Chapter 4

Derivatives and Curve Sketching

When you graph a function you typically plot a few points and connect them with (generally) straight line segments. Most electronic graphing devices use the same approach, and obtain better results by plotting more points and using shorter segments. The more points used, the smoother the graph will appear. This chapter will show you how to choose key points when sketching a graph.

Three properties of the derivative developed in Section 4.1, and proved in Section 4.4, will be used in Section 4.2 to help graph a function. In Section 4.3 we see what the second derivative tells about a graph.

4.1 Three Theorems about the Derivative

This section is based on plausible observations about the graphs of differentiable functions, which we restate as theorems. These ideas will then be combined, in Section 4.2, to sketch graphs of functions.

An effective approach to sketching graphs of functions is to find the extreme values of the function, that is, where the function takes on its largest and smallest values.

OBSERVATION *(Tangent Line at an Extreme Value)* Suppose that a function $f(x)$ attains its largest value when $x = c$, that is, $f(c)$ is the largest value of $f(x)$ over a given open interval that contains c. Figure 4.1.1 illustrates this. The maximum occurs at a point $(c, f(c))$, which we call P. If $f(x)$ is differentiable, at c, then the tangent line at P will exist. What can we say about it?

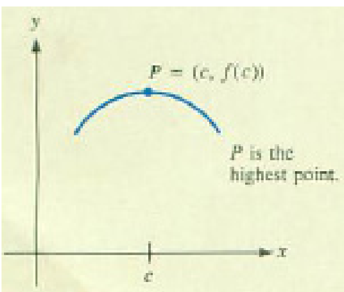
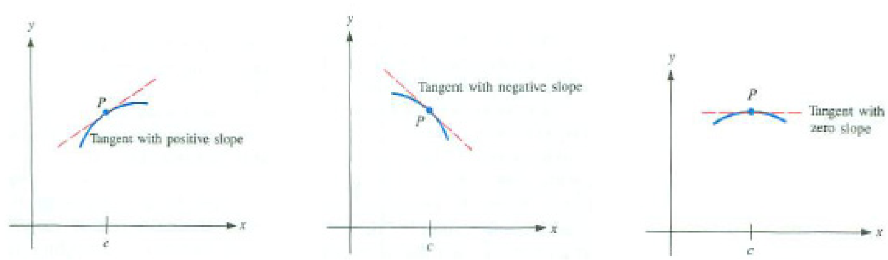


Figure 4.1.1:



(a) (b) (c)

Figure 4.1.2:

If the tangent at P were *not* horizontal (that is, not parallel to the x-axis), then it would be tilted. So a small piece of the graph around P which appears to be almost straight — would look as shown in Figure 4.1.2(a) or (b).

In the first case P could not be the highest point on the curve because there would be higher points *to the right* of P. In the second case P could not be the highest point because there would be higher points *to the left* of P. Therefore the *tangent at P must be horizontal*, as shown in Figure 4.1.2(c). That is, $f'(c) = 0$.

This observation is the foundation for a simple criterion for identifying local extrema.

Theorem of the Interior Extremum

Theorem 4.1.1 (Theorem of the Interior Extremum)**.** *Let f be a function defined at least on the open interval (a, b). If f takes on an extreme value at* *a number c in this interval, then either*

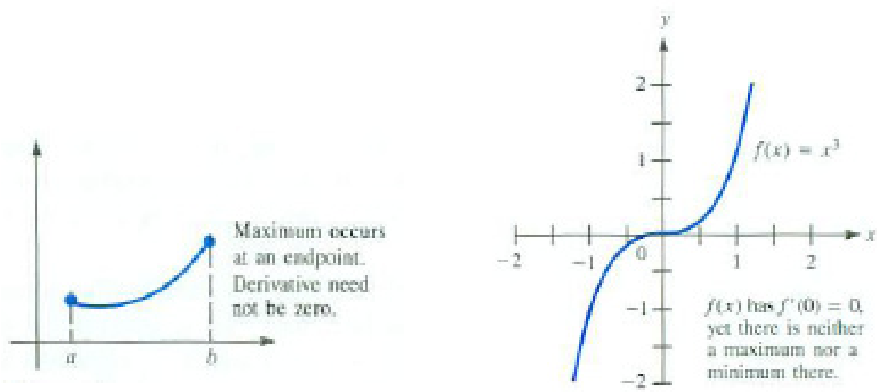
1. $f'(c) = 0$ *or*

2. $f'(c)$ *does not exist.*

If an extreme value occurs within an open interval and the derivative exists there, the derivative must be 0 there. This idea will be used in Section 4.2 to find the maximum and minimum values of a function.

WARNING *(Two Cautions about Theorem 4.1.1)*

1. If in Theorem 4.1.1 the open interval (a, b) is replaced by a closed interval $[a, b]$ the conclusion may not hold. A glance at Figure 4.1.3(a) shows why — the extreme value could occur at an endpoint ($x = a$ or $x = b$).



(a) (b)

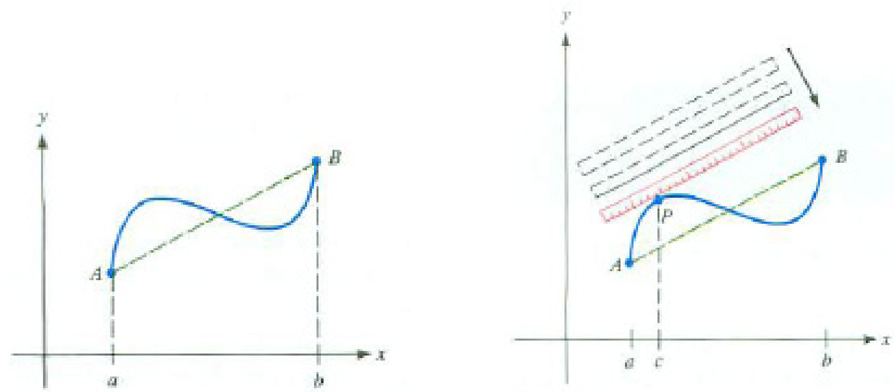
Figure 4.1.3:

2. The converse of Theorem 4.1.1 is not true. Having the derivative equal to 0 at a point does *not* guarantee that there is an extremum at this point. The graph of $y = x^3$, Figure 4.1.3(b), shows why. Since $f'(x) = 3x^2$, $f'(0) = 0$. While the tangent line is indeed horizontal at $(0, 0)$, it crosses the curve at this point. The graph has neither a maximum nor a minimum at the origin.

Though the next observation is phrased in terms of slopes, we will see that it has implications for velocity and any changing quantity.

A line segment that joins two points on the graph of a function f is called a **chord** of f.

OBSERVATION *(Chord and Tangent Line with Same Slope)* Let $A = (a, f(a))$ and $B = (b, f(b))$ be two points on the graph of a differentiable function f defined at least on the interval $[a, b]$, as shown in Figure 4.1.4(a). Draw the line segment AB joining A and B. Assume part of the graph lies above that line. Imagine holding a ruler parallel to AB and lowering it until it just touches the graph of $y = f(x)$, as in Figure 4.1.4(b). The ruler touches the



(a) (b)

Figure 4.1.4:

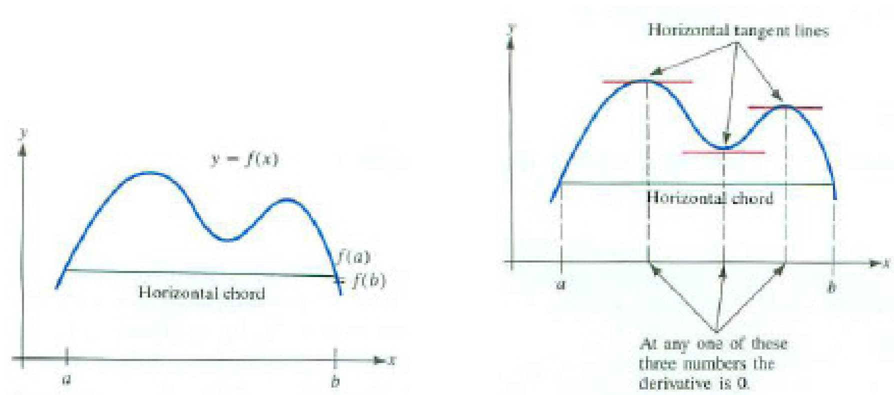
curve at a point P and lies along the tangent at P. At that point $f'(c)$ is equal to the slope of AB. (In Figure 4.1.4(b) there are two such numbers between a and b.)

It is customary to state two separate theorems based on the observation about chords and tangent lines. The first, Rolle's Theorem, is a special case of the second, the Mean-Value Theorem.

Rolle's Theorem

The next theorem is suggested by a special case of the second observation. When the points A and B in Figure 4.1.4(a) have the same y coordinate, the chord AB has slope 0. (See Figure 4.1.5.) In this case, the observation tells us there must a horizontal tangent to the graph. Expressed in terms of derivatives, this gives us Rolle's Theorem[1]

[1]Michel Rolle (1652–1719) was a French mathematician. and an early critic of calculus before later changing his opinion. In addition to his discovery of Rolle's Theorem in 1691, he is the first person known to have placed the index in the opening of a radical to denote the n^{th} root of a number: $\sqrt[n]{x}$. Source: Cajori, A History of Mathematical Notation, Dover Publ., 1993 and `http://en.wikipedia.org/wiki/Michel_Rolle`.



(a) (b)

Figure 4.1.5:

Theorem 4.1.2 (Rolle's Theorem)**.** *Let f be a continuous function on the closed interval $[a, b]$ and have a derivative at all x in the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.*

EXAMPLE 1 Verify Rolle's Theorem for the case with $f(t) = (t^2-1)\ln\left(\frac{t}{\pi}\right)$ on $[1, \pi]$.

SOLUTION The function $f(t)$ is defined for $t > 0$ and is differentiable. In particular, $f(t)$ is differentiable on the closed interval $[1, \pi]$. Notice that $f(1) = 0$ and, because $\ln(1) = 0$, $f(\pi) = 0$. Therefore, by Rolle's Theorem, there must be a value of c between 1 and π where $f'(c) = 0$.

The derivative $f'(t) = 2t\ln\left(\frac{t}{\pi}\right) + \frac{t^2-1}{t}$ is a pretty complicated function. Even though it is not possible to find the exact value of c with $f'(c) = 0$, Rolle's Theorem guarantees that there is at least one such value of c. Figure 4.1.6 confirms that there is only one solution to $f'(c) = 0$ on $[1, \pi]$. In Exercise 6 (at the end of Chapter 10 on page 783) you will find that this critical number is approximately 2.128. ◇

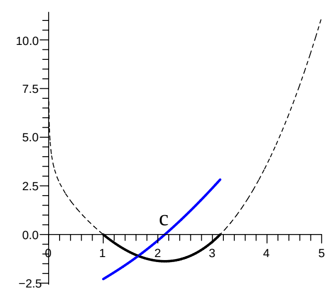


Figure 4.1.6: Graph of $y = f(t)$ (black) and $y = f'(t)$ (blue).

> **Remark:** Assume that $f(x)$ is a differentiable function such that $f'(x)$ is never 0 for x in an interval. Then the equation $f(x) = 0$ *can have at most one solution in that interval.* (If it had two solutions, a and b, then $f(a) = 0$ and $f(b) = 0$, and we could apply Rolle's Theorem on $[a, b]$. (See Figure 4.1.7.)

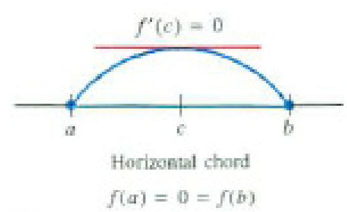


Figure 4.1.7:

This justifies the observation:

In an interval in which the derivative $f'(x)$ is never 0, the graph of $y = f(x)$ can have no more than one x-intercept.

Example 2 applies this.

EXAMPLE 2 Use Rolle's Theorem to determine how many real roots there are for the equation

$$x^3 - 6x^2 + 15x + 3 = 0. \tag{4.1.1}$$

SOLUTION Recall that the Intermediate Value Theorem guarantees that an odd degree polynomial, such as $f(x) = x^3 - 6x^2 + 15x + 3$, has at least one real solution to $f(x) = 0$. Call it r. Could there be another root, s? If so, by Rolle's Theorem, there would be a number c (between r and s) at which $f'(c) = 0$.

To check, we compute the derivative of $f(x)$ and see if it is ever equal to 0. We have $f'(x) = 3x^2 - 12x + 15$. To find when $f'(x)$ is 0, we solve the equation $3x^2 - 12x + 15 = 0$ by the quadratic formula, obtaining

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(15)}}{6} = \frac{12 \pm \sqrt{-36}}{6} = 2 \pm \sqrt{-1}.$$

Thus the equation $x^3 - 6x^2 + 15x + 3$ has only one real root. In Exercise 7 (at the end of Chapter 10) you will find that the sole real solution to (4.1.1) in approximately -0.186. ◇

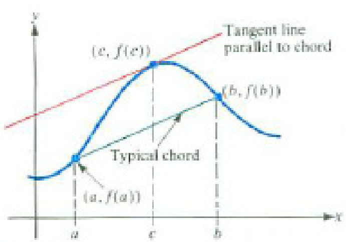


Figure 4.1.8:

Mean-Value Theorem

The "mean-value" theorem, is a generalization of Rolle's Theorem in that it applies to any chord, not just horizontal chords.

In geometric terms, the theorem asserts that if you draw a chord for the graph of a well-behaved function (as in Figure 4.1.8), then somewhere above or below that chord the graph has at least one tangent line parallel to the chord. (See Figure 4.1.4(a).) Let us translate this geometric statement into the language of functions. Call the ends of the chord $(a, f(a))$ and $(b, f(b))$. The slope of the chord is

$$\frac{f(b) - f(a)}{b - a}.$$

Since the tangent line and the chord are parallel, they have the same slopes. If the tangent line is at the point $(c, f(c))$, then

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Specifically, we have

Theorem 4.1.3 (Mean-Value Theorem)**.** *Let f be a continuous function on the closed interval $[a, b]$ and have a derivative at every x in the open interval (a, b). Then there is at least one number c in the open interval (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

EXAMPLE 3 Verify the Mean-Value Theorem for $f(t) = \sqrt{4 - t^2}$ on the interval $[0, 2]$.

SOLUTION Because $4 - t^2 \geq 0$ for t between -2 and 2 (including these two endpoints), f is continuous on $[0, 2]$ and is differentiable on $(0, 2)$. The slope of the chord through $(a, f(a)) = (0, 2)$ and $(b, f(b)) = (2, 0)$ is

$$\frac{f(b) - f(a)}{b - a} = \frac{0 - 2}{2 - 0} = -1.$$

According to the Mean-Value Theorem, there is at least one number c between 0 and 2 where $f'(c)$ is -1.

Let us try to find c. Since $f'(t) = \frac{-2t}{2\sqrt{4 - t^2}}$, we need to solve the equation

$$\begin{aligned}
\frac{-c}{\sqrt{4 - c^2}} &= -1 \\
-c &= -\sqrt{4 - c^2} && \text{multiply both sides by } \sqrt{4 - t^2} \\
c^2 &= 4 - c^2 && \text{square both sides} \\
2c^2 &= 4 \\
c^2 &= 2.
\end{aligned}$$

There are two solutions: $c = \sqrt{2}$ and $c = -\sqrt{2}$. Only $c = \sqrt{2}$ is in $(0, 2)$. This is the number whose existence is guaranteed by the Mean-Value Theorem. (The MVT says nothing about the existence of other numbers satisfying the MVT.) ◇

The interpretation of the derivative as slope suggested the Mean-Value Theorem. What does the Mean-Value Theorem say when the function describes the position of a moving object, and the derivative, its velocity? This is answered in Example 4.

EXAMPLE 4 A car moving on the x-axis has the x-coordinate $x = f(t)$ at time t. At time a its position is $f(a)$. At some later time b its position is $f(b)$. What does the Mean-Value Theorem assert for this car?

SOLUTION In this case the quotient

$$\frac{f(b) - f(a)}{b - a} \quad \text{equals} \quad \frac{\text{Change in position}}{\text{Change in time}}.$$

The Mean-Value Theorem asserts that at some time c, $f'(c)$ is equal to the quotient $\dfrac{f(b)-f(a)}{b-a}$. This says that the velocity at time c is the same as the average velocity during the time interval $[a,b]$. To be specific, if a car travels 210 miles in 5 hours, then at some time its speedometer must read 42 miles per hour. ◇

Consequences of the Mean-Value Theorem

There are several ways of writing the Mean-Value Theorem. For example, the equation

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

is equivalent to

$$f(b) - f(a) = (b-a)f'(c)$$

and hence to

$$f(b) = f(a) + (b-a)f'(c).$$

In this last form, the Mean-Value Theorem asserts that $f(b)$ is equal to $f(a)$ plus a quantity that involves the derivative f' at some number c between a and b. The following important corollaries are based on this alternative view of the Mean-Value Theorem.

Corollary 4.1.4. *If the derivative of a function is 0 throughout an interval I, then the function is constant on the interval.*

Proof

Let a and b be any two numbers in the interval I and let the function be denoted by f. To prove this corollary, it suffices to prove that $f(a) = f(b)$, for that is the defining property of a constant function.

By the Mean-Value Theorem in the form (1), there is a number c between a and b such that

$$f(b) = f(a) + (b-a)f'(c).$$

But $f'(c) = 0$, since $f'(x) = 0$ for all x in I. Hence

$$f(b) = f(a) + (b-a)(0)$$

which proves that

$$f(b) = f(a).$$

•

When Corollary 4.1.4 is interpreted in terms of motion, it is quite plausible. It asserts that if an object has zero velocity for a period of time, then it does not move during that time.

EXAMPLE 5 Use calculus to show that $f(x) = (e^x + e^{-x})^2 - e^{2x} - e^{-2x}$ is a constant. Find the constant.

SOLUTION The function f is differentiable for all numbers x. Its derivative is

$$\begin{aligned} f'(x) &= 2(e^x + e^{-x})(e^x - e^{-x}) - 2e^{2x} + 2e^{-2x} \\ &= 2(e^{2x} - e^{-2x}) - 2e^{2x} + 2e^{-2x} \\ &= 0 \end{aligned}$$

Because $f'(x)$ is always zero, f must be a constant.

To find the constant, just evaluate $f(x)$ for any convenient value of x. For simplicity we choose $x = 0$: $f(0) = (e^0 + e^0)^2 - e^0 - e^0 = 2^2 - 2 = 2$. Thus,

$$(e^x + e^{-x})^2 - e^{2x} - e^{-2x} = 2 \qquad \text{for all numbers } x.$$

This result can also be obtained by squaring $e^x + e^{-x}$. ◇

Corollary 4.1.5. *If two functions have the same derivatives throughout an interval, then they differ by a constant. That is, if $F'(x) = G'(x)$ for all x in an interval, then there is a constant C such that $F(x) = G(x) + C$.*

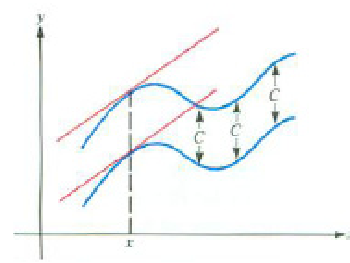


Figure 4.1.9:

Proof

Define a third function h by the equation $h(x) = F(x) - G(x)$. Then

$$h'(x) = F'(x) - G'(x) = 0. \qquad \text{since } F'(x) = G'(x)$$

Since the derivative if h is 0, Corollary 4.1.4 implies that h is constant, that is, $h(x) = C$ for some fixed number C. Thus

$$F(x) - G(x) = C \quad \text{or} \quad F(x) = G(x) + C,$$

and Corollary 4.1.5 is proved. •

Is Corollary 4.1.5 plausible when the derivative is interpreted as slope? In this case, the corollary asserts that if the graphs of two functions have the property that their tangent lines at points with the same x coordinate are parallel, then one graph can be obtained from the other by raising (or lowering) it by a constant amount C. If you sketch two such graphs (as in Figure 4.1.9, you will see that the corollary is reasonable.

EXAMPLE 6 What functions have a derivative equal to $2x$ everywhere?

SOLUTION One such solution is x^2; another is $x^2 + 25$. For any constant

In the language of Section 3.5, any antiderivative of $2x$ must be of the form $x^2 + C$.

C, $D(x^2 + C) = 2x$. Are there any other possibilities? Corollary 4.1.5 tells us there are not, for if F is a function such that $F'(x) = 2x$, then $F'(x) = (x^2)'$ for all x. Thus the functions F and x^2 differ by a constant, say C, that is,

$$F(x) = x^2 + C.$$

The only antiderivatives of $2x$ are of the form $x^2 + C$. ◇

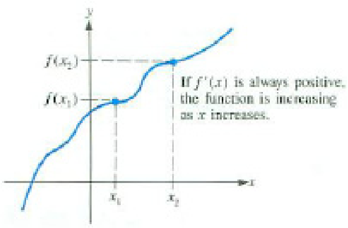


Figure 4.1.10:

Corollary 4.1.4 asserts that if $f'(x) = 0$ for all x, then f is a constant. What can be said about f if $f'(x)$ is *positive* for all x in an interval? In terms of the graph of f, this assumption implies that all the tangent lines slope upward. It is reasonable to expect that as we move from left to right on the graph in Figure 4.1.10, the y-coordinate increases, that is, the function is increasing. (See Section 1.1).)

Corollary 4.1.6. *If f is continuous on the closed interval $[a, b]$ and has a positive derivative on the open interval (a, b), then f is increasing on the interval $[a, b]$.*

If f is continuous on the closed interval $[a, b]$ and has a negative derivative on the open interval (a, b), then f is decreasing on the interval $[a, b]$.

Proof

We prove the "increasing" case; the other case is handled in Exercise 44. Take two numbers x_1 and x_2 such that

$$a \leq x_1 < x_2 \leq b.$$

The goal is to show that $f(x_2) > f(x_1)$.

By the Mean-Value Theorem, there is some number c between x_1 and x_2 such that

$$f(x_2) = f(x_1) + (x_2 - x_1)f'(c).$$

Now, since $x_2 > x_1$, we know $x_2 - x_1$ is positive. Since $f'(c)$ is assumed to be positive, and the product of two positive numbers is positive, it follows that

$$(x_2 - x_1)f'(c) > 0.$$

Thus, $f(x_2) > f(x_1)$, and so $f(x)$ is an increasing function. •

EXAMPLE 7 Determine whether $2x + \sin(x)$ is an increasing function, a decreasing function, or neither.

SOLUTION The funcion $2x + \sin(x)$ is the sum of two simpler functions: $2x$ and $\sin(x)$. The "$2x$" part is an increasing function. The second term, "$\sin(x)$", increases for x between 0 and $\pi/2$ and decreases for x between $\pi/2$

and π. It is not clear what type of function you will get when you add $2x$ and $\sin(x)$. Let's see what Corollary 4.1.6 tells us.

The derivative of $2x + \sin(x)$ is $2 + \cos(x)$. Since $\cos(x) \geq -1$ for all x,

$$(2x + \sin(x))' = 2 + \cos(x) \geq 2 + (-1) = 1.$$

Because $(2x+\sin(x))'$ is positive for all numbers x, $2x+\sin(x)$ is an increasing function. Figure 4.1.11 shows the graph of $2x+\sin(x)$ together with the graphs of $2x$ and $\sin(x)$. ◇

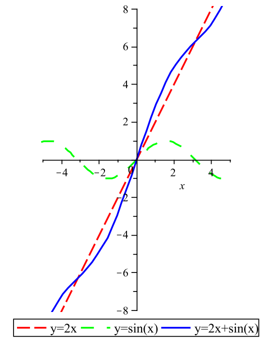


Figure 4.1.11:

Remark: Increasing/Decreasing at a Point

1. Corollary 4.1.6, and the definitions of increasing and decreasing, are stated in terms of intervals. When we talk about a function f increasing (or decreasing) "at a point c," here is what we mean: there is an interval (a, b) with $a < c < b$ where f is increasing.

2. When $f'(c) > 0$ and f' is continuous, the Permanence Property in Section 2.5) tells us there is an interval (a, b) containing c where $f'(x)$ remains positive for all numbers x in (a, b). Thus, f is increasing on (a, b), and hence increasing at c.

"A function is increasing at c" is shorthand for "a function is increasing in an interval that contains c."

More generally, if $f'(x)$ is never negative, that is $f'(x) \geq 0$ for all inputs x, then f is non-decreasing. In the same manner, if $f'(x) \leq 0$ for all inputs x, then f is a non-increasing function.

Summary

This section focused on three theorems, which we state informally. For the assumptions on the functions, see the formal statements in this section.

The Theorem of the Interior Extremum says that at a local extreme the derivative must be zero. (The converse is not true.)

Rolle's Theorem asserts that if a function has equal values at two inputs, its derivative must equal zero at least at one number between them. The Mean-Value Theorem, a generalization of Rolle's Theorem, asserts that for any chord on the graph of a function, there is a tangent line parallel to it. This means that for $a < b$ there is c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, or in a more useful form $f(b) = f(a) + f'(c)(b - a)$.

From the Mean-Value Theorem it follows that where a derivative is positive, a function is increasing; where it is negative it is decreasing; and where it stays at the value zero, it is constant. The last assertion implies that two antiderivatives of the same function differ by a constant (which may be zero).

EXERCISES for Section 4.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] State Rolle's Theorem in words, using as few mathematical symbols as you can.

2.[R] Draw a graph illustrating Rolle's Theorem. Be sure to identify the critical parts of the graph.

3.[R] Draw a graph illustrating the Mean-Value Theorem. Be sure to identify the critical parts of the graph.

4.[R] Express the Mean-Value Theorem in words, using no symbols to denote the function or the interval.

5.[R] Express the Mean-Value Theorem in symbols, where the function is denoted g and the interval is $[e, f]$.

6.[R] Which of the corollaries to the Mean-Value Theorem implies that

(a) if two cars on a straight road have the same velocity at every instant, they remain a fixed distance apart?

(b) If all tangents to a curve are horizontal, the curve is a horizontal line.

Explain each answer.

Exercises 7 to 12 concern the Theorem of the Interior Extremum.

7.[R] Consider the function $f(x) = x^2$ only for x in $[-1, 2]$.

(a) Graph the function $f(x)$ for x in $[-1, 2]$.

(b) What is the maximum value of $f(x)$ for x in the interval $[-1, 2]$?

(c) Does $f'(x)$ exist at the maximum?

(d) Does $f'(x)$ equal zero at the maximum?

(e) Does $f'(x)$ equal zero at the minimum?

8.[R] Consider the function $f(x) = \sin(x)$ only for x in $[0, \pi]$.

(a) Graph the function $f(x)$ for x in $[0, \pi]$.

(b) What is the maximum value of $f(x)$ for x in the interval $[0, \pi]$?

(c) Does $f'(x)$ exist at the maximum?

(d) Does $f'(x)$ equal zero at the maximum?

(e) Does $f'(x)$ equal zero at the minimum?

9.[R]

(a) Repeat Exercise 7 on the interval $[1, 2]$.

(e) Repeat Exercise 8 on the interval $(0, \pi)$.

(f) Repeat Exercise 8 on the interval $(0, 2\pi)$.

10.[R]

(a) Graph $y = -x^2 + 3x + 2$ for x in $[0, 2]$.

(b) Looking at the graph, estimate the x coordinate where the maximum value of y occurs for x in $[0, 2]$.

(c) Find where $dy/dx = 0$.

(d) Using (c), determine exactly where the maximum occurs.

11.[R]

(a) Graph $y = 2x^2 - 3x + 1$ for x in $[0, 1]$.

(b) Looking at the graph, estimate the x coordinate where the maximum value of y occurs for x in $[0, 1]$. At which value of x does it occur?

(c) Looking at the graph, estimate the x coordinate where the minimum value of y occurs for x in $[0, 12]$.

12.[R] For each of the following functions, (a) show that the derivative of the given function is 0 when $x = 0$ and (b) decide whether the function has an extremum at $x = 0$.

(a) $x^2 \sin(x)$

(b) $1 - \cos(x)$

(c) $e^x - x$

(d) $x^2 - x^3$

Exercises 13 to 21 concern Rolle's Theorem.

13.[R]

(a) Graph $f(x) = x^{2/3}$ for x in $[-1, 1]$.

(b) Show that $f(-1) = f(1)$.

(c) Is there a number c in $(-1, 1)$ such that $f'(c) = 0$?

(d) Why does this not contradict Rolle's Theorem?

14.[R]

(a) Graph $f(x) = 1/x^2$ for x in $[-1, 1]$.

(b) Show that $f(-1) = f(1)$.

(c) Is there a number c in $(-1, 1)$ such that $f'(c) = 0$?

(d) Why does this not contradict Rolle's Theorem?

(e) Use the graph to estimate the values of the c's.

In Exercises 15 to 20, verify that the given function satisfies Rolle's Theorem for the given interval. Find all numbers c that satisfy the conclusion of the theorem.

15.[R] $f(x) = x^2 - 2x - 3$ and $[0, 2]$

16.[R] $f(x) = x^3 - x$ and $[-1, 1]$

17.[R] $f(x) = x^4 - 2x^2 + 1$ and $[-2, 2]$

18.[R] $f(x) = \sin(x) + \cos(x)$ and $[0, 4\pi]$

19.[R] $f(x) = e^x + e^{-x}$ and $[-2, 2]$

20.[R] $f(x) = x^2 e^{-x^2}$ and $[-2, 2]$

21.[M] Let $f(x) = \ln(x^2)$. Note that $f(-1) = f(1)$. Is there a number c in $(-1, 1)$ such that $f'(c) = 0$? If so, find at least one such number. If not, why is this not a contradiction of Rolle's Theorem?

Exercises 22 to 27 concern the Mean-Value Theorem. In Exercises 22 to 25, find explicitly all values of c which satisfy the Mean-Value Theorem for the given functions and intervals.

22.[R] $f(x) = x^2 - 3x$ and $[1, 4]$

23.[R] $f(x) = 2x^2 + x + 1$ and $[-2, 3]$

24.[R] $f(x) = 3x + 5$ and $[1, 3]$

25.[R] $f(x) = 5x - 7$ and $[0, 4]$

26.[R]

(a) Graph $y = \sin(x)$ for x in $[\pi/2, 7\pi/2]$.

(b) Draw the chord joining $(\pi/2, f(\pi/2))$ and $(7\pi/2, f(7\pi/2))$.

(c) Draw all tangents to the graph parallel to the chord drawn in (b).

(d) Using (c), determine how many numbers c there are in $(\pi, 7\pi/2)$ such that

$$f'(c) = \frac{f(7\pi/2) - f(\pi/2)}{7\pi/2 - \pi/2}.$$

27.[R]

(a) Graph $y = \cos(x)$ for x in $[0, 9\pi/2]$.

(b) Draw the chord joining $(0, f(0))$ and $(9\pi/2, f(9\pi/2))$.

(c) Draw all tangents to the graph that are parallel to the chord drawn in (b).

(d) Using (c), determine how many numbers c there are in $(0, 9\pi/2)$ such that

$$f'(c) = \frac{f(9\pi/2) - f(0)}{9\pi/2 - 0}.$$

(e) Use the graph to estimate the values of the c's.

28.[R] At time t seconds a thrown ball has the height $f(t) = -16t^2 + 32t + 40$ feet.

(a) What is the initial height? That is, the height when t is zero.

(b) Show that after 2 seconds it returns to its initial height.

(c) What does Rolle's Theorem imply about the velocity of the ball?

(d) Verify Rolle's Theorem in this case by computing the numbers c which it asserts exist.

29.[R] Find all points where $f(x) = 2x^3(x-1)$ can have an extreme value on the following intervals

(a) $(-1/2, 1)$

(b) $[-1/2, 1]$

(c) $[-1/2, 1/2]$

(d) $(-1/2, 1/2)$

30.[R] Let $f(x) = |2x - 1|$.

(a) Explain why $f'(1/2)$ does not exist.

(b) Find $f'(x)$. Hint: Write the absolute value in two parts, one for $x < 1/2$ and the other for $x > 1/2$.

(c) Does the Mean-Value Theorem apply on the interval $[-1, 2]$?

31.[R] The year is 2015. Because a gallon of gas costs six dollars and Highway 80 is full of tire-wrecking potholes, the California Highway Patrol no longer patrols the 77 miles between Sacramento and Berkeley. Instead it uses two cameras. One, in Sacramento, records the license number and time of a car on the freeway, and another does the same in Berkeley. A computer processes the data instantly. Assume that the two cameras show that a car that was in Sacramento at 10:45 reached Berkeley at 11:40. Show that the Mean-Value Theorem justifies giving the driver a ticket for exceeding the 70 mile-per-hour speed limit. (Of course, intuition justifies the ticket, but mentioning the Mean-Value Theorem is likely to impress a judge who studied calculus.)
Note: While it makes a nice story to suggest that mentioning the Mean-Value Theorem will impress a judge who studied calculus, reality is that the California Vehicle Code forbids this way to catch speeders. It reads, "No speed trap shall be used in securing evidence as to the speed of any vehicle. A 'speed trap' is a particlar section of highway measured as to distance in order that the speed of a vehicle may be calculated by securing the time it takes the vehicle to travel the known distance." It sounds as though the lawmakers who wrote this law studied calculus.

32.[M] What is the shortest time for the trip from Berkeley to Sacramento for which the Mean-Value Theorem does not convict the driver of speeding? Note: See Exercise 31.

33.[R] Verify the Mean-Value Theorem for $f(t) = x^2 e^{-x/3}$ on $[1, 10]$. Note: See Example 1.

34.[R] Find all antiderivatives of each of the following functions. Check your answer by differentiation.

(a) $3x^2$

(b) $\sin(x)$

(c) $\frac{1}{1+x^2}$

(d) e^x

35.[R] Find all antiderivatives of each of the following functions. Check your answer by differentiation.

(a) $\cos(x)$

(b) $\sec(x)\tan(x)$

(c) $1/x$ $(x > 0)$

(d) $\sqrt{x}$ $(x > 0)$

36.[R]

(a) Differentiate $\sec^2(x)$ and $\tan^2(x)$.

(b) The derivatives in (a) are equal. Corollary 4.1.5 then asserts that there exists a constant C such that $\sec^2(x) = \tan^2(x) + C$. Find the constant.

37.[R] Show by differentiation that $f(x) = \ln(x/5) - \ln(5x)$ is a constant for all values of x. Find the constant.

38.[M] Find all functions whose second derivative is 0 for all x in $(-\infty, \infty)$.

39.[M] Use Rolle's Theorem to determine how many real roots there are for the equation $x^3 - 6x^2 + 15x + 3 = 0$.

40.[M] Use Rolle's Theorem to determine how many real roots there are for the equation $3x^4 + 4x^3 - 12x^2 + 4 = 0$. Give intervals on which there is exactly one root.

41.[M] Use Rolle's Theorem to determine how many real roots there are for the polynomial $f(x) = 3x^4 + 4x^3 - 12x^2 + A$. That number may depend on A. For which A is there exactly one root? Are there any values of A for which there is an odd number of real roots? NOTE: Exercise 40 uses this equation with $A = 4$.

42.[M] Consider the equation $x^3 - ax^2 + 15x + 3 = 0$. The number of real roots to this equation depends on the value of a.

(a) Find all values of a when the equation has 3 real roots.

(b) Find all values of a when the equation has 1 real root.

(c) Are there any values of a with exactly two real roots?

NOTE: Exercise 39 uses this equation with $a = 6$.

43.[M] If f is differentiable for all real numbers and $f'(x) = 0$ has three solutions, what can be said about the number of solutions of $f(x) = 0$? of $f(x) = 5$?

44.[M] Prove the "decreasing" case of Corollary 4.1.6.

45.[M] For which values of the constant k is the function $7x + k\sin(2x)$ always increasing?

46.[C] If two functions have the same second derivative for all x in $(-\infty, \infty)$, what can be said about the relation between them?

47.[C] If a function f is differentiable for all x and c is a number, is there necessarily a chord of the graph of f that is parallel to the tangent line at $(c, f(c))$? Explain.

48.[C] Sketch a graph of a continuous function $f(x)$ defined for all numbers such that $f'(1)$ is 2, yet there is no open interval around 1 on which f is increasing.

Exercises 49 to 52 involve the **hyperbolic functions**. The **hyperbolic sine** function is $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and the **hyperbolic cosine** function is $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Hyperbolic functions are discussed in greater detail in Section 5.7.

49.[R]

(a) Show that $\frac{d}{dx}\sinh(x) = \cosh(x)$.

(b) Show that $\frac{d}{dx}\cosh(x) = \sinh(x)$.

50.[M] Define $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{2}{e^x + e^{-x}}$ and $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

(a) Show that $\frac{d}{dx}\tanh(x) = (\operatorname{sech}(x))^2$.

(b) Show that $\frac{d}{dx}\operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x)$.

51.[M] Use calculus to show that $(\cosh(x))^2 - (\sinh(x))^2$ is a constant. Find the constant.

52.[M] Use calculus to show that $(\operatorname{sech}(x))^2 + (\tanh(x))^2$ is a constant. Find the constant.

4.2 The First-Derivative and Graphing

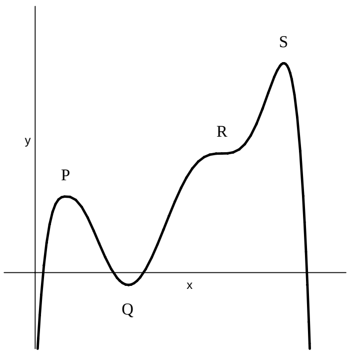


Figure 4.2.1:

Section 4.1 showed the connection between extrema and the places where the derivative is zero. In this section we use this connection to find high and low points on a graph.

The graph of a differentiable function f defined for all real numbers x is shown in Figure 4.2.1. The points P, Q, R, and S are of special interest. S is the highest point on the graph *for all x in the domain*. We call it a *global maximum* or *absolute maximum*. The point P is higher than all points *near it* on the graph; it is called a *local maximum* or *relative maximum*. Similarly, Q is called a *local minimum* or *relative minimum*. The point R is neither a relative maximum nor a relative minimum.

A point that is either a maximum or minimum is called an **extremum**. The plural of extremum is extrema.

If you were to walk left to right along the graph in Figure 4.2.1, you would call P the top of a hill, Q the bottom of a valley, and S the highest point on your walk (it is also a top of a hill). You might notice R, for you get a momentary break from climbing from Q to S. For just this one instant it would be like walking along a horizontal path.

These important aspects of a function and its graph are made precise in the following definitions. These definitions are phrased in terms of a general domain. In most cases the domain of the function will be an interval — open, closed, or half-open.

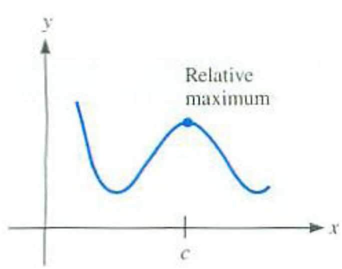


Figure 4.2.2:

DEFINITION *(Relative Maximum (Local Maximum))* The function f has a **relative maximum** (or **local maximum**) at a number c if there is an open interval around c such that $f(c) \geq f(x)$ for all x in that interval that lie in the domain of f.

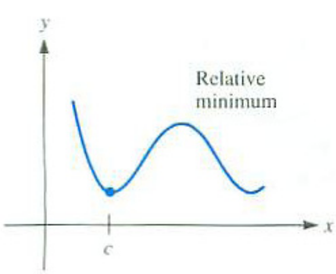


Figure 4.2.3:

DEFINITION *(Relative Minimum (Local Minimum))* The function f has a **relative minimum** (or **local minimum**) at a number c if there is an open interval around c such that $f(c) \leq f(x)$ for all x in that interval that lie in the domain of f.

Each global extremum is also a local extremum.

DEFINITION *(Absolute Maximum (Global Maximum))* The function f has a **absolute maximum** (or **global maximum**) at a number c if $f(c) \geq f(x)$ for all x in the domain of f.

DEFINITION *(Absolute Minimum (Global Minimum))* The function f has a **absolute minimum** (or **global minimum**) at a number c if $f(c) \leq f(x)$ for all x in the domain of f.

A local extremum is like the summit of a single mountain or an individual valley. A global maximum corresponds to Mt. Everest at more than 29,000 feet above sea level; a global minimum corresponds to the Mariana Trench in the Pacific Ocean 36,000 feet below sea level, the lowest point on the Earth's crust.

In this section it is assumed that the functions are differentiable. If a function is not differentiable at an isolated point, this point will need to be considered separately.

DEFINITION *(Critical Number and Critical Point)* A number c at which $f'(c) = 0$ is called a **critical number** for the function f. The corresponding point $(c, f(c))$ on the graph of f is a **critical point**.

Remark: Some texts define a critical number as a number where the derivative is 0 or else is not defined. Since we emphasize differentiable functions, a critical number is defined to be a number where the derivative is 0.

The Theorem of the Interior Extremum, in Section 4.1, says that every local maximum and minimum of a function f occurs where the tangent line to the curve either is horizontal or does not exist.

Some functions have extreme values, and others do not. The following theorem gives simple conditions under which both a global maximum and a global minimum are guaranteed to exist. To convince yourself that this is plausible, imagine drawing the graph of the function. At some point your pencil will reach a highest point and at another point a lowest point.

Theorem 4.2.1 (Extreme Value Theorem)**.** *Let f be a continuous function on a closed interval $[a, b]$. Then f attains an absolute maximum value $M = f(c)$ and an absolute minimum value $m = f(d)$ at some numbers c and d in $[a, b]$.*

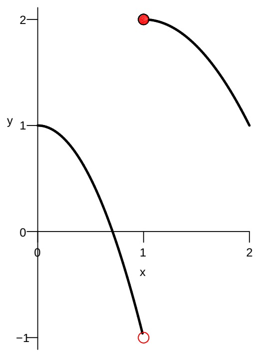


Figure 4.2.4:

EXAMPLE 1 Find the absolute extrema on the interval $[0, 2]$ of the function whose graph is shown in Figure 4.2.4.
SOLUTION The function has an absolute maximum value of 2 but no absolute minimum value. The range is $(-1, 2]$. This function takes on values that are arbitrarily close to -1, but -1 is not in the range of this function. This can occur only because the function is not continuous at $x = 1$. ◇

Recall that Corollary 4.1.6 provides a convenient test to determine if a function is increasing or decreasing at a point: if $f'(c) > 0$ then f is increasing at $x = c$ and if $f'(c) < 0$ then f is decreasing at $x = c$.

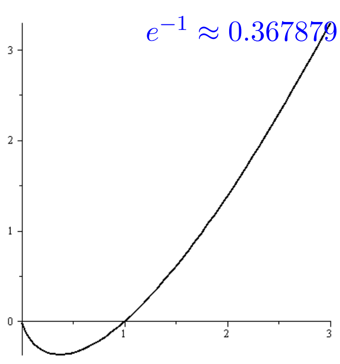
WARNING *D*ifferentiable implies continuous, so "not continuous" implies "not differentiable."

EXAMPLE 2 Let $f(x) = x\ln(x)$ for all $x > 0$. Determine the intervals on which f is increasing, decreasing, or neither.
SOLUTION The function is increasing at numbers x where $f'(x) > 0$ and decreasing where $f'(x) < 0$. More effort is needed to determine the behavior at points where $f'(x) = 0$ (or does not exist). (Observe that the natural domain of f is $x > 0$.) The Product Rule allows us to find

$$f'(x) = \ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1.$$

In order to find where $f'(x)$ is positive or is negative, we first find where it is zero. At such numbers the derivative may switch sign, and the function switch between increasing and decreasing. So we solve the equation:

$$\begin{aligned} f'(x) &= 0 \\ \ln(x) + 1 &= 0 \\ \ln(x) &= -1 \\ e^{\ln(x)} &= e^{-1} \\ x &= e^{-1}. \end{aligned}$$



When x is larger than e^{-1}, $\ln(x)$ is larger than -1 so that $f'(x) = \ln(x)+1$ is positive and f is increasing. Finally, f is decreasing when x is between 0 and e^{-1} because $\ln(x) < -1$, which makes $f'(x) = \ln(x)+1$ negative. The graph of $y = x\ln(x)$ in Figure 4.2.5 confirms these findings. In addition, observe that $x = e^{-1}$ is a minimum of this function. ◇

Using Critical Numbers to Identify Local Extrema

The previous examples show there is a close connection between critical points and local extrema. Notice that, generally, just to the left of a local maximum the function is increasing, while just to the right it is decreasing. The opposite holds for a local minimum. The First-Derivative Test for a Local Extreme Value at $x = c$ gives a precise statement of this result.

First-Derivative Test for a Local Extreme Value at $x = c$

Theorem 4.2.2. *Let f be a function and let c be a number in its domain. Suppose f is continuous on an open interval that contains c and is differentiable on that interval, except possibly at c. Then:*

1. *If f' changes from positive to negative as x moves from left to right through the value c,*
   then f has a local maximum at c.
2. *If f' changes from negative to positive as x moves from left to right through the value c,*
   then f has a local minimum at c.
3. *If f' does not change sign at c,*
   then f does not have a local extremum at $x = c$.

EXAMPLE 3 Classify all critical numbers of $f(x) = 3x^5 - 20x^3 + 10$ as a local maximum, local minimum, or neither.
SOLUTION To identify the critical numbers of f, we find and factor the derivative:

$$f'(x) = 15x^4 - 60x^2 = 15x^2(x^2 - 4) = 15x^2(x - 2)(x + 2).$$

The critical numbers of f are $x = 0$, $x = 2$, and $x = -2$. To determine if any of these numbers provide local extrema it is necessary to know where f is increasing and where it is decreasing.

Because f' is continuous the three critical numbers are the only places the sign of f' can possibly change. All that remains is to determine if f is increasing or decreasing on the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$. This is easily answered from table of function values shown in the first two rows of Table 4.2.1. Observe that $f(-2) = 74 > 10 = f(0)$; this means f is decreasing on $(-2, 0)$. Likewise, f must be decreasing on $(0, 2)$ because

x	$\to -\infty$	-2	0	2	$\to \infty$
$f(x)$	$-\infty$	74	10	-54	∞
$f'(x)$		0	0	0	

Table 4.2.1:

$f(0) = 10 > -54 = f(2)$. For the two unbounded intervals, limits at $\pm\infty$ must be used but the overall idea is the same. Since $\lim_{x\to-\infty} f(x) = -\infty$, the function must be increasing on $(-\infty, -2)$. Likewise, in order to have $\lim_{x\to\infty} f(x) = +\infty$, f must be increasing on $(2, \infty)$. (See Figure 4.2.6.)

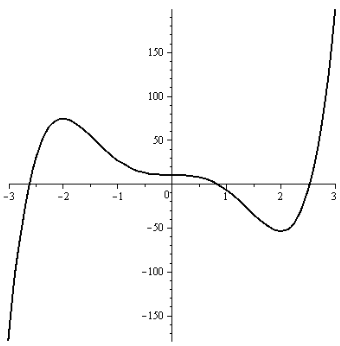


Figure 4.2.6:

To conclude, because the graph of f changes from increasing to decreasing at $x = -2$, there is a local maximum at $(-2, 74)$. At $x = 2$ the graph changes from decreasing to increasing, so a local minimum occurs at $(2, -54)$. Because the derivative does not change sign at $x = 0$, this critical number is not a local extreme. ◇

EXAMPLE 4 Find all local extrema of $f(x) = (x+1)^{2/7}e^{-x}$.
SOLUTION (Observe that the domain of f is $(-\infty, \infty)$.) The Product and Chain Rules for derivatives can be used to obtain

$$\begin{aligned} f'(x) &= \tfrac{2}{7}(x+1)^{-5/7}e^{-x} + (x+1)^{2/7}e^{-x}(-1) \\ &= \tfrac{2}{7}(x+1)^{-5/7}e^{-x} - (x+1)^{2/7}e^{-x} \\ &= (x+1)^{-5/7}e^{-x}\left(\tfrac{2}{7} - (x+1)\right) \\ &= (x+1)^{-5/7}e^{-x}\left(-x - \tfrac{5}{7}\right) \\ &= \tfrac{-x-\frac{5}{7}}{(x+1)^{5/7}e^{x}}. \end{aligned}$$

The only solution to $f'(x) = 0$ is $x = -5/7$, so $c = -5/7$ is the only critical number. In addition, because the denominator of $f'(x)$ is zero when $x = -1$, f is not differentiable for $x = -1$. Using the information in Table 4.2.2, we

x	$\to -\infty$	-1	$-5/7$	$\to \infty$
$f(x)$	∞	0	$(2/7)^{(2/7)}e^{5/7} \approx 1.42811$	0
$f'(x)$		dne	0	

Table 4.2.2: Note that dne means the limit does not exist.

conclude f is decreasing on $(-\infty, -1)$, increasing on $(-1, -5/7)$, and decreasing on $(-5/7, \infty)$. By the First-Derivative Test, f has a local minimum at $(-1, 0)$ and a local maximum at $(-5/7, (2/7)^{(2/7)}e^{5/7}) \approx (-0.71429, 1.42811)$.

Notice that the First-Derivative Test applies at $x = -1$ even though f is not differentiable for $x = -1$. A graph of $y = f(x)$ is shown in Figure 4.2.7. (See also Exercise 27 in Section 4.3.) ◇

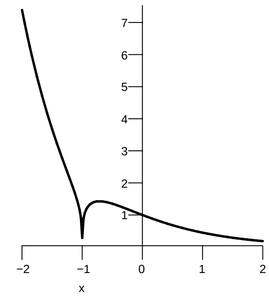


Figure 4.2.7:

Extreme Values on a Closed Interval

Many applied problems involve a continuous function only on a closed interval $[a, b]$. (See Section 4.1.)

The Extreme Value Theorem guarantees the function attains both a maximum and a minimum at some point in the interval. The extreme values occur either at

1. an endpoint ($x = a$ or $x = b$),

2. a critical number ($x = c$ where $f'(c) = 0$), or

3. where f is not differentiable ($x = c$ where $f'(c)$ is not defined).

EXAMPLE 5 Find the absolute maximum and minimum values of $f(x) = x^4 - 8x^2 + 1$ on the interval $[-1, 3]$.

SOLUTION The function is continuous on a closed and bounded interval. The absolute maximum and minimum values occur either at a critical point or at an endpoint of the interval. The endpoints are $x = -1$ and $x = 3$. To find the critical points we solve $f'(x) = 0$:

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2) = 0.$$

There are three critical numbers, $x = 0$, 2, and -2, but only $x = 0$ and $x = 2$ are in the interval. The intervals where the graph of $y = f(x)$ is increasing and decreasing can be determined from the information in Table 4.2.3.

x	-1	0	2	3
$f(x)$	-6	1	-15	10
$f'(x)$		0	0	0

Table 4.2.3:

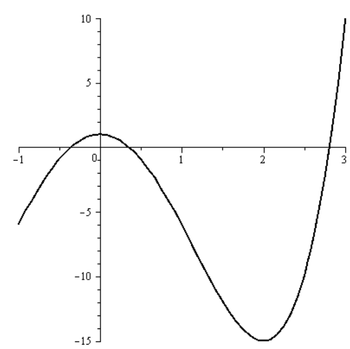


Figure 4.2.8:

Since we are looking only for global extrema on a closed interval, it is unnecessary to determine these intervals or to classify critical points as local extrema. Instead, we simply scan the list of function values at the endpoints and at the critical numbers – row 2 of Table 4.2.3 – for the largest and smallest values of $f(x)$. The largest value is 10, so the global maximum occurs at $x = 3$. The smallest value is -15, so the global minimum occurs at $x = 2$. (See Figure 4.2.8.) ◇

In Example 5 it was not necessary to determine the intervals on which the function is increasing and decreasing, nor did we need to identify the local extreme values. (See also Exercise 5.)

Summary

This section shows how to use the first derivative to find extreme values of a function. Namely, identify when the derivative is zero, positive, and negative, and where it changes sign.

A continuous function on a closed and bounded interval always has a maximum and a minimum. All extrema occur either at an endpoint, a critical number (where $f'(c) = 0$), or where f is not differentiable.

EXERCISES for Section 4.2 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 28, sketch the graph of the given function. Find all intercepts and critical points, determine the intervals where the function is increasing and where it is decreasing, and identify all local extreme values.

1.[R] $f(x) = x^5$

2.[R] $f(x) = (x-1)^4$

3.[R] $f(x) = 3x^4 + x^3$

4.[R] $f(x) = 2x^3 + 3x^2$

5.[R] $f(x) = x^4 - 8x^2 + 1$

6.[R] $f(x) = x^3 - 3x^2 + 3x$

7.[R] $f(x) = x^4 - 4x + 3$

8.[R] $f(x) = 2x^2 + 3x + 5$

9.[R] $f(x) = x^4 + 2x^3 - 3x^2$

10.[R] $f(x) = 2x^3 + 3x^2 - 6x$

11.[R] $f(x) = xe^{-x/2}$

12.[R] $f(x) = xe^{x/3}$

13.[R] $f(x) = e^{-x^2}$

14.[R] $f(x) = xe^{-x^2/2}$

15.[R] $f(x) = x\sin(x) + \cos(x)$

16.[R] $f(x) = x\cos(x) - \sin(x)$

17.[R] $f(x) = \frac{\cos(x)-1}{x^2}$

18.[R] $f(x) = x\ln(x)$

19.[R] $f(x) = \frac{\ln(x)}{x}$

20.[R] $f(x) = \frac{e^x-1}{x}$

21.[R] $f(x) = \frac{e^{-x}}{x}$

22.[R] $f(x) = \frac{x-\arctan(x)}{x^3}$

23.[R] $f(x) = \dfrac{3x+1}{3x-1}$

24.[R] $f(x) = \dfrac{x}{x^2+1}$

25.[R] $f(x) = \dfrac{x}{x^2-1}$

26.[R] $f(x) = \dfrac{1}{2x^2-x}$

27.[R] $f(x) = \dfrac{1}{x^2-3x+2}$

28.[R] $f(x) = \dfrac{\sqrt{x^2+1}}{x}$

In Exercises 29 to 36 sketch the general shape of the graph, using the given information. Assume the function and its derivative are defined for all x and are continuous. Explain your reasoning.

29.[R] Critical point $(1,2)$, $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$.

30.[R] Critical point $(1,2)$ and $f'(x) < 0$ for all x except $x = 1$.

31.[R] x intercept -1, critical points $(1,3)$ and $(2,1)$, $\lim_{x\to\infty} f(x) = 4$, $\lim_{x\to-\infty} f(x) = -1$.

32.[R] y intercept 3, critical point $(1,2)$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 4$.

33.[R] x intercept -1, critical points $(1,5)$ and $(2,4)$, $\lim_{x\to\infty} f(x) = 5$, $\lim_{x\to-\infty} f(x) = -\infty$.

34.[R] x intercept 1, y intercept 2, critical points $(1,0)$ and $(4,4)$, $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to-\infty} f(x) = \infty$.

35.[R] x intercepts 2 and 4, y intercept 2, critical points $(1,3)$ and $(3,-1)$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 1$.

36.[R] No x intercepts, y intercept 1, no critical points, $\lim_{x\to\infty} f(x) = 2$, $\lim_{x\to-\infty} f(x) = 0$.

Exercises 37 to 52 concern functions whose domains are restricted to closed intervals. In each, find the maximum and minimum values for the given function on the given interval.

37.[R] $f(x) = x^2 - x^4$ on $[0,1]$

38.[R] $f(x) = 4x - x^2$ on $[0,5]$

39.[R] $f(x) = 2x^2 - 5x$ on $[-1,1]$

40.[R] $f(x) = x^3 - 2x^2 + 5x$ on $[-1,3]$

41.[R] $f(x) = \dfrac{x}{x^2+1}$ on $[0,3]$

42.[R] $f(x) = x^2 + x^4$ on $[0,1]$

43.[R] $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$ on $[0,3]$

44.[R] $f(x) = \sin(x) + \cos(x)$ on $[0,\pi]$

45.[R] $f(x) = \sin(x) - \cos(x)$ on $[0,\pi]$

46.[R] $f(x) = x + \sin(x)$ on $[-\pi/2, \pi/2]$

47.[R] $f(x) = x + \sin(x)$ on $[-\pi, 2\pi]$

48.[R] $f(x) = x/2 + \sin(x)$ on $[-\pi, 2\pi]$

49.[R] $f(x) = 2\sin(x) - \sin(2x)$ on $[-\pi,\pi]$

50.[R] $f(x) = \sin(x^2) + \cos(x^2)$ on $[0, \sqrt{2\pi}]$

51.[R] $f(x) = \sin(x) - \cos(x)$ on $[-2\pi, 2\pi]$

52.[R] $f(x) = \sin^2(x) - \cos^2(x)$ on $[-2\pi, 2\pi]$

In Exercises 53 to 59 graph the function.

53.[R] $f(x) = \dfrac{\sin(x)}{1+2\cos(x)}$

54.[R] $f(x) = \dfrac{\sqrt{x^2-1}}{x}$

55.[R] $f(x) = \dfrac{1}{(x-1)^2(x-2)}$

56.[R] $f(x) = \dfrac{3x^2+5}{x^2-1}$

57.[R] $f(x) = 2x^{1/3} + x^{4/3}$

58.[R] $f(x) = \dfrac{3x^2+5}{x^2+1}$

59.[R] $f(x) = \sqrt{3}\sin(x) + \cos(x)$

60.[M] Graph $f(x) = (x^2-9)^{1/3}e^{-x}$. HINT: This function is difficult to graph in one picture. Instead, create separate sketches for $x > 0$ and for $x < 0$. Watch out for the points where f is not differentiable.

61.[M] A certain differentiable function has $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$. Moreover, $f(0) = 3$, $f(1) = 1$, and $f(2) = 2$.

(a) What is the minimum value of $f(x)$ for x in $[0, 2]$? Why?

(b) What is the maximum valu $[0, 2]$? Why?

In Exercises 62 to 64 decide if ther meets all of the stated conditions. is such a function, sketch its poss wise, explain why a function can conditions.

62.[M] $f(x) > 0$ for all x, $f'(x) <$

63.[M] $f(3) = 1$, $f(5) = 1$, $f'(x)$

64.[M] $f'(x) \neq 0$ for all x except $f'(x) = 0$ and $f(x) = 0$ for $x = -2$

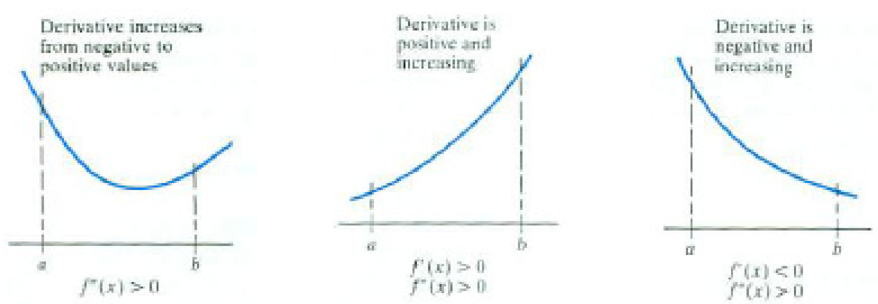
65.[M] What is the minimum v $x)/(x^2-4)$ for $x > 2$?

4.3 The Second Derivative and Graphing

The sign of the first derivative tells whether a function is increasing or decreasing. In this section we examine what the sign of the second derivative tells us about a function and its graph. This information will be used to help graph functions and also to provide an additional way to test whether a critical point is a maximum or minimum.

Concavity and Points of Inflection

The second derivative is the derivative of the first derivative. Thus, the sign of the second derivative determines if the first derivative is increasing or decreasing. For example, if $f''(x)$ is positive for all x in an interval (a, b), then f' is an increasing function throughout the interval (a, b). In other words, the slope of the graph of $y = f(x)$ increases as x increases from left to right on that part of the graph corresponding to (a, b). The slope may increase from



(a) (b) (c)

Figure 4.3.1:

negative values to zero to positive values, as in Figure 4.3.1(a). Or the slope may be positive throughout (a, b), as in Figure 4.3.1(b). Or the slope may be negative throughout (a, b), as in Figure 4.3.1(c).

As you drive along it, going from left to right, you keep turning the steering wheel counterclockwise.

In the same way, if $f''(x)$ is negative on the interval (a, b) then f' is decreasing on (a, b). The slope of the graph of $y = f(x)$ decreases as x increases from left to right on that part of the graph corresponding to (a, b).

DEFINITION *(Concave Up and Concave Down)*

A function f whose first derivative is increasing throughout the open interval (a, b) is called **concave up** in that interval.

A function f whose first derivative is decreasing throughout the open interval (a, b) is called **concave down** in that interval.

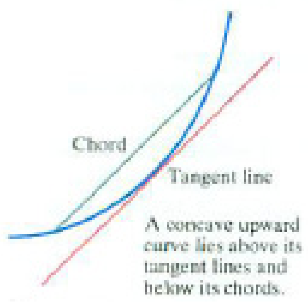


Figure 4.3.2:

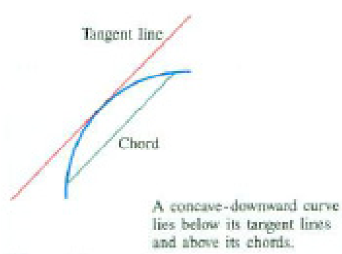


Figure 4.3.3:

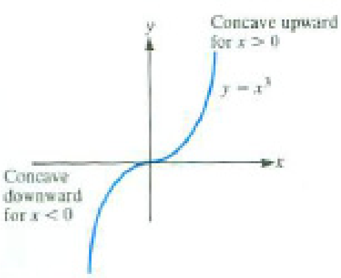


Figure 4.3.4:

When a curve is concave up, it lies above its tangent lines and below its chords. The graph of a concave up function is shaped like a cup. See Figure 4.3.2.

When a curve is concave down, it lies below its tangent lines and above its chords. The graph of a concave down function is shaped like a frown. See Figure 4.3.3.

Convex and Concave Sets

In more advanced courses "concave up" is called "convex." This is because the set in the xy-plane above this part of a graph is a convex set. (A convex set is a set with the property that any two points P and Q in the set the line segment joining them also lies in the set. See also Exercises 26 to 32 in Section 2.5.) In the same way "concave down" is called "concave." For instance, the part of the graph of $y = x^3$ to the right of the x-axis is convex and the part to the left is concave.

EXAMPLE 1 Where is the graph of $f(x) = x^3$ concave up? concave down?

SOLUTION First, compute the second derivative: $f''(x) = 6x$. Clearly, $6x$ is positive when x is positive and negative when x is negative. Thus, the graph is concave up for $x > 0$ and is concave down for $x < 0$. Note that the sense of concavity changes at $x = 0$, where $f''(x) = 0$. (See Figure 4.3.4.) ◇

In an interval where $f''(x)$ is positive, the function $f'(x)$ is increasing, and so the function f is concave up. However, if a function is concave up, $f''(x)$ need not be positive for all x in the interval. For instance, consider $y = x^4$. Even though the second derivative $12x^2$ is zero for $x = 0$, the first derivative $4x^3$ is increasing on any interval, so the graph is concave up over any interval.

Any point where the graph of a function changes concavity is important.

DEFINITION *(Inflection Number and Inflection Point)* Let f be a function and let a be a number. Assume there are numbers b and c such that $b < a < c$ and

1. f is continuous on the open interval (b, c)
2. f is concave up on (b, a) and concave down on (a, c)
   or
   f is concave down on (b, a) and concave up on (a, c).

Then, the point $(a, f(a))$ is called an **inflection point** or **point of inflection** of f. The number a is called an **inflection number** of f.

Notice that having $f''(a) = 0$ does not automatically make a an inflection number of f. To be an inflection number, the concavity has to change at a.

Observe that if the second derivative changes sign at the number a, then a is an inflection number. If the second derivative exists at an inflection number, it must be 0. But there can be an inflection point if $f''(a)$ is not defined. This is illustrated in the next example.

EXAMPLE 2 Examine the concavity of the graph of $y = x^{1/3}$.
SOLUTION Here $y' = \frac{1}{3}x^{-2/3}$ and $y'' = \frac{1}{3}\left(\frac{-2}{9}\right)x^{-5/3}$. Althought $x = 0$ is in the domain of this function, neither y' nor y'' is defined for $x = 0$. When x is negative, y'' is positive; when x is positive, y'' is negative. Thus, the concavity changes from concave up to concave down at $x = 0$. This means $x = 0$ is an inflection number and $(0, 0)$ is an inflection point. See Figure 4.3.5. ◇

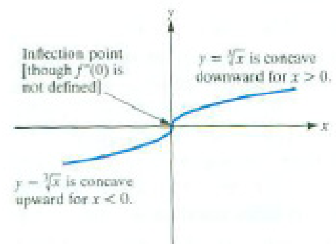


Figure 4.3.5:

The simplest way to look for inflection points is to use both the first and second derivatives:

To find inflection points of $y = f(x)$:

1. Compute $f'(x)$ and $f''(x)$.
2. Look for numbers a such that f'' is not defined at a.
3. Look for numbers a such that $f''(a) = 0$
4. For each interval defined by the numbers found in Steps 2 and 3, determine the sign of $f''(x)$.

This process can be implemented using the same ideas used in Section 4.2 to identify critical points, as Example 3 shows.

EXAMPLE 3 Find the inflection point(s) of $f(x) = x^4 - 8x^3 + 18x^2$.
SOLUTION First, $f'(x) = 4x^3 - 24x^2 + 36x$ and

$$f''(x) \quad = \quad 12x^2 - 48x + 36 = 12(x^2 - 4x + 3) = 12(x - 1)(x - 3).$$

Because f'' is defined for all real numbers, the only candidate for inflection numbers are the solutions to $f''(x) = 0$. Solving $f''(x) = 0$ yields:

$$0 \quad = \quad 12(x - 1)(x - 3).$$

Hence $x - 1 = 0$ or $x - 3 = 0$, and $x = 1$ or $x = 3$.

To decide whether 1 or 3 are inflection numbers of f, look at the sign of $f''(x) = 12(x - 1)(x - 3)$. For $x > 3$ both $x - 1$ and $x - 3$ are positive, so $f''(x)$

is positive. For x in $(1,3)$, $x-1$ is positive and $x-3$ is negative, so $f''(x)$ is negative. For $x<1$, both $x-1$ and $x-3$ are negative, so $f''(x)$ is positive. This is recorded in Table 4.3.1. Since sign changes in $f''(x)$ correspond to changes in concavity of the graph of f, this function has two inflection points: $(1,11)$ and $(3,27)$. (See Figure 4.3.6.)

x	$(-\infty,1)$	1	$(1,3)$	3	$(3,\infty)$
$f''(x)$	+	0	−	0	+

Table 4.3.1:

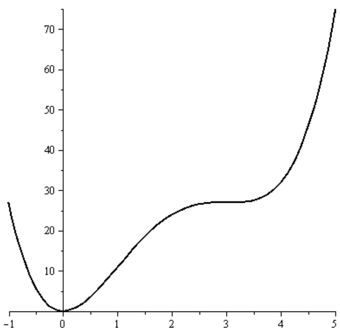


Figure 4.3.6:

◇

Using Concavity in Graphing

The second derivative, together with the first derivative and the other tools of graphing, can help us sketch the graph of a function. Example 4 continues Example 3.

EXAMPLE 4 Graph $f(x)=x^4-8x^3+18x^2$.
SOLUTION Because f is defined for all real numbers and $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty$, it has no asymptotes. Since $f(0)=0^4-8(0^3)+18(0^2)$, its y intercept is 0. To find its x intercepts we look for solutions to the equation

$$\begin{aligned} x^4-8x^3+18x^2 &= 0 \\ x^2(x^2-8x+18) &= 0. \end{aligned}$$

The **discriminant** of ax^2+bx+c is b^2-4ac.

Thus $x=0$ or $x^2-8x+18=0$. The quadratic equation can be solved by the quadratic formula. The discriminant is $(-8)^2-4(1)(18)=-8$ which is negative, so there are no real solutions of $x^2-8x+18=0$. The only x intercept of $y=f(x)$ is $x=0$.

Analysis based on $f'(x)$

In Example 3 the first derivative was found:

$$f'(x) = 4x^3-24x^2+36x = 4x(x^2-6x+9) = 4x(x-3)^2.$$

Thus, $f'(x)=0$ only when $x=0$ and $x=3$. The two critical points are $(0,f(0))=(0,0)$ and $(3,f(3))=(3,27)$. The information in Table 4.3.2 allows us to conclude that the function f is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$ with a local minimum at $(0,0)$.

x	$(-\infty,0)$	0	$(0,3)$	3	$(3,\infty)$
$f'(x)$	−	0	+	0	+

Table 4.3.2:

By Example 3, the graph is concave up on $(-\infty, 1)$ and $(3, \infty)$ and concave down on $(1, 3)$.

To begin to sketch the graph of $y = f(x)$, plot the three points $(0, f(0)) = (0, 0)$, $(1, f(1)) = (1, 11)$, and $(3, f(3)) = (3, 27)$. These three points divide the domain into four intervals. On $(-\infty, 0)$ the function is decreasing and concave up; on $(0, 1)$ it is increasing and concave up; on $(1, 3)$ it is increasing and concave down; and on $(3, \infty)$ it is once again increasing and concave up. The final graph is shown in Figure 4.3.7. ◇

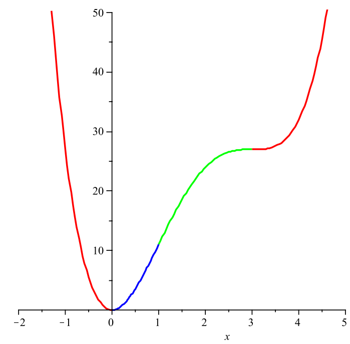


Figure 4.3.7:

(a) (b) (c) (d)

Figure 4.3.8: The general shape of a function that is (a) increasing and concave up, (b) increasing and concave down, (c) decreasing and concave up, and (d) decreasing and concave down

The procedure demonstrated in Example 4 has several advantages. Note that it was necessary to evaluate $f(x)$ only at a few "important" inputs x. These inputs cut the domain into intervals where neither the first derivative nor the second derivative changes sign. On each of these intervals the graph of the function will have one of the four shapes shown in Figure 4.3.8. A graph usually is made up of these four shapes.

Local Extrema and the Second-Derivative Test

The second derivative is also useful in testing whether a critical number corresponds to a relative minimum or relative maximum. For this, we will use the relationships between concavity and tangent lines shown in Figures 4.3.2 and 4.3.3.

Let a be a critical number for the function f. Assume, for instance, that $f''(a)$ is negative. If f'' is continuous in some open interval that contains a, then (by the Permanence Property) $f''(x)$ remains negative for a suitably small open interval that contains a. This means the graph of f is concave down near $(a, f(a))$, hence it lies below its tangent lines. In particular, it lies below the horizontal tangent line at the critical point $(a, f(a))$, as illustrated

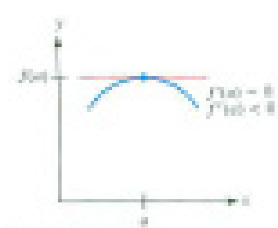


Figure 4.3.9:

in Figure 4.3.9. Thus the function f has a *relative maximum* at the critical number a. Similarly, if $f'(a) = 0$ and $f''(a) > 0$, the critical point $(a, f(a))$ is a relative minimum because the graph of f is concave up and lies above the horizontal tangent line at $(a, f(a))$. These observations suggest the following test for a relative extremum.

Theorem 4.3.1. *Second-Derivative Test for Relative Extrema Let f be a function such that $f'(x)$ is defined at least on some open interval containing the number a. Assume that $f''(x)$ is continuous and $f''(a)$ is defined.*

If $f'(a) = 0$ and $f''(a) < 0$, then f has a relative minimum at $(a, f(a))$.

If $f'(a) = 0$ and $f''(a) > 0$, then f has a relative maximum at $(a, f(a))$.

Compare with Examples 3 and 4.

EXAMPLE 5 Use the Second-Derivative Test to classify all local extrema of the function $f(x) = x^4 - 8x^3 + 18x^2$.

SOLUTION This is the same function analyzed in Examples 3 and 4. The two critical points are $(0, 0)$ and $(3, 27)$. The second derivative is $f''(x) = 12x^2 - 48x + 36$. At $x = 0$ we have

$$f''(0) = 12(0^2) - 48(0) + 36 = 36,$$

which is positive. Since $f'(0) = 0$ and $f''(0) > 0$, f has a local minimum at $(0, 0)$. At $x = 3$ we have

$$f''(3) = 12(3^2) - 48(3) + 36 = 0.$$

Since $f''(3) = 0$, the Second-Derivative Test tells us nothing about the critical number 3.

This is consistent with our previous findings. The point at $(3, 27)$ is an inflection point and not a local extreme point. ◇

Summary

Table 4.3.3 shows the meaning of the signs of $f(x)$, $f'(x)$, and $f''(x)$ in terms of the graph of $y = f(x)$.

The graph has a critical point at $(a, f(a))$ whenever $f'(a) = 0$ (or $f'(a)$ does not exist). This critical point is an extremem of f if the first derivative changes sign at $x = a$; a maximum if the first derivative changes from positive

	is positive (> 0).	is negative (< 0).	changes sign.	is zero ($= 0$).
Where the ordinate $f(x)$	the graph is above the x-axis.	the graph is below the x-axis.	the graph crosses the x-axis.	there is an x intercept.
Where the slope $f'(x)$	the graph slopes upward.	the graph slopes downward.	the graph has a horizontal tangent and a relative extremum.	there is a critical point.
Where $f''(x)$	the graph is concave up (like a cup).	the graph is concave down (like a frown).	the graph has an inflection point.	there may be an inflection point.

Table 4.3.3: EDITOR: This table should appear after the first, short, paragraph of the Summary.

to negative and a minimum if the first derivative changes from negative to positive.

Keep in mind that the graph has an inflection point at $(a, f(a))$ when the sign of $f''(x)$ changes at $x = a$. This can occur when either $f''(a) = 0$ or when $f''(a)$ is not defined. Similarly, a graph can have a maximum or minimum at $(a, f(a))$ when either $f'(a) = 0$ or $f'(a)$ is not defined.

EXERCISES for Section 4.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 16 describe the intervals where the function is concave up and concave down and give any inflection points.

1.[R] $f(x) = x^3 - 3x^2 + 2$

2.[R] $f(x) = x^3 - 6x^2 + 1$

3.[R] $f(x) = x^2 + x + 1$

4.[R] $f(x) = 2x^2 - 5x$

5.[R] $f(x) = x^4 - 4x^3$

6.[R] $f(x) = 3x^5 - 5x^4$

7.[R] $f(x) = \frac{1}{1+x^2}$

8.[R] $f(x) = \frac{1}{1+x^4}$

9.[R] $f(x) = x^3 + 6x^2 - 15x$

10.[R] $f(x) = \frac{x^2}{2} + \frac{1}{x}$

11.[R] $f(x) = e^{-x^2}$

12.[R] $f(x) = xe^x$

13.[R] $f(x) = \tan(x)$

14.[R] $f(x) = \sin(x) + \sqrt{3}\cos(x)$

15.[R] $f(x) = \cos(x)$

16.[R] $f(x) = \cos(x) + \sin(x)$

In Exercises 17 to 29 graph the polynomials, showing critical points, inflection points, and intercepts.

17.[R] $f(x) = x^3 + 3x^2$

18.[R] $f(x) = 2x^3 + 9x^2$

19.[R] $f(x) = x^4 - 4x^3 + 6x^2$

20.[R] $f(x) = x^4 + 4x^3 + 6x^2 - 2$

21.[R] $f(x) = x^4 - 6x^3 + 12x^2$

22.[R] $f(x) = 2x^6 - 10x^4 + 10$

23.[R] $f(x) = 2x^6 + 3x^5 - 10x^4$

24.[R] $f(x) = 3x^4 + 4x^3 - 12x^2 + 4$

25.[R] $f(x) = xe^{-x}$

26.[R] $f(x) = e^{x^3}$

27.[R] $f(x) = 3x^5 - 20x^3 + 10$ NOTE: This function was first encountered in Example 3 in Section 4.2.

28.[R] $f(x) = 3x^4 + 4x^3 - 12x^2 + 4$

29.[R] $f(x) = 2x^6 - 15x^4 + 20x^3 - 20x + 10$

In each of Exercises 30 to 37 sketch the general appearance of the graph of the given function near $(1, 1)$ on the basis of the information given. Assume that f, f', and f'' are continuous.

30.[R] $f(1) = 1$, $f'(1) = 0$, $f''(1) = 1$

31.[R] $f(1) = 1$, $f'(1) = 0$, $f''(1) = -1$

32.[R] $f(1) = 1$, $f'(1) = 0$, $f''(1) = 0$ NOTE: Sketch four quite different possibilities.

33.[R] $f(1) = 1$, $f'(1) = 0$, $f''(1) = 0$, $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$

34.[R] $f(1) = 1$, $f'(1) = 0$, $f''(1) = 1$ and $f''(x) < 0$ for x near 1

35.[R] $f(1) = 1$, $f'(1) = 1$, $f''(1) = -1$

36.[R] $f(1) = 1$, $f'(1) = 1$, $f''(1) = 0$, $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$

37.[R] $f(1) = 1$, $f'(1) = 1$, $f''(1) = 0$ and $f''(x) > 0$ for x near 1

38.[R] Find all inflection points of $f(x) = x\ln(x)$. On what intervals is the graph of $y = f(x)$ concave up? concave down? Graph $y = f(x)$ on an interval large enough to clearly show all interesting features of the graph. On what intervals is the graph increasing? decreasing? NOTE: This graph was first encountered in Example 2.

39.[R] Find all inflection points of $f(x) = x + \ln(x)$. On what intervals is the graph of $y = f(x)$ concave up? concave down? Graph $y = f(x)$ on an interval large enough to show all interesting features of the graph. On what intervals is the function increasing? decreasing?

40.[R] Find all inflection points of $f(x) = (x+1)^{2/7}e^{-x}$. On what intervals is the graph of $y = f(x)$ concave up? concave down? On what intervals is the function increasing? decreasing? NOTE: This function was first encountered in Example 4.

41.[R] Find the critical points and inflection points of $f(x) = x^2e^{-x/3}$. NOTE: See Example 1.

In Exercises 42 to 43 sketch a graph of a hypothetical function that meets the given conditions. Assume f' and f'' are continuous. Explain your reasoning.

42.[R] Critical point $(2,4)$; inflection points $(3,1)$ and $(1,1)$; $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$

43.[R] Critical points $(-1,1)$ and $(3,2)$; inflection point $(4,1)$; $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to 0^-} f(x) = \infty$ $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = \infty$

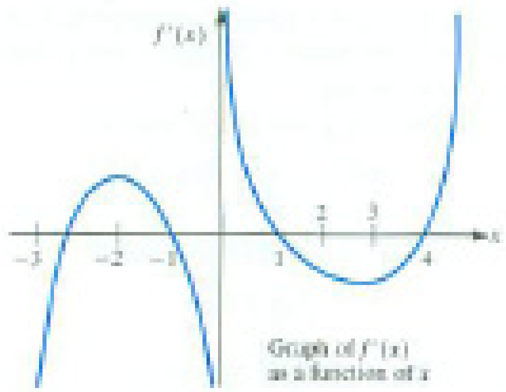


Figure 4.3.10:

44.[M] *(Contributed by David Hayes)* Let f be a function that is continuous for all x and differentiable for all x other than 0. Figure 4.3.10 is the graph of its derivative $f'(x)$ as a function of x.

(a) Answer the following questions about f (*not* about f'). Where is f increasing? decreasing? concave up? concave down? What are the critical numbers? Where do any relative extrema occur? Explain.

(b) Assuming that $f(0) = 1$, graph a hypothetical function f that satisfies the conditions given.

(c) Graph $f''(x)$.

45.[M] Graph $y = 2(x-1)^{5/3} + 5(x-1)^{2/3}$, paying particular attention to points where y' does not exist.

46.[M] Graph $y = x + (x+1)^{1/3}$.

47.[M] Find the critical points and inflection points in $[0, 2\pi]$ of $f(x) = \sin^2(x)\cos(x)$.

48.[M] Can a polynomial of degree 6 have (a) no inflection points? (b) exactly one inflection point? Explain.

49.[M] Can a polynomial of degree 5 have (a) no inflection points? (b) exactly one inflection point? Explain.

50.[C] In the theory of **inhibited growth** it is assumed that the growing quantity y approaches some limiting size M. Specifically, one assumes that the rate of growth is proportional both to the amount present and to the amount left to grow:

$$\frac{dy}{dt} = ky(M - y),$$

where k is a positive number. Prove that the graph of y as a function of time has an inflection point when the amount y is exactly half the limiting amount M.

51.[M] Let f be a function such that $f''(x) = (x - 1)(x - 2)$.

(a) For which x is f concave up?

(b) For which x is f concave down?

(c) List its inflection number(s).

(d) Find a specific function f whose second derivative is $(x - 1)(x - 2)$.

52.[C] A certain function $y = f($
that

$$y' = \sin(y) + 2y +$$

Show that at a critical number the
minimum.

53.[C] Assume that the domain of
axis, and $f'(x)$ and $f''(x)$ are contir
$(1, 1)$ is the only critical point and

(a) Can $f(x)$ be negative for sor

(b) Must $f(x)$ be decreasing for

(c) Must $f(x)$ have an inflection

4.4 Proofs of the Three Theorems

In Section 4.1 two observations about tangent lines led to the Theorem of the Interior Extremum, Rolle's Theorem, and the Mean-Value Theorem. Now, using the definition of the derivative, and no pictures, we prove them. That the proofs go through based only on the definition of the derivative as a limit reassures us that this definition is suitable to serve as part of the foundation of calculus.

Proof of Theorem 4.1.1:

Proof of the Theorem of the Interior Extremum

Suppose the maximum of f on the open interval (a, b) occurs at the number c. This means that $f(c) \geq f(x)$ for each number x between a and b.

$f'(c) = 0$ at the maximum or minimum on an open interval.

Our challenge is to use only this information and the definition of the derivative as a limit to show that $f'(c) = 0$.

Assume that f is differentiable at c. We will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, forcing $f'(c)$ to be zero.

Recall that

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

The assumption that f is differentiable on (a, b) means that $f'(c)$ exists. Consider the difference quotient

$$\frac{f(c + \Delta x) - f(c)}{\Delta x}. \tag{4.4.1}$$

when Δx is so small that $c+\Delta x$ is in the interval (a, b). Then $f(c+\Delta x) \leq f(c)$. Hence $f(c + \Delta x) - f(c) \leq 0$. Therefore, when Δx is positive, the difference quotient in (4.4.1) will be negative, or 0. Consequently, as $\Delta x \to 0$ through *positive* values,

$\frac{\text{negative}}{\text{positive}} = \text{negative}$

$$f'(c) = \lim_{\Delta x \to 0^+} \frac{f(c + \Delta x) - f(c)}{\Delta x} \leq 0. \tag{4.4.2}$$

If, on the other hand, Δx is negative, then the difference quotient in (4.4.3) will be positive, or 0. Hence, as $\Delta x \to 0$ through *negative* values,

$\frac{\text{negative}}{\text{negative}} = \text{positive}$

$$f'(c) = \lim_{\Delta x \to 0^-} \frac{f(c + \Delta x) - f(c)}{\Delta x} \geq 0. \tag{4.4.3}$$

The only way $f'(c) \leq 0$ and $f'(c) \geq 0$ can both hold is when $f'(c) = 0$. This proves that if f has a maximum on (a, b), then $f'(c) = 0$.

See Exercise 12.

The proof for the case when f has a minimum on (a, b) is essentially the same. •

The proofs of Rolle's Theorem and the Mean-Value Theorem are related. Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

Proof of Rolle's Theorem

Proof of Theorem 4.1.2:

If $f(a) = f(b)$, then $f'(c) = 0$ for at least one number between a and b.

The goal here is to use the facts that f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$ to conclude that there must a number c in (a, b) with $f'(c) = 0$.

Since f is continuous on the closed interval $[a, b]$, it has a maximum value M and a minimum value m on that interval. There are two cases to consider: $m < M$ and $m = M$.

Case 1: If $m = M$, f is constant and $f'(x) = 0$ for all x in $[a, b]$. Then any number in (a, b) will serve as the desired number c.

Case 2: Suppose $m < M$. Because $f(a) = f(b)$ the minimum and maximum cannot both occur at the ends of the interval. At least one of the extrema occurs at a number c strictly between a and b. By assumption, f is differentiable at c, so $f'(c)$ exists. Thus, by the Theorem of the Interior Extremum, $f'(c) = 0$. This completes the proof of Rolle's Theorem. •

The idea behind the proof of the Mean-Value Theorem is to define a function to which Rolle's Theorem can be applied.

Proof of the Mean-Value Theorem

Proof of Theorem 4.1.3:

$f'(c) = \frac{f(b)-f(a)}{b-a}$ for at least one number between a and b.

Let $y = L(x)$ be the equation of the chord through the two points $(a, f(a))$ and $(b, f(b))$. The slope of this line is $L'(x) = \dfrac{f(b) - f(a)}{b - a}$. Define $h(x) = f(x) - L(x)$. Note that $h(a) = h(b) = 0$ because $f(a) = L(a)$ and $f(b) = L(b)$.

By assumption, f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). So h, being the difference of f and L, is also continuous on $[a, b]$ and differentiable on (a, b).

Rolle's Theorem applies to h on the interval $[a, b]$. Therefore, there is at least one number c in (a, b) where $h'(c) = 0$. Now, $h'(c) = f'(c) - L'(c)$, so that

$$f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a}.$$

•

Summary

Using only the definition of the derivative and the assumption that a continuous function defined on a closed interval assumes maximum and minimum values, we proved the Theorem of the Interior Extremum, Rolle's Theorem, and the Mean-Value Theorem. Note that we did not appeal to any pictures or to our geometric intuition.

EXERCISES for Section 4.4 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 3 sketch a graph of a differentiable function that meets the given conditions. (Just draw the graph; there is no need to come up with a formula for the function.)

1.[R] $f'(x) < 0$ for all x

2.[R] $f'(3) = 0$ and $f'(x) < 0$ for x not equal to 3

3.[R] $f'(x) = 0$ only when $x = 1$ or 4; $f(1) = 3$, $f(4) = 1$; $f'(x) > 0$ for $x < 1$ and for $x > 4$

In Exercises 4 to 5 explain why no differentiable function satisfies all the conditions.

4.[M] $f(1) = 3$, $f(2) = 4$, $f'(x) < 0$ for all x

5.[M] $f(x) = 2$ only when $x = 0$, 1, and 3; $f'(x) = 0$ only when $x = \frac{1}{4}$, $\frac{3}{4}$, and 4.

6.[M] In "Surely You're Joking, Mr. Feynmann!," Norton, New York, 1985, Nobel laureate Richard P. Feynmann writes:

> I often liked to play tricks on people when I was at MIT. One time, in mechanical drawing class, some joker picked up a French curve (a piece of plastic for drawing smooth curves — a curly funny-looking thing) and said, "I wonder if the curves on that thing have some special formula?"
>
> I thought for a moment and said, "Sure they do. The curves are very special curves. Lemme show ya," and I picked up my French curve and began to turn it slowly. "The French curve is made so that at the lowest point on each curve, no matter how you turn it, the tangent is horizontal."
>
> All the guys in the class were holding their French curve up at different angles, holding their pencil up to it at the lowest point and laying it down, and discovering that, sure enough, the tangent is horizontal.

How was Feynmann playing a trick on his classmates?

7.[M] What can be said about the number of solutions of the equation $f(x) = 3$ for a differentiable function if

(a) $f'(x) > 0$ for all x?

(b) $f'(x) > 0$ for $x < 7$ and $f'(x) < 0$ for $x > 7$?

8.[M] Consider the function $f(x) = x^3 + ax^2 + c$. Show that if $a < 0$ and $c > 0$, then f has exactly one negative root.

9.[M] With the book closed, obtain the Mean-Value Theorem from Rolle's Theorem.

10.[M]

(a) Recall the definition of $L(x)$ in the proof of the Mean-Value Theorem, and show that

$$L(x) = f(a) + \frac{x-a}{b-a}\left(f(b) - f(a)\right).$$

(b) Using (a), show that

$$L'(x) = \frac{f(b) - f(a)}{b-a}.$$

11.[M] Show that Rolle's Theorem is a special case of the Mean-Value Theorem.

12.[C] Prove the Theorem of the Interior Extremum when the minimum of f on (a, b) occurs at c.

13.[C] Show that a polynomial $f(x)$ of degree n, $n \geq 1$, can have at most n distinct real roots, that is, solutions to the equation $f(x) = 0$.

(a) Use algebra to show that the statement holds for $n = 1$ and $n = 2$.

(b) Use calculus to show that the statement then holds for $n = 3$.

(c) Use calculus to show that the statement continues to hold for $n = 4$ and $n = 5$.

(d) Why does it hold for all positive integers n?

14.[C] Is this proposed proof of the Mean-Value Theorem correct?
Proof

Tilt the x and y axes and the graph of the function until the x-axis is parallel to the given chord. The chord is now "horizontal," and we may apply Rolle's Theorem. •

15.[C] Is there a differentiable function f whose domain is the x-axis such that f is increasing and yet the derivative is *not* positive for all x?

16.[C] Prove: If f has a negative derivative on (a, b) then f is decreasing on the interval $[a, b]$.

Exercises 17 to 19 provide analytic justification for the statement in Section 4.3 that "[W]hen a curve is concave up, it lies above its tangent lines and below its chords."

17.[C] Show that in an open interval in which f'' is positive, tangents to the graph of f lie below the curve. HINT: Why do you want to show that if a and x are in the interval, then $f(x) > f(a) + f'(a)(x-a)$? Treat the cases $a < x$ and $x > a$ separately.

18.[C] Assume that $f''(x)$ is positive for x in an open interval. Let $a < b$ be two numbers in the interval. Show that the chord joining $(a, f(a))$ and $(b, f(b))$ lies above the graph of f. HINT: Consider the following three questions:

1. Why does one want to prove that $f(x) < f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)$?
2. How does it help to know that $\dfrac{f(b)-f(a)}{b-a} < \dfrac{f(x)-f(a)}{x-a}$?
3. Show that the function on the right-hand side of the inequality in (b) is increasing for $a < x < b$. Why does this show that the chords lie above the curve?

19.[C]

Sam: I can do Exercise 18 more easily. I'll show that (b) is true. By the Mean-Value Theorem, I can write the left side as $f'(c)$ where c is in $[a, b]$ and the right side as $f'(d)$ where d is in $[a, x]$. Since $b > x$, I know $c > d$, hence $f'(c) > f'(d)$. Nothing to it.

Is Sam's reasoning correct?

20.[C] We stated, in Section 4.3, that if $f(x)$ is defined in an open interval around the critical number a and $f''(a)$ is negative, then $f(x)$ has a relative maximum at a. Explain why this is so, following these steps.

(a) Why is $\lim_{\Delta x \to 0} \dfrac{f'(a+\Delta x) - f'(a)}{\Delta x}$ negative?

(b) Deduce that if Δx is small and positive, then $f'(a+\Delta x)$ is negative.

(c) Show that if Δx is small and negative, then $f'(a+\Delta x)$ is positive.

(d) Show that $f'(x)$ changes sign from positive to negative at a. By the First-Derivative Test for a Relative Maximum, $f(x)$ has a relative maximum at a.

Skill Drill

21.[M] To keep your differentiation skills sharp, differentiate each of the following expressions:

(a) $\sqrt{1-x^2}\sin(3x)$

(b) $\dfrac{\sqrt[3]{x}}{x^2+1}$

(c) $\tan\left(\dfrac{1}{(2x+1)^2}\right)$

(d) $\ln\left(\dfrac{(x^2+1)^3\sqrt{1-x^2}}{\sec^2(x)}\right)$

(e) e^{x^4}

4.S Chapter Summary

In this chapter we saw that the sign of the function and of its first and second derivatives influenced the shape of its graph. In particular the derivatives show where the function is increasing or decreasing and is concave up or down. That enabled us to find extreme points and inflection points. (See Table 4.3.3 on page 277.)

We state here the main ideas informally for a function with continuous first and second derivatives.

If a function has an extremum at a number, then the derivative there is zero, or is not defined, or the number may be an end point of the domain. This narrows the search for extrema. If the derivative is zero and the second derivative is not zero, the function has an extremum there.

The rationale for these tests rests on Rolle's theorem, which says that if a differentiable function vanishes at two inputs on an interval in its domain, its derivative must be zero somewhere between them.

The Mean Value Theorem generalizes this idea. It says that between any two points on its graph there is a point on the graph where the tangent is parallel to the chord through those two points. We used this to show that: If a and b are two numbers, then $f(b) = f(a) + f'(c)(b - a)$ for some number c between a and b.

If $f'(a)$ is positive and if f' is continuous in some open interval containing a, then, by the permanence principle, $f'(x)$ remains positive for some open interval containing a. Typically, if the derivative is positive at some number, then the function is increasing for inputs near that number. (A similar statement holds when $f'(a)$ is negative.)

Sam: Why bother me with limits? The authors say we need them to define derivatives.

Jane: Aren't you curious about why the formula for the derivative of a product is what it is?

Sam: No. It's been true for over three centuries. Just tell me what it is. If someone says the speed of light is 186,000 miles per second am I supposed to find a meter stick and clock and check it out?

Jane: But what if you forget the formula during a test?

Sam: That's not much of a reason.

Jane: But my physics class uses derivatives and limits to define basic concepts.

Sam: Oh?

Jane: Density of mass at a point or density of electric charge are defined as limits. And it uses derivatives all over the place. You will be lost if you don't know their definitions. Just look at the applications in Chapter 5.

Sam: O.K., O.K. enough. I'll look.

EXERCISES for 4.S *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 13 decide if it is possible for a single function to have all of the properties listed. If it is possible, sketch a graph of a differentiable function that meets the given conditions. (Just draw the graph; there is no need to come up with a formula for the function.) If it is not possible, explain why no differentiable function satisfies all of the conditions. **1.**[R] $f(0) = 1$, $f(x) > 0$, and $f'(x) < 0$ for all positive x

2.[R] $f(0) = -1$, $f'(x) < 0$ for all x in $[0, 2]$, and $f(2) = 0$

3.[R] x intercepts at 1 and 5; y intercept at 2; $f'(x) < 0$ for $x < 4$; $f'(x) > 0$ for $x > 4$

4.[R] x intercepts at 2 and 5; y intercept at 3; $f'(x) > 0$ for $x < 1$ and for $x > 3$; $f'(x) < 0$ for x in $(1, 3)$

5.[R] $f(0) = 1$, $f'(x) < 0$ for all positive x, and $\lim_{x\to\infty} f(x) = 1/2$

6.[R] $f(2) = 5$, $f(3) = -1$, $f'(x) \geq 0$ for all x

7.[R] x intercepts only at 1 and 2; $f(3) = -1$, $f(4) = 2$

8.[R] $f'(x) = 0$ only when $x = 1$ or 4; $f(1) = 3$, $f(4) = 1$; $f'(x) < 0$ for $x < 1$; $f'(x) > 0$ for $x > 4$

9.[R] $f(0) = f(1) = 1$ and $f'(0) = f'(1) = 1$

10.[R] $f(0) = f(1) = 1$, $f'(0) = f'(1) = 1$, and $f(x) \neq 0$ for all x in $[0, 1]$

11.[R] $f(0) = f(1) = 1$, $f'(0) = f'(1) = 1$, and $f(x) = 0$ for exactly one number x in $[0, 1]$

12.[R] $f(0) = f(1) = 1$, $f'(0) = f'(1) = 1$, and $f(x)$ has exactly two inflection numbers in $[0, 1]$

13.[R] $f(0) = f(1) = 1$, $f'(0) = f'(1) = 1$, and $f(x)$

has exactly two extrema in $[0, 1]$

14.[R] State the assumptions and conclusions of the Theorem of the Interior Extremum for a function F defined on (a, b).

15.[R] State the assumptions and conclusions of the Mean-Value Theorem for a function g defined on $[c, d]$.

16.[R] The following discussion on higher derivatives in economics appears on page 124 of the College Mathematics Journal **37** (2006):

> Charlie Marion of Shrub Oak, NY, submitted this excerpt from "Curses! The Second Derivative" by Jeremy J. Siegel in the October 2004 issue of *Kiplinger's* (p. 73):
> "... I think what is bugging the market is something that I have seen happen many times before: the Curse of the Second Derivative. The second derivative, for all those readers who are a few years away from their college calculus class, is the rate of change of the rate of change — or, in this case, whether corporate earnings, which are still rising, are rising at a faster or slower pace."
> In the October 1996 issue of the *Notices of the American Mathematical Society*, Hugo Rossi wrote, "IN the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection."

Explain why the third derivative is involved in President Nixon's statement.

17.[M] If you watch the tide come in and go out, you will notice at high tide and at low tide, the height of the tide seems to change very slowly. The same holds when you watch an outdoor thermometer: the temperature seems to change the slowest when it is at its highest or at its lowest. Why is that?

18.[R]

(a) Graph $y = \sin^2(2\theta)\cos(2\theta)$ for θ in $[-\pi/2, \pi/2]$.

(b) What is the maximum value of y?

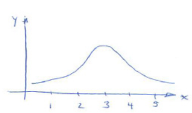
Exercises 19 to 22 display the graph of a function f with continuous f' and f''. Sketch a possible graph of f' and a possible graph of f''.

19.[R] Figure 4.S.1(a)

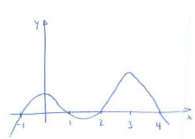
20.[R] Figure 4.S.1(b)

21.[R] Figure 4.S.1(c)

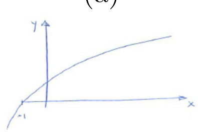
22.[R] Figure 4.S.1(d)



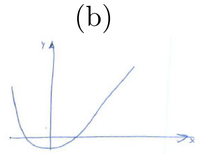
(a)



(b)



(c)



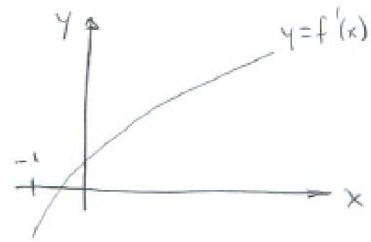
(d)

Figure 4.S.1:

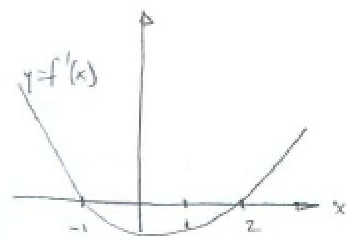
In Exercises 23 and 24 sketch the graphs of two possible functions f whose derivative f' is graphed in the given figure.

23.[R] Figure 4.S.2(a)

24.[R] Figure 4.S.2(b)



(a)



(b)

Figure 4.S.2:

25.[R] Sketch the graph of a function f whose second derivative is graphed in Figure 4.S.3.

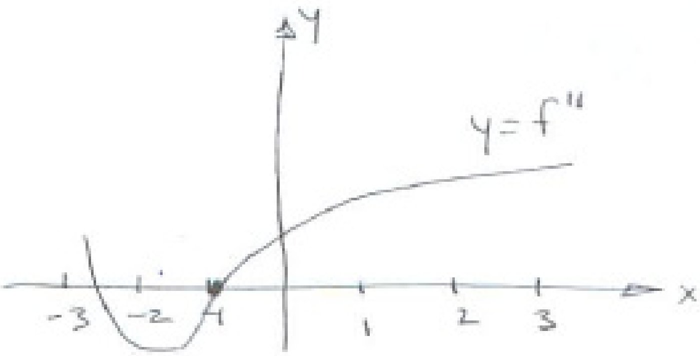


Figure 4.S.3:

26.[R] Figure 4.S.4(a) shows the only x-intercepts of a function f. Sketch the graph of possible f' and f''.

27.[R] Figure 4.S.4(b) shows the only arguments at which $f'(x) = 0$. Sketch the graph of possible f and f''.

28.[R] Figure 4.S.4(c) shows the only arguments at which $f''(x) = 0$. Sketch the graph of possible f and f'.

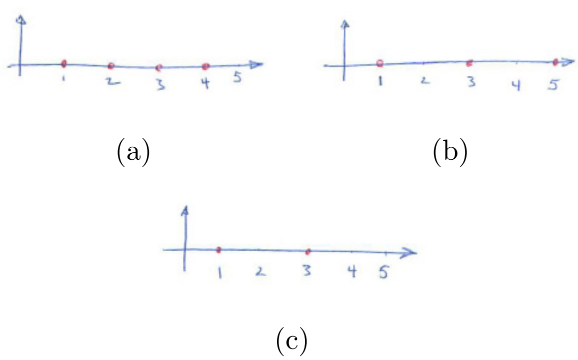


Figure 4.S.4:

In Exercises 29 to 36 graph the given functions, showing extrema, inflection points, and asymptotes.

29.[R] $e^{-2x}\sin(x)$, x in $[0, 4\pi]$

30.[R] $\frac{e^x}{1-e^x}$

31.[R] $x^3 - 9x^2$

32.[R] $x\sqrt{3-x}$

33.[R] $\frac{x-1}{x-2}$

34.[R] $\cos(x) - \sin(x)$, x in $[0, 2\pi]$

35.[R] $x^{1/2} - x^{1/4}$

36.[R] $\frac{x}{4-x^2}$

37.[R] Figure 4.S.5 shows the graph of a function f. Estimate the arguments where

(a) f changes sign,

(b) f' changes sign,

(c) f'' changes sign.

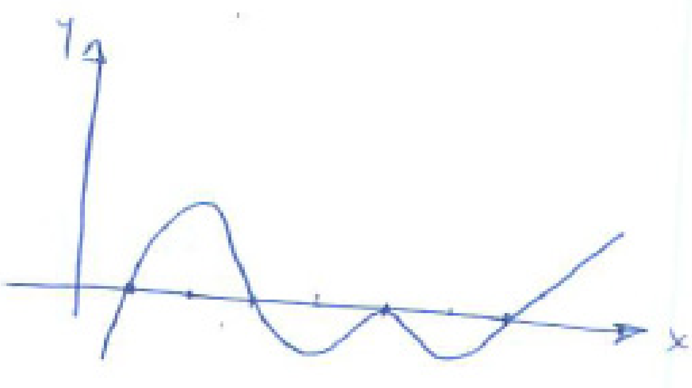


Figure 4.S.5:

38.[R] Assume the function f has continuous f' and f'' defined on an open interval.

(a) If $f'(a) = 0$ and $f''(a) = 0$, does f necessarily have an extrema at a? Explain.

(b) If $f''(a) = 0$, does f necessarily have an inflection point at $x = a$?

(c) If $f'(a) = 0$ and $f''(a) = 3$, does f necessarily have an extremum at a?

39.[R] Find the maximum value of $e^{2\sqrt{3}x}\cos(2x)$ for x in $[0, \pi/4]$.

40.[M]

(a) Show that the equation $5x - \cos(x) = 0$ has exactly one solution.

(b) Find a specific interval which contains the solution.

41.[M] Consider the function f given by the formula $f(x) = x^3 - 3x$.

(a) At which numbers x is $f'(x) = 0$?

(b) Use the theorem of the Interior Extremum to show that the maximum value of $x^3 - 3x$ for x in $[1, 5]$ occurs either at 1 or at 5.

42.[M] Let f and g be polynomials without a common root.

(a) Show that if the degree of g is odd, the graph of f/g has a vertical asymptote.

(b) Show that if the degree of f is less than or equal to the degree of g, then f/g has a horizontal asymptote.

43.[M] If $\lim_{x\to\infty} f'(x) = 0$, does it follow that f has a horizontal asymptote? Explain.

44.[M] Let f be a positive function on $(0, \infty)$ with f' and f'' both continuous. Let $g = f^2$.

(a) If f is increasing, is g?

(b) If f is concave up, is g?

45.[M] Give an example of a positive function on $(0, \infty)$ that is concave down but f^2 is concave up.

46.[M] Graph $\cos(2\theta) + 4\sin(\theta)$ for θ in $[0, 2\pi]$.

47.[M] Graph $\cos(2\theta) + 2\sin(\theta)$ for θ in $[0, 2\pi]$.

48.[M] Figure 4.S.3(b) shows part of a unit circle. The line segment CD is tangent to the circle and has length x. This exercise uses calculus to show that $AB < BC < CD$. (BC is the length of arc joining B and C.)

(a) Express AB and BC in terms of x.

(b) Using (a) and calculus, show that for $x > 0$, $AB < BC < CD$.

49.[M] Show that in an open interval in which f'' is positive, tangents to the graph of f lie below the curve. HINT: Why do you want to show that if a and x are in the interval, then $f(x) > f(a) + f'(a)(x-a)$? It is still necessary to treat the cases $x > a$ and $x < a$ separately. NOTE: This problem appears again as Exercise 96 in Section 5.7, when you have more tools to solve it.

50.[M] Assume that $f''(x)$ is positive for x in an open interval. Let $a < b$ be in the interval. In this exercise you will show that the chord joining $(a, f(a))$ to $(b, f(b))$ lies above the graph of f. ("A concave up curve has chords that lie above the curve.")

(a) Why does one want to prove that

$$f(a)+\frac{f(b)-f(a)}{b-a}(x-a) > f(x), \qquad \text{for } a < x < b?$$

(b) Why does one want to prove that

$$\frac{f(b)-f(a)}{b-a} > \frac{f(x)-f(a)}{x-a}?$$

(c) Show that the function on the right-hand side of the inequality in (b) is increasing for $a < x < b$. Why does this show that chords lie above the curve?

51.[M]

(a) Graph $y = \frac{\sin(x)}{x}$ showing intercepts and asymptotes.

(b) Graph $y = x$ and $y = \tan(x)$ relative to the same axes.

(c) Use (b) to find how many solutions there are to the equation $x = \tan(x)$.

(d) Write a short commentary on the critical points of $\sin(x)/x$. HINT: Part (c) may come in handy.

(e) Refine the graph produced in (a) to show several critical points.

52.[M] Let $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

(a) Show that the graph of $y = f(x)$ always has exactly one inflection point.

(b) Show that the inflection point separates the graph of the cubic polynomial into two parts that are congruent. HINT: Show the graph is symmetric with respect to the inflection point. NOTE: Why can one assume it is enough to show this for $a = 1$ and $d = 0$?

53.[M] Find all functions $f(x)$ such that $f'(x) = 2$ for all x and $f(1) = 4$.

54.[M] Find all differentiable functions such that $f(1) = 3$, $f'(1) = -1$, and $f''(1) = e^x$.

55.[C]

(a) Graph $y = 1/(1 + 2^{-x})$.

(b) The point $(0, 1/2)$ is on the graph and divides it into two pieces. Are the two pieces congruent?

(Curves of this type model the depletion of a finite resource; x is time and y is the fraction used up to time x. See also Exercise 71 in Section 5.7.)

56.[C]

(a) If the graph of f has a horizontal asymptote (say, $\lim_{x\to\infty} f(x) = L$), does it follow that $\lim_{x\to\infty} f'(x)$ exists?

(b) If $\lim_{x\to\infty} f'(x)$ exists in (a), most it be 0?

57.[C] Assume that f is continuous on $[1, 3]$, $f(1) = 5$, $f(2) = 4$, and $f(3) = 5$. Show that the graph of f has a horizontal chord of length 1.

58.[C] A function f defined on the whole x-axis has continuous first- and second-derivatives and exactly one inflection point. In at most how many points can a straight line intersect the graph of f? Explain. (x^n, n an odd integer greater than 1, are examples of such functions.)

59.[C] Let f be an increasing function with continuous f' and f''. What, if anything, can be said about the concavity of $f \circ f$ if

(a) f is concave up?

(b) f is concave down?

60.[C] Assume f has continuous f' and f''. Show that if f and $g = f^2$ have inflection points at the same argument a, then $f'(a) = 0$.

61.[C] Graph $y = x^2 \ln(x)$, showing extrema and inflection points. NOTE: Use the fact that $\lim_{x\to 0^+} x^2 \ln(x) = 0$; see Exercise 20 of Section 5.5.

62.[C] Assume $\lim_{x\to\infty} f'(x) = 3$. Show that for x sufficiently large, $f(x)$ is greater than $2x$. HINT: Review the Mean-Value Theorem.

63.[C] Assume that f is differentiable for all numbers x.

(a) If f is an even function, what, if anything, can be said about $f'(0)$?

(b) If f is an odd function, what, if anything, can be said about $f'(0)$?

Explain your answers.

64.[M] Graph $y = \sin(x^2)$ on the interval $[-\sqrt{\pi}, \sqrt{\pi}]$. Identify the extreme points and the inflection points.

65.[M] Assume that $f(x)$ is a continuous function not identically 0 defined on $(-\infty, \infty)$ and that $f(x + y) = f(x) \cdot f(y)$ for all x and y.

(a) Show that f(0)=1.

(b) Show that $f(x)$ is never 0.

(c) Show that $f(x)$ is positive for all x.

(d) Letting $f(1) = a$, find $f(2)$, $f(1/2)$, and $f(-1)$.

(e) Show that $f(x) = a^x$ for all x.

66.[C] Can a straight line meet the curve $y = x^5$ four times?

67.[C] Assume $y = f(x)$ is a twice differentiable function with $f(0) = 1$ and $f''(x) < -1$ for all x. Is it possible that $f(x) > 0$ for all x in $(1, \infty)$?

68.[C] If $\lim_{x\to\infty} f'(x) = 3$, does it follow that the graph of $y = f(x)$ is asymptotic to some line of the form $y = a + 3x$ for some constant a?

Calculus is Everywhere # 4
Calculus Reassures a Bicyclist

Both authors enjoy bicycling for pleasure and running errands in our flat towns. One of the authors (SS) often bicycles to campus through a parking lot. On each side of his route is a row of parked cars. At any moment a car can back into his path. Wanting to avoid a collision, he wonders where he should ride. The farther he rides from a row, the safer he is. However, the farther he rides from one row, the closer he is to the other row. Where should he ride?

Instinct tells him to ride midway between the two rows, an equal distance from both. But he has second thoughts. Maybe it's better to ride, say, one-third of the way from one row to the other, which is the same as two-thirds of the way from the other row. That would mean he has two safest routes, depending on which row he is nearer. Wanting a definite answer, he resorted to calculus.

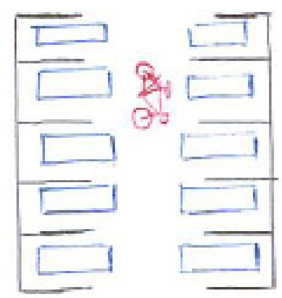


Figure C.4.1: ARTIST:picture of two rows of parked cars, with bicycle

He introduced a function, $f(x)$, which is the probability that he gets through safely when his distance from one row is x, considering only cars in that row. Then he calls the distance between the two rows be d. When he was at a distance x from one row, he was at a distance $d-x$ from the other row. The probability that he did not collide with a car backing out from either row is then the product, $f(x)f(d-x)$. His intuition says that this is maximized when $x=d/2$, putting him midway between the two rows.

What did he know about f? First of all, the farther he rode from one line of cars, the safer he is. So f is an increasing function; thus f' is positive. Moreover, when he was very far from the cars, the probability of riding safely through the lot approached 1. So he assumed $\lim_{x\to\infty} f(x) = 1$ (which it turned out he did not need).

The derivative of f' measured the rate at which he gained safety as he increased his distance from the cars. When x is small, and he rode near the cars, $f'(x)$ was large: he gained a great deal of safety by increasing x. However, when he was far from the cars, he gained very little. That means that f was a decreasing function. In other words f' is negative.

Does that information about f imply that midway is the safest route?

In other words, does the maximum of $f(x)f(d-x)$ occur when $x=d/2$? Symbolically, is

$$f(d/2)f(d/2) \geq f(x)f(d-x)?$$

To begin, he took the logarithm of that expression, in order to replace a product by something easier, a sum. He wanted to see if

$$2\ln(f(d/2)) \geq \ln(f(x)) + \ln(f(d-x)).$$

Letting $g(x)$ denote the composite function $\ln(f(x))$, he faced the inequality,

$$\begin{aligned} 2g(d/2) &\geq g(x) + g(d-x), \\ \text{or} \qquad g(d/2) &\geq \frac{1}{2}(g(x) + g(d-x)). \end{aligned}$$

This inequality asserts that the point $(d/2, g(d/2))$ on the graph of g is at least

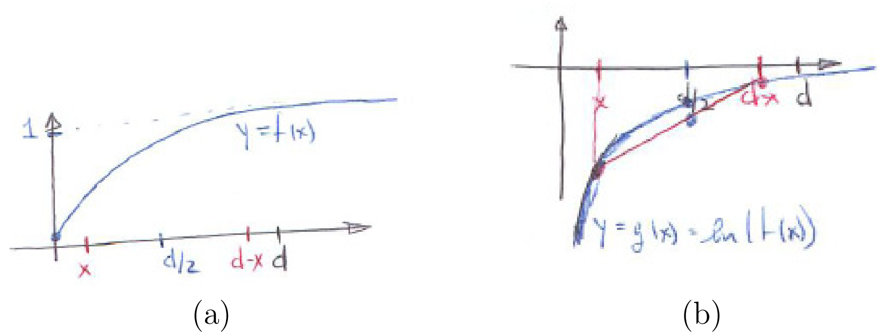


Figure C.4.2:

as high as the midpoint of the chord joining $(x, g(x))$ to $(d-x, g(d-x))$. This would be the case if the second derivative of g were negative, and the graph of g were concave down. He had to compute g'' and hope it is negative. First of all, $g'(x)$ is $f'(x)/f(x)$. Then $g''(x)$ is

$$\frac{f(x)f''(x) - (f'(x))^2}{f(x)^2}.$$

The denominator is positive. Because $f(x)$ is positive and concave down, the numerator is negative. So the quotient is negative. That means that the safest path is midway between the two rows. The bicyclist continues to follow that route, but, after these calculations, with more confidence that it is indeed the safest way.

Calculus is Everywhere # 5
Graphs in Economics

Elementary economics texts are full of graphs. They provide visual images of a variety of concepts, such as production, revenue, cost, supply, and demand. Here we show how economists use graphs to help analyze production as a function of the amount of labor, that is, the number of workers.

Let $P(L)$ be the amount of some product, such as cell phones, produced by a firm employing L workers. Since both workers and wireless network cards come in integer amounts, the graph of $P(L)$ is just a bunch of dots. In practice, these dots suggest a curve, and the economists use that curve in their analysis. So $P(L)$ is viewed as a differentiable function defined for some interval of the form $[0, b]$.

If there are no workers, there is no production, so $P(0) = 0$. When the first few workers are added, production may increase rapidly, but as more are hired, production may still increase, but not as rapidly. Figure C.5.1 is a typical **production curve**. It seems to have an inflection point when the gain from adding more workers begins to decline. The inflection point of $P(L)$ occurs at L_2 in Figure C.5.2.

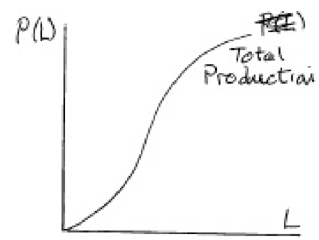


Figure C.5.1:

When the firm employs L workers and adds one more, production increases by $P(L+1) - P(L)$, the marginal production. Economists manage to relate this to the derivative by a simple trick:

$$P(L+1) - P(L) = \frac{P(L+1) - P(L)}{(L+1) - L} \tag{C.5.1}$$

The right-hand side of (C.5.1) is "change in output" divided by "change in input", which is, by the definition of the derivative, an approximation to the derivative, $P'(L)$. For this reason economists define the **marginal production** as $P'(L)$, and think of it as the extra product produced by the L plus first worker. We denote the marginal product as $m(L)$, that is, $m(L) = P'(L)$.

The **average production** per worker when there are L workers is defined as the quotient $P(L)/L$, which we denote $a(L)$. We have three functions: $P(L)$, $m(L) = P'(L)$, and $a(L) = P(L)/L$.

Now the fun begins.

At what point on the graph of the production function is the average production a maximum?

Since $a(L) = P(L)/L$, it is the slope of the line from the origin to the point $(L, P(L))$ on the graph. Therefore we are looking for the point on the graph where the slope is a maximum. One way to find that point is to rotate

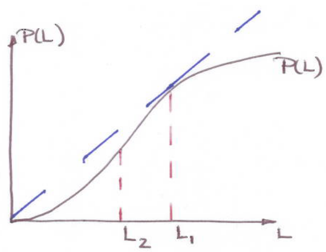


Figure C.5.2:

a straightedge around the origin, clockwise, starting at the vertical axis until it meets the graph, as in Figure C.5.2. Call the point of tangency $(L_1, P(L_1))$. For L less than L_1 or greater than L_q, average productivity is less than $a(L_1)$.

Note that at L_1 the average product is the same as the marginal product, for the slope of the tangent at $(L_1, P(L_1))$ is both the quotient $P(L_1)/L_1$ and the derivative $P'(L_1)$. We can use calculus to obtain the same conclusion:

Since $a(L)$ has a maximum when the input is L_1, its derivative is 0 then. The derivative of $a(L)$ is

$$\frac{d}{dL}\left(\frac{P(L)}{L}\right) = \frac{LP'(L) - P(L)}{L^2}. \tag{C.5.2}$$

At L_1 the quotient in (C.5.2) is 0. Therefore, its numerator is 0, from which it follows that $P'(L_1) = P(L_1)/L_1$. (You might take a few minutes to see why this equation should hold, without using graphs or calculus.)

In any case, the graphs of $m(L)$ and $a(L)$ cross when L is L_1. For smaller values of L, the graph of $m(L)$ is above that of $a(L)$, and for larger values it is below, as shown in Figure C.5.3.

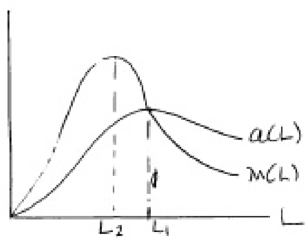


Figure C.5.3:

What does the maximum point on the marginal product graph tell about the production graph?

Assume that $m(L)$ has a maximum at L_2. For smaller L than L_2 the derivative of $m(L)$ is positive. For L larger than L_2 the derivative of $m(L)$ is negative. Since $m(L)$ is defined as $P'(L)$, the second derivative of $P(L)$ switches from positive to negative at L_2, showing that the production curve has an inflection point at $(L_2, P(L_2))$.

Economists use similar techniques to deal with a variety of concepts, such as marginal and average cost or marginal and average revenue, viewed as functions of labor or of capital.