^2} = \frac{48}{25\pi}\frac{d}{dt}\left(\frac{1}{h^2}\right) = \frac{48}{25\pi}\frac{-2}{h^3}\frac{dh}{dt} = \frac{-96}{25\pi h^3}\frac{dh}{dt}. \tag{5.2.11}$$

The last equation expresses the acceleration in terms of h and dh/dt. Substituting (5.2.10) into (5.2.11) gives

$$\frac{d^2h}{dt^2} = \frac{-96}{25\pi h^3}\frac{48}{25\pi h^2}$$

or

$$\frac{d^2h}{dt^2} = \frac{-(96)(48)}{(25\pi)^2h^5} \text{ meters per second per second.} \qquad (5.2.12)$$

Equation (5.2.12) tells us that, since d^2h/dt^2 is negative, the rate at which the water rises in the tank is decreasing.

The problem also asked for the value of d^2h/dt^2 when $h = 2$. To find it, replace h by 2 in (5.2.12), obtaining

$$\frac{d^2h}{dt^2} = \frac{-(96)(48)}{(25\pi)^2 2^5}$$

or

$$\frac{d^2h}{dt^2} = \frac{-144}{625\pi^2} \approx -0.02334 \text{ meters per second per second.}$$

◇

Logarithmic Differentiation

If $\ln(f(x))$ is simpler than $f(x)$, there is a technique for finding $f'(x)$ that saves labor. Example 6 illustrates this method, which depends on implicit differentiation.

EXAMPLE 6 Let $y = \frac{\cos(3x)}{\left(\sqrt[3]{x^2+5}\right)^4}$. Find $\frac{dy}{dx}$.

SOLUTION The solution to this problem by **logarithmic differentiation** begins by simplifying $\ln(y)$ using the properties of logarithms:

$$\begin{aligned} \ln(y) &= \ln(\cos(3x)) - \ln\left(\left(\sqrt[3]{x^2+5}\right)^4\right) && [\ln(A/B) = \ln(A) - \ln(B)] \\ &= \ln(\cos(3x)) - \tfrac{4}{3}\ln\left(x^2+5\right) && [\ln(A^B) = B\ln(A)]. \end{aligned}$$

Next, since $\frac{d}{dx}(\ln(y)) = \frac{1}{y}\frac{dy}{dx}$ by the Chain Rule, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(\ln(\cos(3x)) - \frac{4}{3}\ln\left(x^2+5\right)\right) = \frac{-3\sin(3x)}{\cos(3x)} - \frac{4}{3}\frac{2x}{x^2+5}.$$

Therefore

$$\frac{dy}{dx} = (y)\left(-3\tan(3x) - \frac{4}{3}\frac{2x}{x^2+5}\right).$$

Finally, replace y by its formula, getting

$$\frac{dy}{dx} = \frac{\cos(3x)}{\left(\sqrt[3]{x^2+5}\right)^4}\left(-3\tan(3x) - \frac{4}{3}\frac{2x}{x^2+5}\right).$$

To appreciate logarithmic differentiation, find the derivative directly, as requested in Exercise 53. ◇

If you want to differentiate $\ln(f(x))$ for some function f, first see if you can simplify the expression by using the properties of a logarithm.

Properties of Logarithms

$$\ln(AB) = \ln(A) + \ln(B) \quad \ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B) \quad \ln\left(A^B\right) = B\ln(A)$$

Summary

We described "implicit differentiation," in which you differentiate a function without having an explicit formula for it. The function appears in an equation linking it and another variable. To find its derivative, just differentiate both sides of the equation, using the chain rule.

We applied these techniques in findng extrema and the relation between the rates of change of quantities linked by an equation. We also saw how the properties of logarithms can simplify finding the derivatives of some functions, particularly those involving products, quotients, and powers.

EXERCISES for Section 5.2 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4 find dy/dx at the indicated values of x and y in two ways: explicitly (solving for y first) and implicitly.

1.[R] $xy = 4$ at $(1, 4)$

2.[R] $x^2 - y^2 = 3$ at $(2, 1)$

3.[R] $x^2y + xy^2 = 12$ at $(3, 1)$

4.[R] $x^2 + y^2 = 100$ at $(6, -8)$

In Exercises 5 to 8 find dy/dx at the given points by implicit differentiation.

5.[R] $\frac{2xy}{\pi} + \sin y = 2$ at $(1, \pi/2)$

6.[R] $2y^3 + 4xy + x^2 = 7$ at $(1, 1)$

7.[R] $x^5 + y^3x + yx^2 + y^5 = 4$ at $(1, 1)$

8.[R] $x + \tan(xy) = 2$ at $(1, \pi/4)$

9.[R] Solve Example 3 by implicit differentiation, but differentiate (5.2.2) and (5.2.3) with respect to h instead of r.

10.[R] What is the shape of the cylindrical can of largest volume that can be constructed with a given surface area? Do not find the radius and height of the largest can; find the ratio between them. *Suggestion:* Call the surface area S and keep in mind that it is constant.

11.[M] Using implicit differentiation, find $D(\arctan x)$. *Hint:* Start with $x = \tan(y)$.

12.[M] Using implicit differentiation, find $D(\arcsin x)$. *Hint:* Start with $x = \sin(y)$.

In Exercises 13 to 16 find dy/dx at a general point (x, y) on the given curve.

13.[R] $xy^3 + \tan(x + y) = 1$

14.[R] $\sec(x + 2y) + \cos(x - 2y) + y = 2$

15.[R] $-7x^2 + 48xy + 7y^2 = 25$

16.[R] $\sin^3(xy) + \cos(x + y) + x = 1$

In Exercises 17 to 20 implicit differentiation is used to find a second derivative.

17.[R] Assume that $y(x)$ is a differentiable function of x and that $x^3y + y^4 = 2$. Assume that $y(1) = 1$. Find $y''(1)$, following these steps.

(a) Show that $x^3y' + 3x^2y + 4y^3y' = 0$.

(b) Use (a) to find $y'(1)$.

(c) Differentiate the equation in (a) and show that $x^3y'' + 6x^2y' + 6xy + 4y^3y'' + 12y^2(y')^2 = 0$.

(d) Use the equation in (c) to find $y''(1)$. [*Hint:* $y(1)$ and $y'(1)$ are known.]

18.[R] Find $y''(1)$ if $y(1) = 2$ and $x^5 + xy + y^5 = 35$.

19.[R] Find $y'(1)$ and $y''(1)$ if $y(1) = 0$ and $\sin y = x - x^3$.

20.[R] Find $y''(2)$ if $y(2) = 1$ and $x^3 + x^2y - xy^3 = 10$.

21.[R] Use implicit differentiation to find the highest and lowest points on the ellipse $x^2 + xy + y^2 = 12$. HINT: What do you know about dy/dx at the highest and lowest points on the graph of a function?

22.[M]

(a) What difficulty arises when you use implicit differentiation to maximize $x^2 + y^2$ subject to $x^2 + 4y^2 = 16$?

(b) Show that a maximum occurs when dy/dx is not defined. What is the maximum of x^2+y^2 subject to $x^2 + 4y^2 = 16$?

(c) The problem can be viewed geometrically as "Maximize the square of the distance from the origin for points on the ellipse $x^2 + 4y^2 = 16$." Sketch the ellipse and interpret (b) in terms of it.

23.[R] How fast is the fish in Example 4 moving through the water when it is 1 foot horizontally from the bridge?

24.[R] The angler in Example 4 decides to let the line out as the fish swims away. The fish swims away at a constant speed of 5 feet per second relative to the water. How fast is the angler paying out his line when the horizontal distance from the bridge to the fish is

(a) 1 foot?

(b) 100 feet?

25.[R] A 10-foot ladder is leaning against a wall. A person pulls the base of the ladder away from the wall at the rate of 1 foot per second.

(a) Draw a neat picture of the situation and label the varying lengths by letters and the fixed lengths by numbers.

(b) Obtain an equation involving the variables in (a).

(c) Differentiate it with respect to time.

(d) How fast is the top going down the wall when the base of the ladder is 6 feet from the wall? 8 feet from the wall? 9 feet from the wall?

26.[R] A kite is flying at a height of 300 feet in a horizontal wind.

(a) Draw a neat picture of the situation of label the varying lengths by letters and the fixed lengths by numbers.

(b) When 500 feet of string is out, the kite is pulling the string out at a rate of 20 feet per second. What is the kite's velocity? (Assume the string remains straight.)

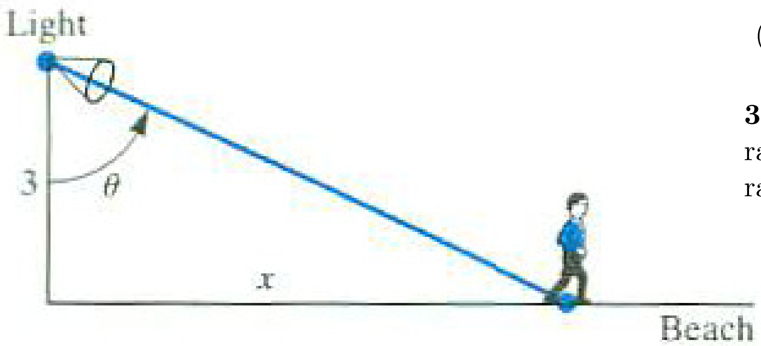


Figure 5.2.5:

27.[R] A beachcomber walks 2 miles per hour along the shore as the beam from a rotating light 3 miles offshore follows him. (See Figure 5.2.5.)

(a) Intuitively, what do you think happens to the rate at which the light rotates as the beachcomber walks further and further along the shore away from the lighthouse?

(b) Let x describe the distance of the beachcomber from the point on the shore nearest the light and θ the angle of the light, obtain an equation relating θ and x.

(c) With the aid of (b), show that $d\theta/dt = 6/(9+x^2)$ (radians per hour).

(d) Does the formula in (c) agree with your guess in (a)?

28.[R] A man 6 feet tall walks at the rate of 5 feet per second away from a street lamp that is 20 feet high. At what rate is his shadow lengthening when he is

(a) 10 feet from the lamp?

(b) 100 feet from the lamp?

29.[R] A large spherical balloon is being inflated at the rate of 100 cubic feet per minute. At what rate is the radius increasing when the radius is

(a) 10 feet?

(b) 20 feet?

(The volume of a sphere of radius r is $V = 4\pi r^3/3$.)

30.[R] A shrinking spherical balloon loses air at the rate of 1 cubic inch per second. At what rate is its radius changing when the radius is

(a) 2 inches

(b) 1 inch?

31.[R] Bulldozers are moving earth at the rate of $1,000$ cubic yards per hour onto a conically shaped hill whose height of the hill increasing when the hill is

(a) 20 yards high?

(b) 100 yards high?

(The volume of a cone of radius r and height h is $V = \pi r^2 h/3$.)

32.[R] The lengths of the two legs of a right triangle depend on time. One leg, whose length is x, increaes at the rate of 5 feet per second, while the other, of length y, decreases at the rate of 6 feet per second. At what rate is the hypotenuse changing when $x = 3$ feet and $y = 4$ feet? Is the hypotenuse increasing or decreasing then?

33.[R] Two sides of a triangle and their included angle are changing with respect to time. The angle increases at the rate of 1 radian per second, one side increases at the rate of 3 feet per second, and the other side decrease at the rate of 2 feet per second. Find the rate at which the area is changing when the angle is $\pi/4$, the first side is 4 feet long, and the second side is 5 long. Is the area decreasing or increasing then?

34.[R] The length of a rectangle is increasing at the rate of 7 feet per second, and the width is decreasing at the rate of 3 feet per second. When the length is 12 feet and the width is 5 feet, find the rate of change of

(a) the area,

(b) the perimeter

(c) the length of the diagonal.

Exercises 35 to 39 concern acceleration.

35.[R] What is the acceleration of the fish described in Example 4 when the length of line is

(a) 300 feet?

(b) 31 feet?

NOTE: The notation $\dot{x}$ for dx/dt, $\dot{\theta}$ for $d\theta/dt$, $\ddot{x}$ for d^2x/dt^2, and $\ddot{\theta}$ for $d^2\theta/dt^2$ was introduced by Newton and is still common in physics.

36.[R] A woman on the ground is watching a jet through a telescope as it approaches at a speed of 10 miles per minute at an altitude of 7 miles. At what rate (in radians per minute) is the angle of the telescope changing when the horizontal distance of the jet from the woman is 24 miles? When the jet is directly above the woman?

37.[R] Find $\ddot{\theta}$ in Example 36 when the horizontal distance from the jet is

(a) 7 miles,

(b) 1 mile.

38.[R] A particle moves on the parabola $y = x^2$ in such a way that $\dot{x} = 3$ throughout the journey. Find the formulas for (a) $\dot{y}$ and (b) $\ddot{y}$.

39.[R] Call one acute angle of a right triangle θ. The adjacent leg has length x and the opposite leg has length y.

40.[R] Call one acute angle of a right triangle θ. The adjacent leg has length x and the opposite leg has length y.

(a) Obtain an equation relating x, y and θ.

(b) Obtain an equation involving $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ (and other variables).

(c) Obtain an equation involving $\ddot{x}$, $\ddot{y}$, and $\ddot{\theta}$ (and other variables).

41.[R] A two-piece extension ladder leaning against a wall is collapsing at the rate of 2 feet per second and the base of the ladder is moving away from the wall at the rate of 3 feet per second. How fast is the top of the ladder moving down the wall when it is 8 feet from the ground and the foot is 6 feet from the wall? (See Figure 5.2.6.)

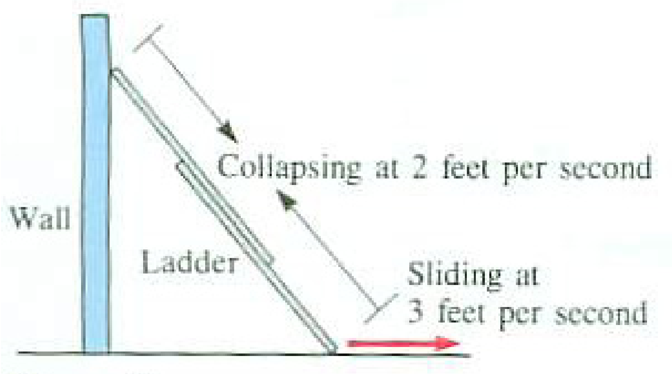


Figure 5.2.6:

42.[R] At an altitude of x kilometers, the atmospheric pressure decreases at a rate of $128(0.88)^x$ millibars per kilometer. A rocket is rising at the rate of 5 kilometers per second vertically. At what rate is the atmospheric pressure changing (in millibars per second) when the altitude of the rocket is (a) 1 kilometer? (b) 50 kilometers?

43.[R] A woman is walking on a bridge that is 20 feet above a river as a boat passes directly under the center of the bridge (at a right angle to the bridge) at 10 feet per second. At that moment the woman is 50 feet from the center and approaching it at the rate of 5 feet per second.

(a) At what rate is the distance between the boat and woman changing at that moment?

(b) Is the rate at which they are approaching or separating increasing or is it decreasing?

44.[R] A spherical raindrop evaporates at a rate proportional to its surface area. Show that the radius shrinks at a constant rate.

45.[R] A couple is on a Ferris wheel when the sun is directly overhead. The diameter of the wheel is 50 feet, and its speed is 0.01 revolution per second.

(a) What is the speed of their shadows on the ground when they are at a two-o'clock position?

(b) A one-o'clock position?

(c) Show that the shadow is moving its fastest when they are at the top or bottom, and its slowest when they are at the three-o'clock or nine-o'clock position.

46.[R] Does the tangent line to the curve $x^3 + xy^2 + x^3y^5 = 3$ at the point $(1, 1)$ pass through the point $(-2, 3)$? (Explain.)

Exercises 47 and 48 obtain by implicit differentiation the formulas for differentiating $x^{1/n}$ and $x^{m/n}$ with the assumption that they are differentiable functions. Here m and n are integers.

47.[M] Let n be a positive integer. Assume that $y = x^{1/n}$ is a differentiable function of x. From the equation $y^n = x$ deduce by implicit differentiation that $y' = (1/n)x^{1/n-1}$.

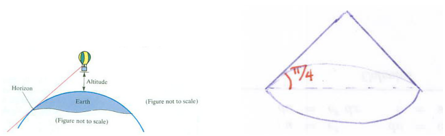
48.[M] Let m be a nonzero integer and n a positive interger. Assume that $y = x^{m/n}$ is a differentiable function of x. From the equation $y^n = x^m$ deduce by implicit differentiation that $y' = (m/n)x^{m/n-1}$.

49.[R] Water is flowing into a hemispherical bowl of radius 5 feet at the constant rate of 1 cubic foot per minute.

(a) At what rate is the top surface of the water rising when it height above the bottom of the bowl is 3 feet? 4 feet? 5 feet?

(b) If $h(t)$ is the depth in feet at time t, find $\ddot{h}$ when $h = 3, 4$, and 5.

50.[R] A man in a hot-air balloon is ascending at the rate of 10 feet per second. How fast is the distance from the balloon to the horizon (that is, the distance the man can see) increasing when the balloon is 1,000 feet high? Assume that the earth is a ball of radius 4,000 miles. (See Figure 5.2.7(a).)



(a) (b)

Figure 5.2.7:

51.[R] The Clean Waste company adds 100 cubic yards of debris to a landfill each day. The operator decides to keep piling it up in the form of a cone whose base angle is $\pi/4$. See Figure 5.2.7(b). (He plans either to turn it into a ski run or put an observation restaurant on top.) At what rate is the height of the cone increasing when it is

(a) 10 yards?

(b) 20 yards ?

(c) 100 yards?

(d) How long will it take to make a cone 30 yards high?

(e) How long to make one 300 yards high, which is the operator's goal?

52.[R] (*Contributed by Keith Sollers, when an undergraduate at the University of California at Davis.*) We quote from his note. "The numbers are ugly, but I think it's a good problem nevertheless. I didn't think it up myself. The Medical Center eye group gave me the problem and asked me to solve it. They were going to put a gas bubble in someone's eye."
The volume of a gas bubble changes from 0.4 cc to 1.6 cc in 74 hours. Assuming that the rate of change of the radius is constant, find,

(a) The rate at which the radius changes;

(b) The rate at which the volume of the bubble is increasing at any volume V;

(c) The rate at which the volume is increasing when the volume is 1 cc.

53.[R] Differentiate the function in Example 6 directly, without taking logarithms first.

In Exercises 54 to 59 differentiate the given function by logarithmic differentiation.

54.[R] $y = x^3 \sin^2(2x)$

55.[R] $y = \sqrt{\sin(2x)}\sqrt[3]{1+x^3}$

56.[R] $y = \frac{x^3\cos(2x)}{(1+x^2)^4}$

57.[R] $y = \frac{\tan^3(5x)}{\sqrt[3]{e^{x^2}}\arcsin(5x)}$

58.[R] $y = \frac{(x^3+2x)(\arctan(3x))}{1+e^{2x}}$

59.[R] $y = \frac{\left(\sqrt{\ln(2x)}\right)^3(\sin(3x))^5}{(x^3+x)^2}$

In Exercises 60 to 64 first simplify the formula for the function with the aid of properties of logarithms. Then, find dy/dx.

60.[M] $y = \ln\left(\frac{\left(\sqrt{1+x^2}\right)^3\left(e^{3x}+1\right)}{1+\sin(2x)}\right)$

61.[M] $y = \ln\left(\left(\sqrt{1+\sin(2x)}\right)^3\right)$

62.[M] $y = \ln\left(\frac{(x^3+2)^5}{(x^2+5)^2}\right)$

63.[M] $y = \ln\left((\sin(2x))^3\sqrt{\arctan(3\right.$

64.[M] $y = \ln\left(\frac{(\ln(x^2))^5(\arcsin(3x))^5}{(\tan(5x))^2}\right)$

65.[M] Find $D(x^k)$, $x>0$, by logarithmic differentiation of $y = x^k$.

66.[M] Let $y = x^x$.

(a) Find y' by logarithmic differentiation. That is, first take the logarithm of both sides.

(b) Find y' by first writing the base as $e^{\ln(x)}$. That is, write $y = x^x = \left(e^{\ln(x)}\right)^x = e^{x\ln(x)}$.

67.[M] Find the first and second derivatives of $y = \sec(x^2)\frac{\sin(x^2)}{x}$.

5.3 Higher Derivatives and the Growth of a Function

The only higher derivative we've used so far is the second derivative. In the study of motion, if y denotes position then y'' is acceleration. In the study of graphs, the second derivative determines whether the graph is concave up ($y'' > 0$) or down ($y'' < 0$). Later, in Section 9.6, the second derivative will appear in a formula that measures the curviness of a curve.

Now we will see how the higher derivatives (including the second derivative) influence the growth of a function. In the next section this will be applied to estimate the error in approximating a function by a polynomial.

Introduction

Imagine that you are in a car motionless at the origin of the x-axis. Then you put your foot to the gas pedal and accelerate. The greater the acceleration, the faster the speed increases; the greater the speed, the further you travel in a given time. So the acceleration, which is the second derivative of the position function, influences the function itself. This illustrates how a higher derivative of a function influences the growth of a function. In this section we examine this influence in more detail.

The following lemma is the basis for our analysis. In terms of daily life, it says, "The faster runner wins the race."

If $a > b$, then $f(x) \geq g(x)$. See Exercise 31.

Lemma 5.3.1. *Let $f(x)$ and $g(x)$ be differentiable functions on an interval I. Let a be a number in I where $f(a) = g(a)$. Assume that $f'(x) \leq g'(x)$ for x in I. Then $f(x) \leq g(x)$ for all x in I to the right of a and $f(x) \geq g(x)$ for all x in I to the left of a.*

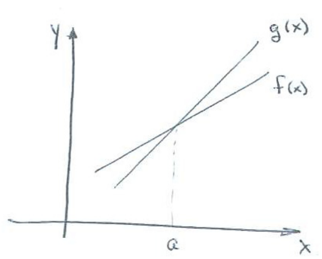


Figure 5.3.1:

Figure 5.3.1 makes this plausible, when the graphs of f and g are straight lines. To the right of $x = a$ the steeper line lies above the other line. To the left of $x = a$ the steeper line lies below the other line.

Proof of Lemma 5.3.1

Consider the case when $x > a$. Let $h(x) = f(x) - g(x)$. Then $h(a) = 0$ and $h'(x) = f'(x) - g'(x) \leq 0$. Thus, h is a non-increasing function. Since $h(a) = 0$, it follows that $h(x) \leq 0$ for $x \geq a$. That is, $f(x) - g(x) \leq 0$, hence $f(x) \leq g(x)$ for $x > a$. •

Repeated application of Lemma 5.3.1 will enable us to establish a connection between higher derivatives and the function itself.

Higher Derivatives and the Growth of a Function

In the following theorem we name the function $R(x)$ because that will be the notation in the next section when $R(x)$ is the "remainder" function. The notation $n!$ (read: "n factorial") for a positive integer n is shorthand for the product of all integers from 1 through n: $n! = n(n-1)\cdots 3\cdot 2\cdot 1$. The symbol 0! is usually defined to be 1.

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120.$

Theorem 5.3.2 (Growth Theorem)**.** *Assume that at a the function R and its first n derivatives are zero,*

$$R(a) = R'(a) = R''(a) = R^{(3)}(a) = \cdots = R^{(n)}(a) = 0.$$

Assume also that $R(x)$ has continuous derivatives up through the derivative of order $n+1$ in some open interval I containing the numbers a and x. Then there is a number c_n in the interval $[a, x]$ such that

$$R(x) = R^{(n+1)}(c_n)\frac{(x-a)^{n+1}}{(n+1)!}. \tag{5.3.1}$$

Before giving the straightforward proof, we illustrate the theorem by several examples.

EXAMPLE 1 Assume that $R(5) = R'(5) = R''(5) = 0$ and $\left|R^{(3)}(x))\right| \le 4$ for x in the interval $(3, 7)$. Show that $|R(x)| \le 2|x-5|^3/3$ for x in $(3, 7)$.
SOLUTION By the Growth Theorem, with $a = 5$ and $n = 2$,

$$R(x) = R^{(3)}(c_3)\frac{(x-5)^3}{3!} \qquad \text{for some number } c_3 \text{ between 5 and } x.$$

Though we do not know c_3, we do know that $|R^{(3)}(c_3)| \le 4$. So

$$|R(x)| = |R^{(3)}(c_3)|\frac{|x-5|^3}{3!} \le 4\frac{|x-5|^3}{3!} = \frac{2}{3}|x-5|^3.$$

◇

The Growth theorem with $n = 1$ and $a = 0$ describes the position of an accelerating car. One has $R(0) = 0$ (at time 0 the car is at position 0), $R'(0) = 0$ (at time 0 the car is not moving) and R'' describes the acceleration. If that acceleration is constant, equal to k, then (5.3.1) gives the car's position at time x as $R(x) = k\frac{x^2}{2!}$. If the acceleration is not constant, it says that $R(x)$ equals the acceleration at some time multiplied by $x^2/2$.

EXAMPLE 2 Show that $|e^x - 1 - x| \le \frac{e}{2}x^2$ for x in $(-1, 1)$.
SOLUTION Let $R(x) = e^x - 1 - x$. Then $R(0) = e^0 - 1 - 0 = 0$. And, since

$R'(x) = e^x - 1$, $R'(0) = e^0 - 1 = 0$ also. $R''(x) = e^x$. By the Growth Theorem, with $a = 0$ and $n = 1$, there is a number c_1 in $(-1, 1)$ such that

$$e^x - 1 - x = e^{c_1}\frac{(x-0)^2}{2!}.$$

We do not know c_1, but, since it is less than 1, $e^{c_1} < e$. Thus

$$|e^x - 1 - x| \leq e\frac{x^2}{2}. \qquad (5.3.2)$$

◇

The inequality (5.3.2) in the preceding example provides a way to estimate e^x when x is small. For instance, $|e^{0.1} - 1 - 0.1| \leq \frac{e}{2}(0.1)^2 = e/200$. The estimate 1.1 for $e^{0.1}$ is off by at most $e/200 \approx 0.013591$.

EXAMPLE 3 Let $R(x) = \cos(x) - 1 + \frac{x^2}{2}$. Show that $|R(x)| \leq \frac{|x^3|}{6}$.
SOLUTION As in Example 2 we use the Growth Theorem with $a = 0$ and $x > 0$. Since powers of $x = (x - 0)$ appear in $R(x)$, this suggests examining $R(x)$ at $a = 0$:

$$\begin{aligned} R(x) &= \cos(x) - 1 - \frac{x^2}{2}, \quad \text{so} \quad R(0) = 1 - 1 + 0 = 0; \\ R'(x) &= -\sin(x) + x, \quad \text{so} \quad R'(0) = 0 + 0 = 0; \\ R''(x) &= -\cos(x) + 1, \quad \text{so} \quad R''(0) = -1 + 1 = 0; \textit{and} \\ R^{(3)}(x) &= \sin(x). \end{aligned}$$

By the Growth theorem, with $a = 0$ and $n = 2$,

$$R(x) = \sin(c_2)\frac{x^3}{3!} \qquad \text{for some number } c_2 \text{ between 0 and } x.$$

Because $|\sin(x)| \leq 1$,

$$|R(x)| \leq \left|(1)\frac{x^3}{6}\right| = \frac{|x|^3}{6}.$$

◇

Example 3 provides a good estimate for values of the cosine function for small angles. For instance, if $x = 0.1$ radians, we have

0.1 radians $= 0.1\frac{180^\circ}{\pi}{}^\circ \approx 5.7^\circ$

$$\left|\cos(0.1) - 1 + \frac{0.1^2}{2}\right| \leq \frac{0.1^3}{6} = 0.00016667 = 1.6667 \times 10^{-4}.$$

Thus, $1 - \frac{0.1^2}{2} = 1 - 0.005 = 0.995$ is an estimate of $\cos(0.1) \approx 0.9950041653$ with an error less than $0.00016667 - \frac{1}{6} \times 10^{-3}$.

Remark: An even better bound on the growth of $R(x)$ in Example 3 is possible. In addition to $R(0) = R'(0) = R''(0) = 0$, notice that $R^{(3)}(0) = \sin(0) = 0$. This means that $|R(x)| \leq \left|M_4 \frac{(x-0)^4}{4!}\right|$ where M_4 is the maximum value of $R^{(4)}(t) = \cos(t)$ in the interval $[0, x]$. As in Example 3, $M \leq 1$. Thus,

$$|R(x)| \leq \left|(1)\frac{x^4}{4!}\right| = \frac{x^4}{24}.$$

This means the difference between the exact value of $\cos(0.1)$ and the estimate $1 - \frac{0.1^2}{2} = 0.995$ is no more than $\frac{0.1^4}{24} = 4.16667 \times 10^{-6}$. This shows the estimate in Example 3 is accurate to five decimal places.

In fact, $|\cos(0.1) - 0.995| \approx 4.16528 \times 10^{-6}$.

In any case, $1 - \frac{x^2}{2}$ is a good estimate of $\cos(x)$ for small values of x.

A Refinement of the Growth Theorem

When proving the Growth theorem we will establish something stronger:

Theorem 5.3.3. *Refined Growth Theorem If $m \leq R^{(n+1)}(t) \leq M$ and all earlier derivatives of R are 0 at a, then*

$$R(x) \text{ is between } m\frac{(x-a)^{n+1}}{(n+1)!} \text{ and } M\frac{(x-a)^{n+1}}{(n+1)!}. \tag{5.3.3}$$

This statement holds even if x is less than a and $(x-a)$ is negative.

EXAMPLE 4 Let $R(x) = e^x - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})$. Show that $\frac{1}{1152} \leq R(\frac{1}{2}) \leq \frac{1}{128}$. Use this estimate to obtain approximations, with error bounds, for $\sqrt{e} = e^{1/2}$ and e.

SOLUTION

$$\begin{aligned}
R(0) &= e^0 - 1 - 0. && \\
R'(x) &= e^x - (1 + x + \frac{x^2}{2!}), &\text{so}\quad & R'(0) = 0. \\
R''(x) &= e^x - (1 + x), &\text{so}\quad & R''(0) = 0. \\
R^{(3)}(x) &= e^x - 1, &\text{so}\quad & R^{(3)}(0) = 0. \\
R^{(4)}(x) &= e^x, &\text{and}\quad & R^{(4)}(0) = 1 \neq 0.
\end{aligned}$$

But, for x in $I = (-1, 1)$, $\frac{1}{3} \le e^{-1} \le e^x \le e^1 < 3$. Theorem 5.3.3, with $a = 0$, $n = 3$, $m = \frac{1}{3}$, $M = 3$, and $x = \frac{1}{2}$ gives

$$\begin{array}{lrcl} & \frac{1}{3}\frac{(1/2)^4}{4!} & \le R(1/2) \le & 3\frac{(1/2)^4}{4!} \\ \text{Then,} & \frac{1}{1152} & \le \sqrt{e} - \left(1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!}\right) \le & \frac{1}{128} \\ \text{or} & \frac{79}{48} + \frac{1}{1152} & \le \sqrt{e} \le & \frac{79}{48} + \frac{1}{128} \\ \text{so} & 1.64670 & \le \sqrt{e} \le & 1.65365. \end{array}$$

As you can check with your calculator, $\sqrt{e} \approx 1.64872$ to five decimal places. ◇

As Example 4 shows, the Growth Theorem provides not only upper bounds on the error in approximating a function by certain polynomials, but lower bounds on that error as well.

Proof of the Growth Theorem

Proof of the Growth Theorem

We illustrate the proof in the case $n = 2$. For convenience, we take the case $x > a$. The case with $x < a$ is complicated by the fact that $x - a$ is then negative and the sign of $(x - a)^n$ depends on whether n is odd or even.

Assume $R(a) = R'(a) = R''(a) = 0$ and $R^{(3)}(x)$ is continuous in the interval $[a, x]$. We want to show there is a number c_2 in $[a, x]$ such that

$$R(x) = R^{(3)}(c_2)\frac{(x-a)^3}{3!}.$$

Let M be the maximum of $R^{(3)}(t)$ and m be the minimum of $R^{(3)}(t)$ on the closed interval $[a, x]$. Thus

$$m \le R^{(3)}(t) \le M \qquad \text{for all } t \text{ in } [a, x].$$

We will see first what the inequality $R^{(3)}(t) \le M$ implies about $R(x)$.

We rewrite that inequality as

$$\frac{d}{dt}\left(R^{(2)}(t)\right) \le \frac{d}{dt}\left(M(t-a)\right). \tag{5.3.4}$$

Now apply Lemma 5.3.1 with $f(t) = R^{(2)}(t)$ and $g(t) = M(t - a)$. Note that $f(a) = 0$ and $g(a) = M(a - a) = 0$. (That is why we used the antiderivative $M(t - a)$ rather than the expected Mt.) Also $f''(a) = 0 = g''(a)$. By the lemma

$$R^{(2)}(t) \le M(t-a). \tag{5.3.5}$$

Next, rewrite (5.3.5) as

$$\frac{d}{dt}\left(R'(t)\right) \le \frac{d}{dt}\left(M\frac{(t-a)^2}{2}\right).$$

Applying the lemma again shows that

$$R'(t) \le M\frac{(t-a)^2}{2}. \tag{5.3.6}$$

Finally, rewrite (5.3.6) as

$$\frac{d}{dt}\left(R(t)\right) \le \frac{d}{dt}\left(M\frac{(t-a)^3}{3\cdot 2}\right).$$

The lemma asserts that

$$R(t) \le M\frac{(t-a)^3}{3!}. \tag{5.3.7}$$

Similar reasoning, starting with $m \le R^{(3)}(t)$ shows that

$$m\frac{(t-a)^3}{3!} \le R(t). \tag{5.3.8}$$

Combining (5.3.7) and (5.3.8) gives two bounds on $R(t)$; in particular on $R(x)$:

$$m\frac{(x-a)^3}{3!} \le R(x) \le M\frac{(x-a)^3}{3!}.$$

Because $R^{(3)}$ is continuous on $[a,x]$ it assumes all values between m and M. Thus there is a number c_2 in $[a,x]$ such that

$$R(x) = R^{(3)}(c_2)\frac{(x-a)^3}{3!}.$$

•

Summary

We showed that the bound on the size of the derivative of a function limits the growth of the function itself. When this observation is applied repeatedly we showed that if a function $R(x)$ and its first n derivatives are all zero at a, then

$$R(x) = R^{(n+1)}(c_n)\frac{(x-a)^{n+1}}{(n+1)!} \qquad \text{for some } c_n \text{ between } a \text{ and } x.$$

The number c_n depends on n, not just on a, x, and the function $R(x)$.

EXERCISES for Section 5.3 *Key:* R–routine, M–moderate, C–challenging

1.[R] If $f'(x) \geq 3$ for all $x \in (-\infty, \infty)$ and $f(0) = 0$, what can be said about $f(2)$? about $f(-2)$?

2.[R] If $f'(x) \geq 2$ for all $x \in (-\infty, \infty)$ and $f(1) = 0$, what can be said about $f(3)$? about $f(-3)$?

3.[R] What can be said about $f(2)$ if $f(1) = 0$, $f'(1) = 0$, and $2.5 \leq f''(x) \leq 2.6$ for all x?

4.[R] What can be said about $f(4)$ if $f(1) = 0$, $f'(1) = 0$, and $2.9 \leq f''(x) \leq 3.1$ for all x?

5.[R] A car starts from rest and travels for 4 hours. Its acceleration is always at least 5 miles per hour per hour, but never exceeds 12 miles per hour per hour. What can you say about the distance traveled during those 4 hours?

6.[R] A car starts from rest and travels for 6 hours. Its acceleration is always at least 4.1 miles per hour per hour, but never exceeds 15.5 miles per hour per hour. What can you say about the distance traveled during those 6 hours?

7.[R] State the Growth Theorem for $x \geq a$ in the case where R has at least five continuous derivatives and $R(a) = R'(a) = R''(a) = R^{(3)}(a) = R^{(4)}(a) = 0$.

8.[R] State the Growth Theorem in words, using as little math notation as possible.

9.[R] If $R(1) = R'(1) = R''(1) = 0$ and $R^{(3)}(x)$ is continuous on an interval that includes 1 and $R^{(3)}(x) \leq 2$, what can be said about $R(4)$?

10.[R] If $R(3) = R'(3) = R''(3) = R^{(3)}(3) = R^{(4)}(3) = 0$ and $R^{(5)}(x) \leq 6$, what can be said about $R(3.5)$?

11.[R] Let $R(x) = \sin(x) - \left(x - \frac{x^3}{6}\right)$. Show that

(a) $R(0) = R'(0) = R''(0) = R^{(3)}(0) = 0$.

(b) $R^{(4)}(x) = \sin(x)$.

(c) $|R(x)| \leq \frac{x^4}{24}$.

(d) Use $x - \frac{x^3}{6}$ to approximate $\sin(x)$ for $x = 1/2$.

(e) Use (c) to estimate the difference between the exact value for $\sin\left(\frac{1}{2}\right)$ and the approximation obtained in (d).

(f) Explain why $|R(x)| \leq \frac{|x|^5}{120}$. How can this be used to obtain a better estimate of the difference between the exact value for $\sin\left(\frac{1}{2}\right)$ and the approximation obtained in (d)?

(g) By how much does the estimate in (d) differ from $\sin\left(\frac{1}{2}\right)$?

Incidentally, an angle of $\frac{1}{2}$ radian is about 29°.

12.[R] Let $R(x) = \cos(x) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)$. Show that

(a) $R(0) = R'(0) = R''(0) = R^{(3)}(0) = R^{(4)}(0) = R^{(5)}(0) = 0$.

(b) $R^{(6)}(x) = -\cos(x)$.

(c) $|R(x)| \leq \frac{x^6}{6!}$.

(d) Use $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ to estimate $\cos(x)$ for $x = 1$.

(e) By how much does the estimate in (d) differ from $\cos(1)$?

Incidentally, an angle of 1 radian is about 57°.

13.[R] Let $R(x) = (1 + x)^5 - (1 + 5x + 10x^2)$. Show that

(a) $R(0) = R'(0) = R''(0) = 0$.

(b) $R^{(3)}(x) = 60(1 + x)^2$.

(c) $|R(x)| \leq 80x^3$ (on $[-1, 1]$)

(d) Use $1+5x+10x^2$ to estimate $(1+x)^5$ for $x = 0.2$.

(e) By how much does the estimate in (d) differ from $(1.2)^5$?

14.[M] If $f(3) = 0$ and $f'(x) \geq 2$ for all $x \in (-\infty, \infty)$, what can be said about $f(1)$? Explain.

15.[M] If $f(0) = 3$ and $f'(x) \geq -1$ for all $x \in (-\infty, \infty)$, what can be said about $f(2)$ and about $f(-2)$? Explain.

In Example 3 the polynomial $1 - \frac{x^2}{2}$ was shown to be a good approximation to $\cos(x)$ for x near 0. You may wonder how that polynomial was chosen. Exercise 16 shows how.

16.[M] Let $P(x) = a_0 + a_1x + a_2x^2$ be an arbitrary quadratic polynomial. For which values of a_0, a_1, and a_2 is:

(a) $\cos(0) - P(0) = 0$?

(b) $\cos'(0) - P'(0) = 0$?

(c) $\cos''(0) - P''(0) = 0$?

(d) Let $R(x) = \cos(x) - P(x)$. For which $P(x)$ is $R(0) = R'(0) = R''(0) = 0$?

17.[M] Find constants a_0, a_1, a_2, and a_3 such that if $R(x) = \tan(x) - \left(a_0 + a_1x + a_2x^2 + a_3x^3\right)$ then $R(0) = R'(0) = R''(0) = R^{(3)}(0) = 0$.

18.[M] Find constants a_0, a_1, a_2, and a_3 such that if $R(x) = \sqrt{1+x} - \left(a_0 + a_1x + a_2x^2 + a_3x^3\right)$ then $R(0) = R'(0) = R''(0) = R^{(3)}(0) = 0$.

19.[M] Find constants a_0, a_1, a_2, and a_3 such that if

$$R(x) = \sin x - \left(a_0 + a_1\left(x - \frac{\pi}{6}\right) + a_2\left(x - \frac{\pi}{6}\right)^2 + a_3\left(x - \frac{\pi}{6}\right)^3\right)$$

then $R\left(\frac{\pi}{6}\right) = R'\left(\frac{\pi}{6}\right) = R''\left(\frac{\pi}{6}\right) = R^{(3)}\left(\frac{\pi}{6}\right) = 0$.

Exercises 20 to 24 are related.

20.[M] Because $e > 1$, it is known that $e^x \geq 1$ for every $x \geq 0$.

(a) Use Lemma 5.3.1 to deduce that $e^x > 1 + x$, for $x > 0$.

(b) Use (a) and Lemma 5.3.1 to deduce that, for $x > 0$, $e^x > 1 + x + \frac{x^2}{2!}$.

(c) Use (b) and Lemma 5.3.1 to deduce that, for $x > 0$, $e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

(d) In view of (a), (b), and (c), what is the general inequality that can be proved by this approach?

21.[M] Let k be a fixed positive number. For x in $[0, k]$, $e^x \leq e^k$.

(a) Deduce that $e^x \leq 1 + e^kx$ for x in $[0, k]$.

(b) Deduce that $e^x \leq 1 + x + e^k\frac{x^2}{2!}$ for x in $[0, k]$.

(c) Deduce that $e^x \leq 1 + x + \frac{x^2}{2!} + e^k\frac{x^3}{3!}$ for x in $[0, k]$.

(d) In view of (a), (b), and (c), what is the general inequality that can be proved by this approach?

22.[M] Combine the results of Exercises 20 and 21 to estimate $e = e^1$ to two decimal places. NOTE: Assume $e \leq 3$.

23.[M] What properties of e^x did you use in Exercises 20 and 21?

24.[M] Let $E(x)$ be a function such that $E(0) = 1$ and $E'(x) = E(x)$ for all x.

(a) Show that $E(x) \geq 1$ for all $x \geq 0$.

(b) Use (a) to show that $E(x)$ is an increasing function for all $x \geq 0$. HINT: Show that $E'(x) \geq 1$, for all $x \geq 0$.

(c) Show $E(x) \geq 1 + x + \frac{x^2}{2}$ for all $x \geq 0$.

Exercises 25 to 30 show that $\lim_{x\to\infty} \frac{x}{e^x}$, $\lim_{x\to\infty} \frac{\ln(y)}{y}$, $\lim_{x\to 0^+} x\ln(x)$, $\lim_{x\to\infty} \frac{x^k}{b^x}$ $(b > 1)$, and $\lim_{x\to 0^+} x^x$ are closely connected. (If you know one of them you can deduce the other three.)

Exercises 25 to 26 use the fact that $e^x > 1 + x + \frac{x^2}{2}$ for all $x > 0$ (see Exercise 20).

25.[M] Evaluate $\lim_{x\to\infty} \frac{x}{e^x}$.

26.[M] Evaluate $\lim_{y\to\infty} \frac{\ln(y)}{y}$. HINT: Let $y = e^x$ and compare with Exercise 25.

Exercise 27 provides a proof of the fact that the exponential function grows faster than any power of x.

27.[M]

(a) Let n be a positive integer. Write $\frac{x^n}{e^x} = \left(\frac{x}{e^{x/n}}\right)\left(\frac{x}{e^{x/n}}\right)\cdots\left(\frac{x}{e^{x/n}}\right)$. Let $y = x/n$ so that $\frac{x}{e^{x/n}} = \frac{ny}{e^y}$. Use Exercise 25 ($n$ times) to show that $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$.

(b) Deduce that for any fixed number k, $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$.

28.[M] Evaluate $\lim_{x\to 0^+} x\ln(x)$ as follows: Let $x = 1/t$, where $t \to \infty$. Then $x\ln(x) = \frac{1}{t}\ln\left(\frac{1}{t}\right) = \frac{-\ln(t)}{t}$. and refer to Exercise 26.

29.[M] Evaluate $\lim_{x\to 0^+} x^x$ as follows: Let $y = x^x$. Then $\ln(y) = x\ln(x)$, a limit that was evaluated in Exercise 28. Explain why $\ln(y) \to 0$ implies $y \to 1$.

30.[M] Evaluate $\lim_{x\to\infty} \frac{x^k}{b^x}$ for any $b > 1$ and k is a positive integer, HINT: Use the result obtained in Exercise 27.

31.[M] Explain why $f(a) = g(a)$ and $f'(x) \leq g'(x)$ on $[a, b]$ with $a > b$ implies $f(x) \geq g(x)$ for all x in $[a, b]$.

32.[M] In Example 2 it is shown that $|e^x - 1 - x| \leq \frac{e}{2}x^2$ for all x in $(-1, 1)$. Find a bound for

(a) $R(x) = e^x - 1 - x - \frac{x^2}{2}$ on $(-1, 1)$.

(b) $R(x) = e^x - 1 - x$ on $(-2, 1)$.

(c) $R(x) = e^x - 1 - x$ on $(-1, 2)$.

(d) $R(x) = e^x - 1 - x - \frac{x^2}{2}$ on $(-2, 1)$.

(e) $R(x) = e^x - 1 - x - \frac{x^2}{2}$ on $(-1, 2)$.

33.[C] Apply Lemma 5.3.1 for $x > a$ to the case when $R(a) = R''(a) = 0$, $R^{(3)}(t) \leq M$, (for all t in $[a, x]$) but $R'(a) = 5$.

34.[C] Consider the following pro
usual, I can do things more sim
For instance, say $R(a) = R'(a)$
$R^{(3)}(x) \leq M$. I'll show how M affe
for $x > a$.
By the Mean-Value Theorem, $R(x$
$R'(c_1)(x - a)$ for some c_1 in $[a, x]$.
MVT again, this time finding $R'(c_1$
$R''(c_2)(c_1 - a)$ for some c_2 in $[a, c_1]$
tion of this idea then gives $R''(c_2)$
$R^{(3)}(c_3)(c_3 - a)$.
Then I put these all together, gett

$$R(x) \leq M(x - a)(c_2 - a$$

Since c_1, c_2, and c_3 are in $[a, x]$, I ca

$$R(x) \leq M(x - a$$

I didn't need that lemma about tw
Is Sam correct? Is this a valid subs
treatment? Explain.

35.[C] The proof of the Growth
less than a is slightly different than
greater than a. Prove it for the ca
in this case $(x - a)^3$ and $(x - a)$ ar

5.4 Taylor Polynomials and Their Errors

We spend years learning how to add, subtract, multiply, and divide. These same operations are built into any calculator or computer. Both we and machines can evaluate a polynomial, such as

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

when x and the coefficients a_0, a_1, a_2, ..., a_n are given. Only multiplication and addition are needed. But how do we evaluate e^x? We resort to our calculators or look in a table that lists values of e^x. If e^x were a polynomial in disguise, then it would be easy to evaluate it by finding the polynomial and evaluating it instead. But e^x cannot be a polynomial, as the reasons in the margin show.

Three different reasons:
1. Because e^x equals its own derivative and no polynomial equals its own derivative (other than the polynomial that has constant value 0).
2. When you differentiate a non-constant polynomial, you get a polynomial with a lower degree.
3. Also, $e^x \to 0$ as $x \to -\infty$ and no non-constant polynomial has this property.

Since we cannot write e^x as a polynomial, we settle for the next best thing. Let's look for a polynomial that closely *approximates* e^x. However, no polynomial can be a good approximation of e^x for *all* x, since e^x grows too fast as $x \to \infty$. We search, instead, for a polynomial that is close to e^x for x in some short interval.

In this section we develop a method to construct polynomial approximations to functions. The accuracy of these approximations can be determined using the Growth Theorem from the previous section. Higher derivatives play a pivotal role.

Fitting a Polynomial, Near 0

Suppose we want to find a polynomial that closely approximates a function $y = f(x)$ for x near the input 0. For instance, what polynomial $p(x)$ of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ might produce a good fit?

First we insist that

$$p(0) = f(0) \tag{5.4.1}$$

so the approximation is exact when $x = 0$.

Second, we would like the slope of the graph of $p(x)$ to be the same as that of $f(x)$ when x is 0. Therefore, we require

$$p'(0) = f'(0). \tag{5.4.2}$$

There are many polynomials that satisfy these two conditions. To find the best choices for the four numbers a_0, a_1, a_2, and a_3 we need four equations. To get them we continue the pattern started by (5.4.1) and (5.4.2). So we also insist that

$$p''(0) = f''(0) \tag{5.4.3}$$

and

$$p^{(3)})(0) = f^{(3)}(0). \qquad (5.4.4)$$

Equation (5.4.3) forces the polynomial $p(x)$ to have the same sense of concavity as the function $f(x)$ at $x = 0$. We expect the graphs of $f(x)$ and such a polynomial $p(x)$ to resemble each other for x close to a.

To find the unknowns a_0, a_1, a_2, and a_3 we first compute $p(x)$, $p'(x)$, $p''(x)$, and $p^{(3)}(x)$ at 0. Table 5.4.1 displays the computations that express the unknowns, a_0, a_1, a_2, and a_3, in terms of $f(x)$ and its derivatives. For example, note how we compute $p''(x) = 2a_2 + 3 \cdot 2a_3x$ and evaluate it at 0 to obtain $p''(0) = 2a_2 + 3 \cdot 2a_3 \cdot 0 = 2a_2$. Then we obtain an equation for a_2 by equating $p''(0)$ and $f''(0)$; that is, $2a_2 = f''(0)$, so $a_2 = \frac{1}{2}f''(0)$.

$p(x)$ and its derivatives	Their values at 0	Equation for a_k	Forn
$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$	$p(0) = a_0$	$a_0 = f(0)$	$a_0 =$
$p^{(1)}(x) = a_1 + 2a_2x + 3a_3x^2$	$p^{(1)}(0) = a_1$	$a_1 = f^{(1)}(0)$	$a_1 =$
$p^{(2)}(x) = 2a_2 + 3 \cdot 2a_3x$	$p^{(2)}(0) = 2a_2$	$2a_2 = f^{(2)}(0)$	$a_2 =$
$p^{(3)}(x) = 3 \cdot 2a_3$	$p^{(3)}(0) = 3 \cdot 2a_3$	$3 \cdot 2a_3 = f^{(3)}(0)$	$a_3 =$

Table 5.4.1:

Factorials appear in the denominators.

We can write a general formula for a_k if we let $f^{(0)}(x)$ denote $f(x)$ and recall that $0! = 1$ (by definition), $1! = 1$, $2! = 2 \cdot 1 = 2$, and $3! = 3 \cdot 2$. According to Table 5.4.1,

$$a_k = \frac{f^{(k)}(0)}{k!}, \qquad k = 0, 1, 2, 3.$$

Therefore

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)})(0)}{3!}x^3.$$

The coefficient of x^k is *completely determined* by the k^{th} derivative of f evaluated at 0. It equals the k^{th} derivative of f at 0 divided by $k!$.

The n^{th}-order Taylor polynomial has degree at most n.

DEFINITION *(Taylor Polynomials at 0)* Let n be a non-negative integer and let f be a function with derivatives at 0 of all orders through n. Then the polynomial

$$f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \qquad (5.4.5)$$

is called the **n^{th}-order Taylor polynomial of f centered at 0** and is denoted $P_n(x; 0)$. It is also called a **Maclaurin polynomial**.

Whether $P_n(x;0)$ approximates $f(x)$ for x near 0 is not obvious. We will show that the Macaurin polynomials for e^x do provide good approximations of the functions when x is not too large.

EXAMPLE 1 Find the Maclaurin polynomial $P_4(x;0)$ that agrees with $1/(1-x)$ and its first four derivatives at 0.

SOLUTION The first step is to compute $1/(1-x)$ and its first four derivatives, then evaluate them at $x = 0$. Dividing them by suitable factorials gives the coefficients of the Maclaurin polynomial. Table 5.4.2 records the computations.

		at x	at 0
$f(x)$	$=$	$\frac{1}{1-x}$	1
$f'(x)$	$=$	$\frac{1}{(1-x)^2}$	1
$f''(x)$	$=$	$\frac{2}{(1-x)^3}$	2
$f^{(3)}(x)$	$=$	$\frac{3\cdot 2}{(1-x)^4}$	$3 \cdot 2$
$f^{(4)}(x)$	$=$	$\frac{4\cdot 3\cdot 2}{(1-x)^5}$	$4 \cdot 3 \cdot 2$

Table 5.4.2:

So the fourth-degree Maclaurin polynomial is

$$P_4(x;0) = 1 + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{3\cdot 2}{3!}x^3 + \frac{4\cdot 3\cdot 2}{4!}x^4,$$

which simplifies to

$$P_4(x;0) = 1 + x + x^2 + x^3 + x^4.$$

Figure 5.4.1 suggests that $P_4(x;0)$ does a fairly good job of approximating $1/(1-x)$ for x near 0. ◇

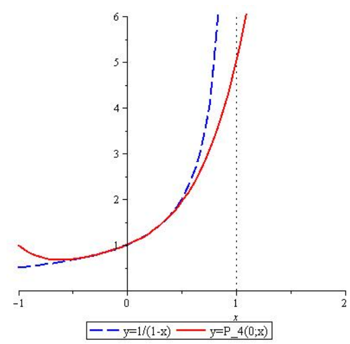


Figure 5.4.1:

The calculations in Example 1 suggest that

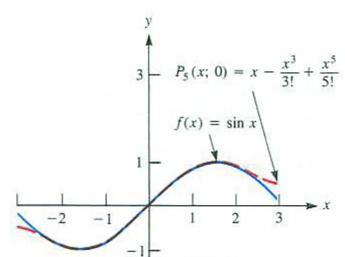
> The Maclaurin polynomial $P_n(x;0)$ associated with $1/(1-x)$ is
>
> $$1 + x + x^2 + x^3 + \cdots + x^n.$$

Because all the derivatives of e^x at 0 are 1,

> The Maclaurin polynomial $P_n(x;0)$ associated with e^x is
>
> $$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

EXAMPLE 2 Find the Maclaurin polynomial of degree 5 for $f(x) = \sin(x)$.

SOLUTION Again we make a table for computing the coefficients of the Taylor polynomial centered at 0. (See Table 5.4.3.)



at x			at 0		
$f^{(0)}(x)$	$=$	$\sin(x)$	$f^{(0)}(0)$	$=$	$\sin(0) = 0$
$f^{(1)}(x)$	$=$	$\cos(x)$	$f^{(1)}(0)$	$=$	$\cos(0) = 1$
$f^{(2)}(x)$	$=$	$-\sin(x)$	$f^{(2)}(0)$	$=$	$-\sin(0) = 0$
$f^{(3)}(x)$	$=$	$-\cos(x)$	$f^{(3)}(0)$	$=$	$-\cos(0) = -1$
$f^{(4)}(x)$	$=$	$\sin(x)$	$f^{(4)}(0)$	$=$	$\sin(0) = 0$
$f^{(5)}(x)$	$=$	$\sin(x)$	$f^{(5)}(0)$	$=$	$\sin(0) = 0$

Table 5.4.3:

Thus

$$\begin{aligned} P_4(x;0) &= f^{(0)}(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \frac{f^{(5)}(0)}{5!}x^5 \\ &= 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!}. \end{aligned}$$

Figure 5.4.2 illustrates the graphs of $P_5(x;1)$ and $\sin(x)$ near 0. ◇

Having found the fifth-order Maclaruin polynomial for $\sin(x)$, let us see how good an approximation it is of $\sin(x)$. Table 5.4.4 compares their values to six-decimal-place accuracy for inputs both near 0 and far from 0. As we see, the closer x is to 0, the better the Taylor approximation is. When x is large, $P_5(x;0)$ gets very large, but the value of $\sin(x)$ stays between -1 and 1.

x	$\sin(x)$	$P_5(x;0)$
0.0	0.000000	0.000000
0.1	0.099833	0.099833
0.5	0.479426	0.479427
1.0	0.841471	0.841667
2.0	0.909297	0.933333
π	0.000000	0.524044
2π	0.000000	46.546732

Table 5.4.4:

A Shorthand Notation

The Maclaurin polynomials associated with $\sin(x)$ have only odd powers and its terms alternate in sign. For m odd,

$$P_m(x;0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \pm \frac{x^m}{m!}.$$

The $\pm$ in front of $x^m/m!$ indicates the coefficient is either positive or negative. For the terms involving x, x^5, x^9, ..., the coefficient is $+1$. For x^3, x^7, x^{11}, ...it is -1. Because m is odd, it can be written as $2n+1$. If n is even, the coefficient of x^{2n+1} is $+1$. If n is odd, the coefficient of x^{2n+1} is -1. The shorthand notation to write the typical summand is

$$(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

So we may write

$$P_{2n+1}(x;0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1_!}.$$

Taylor Polynomials Centered at a

We may be interested in estimating a function $f(x)$ near a number a, not just near 0. In that case, we express the approximating polynomial in terms of powers of $x-a$ instead of powers of $x = x - 0$ and make the derivatives of the approximating polynomial, evaluated at a, coincide with the derivatives of the function at a. Calculuations similar to those that gave us the polynomial (5.4.5) produce the polynomial called a "Taylor polynomial centered at a". (If a is not 0, it is not called a Maclaurin polynomial.)

DEFINITION *(Taylor Polynomials of degree n, $P_n(x;a)$)* If the function f has derivatives through order n at a, then the **n^{th}-order Taylor polynomial of f centered at** a is defined as

$$f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and is denoted $P_n(x;a)$.

The n^{th}-order Taylor polynomial of f centered at a is denoted $P_n(x;a)$. It's degree is at most n.

EXAMPLE 3 Find the n^{th}-order Taylor polynomial centered at a for $f(x) = e^x$.

SOLUTION All the derivatives of e^x evaluated at a are e^a. Thus

$$P_n(x;a) = e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + \frac{e^a}{3!}(x-a)^3 + \cdots + \frac{e^a}{n!}(x-a)^n.$$

◇

The Error in Using A Taylor Polynomial

There is no point using $P_n(x;a)$ to estimate a function $f(x)$ if we have no idea how large the difference between $f(x)$ and $P_n(x;a)$ may be. So let us take a look at the difference.

Define the **remainder** to be the difference between the function, $f(x)$, and the Taylor polynomial, $P_n(x;a)$. Denote the remainder as $R_n(x;a)$. Then,

$$f(x) = P_n(x;a) + R_n(x;a).$$

We will be interested in the absolute value of the remainder. We call $|R_n(x;a)|$ the **error** in using $P_n(x;a)$ to approximate $f(x)$. We do not care whether $P_n(x;a)$ is larger or smaller than the exact value.

Theorem 5.4.1 (The Lagrange Form of the Remainder)**.** *Assume that a function $f(x)$ has continuous derivatives of orders through $n+1$ in an interval that includes the numbers a and x. Let $P_n(x;a)$ be the n^{th}-order Taylor polynomial associated with $f(x)$ in powers of $x-a$. Then there is a number c_n between a and x such that*

$$R_n(x;a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}.$$

Proof of Theorem 5.4.1

For simplicity, we denote the remainder $R_n(x;a) = f(x) - P_n(x;a)$ by $R(x)$. Since $P_n(a;a) = f(a)$,

$$R(a) = f(a) - P_n(a;a) = f(a) - f(a) = 0.$$

Similarly, repeated differentiation of $R(x)$, leads to

$$R^{(k)}(x) = f^{(k)}(x) - P_n^{(k)}(x;a), \tag{5.4.6}$$

$P_n^{(k)}(a;a) = f^{(k)}(a)$, $k = 0, 1, \ldots, n$.

for each integer k, $1 \le k \le n$. From the definition of $P_n(x;a)$,

$$R^{(k)}(a) = f^{(k)}(a) - P_n^{(k)}(a;a) = 0.$$

$R^{(n+1)}(x) = f^{(n+1)}(x)$

Since $P_n(x;a)$ is a polynomial of degree at most n, its $(n+1)^{\text{st}}$ derivative is 0. As a result, the $(n+1)^{\text{st}}$ derivative of $R(x)$ is the same as the $(n+1)^{\text{st}}$ derivative of $f(x)$. Thus, $R(x)$ satisfies all the assumptions of the Growth Theorem. Recalling (5.3.1) from Section 5.3, we see

See Theorem 5.3.2 in Section 5.3.

Lagrange Form of the Remainder
There is a number c_n between a and x such that

$$R_n(x;a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}.$$

•

EXAMPLE 4 Discuss the error in using $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ to estimate $\sin(x)$ for $x > 0$.

SOLUTION Example 2 showed that $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is the Maclaurin polynomial, $P_5(x;0)$, associated with $\sin(x)$. In this case $f(x) = \sin(x)$ and each derivative of $f(x)$ is either $\pm\sin(x)$ or $\pm\cos(x)$. Therefore, $|f^{n+1}(c_n)|$ is at most 1, and we have

$$\frac{|f^{5+1}(c_5)|}{6!}x^6 \leq \frac{x^6}{6!}.$$

Then

$$\left|\sin(x) - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right| \leq \frac{|x|^6|}{6!} = \frac{x^6}{720}.$$

For instance, with $x = 1/2$,

$$\left|\sin\left(\frac{1}{2}\right) - \left(\left(\frac{1}{2}\right) - \frac{\left(\frac{1}{2}\right)^3}{6} + \frac{\left(\frac{1}{2}\right)^5}{120}\right)\right| \leq \frac{\left(\frac{1}{2}\right)^6}{720} = \frac{1}{(64)(720)} = \frac{1}{46,080} \approx 0.0000217 = 2.17\times 10^{-5}$$

So the approximation

$$P_5(\frac{1}{2};0) = \frac{1}{2} - \frac{1}{3!}\left(\frac{1}{2}\right)^3 + \frac{1}{5!}\left(\frac{1}{2}\right)^5 = \frac{1}{2} - \frac{1}{48} + \frac{1}{3840} = \frac{1841}{3840} \approx 0.4794271$$

differs from $\sin(1/2)$ (the sine of half a radian) by less than 2.17×10^{-5}; this means at least the first four decimal places are correct. The exact value of $\sin(1/2)$, to ten decimal places is 0.4794255386 and our estimate is correct to five decimal places. By comparison, a calculator gives $\sin(1/2) \approx 0.479426$, which is also correct to five decimal places. ◇

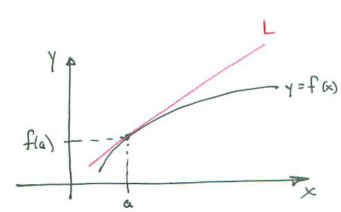


Figure 5.4.3: (Insert label for point $(a, f(a))$.)

The Linear Approximation $P_1(x;a)$

The graph of the Taylor polynomial $P_1(x;a) = f(a) + f'(a)(x-a)$ is a line that passes through the point $(a, f(a))$ and has the same slope as f does at a. That means that the graph of $P_1(x;a)$ is the *tangent line* to the graph of f

at $(a, f(a))$. It is customary to call $P_1(x; a) = f(a) + f'(a)(x - a)$ the **linear approximation** to $f(x)$ for x near a. It is often denoted $L(x)$. Figure 5.4.3 shows the graphs of f and L near the point $(a, f(a))$.

Let x be a number close to a and define $\Delta x = x - a$ and $\Delta y = f(a+\Delta x) - f(a)$, quantities used in the definition of the derivative: $f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Often Δx is denoted by dx and $f'(a)dx$ is defined to be "dy", as shown in Figure 5.4.4. Note that dy is an approximation to Δy, and $f(a) + dy$ is an approximation to $f(a + \Delta x) = f(a) + \Delta y$.

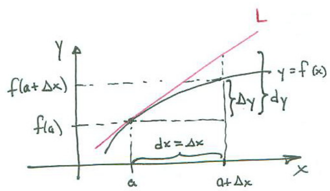


Figure 5.4.4:

In Section 8.2 we will use $dy = f'(x)dx$ and dx as bookkeeping tools to simplify the search for antiderivatives.

The expressions "dx" and "dy" are called **differentials**. In the seventeenth century, dx and dy referred to "infinitesimals", infinitely small numbers. Leibniz viewed the derivative as the quotient $\frac{dy}{dx}$, and that notation for the derivative persists more than three centuries later.

> **WARNING** *(The derivative is not a quotient.)* The derivative is the limit of a quotient.

The next example uses the linear approximation to estimate $\sqrt{x}$ near $x = 1$.

EXAMPLE 5 Use $P_1(x; 1)$ to estimate $\sqrt{x}$ for x near 1. Then discuss the error.

SOLUTION In this case $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, and $f'(1) = 1/2$. The linear approximation of $f(x)$ near $a = 1$ is

$$P_1(x; 1) = f(1) + f'(1)(x - 1) = 1 + \frac{1}{2}(x - 1)$$

and the remainder is

$$R_1(x; 1) = \sqrt{x} - \left(1 + \frac{1}{2}(x - 1)\right).$$

Table 5.4.5 shows how rapidly $R_1(x; 1)$ approaches 0 as $x \to 1$ and compares

x	$R_1(x;1)$					$(x-1)^2$	$R_1(x;1)/(x-$
2.0	$\sqrt{2}$	$-$	$\left(1+\frac{1}{2}(2-1)\right)$	$\approx$	-0.08578643	1	-0.08579
1.5	$\sqrt{1.5}$	$-$	$\left(1+\frac{1}{2}(1.5-1)\right)$	$\approx$	-0.02525512	0.25	-0.10102
1.1	$\sqrt{1.1}$	$-$	$\left(1+\frac{1}{2}(1.1-1)\right)$	$\approx$	-0.00119115	0.01	-0.11912
1.01	$\sqrt{1.01}$	$-$	$\left(1+\frac{1}{2}(1.01-1)\right)$	$\approx$	-0.00001243	0.0001	-0.12438

Table 5.4.5:

this difference with $(x-1)^2$.

The final column in Table 5.4.5 shows that $\frac{R_1(x;1)}{(x-1)^2}$ is nearly constant. Because $(x-1)^2 \to 0$ as $x \to 0$, this means $R_1(x;1)$ approaches 0 at the same rate as the square of $(x-1)$.

Since the Lagrange form for $R_1(x;1)$ is approximately $\frac{1}{2}f''(1)(x-1)^2$ when x is near 1, $\frac{R_1(x;1)}{(x-1)^2}$ should be near $\frac{1}{2}f''(1)$ when x is near 1. Just as a check, compute $\frac{1}{2}f''(1)$. We have $f''(x) = \frac{-1}{4}x^{-3/2}$. Thus $\frac{1}{2}f''(1) = \frac{1}{2}\left(\frac{-1}{4}\right) = \frac{-1}{8} = -0.125$. This is consistent with the final column of Table 5.4.5. ◇

Summary

Given a function f with n derivatives on an interval that contains the number a we defined the n^{th}-order Taylor polynomial at a, $P_n(x;a)$. The first n derivatives of the Taylor polynomial of degree n coincide with the first n derivatives of the given function f at a. Also, $P_n(x;a)$ has the same function value at a that f does.

We define the "zeroth derivative" of a function to be the function itself *and* start counting from 0. This allows us to say simply that the derivatives $P_n^{(k)}(x;a)$ coincide with $f^{(k)}(a)$ for $k = 0, 1, \ldots, n$.

$$P_n(x;a) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $a = 0$, $P_n(x;0)$ is call a Maclaurin polynomial. The general Maclaurin polynomial associated with

$$\begin{array}{rcl} e^x & \text{is} & 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \\ \sin(x) & \text{is} & x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(n+1)!} \\ \cos(x) & \text{is} & 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \\ 1/(1-x) & \text{is} & 1 + x + x^2 + x^3 + \cdots + x^n \end{array}$$

The remainder in using the Taylor polynomial of degree n to estimate a function involves the $(n+1)^{\text{st}}$ derivative of the function:

$$R_n(x;a) = f(x) - P_n(x;a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}$$

where c_n is a number between a and x . The error is the absolute value of the remainder, $|R_n(x;a)|$.

The linear approximation to a function near a is

$$L(x) = P_1(x;a) = f(a) + f'(a)(x-a).$$

The differentials are $dx = x - a$ and $dy = f'(a)dx$. While $dx = \Delta x$, $dy \approx \Delta y = f(x+\Delta x) - f(x)$.

EXERCISES for Section 5.4 *Key:* R–routine, M–moderate, C–challenging

Use a graphing calculator or computer algebra computer algebra system to assist with the computations and with the graphing.

1.[R] Give at least three reasons $\sin(x)$ cannot be a polynomial.

In Exercises 2 to 13 compute the Taylor polynomials. Graph $f(x)$ and $P_n(x;a)$ on the same axes on a domain centered at a. Keep in mind that the graph of $P_1(x;a)$ is the tangent line at the point $(a, f(a))$.

2.[R] $f(x) = 1/(1+x)$, $P_1(x;0)$ and $P_2(x;0)$

3.[R] $f(x) = 1/(1+x)$, $P_1(x;1)$ and $P_2(x;1)$

4.[R] $f(x) = \ln(1+x)$, $P_1(x;0)$, $P_2(x;0)$ and $P_3(x;0)$

5.[R] $f(x) = \ln(1+x)$, $P_1(x;1)$, $P_2(x;1)$ and $P_3(x;1)$

6.[R] $f(x) = e^x$, $P_1(x;0)$, $P_2(x;0)$, $P_3(x;0)$, and $P_4(x;0)$

7.[R] $f(x) = e^x$, $P_1(x;2)$, $P_2(x;2)$, $P_3(x;2)$, and $P_4(x;2)$

8.[R] $f(x) = \arctan(x)$, $P_1(x;0)$, $P_2(x;0)$, and $P_3(x;0)$

9.[R] $f(x) = \arctan(x)$, $P_1(x;-1)$, $P_2(x;-1)$, and $P_3(x;-1)$

10.[R] $f(x) = \cos(x)$, $P_2(x;0)$ and $P_4(x;0)$

11.[R] $f(x) = \sin(x)$, $P_7(x;0)$

12.[R] $f(x) = \cos(x)$, $P_6(x;\pi/4)$

13.[R] $f(x) = \sin(x)$, $P_7(x;\pi/4)$

14.[R] Can there be a polynomial $p(x)$ such that $\sin(x) = p(x)$ for all x in the interval $[1, 1.0001]$? Explain.

15.[R] Can there be a polynomial $p(x)$ such that $\ln(x) = p(x)$ for all x in the interval $[1, 1.0001]$? Explain.

16.[R] State the Lagrange formula for the error in using a Taylor polynomial as an estimate of the value of a function. Use as little mathematical notation as you can.

In Exercises 17 to 22 obtain the Maclaurin polynomial of order n associated with the given function.

17.[R] $1/(1-x)$

18.[R] e^x

19.[R] e^{-x}

20.[R] $\sin(x)$

21.[R] $\cos(x)$

22.[R] $1/(1+x)$

23.[R] Let $f(x) = \sqrt{x}$.

(a) What is the linear approximation, $P_1(x;4)$, to $\sqrt{x}$ at $x = 4$?

(b) Fill in the following table.

x	$R_1(x;4) = f(x) - P_1(x;4)$	$(x-4)^2$	$\frac{R_1(x;4)}{(x-4)^2}$
5.0			
4.1			
4.01			
3.99			

(c) Compute $f''(4)/2$. Explain the relationship between this number and the entries in the fourth column of the table in (b).

24.[R] Repeat Exercise 23 for the linear approximation to $\sqrt{x}$ at $a = 3$. Use $x = 4$, 3.1, 3.01, and 2.99.

25.[R] Assume $f(x)$ has continuous first and second derivatives and that $4 \le f''(x) \le 5$ for all x.

(a) What can be said in general about the error in using $f(2) + f'(2)(x-2)$ to approximate $f(x)$?

(b) How small should $x - 2$ be to be sure that the error — the absolute value of the remainder — is less than or equal to 0.005? NOTE: This ensures the approximate value is correct to 2 decimal places.

26.[R] Let $f(x) = 2 + 3x + 4x^2$.

(a) Find $P_2(x;0)$.

(b) Find $P_3(x;0)$.

(c) Find $P_2(x;5)$.

(d) Find $P_3(x;5)$.

27.[R]

(a) What can be said about the degree of the polynomial $P_n(x;0)$?

(b) When is the degree of $P_n(x;0)$ less than n?

(c) When is the degree of $P_n(x;a)$ less than n? $(a \neq 0)$

28.[M] In the case of $f(x) = 1/(1-x)$ the error $R_n(x;0)$ in using a Maclaurin polynomial $P_n(x;0)$ to estimate the function can be calculated exactly. Show that it equals $\left|x^{n+1}/(1-x)\right|$.

Exercises 29 to 32 are related.

29.[R] Let $f(x) = (1+x)^3$.

(a) Find $P_3(x;0)$ and $R_3(x;0)$.

(b) Check that your answer to (a) is correct by multiplying out $(1+x)^3$.

30.[R] Let $f(x) = (1+x)^4$.

(a) Find $P_4(x;0)$ and $R_4(x;0)$.

(b) Check that your answer to (a) is correct by multiplying out $(1+x)^4$.

31.[R] Let $f(x) = (1+x)^5$. Using $P_5(x;0)$, show that

$$(1+x)^5 = 1+5x+\frac{5\cdot 4}{1\cdot 2}x^2+\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}x^3+\frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}x^4+\frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}$$

For a positive integer n and a non-negative integer k, with $k \leq n$, the symbol $\binom{n}{k}$ denotes the **binomial coefficient**:

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-}{1\cdot 2\cdot 3\cdots k}$$

Thus

$$(1+x)^5 = \binom{5}{0}+\binom{5}{1}x+\binom{5}{2}$$

Using $P_n(x;0)$ one can show that, for any positive integer n,

$$(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}$$

This is the basis for the **Binomial Theorem**,

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}.$$

NOTE: Recall that $\binom{n}{0} = \frac{n!}{0!n!} = 1$ and $\binom{n}{n} = \frac{n!}{n!0!} = 1$.

32.[M]

(a) Using algebra (no calculus) derive the binomial theorem for $(a+b)^3$ from the binomial theorem for $(1+x)^3$.

(b) Obtain the binomial theorem for $(a+b)^{12}$ from the special case $(1+x)^{12} = \sum_{k=0}^{12}\binom{12}{k}x^k$.

In Exercises 33 and 34, use a calculator or computer to help evaluate the Taylor polynomials

33.[M] Let $f(x) = e^x$.

(a) Find $P_{10}(x; 0)$.

(b) Compute $f(x)$ and $P_{10}(x; 0)$ at $x = 1$, $x = 2$, and $x = 4$.

(a) Find $P_{10}(x; 1)$.

(b) Compute $f(x)$ and $P_{10}(x; 1)$ at $x = 1$, $x = 2$, and $x = 4$.

34.[M] Let $f(x) = \ln(x)$.

Exercises 35 to 38 involve even and odd functions Recall, from Section 2.6, that a function is even if $f(-x) = f(x)$ and is odd if $f(-x) = -f(x)$.

35.[M] Show that if f is an odd function, f' is an even function.

36.[M] Show that if f is an even function, f' is an odd function.

37.[M]

(a) Which polynomials are even functions?

(b) If f is an even function, are its associated Maclaurin polynomials necessarily even functions? Explain.

38.[M]

(a) Which polynomials are odd functions?

(b) If f is an odd function, are its associated Maclaurin polynomials necessarily odd functions? Explain.

39.[C] This exercise constructs Maclaurin polynomials that do not approximate the associated function. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$.

(a) Find $f'(0)$.

(b) Find $f''(0)$.

(c) Find $P_2(x; 0)$.

(d) What is $P_{100}(x; 0)$.

HINT: Recall the definition of the derivative.

40.[C] Show that in an open interval in which f''' is positive, that $f(x) > f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$. HINT: Treat the cases $a < x$ and $x > a$ separately. NOTE: See also Exercise 17 in Section 4.4.

41.[C]

(a) Show that in an open interval in which $f^{(n+1)}$ is positive (n a positive integer), that $f(x)$ is greater than $P_n(x;0)$.

(b) What additional information is needed to make this a true statement for $x < a$?

NOTE: See also Exercise 40.

42.[C] The quantity $\sqrt{1 - v^2/c^2}$ occurs often in the theory of relativity. Here v is the velocity of an object and c the velocity of light. Justify the following approximations that physicists use:

(a) $\sqrt{1 - \frac{v^2}{c^2}} \approx 1 - \frac{1}{2}\frac{v^2}{c^2}$

(b) $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$g

NOTE: Even for a rocket v/c is very small.

43.[C] Using the formula for the
ometric series, justify the factoriz
tion 2.2. (See Exercise 41, Section

$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x$

44.[C] If $P_n(x;0)$ is the Maclaurin
ated with $f(x)$, is $P_n(-x;0)$ the M
associated with $f(-x)$? Explain.

45.[C] Let $P(x)$ be the Maclauri
second-order associated with $f(x)$
Maclaurin polynomial of the seco
with $g(x)$. What part, if any, of $P($
rin polynomial associated with $f(x$

5.5 L'Hôpital's Rule for Finding Certain Limits

There are two types of limits in calculus: those that you can evaluate at a glance, and those that require some work to evaluate. In Section 2.4 we learned to call the limits that can be evaluated easily **determinate** and those that require some work to evaluate are called **indetermnate**.

For instance $\lim_{x\to\pi/2} \frac{\sin(x)}{x}$ is clearly $1/(\pi/2) = 2/\pi$. That's easy. But $\lim_{x\to 0}(\sin(x))/x$ is not obvious. Back in Section 2.2 we used a diagram of circles, sectors, and triangles, to show that this limit is 1.

In this section we describe a technique for evaluating more indeterminate limits, for instance

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

when both $f(x)$ and $g(x)$ approach 0 as x approaches a. The numerator is trying to drag $f(x)/g(x)$ toward 0, at the same time as the denominator is trying to make the quotient large. L'Hôpital's rule helps determine which term wins or whether there is a compromise.

L'Hôpital is pronounced lope-ee-tall.

Indeterminate Limits

The following limits are called **indeterminate** because you can't determine them without knowing more about the functions of f and g.

$$\lim_{x\to a} \frac{f(x)}{g(x)}, \text{ where } \lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

$$\lim_{x\to a} \frac{f(x)}{g(x)}, \text{ where } \lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty$$

L'Hôpital's Rule provides a way for dealing with these limits (and limits that can be transformed to those forms.) In short, l'Hôpital's rule applies only when you need it.

Theorem 5.5.1 (L'Hôpital's Rule (zero-over-zero case))**.** *Let a be a number and let f and g be differentiable over some open interval that contains a. Assume also that $g'(x)$ is not 0 for any x in that interval except perhaps at a. If*

$$\lim_{x\to a} f(x) = 0, \ \lim_{x\to a} g(x) = 0, \text{ and } \lim_{x\to a} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

In short, "to evaluate the limit of a quotient that is indeterminant, evaluate the limit of the quotient of their derivatives." You evaluate the limit of the quotient of the derivatives, not the derivative of the quotient. We will discuss the proof after some examples.

EXAMPLE 1 Find $\lim_{x\to 1}(x^5-1)/(x^3-1)$.
SOLUTION In this case,

$$a = 1,\ f(x) = x^5 - 1,\ \text{and}\ g(x) = x^3 - 1.$$

Remember to check that the hypotheses of l'Hôpital's Rule are satisfied.

All the assumptions of l'Hôpital's rule are satisfied. In particular,

$$\lim_{x\to 1}(x^5-1) = 0 \text{ and } \lim_{x\to 1}(x^3-1) = 0.$$

According to l'Hôpital's rule,

$$\lim_{x\to 1}\frac{x^5-1}{x^3-1} \overset{\text{l'H}}{=} \lim_{x\to 1}\frac{(x^5-1)'}{(x^3-1)'}$$

if the latter limit exists. Now,

$\lim_{x\to 1}\frac{(x^5-1)'}{(x^3-1)'}$	$=$	$\lim_{x\to 1}\frac{5x^4}{3x^2}$	differentiation of numerator and differentiation of denominator
	$=$	$\lim_{x\to 1}\frac{5}{3}x^2$	algebra
	$=$	$\frac{5}{3}$.	

Thus

$$\lim_{x\to 1}\frac{x^5-1}{x^3-1} = \frac{5}{3}.$$

◇

Sometimes it may be necessary to apply l'Hôpital's Rule more than once, as in the next example.

EXAMPLE 2 Find $\lim_{x\to 0}(\sin(x)-x)/x^3$.
SOLUTION As $x \to 0$, both numerator and denominator approach 0. By l'Hôpital's Rule,

$$\lim_{x\to 0}\frac{\sin(x)-x}{x^3} \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{(\sin(x)-x)'}{(x^3)'} = \lim_{x\to 0}\frac{\cos(x)-1}{3x^2}.$$

But as $x \to 0$, both $\cos(x) - 1 \to 0$ and $3x^2 \to 0$. So use l'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{\cos(x)-1}{3x^2} \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{(\cos(x)-1)'}{(3x^2)'} = \lim_{x\to 0}\frac{-\sin(x)}{6x}.$$

Both $\sin(x)$ and $6x$ approach 0 as $x \to 0$. Use l'Hôpital's Rule yet another time:

Or recall from Section 2.2 that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

$$\lim_{x\to 0} \frac{-\sin(x)}{6x} \overset{\text{l'H}}{=} \lim_{x\to 0} \frac{(-\sin(x))'}{(6x)'} = \lim_{x\to 0} \frac{-\cos(x)}{6} = \frac{-1}{6}.$$

So after three applications of l'Hôpital's Rule we find that

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = -\frac{1}{6}.$$

◇

Sometimes a limit may be simplified before l'Hôpital's Rule is applied. For instance, consider

$$\lim_{x\to 0} \frac{(\sin(x) - x)\cos^5(x)}{x^3}.$$

Since $\lim_{x\to 0} \cos^5(x) = 1$, we have

$$\lim_{x\to 0} \frac{(\sin(x) - x)\cos^5(x)}{x^3} = \left(\lim_{x\to 0} \frac{\sin(x) - x}{x^3}\right) \cdot 1,$$

which, by Example 2, is $-\frac{1}{6}$. This shortcut saves a lot of work, as may be checked by finding the limit using l'Hôpital's Rule without separating $\cos^5(x)$.

Theorem 5.5.1 concerns limits as $x \to a$. L'Hôpital's Rule also applies if $x \to \infty$, $x \to -\infty$, $x \to a^+$, or $x \to a^-$. In the first case, we would assume that $f(x)$ and $g(x)$ are differentiable in some interval (c, ∞) and $g'(x)$ is not zero there. In the case of $x \to a^+$, assume that $f(x)$ and $g(x)$ are differentiable in some open interval (a, b) and $g'(x)$ is not 0 there.

Infinity-over-Infinity Limits

"Infinity-over-infinity" is indeterminate.

Theorem 5.5.1 concerns the limit of $f(x)/g(x)$ when both $f(x)$ and $g(x)$ approach 0. But a similar problem arises when both $f(x)$ and $g(x)$ get arbitrarily large as $x \to a$ or as $x \to \infty$. The behavior of the quotient $f(x)/g(x)$ will be influenced by how rapidly $f(x)$ and $g(x)$ become large.

In short, if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a}(f(x)/g(x))$ is an indeterminate form.

The next theorem presents a form of l'Hôpital's Rule that covers the case in which $f(x) \to \infty$ and $g(x) \to \infty$.

Theorem 5.5.2 (L'Hôpital's Rule (infinity-over-infinity case)**.** *Let f and g be defined and differentiable for all x larger than some number. Then, if $g'(x)$ is not zero for all x larger*

$$\lim_{x\to\infty} f(x) = \infty, \ \lim_{x\to\infty} g(x) = \infty, \ \textit{and} \ \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L,$$

it follows that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

A similar result holds for $x \to a$, $x \to a^-$, $x \to a^+$, or $x \to -\infty$. Moreover, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ could both be $-\infty$, or one could be ∞ and the other $-\infty$.

EXAMPLE 3 Find $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$.

SOLUTION Since $\ln(x) \to \infty$ and $x^2 \to \infty$ as $x \to \infty$, we may use l'Hôpital's Rule in the "infinity-over-infinity" form.

We have

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} \stackrel{\text{l'H}}{=} \lim_{x\to\infty} \frac{(\ln(x))'}{(x^2)'} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Hence $\lim_{x\to\infty}((\ln(x))/x^2) = 0$. This says that $\ln(x)$ grows much more slowly than x^2 does as x gets large. ◇

EXAMPLE 4 Find

$$\lim_{x\to\infty} \frac{x - \cos(x)}{x}. \tag{5.5.1}$$

SOLUTION Both numerator and denominator approach ∞ and $x \to \infty$. Trying l'Hôpital's Rule, we obtain

$$\lim_{x\to\infty} \frac{x - \cos(x)}{x} \stackrel{\text{l'H}}{=} \lim_{x\to\infty} \frac{(x - \cos(x))'}{x'} = \lim_{x\to\infty} \frac{1 + \sin(x)}{1}.$$

L'Hôpital's Rule may fail to provide an answer.

But $\lim_{x\to\infty}(1+\sin(x))$ does not exist, since $\sin(x)$ oscillates back and forth from -1 to 1 as $x \to \infty$

What can we conclude about the limit in (5.5.1)? Nothing at all. L'Hôpital's Rule says that if $\lim_{x\to\infty} f'(x)/g'(x)$ exists, then $\lim_{x\to\infty} f(x)/g(x)$ exists and has the same value. It say nothing about the case when $\lim_{x\to\infty} f'(x)/g'(x)$ does not exist.

It is not difficult to evaluate (5.5.1) directly, as follows:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x - \cos(x)}{x} &= \lim_{x\to\infty} \left(1 - \frac{\cos(x)}{x}\right) && \text{algebra} \\
&= 1 - 0 && \text{since } |\cos(x)| \le 1 \\
&= 1.
\end{aligned}$$

◇

Moral: Look carefully at a limit before you decide to use l'Hôpital's Rule.

Two cars can help make Theorem 5.5.2 plausible. Imagine that $f(t)$ and $g(t)$ describe the locations on the x-axis of two cars at time t. Call the cars the f-car and the g-car. See Figure 5.5.1. Their velocities are therefore $f'(t)$ and $g'(t)$. These two cars are on endless journeys. But assume that as time $t \to \infty$ the f-car tends to travel at a speed closer and closer to L times the speed of the g-car. That is, assume that

$$\lim_{t\to\infty} \frac{f'(t)}{g'(t)} = L.$$

No matter how the two cars move in the short run, it seems reasonable that in the long run the f-car will tend to travel about L times as far as the g-car; that is,

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = L.$$

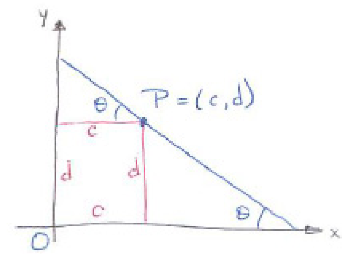


Figure 5.5.1:

Transforming Limits So You Can Use l'Hôpital's Rule

Many limits can be transformed to limits to which l'Hôpital's Rule applies. For instance, the problem of finding

$$\lim_{x\to 0^+} x\ln(x)$$

"zero-times-infinity" is indeterminate

does not fit into l'Hôpital's Rule, since it does not involve the quotient of two functions. As $x \to 0^+$, one factor, x, approaches 0 and the other factor $\ln(x)$, approaches $-\infty$. So this is another type of indeterminate limit, involving a small number times a large number ("zero-times-infinity"). It is not obvious how this product, $x\ln(x)$, behaves as $x \to 0^+$. (Such a limit can turn out to be "zero, medium, large, or infinite"). A little algebra transforms the zero-times-infinity case into a problem to which l'Hôpital's Rule applies, as the next example illustrates.

EXAMPLE 5 Find $\lim_{x\to 0^+} x\ln(x)$.
SOLUTION Rewrite $x\ln(x)$ as a quotient, $\frac{\ln(x)}{(1/x)}$. Note that

$$\lim_{x\to 0^+} \ln(x) = -\infty \text{ and } \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

By l'Hôpital's Rule,

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} \overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$$

Thus

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = 0,$$

The factor x, which approaches 0, dominates the factor $\ln(x)$ which "slowly grows towards $-\infty$."

from which it follows that $\lim_{x\to 0^+} x\ln(x) = 0$. ◇

The final example illustrates another type of limit that can be found by first relating it to limits to which l'Hôpital's Rule applies.

Try this on your calculator first.

EXAMPLE 6 $\lim_{x\to 0^+} x^x$.

SOLUTION Since this limit involves an exponential, not a quotient, it does not fit directly into l'Hôpital's Rule. But a little algebra changes the problem to one covered by l'Hôpital's Rule.

Let	y	$=$	x^x.
Then	$\ln(y)$	$=$	$\ln(x^x) = x\ln(x)$
By Example 5,	$\lim_{x\to 0^+} x\ln(x)$	$=$	0.

Therefore, $\lim_{x\to 0^+} \ln(y) = 0$. By the definition of $\ln(y)$ and the continuity of $e^x = \exp(x)$,

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} \exp(\ln(y)) = \exp(\lim_{x\to 0^+} (\ln(y)) = e^0 = 1.$$

Hence $x^x \to 1$ as $x \to 0^+$. ◇

Concerning the Proof

A complete proof of Theorem 5.5.1 may be found in Exercises 71 to 73. The following argument is intended to make the theorem plausible. To do so, consider the *special case* where f, f', g, and g' are all continuous throughout an open interval containing a — in particular, all four functions are defined at a. Assume that $g'(x) \neq 0$ throughout the interval. Since we have $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, it follows by continuity that $f(a) = 0$ and $g(a) = 0$.

Assume that $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Then

$\lim_{x\to a}\dfrac{f(x)}{g(x)}$	$= \lim_{x\to a}\dfrac{f(x)-f(a)}{g(x)-g(a)}$	since $f(a)=0$ and $g(a)=0$
	$= \lim_{x\to a}\dfrac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$	algebra
	$= \dfrac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}$	limit of quotient equals quotient of limits
	$= \dfrac{f'(a)}{g'(a)}$	definitions of $f'(a)$ and $g'(a)$
	$= \dfrac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}$	f' and g' are continuous, by assumption
	$= \lim_{x\to a}\dfrac{f'(x)}{g'(x)}$	quotient of limits equals limit of quotients
	$= L$	by assumption.

Consequently,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = L.$$

Summary

We described l'Hôpital's Rule, which is a technique for dealing with limits of the indeterminate form "zero-over-zero" and "infinity-over-infinity" . In both of these cases

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

if the latter limit exists. Note that it concerns the quotient of two derivatives, *not* the derivative of the quotient.

Table 5.5.1 shows how some limits of other indeterminate forms can be converted into either of these two forms.

L'Hôpital's rule comes in handy during our study of a uniform sprinkler in the Calculus is Everywhere section at the end of this chapter.

Indeterminate Forms	Name	Conversion Method	New Form
$f(x)g(x)$; $f(x)\to 0$, $g(x)\to 0$	Zero-times-infinity ($0\cdot\infty$)	Write as $\frac{f(x)}{1/g(x)}$ or $\frac{g(x)}{1/f(x)}$	$\frac{0}{0}$ or $\frac{\infty}{\infty}$
$f(x)^{g(x)}$; $f(x)\to 1$, $g(x)\to\infty$	One-to-infinity (1^∞)	Let $y = f(x)^{g(x)}$; take $\ln(y)$, find limit of $\ln(y)$, and then find limit of $y = e^{\ln(y)}$	$\ln(y)$ has form $\infty\cdot 0$
$f(x)^{g(x)}$; $f(x)\to 0$, $g(x)\to 0$	Zero-to-zero (0^0)	Same as for 1^∞	$\ln(y)$ has form $0\cdot\infty$.

Table 5.5.1:

EXERCISES for Section 5.5 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 16 check that l'Hôpital's Rule applies and use it to find the limits. *Identify all uses of l'Hôpital's Rule, including the type of indeterminant form.*

1.[R] $\lim_{x\to 2} \frac{x^3-8}{x^2-4}$

2.[R] $\lim_{x\to 1} \frac{x^7-1}{x^3-1}$

3.[R] $\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)}$

4.[R] $\lim_{x\to 0} \frac{\sin(x^2)}{(\sin(x))^2}$

5.[R] $\lim_{x\to 0} \frac{\sin(5x)\cos(3x)}{x}$

6.[R] $\lim_{x\to 0} \frac{\sin(5x)\cos(3x)}{x-\frac{\pi}{2}}$

7.[R] $\lim_{x\to \frac{\pi}{2}} \frac{\sin(5x)\cos(3x)}{x}$

8.[R] $\lim_{x\to \frac{\pi}{2}} \frac{\sin(5x)\cos(3x)}{x-\frac{\pi}{2}}$

9.[R] $\lim_{x\to \infty} \frac{x^3}{e^x}$

10.[R] $\lim_{x\to \infty} \frac{x^5}{3^x}$

11.[R] $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

12.[R] $\lim_{x\to 0} \frac{\sin(x)-x}{(\sin(x))^3}$

13.[R] $\lim_{x\to 0} \frac{\tan(3x)}{\ln(1+x)}$

14.[R] $\lim_{x\to 1} \frac{\cos(\pi x/2)}{\ln(x)}$

15.[R] $\lim_{x\to 2} \frac{(\ln(x))^2}{x}$

16.[R] $\lim_{x\to 0} \frac{\arcsin(x)}{e^{2x}-1}$

In each of Exercises 17 to 22 transform the problem into one to which l'Hôpital's Rule applies; then find the limit. *Identify all uses of l'Hôpital's Rule, including the type of indeterminant form.*

17.[R] $\lim_{x\to 0}(1-2x)^{1/x}$

18.[R] $\lim_{x\to 0}(1+\sin(2x))^{\csc(x)}$

19.[R] $\lim_{x\to 0^+}(\sin(x))^{(e^x-1)}$

20.[R] $\lim_{x\to 0^+} x^2\ln(x)$

21.[R] $\lim_{x\to 0^+}(\tan(x))^{\tan(2x)}$

22.[R] $\lim_{x\to 0^+}(e^x-1)\ln(x)$

WARNING *(Do Not Overuse l'Hôpital's Rule)* Remember that l'Hôpital's Rule, carelessly applied, may give a wrong answer or no answer.

In Exercises 23 to 51 find the limits. Use l'Hôpital's Rule only if it applies. *Identify all uses of l'Hôpital's Rule, including the type of indeterminant form.*

23.[R] $\lim_{x\to\infty}\frac{2^x}{3^x}$

24.[R] $\lim_{x\to\infty}\frac{2^x+x}{3^x}$

25.[R] $\lim_{x\to\infty}\frac{\log_2(x)}{\log_3(x)}$

26.[R] $\lim_{x\to 1}\frac{\log_2(x)}{\log_3(x)}$

27.[R] $\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)$

28.[R] $\lim_{x\to\infty}\left(\sqrt{x^2+3}-\sqrt{x^2+4x}\right)$

29.[R] $\lim_{x\to\infty}\frac{x^2+3\cos(5x)}{x^2-2\sin(4x)}$

30.[R] $\lim_{x\to\infty}\frac{e^x-1/x}{e^x-1/x}$

31.[R] $\lim_{x\to 0}\frac{3x^3+x^2-x}{5x^3+x^2+x}$

32.[R] $\lim_{x\to\infty}\frac{3x^3+x^2-x}{5x^3+x^2+x}$

33.[R] $\lim_{x\to\infty}\frac{\sin(x)}{4+\sin(x)}$

34.[R] $\lim_{x\to\infty} x\sin(3x)$

35.[R] $\lim_{x\to 1^+}(x-1)\ln(x-1)$

36.[R] $\lim_{x\to\pi/2}\frac{\tan(x)}{x-(\pi/2)}$

37.[R] $\lim_{x\to 0}(\cos(x))^{1/x}$

38.[R] $\lim_{x\to 0^+} x^{1/x}$

39.[R] $\lim_{x\to 0}(1+x)^{1/x}$

40.[R] $\lim_{x\to 0}\left(1+x^2\right)^x$

41.[R] $\lim_{x\to 1}\frac{x^2-1}{x^3-1}$

42.[R] $\lim_{x\to 0}\frac{xe^x(1+x)^3}{e^x-1}$

43.[R] $\lim_{x\to 0}\frac{xe^x\cos^2(6x)}{e^{2x}-1}$

44.[R] $\lim_{x\to 0}(\csc(x)-\cot(x))$

45.[R] $\lim_{x\to 0}\frac{\csc(x)-\cot(x)}{\sin(x)}$

46.[R] $\lim_{x\to 0}\frac{5^x-3^x}{\sin(x)}$

47.[R] $\lim_{x\to 0}\frac{(\tan(x))^5-(\tan(x))^3}{1-\cos(x)}$

48.[R] $\lim_{x\to 2}\frac{x^3+8}{x^2+5}$

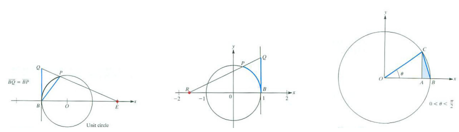
49.[R] $\lim_{x\to\pi/4}\frac{\sin(5x)}{\sin(3x)}$

50.[R] $\lim_{x\to 0}\left(\frac{1}{1-\cos(x)}-\frac{2}{x^2}\right)$

51.[R] $\lim_{x\to 0}\frac{\arcsin(x)}{\arctan(2x)}$

52.[M] In Figure 5.5.2(a) the unit circle is centered at O, BQ is a vertical tangent line, and the length of BP is the same as the length of BQ. What happens to the point E as $Q\to B$?

53.[M] In Figure 5.5.2(b) the unit circle is centered at the origin, BQ is a vertical tangent line, and the length of BQ is the same as the arc length $\widehat{BP}$. Show that the x-coordinate of R approaches -2 as $P \to B$.



(a) (b) (c)

Figure 5.5.2:

54.[M] Exercise 43 of Section 2.2 asked you to guess a certain limit. Now that limit will be computed.

> **WARNING** *(Common Sense)* As Albert Einstein observed, "Common sense is the deposit of prejudice laid down in the mind before the age of 18."

In Figure 5.5.2(c), which shows a circle, let $f(\theta)$ be the area of triangle ABC and $g(\theta)$ be the area of the shaded region formed by deleting triangle OAC from sector OBC.

(a) Why is $f(\theta)$ smaller than $g(\theta)$?

(b) What would you guess is the value of $\lim_{\theta\to 0} f(\theta)/g(\theta)$?

(c) Find $\lim_{\theta\to 0} f(\theta)/g(\theta)$.

55.[M] The following argument appears in an economics text:
"Consider the production function

$$y = k\left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-1/\rho},$$

where k, α, x_1, and x_2 are positive constants and $\alpha < 1$. Taking the limit as $\rho \to 0^+$, we find that

$$\lim_{\rho\to 0^+} y = k x_1^{\alpha} x_2^{1-\alpha},$$

which is the Cobb-Douglas function, as expected."
Fill in the details.

56.[M] Sam proposes the following proof for Theorem 5.5.1: "Since

$$\lim_{x\to a^+} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a+} g(x) = 0,$$

I will define $f(a) = 0$ and $g(a) = 0$. Next I consider $x > a$ but near a. I now have continuous functions f and g defined on the closed interval $[a, x]$ and differentiable on the open interval (a, x). So, using the Mean-Value Theorem, I conclude that there is a number c, $a < c < x$, such that

$$\frac{f(x) - f(a)}{x - a} = f'(c) \qquad \text{and} \qquad \frac{g(x) - g(a)}{x - a} = g'(c).$$

Since $f(a) = 0$ and $g(a) = 0$, these equations tell me that

$$f(x) = (x-a)f'(c) \qquad \text{and} \qquad g(x) = (x-a)g'(c)$$

$$\text{Thus} \qquad \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

$$\text{Hence} \qquad \lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(c)}{g'(c)}.$$

Sam made one error. What is it?

57.[C] Find $\lim_{x\to 0} \left(\frac{1+2^x}{x}\right)^{1/x}$.

58.[C] R. P. Feynman, in *Lectures in Physics*, wrote: "Here is the quantitative answer of what is right instead of kT. This expression

$$\frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

should, of course, approach kT as $\omega \to 0$. .... See if you can prove that it does — learn how to do the mathematics."
Do the mathematics. NOTE: All symbols, except T, denote constants.

59.[M] Graph $y = x^x$ for $0 < x \leq 1$, showing its minimum point.

In Exercises 60 to 62 graph the specified function, being sure to show (a) where the function is increasing and decreasing, (b) where the function has any asymptotes, and (c) how the function behaves for x near 0.

60.[M] $f(x) = (1+x)^{1/x}$ for $x > -1$, $x \neq 0$

61.[M] $y = x\ln(x)$

62.[M] $y = x^2\ln(x)$

63.[M] In which cases below is it possible to determine $\lim_{x\to a} f(x)^{g(x)}$ without further information about the functions?

(a) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = 7$

(b) $\lim_{x\to a} f(x) = 2$; $\lim_{x\to a} g(x) = 0$

(c) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = 0$

(d) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = \infty$

(e) $\lim_{x\to a} f(x) = \infty$; $\lim_{x\to a} g(x) = 0$

(f) $\lim_{x\to a} f(x) = \infty$; $\lim_{x\to a} g(x) = -\infty$

64.[M] In which cases below is it possible to determine $\lim_{x\to a} f(x)/g(x)$ without further information about the functions?

(a) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = \infty$

(b) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = 1$

(c) $\lim_{x\to a} f(x) = 0$; $\lim_{x\to a} g(x) = 0$

(d) $\lim_{x\to a} f(x) = \infty$; $\lim_{x\to a} g(x) = -\infty$

65.[M] Sam is angry. "Now I know why calculus books are so long. They spend all of page 72 showing that $\lim_{x\to 0}\frac{\sin(x)}{x}$ is 1. They could have saved space (and me a lot of trouble) if they had just used l'Hôpital's approach."
Is Sam right, for once?

66.[M] Jane says, "I can get $\lim_{x\to 0}\frac{e^x-1}{x}$ easily. It's just the derivative of e^x evaluated at 0. I don't need l'Hôpital's Rule." Is Jane right, or has Sam's influence affected her ability to reason?

67.[M]

If $\lim_{t\to\infty} f(t) = \infty = \lim_{t\to\infty} g(t)$
and $\lim_{t\to\infty}\frac{f(t)}{g(t)} = 3$,

what can be said about

$$\lim_{t\to\infty}\frac{\ln(f(t))}{\ln(g(t))}?$$

NOTE: Do *not* assume f and g are differentiable.

68.[C] Give an example of a pair of functions f and g such that we have $\lim_{x\to 0} f(x) = 1$, $\lim_{x\to 0} g(x) = \infty$, and $\lim_{x\to 0} f(x)^{g(x)} = 2$.

69.[C] Obtain l'Hôpital's Rule for $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ from the case $\lim_{t\to 0^+}\frac{f(t)}{g(t)}$.
HINT: Let $t = 1/x$.

70.[C] Find the limit of $(1^x + 2^x + 3^x)^{1/x}$ as

(a) $x \to 0$

(b) $x \to \infty$

(c) $x \to -\infty$.

The proof of Theorem 5.5.1, to be outlined in Exercise 73, depends on the following generalized mean-value theorem.
Generalized Mean-Value Theorem. Let f and g be two functions that are continuous on $[a, b]$ and differentiable on (a, b). Furthermore, assume that $g'(x)$ is never 0 for x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

71.[M] During a given time interval one car travels twice as far as another car. Use the Generalized Mean-Value Theorem to show that there is at least one instant when the first car is traveling exactly twice as fast as the second car.

72.[C] To prove the Generalized Mean-Value Theorem, introduce a function h defined by

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)). \quad (5.5.2)$$

Show that $h(b) = 0$ and $h(a) = 0$. Then apply Rolle's Theorem to h on (a, b). NOTE: Rolle's Theorem is Theorem 4.1.2 in Section 4.1.

> **Remark:** The function h in (5.5.2) is similar to the function h used in the proof of the Mean-Value Theorem (Theorem 4.1.3 in Section 4.1). Check that $h(x)$ is the vertical distance between the point $(g(x), f(x))$ and the line through $(g(a), f(a))$ and $(g(b), f(b))$.

73.[C] Assume the hypotheses of Theorem 5.5.1. Define $f(a) = 0$ and $g(a) = 0$, so that f and g are continuous at a. Note that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)},$$

and apply the Generalized Mean-Value Theorem from Exercise 71. NOTE: This Exercise proves Theorem 5.5.1, l'Hôpital's Rule in the zero-over-zero case.

74.[C]

If	$\lim_{t\to\infty} f(t) =$	∞	$= \lim_{t\to\infty} g(t)$
and	$\lim_{t\to\infty} \frac{\ln(f(t))}{\ln(g(t))}$	$=$	$1,$
must	$\lim_{t\to\infty} \frac{f(t)}{g(t)}$	$=$	$1?$

Explain.

75.[C] Assume that f, f', and f'' are defined in $[-1, 1]$ and are continuous. Also, $f(0) = 0$, $f'(0) = 0$, and $f''(0) > 0$.

(a) Sketch what the graph of f may look like for x in $[0, a]$, where a is a small positive number.

(b) Interpret the quotient

$$Q(a) = \frac{\int_0^a f(x)\, dx}{af(a) - \int_0^a f(x)\, dx}$$

in terms of the graph in (a).

(c) What do you think happens to $Q(a)$ as $a \to 0$?

(d) Find $\lim_{a\to 0} Q(a)$.

HINT: Because f''' might not be continuous at 0, you need to use $\lim_{a\to 0} \frac{f'(a)}{a} = f''(0)$.

76.[C]

Sam: I bet I can find $\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$ by using the Taylor polynomial $P_2(x; 0)$ for e^x and paying attention to the error.

Is Sam right?

5.6 Natural Growth and Decay

In 2009 the population of the United States was about 306 million and growing at a rate of about 1% (roughly 3 million people) a year. The world population was about 6.79 billion and growing at a rate of about 1.5% (roughly 100 million people) a year.

The United States population has been increasing at about 1% a year for years. It is an example of natural growth.

Natural Growth

Let $P(t)$ be the size of a population at time t. If its rate of growth is proportional to its size, there is a positive constant k such that

$$\frac{dP(t)}{dt} = kP(t). \tag{5.6.1}$$

To find an explicit formula for $P(t)$ as a function of t, rewrite (5.6.1) as

$$\frac{\frac{dP(t)}{dt}}{P(t)} = k. \tag{5.6.2}$$

The left-hand side can be rewritten as the derivative of $\ln(P(t))$ and so (5.6.2) can be rewritten as

$$\frac{d(\ln(P(t))}{dt} = \frac{d(kt)}{dt}.$$

Therefore there is a constant C such that

$$\ln(P(t)) = kt + C. \tag{5.6.3}$$

From (5.6.3) it follows, by the definition of a logarithm, that

$$P(t) = e^{kt+C},$$

hence

$$P(t) = e^C e^{kt}.$$

Since C is a constant, so is e^C, which we give a simpler name: A. We have the following simple explicit formula for $P(t)$:

The equation for **natural growth** is

$$P(t) = Ae^{kt}$$

where k is a positive constant. Because $P(0) = Ae^{k(0)} = A$, the coefficient A is the **initial population**.

Because of the presence of the exponential e^{kt}, natural growth is also called **exponential growth**.

EXAMPLE 1 The size of the world population at the beginning of 1988 was approximately 5.14 billion. At the beginning of 1989 it was 5.23 billion. Assume that the growth rate remains constant.

(a) What is the growth constant k?

(b) What would the population be in 2009?

(c) When will the population double is size?

SOLUTION Let $P(t)$ be the population in billions at time t. For convenience, measure time starting in the year 1988; that is, $t = 0$ corresponds to 1988 and $t = 1$ to 1989. Thus $P(0) = 5.14$ and $P(1) = 5.23$. The natural growth equation describing the population in billions at time is

$$P(t) = 5.14e^{kt}. \tag{5.6.4}$$

(a) To find k, we note that

$$P(1) = 5.14e^{k\cdot 1},$$

so

$$\begin{aligned} 5.14e^k &= 5.23 \\ e^k &= \frac{5.23}{5.14} \\ k &= \ln\left(\frac{5.23}{5.14}\right) \approx 0.174. \end{aligned}$$

Hence (5.6.4) takes the form

$$P(t) = 5.14e^{0.174t}.$$

This equation is all that we need to answer the remaining questions.

(b) The year 2009 corresponds to $t = 21$, so in the year 2009 the population, in billions, would be

$$P(21) = 5.14e^{0.174\cdot 21} = 5.14e^{0.3654} \approx 5.14(1.441) \approx 7.41.$$

The population would be approximately 7.41 billion in 2009. (Recall from the introduction of this section that the actual estimate of the world population in 2009 is about 6.79 billion. This suggests that the actual growth rate has not been constant; it has increased during the past 21 years.)

(c) The population will double when it reaches $2(5.14) = 10.28$ billion. We need to solve for t in the equation $P(t) = 10.28$. We have

$$\begin{aligned} 5.14e^{kt} &= 10.28 \\ e^{kt} &= 2 \\ kt &= \ln(2) \\ t &= \frac{\ln(2)}{k} \approx \frac{0.6931}{0.0174} \approx 39.8360. \end{aligned}$$

The world population will double approximately 40 years after 1988, which corresponds to the year 2028.

◇

The time it takes for a population to double is called the **doubling time** and is denoted t_2. Exponential growth is often described by its doubling time t_2 rather than by its growth constant k. However, if you know either t_2 or k you can figure out the other, as they are related by the equation

$$t_2 = \frac{\ln(2)}{k}$$

which appeared during part (c) of the solution to Example 1.

Exponential growth may also be described in terms of an annual percentage increase, such as "The population is growing 6 percent per year." That is, each year the population is multiplied by the factor 1.06: $P(t+1) = P(t)(1.06)$.

On the other hand, from the exponential growth function, we see that

$$P(t+1) = P(0)e^{k(t+1)} = P(0)e^{kt}e^{k} = P(t)e^{k}.$$

That is, during each unit of time the population increases by a factor of e^k. Now, when k is small, $e^k \approx 1+k$. Consequently we can approximate 6 percent annual growth by letting $k = 0.06$. This approximation is valid whenever the growth rate is only a few percent. Since population figures are themselves only an approximation, setting the growth constant k equal to the annual percentage rate is a reasonable tactic.

EXAMPLE 2 Find the doubling time if the growth rate is 2 percent per year.

SOLUTION The growth rate is 2 percent, so we set $k = 0.02$. Then

$$t_2 = \frac{\ln(2)}{k} \approx \frac{0.693}{0.02} = 34.65 \text{ years.}$$

◇

The Mathematics of Natural Decay

As Glen Seaberg observes in the conversation given on page 381, some radioactive elements decay at a rate proportional to the amount present. The time it takes for half the initial amount to decay is denoted $t_{1/2}$ and is called the element's **half-life**.

Similarly, in medicine one speaks of the half-life of a drug administered to a patient: the time required for half the drug to be removed from the body. This half-life depends both on the drug and the patient, and can from 20 minutes for penicillin to 2 weeks for quinacrine, an antimalarial drug. This half-life is critical to determining how frequently a drug can be administered. Some elderly patients died from overdoses before it was realized that the half-life of some drugs is longer in the elderly than in the young.

Letting $P(t)$ again represent the amount present at time t, we have

Now k is negative.

$$P'(t) = kP(t) \qquad k < 0$$

where k is the **decay constant**. This is the same equation as (5.6.1), so

$$P(t) = P(0)e^{kt},$$

as before, except now k is a *negative number*. Since k is negative, the factor e^{kt} is a decreasing function of t.

Just as the doubling time is related to (positive) k by the equation $t_2 = (\ln(2))/k$, the half-life is related to (negative) k by the equation $t_{1/2} = (\ln(1/2))/k$, which can be rewritten as $t_{1/2} = -(\ln(2))/k$.

EXAMPLE 3 The Chernobyl nuclear reactor accident, in April 1986, released radioactive cesium 137 into the air. The half-life of ^{137}Cs is 27.9 years.

(a) Find the decay constant k of ^{137}Cs.

(b) When will only one-fourth of an initial amount remain?

(c) When will only 20 percent of an initial amount remain?

SOLUTION

(a) The formula for the half-life can be solved for k to give:

$$k = \frac{-\ln(2)}{t_{1/2}} \approx \frac{-0.693}{27.9} \approx -0.0248.$$

(b) This can be done without the aid of any formulas. Since $\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$, in two half-lives only one-quarter of an initial amount remains. The answer is $2(27.9) = 55.8$ years.

(c) We want to find t such that only 20 percent remains. While we know the answer is greater than 55.8 years (since 20% is less than 25%), finding the exact time requires using the formula for $P(t)$.

We want

$$P(t) = 0.20P(0).$$

That is, we want to solve

$$\begin{aligned} P(0)e^{kt} &= 0.20P(0). \\ \text{Then} \qquad e^{kt} &= 0.20 \\ kt &= \ln(0.20) \\ t &= \frac{\ln(0.20)}{k}. \end{aligned}$$

Since $k \approx -0.0248$, this gives

$$t \approx \frac{-1.609}{-0.0248} = 64.9 \text{ years.}$$

After 64.9 years (that is, 2051) only 20% of the original amount remains.

◇

Summary

We developed the mathematics of growth or decay that is proportional to the amount present. This required solving the differential equation

$$\frac{dP}{dt} = kP$$

where k is a constant, positive in the case of growth and negative in the case of decay. The solution is

$$P(t) = Ae^{kt}$$

where A is $P(0)$, the amount of the substance present when $t = 0$.

In the case of growth, the time for the quantity to double (the "doubling time") is denoted t_2. In the case of decay, the time when only half the original amount survives is denoted $t_{1/2}$, the "half-life." One has

$$t_2 = \frac{\ln(2)}{k} \qquad \text{and} \qquad t_{1/2} = \frac{\ln(1/2)}{k} = -\frac{\ln(2)}{k}.$$

The Scientist, The Senator, and Half-Life

During the hearings in 1963 before the Senate Foreign Relations Committee on the nuclear test ban treaty, this exchange took place between Glen Solborg, winner of the Nobel prize for chemistry in 1951, and Senator James W. Fulbright.

Seaborg: Tritium is used in a weapon, and it decays with a half-life of about 12 years. But the plutonium and uranium have such long half-lives that there is no detectable change in a human lifetime.

Fulbright: I am sure this seems to be a very naive question, but why do you refer to half-life rather than whole life? Why do you measure by half-lives?

Seaborg: Here is something that I could go into a very long discussion on.

Fulbright: I probably wouldn't benefit adequately from a long discussion. It seems rather odd that you should call it a half-life rather than its whole life.

Seaborg: Well, I will try. If we have, let us say, one million atoms of a material like tritium, in 12 years half of those will be transformed into a decay product and you will have 500,000 atoms.

Then, in another 12 years, half of what remains transforms, so you have 250,000 atoms left. And so forth.

On that basis it never all decays, because half is always left, but of course you finally get down to where your last atom is gone.

EXERCISES for Section 5.6 *Key:* R–routine, M–moderate, C–challenging

1.[R]

(a) Show that exponential growth can be expressed as $P = Ab^t$ for some constants A and b.

(b) What can be said about b?

2.[R]

(a) Show that exponential decay can be expressed as $P = Ab^t$ for some constants A and b.

(b) What can be said about b?

3.[R] If $P(t) = 30e^{0.2t}$ what are the initial size and the doubling time?

4.[R] If $P(t) = 30e^{-0.2t}$ what are the initial size and the half life?

5.[R] What is the doubling time for a population always growing at 1% a year?

6.[R] What is the half life for a population always shrinking at 1% a year?

7.[R] A quantity is increasing according to the law of natural growth. The amount present at time $t = 0$ is A. It will double when $t = 10$.

(a) Express the amount at time t in the form Ae^{kt} for a suitable k.

(b) Express the amount at time t in the form Ab^t for a suitable b.

8.[R] The mass of a certain bacterial culture after t hours is $10 \cdot 3^t$ grams.

(a) What is the initial amount?

(b) What is the growth constant k?

(c) What is the percent increase in any period of 1 hour?

9.[R] Let $f(t) = 3 \cdot 2^t$.

(a) Solve the equation $f(t) = 12$.

(b) Solve the equation $f(t) = 5$.

(c) Find k such that $f(t) = 3e^{kt}$.

10.[R] In 1988 the world population was about 5.1 billion and was increasing at the rate of 1.7 percent per year. If it continues to grow at that rate, when will it (a) double? (b) quadruple? (c) reach 100 billion?

11.[R] The population of Latin America has a doubling time of 27 years. Estimate the percent it grows per year.

12.[R] At 1:00 P.M. a bacterial culture weighed 100 grams. At 4:30 P.M. it weighed 250 grams. Assuming that it grows at a rate proportional to the amount present, find (a) at what time it will grow to 400 grams, (b) its growth constant.

13.[R] A bacterial culture grows from 100 to 400 grams in 10 hours according to the law of natural growth.

(a) How much was present after 3 hours?

(b) How long will it take the mass to double? quadruple? triple?

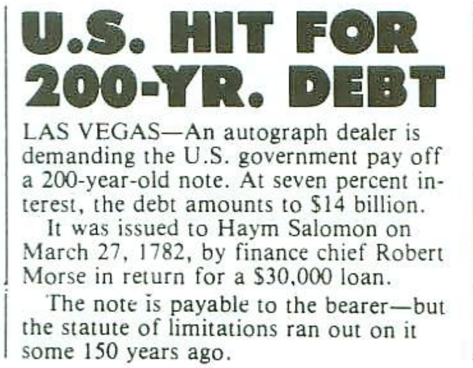
14.[R] A radioactive substance disintegrates at the rate of 0.05 grams per day when its mass is 10 grams.

(a) How much of the substance will remain after t days if the initial amount is A?

(b) What is its half-life?

15.[R] In 2009 the population of Mexico was 111 million and of the United States 308 million. If the population of Mexico increases at 1.15% per year and the population of the United States at 1.0% per year, when would the two nations have the same size population?

16.[R] The size of the population in India was 689 million in 1980 and 1,027 million in 2007. What is its doubling time t_2?



U.S. HIT FOR 200-YR. DEBT

LAS VEGAS—An autograph dealer is demanding the U.S. government pay off a 200-year-old note. At seven percent interest, the debt amounts to $14 billion.

It was issued to Haym Salomon on March 27, 1782, by finance chief Robert Morse in return for a $30,000 loan.

The note is payable to the bearer—but the statute of limitations ran out on it some 150 years ago.

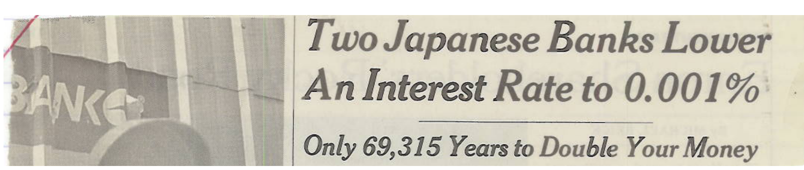
Figure 5.6.1:

17.[R] The newspaper article shown in Figure 5.6.1 illustrates the rapidity of exponential growth.

(a) Is the figure of $14 billion correct? Assume that the interest is compounded annually.

(b) What interest rate would be required to produce an account of $14 billion if interest were compounded once a year?

(c) Answer (b) for "continuous compounding," which is another term for natural growth (a bank account increases at a rate proportional to the amount in the account at any instant).



Two Japanese Banks Lower An Interest Rate to 0.001%

Only 69,315 Years to Double Your Money

Figure 5.6.2:

18.[R] The headline shown in Figure 5.6.2 appeared in 2002. Is the number 69,315 correct? Explain.

19.[R] Carbon 14 (chemical symbol ^{14}C), an isotope of carbon, is radioactive and has a half-life of approximately 5,730 years. If the ^{14}C concentration in a piece of wood of unknown age is half of the concentration in a present-day live specimen, then it is about 5,730 years old. (This assumes that ^{14}C concentrations in living objects remain about the same.) This gives a way of estimating the age of an undated specimen. Show that if A_C is the concentration of ^{14}C in a live (contemporary) specimen and A_u is the concentration of ^{14}C in a specimen of unknown age, then the age of the undated material is about $8,300 \ln(A_C/A_u)$ years. NOTE: This method, called **radiocarbon dating** is reliable up to about 70,000 years.

20.[R] From a letter to an editor in a newspaper:

> **I've been hearing bankers and investment advisers talk about something called the "rule of 72." Could you explain what it means?**
>
> How quickly would you like to double your money? That's what the "rule of 72" will tell you. To find out how fast your money will double at any given interest rate or yield, simply divide that yield into 72. This will tell you how many years doubling will take.
>
> Let's say you have a long-term certificate of deposit paying 12 percent [annually]. At that rate your money would double in six years. A money-market fund paying 10 percent would take 7.2 years to double your investment.

(a) Explain the rule of 72 and what number should be used instead of 72.

(b) Why do you think 72 is used?

21.[R] Benjamin Franklin conjectured that the population of the United States would double every 20 years, beginning in 1751, when the population was 1.3 million.

(a) If Franklin's conjecture were right, what would the population of the United States be in 2010?

(b) In 2010 the population was 310 million. Assuming natural growth, what would the doubling time be?

22.[M] (Doomsday equation) A differential equation of the form $dP/dt = kP^{1.01}$ is called a **doomsday equation**. The rate of growth is just slightly higher than that for natural growth. Solve the differential equation to find $P(t)$. How does $P(t)$ behave as t increases? Does $P(t)$ increase forever?

23.[M] The following situations are all mathematically the same:

1. A drug is administered in a dose of A grams to a patient and gradually leaves the system through excretion.

2. Initially there is an amount A of smoke in a room. The air conditioner is turned on and gradually the smoke is removed.

3. Initially there is an amount A of some pollutant in a lake, when further dumping of toxic materials is prohibited. The rate at which water enters the lake equals the rate at which it leaves. (Assume the pollution is thoroughly mixed.)

In each case, let $P(t)$ be the amount present at time t (whether drug, smoke, or pollution).

(a) Why is it reasonable to assume that there is a constant k such that for small intervals of time, Δt, $\Delta P \approx kP(t)\Delta t$?

(b) From (a) deduce that $P(t) = Ae^{kt}$.

(c) Is k positive or negative?

24.[M] **Newton's law of cooling** assumes that an object cools at a rate proportional to the difference between its temperature and the room temperature. Denote the room temperature as A. The differential equation for Newton's law of cooling is $dy/dt = k(y - A)$ where k and A are constants.

(a) Explain why k is negative.

(b) Draw the slope field for this differential equation when $k = -1/2$.

(c) Use (b) to conjecture the behavior of $y(t)$ as $t \to \infty$.

(d) Solve for y as a function of t.

(e) Draw the graph of $y(t)$ on the slope field produced in (b).

(f) Find $\lim_{t\to\infty} y(t)$.

25.[M] Let $I(x)$ be the intensity of sunlight at a depth of x meters in the ocean. As x increases, $I(x)$ decreases.

(a) Why is it reasonable to assume that there is a constant k (negative) such that $\Delta I \approx kI(x)\Delta x$ for small Δx?

(b) Deduce that $I(x) = I(0)e^{kx}$, where $I(0)$ is the intensity of sunlight at the surface. Incidentally, sunlight at a depth of 1 meter is only one-fourth as intense as at the surface.

26.[M] A particle moving through a liquid meets a "drag" force proportional to the velocity; that is, its acceleration is proportional to its velocity. Let x denote its position and v its velocity at time t. Assume $v > 0$.

(a) Show that there is a positive constant k such that $dv/dt = -kv$.

(b) Show that there is a constant A such that $v = Ae^{-kt}$.

(c) Show that there is a constant B such that $x = -\frac{1}{k}Ae^{-kt} + B$.

(d) How far does the particle travel as t goes from 0 to ∞? (Is this a finite or infinite distance?)

27.[M]

(a) Show that the natural growth function $P(t) = Ae^{kt}$ can be written in terms of A and t_2 as $P(t) = A \cdot 2^{t/t_2}$.

(b) Check that the function found in (a) is correct when $t = 0$ and $t = t_2$.

28.[M]

(a) Express the natural decay function $P(t) = Ae^{kt}$ in terms of A and $t_{1/2}$.

(b) Check that the function found in (a) is correct when $t = 0$ and $t = t_{1/2}$.

29.[M] A population is growing exponentially. Initially, at time 0, it is P_0. Later, at time u it is P_u.

(a) Show that at time t it is $P_0(P_u/P_0)^{t/u}$.

(b) Check that the formula in (a) gives the correct population when $t = 0$ and $t = u$.

30.[M] Let $P(t) = Ae^{kt}$. Then $\frac{P(t+1)-P(t)}{P(t)} = e^k - 1$. Show that when k is small, $e^k - 1 \approx k$. That means the relative change in one unit of time is approximately k.

31.[C] A certain fish population increases in number at a rate proportional to the size of the population. In addition, it is being harvested at a constant rate. Let $P(t)$ be the size of the fish population at time t.

(a) Show that there are positive constants h and k such that for small Δt, $\Delta P \approx kP\Delta t - h\Delta t$.

(b) Find a formula for $P(t)$ in terms of $P(0)$, h, and k. HINT: First divide by Δt in (a) and then take limits as $\Delta t \to 0$.

(c) Describe the behavior of $P(t)$ in the three cases $h = kP(0)$, $h > kP(0)$, and $h < kP(0)$

32.[C] The half-life of a drug administered to a certain patient is 8 hours. It is given in a 1-gram dose every 8 hours.

(a) How much is there in the patient just after the second dose is administered?

(b) How much is there in the patient just after the third dose? The fourth dose?

(c) Let $P(t)$ be the amount in the patient at t hours after the first dose. Graph $P(t)$ for a period of 48 hours. NOTE: $P(t)$ has meaning for all values of t, not just at the integers.

(d) Does the amount in the patient get arbitrarily large as time goes on?

33.[C] The half-life of the drug in Exercise 32 is 16 hours when administered to a different patient. Answer, for this patient, the questions in Exercise 32.

34.[C] The half-life of a drug in a certain patient is $t_{1/2}$ hours. It is administered every h hours. Can it happen that the concentration of the drug gets arbitrarily high? Explain your answer.

Exercises 35 to 37 introduce and analyze the **inhibited or logistic growth** model. This model will be encountered in the CIE for Chapter 10.

35.[C] In many cases of growth there is obviously a finite upper bound M which the population cannot exceed. Why is it reasonable to assume (or to take as a model) that

$$\frac{dP}{dt} = kP(t)(M - P(t)) \qquad 0 < P(t) < M \qquad (5.6.5)$$

for some constant k?

36.[C]

(a) Solve the differential equation in Exercise 35. HINT: You will need the partial fraction identity

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

and the property of logarithms: $\ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$. After simplification, your answer should have the form

$$P(t) = \frac{M}{1 + ae^{-Mkt}}$$

for a suitable constant a.

(b) Find $\lim_{t\to\infty} P(t)$. Is this reasonable?

(c) Express a in terms of $P(0)$, M, and k.

37.[C] By considering (5.6.5) in Exercise 35 directly (not the explicit formula in Exercise 36), show that

(a) P is an increasing function.

(b) The maximum rate of change of P occurs when $P(t) = M/2$.

(c) The graph of $P(t)$ has an inflection point.

38.[C] A salesman, trying to persuade a tycoon to invest in Standard Coagulated Mutual Fund, shows him the accompanying graph which records the value of a similar investment made in the fund in 1965. "Look! In the first 5 years the investment increased \$1,000," the salesman observed, "but in the past 5 years it increased by \$2,000. It's really improving. Look at the graph of the graph from 1985 to 1990, which you can see clearly in Figure 5.6.3."

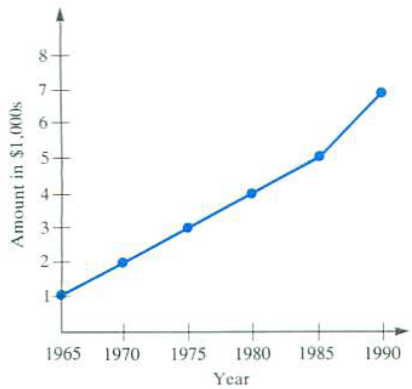


Figure 5.6.3:

The tycoon replied, "Hogwash. Though your graph is steeper from 1985 to 1990, in fact, the rate of return is less than from 1965 to 1970. Indeed, that was your best period."

(a) If the percentage return on the accumulated investment remains the same over each 5-year period as the first 5-year period, sketch the graph.

(b) Explain the tycoon's reasoning.

39.[C] Each of two countries is growing exponentially but at different rates. One is describe by the function $A_1e^{k_1t}$, the other by $A_2e^{k_2t}$, and k_1 is not equal to k_2. Is their total population growing exponentially? That is, are there constants A and k such that the formula describing their total population has the form Ae^{kt}. Explain your answer.

40.[C] Assume c_1, c_2, and c_3 are distinct constants. Can there be constants A_1, A_2, and A_3, not all 0, such that $A_1e^{c_1x} + A_2e^{c_2x} + A_3e^{c_3x} = 0$ for all x?

41.[C] If each of two functions describes natural growth does their (a) product? (b) quotient? (c) sum?

5.7 The Hyperbolic Functions and Their Inverses

Certain combinations of the exponential functions e^x and e^{-x} occur often in differential equations and engineering — for instance, in the study of the shape of electrical transmission or suspension cables — to be given names. This section defines these **hyperbolic functions** and obtains their basic properties. Since the letter x will be needed later for another purpose, we will use the letter t when writing the two preceding exponentials, namely, e^t and e^{-t}.

The Hyperbolic Functions

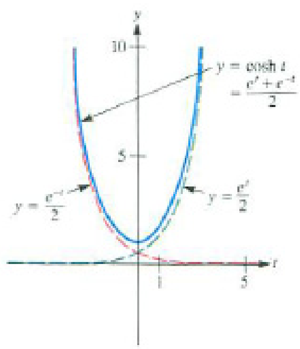


Figure 5.7.1:

DEFINITION *(The hyperbolic cosine.)* Let t be a real number. The **hyperbolic cosine** of t, denoted $\cosh(t)$, is given by the formula

$$\cosh(t) = \frac{e^t + e^{-t}}{2}.$$

To graph $\cosh(t)$, note first that

$$\cosh(-t) = \frac{e^{-t} + e^{-(-t)}}{2} = \frac{e^t + e^{-t}}{2} = \cosh(t).$$

Pronounced as written, "cosh," rhyming with "gosh."

Since $\cosh(-t) = \cosh(t)$, the cosh function is even, and so its graph is symmetric with respect to the vertical axis. Furthermore, $\cosh(t)$ is the sum of the two terms

$$\cosh(t) = \frac{e^t}{2} + \frac{e^{-t}}{2}.$$

For $|t| \to \infty$, the graph of $y = \cosh(t)$ is asymptotic to the graph of $y = e^t/2$ or $y = e^{-t}/2$.

As $t \to \infty$, the second term, $e^{-t}/2$, is positive and approaches 0. Thus, for $t > 0$ and large, the graph of $\cosh(t)$ is just a little above the graph of $e^t/2$. This information, together with the fact that $\cosh(0) = (e^0 + e^{-0})/2 = 1$, is the basis for Figure 5.7.1.

The curve $y = \cosh(t)$ in Figure 5.7.1 is called a **catenary** (from the Latin *catena* meaning "chain"). It describes the shape of a free-hanging chain. (See the CIE on the Suspension Bridge and the Hanging Cable for Chapter 15.)

"sinh" is pronounced "sinch," rhyming with "pinch."

DEFINITION *(The hyperbolic sine.)* Let t be a real number. The **hyperbolic sine** of t, denoted $\sinh(t)$, is given by the formula

$$\sinh(t) = \frac{e^t - e^{-t}}{2}.$$

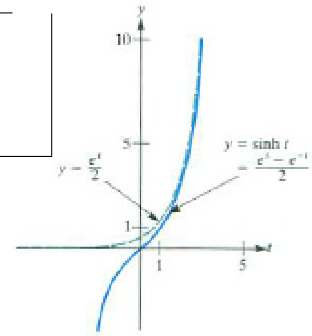


Figure 5.7.2:

It is a simple matter to check that $\sinh(0) = 0$ and $\sinh(-t) = -\sinh(t)$, so that the graph of $\sinh(t)$ is symmetric with respect to the origin. Moreover, it lies below the graph of $e^t/2$. However, the graphs of $\sinh(t)$ and $e^t/2$ approach each other since $e^{-t}/2 \to 0$ as $t \to \infty$. Figure 5.7.2 shows the graph of $\sinh(t)$.

Note the contrast between $\sinh(t)$ and $\sin(t)$. As $|t|$ becomes large, the hyperbolic sine becomes large, $\lim_{t\to\infty} \sinh(t) = \infty$ and $\lim_{t\to-\infty} \sinh(t) = -\infty$. There is a similar contrast between $\cosh(t)$ and $\cos(t)$. While the trigonometric functions are periodic, the hyperbolic functions are not.

Example 1 shows why the functions $(e^t + e^{-t})/2$ and $(e^t - e^{-t})/2$ are called **hyperbolic**.

EXAMPLE 1 Show that for any real number t the point with coordinates

$$x = \cosh(t), \qquad y = \sinh(t)$$

lie on the hyperbola $x^2 - y^2 = 1$.

SOLUTION Compute $x^2 - y^2 = \cosh^2(t) - \sinh^2(t)$ and see whether it simplifies to 1. We have

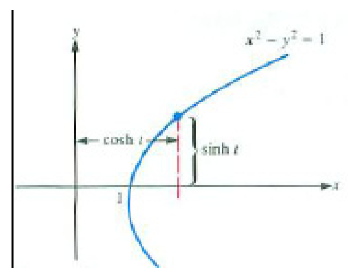


Figure 5.7.3:

$$\begin{aligned}
\cosh^2(t) - \sinh^2(t) &= \left(\tfrac{e^t+e^{-t}}{2}\right)^2 - \left(\tfrac{e^t-e^{-t}}{2}\right)^2 \\
&= \tfrac{e^{2t}+2e^te^{-t}+e^{-2t}}{4} - \tfrac{e^{2t}-2e^te^{-t}+e^{-2t}}{4} \\
&= \tfrac{2+2}{4} \qquad \text{cancellation} \\
&= 1.
\end{aligned}$$

Observe that since $\cosh(t) \geq 1$, the point $(\cosh(t), \sinh(t))$ is on the right half of the hyperbola $x^2 - y^2 = 1$, as shown in Figure 5.7.3. ◇

Example 1 proves a fundamental identity for hyperbolic functions: $\cosh^2(t) - \sinh^2(t) = 1$

By contrast, $(\cos(\theta), \sin(\theta))$ lies on the circle $x^2 + y^2 = 1$, so the trigonometric functions are also called **circular functions**.

There are four more hyperbolic functions, namely, the hyperbolic tangent, hyperbolic secant, hyperbolic cotangent, and hyperbolic cosecant. They are defined as follows:

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} \qquad \operatorname{sech}(t) = \frac{1}{\cosh(t)} \qquad \coth(t) = \frac{\cosh(t)}{\sinh(t)} \qquad \operatorname{csch}(t) = \frac{1}{\sinh(t)}.$$

Each can be expressed explicitly in terms of exponentials. For instance,

$$\tanh(t) = \frac{(e^t - e^{-t})/2}{(e^t + e^{-t})/2} = \frac{e^t - e^{-t}}{e^t + e^{-t}}.$$

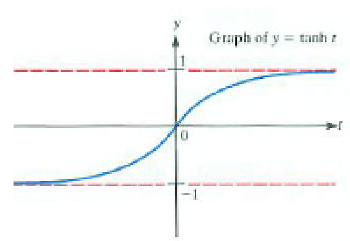


Figure 5.7.4:

As $t \to \infty$, $e^t \to \infty$ and $e^{-t} \to 0$. Thus $\lim_{t\to\infty} \tanh(t) = 1$. Similarly, $\lim_{t\to-\infty} \tanh(t) = -1$. Figure 5.7.4 is a graph of $y = \tanh(t)$.

The Derivatives of the Hyperbolic Functions

The derivatives of the six hyperbolic functions can be computed directly. For instance,

$$(\cosh(t))' = \left(\frac{e^t + e^{-t}}{2}\right)' = \frac{e^t - e^{-t}}{2} = \sinh(t).$$

Table 5.7.1 lists the derivatives of the six hyperbolic functions. Notice that the formulas, except for the signs, are like those for the derivatives of the trigonometric functions.

Function	Derivative
$\cosh(t)$	$\sinh(t)$
$\sinh(t)$	$\cosh(t)$
$\tanh(t)$	$\operatorname{sech}^2(t)$
$\coth(t)$	$-\operatorname{csch}^2(t)$
$\operatorname{sech}(t)$	$-\operatorname{sech}(t)\tanh(t)$
$\operatorname{csch}(t)$	$-\operatorname{csch}(t)\coth(t)$

Table 5.7.1:

The Inverses of the Hyperbolic Functions

Inverse hyperbolic functions appear on some calculators and in tables of mathematical functions. Just as the hyperbolic functions are expressed in terms of the exponential function, each inverse hyperbolic function can be expressed in terms of a logarithm. They provide useful antiderivatives as well as solutions to some differential equations.

Consider the inverse of $\sinh(t)$ first. Since $\sinh(t)$ is increasing, it is one-to-one; there is no need to restrict its domain. To find its inverse, it is necessary to solve the equation

$$x = \sinh(t)$$

Finding the inverse of the hyperbolic sine

for t as a function of x. The steps are straightforward:

$$\begin{array}{rrcll} & x &=& \frac{e^t - e^{-t}}{2}, & \text{definition of } \sinh(t) \\ & 2x &=& e^t - \frac{1}{e^t}, & e^{-t} = 1/e^t \\ & 2xe^t &=& (e^t)^2 - 1, & \text{multiply by } e^t \\ \text{or} & (e^t)^2 - 2xe^t - 1 &=& 0. & \end{array}$$

Equation (5.7) is quadratic in the unknown e^t. By the quadratic formula,

$$e^t = \frac{2x \pm \sqrt{(2x)^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since $e^t > 0$ and $\sqrt{x^2+1} > x$, the plus sign is kept and the minus sign is rejected. Thus

$$e^t = x + \sqrt{x^2 + 1} \qquad \text{and} \qquad t = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Consequently, the inverse of the function $\sinh(t)$ is given by the formula

$$\operatorname{arcsinh}(x) = \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right).$$

Formula for $\operatorname{arcsinh}(x)$

Computation of $\operatorname{arctanh}(x)$ is a little different. Since the derivative of $\tanh(t)$ is $\operatorname{sech}^2(t)$, the function $\tanh(t)$ is increasing and has an inverse. However, $|\tanh(t)| < 1$, and so the inverse function will be defined only for $|x| < 1$. Computations similar to those for $\operatorname{arcsinh}(x)$ show that

$$\operatorname{arctanh}(x) = \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad |x| < 1.$$

Formula for $\operatorname{arctanh}(x)$

Inverses of the other four hyperbolic functions are computed similarly. The functions $\operatorname{arccosh}(x)$ and $\operatorname{arcsech}(x)$ are chosen to be positive. Their formulas are included in Table 5.7.2.

Function	Formula	Derivative	Domain
$\operatorname{arccosh}(x)$	$\ln(x+\sqrt{x^2-1})$	$\frac{1}{\sqrt{x^2-1}}$	$x \geq 1$
$\operatorname{arcsinh}(x)$	$\ln(x+\sqrt{x^2+1})$	$\frac{1}{\sqrt{x^2+1}}$	x-axis
$\operatorname{arctanh}(x)$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$\frac{1}{1-x^2}$	$\|x\| < 1$
$\operatorname{arccoth}(x)$	$\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$	$\frac{1}{1-x^2}$	$\|x\| > 1$
$\operatorname{arcsech}(x)$	$\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$\frac{-1}{x\sqrt{1-x^2}}$	$0 < x \leq 1$
$\operatorname{arccsch}(x)$	$\ln\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right)$	$\frac{-1}{\|x\|\sqrt{1+x^2}}$	$x \neq 0$

Table 5.7.2:

The derivatives are found by differentiating the formulas in the second column.

Summary

We introduced the six hyperbolic functions and their inverses, including $\sinh(x)$ (pronounced *sinch*), $\cosh(x)$ (pronounced *cōsh*), $\tanh(x)$ (pronounced *tanch* or rhymes with "ranch") and their inverses $\operatorname{arcsinh}(x)$, arccosh, and arctanh. Because they are all expressible in terms of exponentials, square roots, and logarithms, they do not add to the collection of elementary functions. However, some of them are especially convenient.

The points $(\cosh(t), \sinh(t))$ lie on the graph of the hyperbola $x^2 - y^2 = 1$. (See Example 1.) The parameter t, which can be any number, has a geometric interpretation: it is the area of the shaded region in Figure 5.7.5(a). This corresponds to the fact that a sector of the unit circle with angle 2θ has area θ, as shown in Figure 5.7.5(b). (See Exercise 78.)

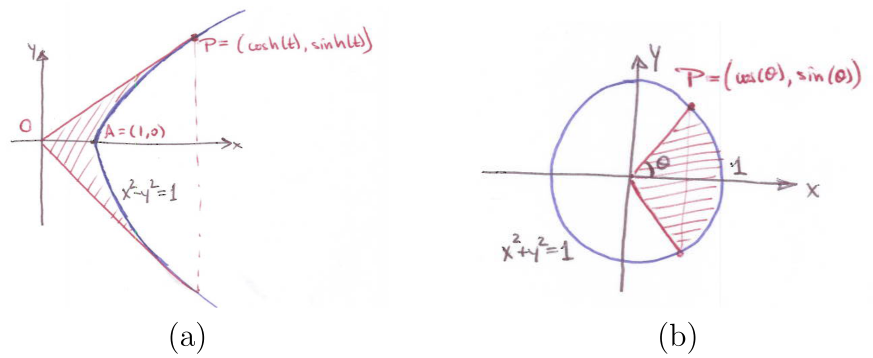


Figure 5.7.5:

EXERCISES for Section 5.7 *Key:* R–routine, M–moderate, C–challenging

1.[R]

(a) Compute $\cosh(t)$ and $e^t/2$ for $t = 0$, 1, 2, 3, and 4.

(b) Using the data in (a), graph $y = \cosh(t)$ and $y = e^t/2$ relative to the same axes.

2.[R]

(a) Compute $\tanh(t)$ for $t = 0$, 1, 2, and 3.

(b) Using the data in (a), and the fact that $\tanh(-t) = \tanh(t)$, graph $y = \tanh(t)$.

In Exercises 3 to 5 obtain the derivatives of the given functions and express them in terms of hyperbolic functions.

3.[R] $\tanh(x)$

4.[R] $\sinh(x)$

5.[R] $\cosh(x)$

6.[R]

(a) Compute $\sinh(t)$ and $\cosh(t)$ for $t = -3, -2, -1$, 0, 1, 2, and 3.

(b) Plot the seven points $(x, y) = (\cosh(t), \sinh(t))$ found in (a).

(c) Explain why the point plotted in (b) lie on the hyperbola $x^2 - y^2 = 1$.

7.[R]

(a) Show that $\text{sech}^2(x) + \tanh^2(x) = 1$.

(b) What equation links $\sec(\theta)$ and $\tan(\theta)$?

In Exercises 8 to 16 use the definitions of the hyperbolic functions to verify the given identities. Notice how they differ from the corresponding identities for the trigonometric functions. In Section 12.6, with the aid of complex numbers, the hyperbolic functions are expressed in terms of the trigonometric functions.

8.[R] $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

9.[R] $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

10.[R] $\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}$

11.[R] $\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$

12.[R] $\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$

13.[R] $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$

14.[R] $\sinh(2x) = 2\sinh(x)\cosh(x)$

15.[R] $2\sinh^2(x/2) = \cosh(x) - 1$

16.[R] $2\cosh^2(x/2) = \cosh(x) + 1$

In Exercises 17 to 19 obtain a formula for the given function.

17.[M] $\text{arctanh}(x)$

18.[M] $\text{arcsech}(x)$

19.[M] $\text{arccosh}(x)$

In Exercises 20 to 23 show that the derivative of the first function is the second function.

20.[M] $\text{arccosh}(x)$; $1/\sqrt{x^2-1}$

21.[M] $\text{arcsinh}(x)$; $1/\sqrt{x^2+1}$

22.[M] $\text{arcsech}(x)$; $1/(x\sqrt{1-x^2})$

23.[M] $\text{arccsch}(x)$; $1/(x\sqrt{1+x^2})$

24.[M] Find the inflection points on the curve $y = \tanh(x)$.

25.[M] Graph $y = \sinh(x)$ and $y = \text{arcsinh}(x)$ relative to the same axes. Show any inflection points.

26.[C] One of the applications of hyperbolic functions is to the study of motion in which the resistance of the medium is proportional to the square of the velocity. Suppose that a body starts from rest and falls x meters in t seconds. Let g (a constant) be the acceleration due to gravity. It can be shown that there is a constant $V > 0$ such that

$$x = \frac{V^2}{g} \ln\left(\cosh\left(\frac{gt}{V}\right)\right).$$

(a) Find the velocity $v(t) = dx/dt$ as a function of t.

(b) Show that $\lim_{t\to\infty} v(t) = V$.

(c) Compute the acceleration $a(t) = dv/dt$ as a function of t.

(d) Show that the acceleration equals $g - g(v/V)^2$.

(e) What is the limit of the acceleration as $t \to \infty$?

27.[C] In this exercise you will discover two different formulas for an antiderivative of $f(x) = \frac{1}{\sqrt{ax+b}\sqrt{cx+d}}$. The correct formula to use depends on the signs of a and c.

(a) Show that $\frac{2}{\sqrt{-ac}} \arctan\sqrt{\frac{-c(ax+b)}{a(cx+d)}}$ is an antiderivative of $f(x)$ when $a > 0$ and $c < 0$.

(b) Show that $\frac{2}{\sqrt{ac}} \operatorname{arctanh}\sqrt{\frac{c(ax+b)}{a(cx+d)}}$ is an antiderivative of $f(x)$ when $a > 0$ and $c > 0$.

5.S Chapter Summary

This chapter shows the derivative at work; applying it to practical problems, estimating errors, and evaluating some limits.

To determine the extrema of some quantity one must determine a function that represents how the quantity depends on other quantities. Then, finding the extrema is like finding the highest or lowest points on the graph of the function.

When two varying quantities are related by an equation, the derivative can tell the relation between the rates at which they change: just differentiate both sides of the equation that relates them. That differentiation depends on the chain rule and is called implicit differentiation because one differentiates a function without having an explicit formula for it.

The next two sections form a unit that rests one of the main uses of higher derivatives: to estimate errors when approximating a function by a polynomial and later, in Section 6.5, to estimate errors in approximating area under a curve by trapezoids and parabolas.

The key to the Growth Theorem is that if R is a function such that

$$0 = R(a) = R'(a) = R''(a) = R^{(n)}(a)$$

and in some interval around a we know $R^{(n+1)}(x)$ is continuous, then there is a number c_n in $[a, x]$ such that

$$|R(x)| \leq R^{(n+1)}(c_n)\frac{(x-a)^{n+1}}{(n+1)!} \qquad \text{for all } x \text{ in that interval.}$$

That means we have information on how rapidly $R(x)$ can grow for x near a. This information was used to control the error when using a polynomial to approximate a function.

A likely candidate for the polynomial of degree n that closely resembles a given function f near $x = a$ is the one whose derivatives at a, up through order n, agree with those of f there. That polynomial is

$$P(x) = P_n(x; a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Because the polynomial was chosen so that $P^{(k)}(a) = f^{(k)}(a)$ for all k up through n, the remainder function function $R(x) = f(x) - P(x)$ has all its derivatives up through order n at a equal to 0. Moreover, since the $(n+1)^{\text{st}}$ derivative of any polynomial of degree at most n is identically 0, $R^{(n+1)}(x) = f^{(n+1)}(x)$. Thus the error $|f(x) - P(x)|$ is at most $M\frac{|x-a|^{n+1}}{(n+1)!}$, if $|f^{(n+1)}(t)|$ stays less than or equal to M for t between a and x. A similar conclusion holds if

$|f^{n+1}(t)|$ stays larger than a fixed number. Using these facts we obtained Lagrange's formula for the error:

$$\frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c_n \text{ between } a \text{ and } x.$$

The case $n = 1$ reduces to the linear approximation of a curve by the tangent line at $(a, f(a))$. In this case the error is controlled by the second derivative.

We return to Taylor polynomials in Chapter 12, where we express e^x, $\sin(x)$, and $\cos(x)$ as "polynomials of infinite degree," and use them and complex numbers to express $\sin(x)$ and $\cos(x)$ in terms of exponential functions.

Section 5.5 concerns l'Hôpital's rule, a tool for computing certain limits, such as the limit of a quotient whose numerator and denominator both approach zero.

The final two sections, on natural growth and decay and the hyperbolic functions, conclude the chapter. While these sections are not needed in future chapters of this book, they are important applications in a wide variety of disciplines, including biology and engineering.

EXERCISES for 5.S *Key:* R–routine, M–moderate, C–challenging

1.[R] Arrange the following numbers by order of increasing size as $x \to \infty$.

(a) $1000x$

(b) $\log_2(x)$

(c) $\sqrt{x}$

(d) $(1.0001)^x$

(e) $\log_{1000}(x)$

(f) $0.01x^3$

In Exercises 2 to 28 find the limits, if they exist.

2.[R] $\lim_{u\to\infty} \left(\frac{u+1}{u}\right)^{u+1} \frac{1}{\sqrt{u}}$

3.[R] $\lim_{x\to\infty} \left(\frac{x+2}{x+1}\right)^{x+3}$

4.[R] $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{x+1}$

5.[R] $\lim_{x\to 3} \frac{x-2}{\cos(\pi x)}$

6.[R] $\lim_{x\to 3} \frac{x-2}{\sin(\pi x)}$

7.[R] $\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x}$

8.[R] $\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$

9.[R] $\lim_{x\to\infty} \frac{(1+x^2)^{1/2}}{(2+x^2)^{1/3}}$

10.[R] $\lim_{x\to\infty} \frac{1+x+x^2}{2+3x+4x^2}$

11.[R] $\lim_{x\to 1} \frac{\ln(x)\tan\left(\frac{\pi x}{4}\right)}{\cos\left(\frac{\pi x}{2}\right)}$

12.[R] $\lim_{x\to 0} \frac{f(3+x)-f(3)}{x}$ where, $f(t) = (x^2 + 5)\sin^2(3x)$.

13.[R] $\lim_{x\to\infty} \frac{\ln(6x)-\ln(5x)}{\ln(7x)-\ln(6x)}$

14.[R] $\lim_{x\to\infty} \frac{\ln(6x)-\ln(5x)}{x\ln(7x)-x\ln(6x)}$

16.[R] $\lim_{x\to\pi} \frac{\ln(x^3-\sin(x))-3\ln(\pi)}{x-\pi}$

17.[R] $\lim_{x\to 0} \frac{(x+2)}{(x+3)} \frac{(\cos(5x)-1)}{\cos(7x)-1)}$

18.[R] $\lim_{x\to\infty} \left(\frac{x+2}{x+1}\right)^{2x}$

19.[R] $\lim_{x\to\pi} \frac{\sin^4(x)}{(\pi^4-x^4)^2}$

20.[R] $\lim_{x\to\infty} \frac{\sec^4(x)\tan(3x)}{\sin(2x)}$

21.[R] $\lim_{x\to 1} \frac{e^{3x}(x^2-1)}{\cos(\sqrt{2}x)\tan(3x-3)}$

22.[R] $\lim_{x\to 0}(1 + 0.005x)^{20x}$

23.[R] $\lim_{t\to 0} \frac{e^{3(x+t)}-e^{3x}}{5t}$

24.[R] $\lim_{t\to 0} \frac{e^{3(x+t)}-e^{3x}}{5t}$

25.[R] $\lim_{x\to 0} \left(\frac{1+2^x}{2}\right)^{1/x}$

26.[R] $\lim_{x\to 0} \left(\frac{1+2^x}{1+3^x}\right)^{1/x}$

27.[R] $\lim_{x\to\infty}(1 + 0.003x)^{20/x}$

In Exercises 29 to 36 find the derivative of the given function.

29.[R] $(\cos(x))^{1/x^2}$

30.[R] $\ln\left(\sec^2(3x)\sqrt{1+x^2}\right)$

31.[R] $\ln\left(\sqrt{e^{x^3}}\right)$

32.[R] $\frac{5+3x+7x^2}{58-4x+x^2}$

33.[R] $\frac{\tan^2(2x)}{(1+\cos(2x))^4}$

34.[R] $(\cos^2(3x))^{\cos^2(2x)}$

35.[R] $f(x) = \begin{cases} x^2\sin(\pi/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
HINT: Use the definition of the derivative to find $f'(0)$.

36.[R] $f(x) = \begin{cases} \frac{\sin(\pi x)}{x} & \text{if } x \neq 0 \\ \pi & \text{if } x = 0 \end{cases}$

37.[R]

(a) Find $P_1(x;64)$ for $f(x)=\sqrt{x}$.

(b) Use $P_1(x;64)$ to estimate $\sqrt{67}$.

(c) Put bounds on the error in the estimate in (b).

38.[R]

(a) Show that when x is small $\sqrt[3]{1+x}$ is approximately $1+x/3$.

(b) Use (a) to estimate $\sqrt[3]{0.94}$ and $\sqrt[3]{1.06}$.

39.[R]

(a) Show that when x is small $1/\sqrt[3]{1+x}$ is approximately $1-x/3$.

(b) Use (a) to estimate $\sqrt[3]{0.94}$ and $\sqrt[3]{1.06}$.

40.[R]

(a) Find the Maclaurin polynomial of degree 6 associated with $\cos(x)$.

(b) Use (a) to estimate $\cos(\pi/4)$.

(c) What is the error between the estimate found in (b) and the exact value, $\sqrt{2}/2$.

(d) What is the Lagrange bound for the error?

In Exercises 41 to 52, examine the limit, determine whether it exists, and, if it does exist, find its value.

41.[R] $\lim_{x\to 1}\frac{1-e^x}{1-e^{2x}}$

42.[R] $\lim_{x\to 0}\frac{x}{\sqrt{1+x^2}}$

43.[R] $\lim_{x\to 0}\frac{1-e^x}{1-e^{2x}}$

44.[R] $\lim_{x\to\infty}\frac{x^2}{(1+x^3)^{2/3}}$

45.[R] $\lim_{x\to\infty}x^2\sin(x)$

46.[R] $\lim_{x\to 8}\frac{2^x-2^8}{x-8}$

47.[R] $\lim_{x\to 1}\frac{e^{x^2}-e^x}{x-1}$

48.[R] $\lim_{x\to 4}\frac{2^x+2^4}{x+4}$

49.[R] $\lim_{x\to 0}\frac{\sin(x)-e^{2x}}{x}$

50.[R] $\lim_{x\to 0}\frac{e^{3x}\sin(2x)}{\tan(3x)}$

51.[R] $\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{\sqrt[3]{1+x^2}-1}$

52.[R] $\lim_{x\to\pi/2}\frac{\sin 9x)\cos(x)}{x-\pi/2}$

53.[R] If $\lim_{x\to\infty}f'(x)=3$ and $\lim_{x\to\infty}g'(x)=3$, what, if anything, can be said about

(a) $\lim_{x\to\infty}\frac{f(x)}{3x}$

(b) $\lim_{x\to\infty}(g(x)-f(x))$

(c) $\lim_{x\to\infty}\frac{f(x)}{g(x)}$

(d) $\lim_{x\to\infty}(f(x)-3x)$

(e) $\lim_{x\to\infty}\frac{(f(x))^3}{(g(x))^3}$

54.[M] The point $P = (c, d)$ lies in the first quadrant. Each line through P of negative slope determines a triangle whose vertices are the intercepts of the line on the axes, and the origin.

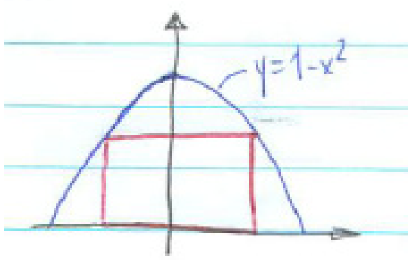
(a) Find the slope of the line that minimizes the area.

(b) Find the minimum area.

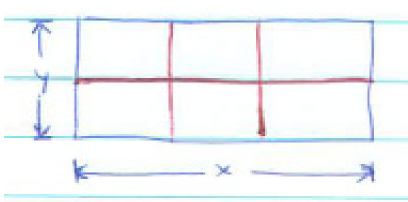
55.[M] Figure 5.S.1(a) shows a typical rectangle whose base is the x-axis, inscribed in the parabola $y = 1 - x^2$.

(a) Find the rectangle of largest perimeter.

(b) Find the rectangle of largest area.



(a)



(b)

Figure 5.S.1:

56.[M] A rectangle of perimeter 12 inches is spun around one of its edges to produce a circular cylinder.

(a) For which rectangle is the area of the curved surface of the cylinder a maximum?

(b) For which rectangle is the volume of the cylinder a maximum?

57.[M] Consider isosceles triangles whose equal sides have length a and the angle where these two sides meet is θ. For which angle θ is the area of the triangle a maximum?

(a) Solve this problem using calculus.

(b) Solve the same problem without calculus.

58.[M] A farmer has 200 feet of fence which he wants to use to enclose a rectangle divided into six congruent rectangles, as shown in Figure 5.S.1(b). He wishes to enclose a maximum area.

(a) If x is near 0, what is the area, approximately?

(b) How large can x be?

(c) In the case that produces the maximum area, which do you think will be larger x or y? Why?

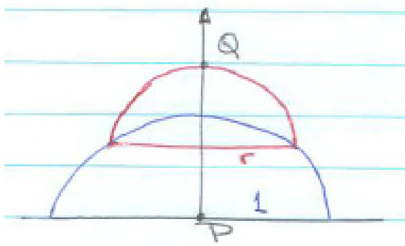
(d) Find the dimensions x and y that maximizes the area.

59.[M] A semicircle of radius $a < r \leq 1$ rests upon a semicircle of radius 1, as shown in Figure 5.S.2(a). The length of PQ, the segment from the origin of the lower circle to the top of the upper circle is a function of r, $f(r)$.

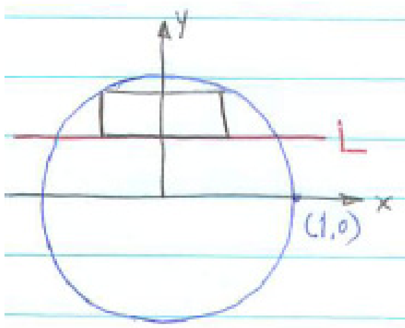
(a) Find $f(0)$ and $f(1)$.

(b) Find $f(r)$.

(c) Maximize $f(r)$, testing the maximum by the second derivative.



(a)



(b)

Figure 5.S.2:

Exercises 60 to 62 are independent, but related. They contain a surprise.

60.[M] Figure 5.S.2(b) shows the unit circle $x^2+y^2=1$, the line L whose equation is $y=1/3$, and a typical rectangle with base on L, inscribed in the circle. Find the rectangle with base on L that has (a) minimum perimeter and (b) maximum perimeter.

61.[M] Like Exercise 60 but this time the line L has the equation $y=1/2$.

62.[M] The analyses in Exercises 60 to 61 are different. Let the line L have the equation $y=c$, $0<c<1$. For which values of c is the analysis like that for (a) Exercise 60? (b) Exercise 61?

63.[M] A. Bellemans, in "Power Demand in Walking and Pace Optimization," Amer. J. Physics 49(1981) pp. 25–27, modeling the work spent on walking writes "$H = L(1-\cos(\gamma)$ or, to a sufficient approximation for the present purpose, $H = L\gamma^2/2$." Justify this approximation.

64.[C] Let k be a constant. Determine $\lim_{x\to\infty} x\left(e^{-k} - \left(1-\frac{k}{x}\right)^x\right)$.

65.[C] Let k be a constant. Determine $\lim_{x\to\infty} x\left(e^{k} - \left(1+\frac{k}{x}\right)^x\right)$.

66.[M] Let $p_n(x)$ be the Maclaurin polynomial of degree n associated with e^x. Because $e^x \cdot e^{-x} = 1$, we might expect that $p_n(x)p_n(-x)$ would also be 1. But that cannot be because the degree of the product is $2n$.

(a) Compute $p_2(x)p_2(-x)$ and $p_3(x)p_3(-x)$.

(b) Make a conjecture about $p_n(e^x)p_n(e^{-x})$ based on (a).

67.[M] Let $p_n(x)$ be the Maclaurin polynomial of degree n associated with e^x. Because $e^{2x} = e^x \cdot e^x$, we might expect that $p_{2n}(x) = p_n(x)p_n(x)$.

(a) Why is that equation false for $n \geq 1$?

(b) To what extent does $p_2(x)p_2(x)$ resemble $p_2(2x)$ and $p_3(x)p_3(x)$ resemble $p_3(2x)$?

(c) Make a conjecture based on (a) and (b).

68.[M] Let $p_n(x)$ be the Maclaurin polynomial of degree n associated with e^x. The equation $e^{x+y} = e^x \cdot e^y$ suggests that $p_n(x+y)$ might equal $p_n(x)p_n(y)$.

(a) Why is that hope not realistic?

(b) To what extent does $p_2(x)p_2(y)$ resemble $p_2(x+y)$?

69.[M] What can be said about $f(10)$ if $f(1)=5$, $f'(1)=3$ and $2, f''(x) < 4$ for x in $(-10,20)$?

70.[M] The demand for a product is influenced by its price. In one example an economics text links the amount sold (x) to the price (P) by the equation $x = b - aP$, where b and a are positive constants. As the price increases the sales go down. The cost of producing x items is an increasing function $C(x) = c+kx$, where c and k are positive constants.

(a) Express P in terms of x.

(b) Express the total revenue $R(x)$ in terms of x.

(c) Note that $C(0) = c$. So what is the economic significance of c?

(d) What is the economic significance of k?

(e) Let $E(x)$ be the profit, that is, the revenue minus the cost. Express $E(x)$ as a function of x.

(f) Which value of x produces the maximum profit?

(g) The marginal revenue is defined as dR/dx and the marginal cost as dC/dx. Show that for the value of x that produces the maximum profit, $dR/dx = dC/dx$.

(h) What is the economic significance of $dR/dx = dC/dx$ in (g)?

71.[M] This exercise concerns a function used to describe the consumption of a finite resource, such as petroleum. Let Q be the amount initially available. Let a be a positive constant and b be a negative constant. Let $y(t)$ be the amount used up by the time t. The function $Q/(1 + ae^{bt})$ is often used to represent $y(t)$.

(a) Show that $\lim_{t\to\infty} y(t) = Q$ and $\lim_{t\to-\infty} y(t) = 0$. Why are these realistic?

(b) Show that $y(t)$ has an inflection point when $t = -\ln(a)/b$.

(c) Show that at the inflection point, $y(t) = Q/2$, that is, half the resource has been used up.

(d) Sketch the graph of $y(t)$.

(e) Where is $y'(t)$, the rate of using the resource, greatest?

NOTE: The same function describes limited growth that is bounded by Q, so called **logistic growth**.

72.[M] About 100 cubic yards are added to a land fill every day. The operator decides to pile the debris up in the form of a cone whose base angle is $\pi/4$. (He hopes to make a ski run where it never snows.) At what rate is the height of the cone increasing when the height is (a) 10 yards? (b) 20 yards? (c) 100 yards? (d) How long will it take to make a cone 100 yards high? 300 yards high? NOTE: The volume of a circular cone is one third the product of its height and the area of its base.

73.[M] A wine dealer has a case of wine that he could sell today for \$100. Or, he could decide to store it, letting it mellow, and sell later for a higher price. Assume he could sell in t years for \$ $100e^{\sqrt{t}}$. In order to decide which option to choose he computes the present value of the sale. If the interest rate is r, the present value of one dollar t years hence is e^{-rt}. When should he sell the wine?

74.[M]

(a) Estimate $\int_0^1 \frac{\sin(x)}{x} dx$ by using the Maclaurin polynomial $P_6(x;0)$ associated with $\sin(x)$ to approximate $\sin(x)$.

(b) Use the Lagrange form of the error to put an upper bound on the error in (a).

75.[M] A differentiable function is defined throughout $(-\infty, \infty)$. Its derivative is 0 at exactly two inputs.

(a) Can there be exactly one relative extremum?

(b) Could it have two relative maxima?

(c) What is the maximum number of relative extrema possible?

(d) What is the minimum number?

HINT: Sketch graphs, then explain.

76.[M] A differentiable function is defined throughout $(-\infty, \infty)$. Its derivative is 0 at exactly three inputs. and the function approaches 0 as x approaches ∞

(a) Can there be exactly two relative extremum?

(b) Could it have three relative maxima?

(c) What is the maximum number of relative extrema possible?

(d) What is the minimum number?

HINT: Sketch graphs, then explain.

77.[M] A differentiable function is defined throughout $(-\infty, \infty)$. Its derivative is 0 at exactly two inputs. and the function approaches the same finite limit as x approaches ∞ and $-\infty$.

(a) Can there be exactly one relative extremum?

(b) Could it have two relative maxima?

(c) What is the greatest number of relative extrema possible?

(d) What is the least number?

Hint: Sketch graphs, then explain.

78.[C] In the paper cited in the Exercise 63, Bellemans writes "The total mechanical power required for walking is $P(v, a) = \alpha M v^3/a + (\beta M g v)/L)a$. Enlarging the pace, a, at a constant speed v, lowers the first term and increases the second one so that the formula predicts an optimal pace $a^*(v)$, minimizing $P(v, a)$." In the formula, α, M, v, β, g, and L are constants.

(a) Show that $a^*(v) = \left(\frac{\alpha}{\beta}\right)^{1/2} \left(\frac{L}{g}\right)^{1/2} v$

(b) Verify that the "corresponding minimum power" is

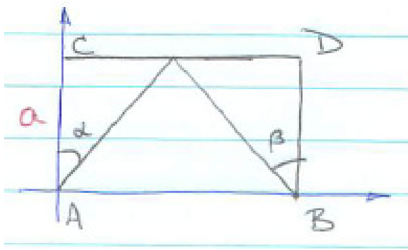
$$P(v, a^*(v)) = 2(\alpha\beta)^{1/2} \left(\frac{g}{L}\right)^{1/2} M v^2.$$

"One would therefore expect that, when walking naturally on the flat at a fixed velocity, a subject will adjust its pace automatically to the optimum value corresponding to the minimum work expenditure. This has indeed been verified experimentally."

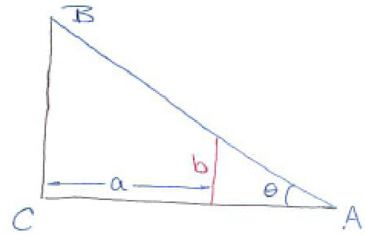
79.[C] Figure 5.S.3(a) shows two points A and B a mile apart and both at a distance a from the river CD. Sam is at A. He will walk in a straight line to the river at 4 mph, fill a pail, then continue on to B at 3 mph. He wishes to do this in the shortest time.

(a) For the fastest route which angle in Figure 5.S.3 do you expect to be larger, α or β?

(b) Show that for the fastest route $\sin(\alpha)/\sin(\beta)$ equals 4/3.



(a)



(b)

Figure 5.S.3:

80.[C] A fence b feet high is a feet from a tall building, whose wall contains BC. as shown in Figure 5.S.3(b). Find the angle θ that minimizes the length of AB. (That angle produces the shortest ladder to reach the building and stay above the fence.)

81.[C]

(a) Show that if a differentiable function f is even, then f' is odd, by differentiating both sides of the equation $f(-x) = f(x)$.

(b) Explain why the conclusion in (a) is to be expected by interpreting it in terms of the graph of f.

82.[C] Show that if a differentiable function is odd, then its derivative is even.

83.[C] What do the previous two exercises imply about a Maclaurin polynomial associated with an odd function? associated with an even function?

84.[C] Show that

(a) If $p_n(x)$ is a Maclaurin polynomial associated with $f(x)$, then $p_n'(x)$ is a Maclaurin polynomial associated with $f'(x)$.

(b) Use (a) to find the 6^{th}-order Maclaurin polynomial for $1/(1-x)^2$.

85.[C] (Assume $e < 3$.) Let $P_1(x)$ be $P_1(x;0)$ for e^x. For how large an x can you be sure that

(a) $|e^x - P_1(x)| < 0.01$?

(b) $|e^x - P_2(x)| < 0.01$?

(c) $|e^x - P_3(x)| < 0.01$?

86.[C] A number b is **algebraic** if there is a non-zero polynomial $\sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, with coefficients a_i that are rational numbers, such that $\sum_{i=0}^{n} a_i b^i = 0$. In other words, b is algebraic if there is a function f that satisfies (a) $f(b) = 0$, (b) all derivatives of f at 0 are rational, but not all zero, and (c) there is a positive integer m such that $D^m(f) = 0$. (Recall that D is the differentiation operator.)

We call a number b **almost algebraic** if (a) b is not algebraic and there is a function f with (b) $f(b) = 0$, (c) all derivatives of f at 0 are rational, but not all zero, and (d) there is a non-zero polynomial $p(D)$ such that $p(D)(f) = 0$. For example, if $p(x) = x^2 + 1$ then $p(D)(f) = D^2(f) + f = f'' + f$.

Show that π is almost algebraic. (Assume it is not algebraic.)

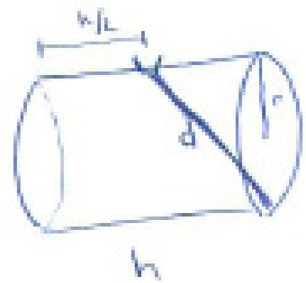


Figure 5.S.4: ARTIST: Show wine level inside the barrel.

87.[M] Kepler, the astrologer and astronomer, to celebrate his wedding in 1613, ordered some wine, which was available in cylindrical barrels of various shapes. He was surprised by the way the merchant measure the volume of a barrel. A ruler was pushed through the opening in the side of the barrel (used to fill the barrel) until it came to a stop at the edge of a circular base. The merchant used the length of the part of the ruler inside the barrel to determine the volume of the barrel. Figure 5.S.4 shows the method.

The barrel in Figure 5.S.4 has radius r, height h, and volume V. The length of the ruler inside the barrel is d.

(a) Using common sense, show that d does not determine V.

(b) How small can V be for a given value of d?

(c) Using calculus, show that the maximum volume for a given d occurs when $h = 2\sqrt{2}d/\sqrt{6}$ and $r = d/\sqrt{6}$.

(d) Show that to maximize the volume the height must be $\sqrt{2}$ times the diameter. (This is what Kepler showed.)

NOTE: Try to solve this problem two different ways. One without implicit differentiation and the other with implicit differentiation.

88.[M] Let m and n be positive numbers. Find the maximum and minimum values of $m\sin(x) + n\cos(x)$.

89.[M] Let m and n be positive integers. Let $f(x) = \sin^m(x)\cos^n(x)$ for x in $[0, \pi/2]$.

(a) For which x is $f(x)$ a minimum?

(b) For which x is $f(x)$ a maximum?

(c) What is the maximum value of $f(x)$?

90.[M]

(a) Let $P(x)$ be a polynomial such that $D^2(x^2P(x)) = 0$. Show that $P(x) = 0$.

(b) Does the same conclusion follow if instead we assume $D^2(xP(x)) = 0$?

HINT: If $P(x)$ has degree n, what are the degrees of $xP(x)$ and $x^2P(x)$?

91.[M] Translate this news item into the language of calculus: "The one positive sign during the quarter was a slowing in the rate of increase in home foreclosures."

92.[M] In May 2009 it was reported that "the nation's industrial production fell in April by the smallest amount in six months, fresh evidence that the pace of the economy's decline is slowing."
Let $P(t)$ denote the total production up to time t with t representing the number of months since January 2000 $(t = 0)$.

(a) Translate the above statement into the language of calculus, that is, in terms of $P(t)$ and its derivatives (evaluated at appropriate values of t).

(b) Sketch a possible graph of $P(t)$ for November 2008 through April 2009.

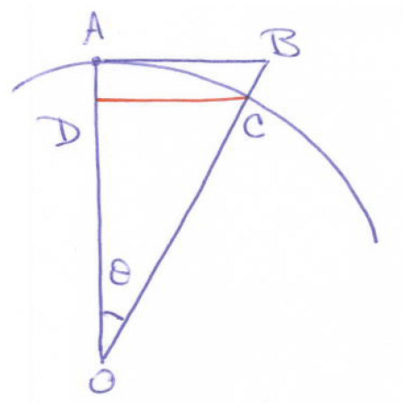


Figure 5.S.5:

93.[M] (A challenge to your intuition.) In Figure 5.S.5 AB is tangent to an arc of a circle, OA is a radius and DC is parallel to AB.

(a) What do you think happens to the ratio of the area of ABC to the area of ADC as $\theta \to 0$?

(b) Using calculus, find the limit of that ratio as $\theta \to 0$.

(c) In view of (b), which provides a better estimate of the area of a disk, the circumscribed regular n-gon or the inscribed regular n-gon?

(d) In view of the limit in (b), what combination of the estimates by the inscribed regular n-gon and the circumscribed regular n-gon, would likely provide a very good estimate of the area of the disk?

94.[M] Let $f(x)$ be a function having a second derivative at a. Supply all the steps to show that the second-order polynomial $g(x)$ such that $g(a) = f(a)$, $g'(a) = f'(a)$, and $g''(a) = f''(a)$ is given by $g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$.

95.[M] Let f and g be differentiable.

(a) If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 3$, must $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exist and be 3?

(b) If the second limit in (a) exists, can it have a value other than 3?

96.[M] Use Taylor polynomials, and their errors, to show that in an open interval in which f'' is positive, tangents to the graph of f lie below the curve. As in Exercise 49 in Section 4.4, you want to show that if a and x are in the interval, then $f(x) > f(a) + f'(a)(x - a)$. It is necessary to treat the cases $x > a$ and $x < a$ separately.

97.[M] Evaluate each limit, indicating the indeterminate form each time l'Hôpital's Rule is applied.

(a) $\lim_{x \to 0} \left(\frac{1 + 2^x}{2} \right)^{1/x}$

(b) $\lim_{x \to 0} \left(\frac{1 + 2^x}{1 + 3^x} \right)^{1/x}$

98.[C]

Sam: I can use Taylor polynomials to get l'Hôpital's theorem.

Jane: How so?

Sam: I write $f(x) = f(0) + f'(0)x + f''(c)x^2/2$ and $g(x) = g(0) + g'(0)x + g''(d)x^2/2$.

Jane: O.K.

Sam: Since $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ are both zero I have $f(0) = g(0) = 0$. I can write, after canceling some x's

$$\frac{f(x)}{g(x)} = \frac{f'(0) + f''(c)x/2}{g'(0) + g''(d)x/2}.$$

Jane: But you don't know the second derivatives.

Sam: It doesn't matter. I just take limits and get

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(0) + f''(c)x/2}{g'(0) + g''(d)x/2}.$$

So

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}.$$

There you have it.

Jane: Let me check your steps.

Check the steps and comment on Sam's proof.

When you throw a fair six-sided die many times, you would expect a 5 to show about 1/6 of the times. That is, if you throw it n times and get k 5's, you would expect k/n to be near 1/6.
More generally, if a certain trial has probability p of success and $q = 1 - p$ of failure, and is repeated n times, with k successes, you would expect k/n to be near p. That means that if n is large you would expect $(k/n) - p$ to be small. In other words, let $\epsilon = (k/n) - p$, where ϵ approaches 0 as $n \to \infty$. This means that in most cases $= np + \epsilon n$, or $k = np + z$, where $z/n \to 0$ as $n \to \infty$.
The probability of exactly k successes (and $n - k$ failures) in n trials is

$$\frac{n!}{k!(n-k)!}p^k q^{n-k}. \tag{5.S.1}$$

Exercises 99 to 103 show that for large n (and k) (5.S.1) is approximately

$$\frac{1}{\sqrt{2\pi npq}} \exp\left(\frac{-z^2}{2npq}\right). \tag{5.S.2}$$

Note that (5.S.2) involves $\exp(-x^2)$, whose graph has the shape of the famous bell curve associated with the normal (or Gaussian) distribution in probability and statistics.

99.[C] In Exercise 9 in Section 11.6 we will derive Stirling's formula for an approximation to $n!$:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Use Stirling's formula to show that (5.S.1) is approximately

$$\left(\frac{n}{2\pi k(n-k)}\right)^{1/2}\left(\frac{np}{k}\right)^k\left(\frac{nq}{n-k}\right)^{n-k} \tag{5.S.3}$$

in the sense that (5.S.2) divided by (5.S.3) approaches 1 as $n \to \infty$.

100.[M] Show that as $n \to \infty$, the first factor in (5.S.3) is asymptotic to

$$\left(\frac{1}{2\pi pqn}\right)^{1/2} \tag{5.S.4}$$

in the sense that the ratio between it and (5.S.4) approaches 1 as $n \to \infty$.

101.[M] To relate the rest of (5.S.3) to the exponential function, $\exp(x)$, we take its logarithm. Show that

$$\ln\left(\left(\frac{np}{k}\right)^k\left(\frac{nq}{n-k}\right)^{n-k}\right) = -(np+z)\ln\left(1+\frac{z}{np}\right) - (nq-z)\ln\Bigg(\tag{5.S.5}$$

102.[M] Using the Maclaurin polynomial of degree two to approximate $\ln(1+t)$, show that for large n, (5.S.5) is approximately

$$\frac{-z^2}{2pqn}.$$

103.[M] Conclude that for large n, (5.S.1) is approximately (5.S.2).

104.[M] When studying the normal distribution in statistics one will meet an equation that amounts to

$$\frac{\int_{-\infty}^{\infty} x\exp(-(x-\mu)^2)\,dx}{\int_{-\infty}^{\infty} \exp(-(x-\mu)^2)\,dx} = \mu,$$

where μ is a constant. Show that the equation is correct. Hint: Make the substitution $t = x - \mu$.

105.[M] Show that $\int_1^{\infty} x\exp(-x^2)\,dx$ is less than $\int_0^1 x\exp(-x^2)\,dx$. This implies that the area in the "tail" of the bell curve is fairly small in spite of the growth of the coefficient x. As a result, economic predictions based on the bell curve may downplay the likelihood of rare events. This bias may have been one of the several factors that combined to produce the credit crisis and recession that began in 2007.

106.[M] If $P(x)$ is a Maclaurin polynomial associated with $f(x)$, is $P(-x)$ a Maclaurin polynomial associated with $f(-x)$?

107.[M] If $P(x)$ is a Maclaurin polynomial associated with $f(x)$, what is the Maclaurin polynomial of the same degree associated with $f(2x)$?

108.[M] Find the Maclaurin polynomial of degree 6 associated with $1/e^x$.

109.[M] Find the Maclaurin polynomial of degree 6 associated with $\sin(x)\cos(x)$.

110.[M] The center $(x, 0)$, $x > 0$, of a circle C_1 of radius 1 is at a distance x from the center $(0, 0)$ of a circle C_2 of radius 2. AB is the chord joining their two points in common. Let A_1 be the area within C_1 to the left of that chord and A_2 the area within C_2 to the right of that chord.

(a) Which is larger, A_1 or A_2? Hint: Sketch a diagram of these circles and the chord.

(b) If $\lim_{x\to 3^-} A_2/A_1$ exists, what do you think it is?

(c) Determine whether the limit in (b) exists. If it does, find it.

111.[M] In the setup of Exercise 110, let O_1 be the center of C_1 and O_2 the center of C_2. What happens to the ratio of the area common to the two disks and the area of the quadrilateral AO_1BO_2 as $x \to 3^-$?

112.[M] Let $g(x) = f(x^2)$.

(a) Express the Maclaurin polynomial for $g(x)$ up through the term of degree 4 in terms of f and its derivatives.

(b) How is the answer in (a) related to a Maclaurin polynomial associated with f?

113.[M] Find $\lim_{x\to\pi/2^-}(\sec(x) - \tan(x))$

(a) Using l'Hôpital's rule

(b) Without using l'Hôpital's rule

114.[M] Assume that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$.

(a) If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, what, if anything, can be said about $\lim_{x\to\infty} \frac{ln(f(x))}{ln(g(x))}$?

(b) If $\lim_{x\to\infty} \frac{ln(f(x))}{ln(g(x))} = 1$, what, if anything, can be said about $\lim_{x\to\infty} \frac{f(x)}{g(x)}$?

115.[C] Assume that the function $f(x)$ is defined on $[0, \infty)$, has a continuous positive second derivative and $\lim_{x\to\infty} f(x) = 0$.

(a) Can $f(x)$ ever be negative?

(b) Can $f'(x)$ ever be positive?

(c) What are the possible general shapes for the graph of f?

(d) Give an explicit formula for an example of such a function.

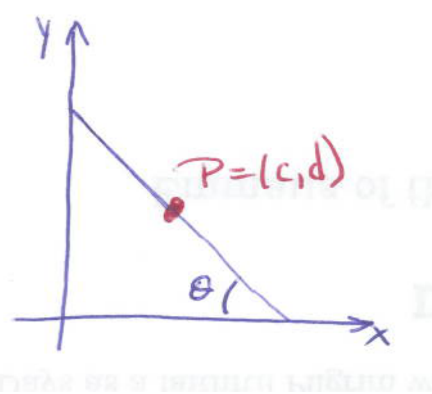


Figure 5.S.6:

116.[C] Let c and d be fixed positive numbers. Consider line segments through $P = (c, d)$ whose ends are on the positive x- and y-axes, as in Figure 5.S.6.

Let θ be the acute angle between the line and the x-axis. Show that the angle α that produces the shortest line segment through P has $\tan^3(\alpha) = d/c$.

117.[C] (See Exercise 116.)

(a) Show that for the angle β such that the the area of the triangle determined by the line segment and the two axes is a minimum, $\tan(\beta) = d/c$.

(b) Show that for β as in (a), OP bisects the line into two parts of equal length.

118.[C] An adventurous bank decides to compound interest twice a year, at time x $(0 < x < 1)$ and at time 1 (instead of at the usual 1/2 and 1). Assume that the annual interest rate is r. Is there a time, x, such that the account grows to more than if the interest was computed at 1/2 and 1?

119.[C] Every six hours a patient takes an amount A of a medicine. Once in the patient, the medicine decays exponentially. In six hours the amount declines from A to kA, where k is less than 1 (and positive). Thus, in 12 hours, the amount in the system is $kA + k^2A$. At exactly 12 hours, the patient takes another pill and the amount in her system is $A + kA + k^2A$.

(a) Graph the general shape of the sketch showing the amount of medicine in the patient as a function of time.

(b) When a pill is taken at the end of n six-hour periods how much is in the system?

(c) Does the amount in the system become arbitrarily large? (If so, this could be dangerous.)

The constant k depends on many factors, such as the age of the patient. For this reason, a dosage tested on a 20-year old may be lethal on a 70-year.

Skill Drill: Derivatives

The remaining exercises offer an opportunity to practice differentiating. In each case show that the derivative of the first function is the second function.

120.[M] $\arctan\left(\frac{x}{a}\right)$; $\frac{a}{x^2+a^2}$.

121.[M] $\frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$; $x\sqrt{ax+b}$.

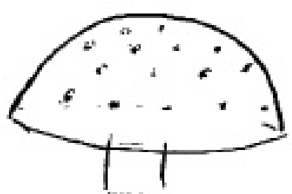
122.[M] $\sin(ax) - \frac{1}{3}\sin^3(ax)$; $a\cos^3(ax)$.

123.[M] $e^{ax}(a\cos(bx) + b\sin(bx))$; $(a^2 + b^2)e^{ax}\cos(bx)$.

124.[M] Let $f(x) = (5x^3 + x + 2)^{20}$. Find (a) $f^{(60)}(4)$ and (b) $f^{(61)}(2)$.

Calculus is Everywhere # 6
The Uniform Sprinkler

One day one of the authors (S.S.) realized that the sprinkler did not water his lawn evenly. Placing empty cans throughout the lawn, he discovered that some places received as much as nine times as much water as other places. That meant some parts of the lawn were getting too much water and other parts not enough water.



The sprinkler, which had no moving parts, consisted of a hemisphere, with holes distributed uniformly on its surface, as in Figure C.6.1. Even though the holes were uniformly spaced, the water was not supplied uniformly to the lawn. Why not?

A little calculus answered that question and advised how the holes should be placed to have an equitable distribution. For convenience, it was assumed that the radius of the spherical head was 1, that the speed of the water as it left the head was the same at any hole, and air resistance was disregarded.

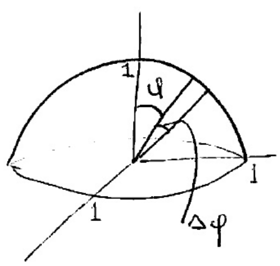


Figure C.6.2:

Consider the water contributed to the lawn by the uniformly spaced holes in a narrow band of width $d\phi$ near the angle ϕ, as shown in Figure C.6.2. To be sure the jet was not blocked by the grass, the angle ϕ is assumed to be no more than $\pi/4$.

Water from this band wets the ring shown in Figure C.6.3.

The area of the band on the sprinkler is roughly $2\pi\sin(\phi)\,d\phi$. As shown in Section 9.3, see Exercises 25 and 26, water from this band lands at a distance from the sprinkler of about

$$x = kv^2\sin(2\phi).$$

Here k is a constant and v is the speed of the water as it leaves the sprinkler. The width of the corresponding ring on the lawn is roughly

$$dx = 2kv^2\cos(2\phi)d\phi.$$

Since its radius is approximately $kv^2\sin(2\phi)$, its area is approximately

$$2\pi\left(kv^2\sin(2\phi)\right)\left(2kv^2\cos(2\phi)\ d\phi\right),$$

which is proportional to $\sin(2\phi)\cos(2\phi)$, hence to $\sin(4\phi)$.

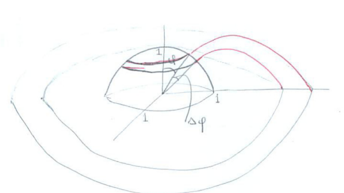


Figure C.6.3:

Thus the water supplied by the band was proportional to $\sin(\phi)$ but the area watered by that band was proportional to $\sin(4\phi)$. The ratio

$$\frac{\sin(4\phi)}{\sin(\phi)}$$

is the key to understanding both why the distribution was not uniform and to finding out how the holes should be placed to water the lawn uniformly.

By l'Hôpital's rule, this fraction approaches 4 as ϕ approaches zero:

$$\lim_{\phi \to 0} \frac{\sin(4\phi)}{\sin(\phi)} = 4. \tag{C.6.1}$$

This means that for angles ϕ near 0 that ratio is near 4. When ϕ is $\pi/4$, that ratio is $\frac{\sin(\pi)}{\sin(\pi/4)} = 0$, and water was supplied much more heavily far from the sprinkler than near it. To compensate for this bias the number of holes in the band should be proportional to $\sin(4\phi)/\sin(\phi)$. Then the amount of water is proportional to the area watered, and watering is therefore uniform.

Professor Anthony Wexler of the Mechanical Engineering Department of UC-Davis calculated where to drill the holes and made a prototype, which produced a beautiful fountain and a much more even supply of water. Moreover, if some of the holes were removed, it would water a rectangular lawn.

We offered the idea to the firm that made the biased sprinkler. After keeping the prototype for half a year, it turned it down because "it would compete with the product we have."

Perhaps, when water becomes more expensive our uniform sprinkler may eventually water many a lawn.

EXERCISES

1.[R] Show that the limit (C.6.1) is 4

(a) using only trigonometric identities.

(b) using l'Hôpital's rule.

2.[R] Show that $\sin(4x)/\sin(x)$ is a decreasing function for x in the interval $[0, \pi/4]$. HINT: Use trigonometric identities and no calculus. (However, you may be amused if you also do this by calculus.)

3.[R] An oscillating sprinkler goes back and forth at a fixed angular speed.

(a) Does it water a lawn uniformly?

(b) If not, how would you modify it to provide more uniform coverage?