Chapter 3

The Derivative

In this chapter we meet one of the two main concepts of calculus, the **derivative of a function**. The derivative tells how rapidly or slowly a function changes. For instance, if the function describes the position of a moving particle, the derivative tells us its velocity.

The definition of a derivative rests on the notion of a limit. The particular limits examined in Chapter 2 are the basis for finding the derivatives of all functions of interest.

The goal of this chapter is twofold: to develop those techniques and also an understanding of the meaning of a derivative.

3.1 Velocity and Slope: Two Problems with One Theme

This section discusses two problems which at first glance may seem unrelated. The first one concerns the slope of a tangent line to a curve. The second involves velocity. A little arithmetic will show that they are both just different versions of one mathematical idea: the *derivative*.

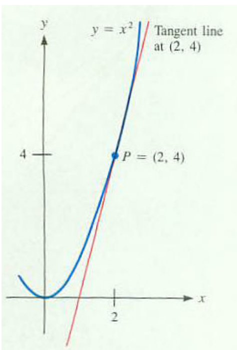


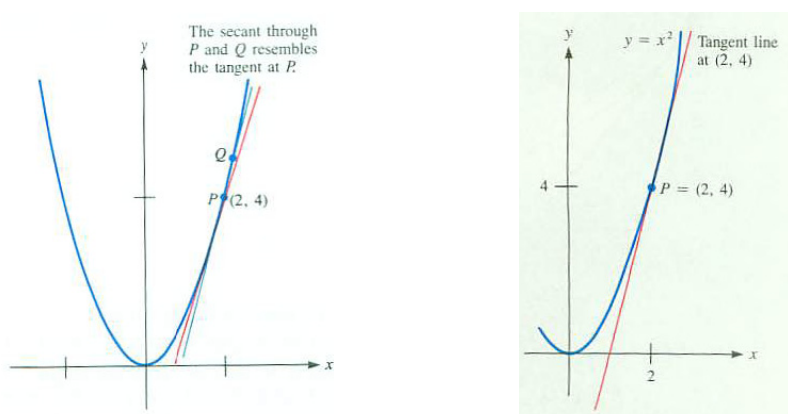
Figure 3.1.1:

Slope

Our first problem is important because it is related to finding the straight line that most closely resembles a given graph near a point on the graph.

EXAMPLE 1 What is the slope of the tangent line to the graph of $y = x^2$ at the point $P = (2, 4)$, as shown in Figure 3.1.1

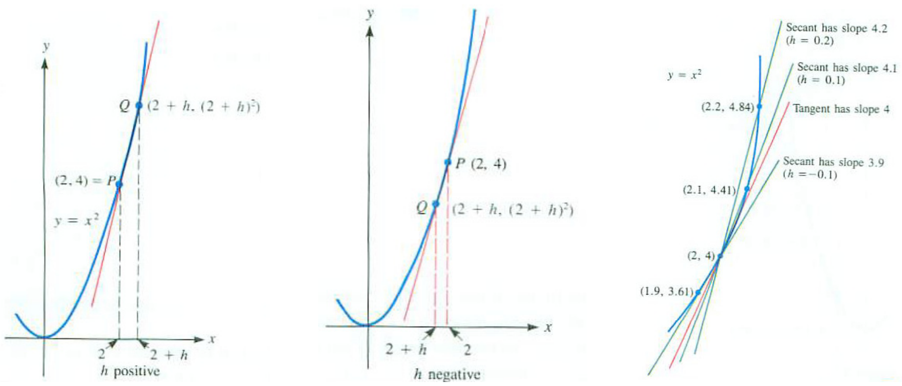
In Section 2.1 we used a point Q on the curve near P to determine a line that closely resembles the tangent line at $(2, 4)$. Using $Q = (2.01, 2.01^2)$ and also $Q = (1.99, 1.99^2)$, we found that the slope of the tangent line is between 4.01 and 3.99. We did not find the slope of the tangent at $(2, 4)$. Rather than making more estimates by choosing specific points nearer $(2, 4)$, such as $(2.00001, 2.00001^2)$, it is simpler to consider a typical point.



(a) (b)

Figure 3.1.2:

SOLUTION Consider the line through $P = (2, 4)$ and $Q = (x, x^2)$ when x is close to 2 — but not equal to 2. (See Figures 3.1.2(a) and (b).) This line has



(a) (b) (c)

Figure 3.1.3:

slope

$$\frac{x^2 - 2^2}{x - 2}.$$

To find out what happens to this quotient as Q moves closer to P (and x moves closer to 2) apply the techniques of limits developed in Chapter 2. We have

Recall $a^2 - b^2 = (a+b)(a-b)$.

$$\lim_{x\to 2} \frac{x^2 - 2^2}{x - 2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

Thus, we expect the tangent line to $y = x^2$ at $(2,4)$ to have slope 4.

Figure 3.1.3(c) shows how secant lines approximate the tangent line. It suggests a blowup of a small part of the curve $y = x^2$. ◇

Note that we never had to make any estimates with specific choices of the nearby point Q. We did not even have to draw the curve.

Velocity

If an airplane or automobile is moving at a constant velocity, we know that "distance traveled equals velocity times time." Thus

$$\text{velocity} = \frac{\text{distance traveled}}{\text{elapsed time}}.$$

If the velocity is *not* constant, we still may speak of its "average velocity," which is defined as

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{elapsed time}}.$$

For instance, if you drive from San Francisco to Los Angeles, a distance of 400 miles, in 8 hours, the average velocity is 400/8 or 50 miles per hour.

Suppose that up to time t_1 you have traveled a distance D_1, while up to time t_2 you have traveled a distance D_2, where $t_2 > t_1$. Then during the time interval $[t_1, t_2]$ the distance traveled is $D_2 - D_1$. Thus the average velocity during the time interval $[t_1, t_2]$, which has duration $t_2 - t_1$, is

$$\text{average velocity} = \frac{D_2 - D_1}{t_2 - t_1}.$$

The arithmetic of average velocity is the same as that for the slope of a line.

The next problem shows how to find the velocity at any instant for an object whose velocity is not constant.

EXAMPLE 2 A rock initially at rest falls $16t^2$ feet in t seconds. What is its velocity after 2 seconds? Whatever it is, it will be called the **instantaneous velocity**.

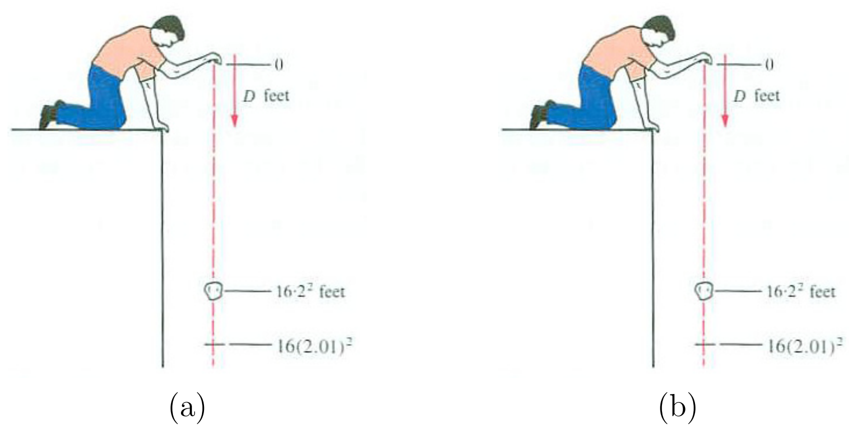


Figure 3.1.4: Note: (b) needs to have 2.01 replaced by t.

SOLUTION

To start, make an estimate by finding the average velocity of the rock during a short time interval, say from 2 to 2.01 seconds. At the start of this interval the rock has fallen $16(2^2) = 64$ feet. By the end it has fallen $16(2.01^2) = 16(4.0401) = 64.6416$ feet. So, during this interval of 0.01 seconds the rock fell 0.6416 feet. Its average velocity during this time interval is

$$\text{average velocity} = \frac{64.6416 - 64}{2.01 - 2} = \frac{0.6416}{0.01} = 64.16 \text{ feet per second.}$$

This is an estimate of the velocity at time $t = 2$ seconds. (See Figure 3.1.4(a).)

Rather than make another estimate with the aid of a still shorter interval of time, let us consider the typical time interval from 2 to t seconds, $t > 2$. (Although we will keep $t > 2$, estimates could just as well be made with $t < 2$.) During this short time of $t - 2$ seconds the rock travels $16(t^2) - 16(2^2) = 16(t^2 - 2^2)$ feet, as shown in Figure 3.1.4(b). The average velocity of the rock during this period is

$$\text{average velocity} = \frac{16t^2 - 16(2^2)}{t-2} = \frac{16(t^2 - 2^2)}{t-2} \text{ feet per second.}$$

When t is close to 2, what happens to the average velocity? It approaches

$$\lim_{t\to 2} \frac{16(t^2 - 2^2)}{t-2} = 16 \lim_{t\to 2} \frac{t^2 - 2^2}{t-2} = 16 \lim_{t\to 2}(t+2) = 16 \cdot 4 = 64 \text{ feet per second.}$$

We say that the (instantaneous) velocity at time $t = 2$ is 64 feet per second. ◇

Even though Examples 1 and 2 seem unrelated, their solutions turn out to be practically identical: The slope in Example 1 is approximated by the quotient

$$\frac{x^2 - 2^2}{x - 2}$$

and the velocity in Example 2 is approximated by the quotient

$$\frac{16t^2 - 16(2^2)}{t-2} = 16 \cdot \frac{t^2 - 2^2}{t-2}.$$

The only difference between the solutions is that the second quotient has an extra factor of 16 and x is replaced with t. This may not be too surprising, since the functions involved, x^2 and $16t^2$ differ by a factor of 16. (That the independent variable is named t in one case and x in the other does not affect the computations.)

A variable by any name is a variable.

The Derivative of a Function

In both the slope and velocity problems we were lead to studying similar limits. For the function x^2 it was

$$\frac{x^2 - 2^2}{x - 2} \quad \text{as } x \text{ approaches 2.}$$

For the function $16t^2$ it was

$$\frac{16t^2 - 16(2^2)}{t-2} \quad \text{as } t \text{ approaches 2.}$$

In both cases we formed "change in outputs divided by change in inputs" and then found the limit as the change in inputs became smaller and smaller. This can be done for other functions, and brings us to one of the two key ideas in calculus, the *derivative of a function*.

> **DEFINITION** *(Derivative of a function at a number a)* Let f be a function that is defined at least in some open interval that contains the number a. If
> $$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$
> exists, it is called the **derivative of** f **at** a, and is denoted $f'(a)$. In this case the function f is said to be **differentiable at** a.

Read $f'(a)$ as "f prime at a" or "the derivative of f at a."

EXAMPLE 3 Find the derivative of $f(x) = 16x^2$ at 2.

SOLUTION In this case, $f(x) = 16x^2$ for any input x. By definition, the derivative of this function at 2 is

$$\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{16x^2 - 16(2^2)}{x - 2} = 16 \lim_{x\to 2} \frac{x^2 - 2^2}{x - 2} = 16 \lim_{x\to 2}(x + 2) = 64.$$

We say that "the derivative of the function $f(x)$ at 2 is 64" and write $f'(2) = 64$. ◇

Now that we have the derivative of f, we can define the slope of its graph at a point $(a, f(a))$ as the value of the derivative, $f'(a)$. Then we define the **tangent line** at $(a, f(a))$ as the line through $(a, f(a))$ whose slope is $f'(a)$.

EXAMPLE 4 Find the derivative of e^x at a.

SOLUTION We must find

$$\lim_{x\to a} \frac{e^x - e^a}{x - a}. \tag{3.1.1}$$

The limit is hard to see. However, it is easy to calculate if we write x as $a + h$, and find what happens as h approaches 0. The denominator $x - a$ is just h. Then (3.1.1) now reads

$$\lim_{h\to 0} \frac{e^{a+h} - e^a}{h}.$$

This form of the limit is more convenient:

$$\begin{aligned}
\lim_{h\to 0} \frac{e^{a+h} - e^a}{h} &= \lim_{h\to 0} \frac{e^a e^h - e^a}{h} && \text{law of exponents} \\
&= e^a \lim_{h\to 0} \frac{e^h - 1}{h} && \text{factor out a constant} \\
&= e^a \cdot 1 && \text{Section 2.2} \\
&= e^a.
\end{aligned}$$

So the limit is e^a. In short, "the derivative of e^x is e^x itself." ◇

Differentiability and Continuity

If a function is differentiable at each point in its domain the function is said to be **differentiable**.

A small piece of the graph of a differentiable function at a looks like part of a straight line. You can check this by zooming in on the graph of a function of your choice. Differential calculus can be described as the study of functions whose graphs locally look almost like a line.

It is no surprise that a differentiable function is always continuous. To show that a function is continuous at an argument a in its domain We must show that $\lim x \to a f(x)$ equals $f(a)$, which amounts to showing $\lim_{x\to a}(f(x)-f(a))$ equals 0. To relate this limit to $f'(a)$ we rewrite the limit as

$$\begin{aligned}
\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}(x-a)\right)\\
&= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\lim_{x\to a}(x-a)\\
&= f'(a)\cdot 0 && \text{definition of the derivative}\\
&= 0.
\end{aligned}$$

So, f is continuous at a.

A function can be continuous yet not differentiable. For instance, $f(x) = |x|$ is continuous but not differentiable at 0, as Figure 3.1.5 suggests.

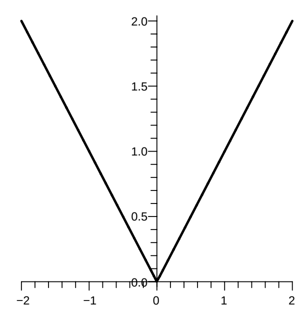


Figure 3.1.5:

Summary

From a mathematical point of view, the problems of finding the slope of the tangent line and the velocity of the rock are the same. In each case estimates lead to the same type of quotient, $\frac{f(x)-f(a)}{x-a}$. The behavior of this *difference quotient* is studied as x approaches a. In each case the answer is a limit, called the derivative of the function at the given number, a. Finding the derivative of a function is called "differentiating" the function.

EXERCISES for Section 3.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] Let g be a function and b a number. Define the "derivative of g at b".

2.[R] How is the tangent line to the graph of f at $(a, f(a))$ defined?

3.[R]

(a) Find the slope of the tangent line to $y = x^2$ at $(4, 16)$.

(b) Use it to draw the tangent line to the curve at $(4, 16)$.

4.[R]

(a) Find the slope of the tangent line to $y = x^2$ at $(-1, 1)$.

(b) Use it to draw the tangent line to the curve at $(-1, 1)$.

Exercises 5 to 17 concern slope. In each case use the technique of Example 1 to find the slope of the tangent line to the curve at the point.

5.[R] $y = x^2$ at the point $(3, 3^2) = (3, 9)$

6.[R] $y = x^2$ at the point $(\frac{1}{2}, (\frac{1}{2})^2) = (\frac{1}{2}, \frac{1}{4})$

7.[R] $y = x^3$ at the point $(2, 2^3) = (2, 8)$

8.[R] $y = x^3$ at the point $(-2, (-2)^3) = (-2, -8)$

9.[R] $y = \sin(x)$ at the point $(0, \sin(0)) = (0, 0)$

10.[R] $y = \sin(x)$ at the point $(0, \cos(0)) = (0, 1)$

11.[R] $y = \cos(x)$ at the point $(\pi/4, \cos(\pi/4)) = (\pi/4, \sqrt{2}/2)$

12.[R] $y = \cos(x)$ at the point $(\pi/6, \sin(\pi/6)) = (\pi/6, 1/2)$

13.[R] $y = 2^x$ at the point $(1, 2^1) = (1, 2)$

14.[R] $y = 4^x$ at the point $(1/2, 4^{1/2}) = (1/2, 2)$

15.[R]

(a) Graph $y = 1/x$ and, by eye, draw the tangent at the point $(2, 1/2)$.

(b) Using a ruler, measure a rise-run triangle to estimate the slope of the tangent line drawn in (a).

(c) Using no pictures at all, find the slope of the tangent line to the curve $y = 1/x$ at $(2, 1/2)$.

16.[R]

(a) Sketch the graph of $y = x^3$ and the tangent line at $(0, 0)$.

(b) Find the slope of the tangent line to the curve $y = x^3$ at the point $(0, 0)$

NOTE: Be particularly careful when sketching the graph near $(0, 0)$. In this case the tangent line crosses the curve.

17.[R]

(a) Sketch the graph of $y = x^2$ and the tangent line at $(1, 1)$.

(b) Find the slope of the tangent line to the curve $y = x^2$ at the point $(0, 0)$

In Exercises 18 to 21 use the method of Example 2 to find the velocity of the rock after

18.[R] 3 seconds

19.[R] $\frac{1}{2}$ second

20.[R] 1 second

21.[R] $\frac{1}{4}$ second

22.[R] A certain object travels t^3 feet in the first t seconds.

(a) How far does it travel during the time interval from 2 to 2.1 seconds?

(b) What is the average velocity during that time interval?

(c) Let h be any positive number. Find the average velocity of the object from time 2 to $2+h$ seconds. HINT: To find $(2+h)^3$, just multiply out the product $(2+h)(2+h)(2+h)$.

(d) Find the velocity of the object at 2 seconds by letting h approach 0 in the result found in (c).

23.[R] A certain object travels t^3 feet in the first t seconds.

(a) Find the average velocity during the time interval from 3 to 3.01 seconds?

(b) Find its average velocity during the time interval from 3 to t seconds, $t > 3$.

(c) By letting t approach 3 in the result found in (b), find the velocity of the object at 3 seconds.

Exercises 24 and 25 illustrate a different notation to find the slope of the tangent.

24.[R] Consider the parabola $y = x^2$.

(a) Find the slope of the line through $P = (2, 4)$ and $Q = (2 + h, (2 + h)^2)$, where $h \neq 0$.

(b) Show that as h approaches 0, the slope in (a) approaches 4.

25.[R] Consider the curve $y = x^3$.

(a) Find the slope of the line through $P = (2, 8)$ and $Q = (1.9, 1.9^3)$.

(b) Find the slope of the line through $P = (2, 8)$ and $Q = (2.01, 2.01^3)$.

(c) Find the slope of the line through $P = (2, 8)$ and $Q = (2 + h, (2 + h)^3)$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (a) approaches 12.

26.[R] Consider the curve $y = \sin(x)$.

(a) Find the slope of the line through $P = (0, 0)$ and $Q = (-0.1, \sin(-0.1))$.

(b) Find the slope of the line through $P = (0, 0)$ and $Q = (0.01, \sin(0.01))$.

(c) Find the slope of the line through $P = (0, 0)$ and $Q = (h, \sin(h))$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (c) approaches 1.

(e) Use (d) to draw the tangent line to $y = \sin(x)$ at $(0, 0)$.

27.[R] Consider the curve $y = \cos(x)$.

(a) Find the slope of the line through $P = (0, 1)$ and $Q = (-0.1, \cos(-0.1))$.

(b) Find the slope of the line through $P = (0, 1)$ and $Q = (0.01, \cos(0.01))$.

(c) Find the slope of the line through $P = (0, 1)$ and $Q = (h, \cos(h))$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (c) approaches 0.

(e) Use (d) to draw the tangent line to $y = \cos(x)$ at $(0, 1)$.

28.[R] Consider the curve $y = 2^x$.

(a) Find the slope of the line through $P = (2, 2^2)$ and $Q = (1.9, 2^{1.9})$.

(b) Find the slope of the line through $P = (2, 2^2)$ and $Q = (2.1, 2^{2.1})$.

(c) Find the slope of the line through $P = (2, 2^2)$ and $Q = (2 + h, 2^{2+h})$, where $h \neq 0$.

(d) Show that the slope of the curve $y = 2^x$ at $(2, 2^2)$ is approximately $4(0.693) = 2.772$.

(e) Use (d) to draw the tangent line to $y = 2^x$ at $(2, 4)$.

29.[R] Consider the curve $y = e^x$.

(a) Find the slope of the line through $P = (-0.5, e^{-0.5})$ and $Q = (-0.6, e^{-0.6})$.

(b) Find the slope of the line through $P = (-0.5, e^{-0.5})$ and $Q = (-0.49, e^{-0.49})$.

(c) Find the slope of the line through $P = (-0.5, e^{-0.5})$ and $Q = (-0.5 + h, e^{-0.5+h})$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (c) approaches $e^{-0.5}$.

30.[R] Show that the slope of the curve $y = 2^x$ at $(3, 8)$ is approximately $8(0.693) = 5.544$.

31.[R]

(a) Use the method of this section to find the slope of the curve $y = x^3$ at $(1, 1)$.

(b) What does the graph of $y = x^3$ look like near $(1, 1)$?

32.[R]

(a) Use the method of this section to find the slope of the curve $y = x^3$ at $(-1, -1)$.

(b) What does the graph of $y = x^3$ look like near $(-1, -1)$?

33.[R]

(a) Draw the curve $y = e^x$ for x in the interval $[-2, 1]$.

(b) Draw as well as you can, using a straightedge, the tangent line at $(1, e)$.

(c) Estimate the slope of the tangent line by measuring its "rise" and its "run."

(d) Using the derivative of e^x, find the slope of the curve at $(1, e)$.

34.[R]

(a) Sketch the curve $y = e^x$ for x in $[-1, 1]$.

(b) Where does the curve in (a) cross the y-axis?

(c) What is the (smaller) angle between the graph of $y = e^x$ and the y-axis at the point found in (b)?

35.[R] With the aid of a calculator, estimate the slope of $y = 2^x$ at $x = 1$, using the intervals

(a) $[1, 1.1]$

(b) $[1, 1.01]$

(c) $[0.9, 1]$

(d) $[0.99, 1]$

36.[R] With the aid of a calculator, estimate the slope of $y = \frac{x+1}{x+2}$ at $x = 2$, using the intervals

(a) $[2, 2.1]$

(b) $[2, 2.01]$

(c) $[2, 2.001]$

(d) $[1.999, 2]$

37.[M] Estimate the derivative of $\sin(x)$ at $x = \pi/3$

(a) to two decimal places.

(b) to three decimal places.

38.[M] Estimate the derivative of $\ln(x)$ at $x = 2$

(a) to two decimal places.

(b) to three decimal places.

The ideas common to both slope and velocity also appear in other applications. Exercises 39 to 42 present the same ideas in biology, economics, and physics.

39.[M] A certain bacterial culture has a mass of t^2 grams after t minutes of growth.

(a) How much does it grow during the time interval $[2, 2.01]$?

(b) What is the average rate of growth during the time interval $[2, 2.01]$?

(c) What is the "instantaneous" rate of growth when $t = 2$?

40.[M] A thriving business has a profit of t^2 million dollars in its first t years. Thus from time $t = 3$ to time $t = 3.5$ (the first half of its fourth year) it has a profit of $(3.5)^2 - 3^2$ million dollars, giving an annual rate of

$$\frac{(3.5)^2 - 3^2}{0.5} = 6.5 \text{ million dollars per}$$

(a) What is its annual rate of profit during the time interval $[3, 3.1]$?

(b) What is its annual rate of profit during the time interval $[3, 3.01]$?

(c) What is its instantaneous rate of profit after 3 years?

Exercises 41 and 42 concern density.

41.[M] The mass of the left-hand x centimeters of a nonhomogeneous string 10 centimeters long is x^2 grams, as shown in Figure 3.1.6. For instance, the string in the interval $[0, 5]$ has a mass of $5^2 = 25$ grams and the string in the interval $[5, 6]$ has mass $6^2 - 5^2 = 11$ grams. The **average density** of any part of the string is its mass divided by its length. ($\frac{\text{total mass}}{\text{length}}$ grams per centimeter)

(a) Consider the leftmost 5 centimeters of the string, the middle 2 centimeters of the string, and the rightmost 2 centimeters of the string. Which piece of the string has the largest mass?

(b) Of the three pieces of the string in (a), which part of the string is densest?

(c) What is the mass of the string in the interval $[3, 3.01]$?

(d) Using the interval $[3, 3.01]$, estimate the density at 3.

(e) Using the interval $[2.99, 3]$, estimate the density at 3.

(f) By considering intervals of the form $[3, 3+h]$, h positive, find the density at the point 3 centimeters from the left end.

(g) By considering intervals of the form $[3+h, 3]$, h negative, find the density at the point 3 centimeters from the left end.

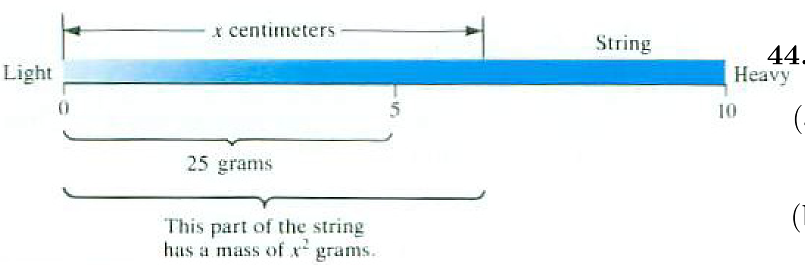


Figure 3.1.6:

42.[M] The left x centimeters of a string have a mass of x^2 grams.

(a) What is the mass of the string in the interval $[2, 2.01]$?

(b) Using the interval $[2, 2.01]$, estimate the density at 2.

(c) Using the interval $[1.99, 2]$, estimate the density at 2.

(d) By considering intervals of the form $[2, 2+h]$, h positive, find the density at the point 2 centimeters from the left end.

(e) By considering intervals of the form $[2+h, 2]$, h negative, find the density at the point 2 centimeters from the left end.

43.[M]

(a) Graph the curve $y = 2x^2 + x$.

(b) By eye, draw the tangent line to the curve at the point $(1, 3)$. Using a ruler, estimate its slope.

(c) Sketch the line that passes through the point $(1, 3)$ and the point $(x, 2x^2 + x)$.

(d) Find the slope of the line in (c).

(e) Letting x get closer and closer to 1, find the slope of the tangent line at $(1, 3)$.

(f) How close was your estimate in (b)?

44.[M] An object travels $2t^2 + t$ feet in t seconds.

(a) Find its average velocity during the interval of time $[1, x]$, where x is positive.

(b) Letting x get closer and closer to 1, find the velocity at time 1.

(c) How close was your estimate in (a)?

45.[M] Find the slope of the tangent line to the curve $y = x^2$ of Example 1 at the typical point $P = (x, x^2)$. To do this, consider the slope of the line through P and the nearby point $Q = (x+h, (x+h)^2)$ and let h approach 0.

46.[M] Find the velocity of the falling rock of Example 2 at any time t. To do this, consider the average velocity during the time interval $[t, t+h]$ and then let h approach 0.

47.[M] Does the tangent line to the curve $y = x^2$ at the point $(1, 1)$ pass through the point $(6, 12)$?

48.[M]

(a) Graph the curve $y = 2^x$ as well as you can for $-2 \le x \le 3$.

(b) Using a straight edge, draw as well as you can a tangent to the curve at $(2, 4)$. Estimate the slope of this tangent by using a ruler to draw and measure a "rise-and-run" triangle.

(c) Using a secant through $(2, 4)$ and $(x, 2^x)$, for x near 2, estimate the slope of the tangent to the curve at $(2, 4)$. HINT: Choose particular values of x and use your calculator to create a table of your results.

49.[C]

(a) Using your calculator estimate the slope of the tangent line to the graph of $f(x) = \sin(x)$ at $(0, 0)$.

(b) At what (famous) angle do you think the curve crosses the x-axis at $(0, 0)$?

50.[C]

(a) Sketch the curve $y = x^3 - x^2$.

(b) Using the method of the nearby point, find the slope of the tangent line to the curve at the point $(a, a^3 - a^2)$.

(c) Find all points on the curve where the tangent line is horizontal.

(d) Find all points on the curve where the tangent line has slope 1.

51.[C] Repeat Exercise 50 for the curve $y = x^3 - x$.

52.[C] An astronaut is traveling from left to right along the curve $y = x^2$. When she shuts off the engine, she will fly off along the line tangent to the curve at the point where she is at the moment the engines turn off. At what point should she shut off the engine in order to reach the point

(a) $(4, 9)$?

(b) $(4, -9)$?

53.[C] See Exercise 52. Where can an astronaut who is traveling from left to right along $y = x^3 - x$ shut off the engine and pass through the point $(2, 2)$?

54.[C]

Sam: I don't like the book's definition of the derivative.

Jane: Why not?

Sam: I can do it without limits, and more easily.

Jane: How?

Sam: Just define the derivative off at a as the slope of the tangent line at $(a, f(a))$ on the graph of f.

Jane: Something must be wrong with that.

Who is right, Sam or Jane?

3.2 The Derivatives of the Basic Functions

In this section we use the definition of the derivative to find the derivatives of the important functions x^a (a rational), e^x, $\sin x$, and $\cos x$. We also introduce some of the standard notations for the derivative. For convenience, we begin by repeating the definition of the derivative.

> **DEFINITION** *(Derivative of a function at a number)* Assume that the function f is defined at least in an open interval containing a. If
> $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.2.1}$$
> exists, it is called the **derivative of f at a**.

There are several notations for the quotient that appears in (3.2.1) and also for the derivative. Sometimes it is convenient to use $a + h$ instead of x and let h approach 0. Then, (3.2.1) reads

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{3.2.2}$$

Expression (3.2.2) says the same thing as (3.2.1). See how the quotient, "change in output" divided by "change in input", behaves as the change in input gets smaller and smaller.

The symbol Δ is Greek for "D"; it is pronounced "delta". So Δf is read 'delta eff." In mathematics, "Δ" generally indicates difference or change.

Sometimes it is useful to call the change in output "Δf" and the change in input "Δx." That is, $\Delta f = f(x) - f(a)$ and $\Delta x = x - a$. Then

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}. \tag{3.2.3}$$

There is nothing sacred about the letters a, x, and h. One could say

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \tag{3.2.4}$$

or

$$f'(x) = \lim_{u \to x} \frac{f(u) - f(x)}{u - x}. \tag{3.2.5}$$

The symbol "$f'(a)$" is read aloud as "f prime at a" or "the derivative of f at a." The symbol $f'(x)$ is read similarly. However, the notation $f'(x)$ reminds us that f', like f, is a function. For each input x the derivative, $f'(x)$, is the output. The derivative of the function f is also written as $D(f)$.

The derivative of a specific function, such as x^2, is denoted $(x^2)'$ or $D(x^2)$. Then, $D(x^2) = 2x$ is read aloud as "the derivative of x^2 is $2x$." This is shorthand for "the derivative of the function that assigns x^2 to x is the function that assigns $2x$ to x." Since the value of derivative depends on x, the derivative is a function.

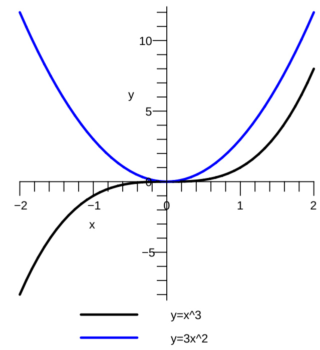


Figure 3.2.1:

EXAMPLE 1 Find the derivative of x^3 at a.
SOLUTION

$$(x^3)' = \lim_{x \to a} \frac{x^3 - a^3}{x - a} = 3a^2.$$

This limit was evaluated by noticing that it is one of the four limits in Section 2.2 (page 67). Using (2.2.6), we can write $(x^3)' = 3x^2$ or $D(x^3) = 3x^2$. ◇

In the same manner, $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$ implies that for any positive integer n, the derivative of x^n is nx^{n-1}. The exponent n becomes the coefficient and the exponent of x shrinks from n to $n-1$:

Derivative of x^n

$(x^n)' = nx^{n-1}$ where n is a positive integer.

The next example treats an exponential function with a fixed base.
EXAMPLE 2 Find the derivative of 2^x.

SOLUTION

$$\begin{aligned} D(2^x) &= \lim_{h \to 0} \frac{2^{(x+h)} - 2^x}{h} \\ &= \lim_{h \to 0} \frac{2^x 2^h - 2^x}{h} \\ &= \lim_{h \to 0} 2^x \frac{2^h - 1}{h} \\ &= 2^x \lim_{h \to 0} \frac{2^h - 1}{h}. \end{aligned}$$

In Section 2.2, page 67, we found that $\lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.693$. Thus,

$$D(2^x) \approx (0.693)2^x.$$

◇

No one wants to remember the (approximate) constant 0.693, which appears when we use base 2. Recall that in Section 3.1 we found that the derivative of e^x is e^x. There is no need to memorize some fancy constant, such as 0.693.

We emphasize this important, and simple, formula

Derivative of e^x

$$D(e^x) = e^x.$$

The function e^x has the remarkable property that it equals its derivative.

Next, we turn to trigonometric functions.

EXAMPLE 3 Find the derivative of $\sin(x)$.

Recall that $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$.

SOLUTION

$$\begin{aligned} D(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\ &= \lim_{h\to 0} \sin x \frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h}. \end{aligned}$$

In Section 2.2 we found that: $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0} \frac{1-\cos(h)}{h} = 0$. Thus $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$ and

$$D(\sin x) = (\sin x)(0) + (\cos x)(1) = \cos(x).$$

◇

We have the important formula

Derivative of $\sin(x)$

$$D(\sin(x)) = \cos(x).$$

If we graph $y = \sin(x)$ (see Figure 3.2.2), and consider its shape, the formula $D(\sin(x)) = \cos(x)$ is not a surprise. For instance, for x in $(-\pi/2, \pi/2)$ the slope is positive. So is $\cos(x)$. For x in $(\pi/2, 3\pi/2)$ the slope of the sine curve is negative. So is $\cos(x)$. Since $\sin(x)$ has period 2π, we would expect its derivatve also to have period 2π. Indeed, $\cos(x)$ does have period 2π.

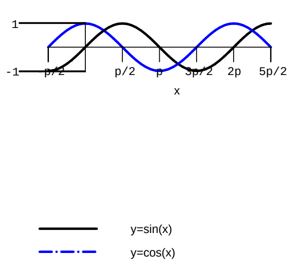


Figure 3.2.2:

In a similar manner, using the definition of the derivative and the identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, one can show that

Derivative of $\cos(x)$

$$D(\cos(x)) = -\sin(x).$$

Derivatives of Other Power Functions

We showed that if n is a positive integer, $D(x^n) = nx^{n-1}$. Now let us find the derivative of power functions x^n where n is not a positive integer.

EXAMPLE 4 Find the derivative of $x^{-1} = \frac{1}{x}$.
SOLUTION Before we calculate the necessary limit, let's pause to see how the slope of $y = 1/x$ behaves. A glance at Figure 3.2.3 shows that the slope is always negative. Also, for x near 0, the absolute value of the slope is large, but when $|x|$ is large, the slope is near 0.

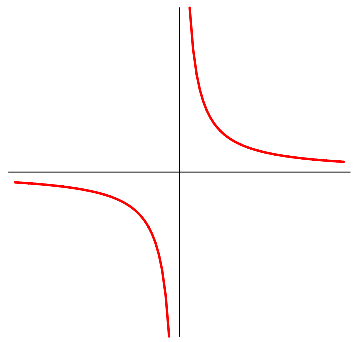


Figure 3.2.3:

Now, let's find the derivative of $1/x$:

$$\begin{aligned} D(1/x) &= \lim_{t\to x} \frac{1/t - 1/x}{t - x} \\ &= \lim_{t\to x} \frac{1}{t-x}\left(\frac{x-t}{xt}\right) \\ &= \lim_{t\to x} \frac{-1}{xt} \\ &= -\frac{1}{x^2}. \end{aligned}$$

As a check, note that $-1/x^2$ is always negative, has large absolute value when x is near 0, and is near 0 when $|x|$ is large. ◇

It is worth memorizing that

Derivative of x^{-1}

$$D\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Or, written in exponential notation,

$$D(x^{-1}) = -x^{-2}.$$

The second form fits into the pattern established for positive integers n, $D(x^n) = nx^{n-1}$.

EXAMPLE 5 Find the derivative of $x^{2/3}$.
SOLUTION Once again we use the definition of the derivative:

$$D(x^{2/3}) = \lim_{t\to x} \frac{t^{2/3} - x^{2/3}}{t - x}.$$

A bit of algebra will help us find that limit. We write the four terms $t^{2/3}$, $x^{2/3}$, t, and x as powers of $t^{1/3}$ and $x^{1/3}$. Thus

$$D(x^{2/3}) = \lim_{t\to x} \frac{\left(t^{1/3}\right)^2 - \left(x^{1/3}\right)^2}{\left(t^{1/3}\right)^3 - \left(x^{1/3}\right)^3}.$$

If you don't recall these formulas, multiply out $(a-b)(a+b)$ and $(a-b)(a^2+ab+b^2)$.

Recalling that $a^2-b^2=(a-b)(a+b)$ and $a^3-b^3=(a-b)(a^2+ab+b^2)$, we find

$$\begin{aligned} D(x^{2/3}) &= \lim_{t\to x} \frac{\left(\left(t^{1/3}\right)-\left(x^{1/3}\right)\right)\left(\left(t^{1/3}\right)+\left(x^{1/3}\right)\right)}{\left(\left(t^{1/3}\right)-\left(x^{1/3}\right)\right)\left(\left(t^{1/3}\right)^2+\left(t^{1/3}\right)\left(x^{1/3}\right)+\left(x^{1/3}\right)^2\right)} \\ &= \lim_{t\to x} \frac{\left(t^{1/3}\right)+\left(x^{1/3}\right)}{\left(t^{1/3}\right)^2+\left(t^{1/3}\right)\left(x^{1/3}\right)+\left(x^{1/3}\right)^2} \\ &= \frac{\left(x^{1/3}\right)+\left(x^{1/3}\right)}{\left(x^{1/3}\right)^2+\left(x^{1/3}\right)\left(x^{1/3}\right)+\left(x^{1/3}\right)^2} \\ &= \tfrac{2x^{1/3}}{3x^{2/3}} = \tfrac{2}{3}x^{-1/3}. \end{aligned}$$

In short,

$$D(x^{2/3}) = \frac{2}{3}x^{-1/3}.$$

Note that this formula follows the pattern we found for $D(x^n)$ for $n=1$, 2, 3, ... and -1. The exponent of x becomes the coefficient and the exponent of x is lowered by 1. ◇

The method used in Example 5 applies to any positive rational exponent. In the next two sections we will show how this result extends first to negative rational exponents (Section 3.3) and then to irrational exponents (Section 3.5). In all three cases the formula will be the same. We state the general result here, but remember that — so far — we have justified it only for positive rational exponents and -1.

Derivative of Power Functions x^a

For any fixed number a, $D(x^a) = ax^{a-1}$. (3.2.6)

This formula holds for values of x where both x^a and x^{a-1} are defined. For instance, $x^{1/2}=\sqrt{x}$ is defined for $x\geq 0$, but its derivative $\frac{1}{2}x^{-1/2}$ is defined only for $x>0$.

The derivative of the square root function occurs so often, we emphasize its formula

Derivative of Square Root Function (as Power Function)

$$D(x^{1/2}) = \frac{1}{2}x^{-1/2}$$

or, in terms of the usual square root sign,

Derivative of Square Root Function (Square Root Sign)

$$D(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

Another Notation for the Derivative

We have used the notations f' and $D(f)$ for the derivative of a function f. There is another notation that is also convenient.

If $y = f(x)$, the derivative is denoted by the symbols

$$\frac{dy}{dx} \text{ or } \frac{df}{dx}.$$

The symbol $\frac{dy}{dx}$ is read as "the derivative of y with respect to x" or "dee y, dee x."

In this notation the derivative of x^3, for instance, is written

$$\frac{d\,(x^3)}{dx}.$$

If the function is expressed in terms of another letter, such as t, we would write

$$\frac{d\,(t^3)}{dt}.$$

In Section 5.4 a meaning will be given to dx and dy.

Keep in mind that in the notations df/dx and dy/dx, the symbols df, dy, and dx have no meaning by themselves. The symbol dy/dx should be thought of as a signle entity, just like the numeral 8, which we do not think of as formed of two 0's.

In the study of motion, Newton's **dot notation** is often used. If x is a function of time t, then $\dot{x}$ denotes the derivative dx/dt.

Summary

In this section we see why limits are important in calculus. We need them to define the derivative of a function. The definition can be stated in several ways, but each one says, informally, "look at how a small change in input changes the output." Here is the formal definition, in various costumes:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad f'(x) = \lim_{x\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} \qquad f'(x) = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}.$$

The following derivatives should be memorized. However, if you forget a formula, you should be able to return to the definition and evaluate the necessary limit.

Function	Derivative
$f(x)$	$f'(x)$
x^a	ax^{a-1}
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

EXERCISES for Section 3.2 *Key:* R–routine, M–moderate, C–challenging

1.[R] Show that $D(\cos(x)) = -\sin(x)$. HINT: $\cos(A+B) = \cos(a)\cos(b) - \sin(a)\sin(b)$

Using the definition of the derivative, compute the appropriate limit to find the derivatives of the functions in Exercises 2 to 12.

2.[R] $1/(x+2)$

3.[R] $2x - x^2$

4.[R] 3^x. HINT: use your calculator to estimate the messy coefficient that appears

5.[R] $6x^3$

6.[R] $x^{4/3}$

7.[R] $5x^2$

8.[R] $4\sin(x)$

9.[R] $2e^x + \sin(x)$

10.[R] $x^2 + x^3$

11.[R] $1/(2x+1)$

12.[R] $1/x^2$

13.[R] Use the formulas obtained for the derivatives of e^x, x^a, $\sin(x)$, and $\cos(x)$ to evaluate the derivatives of the given function at the given input.

(a) e^x at -1

(b) $x^{1/3}$ at -8

(c) $\sqrt[3]{x}$ at 27

(d) $\cos(x)$ at $\pi/4$

(e) $\sin(x)$ at $2\pi/3$

14.[R] Use the formulas obtained for the derivatives of e^x, x^a, $\sin(x)$, and $\cos(x)$ to evaluate the derivatives of the given function at the given input.

(a) e^x at 0

(b) $x^{2/3}$ at -1

(c) $\sqrt{x}$ at 25

(d) $\cos(x)$ at $-\pi$

(e) $\sin(x)$ at $\pi/3$

15.[R] State the definition of the derivative of a function in words, using no mathematical symbols.

16.[R] State the definition of the derivative of $g(t)$ at b as a mathematical formula, with no words.

In Exercises 17 to 22 use the definition of the derivative to show that the given equation is correct. Later in this chapter we will develop shortcuts for finding these derivatives.

17.[M] $D\left(e^{-x}\right) = -e^{-x}$

18.[M] $D\left(e^{3x}\right) = 3e^{3x}$

19.[M] $D(1/\cos(x)) = \sin(x)/\cos^2(x)$

20.[M] $D(\tan(x)) = 1 + \tan^2(x) = \sec^2(x)$ HINT: use the identity $\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$

21.[M] $D(\sin(2x)) = 2\cos(2x)$

22.[M] $D(\cos(x/2) = -1/2\sin(x/2)$

23.[M] This Exercise shows why, in calculus, angles are measured in radians. Let $\mathrm{Sin}(x)$ denote the sine of an angle of x degrees and let $\mathrm{Cos}(x)$ denote the cosine of an angle of x degrees.

(a) Graph $y = \mathrm{Sin}(x)$ on the interval $[-180, 360]$, using the same scale on both the x- and y-axes.

(b) Find $\lim_{x\to 0} \frac{\mathrm{Sin}(x)}{x}$.

(c) Find $\lim_{x\to 0} \frac{1-\mathrm{Cos}(x)}{x}$.

(d) Using the definition of the derivative, differentiate $\mathrm{Sin}(x)$.

24.[C] Use the limit process to show that $D\left((x^{-5}\right) = -5x^{-6}$.

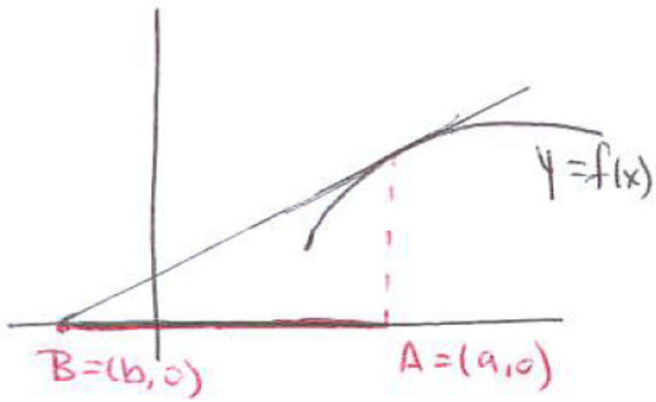


Figure 3.2.4:

Let f be a differentiable function and a a number such that $f'(a)$ is not zero. The tangen
at $A = (a, f(a))$ meets the x-axis a
see Figure 3.2.4. The **subtangent**
Its length is $|a - b|$.

Exercises 25 and 26 involve the su
tion.

25.[C] Show that for the function
subtangent is the same for all valu

26.[C] Find the length of the sub
for any differentiable function f. I
is not zero.

3.3 Shortcuts for Computing Derivatives

This section develops methods for finding the derivative of a function, or what is called **differentiating** a function. With these methods it will he a routine matter to find, for instance, the derivative of

The verb is "differentiate."

$$\frac{(3+4x+5x^2)e^x}{\sin(x)}$$

without going back to the definition of the derivative and (at great effort) finding the limit of a complicated quotient.

Before developing the methods in this and the next two sections, it will be useful to find the derivative of any constant function.

The Derivative of a Constant Function

In other symbols, $\frac{d(C)}{dx} = 0$ and $D(C) = 0$.

Constant Rule

The derivative of a constant function $f(x) = C$ is 0.

$$(C)' = 0$$

Proof

Let C be a fixed number and let f be the constant function, $f(x) = C$ for all inputs x. By the definition of a derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Since the function f has the same output C for all inputs,

Δx is another name for h

$$f(x+\Delta x) = C \text{ and } f(x) = C.$$

Thus

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{C - C}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{0}{\Delta x} \\ &= \lim_{\Delta x \to 0} 0 \qquad \text{since } \Delta x \neq 0 \\ &= 0. \end{aligned}$$

This shows the derivative of any constant function is 0 for all x. •

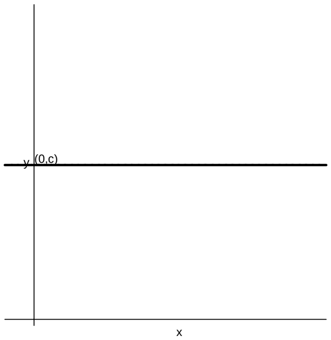


Figure 3.3.1:

From two points of view, the Constant Rule is no surprise: Since the graph of $f(x) = C$ is a horizontal line, it coincides with each of its tangent lines, as can be seen in Figure 3.3.1. Also, if we think of x as time and $f(x)$ as the position of a particle at time x, the Constant Rule implies that a stationary particle has zero velocity.

Derivatives of $f + g$ and $f - g$

The next theorem asserts that if the functions, f and g have derivatives at a certain number, so does their sum $f + g$ and

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

In other words, "the derivative of the sum is the sum of the derivatives." Equivalently, $(f + g)' = f' + g'$ and $D(f + g) = D(f) + D(g)$. A similar formula holds for the derivative of $f - g$.

Sum Rule and Difference Rule

If f and g are differentiable functions, then so are $f + g$ and $f - g$. The **Sum Rule** and **Difference Rule** for computing their derivatives are

$$\begin{aligned} (f+g)' &= f' + g' && \text{Sum Rule} \\ (f-g)' &= f' - g' && \text{Difference Rule} \end{aligned}$$

Proof

To justify this we must go back to the definition of the derivative. To begin, we give the function $f + g$ the name u, that is, $u(x) = f(x) + g(x)$. We have to examine

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \tag{3.3.1}$$

or, equivalently,

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}. \tag{3.3.2}$$

In order to evaluate (3.3.2), we will express Δu in terms of Δf and Δg. Here are the details:

$$\begin{aligned} \Delta u &= u(x+\Delta x) - u(x) \\ &= (f(x+\Delta x) + g(x+\Delta x)) - (f(x) + g(x)) && \text{definition of } u \\ &= (f(x)+\Delta f) + (g(x)+\Delta g) - (f(x)+g(x)) && \text{definition of } \Delta f \text{ and } \Delta g \\ &= \Delta f + \Delta g \end{aligned}$$

All told, $\Delta u = \Delta f + \Delta g$. The change in u is the change in f plus the change in g.

The hard work is over. We can now evaluate (3.3.2):

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f + \Delta g}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} = f'(x) + g'(x).$$

Thus, $u = f + g$ is differentiable and

$$u'(x) = f'(x) + g'(x).$$

A similar argument applies to $f - g$. •

The Sum and Difference Rules extend to any finite number of differentiable functions. For example.

$$\begin{aligned} (f+g+h)' &= f' + g' + h' \\ (f-g+h)' &= f' - g' + h' \end{aligned}$$

EXAMPLE 1 Using the Sum Rule, differentiate $x^2 + x^3 + \cos(x) + 3$.
SOLUTION

$$\begin{aligned} D\left(x^2 + x^3 + \cos(x) + 3\right) &= D(x^2) + D(x^3) + D(\cos(x)) + D(3) \\ &= 2x^{2-1} + 3x^{3-1} + (-\sin(x)) + 0 \\ &= 2x + 3x^2 - \sin(x). \end{aligned}$$

◇

EXAMPLE 2 Differentiate $x^4 - \sqrt{x} - e^x$.
SOLUTION

$$\begin{aligned} \tfrac{d}{dx}\left(x^4 - \sqrt{x} - e^x\right) &= \tfrac{d}{dx}\left(x^4\right) - \tfrac{d}{dx}\left(\sqrt{x}\right) - \tfrac{d}{dx}\left(e^x\right) \\ &= 4x^3 - \tfrac{1}{2\sqrt{x}} - e^x \end{aligned}$$

◇

The Derivative of fg

The following theorem, concerning the derivative of the product of two functions, may be surprising, for it turns out that the derivative of the product is *not* the product of the derivatives. The formula is more complicated than the one for the derivative of the sum. It asserts that *"the derivative of the product is the derivative of the first function times the second plus the first function times the derivative of the second."*

Product Rule

If f and g are differentiable functions, then so is their product fg. Its derivative is given by the formula

$$(fg)' = f'g + fg'$$

Proof

The proof is similar to that for the Sum and Difference Rules. This time we give the product fg the name u. Then we express Δu in terms of Δf and Δg. Finally, we determine $u'(x)$ by examining $\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$. These steps are practically forced upon us.

We have

$$u(x) = f(x)g(x) \quad \text{and} \quad u(x + \Delta x) = f(x + \Delta x)g(x + \Delta x).$$

Rather than subtract $u(x)$ from $u(x + \Delta x)$ directly, first write

$$f(x + \Delta x) = f(x) + \Delta f \quad \text{and} \quad g(x + \Delta x) = g(x) + \Delta g.$$

Then

$$\begin{aligned} u(x + \Delta x) &= (f(x + \Delta x))\,(g(x + \Delta x)) \\ &= (f(x) + \Delta f)\,(g(x) + \Delta g) \\ &= f(x)g(x) + (\Delta f)g(x) + f(x)\Delta g + (\Delta f)(\Delta g). \end{aligned}$$

Hence

$$\begin{aligned} \Delta u &= u(x + \Delta x) - u(x) \\ &= f(x)g(x) + (\Delta f)g(x) + f(x)(\Delta g) + (\Delta f)(\Delta g) - f(x)g(x) \\ &= (\Delta f)g(x) + f(x)(\Delta g) + (\Delta f)(\Delta g) \end{aligned}$$

and

$$\begin{aligned} \frac{\Delta u}{\Delta x} &= \frac{(\Delta f)g(x) + f(x)(\Delta g) + (\Delta f)(\Delta g)}{\Delta x} \\ &= \frac{\Delta f}{\Delta x}g(x) + f(x)\frac{\Delta g}{\Delta x} + \Delta f\frac{\Delta g}{\Delta x} \end{aligned}$$

As $\Delta x \to 0$, $\Delta g/\Delta x \to g'(x)$ and $\Delta f/\Delta x \to f'(x)$. Furthermore, because f is differentiable, hence continuous, $\Delta f \to 0$ as $x \to 0$. It follows that

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = f'(x)g(x) + f(x)g'(x) + 0 \cdot g'(x).$$

The formula for $(fg)'$ was discovered by Leibniz in 1676. His first guess was wrong.

Therefore, u is differentiable and

$$u' = f'g + fg'.$$

•

Remark: Figure 3.3.2 illustrates the Product Rule and its proof. With f, Δf, g, and Δg taken to be positive, the inner rectangle has area $u = fg$ and the whole rectangle has area $u + \Delta u = (f+\Delta f)(g+\Delta g)$. The shaded region whose area is Δu is made up of rectangles of areas $f\cdot(\Delta g)$, $(\Delta f)\cdot g$, and $(\Delta f)\cdot(\Delta g)$. The little corner rectangle, of area $(\Delta f)\cdot(\Delta g)$, is negligible in comparison with the other two rectangles. Thus, $\Delta u \approx (\Delta f)g + f(\Delta g)$, which suggests the formula for the derivative of a product.

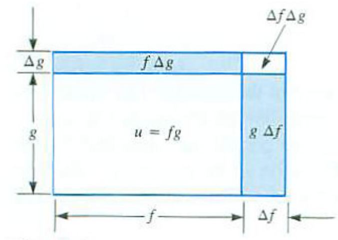


Figure 3.3.2:

Note that the function to be differentiated is the product of the functions differentiated in Examples 1 and 2.

EXAMPLE 3 Find $D\left((x^2+x^3+\cos(x)+3)\left(x^4-\sqrt{x}-e^x\right)\right)$.
SOLUTION By the Product Rule,

$$\begin{aligned} D&\left((x^2+x^3+\cos(x)+3)\left(x^4-\sqrt{x}-e^x\right)\right)\\ &= \left(D\left(x^2+x^3+\cos(x)+3\right)\right)\left(x^4-\sqrt{x}-e^x\right)\\ &\quad+\left(x^2+x^3+\cos(x)+3\right)\left(D\left(x^4-\sqrt{x}-e^x\right)\right)\\ &= \left(2x+3x^2-\sin(x)\right)\left(x^4-\sqrt{x}-e^x\right)\\ &\quad+\left(x^2+x^3+\cos(x)+3\right)\left(4x^3-\frac{1}{2\sqrt{x}}-e^x\right)\end{aligned}$$

◇

Derivative of Constant Times f

A special case of the formula for the product rule occurs so frequently that it is singled out in the Constant Multiple Rule.

Constant Multiple Rule

If C is a constant function and f is a differentiable function, the Cf is differentiable and its derivative is given by the formula

$$(Cf)' = C(f').$$

In other notations, $\frac{d(Cf)}{dx} = C\frac{df}{dx}$ and $D(Cf) = CD(f)$.

The derivative of a constant times a function is the constant times the derivative of the function.

Proof

Because we are dealing with a product of two differentiable functions, C and f, we may use the Product Rule. We have

$$\begin{aligned}(Cf)' &= (C')f + C(f') && \text{derivative of a product}\\ &= 0\cdot f + Cf' && \text{derivative of constant is 0}\\ &= C(f').\end{aligned}$$

•

The Constant Multiple Rule asserts that that "it is legal to move a constant factor outside the derivative symbol."

EXAMPLE 4 Find $D(6x^3)$.
SOLUTION

$$\begin{aligned} D(6x^3) &= 6D(x^3) && \text{6 is a constant} \\ &= 6 \cdot 3x^2 && D(x^n) = nx^{n-1} \\ &= 18x^2. \end{aligned}$$

With a little practice, one would simply write $D(6x^3) = 18x^2$. ◇

EXAMPLE 5 Find $D\left(\sqrt{x}/11\right)$.
SOLUTION

$$D\left(\frac{\sqrt{x}}{11}\right) = D\left(\frac{1}{11}\sqrt{x}\right) = \frac{1}{11}D(\sqrt{x}) = \frac{1}{11}\frac{1}{2\sqrt{x}} = \frac{1}{22}x^{-1/2}$$

◇

Example 5 generalizes to the fact that for a nonzero C,

Constant Division Rule

$$\left(\frac{f}{C}\right)' = \frac{f'}{C} \qquad C \neq 0.$$

The formula for the derivative of the product extends to the product of several differentiable functions. For instance,

$$(fgh)' = (f')gh + f(g')h + fg(h')$$

See Exercise 45. In each summand only one derivative appears. The next example illustrates the use of this formula.

EXAMPLE 6 Differentiate $\sqrt{x}e^x \sin(x)$.
SOLUTION

$$\begin{aligned} &\left(\sqrt{x}e^x \sin(x)\right)' \\ &= (\sqrt{x})'e^x \sin(x) + \sqrt{x}(e^x)' \sin(x) + \sqrt{x}e^x(\sin(x))' \\ &= \left(\frac{1}{2\sqrt{x}}\right) e^x \sin(x) + \sqrt{x}e^x \sin(x) + \sqrt{x}e^x \cos(x) \end{aligned}$$

◇

Any polynomial can be differentiated by the methods already developed.

EXAMPLE 7 Differentiate $6t^8 - t^3 + 5t^2 + \pi^3$.
SOLUTION Notice that the independent variable in this polynomial is t, and the polynomial is to be differentiated with respect to t.

Differentiate a polynomial "term-by-term". Note that π^3 is a constant.

$$\begin{aligned} \frac{d}{dt}\left(6t^8 - t^3 + 5t^2 + \pi^3\right) &= \frac{d}{dt}\left(6t^8\right) - \frac{d}{dt}\left(t^3\right) + \frac{d}{dt}\left(5t^2\right) + \frac{d}{dt}\left(\pi^3\right) \\ &= 48t^7 - 3t^2 + 10t + 0 \\ &= 48t^7 - 3t^2 + 10t \end{aligned}$$

◇

Derivative of $1/g$

Often one needs the derivative of the reciprocal of a function g, that is, $(1/g)'$.

Reciprocal Rule

If g is a differentiable function, then

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}, \qquad \text{where } g(x) \neq 0$$

Proof

Again we must go back to the definition of the derivative.

Assume $g(x) \neq 0$ and let $u(x) = 1/g(x)$. Then $u(x+\Delta x) = 1/g(x+\Delta x) = 1/(g(x) + \Delta g)$. Thus

$$\begin{aligned} \Delta u &= u(x+\Delta x) - u(x) \\ &= \frac{1}{g(x)+\Delta g} - \frac{1}{g(x)} \\ &= \frac{g(x) - (g(x)+\Delta g)}{g(x)(g(x)+\Delta g)} && \text{common denominator} \\ &= \frac{-\Delta g}{g(x)(g(x)+\Delta g)} && \text{cancellation.} \end{aligned}$$

Then

$$
\begin{aligned}
u'(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} &= \lim_{\Delta x \to 0} \frac{-\Delta g / \left(g(x)(g(x) + \Delta g) \right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta g / \Delta x}{g(x)(g(x) + \Delta g)} && \text{algebra: } \frac{(a/b)}{c} = \frac{(a/c)}{b} \\
&= \frac{\lim\limits_{\Delta x \to 0} \left(\frac{-\Delta g}{\Delta x} \right)}{\lim\limits_{\Delta x \to 0} \left(g(x)(g(x) + \Delta g) \right)} && \text{quotient rule for limits} \\
&= \frac{-g'(x)}{g(x)^2} && \text{because } g(x) \text{ is continuo} \\
& && \lim_{\Delta x \to 0} \Delta g = 0.
\end{aligned}
$$

•

EXAMPLE 8 Find $D\left(\frac{1}{\cos(x)}\right)$.

SOLUTION In this case, $g(x) = \cos(x)$ and $g'(x) = -\sin(x)$. Therefore,

$$
\begin{aligned}
D\left(\frac{1}{\cos(x)}\right) &= \frac{-(-\sin(x))}{(\cos(x))^2} \\
&= \frac{\sin(x)}{\cos^2(x)} && \text{for all } x \text{ with } \cos(x) \neq 0
\end{aligned}
$$

◇

Example 8 gives a formula for the derivative of $\sec(x)$, which is defined as $1/\cos(x)$.

$$
D(\sec(x)) = D\left(\frac{1}{\cos(x)}\right) = \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)} \frac{1}{\cos(x)} = \tan(x)\sec(x)
$$

Therefore,

Memorize this formula.

Derivative of $\sec(x)$

$$
D(\sec(x)) = \sec(x)\tan(x)
$$

The reciprocal rule allows us to complete the justification of the power rule for exponents that are negative rational numbers.

EXAMPLE 9 Show that the Power Rule, (3.2.6) in Section 3.2, is valid when a is a negative rational number. That is, show that $D(x^{-p/q}) = (-p/q)x^{(-p/q)-1}$ for any integers p and q, with $q \neq 0$.

SOLUTION The key is to notice that the Reciprocal Rule can be applied to find the derivative of $x^{-p/q} = 1/x^{p/q}$.

$$D\left(x^{-p/q}\right) = D\left(\frac{1}{x^{p/q}}\right) = \frac{-D(x^{p/q})}{(x^{p/q})^2} = \frac{-\frac{p}{q}x^{\frac{p}{q}-1}}{x^{2\frac{p}{q}}} = -\frac{p}{q}x^{\left(\frac{p}{q}\right)-1-2\left(\frac{p}{q}\right)} = -\frac{p}{q}x^{-(p/q)-1}.$$

◇

The Derivative of f/g

EXAMPLE 10 Derive a formula for the derivative of the quotient f/g.
SOLUTION The quotient f/g can be written as a product $f \cdot \frac{1}{g}$. Assuming f and g are differentiable functions, we may use the product and reciprocal rules to find

$$\begin{aligned}
\left(\tfrac{f(x)}{g(x)}\right)' &= \left(f(x)\tfrac{1}{g(x)}\right)' && \text{rewrite quotient as product} \\
&= f'(x)\left(\tfrac{1}{g(x)}\right) + f(x)\left(\tfrac{1}{g(x)}\right)' && \text{product rule} \\
&= f'(x)\left(\tfrac{1}{g(x)}\right) + f(x)\left(\tfrac{-g'(x)}{g(x)^2}\right) && \text{reciprocal rule, assuming } g(x) \neq 0 \\
&= \tfrac{f'(x)}{g(x)} - \tfrac{f(x)g'(x)}{g(x)^2} && \text{algebra} \\
&= \tfrac{g(x)f'(x)-f(x)g'(x)}{g(x)^2} && \text{algebra: common denominator.}
\end{aligned}$$

◇

Example 10 is the proof of the **quotient rule**. The quotient rule should be committed to memory. A simple case of the quotient rule has already been used to find the derivative of $\sec(x) = \frac{1}{\cos(x)}$ (Example 8). The full quotient rule will be used to find the derivative of $\tan(x) = \frac{\sin(x)}{\cos(x)}$ (Example 11). Because the quotient rule is used so often, it should be memorized.

Quotient Rule

Let f and g be differentiable functions at x, and assume $g(x) \neq 0$. Then the quotient f/g is differentiable at x, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \quad \text{where } g(x) \neq 0.$$

A memory device for $(f/g)'$

Remark: Because the numerator in the quotient rule is a difference, it is important to get the terms in the numerator in the correct order. Here is an easy way to remember the quotient rule.

Step 1. Write down the parts where g^2 and g appear:

$$\frac{g\qquad\quad}{g^2}.$$

This ensures that you get the denominator correct and have a good start on the numerator.

Step 2. To complete the numerator, remember that it has a minus sign:

$$\frac{gf' - fg'}{g^2}.$$

EXAMPLE 11 Find the derivative of the tangent function.
SOLUTION

$$\begin{aligned}
(\tan(x))' &= \left(\frac{\sin(x)}{\cos(x)}\right)' \\
&= \frac{\cos(x)(\sin(x))' - \sin(x)(\cos(x))'}{(\cos(x))^2} && \text{quotient rule} \\
&= \frac{(\cos(x))\cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} && \sin^2(x) + \cos^2(x) = 1 \\
&= \sec^2(x) && \sec(x) = 1/\cos(x)
\end{aligned}$$

This result is valid whenever $\cos(x) \neq 0$, and should be memorized. ⋄

Derivative of $\tan(x)$

$$D(\tan(x)) = \sec^2(x) \qquad \text{for all } x \text{ in the domain of } \tan(x).$$

EXAMPLE 12 Compute $(x^2/(x^3+1))'$, showing each step.
SOLUTION

$$\left(\frac{x^2}{x^3+1}\right)' = \frac{(x^3+1)\cdots}{(x^3+1)^2} \quad \text{write denominator and start numerator}$$

$$= \frac{(x^3+1)(x^2)' - (x^2)(x^3+1)'}{(x^3+1)^2} \quad \text{complete numerator, remembering the minus sign}$$

$$= \frac{(x^3+1)(2x) - (x^2)(3x^2)}{(x^3+1)^2} \quad \text{compute derivatives}$$

$$= \frac{2x^4 + 2x - 3x^4}{(x^3+1)^2} \quad \text{algebra}$$

$$= \frac{2x - x^4}{(x^3+1)^2} \quad \text{algebra: collecting}$$

◇

As Example 12 illustrates, the techniques for differentiating polynomials and quotients can be combined to differentiate any **rational function**, that is, any quotient of polynomials.

Summary

Let f and g be two differentiable functions and let C be a constant function. We obtained formulas for differentiating $f+g$, $f-g$, fg, Cf, $1/f$, and f/g.

Rule	Formula	Comment
Constant Rule	$C' = 0$	C a constant
Sum Rule	$(f+g)' = f' + g'$	
Difference Rule	$(f-g)' = f' - g'$	
Product Rule	$(fg)' = f'g + fg'$	
Constant Multiple Rule	$(Cf)' = Cf'$	
Reciprocal Rule	$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$	$g(x) \neq 0$
Quotient Rule	$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$	$g(x) \neq 0$

Table 3.3.1:

With the aid of the formulas in Table 3.3.1, we can differentiate $\sec(x)$, $\csc(x)$, $\tan(x)$, and $\cot(x)$ using $(\sin(x))' = \cos(x)$ and $(\cos(x))' = -\sin(x)$. We also have shown that $D(x^a) = ax^{a-1}$ for any fixed rational number a. (In Section 3.5 we will show it holds for any fixed exponent a.)

Function	Derivative	Comment
x^a	ax^{a-1}	a is a fixed number
$\tan(x)$	$\sec^2(x)$	for all x except odd multiples of $\pi/2$
$\sec(x)$	$\sec(x)\tan(x)$	for all x except odd multiples of $\pi/2$

Table 3.3.2:

EXERCISES for Section 3.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 15 differentiate the given function. *Use only the formulas presented in this and earlier sections.*

1.[R] $5x^3$

2.[R] $5x^3 - 7x + 2^3$

3.[R] $3\sqrt{x} - \sqrt[3]{x}$

4.[R] $1/\sqrt{x}$

5.[R] $(5 + x)(x^2 - x + 7)$

6.[R] $\sin(x)\cos(x)$

7.[R] $3\tan(x)$

8.[R] $3(\tan(x))^2$ HINT: Write $(\tan(x))^2$ as $\tan(x)\tan(x)$

9.[R] $\dfrac{x^3-1}{2x+1}$

10.[R] $\dfrac{\sin(x)}{e^x}$

11.[R] $\dfrac{3x^2 + x + \sqrt{2}}{\cos(x)}$

12.[R] $\dfrac{2}{x^3} + \dfrac{3}{x^4}$

13.[R] $x^2\sin(x)e^x$

14.[R] $\sqrt{x}\sin(x)$

15.[R] $\sqrt{x}/e^x$

16.[R] Differentiate the following functions:

(a) $\dfrac{(1+\sqrt{x})(x^3 + \sin(x))}{x^2 + 5x + 3e^x}$ (b) $\dfrac{(3 + 4x + 5x^2)e^x}{\sin(x)}$

17.[R] Use the quotient rule to obtain the following derivatives.

(a) $D(\tan(x)) = (\sec(x))^2$

(b) $D(\cot(x)) = -(\csc(x))^2$

(c) $D(\sec(x)) = \sec(x)\tan(x)$

(d) $D(\csc(x)) = -\csc(x)\cot(x)$

NOTE: There is a pattern here. The minus sign goes with each "co" function (cos, cot, csc).

18.[R] Find $\left(e^{2x}\right)'$ by writing e^{2x} as $e^x e^x$.

19.[R] Find $\left(e^{3x}\right)'$ by writing e^{3x} as $e^x e^x e^x$.

20.[R] Find $(e^{-x})'$ by writing e^{-x} as $\frac{1}{e^x}$.

21.[R] Find $\left(e^{-2x}\right)'$ by writing $e^{-2x} = e^{-x} \cdot e^{-x}$. (See Exercise 20.)

22.[R] Find $\left(e^{-2x}\right)'$ by writing $e^{-2x} = \frac{1}{e^{2x}}$. (See Exercise 18.)

In Exercises 23 to 41 find the derivative of the function using formulas from this section.

23.[R] $2^3 - \sqrt{\pi}$

24.[R] $(x - x^{-1})^2$

25.[R] $3\sin(9x) - 5\cos(x)$

26.[R] $5\tan(x)$

27.[R] $u^5 - 6u^3 + u - 7$

28.[R] $t^8/8$

29.[R] $s^{-7}/(-7)$

30.[R] $\sqrt{t}(t+4)$

31.[R] $5/u^5$

32.[R] $(x^3)^{1/2}$

33.[R] $6\tan(x)$

34.[R] $3\sec(x) - 4\cos(x)$

35.[R] $\sec^2(\theta) - \tan^2(\theta)$ NOTE: remember to simplify your answer

36.[R] $(3x)^4$

37.[R] $u^2 e^u$

38.[R] $e^t\sin(t)/\sqrt{t}$

39.[R] $(3 + x^5)e^{-x}\tan(x)$

40.[R] $(x - x^2)^3$ HINT: multiply it out first

41.[R] $\sqrt[3]{x}/\sqrt[5]{x}$

42.[R] In Section 3.1 we showed that $D(1/x) = -1/x^2$. Obtain this same formula by using the Quotient Rule.

43.[R] If you had lots of time, how would you differentiate $(1+2x)^{100}$ using the formulas developed so far? NOTE: In Section 3.5 we will obtain a shortcut for differentiating $(1+2x)^{100}$.

44.[M] At what point on the graph of $y = xe^{-x}$ is the tangent horizontal?

45.[M] Using the formula for the derivative of a product, obtain the formula for $(fgh)'$. HINT: First write fgh as $(f)(gh)$. Then use the Product Rule twice.

46.[M] Obtain the formula for $(f-g)'$ by first writing $f - g$ as $f + (-1)g$.

47.[M] Using the definition of the derivative, show that $(f-g)' = f' - g'$.

48.[M] Using the version of the definition of the derivative that makes use of both x and $x+h$, obtain the formula for differentiating the sum of two functions.

49.[C] Using the version of the definition of the derivative in the form $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, obtain the formula for differentiating the product of two functions.

Exercises 50 to 52 are examples of **proof by mathematical induction**. In this technique the truth of the statement for n is used to prove the truth of the statement of $n+1$.

50.[C] In Section 3.2 we show that $D(x^n) = nx^{n-1}$, when n is a positive integer. Now that we have the formula for the derivative of a product of two functions we can obtain this result much more easily.

(a) Show, using the definition of the derivative, that the formula $D(x^n) = nx^{n-1}$ holds when $n = 1$.

(b) Using (a) and the formula for the derivative of a product, show that the formula holds when $n = 2$. HINT: $x^2 = x \cdot x$.

(c) Using (b) and the formula for the derivative of a product, show that it holds when $n = 3$.

(d) Show that if it holds for som
it also holds for the integer n

(e) Combine (c) and (d) to sho
holds for $n = 4$.

(f) Why must it hold for $n = 5$?

(g) Why must it hold for all pos

51.[C] Using induction, as in Exe
for each positive integer n, $D(x^{-n}$

52.[C] Using induction, as in
that for each positive integer
$n \sin^{n-1}(x)\cos(x)$.

53.[C] We obtained the formula f
f/g as the product of f and $1/g$.
rectly from the definition of the der
review how we obtained the formu
of a product.

3.4 The Chain Rule

We come now to the most frequently used formula for computing derivatives. For example, it will help us to find the derivative of $(1+x^2)^{100}$ without having to multiply out one hundred copies of $(1+x^2)$. You might be tempted to guess that the derivative of $(1+x^2)^{100}$ would be $100(1+x^2)^{99}$. *This cannot be right.* After all, when you expand $(1+x^2)^{100}$ you get a polynomial of degree 200, so its derivative is a polynomial of degree 199. But when you expand $(1+x^2)^{99}$ you get a polynomial of degree 198. Something is wrong.

At this point we know the derivative of $\sin(x)$, but what is the derivative of $\sin(x^2)$? It is *not* the cosine of x^2. In this section we obtain a way to differentiate these functions easily — and correctly.

The key is that both $(1+x^2)^{100}$ and $\sin(x^2)$ are composite functions. This section shows how to differentiate composite functions.

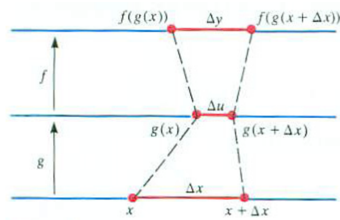


Figure 3.4.1:

How to Differentiate a Composite Function

Recall that $y = (f \circ g)(x) = f(g(x))$ can be built up by setting $u = g(x)$ and $y = f(u)$. The derivative of y with respect to x is the limit of $\Delta y/\Delta x$ as Δx approaches 0. Now, the change in Δx causes a change Δu in u, which, in turn, causes the change Δy in y. (See Figure 3.4.1.) If Δu is not zero, then we may write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}. \tag{3.4.1}$$

It could happen that $\Delta u = 0$, as it would, for instance, if g were a constant function. This special case is treated in Exercise 75.

Then,

$$(f\circ g)'(x) \quad = \quad \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}.$$

Since g is continuous, $\Delta u \to 0$ as $\Delta x \to 0$. So we have

$$(f\circ g)'(x) \quad = \quad \lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = f'(u)g'(x).$$

Which gives us

Chain Rule

Let g be differentiable at x and f be differentiable at $g(x)$, then

$$(f\circ g)'(x) = f'(g(x))g'(x).$$

The Chain Rule is the technique most frequently used in finding derivatives.

This formula tells us how to differentiate a composite function, $f \circ g$:

Step 1. Compute the derivative of the outer function f, evaluated at the inner function. This is $f'(g(x))$.

Step 2. Compute the derivative of the inner function, $g'(x)$.

Step 3. Multiply the derivatives found in Steps 1. and 2., obtaining $f'(g(x))g'(x)$.

In short, to differentiate $f(g(x))$, think of g as the "inner function" and f as the "outer function." Then the derivative of $f \circ g$ is

$$\underbrace{f'(g(x))}_{\text{derivative of outer function evaluated at inner function}} \quad \text{times} \quad \underbrace{g'(x)}_{\text{derivative of inside function}}$$

Examples

EXAMPLE 1 Find $D\left((1+x^2)^{100}\right)$.

SOLUTION Here $g(x) = 1 + x^2$ (the inside function) and $f(u) = u^{100}$ (the outside function). The first step is to compute $f'(u) = 100u^{99}$, which gives us $f'(g(x)) = 100(1 + x^2)^{99}$. The second step is to find $g'(x) = 2x$. Then,

$$(f\circ g)'(x) = f'(\underbrace{u}_{u=g(x)})g'(x) = \underbrace{100u^{99}}_{f'(g(x))}\cdot\underbrace{2x}_{g'(x)} = 100(1+x^2)^{99}\cdot 2x = 200x(1+x^2)^{99}.$$

The answer is not just $100(1+x^2)^{99}$. There is an extra factor of $2x$ that comes from the derivative of the inner function, so its degree is 199, as expected. ⋄

The same example, done with Leibniz notation, looks like this:

$$y = (1 + x^2)^{100} = u^{100}, \qquad u = 1 + x^2.$$

Then the Chain Rule reads simply

$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}\frac{du}{dx}}_{\text{Chain Rule}} = 100u^{99}\cdot 2x = \underbrace{100(1+x^2)^{99}(2x)}_{\text{Using } u=1+x^2} = 200x(1+x^2)^{99}.$$

George Berkeley, 1734, *The Analyst: A Discourse Addressed to an Infidel Mathematician*. See also `http://muse.jhu.edu/journals/configurations/v004/4.1paxson.html`.

WARNING *(Notation)* We avoided using Leibniz notation earlier, in particular, during the derivation of the Chain Rule, because it tempts the reader to cancel the du's in (3.4.1). However, the expressions dy, du, and dx are meaningless — in themselves. In Leibniz's time in the late seventeenth century their meaning was fuzzy, standing for a quantity that was zero and also vanishingly small at the same time. Bishop Berkeley poked fun at this, asking "may we not call them the ghosts of departed quantities?"

With practice, you will be able to do the whole calculation without introducing extra symbols, such as u, which do not apear in the final answer. You will be writing just

$$D\left((1+x^2)^{100}\right) = 100(1+x^2)^{99}\cdot 2x = 200x(1+x^2)^{99}.$$

But this skill, like playing the guitar, takes practice, which the exercises at the end of this section (and chapter) provide.

When we write $\frac{dy}{du}$ and $\frac{du}{dx}$, the u serves two rolls. In $\frac{dy}{du}$ it denotes an independent variable while in $\frac{du}{dx}$, u is a dependent variable. This double role usually causes no problem in computing derivatives.

EXAMPLE 2 If $y = \sin(x^2)$, find $\frac{dy}{dx}$.
SOLUTION Starting from the outside, let $y = \sin(u)$ and $u = x^2$. Then, be the Chain Rule,

$$\left(\sin(x^2)\right)' = \frac{dy}{dx} = \underbrace{\frac{dy}{du}\frac{du}{dx}}_{\text{Chain Rule}} = \cos(u)\cdot 2x = \cos(x^2)\cdot 2x = 2x\cos(x^2).$$

In this case the outside function is the sine and the inside function is x^2. So we have

$$(\underbrace{\sin}_{\text{outside}}\ \underbrace{(x^2)}_{\text{inside}})' = \underbrace{\cos(x^2)}_{\text{derivative of outside function evaluated at inside function}} \quad \text{times} \quad \underbrace{2x}_{\text{derivative of inside function}} = 2x\cos(x^2).$$

◇

The Chain Rule holds for compositions of more than two functions. We illustrate this in the next example.

EXAMPLE 3 Differentiate $y = \sqrt{\sin(x^2)}$.
SOLUTION In this case the function is the composition of three functions:

$$u = x^2 \qquad v = \sin(u) \qquad y = \sqrt{v} \text{ (provided } v \geq 0).$$

Then

Do this example yourself *without introducing any auxiliary symbols (u, v, and y).*

$$\begin{aligned}\frac{dy}{dx} &= \underbrace{\frac{dy}{dv}\frac{dv}{dx}}_{\text{Chain Rule}} = \underbrace{\frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}}_{\text{Chain Rule, again}} = \frac{1}{2\sqrt{v}}\cdot\cos(u)\cdot 2x \\ &= \frac{1}{2\sqrt{\sin(x^2)}}\cdot\cos(x^2)\cdot 2x = \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}\end{aligned}$$

◇

EXAMPLE 4 Let $y = 2^x$. Find y'.

$b = e^{\ln(b)}$ for any $b > 0$

SOLUTION As it stands, 2^x is not a composite function. However, we can write $2 = e^{\ln(2)}$ and then 2^x equals $(e^{\ln(2)})^x = e^{\ln(2)x}$. Now we see that 2^x can be expressed as the composite function:

$$y = e^u, \text{ where } u = (\ln(2))x.$$

Then

$$y' = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot \ln(2) = e^{\ln(2)x}\ln(2) = 2^x \ln(2).$$

In Example 2 (Section 3.2), using a calculator, we found $D(2^x) \approx (0.693)2^x$. We have just seen that the exact formula for this derivative is $D(2^x) = 2^x \ln(2)$. This means that 0.693 is an approximation of $\ln(2)$. Does your calculator agree that $\ln(2) \approx 0.693$? ◇

Sometimes it is convenient to introduce an intermediate variable when using the Chain Rule. The next Example illustrates this idea, which will be used extensively in the next section.

The next Example shows how the Chain Rule can be combined with other differentiation rules such as the Product and Quotient Rules.

EXAMPLE 5 Find $D\left(x^3\tan(x^2)\right)$.

SOLUTION The function $x^3\tan(x^2)$ is the product of two functions. We first apply the Product Rule to obtain:

Product Rule:
$(fg)' = f' \cdot g + f \cdot g'$

$$\begin{aligned} D\left(x^3\tan(x^2)\right) &= \left(x^3\right)'\tan(x^2) + x^3\left(\tan(x^2)\right)' \\ &= 3x^2\tan(x^2) + x^3\left(\tan(x^2)\right)'. \end{aligned}$$

$(\tan(x))' = \sec^2(x)$

Since "the derivative of the tangent is the square of the secant," the Chain Rule tells us that

$$\left(\tan(x^2)\right)' = \sec^2(x^2)(x^2)' = 2x\sec^2(x^2).$$

Thus,

$$\begin{aligned} D\left(x^3\tan(x^2)\right) &= 3x^2\tan(x^2) + x^3\left(\tan(x^2)\right)' \\ &= 3x^2\tan(x^2) + x^3\left(2x\sec^2(x^2)\right) \\ &= 3x^2\tan(x^2) + 2x^4\sec^2(x^2). \end{aligned}$$

◇

In the computation of $D(\tan(x^2))$ we did not introduce any new symbols. That is how your computations will look, once you get the rhythm of the Chain Rule.

Famous Composite Functions

Certain types of composite functions occur so often that it is worthwhile memorizing their derivatives. Here is a list:

Function	Derivative	Example
$(g(x))^n$	$ng(x)^{n-1}g'(x)$	$\left((1+x^2)^{100}\right)' = 100(1+x^2)^{99}(2x)$
$\frac{1}{g(x)}$	$\frac{-g'(x)}{(g(x))^2}$	$D\left(\frac{1}{\cos(x)}\right) = \frac{-(-\sin(x))}{(\cos(x))^2}$
$\sqrt{g(x)}$	$\frac{g'(x)}{2\sqrt{g(x)}}$	$\left(\sqrt{\tan(x)}\right)' = \frac{(\sec(x))^2}{2\sqrt{\tan(x)}}$
$e^{g(x)}$	$e^{g(x)}g'(x)$	$\left(e^{x^2}\right)' = e^{x^2}(2x)$

Table 3.4.1:

Summary

This section presented the single most important tool for computing derivatives: the Chain Rule, which says that the derivative of $f \circ g$ at x is

$$\underbrace{f'(g(x))}_{\text{derivative of outer function evaluated at the inner function}} \quad \text{times} \quad \underbrace{g'(x)}_{\text{derivative of inner function}}$$

Introducing the symbol u, we described the Chain Rule for $y = f(u)$ and $u = g(x)$ with the brief notation

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

When the function is built up from more than two functions, such as $y = f(u)$, $u = g(v)$, and $v = h(x)$. Then we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx},$$

a chain of more derivatives.

With practice, applying the chain rule can become second nature.

All that remains to describe how to differentiate $\ln(x)$ and the inverse trigonometric functions. The next section, with the aid of the chain rule, determines their derivatives.

@ With practice, applying the Chain Rule can become second nature.

EXERCISES for Section 3.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4, repeat the specified example from this section *without* introducing an extra variable (such as u).

1.[R] Example 1.
2.[R] Example 2.
3.[R] Example 3.
4.[R] Example 4.

In Exercises 5 to 18 find the derivative of each function.

5.[R] $(x^3+2)^5$
6.[R] $(x^2+3x+1)^4$
7.[R] $\sqrt{\cos(x^3)}$
8.[R] $\sqrt{\tan(x^2)}$
9.[R] $\left(\frac{1}{x}\right)^{10}$
10.[R] $\cos(3x)\sin(2x)$
11.[R] $x^2\tan(x^3)$
12.[R] $(1+2x)\sin(x^4)$
13.[R] $5(\tan(x^3))^2$
14.[R] $\frac{\cos^3(2x)}{x^5}$
15.[R] $\sin(2\exp(x))$
16.[R] $e^{\cos(x)}$
17.[R] $\frac{(1+2x)^2}{x^3}$
18.[R] $(\sec(5x))(\cos(5x))$ Hint: simplify your answer

In Exercises 19 to 40 differentiate the given function.

19.[R] $(5x^2+3)^{10}$
20.[R] $(\sin(3x))^3$
21.[R] $\frac{1}{5t^2+t+2}$
22.[R] $\frac{1}{e^{5s}+s}$
23.[R] $\sqrt{4+u^2}$
24.[R] $(\sqrt{\cos(2\theta)})^3$
25.[R] e^{5x^3}
26.[R] $\sin^2(3x)$
27.[R] $e^{\tan(3t)}$
28.[R] $\sqrt{\tan(2u)}$
29.[R] $\sqrt[3]{\tan(s^2)}$
30.[R] $v^3\tan(2v)$
31.[R] $e^{2r}\sin(3r)$
32.[R] $\frac{\sec(2x)}{x^2}$
33.[R] $\exp(\sin(2x))$
34.[R] $\frac{(3t+2)^4}{\sin(2t)}$
35.[R] $e^{-5s}\tan(3s)$
36.[R] e^{x^2}
37.[R] $(\sin(2u))^5(\cos(3u))^6$
38.[R] $\left(x+3^{3x}\right)^2(\sin(\sqrt{x}))^3$
39.[R] $\frac{t^3}{(t+\sin^2(3t))}$
40.[R] $\frac{(3x+2)^4}{(x^3+x+1)^2}$

Learning to use the chain rule takes practice. Exercises 41 to 68 offer more opportunities to practice that skill. They also show that sometimes the derivative of a function can be much simpler than the function. In each case show that the derivative of the first function is the second function. (The two functions are separated by a semi-colon.) The letters a, b, and c denote constants.

41.[M] $\frac{b}{2a^2(ax+b)^2}-\frac{1}{a^2(ax+b)}$; $\frac{x}{(ax+b)^2}$
42.[M] $\frac{-1}{2a(ax+b)^2}$; $\frac{1}{(ax+b)^3}$
43.[M] $\frac{2}{3a}\sqrt{(ax+b)^3}$; $\sqrt{ax+b}$
44.[M] $\frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$; $x\sqrt{ax+b}$
45.[M] $\frac{-\sqrt{ax^2+c}}{cx}$; $\frac{1}{x^2\sqrt{ax^2+c}}$
46.[M] $\frac{x}{c\sqrt{ax^2+c}}$; $(ax^2+c)^{-3/2}$
47.[M] $\frac{1}{a}\sin(ax)-\frac{1}{3a}\sin^3(ax)$; $\cos^3(ax)$
48.[M] $\frac{1}{a(n+1)}\sin^{n+1}(ax)$; $\sin^n(ax)\cos(ax)$
49.[M] $\frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$; $\frac{x}{\sqrt{ax+b}}$
50.[M] $\frac{2(3a^2x^2-4abx+8b^2)}{15a^3}\sqrt{ax+b}$; $\frac{x^2}{\sqrt{ax+b}}$
51.[M] $\frac{-\sqrt{ax^2+c}}{cx}$; $\frac{1}{x^2\sqrt{ax^2+c}}$
52.[M] $\frac{-x^2}{a\sqrt{ax^2+c}}+\frac{2}{a^2}\sqrt{ax^2+c}$; $\frac{x^3}{(ax^2+c)^{3/2}}$
53.[M] $\frac{-1}{a}\cos(ax)+\frac{1}{3a}\cos^3(ax)$; $\sin^3(ax)$
54.[M] $\frac{3x}{8}-\frac{3\sin(2ax)}{16a}-\frac{\sin^3(ax)\cos(ax)}{4a}$; $\sin^4(ax)$
55.[M] $\frac{\sin((a-b)x)}{2(a-b)}-\frac{\sin((a+b)x)}{2(a+b)}$; $\sin(ax)\sin(bx)$ (Assume $a^2\neq b^2$.)
56.[M] $\frac{x}{2}+\frac{\sin(2ax)}{3a}$; $\cos^3(ax)$
57.[M] $\frac{1}{a}\tan(ax)$; $\frac{1}{\cos^2(ax)}$
58.[M] $\frac{1}{a}\tan\left(\frac{ax}{2}\right)$; $\frac{1}{1+\cos(ax)}$
59.[M] $2\sqrt{2}\sin\left(\frac{x}{2}\right)$; $\sqrt{1+\cos(x)}$ Note: You will need to use a trigonometric identity.
60.[M] $\frac{\sin((a-b)x)}{2(a-b)}+\frac{\sin((a+b)x)}{2(a+b)}$; $\cos(ax)\cos(bx)$ (Assume $a^2\neq b^2$.)
61.[M] $\frac{1}{a}(\tan(ax)-\cot(ax))$; $\frac{1}{\sin^2(ax)\cos^2(ax)}$
62.[M] $\frac{1}{a}\tan(ax)-1$; $\tan^2(ax)$
63.[M] $\frac{\sec^n(ax)}{an}$; $\tan(ax)\sec^n(ax)$ (Assume $n\neq 0$.)
64.[M] $\frac{\sin(ax)}{a^2}-\frac{x\cos(ax)}{a}$; $x\sin(ax)$
65.[M] $\frac{\cos(ax)}{a^2}+\frac{x\sin(ax)}{a}$; $x\cos(ax)$
66.[M] $\frac{1}{a^2}e^{ax}(ax-1)$; xe^{ax}
67.[M] $\frac{1}{a^3}e^ax(a^2x62-2ax+2)$; x^2e^{ax}
68.[M] $\frac{e^{ax}(a\sin(bx)-b\cos(bx))}{a^2+b^2}$; $e^{ax}\sin(bx)$

Exercises 69 and 70 illustrate how differentiation can be used to obtain one trigonometry identity from another.

69.[M]

(a) Differentiate both sides of the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$. What trigonometric identity do you get?

(b) Differentiate the identity found in (a) to obtain another trigonometric identity. What identity is obtained?

(c) Does this process continued forever produce new identities?

70.[M] Let k be a constant. Differentiate both sides of the identity $\sin(x + k) = \sin(x)\cos(k) + \cos(x)\sin(k)$ to obtain the corresponding identity for $\cos(x + k)$.

71.[M] Differentiate $(e^x)^3$

(a) directly, by the Chain Rule

(b) after writing the function as $e^x \cdot e^x \cdot e^x$ and using the product rule

(c) after writing the function as e^{3x} and using the chain rule

(d) Which of these approaches to you prefer? Why?

72.[M] In Section 3.3 we obtain the derivative of $1/g(x)$ by using the definition of the derivative. Obtain that formula for the Reciprocal Rule by using the Chain Rule.

73.[C] In our proof of the Chain Rule we had to assume that Δu is not 0 when Δx is sufficiently small. Show that if the derivative of g is not 0 at the argument x, then the proof is valid.

74.[C] Here is an example of a
covered by the proof of the Chai
text. Define $g(x)$ to be $x^2 \sin\left(\frac{1}{x}\right)$
0 and $g(0)$ to be 0.

(a) Sketch the part of the graph

(b) Show that there are arbitra
Δx such that $\Delta u = g(\Delta x) -$

(c) Show that g is differentiable

75.[C] Here is a proof of the Ch
ages to avoid division by $\Delta u =$
ferentiable at $g(a)$, where g is diff
$\Delta f = f(g(a) + \Delta u) - f(g(a))$. Th
a function of Δu, which we call $p($
is defined for $\Delta u \neq 0$. By the def
tends to 0 as Δu approaches 0. I
Note that p is continuous at 0.

(a) Show that $\Delta f = f'(g(a))\Delta$
Δu is different than 0, and a

(b) Define $q(\Delta x) = \frac{\Delta u}{\Delta x} - g'(a)$.
approaches 0 as Δx approa
$\Delta u = g'(a)\Delta x + q(\Delta x)\Delta x$ w

(c) Combine (a) and (b) to show

$$\Delta f = f'(g(a))\left(g'(a)\Delta x + q(\right.$$

(d) Using (c), show that

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(g($$

(e) Why did we have to define p

3.5 Derivative of an Inverse Function

In this section we obtain the derivatives of the inverse functions of e^x and of the six trigonometric functions. This will complete the inventory of basic derivatives. The Chain Rule will be our main tool.

Differentiability of Inverse Functions

As mentioned in Section 1.1, the graph of an inverse function is an exact copy of the graph of the original function. One graph is obtained from the other by reflection across the line $y = x$. If the original function, f, is differentiable at a point (a, b), $b = f(a)$, then the graph of $y = f(x)$ has a tangent line at (a, b). In particular, the reflection of the tangent line to the graph of f is the tangent line to the inverse function at (b, a). Thus, we expect that the inverse function, f^{-1}, is differentiable at (b, a), and we will assume it is.

$b = f(a)$ means $a = f^{-1}(b)$

First, the Chain Rule will be used to find the derivative of $\log_e(x)$.

The Derivative of $\log_e(x)$

Let $y = \log_e(x)$. Figure 3.5.1 shows the graphs of $y = e^x$ and inverse function $y = \log_e(x)$. We want to find $y' = \frac{dy}{dx}$. By the definition of logarithm as the inverse of the exponential function

$$x = e^y. \tag{3.5.1}$$

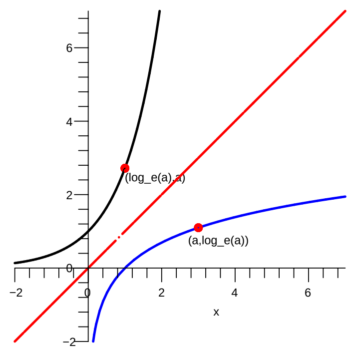


Figure 3.5.1:

We differentiate both sides of (3.5.1) with respect to x:

$$\frac{d(x)}{dx} = \frac{d(e^y)}{dx}$$ e^y is a function of x, since y is a function of x

$$1 = \frac{d(e^y)}{dx}$$ observe that $\frac{dx}{dx} = 1$

$$1 = e^y \frac{dy}{dx}$$ Chain Rule.

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

This is another differentiation rule that should be memorized.

Derivative of e^x
$(\log_e(x))' = \dfrac{1}{x}, \qquad x > 0.$

It may come as a surprise that such a "complicated" function has such a simple derivative. It may also be a surprise that $\log_e(x)$ is one of the most important functions in calculus, mainly because it has the derivative $1/x$.

EXAMPLE 1 Find $(\log_b)'$ for any $b > 0$.
SOLUTION The function $\log_b x$ is just a constant times $\log_e(x)$:

$$\log_b(x) = (\log_b(e)) \log_e(x).$$

Therefore

$$(\log_b(x))' = (\log_b(e)) \frac{1}{x}. \tag{3.5.2}$$

If b is not e, then $\log_b(e)$ is not 1. If e is chosen as the base for logarithms, then the coefficient in front of the $\frac{1}{x}$ becomes $\log_e(e) = 1$. That is why we prefer e as the base for logarithms in calculus ◇

We call $\log_e(x)$ the **natural logarithm**, denoted $\ln(x)$.

> **WARNING** *(Logarithm Notation)* $\ln(x)$ is often written simply as $\log(x)$, with the base understood to be e. All references to the base-10 logarithm will use the notation $\log_{10}$.

The Derivative of $\arcsin(x)$

Inverse trigonometric functions are introduced in Section 1.2.

For x in $[-\pi/2, \pi/2]$ $\sin(x)$ is one-to-one and therefore has an inverse function, $\arcsin(x)$. This function gives the angle, in radians, if you know the sine of the angle. For instance, $\arcsin(1) = \pi/2$, $\arcsin(\sqrt{2}/2) = \pi/4$, $\arcsin(-1/2) = -\pi/6$, and $\arcsin(-1) = -\pi/2$. The domain of $\arcsin(x)$ is $[-1, 1]$; its range is $[-\pi/2, \pi/2]$. For convenience we include the graphs of $y = \sin(x)$ and $y = \arcsin(x)$ in Figure 3.5.2, but will not need them as we find $(\arcsin(x))'$.

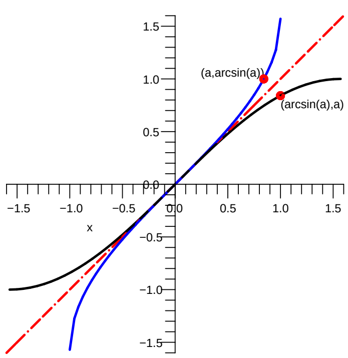


Figure 3.5.2:

To find $(\arcsin(x))'$, we proceed exactly we did when finding $(\log_e(x))'$. Let $y = \arcsin(x)$, then

$$x = \sin(y). \tag{3.5.3}$$

$$\begin{aligned}
x &= \sin(y). \\
\frac{d(x)}{dx} &= \frac{d(\sin(y))}{dx} && \text{differentiate } \textit{with respect to } x \\
1 &= (\cos(y))\, y' && \text{Chain Rule} \\
y' &= \frac{1}{\cos(y)} && \text{algebra} \\
y' &= \frac{1}{1+\tan^2(y)} && \text{trigonometric identity} \\
y' &= \frac{1}{1+x^2} && x = \tan(y).
\end{aligned}$$

The relationship $\sin(y) = x$ can be used to express $\cos(y)$ in terms of x.

Figure 3.5.3 displays the diagram that defines the sine of an angle. The line segment AB represents $\cos(y)$ and the line segment BC represents $\sin(y)$. Observe that the cosine is positive for angles y in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, the first and fourth quadrants. When $x = \sin(y)$, $x^2 + \cos^2(y) = 1$ gives $\cos(y) = \pm\sqrt{1-x^2}$. We use the positive value: $\cos(y) = \sqrt{1-x^2}$ because arcsin is an increasing function. Consequently, we find

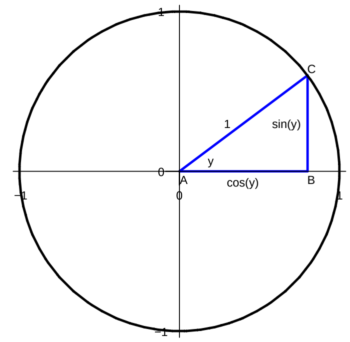


Figure 3.5.3:

Derivative of $\arcsin(x)$

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}, \qquad |x| < 1.$$

The formula for the derivative of the inverse sine should be memorized.

Note at $x = 1$ or at $x = -1$, the derivative is not defined. However, for x near 1 or -1 the derivative is very large (in absolute value), telling us that the graph of the arcsine function is very steep near its two ends. That is a reflection of the fact that the graph of $\sin(x)$ is horizontal at $x = -\pi/2$ and $x = \pi/2$.

Functions such as x^3-x, $x^{2/7}$, and $\frac{1}{\sqrt{1-x^2}}$ that can be written in terms of the algebraic operations of addition, subtraction, multiplication, division, raising to a power, and extracting a root are called **algebraic functions**. Functions that cannot be written in this way, including e^x, $\cos(x)$, and $\arcsin(x)$, are known as **transcendental functions**. The derivative of $\arcsin(x)$ shows that the derivative of a transcendental function can be an algebraic function. But the derivative of an algebraic function will always be algebraic.

An algebraic function always has an algebraic derivative.

EXAMPLE 2 Differentiate $\arcsin\left(x^2\right)$.
SOLUTION This Chain Rule is used to find this derivative:

$$\frac{d}{dx}\left(\arcsin\left(x^2\right)\right) = \frac{1}{\sqrt{1-\left(x^2\right)^2}} \cdot \frac{d}{dx}\left(x^2\right) = \frac{2x}{\sqrt{1-x^4}}.$$

◇

EXAMPLE 3 Differentiate $\frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\arcsin\left(\frac{x}{a}\right)\right)$ where a is a constant.

SOLUTION

$$
\begin{aligned}
D\left(\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\left(\frac{x}{a}\right)\right)\right)
&= \tfrac{1}{2}D\left(\left(x\sqrt{a^2-x^2}+a^2\arcsin\left(\tfrac{x}{a}\right)\right)\right)\\
&= \tfrac{1}{2}\left(D\left(x\sqrt{a^2-x^2}\right)+a^2D\left(\arcsin\left(\tfrac{x}{a}\right)\right)\right)\\
&= \tfrac{1}{2}\left(\left((1)\sqrt{a^2-x^2}\right)+\left(x\left(\frac{\left(\frac{1}{2}\right)(-2x)}{\sqrt{a^2-x^2}}\right)\right)\right.\\
&\quad \left. +a^2\left(\frac{\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right)\right) \qquad D(\arcsin(x))=\tfrac{1}{\sqrt{1-x^2}}\\
&= \tfrac{1}{2}\left(\sqrt{a^2-x^2}+\frac{-x^2}{\sqrt{a^2-x^2}}+\frac{a^2}{\sqrt{a^2-x^2}}\right)\\
&= \tfrac{1}{2}\left(\frac{a^2-x^2-x^2+a^2}{\sqrt{x^2-a^2}}\right)\\
&= \sqrt{a^2-x^2}
\end{aligned}
$$

Note that a rather complicated-looking function can have a simple derivative.
◇

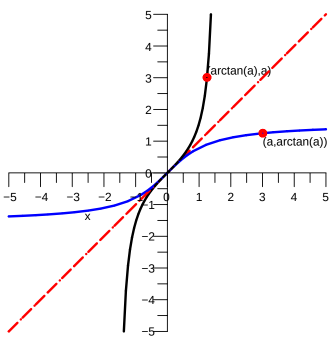


Figure 3.5.4:

See Exercise 82.

The Derivative of $\arctan(x)$

For x in $(-\pi/2, \pi/2)$ $\tan(x)$ is one-to-one and has an inverse function, $\arctan(x)$. This inverse function tells us the angle, in radians, if we know the tangent of the angle. For instance, $\arctan(1) = \pi/4$, $\arctan(0) = 0$, and $\arctan(-1) = -\pi/4$. When x is a large positive number, $\arctan(x)$ is near, and smaller than, $\pi/2$. When x is a large negative number, $\arctan(x)$ is near, and larger than, $-\pi/2$. Figure 3.5.4 shows the graph of $y = \arctan(x)$ and $y = \tan(x)$. We will not need this graph when differentiating $\arctan(x)$, but it serves as a check on the formula.

To find $(\arctan(x))'$, we again call on the Chain Rule. Starting with

$$y = \arctan(x),$$

we proceed as before:

$$
\begin{aligned}
x &= \tan(y). \\
\frac{d(x)}{dx} &= \frac{d(\tan(y))}{dx} && \text{differentiate } \textit{with respect to } x\\
1 &= \left(\sec^2(y)\right)y' && \text{Chain Rule}\\
y' &= \frac{1}{\sec^2(y)} && \text{algebra}\\
y' &= \frac{1}{1+\tan^2(y)} && \text{trigonometric identity}\\
y' &= \frac{1}{1+x^2} && x=\tan(y).
\end{aligned}
$$

This derivation is summarized by a simple formula, which should be memorized.

Derivative of $\arctan(x)$

$$D(\arctan(x)) = \frac{1}{1+x^2} \qquad \text{for all inputs } x$$

EXAMPLE 4 Find $D(\arctan(3x))$.
SOLUTION By the Chain Rule

$$D(\arctan(3x)) = \frac{1}{1+(3x)^2}\frac{d(3x)}{dx} = \frac{3}{1+9x^2}.$$

◇

EXAMPLE 5 Find $D\left(x\tan^{-1}(x) - \frac{1}{2}\ln\left(1+x^2\right)\right)$.
SOLUTION

$$\begin{aligned} D\left(x\tan^{-1}(x) - \tfrac{1}{2}\ln\left(1+x^2\right)\right) &= D\left(x\tan^{-1}(x)\right) - \tfrac{1}{2}D\left(\ln\left(1+x^2\right)\right) \\ &= \left(\tan^{-1}(x) + \frac{x}{1+x^2}\right) - \frac{1}{2}\frac{2x}{1+x^2} \\ &= \tan^{-1}(x). \end{aligned}$$

◇

More on $\ln(x)$

An **antiderivative** of a function, $f(x)$, is another function, $F(x)$, whose derivative is equal to $f(x)$. That is, $F'(x) = f(x)$, and so $\ln(x)$ is an antiderivative of $1/x$. We showed that for $x > 0$, $\ln(x)$ is an antiderivative of $1/x$. But what if we needed an antiderivative of $1/x$ for negative x? The next example answers this question.

Recall that $\ln(x)$ is not defined for $x < 0$.

EXAMPLE 6 Show that for negative x, $\ln(-x)$ is an antiderivative of $1/x$.
SOLUTION Let $y = \ln(-x)$. By the Chain Rule,

$$\frac{dy}{dx} = \left(\frac{1}{-x}\right)\frac{d(-x)}{dx} = \frac{1}{-x}(-1) = \frac{1}{x}.$$

So $\ln(-x)$ is an antiderivative of $1/x$ when x is negative. ◇

In view of Example 6, $\ln|x|$ is an antiderivative of $1/x$, whether x is positive or negative.

Derivative of $\ln|x|$

$$D(\ln|x|) = \frac{1}{x} \qquad \text{for } x \neq 0.$$

We know the derivative of x^a for any rational number a. To extend this result to x^k for any number k, and positive x, we write x as $e^{\ln(x)}$.

EXAMPLE 7 Find $D(x^k)$ for $x > 0$ and any constant $k \neq 0$, rational or irrational.

SOLUTION For $x > 0$ we can write $x = e^{\ln(x)}$. Then

$$x^k = \left(e^{\ln(x)}\right)^k = e^{k\ln(x)}.$$

Now, $y = e^{k\ln(x)}$ is a composite function, $y = e^u$ where $u = k\ln(x)$. Thus,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\frac{k}{x} = x^k\frac{k}{x} = kx^{k-1}.$$

◇

The preceding example shows that for positive x and any fixed exponent k, $\left(x^k\right)' = kx^{k-1}$. It probably does not come as a surprise. In fact you may wonder why we worked so hard to get the derivative of x^a when a is an integer or rational number when this example covers all exponents. We had two reasons for treating the special cases. First, they include cases when x is negative. Second, they were simpler and helped introduce the derivative.

The Derivatives of the Six Inverse Trigonometric Functions

Of the six inverse trigonometric functions, the most important are arcsin and arctan. The other four are treated in Exercises 71 to 74. Table 3.5.1 summarizes all six derivatives. There is no reason to memorize all six of these formulas. If we need, say, an antiderivative of $\frac{-1}{1+x^2}$, we do not have to use $\operatorname{arccot}(x)$. Instead, $-\arctan(x)$ would do. So, for finding antiderivatives, we don't need arccot — or any of the inverse co-functions. You should memorize the formulas for the derivatives of arcsin, arctan, and arcsec.

Note that the negative signs go with the "co-" functions.

$D(\arcsin(x)) = \dfrac{1}{\sqrt{1-x^2}}$	$D(\arccos(x)) = -\dfrac{1}{\sqrt{1-x^2}}$	$(-1 < x < 1)$
$D(\arctan(x)) = \dfrac{1}{1+x^2}$	$D(\text{arccot}(x)) = -\dfrac{1}{1+x^2}$	$(-\infty < x < \infty)$
$D(\text{arcsec}(x)) = \dfrac{1}{x\sqrt{x^2-1}}$	$D(\text{arccsc}(x)) = -\dfrac{1}{x\sqrt{x^2-1}}$	$(x > 1 \text{ or } x < -1)$

Table 3.5.1: Derivatives of the six inverse trigonometric functions.

Another View of e

For each choice of the base b ($b > 0$), we obtain a certain value for $\lim_{x\to 0} \frac{b^x - 1}{x}$.
We defined e to be the base for which that limit is as simple as possible, namely 1: $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$.
Now that we know that the derivative of $\ln x = \log_e x$ is $1/x$, we can obtain a new view of e.
We know that the derivative of $\ln(x)$ at 1 is $1/1 = 1$. By the definition of the derivative, that means

$$\lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h} = 1.$$

Since $\ln(1) = 0$, we have

$$\lim_{h\to 0} \frac{\ln(1+h)}{h} = 1.$$

By a property of logarithms, we may rewrite the limit as

$$\lim_{h\to 0} \ln\left((1+h)^{1/h}\right) = 1.$$

Writing e^x as $\exp(x)$ for convenience, we conclude that

$$\exp\left(\lim_{h\to 0} \ln\left((1+h)^{1/h}\right)\right) = \exp(1) = e.$$

Since exp is a continuous function, we may switch the order of exp and lim, getting

$$\lim_{h\to 0} \left(\exp\left(\ln\left((1+h)^{1/h}\right)\right)\right) = e.$$

But, $\exp(\ln(p)) = p$ for any positive number, by the very definition of a logarithm. That tells us that

$$\lim_{h\to 0} (1+h)^{1/h} = e.$$

This is a much more direct view of e than the one we had in Section 2.2. As a check, let $h = 1/1000 = 0.001$. Then $(1 + 1/1000)^{1000} \approx 2.717$, and values of h that are closer to 0 give even better estimates for e, whose decimal expansion begins 2.718.

Summary

A geometric argument suggests that the inverse of every differentiable function is differentiable. The Chain Rule then helps find the derivatives of $\ln(x)$, $\arcsin(x)$, and $\arctan(x)$ and of the other four inverse trigonometric functions.

EXERCISES for Section 3.5 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 6 evaluate the function and its derivative at the given argument.

1.[R] $\arcsin(x)$; $1/2$

2.[R] $\arcsin(x)$; $-1/2$

3.[R] $\arctan(x)$; -1

4.[R] $\arctan(x)$; $\sqrt{3}$

5.[R] $\ln(x)$; e

6.[R] $\ln(x)$; 1

In Exercises 7 to 28 differentiate the function.

7.[R] $\arcsin(3x)\sin(3x)$

8.[R] $\arctan(5x)\tan(5x)$

9.[R] $e^{2x}\ln(3x)$

10.[R] $e^{\left(\ln(3x)x^{\sqrt{2}}\right)}$

11.[R] $x^2\arcsin(x^2)$

12.[R] $(\arcsin(3x))^2$

13.[R] $\frac{\arctan(2x)}{1+x^2}$

14.[R] $\frac{x^3}{\arctan(6x)}$

15.[R] $\log_{10}(x)$
HINT: Express $\log_{10}$ in terms of the natural logarithm.

16.[R] $\log_x(10)$
HINT: Express $\log_x$ in terms of the natural logarithm.

17.[R] $\arcsin(x^3)$

18.[R] $\arctan(x^2)$

19.[R] $(\arctan(3x))^2$

20.[R] $(\arccos(5x))^3$

21.[R] $\frac{\arcsin(1+x^2)}{1+3x}$

22.[R] $\operatorname{arcsec}(x^3)$

23.[R] $x^2\arcsin(3x)$

24.[R] $\frac{\arctan(3x)}{\tan(2x)}$

25.[R] $\frac{\arctan(x^3)}{\arctan(x)}$

26.[R] $\ln(\sin(3x))$

27.[R] $\ln(\sin(x)^3)$

28.[R] $\ln(\exp(4x))$

In Exercises 29 to 65 check that the derivative of the first function is the second. (A semi-colon separates the two functions.) The letters a, b, and c denote constants.

29.[R] $\frac{1}{cn}\ln\left(\frac{x^n}{ax^n+c}\right)$; $\frac{1}{x(ax^n+c)}$
HINT: To simplify the calculation, first use the fact that $\ln(p/q) = \ln(p) - \ln(q)$.

30.[R] $\frac{1}{nc}\ln\left(\frac{\sqrt{ax^n+c}-\sqrt{c}}{\sqrt{ax^n+c}+\sqrt{c}}\right)$; $\frac{1}{x\sqrt{ax^n+c}}$ (Assume $c > 0$.)

31.[R] $\frac{2}{n\sqrt{-c}}\operatorname{arcsec}\left(\sqrt{\frac{ax^n}{-c}}\right)$; $\frac{1}{x\sqrt{ax^n+c}}$ (Assume $c < 0$.)

32.[R] $\sqrt{ax^2+c}+\sqrt{c}\ln\left(\frac{\sqrt{ax^2+c}-\sqrt{c}}{x}\right)$; $\frac{\sqrt{ax^2+c}}{x}$ (Assume $c > 0$.)

33.[R] $\sqrt{ax^2+c}-\sqrt{-c}\arctan\left(\frac{\sqrt{ax^2+c}}{\sqrt{-c}}\right)$; $\frac{\sqrt{ax^2+c}}{x}$ (Assume $c < 0$.)

34.[R] $\frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)$; $\frac{1}{ax^2+bx+c}$ (Assume $b^2 < 4ac$.)

35.[R] $\frac{-2}{2ax+b}$; $\frac{1}{ax^2+bx+c}$ (Assume $b^2 = 4ac$.)

36.[R] $\frac{1}{\sqrt{b^2-4ac}}\ln\left(\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right)$; $\frac{1}{ax^2+bx+c}$ (Assume $b^2 > 4ac$)
HINT: Use properties of ln *before* differentiating.

37.[R] $\frac{1}{2}\left((x-a)\sqrt{2ax-x^2}+a^2\arcsin\left(\frac{x-a}{a}\right)\right)$; $\sqrt{2ax-x^2}$

38.[R] $\arccos\left(\frac{a-x}{a}\right)$; $\frac{1}{\sqrt{2ax-x^2}}$

39.[R] $\arcsin(x)-\sqrt{1-x^2}$; $\sqrt{\frac{1+x}{1-x}}$

40.[R] $2\arcsin\left(\sqrt{\frac{x-b}{a-b}}\right)$; $\frac{1}{\sqrt{x-b}\sqrt{x-a}}$

41.[R] $\frac{1}{a}\ln\left(\tan\left(\frac{ax}{2}\right)\right)$; $\frac{1}{\sin(ax)}$

42.[R] $\ln(\ln(ax))$; $\frac{1}{x\ln(ax)}$

43.[R] $\frac{-1}{(n-1)(\ln(ax))^{n-1}}$; $\frac{1}{x(\ln(ax))^n}$

44.[R] $x\arcsin(ax)+\frac{1}{a}\sqrt{1-a^2x^2}$; $\arcsin(ax)$

45.[R] $x(\arcsin(ax))^2-2x+\frac{2}{a}\sqrt{1-a^2x^2}\arcsin(ax)$; $(\arcsin(ax))^2$

46.[R] $\frac{1}{ab}\left(ax-\ln\left(b+ce^{ax}\right)\right)$; $\frac{1}{b+ce^{ax}}$

47.[R] $\frac{1}{a\sqrt{bc}}\arctan\left(e^{ax}\sqrt{\frac{b}{c}}\right)$; $\frac{1}{be^ax+ce^{-ax}}$ (Assume b, $c > 0$.)

48.[R] $x(\ln(ax))^2-2x\ln(ax)+2x$; $\ln^2(ax) = (\ln(ax))^2$

49.[R] $-\frac{1}{2}\ln\left(\frac{1+\cos(x)}{1-\cos(x)}\right)$; $\frac{1}{\sin(x)} = \csc(x)$

50.[R] $\frac{1}{b^2}\left(a+bx-a\ln(a+bx)\right)$; $\frac{x}{ax+b}$ (Assume $a+bx > 0$.)

51.[R] $\frac{1}{b^3}\left(a+bx-2a\ln(a+bx)-\frac{a^2}{a+bx}\right)$; $\frac{x^2}{(a+bx)^2}$, (Assume $a+bx > 0$.)

52.[R] $\frac{1}{ab}\arctan\left(\frac{bx}{a}\right)$; $\frac{1}{a^2+b^2x^2}$

53.[R] $\frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^2}\arctan\left(\frac{x}{a}\right)$; $\frac{1}{(a^2+x^2)^2}$

54.[R] $\frac{1}{2a^2}\arctan\left(\frac{x^2}{a^2}\right)$; $\frac{x}{a^4+x^4}$

55.[R] $\frac{2\sqrt{x}}{b} - 2\frac{a}{b^3}\arctan\left(\frac{b\sqrt{x}}{a}\right)$; $\frac{\sqrt{x}}{a^2+b^2x}$

56.[R] $x\,\mathrm{arcsec}(ax) - \frac{1}{a}\ln\left(ax+\sqrt{a^2x^2-1}\right)$; $\mathrm{arcsec}(ax)$

57.[R] $x\arctan(ax) - \frac{1}{2a}\ln\left(1+a^2x^2\right)$; $\arctan(ax)$

58.[R] $x\arccos(ax) - \frac{1}{a}\sqrt{1-a^2x^2}$; $\arccos(ax)$

59.[R] $\frac{x^2}{2}\arcsin(ax) - \frac{1}{4a^2}\arcsin(ax) + \frac{x}{2a}\sqrt{1-a^2x^2}$; $x\arcsin(ax)$

60.[R] $x\left(\arcsin(ax)\right)^2 - 2x + \frac{2}{a}\sqrt{1-a^2x^2}\arcsin(ax)$; $(\arcsin(ax))^2$

61.[R] $\frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$; $x\cos(ax)$

62.[R] $\frac{1}{a^3}e^{ax}\left(a^2x^2-2ax+2\right)$; x^2e^{ax}

63.[R] $\frac{1}{ab}\left(ax - \ln\left(b+ce^{ax}\right)\right)$; $\frac{1}{b+ce^{ax}}$

64.[R] $\frac{1}{a^2+b^2}e^{ax}\left(a\sin(bx) - b\cos(bx)\right)$; $e^{ax}\sin(bx)$

65.[R] $\ln\left(\sec(x)+\tan(x)\right)$; $\sec(x)$

66.[M] Find $D(\ln^3(x))$

(a) by the Chain Rule and

(b) by first writing $\ln^3(x)$ as $\ln(x)\cdot\ln(x)\cdot\ln(x)$.

Which method do you prefer? Why?

67.[M] We have used the equation $\sec^2(x) = 1+\tan^2(x)$.

(a) Derive this equation from the equation $\cos^2(x)+\sin^2(x) = 1$.

(b) Derive the equation $\cos^2(x)+\sin^2(x) = 1$ from the Pythagorean Theorem.

68.[M] Find two antiderivatives of each of the following functions:

(a) $2x$

(b) x^2

(c) $1/x$

(d) $\sqrt{x}$

69.[M] Find two antiderivatives of each of the following functions:

(a) e^{3x}

(b) $\cos(x)$

(c) $\sin(x)$

(d) $1/(1+x^2)$

70.[M] This problem provides some additional experience with the development of the formula $\log_b(x) = \log_b(e)\log_e(x)$. Let $b > 0$. Recall that $\log_b(a) = \frac{\log_e(a)}{\log_e(b)}$.

(a) Show that $\log_b(e) = 1/\log_e(b)$.

(b) Conclude that $\log_b(x) = \log_b(e)\log_e(x)$.

NOTE: This result is used in Example 1.

In Exercises 71 to 74 use the Chain Rule to obtain the given derivative.

71.[M] $(\arccos(x))' = \frac{-1}{\sqrt{1-x^2}}$

72.[M] $(\mathrm{arcsec}(x))' = \frac{1}{x\sqrt{x^2-1}}$

73.[M] $(\mathrm{arccot}(x))' = \frac{-1}{1+x^2}$

74.[M] $(\mathrm{arccsc}(x))' = \frac{-1}{x\sqrt{x^2-1}}$

75.[M] Verify that $D\left(2(\sqrt{x}-1)e^{\sqrt{x}}\right) = e^{\sqrt{x}}$.

76.[M]

Sam: I say that $D(\log_b(x)) = \frac{1}{x\ln(b)}$. It's simple. Let $y = \log_b(x)$. That tells me $x = b^y$. I differentiate both sides of that, getting $1 = b^y(\ln(b))y'$. So $y' = \frac{1}{b^y\ln(b)} = \frac{1}{x\ln(b)}$.

Jane: Well, not so fast. I start with the equation $\log_b(x) = (\log_b(e))\ln(x)$. So $D(\log_b(x)) = \frac{\log_b(e)}{x}$.

Sam: Something is wrong. Where did you get that equation you started with?

Jane: Just take $\log_b$ of both sides of $x = e^{\ln(x)}$.

Sam: I hope this won't be on the next midterm.

Settle this argument.

We did not need the Chain Rule to find the derivatives of inverse functions. Instead, we could have taken a geometric approach, using the "slope of the tangent line" interpretation of the derivative. When we reflect the graph of f around the line $y = x$ to obtain the graph of f^{-1}, the reflection of the tangent line to the graph of f with slope m is the tangent line to the graph of f^{-1} with slope $1/m$. (See Section 1.1.) Exercises 77 to 81 use this approach to develop formulas obtained in this section.

77.[C] Let $f(x) = \ln(x)$. The slope of the graph of $y = \ln(x)$ at $(a, \ln(a))$, $a > 0$, is the reciprocal of the slope of the graph of $y = e^x$ at $(\ln(a), a)$. Use this fact to show that the slope of the graph of $y = \ln(x)$ when $x = a$ is $1/a$.

In Exercises 78 to 81 use the technique illustrated in Exercise 77 to differentiate the given function.

78.[C] $f(x) = \operatorname{arcsec}(x)$.
$f(x) = \arcsin(x)$.
$\arctan(x)$.
81.[C]
80.[C] $f(x) =$
79.[C] $f(x) = \arccos(x)$.

82.[M]

(a) Evaluate $\lim_{x\to\infty} \frac{1}{1+x^2}$ and $\lim_{x\to-\infty} \frac{1}{1+x^2}$.

(b) What do these results tell you about the graph of the arctangent function?

83.[C] Use the assumptions and methods in Exercise 85 to find $D(f/g)$.

84.[C] Use the approach described before Exercise 77 to find $D(x^a)$ for positive x.

85.[C]

Sam: I can get the formula for $(fg)'$ real easy.

Jane: How?

Sam: Start with $\ln(fg) = \ln(f) + \ln(g)$. Then differentiate like mad, using the chain rule:

$$\frac{1}{fg}(fg)' = \frac{f'}{f} + \frac{g'}{g}.$$

Jane: So?

Sam: Then solve for $(fg)'$ and out pops $(fg)' = fg' + gf'$.

Jane: I wonder why the book used all those Δs instead.

Why didn't the book use Sam's approach? HINT: There are two problems with Sam's approach.

86.[C]

Sam: In Exercise 85 they assumed that fg is differentiable if f and g are. I can get around that by using the fact that exp and ln are differentiable.

Jane: How so?

Sam: I write fg as $\exp(\ln(fg))$.

Jane: So?

Sam: But $\ln(fg) = \ln(f) + \ln(g)$, and that does it.

Jane: I'm lost.

Sam: Well, $fg = \exp(\ln(f) + \ln(g))$ and just use the chain rule. It's good for more than grinding out derivatives. In fact, when you differentiate both sides of my equation, you get that fg is differentiable and $(fg)'$ is $f'g + fg'$.

Jane: Why wouldn't the authors use this approach?

Sam: It would make things too easy and reveal that calculus is all about e, exponentials, and logarithms. (I peeked at Chapter 12 and saw that you can even get sine and cosine out of e^x.)

Is Sam's argument correct? If not, identify where it is incorrect.

3.6 Antiderivatives and Slope Fields

So far in this chapter we have started with a function and found its derivative. In this section we will go in the opposite direction: Given a function f, we will be interested in finding a function F whose derivative is f. Why? Because this procedure of going from the derivative back to the function plays a central role in **integral calculus**, as we will see in Chapter 5. Chapter 6 describes several ways to find antiderivatives.

Some Antiderivatives

EXAMPLE 1 Find an antiderivative of x^6.

SOLUTION When we differentiate x^a we get ax^{a-1}. The exponent in the derivative, $a-1$, is one less than the original exponent, a. So we expect an *anti*derivative of x^6 to involve x^7.

Now, $(x^7)' = 7x^6$. This means x^7 is an antiderivative of $7x^6$, not of x^6. We must get rid of that coefficient of 7 in front of x^6. To accomplish this, divide x^7 by 7. We then have

$$\left(\frac{x^7}{7}\right)' = \frac{7x^6}{7} \quad \text{because } \left(\frac{f}{C}\right)' = \frac{f'}{C}$$
$$= x^6 \quad \text{canceling common factor 7 from numerator and denominator.}$$

We can state that $\frac{1}{7}x^7$ is an antiderivative of x^6.

However, $\frac{1}{7}x^7$ is not the only antiderivative of x^6. For instance,

$$\left(\frac{1}{7}x^7 + 2011\right)' = \frac{1}{7}7x^6 + 0 = x^6.$$

We can add any constant to $\frac{1}{7}x^7$ and the result is always an antiderivative of x^6. ◇

A constant added to any antiderivative of a function f gives another antiderivative of f.

As Example 1 suggests, if $F(x)$ is an antiderivative of $f(x)$ so is $F(x) + C$ for any constant C.

The reasoning in this example suggests that $\frac{1}{a+1}x^{a+1}$ is an antiderivative of x^a. This formula is meaningless when $a + 1 = 0$. We have to expect a different formula for antiderivatives of $x^{-1} = \frac{1}{x}$. In Section 3.5 we saw that $(\ln(x))' = 1/x$. That's one reason the function $\ln(x)$ is so important: it provides an antiderivative for $1/x$.

Power Rule for Antiderivatives

For any number a, except -1, the antiderivatives of x^a are

$$\frac{1}{a+1}x^{a+1} + C \quad \text{for any constant } C.$$

The antiderivatives of $x^{-1} = \frac{1}{x}$ are, when $x > 0$,

$$\ln(x) + C \quad \text{for any constant } C.$$

Every time you compute a derivative, you are also finding an antiderivative. For instance, since $D(\sin(x)) = \cos(x)$, $\sin(x)$ is an antiderivative of $\cos(x)$. So is $\sin(x)+C$ for any constant C. There are tables of antiderivatives that go on for hundreds of pages. Here is a miniature table with entries corresponding to the derivatives that we have found so far.

Search Google for "antiderivative table".

Function (f)	Antiderivative (F)	Comment
x^a	$\frac{1}{a+1}x^{a+1}$	for $a \neq -1$
$x^{-1} = \frac{1}{x}$	$\ln(x)$	
e^x	e^x	
$\cos(x)$	$\sin(x)$	
$\sin(x)$	$-\cos(x)$	
$\sec^2(x)$	$\tan(x)$	see Example 8 in Section 3.3
$\sec(x)\tan(x)$	$\sec(x)$	see Example 11 in Section 3.3
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$	see Section 3.4
$\frac{1}{1+x^2}$	$\arctan(x)$	see Section 3.4

Table 3.6.1: Miniature table of antiderivatives ($F' = f$).

An **elementary function** is a function that can be expressed in terms of polynomials, powers, trigonometric functions, exponentials, logarithms, and compositions. The derivative of an elementary function is elementary. We might expect that every elementary function would have an antiderivative that is also elementary.

Joseph Liouville (1809–1882)

In 1833 Joseph Liouville proved beyond a shadow of a doubt that there are elementary functions that do not have elementary antiderivatives. Here are five examples of such functions:

e^{-x^2} is important in statisticians' **bell curve**

$$e^{x^2} \qquad \frac{\sin(x)}{x} \qquad x\tan(x) \qquad \sqrt{x}\sqrt[3]{1+x} \qquad \sqrt[4]{1+x^2}$$

There are two types of elementary functions: the **algebraic** and the **transcendental**. Algebraic functions, defined in Section 3.5, consist of polynomials, quotients of polynomials (the rational functions), and all functions that can be built up by the four operations of algebra and taking roots. For instance, $\frac{\sqrt{x+\sqrt[3]{x}+x^2}}{(1+2x)^5}$ is algebraic; while functions such as $\sin(x)$ and 2^x are not algebraic. These functions are called **transcendental**.

The four operations of algebra are $+$, $-$, $\times$ and $/$.

It is difficult to tell whether a given elementary function has an elementary antiderivative. For instance, $x\sin(x)$ does, namely $-x\cos(x)+\sin(x)$, as you may readily check; but $x\tan(x)$ does not. The function e^{x^2} does not, as mentioned earlier. However, $e^{\sqrt{x}}$, which looks more frightening, doaes have an elementary antiderivative. (See Exercise 75.)

The table of antiderivatives will continue to expand as more derivatives are obtained in the rest of Chapter 3. The importance of antiderivatives will be revealed in Chapter 5. Specific techniques for finding them are developed in Chapter 8. (See Exercise 1.)

Picturing Antiderivatives

If it is not possible to find an explicit formula for the antiderivative of many (most) elementary functions, why do we believe that these functions have antiderivatives? This section puts the answer directly in front of your eyes.

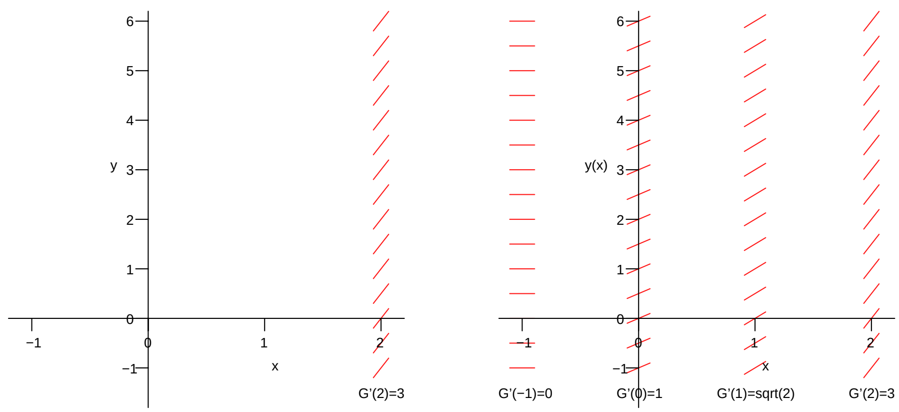
The **slope field** for a function $f(x)$ is made of short line segments with slope $f(x)$ at a few points whose x-coordinate is x. By drawing a slope field you will not only convince yourself that an antiderivative exists, but will see the shape of its graph.

EXAMPLE 2 Imagine that you are looking for an antiderivative $F(x)$ of $\sqrt{1+x^3}$. You want $F'(x)$ to be $\sqrt{1+x^3}$. Or, to put it geometrically, you want the slope of the curve $y=F(x)$ to be $\sqrt{1+x^3}$. For instance, when $x=2$, you want the slope to be $\sqrt{1+x^3}=3$. We do not know what $F(2)$ is, but at least we can draw a short piece of the tangent line at all points for which $x=2$; they all have slope 3. (See Figure 3.6.1(a).) When $x=1$, $\sqrt{1+x^3}=\sqrt{2}\approx 1.4$. So we draw short lines with slope $\sqrt{2}$ on the vertical line $x=1$. When $x=0$, $\sqrt{1+x^3}=1$; the tangent lines for $x=0$ all have slope 1. When $x=-1$, the slopes are $\sqrt{1+x^3}=0$ so the tangent lines are all horizontal. (See Figure 3.6.1(b).)

The plot of a slope field is most commonly made with the aid of a specialized software on a graphing calculator or computer. A typical slope field, showing more segments of tangent curves than we have the patience to draw by hand, is shown in Figure 3.6.2(a) shows a computer-generated direction field for $f(x)=\sqrt{1+x^3}$, which has many more segments of tangent lines than Figure 3.6.1(a).

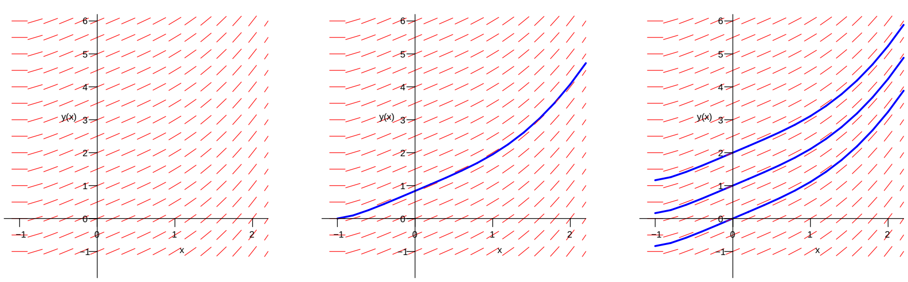
For a sample of available resources, search Google for "calculus slope field plot".

You can almost see the curves that follow the slope field for $f(x)=\sqrt{1+x^3}$. Start at a point, say $(-1,0)$. At this point the slope is $F'(-1)=f(-1)=0$,



(a) (b)

Figure 3.6.1: ARTIST: All references to G should be changed to F.



(a) (b) (c)

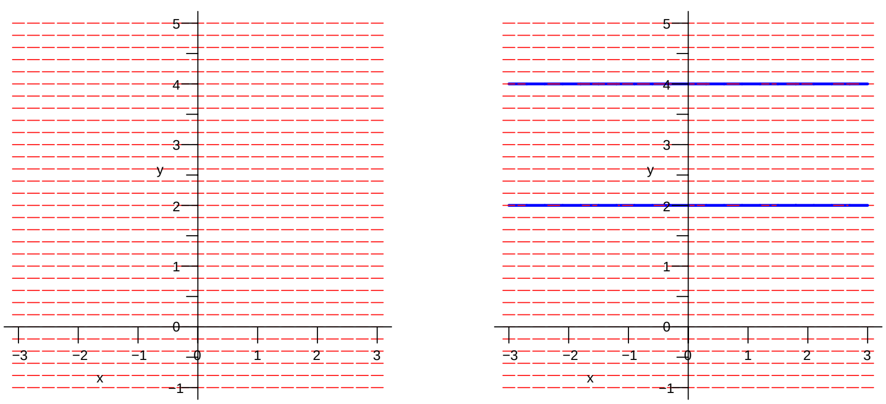
Figure 3.6.2: (a) Slope field for $f(x) = \sqrt{1+x^3}$. (b) Includes the antiderivative with $F(-1) = 0$. (c) Shows three more antiderivatives of $f(x)$.

and the curve starts moving horizontally to the right. As soon as the curve leaves this initial point the slope, as given by $F'(x) = f(x)$, becomes slightly positive. This pushes the curve upward. The slope continues to increase as x increases. The curve in Figure 3.6.2(b) is the graph of the antiderivative of $f(x) = \sqrt{1 + x^3}$ which equals 0 when x is -1.

If you start from a different initial point, you will obtain a different antiderivative. Three antiderivatives are shown in Figure 3.6.2(c). Many other antiderivatives for $f(x) = \sqrt{1 + x^3}$ are visible in the slope field. None of these functions is elementary. ◇

Example 2 suggests that different antiderivatives of a function differ by a constant: the graph of one is simply the graph of the other raised or lowered by their constant difference. The next example reinforces the idea that the constant functions are the only antiderivatives of the zero function.

EXAMPLE 3 Draw the slope field for $\frac{dy}{dx} = 0$.
SOLUTION Since the slope is 0 everywhere, each of the tangent lines is represented by a horizontal line segment, as in Figure 3.6.3(a). In Figure 3.6.3(b)



(a) (b)

Figure 3.6.3:

two possible antiderivatives of 0 are shown, namely the constant functions $f(x) = 2$ and $g(x) = 4$. ◇

We will assume from now on that

- Every antiderivative of the zero function on an interval is constant. That is, if $f'(x) = 0$ for all x in an interval, then $f(x) = C$ for some constant C.
- Two antiderivatives of a function on an interval differ by a constant. That is, if $F'(x) = G'(x)$ for all x in an interval, then $F(x) = G(x) + C$ for some constant C.

These basic results will be established using the definitions and theorems of calculus in Section 3.7.

How computers find antiderivatives

There are algorithms implemented in software on computers, hand-held devices, and calculators that can determine if a given elementary function has an elementary antiderivative. The most well-known is the **Risch algorithm**, developed in 1968, based on differential equations and abstract algebra. A Google search for "risch antiderivative elementary symbolic" produces links related to the Risch algorithm.

Reference: http://en.wikipedia.org/wiki/Risch_algorithm

Summary

The *antiderivative* was introduced as the inverse operation of differentiation. If $F' = f$, then F is an antiderivative of f; so is $F + C$ for any constant C. Alternatively, if F and G are antiderivatives of the same function, then their difference, $F - G$, is constant.

We introduced the notion of an *elementary function.* Such a function is built up from polynomials, logarithms, exponentials, and the trigonometric functions by the four operations $+$, $-$, $\times$, $/$, and the most important operation, composition. While the derivative of an elementary function is elementary, its antiderivative does not need to be elementary. Each elementary function is either algebraic or transcendental.

We showed how a *slope field* can help analyze an antiderivative even though we may not know a formula for it. Slope fields appear later, in Section 6.4 when we discover one of the most important theorems of calculus and when we study differential equations in Chapter 13.

EXERCISES for Section 3.6 *Key:* R–routine, M–moderate, C–challenging

1.[R]

(a) Verify that $-x\cos(x)+\sin(x)$ is an antiderivative of $x\sin(x)$.

(b) Spend at least one minute and at most ten minutes trying to find an antiderivative of $x\tan(x)$.

In Exercises 2 to 11 give two antiderivatives for each given function.

2.[R] x^3

3.[R] x^4

4.[R] x^{-2}

5.[R] $\frac{1}{x^3}$

6.[R] $\sqrt[3]{x}$

7.[R] $\frac{2}{x}$

8.[R] $\sec(x)\tan(x)$

9.[R] $\sin(x)$

10.[R] e^{-x}

11.[R] $\sin(2x)$

In Exercises 12 to 20

(a) draw the slope field for the given derivative,

(b) then use it to draw the graphs of two possible antiderivatives $F(x)$.

12.[R] $F'(x) = 2$

13.[R] $F'(x) = x$

14.[R] $F'(x) = \frac{-x}{2}$

15.[R] $F'(x) = \frac{1}{x}$, $x > 0$

16.[R] $F'(x) = \cos(x)$

17.[R] $F'(x) = \sqrt{x}$

18.[R] $F'(x) = e^{-x}$, $x > 0$

19.[R] $F'(x) = 1/x^2$, $x \neq 0$

20.[R] $F'(x) = 1/(x-1)$, $x \neq 1$

In Exercises 21 to 30 use differentiation to check that the first function is an antiderivative of the second function.

21.[R] $2x\sin(x) - (x^2 - 2)\cos(x)$; $x^2\sin(x)$

22.[R] $(4x^3 - 24x)\sin(x) - (x^4 - 12x^2 + 24)\cos(x)$; $x^4\sin(x)$

23.[R] $\frac{-1}{2x^2}$; $\frac{1}{x^3}$

24.[R] $\frac{-2}{\sqrt{x}}$; $\frac{1}{x^{3/2}}$

25.[R] $(x-1)e^x$; xe^x

26.[R] $(x^2 - 2x + 2)e^x$; x^2e^x

27.[R] $\frac{1}{2}e^u(\sin(u) - \cos(u))$; $e^u\sin(u)$

28.[R] $\frac{1}{2}e^u(\sin(u) + \cos(u))$; $e^u\cos(u)$

29.[R] $\frac{x}{2} - \frac{\sin(x)\cos(x)}{2}$; $\sin^2(x)$

30.[R] $2x\cos(x) - (x^2 - 2)\sin(x)$; $x^2\cos(x)$

31.[M]

(a) Draw the slope field for $\frac{dy}{dx} = e^{-x^2}$.

(b) Draw the graph of the antiderivative of e^{-x^2} that passes through the point $(0, 1)$.

32.[M]

(a) Draw the slope field for $\frac{dy}{dx} = \frac{\sin(x)}{x}$, $x \neq 0$, and $\frac{dy}{dx} = 1$ for $x = 0$.

(b) What is the slope for any point on the y-axis?

(c) Draw the graph of the antiderivative of $f(x)$ that passes through the point $(0, 1)$.

33.[C] A table of antiderivatives lists two antiderivatives of $\frac{1}{x^2(a+bx)}$, where a and b are constants, namely

$$\frac{-1}{a^2}\left(\frac{a+bx}{x} - b\ln\left(\frac{a+bx}{x}\right)\right) \quad \text{and} \quad -\frac{1}{ax}+\frac{b}{a^2}\ln\left(\frac{a+bx}{x}\right).$$

Assume $\frac{a+bx}{x} > 0$.

(a) By differentiating both expressions, show that both are correct.

(b) Show that the two expressions differ by a constant, by finding their difference.

34.[C] If $F(x)$ is an antiderivative of $f(x)$, find a function that is an antiderivative of

(a) $g(x) = 2f(x)$,

(b) $h(x) = f(2x)$.

35.[C]

(a) Draw the slope field for $dy/dx = -y$.

(b) Draw the graph of the function $y = f(x)$ such that $f(0) = 1$ and $dy/dx = -y$.

(c) What do you think $\lim_{x\to\infty} f(x)$ is?

3.7 Motion and the Second Derivative

In an official drag race Melanie Troxel reached a speed of 324 miles per hour, which is about 475 feet per second, in a mere 4.539 seconds. By comparison, a 1968 Fiat 850 Idromatic could reach a speed of 60 miles per hour in 25 seconds and a 1997 Porsche 911 Turbo S in a mere 3.6 seconds.

Source: http://web.missouri.edu/~apcb20/times.html. Numerical acceleration data for other cars can be found with a web search for "automobile acceleration."

Since Troxel increased her speed from 0 feet per second to 475 feet per second in 4.539 seconds her speed was increasing at the rate of $\frac{475}{4.539} \approx 105$ feet per second per second, assuming she kept the motor at maximum power throughout the time interval. That acceleration is more than three times the acceleration due to gravity at sea level (32 feet per second per second). Ms. Troxel must have felt quite a force as her seat pressed against her back.

This brings us to the formal definition of **acceleration** and an introduction to higher derivatives.

In Sections 3.1 and 3.2 we saw that the velocity of an object moving on a line is represented by a derivative. In this section we examine the acceleration mathematically.

Acceleration

The sign of the velocity indicates direction. **Speed**, the absolute value of velocity, does not indicate direction.

Velocity is the rate at which position changes. The rate at which velocity changes is called **acceleration**, denoted a. Thus if $y = f(t)$ denotes position on a line at time t, then the derivative $\frac{dy}{dt}$ equals the velocity, and the derivative of the derivative equals the acceleration. That is,

$$v = \frac{dy}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

The derivative of the derivative of a function $y = f(x)$ is called the **second derivative**. It is denoted many different ways, including:

$$\frac{d^2y}{dx^2}, \quad D^2y, \quad y'', \quad f'', \quad D^2f, \quad f^{(2)}, \quad \text{or} \quad \frac{d^2f}{dx^2}.$$

If $y = f(t)$, where t denotes the time, the first and second derivatives dy/dt, and d^2y/dt^2 are sometimes denoted $\dot{y}$ and $\ddot{y}$, respectively.

y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
x^3	$3x^2$	$6x$
$\frac{1}{x}$	$\frac{-1}{x^2}$	$\frac{2}{x^3}$
$\sin(5x)$	$5\cos(5x)$	$-25\sin(5x)$

For instance, if $y = x^3$,

$$\frac{dy}{dx} = 3x^2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x.$$

Other ways of denoting the second derivative of this function are

$$D^2(x^3) = 6x, \; \frac{d^2(x^3)}{dx^2} = 6x, \quad \text{and} \quad (x^3)'' = 6x.$$

The table in the margin lists dy/dx, the first derivative, and d^2y/dx^2, the second derivative, for a few functions.

Most functions f met in applications of calculus can be differentiated repeatedly in the sense that Df exists, the derivative of Df, namely, D^2f, exists, the derivative of D^2f exists, and so on.

The derivative of the second derivative is called the **third derivative** and is denoted many ways, such as

$$\frac{d^3y}{dx^3}, \quad D^3y, \quad y''', \quad f''', \quad f^{(3)}, \quad \text{or} \quad \frac{d^3f}{dx^3}.$$

The fourth derivative is defined similarly, as the derivative of the third derivative. In the same way we can define the n^{th} derivative for any positive integer n and denote this by such symbols as

$$\frac{d^ny}{dx^n}, \quad D^ny, \quad f^{(n)}, \quad \text{or} \quad \frac{d^nf}{dx^n}.$$

It is read as "the n^{th} derivative with respect to x." For instance, if $f(x) = 2x^3 + x^2 - x + 5$, we have

$$\begin{array}{rcll} f^{(1)}(x) & = & 6x^2 + 2x - 1 & \\ f^{(2)}(x) & = & 12x + 2 & \\ f^{(3)}(x) & = & 12 & \\ f^{(4)}(x) & = & 0 & \\ f^{(n)}(x) & = & 0 & \text{for } n \geq 5. \end{array}$$

EXAMPLE 1 Find $D^n(e^{-2x})$ for each positive integer n.
SOLUTION

$$\begin{array}{rcl} D^1\left(e^{-2x}\right) & = & D\left(e^{-2x}\right) = -2e^{-2x} \\ D^2\left(e^{-2x}\right) & = & D\left(-2e^{-2x}\right) = (-2)^2e^{-2x} \\ D^3\left(e^{-2x}\right) & = & D\left((-2)^2e^{-2x}\right) = (-2)^3e^{-2x} \end{array}$$

At each differentiation another (-2) becomes part of the coefficient. Thus

$$D^n\left(e^{-2x}\right) = (-2)^ne^{-2x}.$$

This can also be written

$$D^n\left(e^{-2x}\right) = (-1)^n2^ne^{-2x}.$$

The power $(-1)^n$ records a "plus" if n is even and a "minus" if n is odd.

◇

Finding Velocity and Acceleration from Position

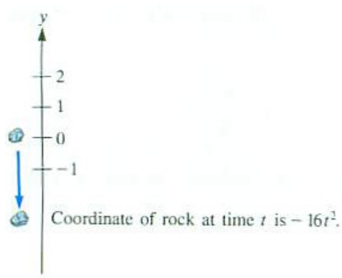


Figure 3.7.1:

EXAMPLE 2 A falling rock drops $16t^2$ feet in the first t seconds. Find its velocity and acceleration.

SOLUTION Place the y-axis in the usual position, with 0 at the beginning of the fall and the part with positive values above 0, as in Figure 3.7.1. At time t the object has the y coordinate

$$y = -16t^2.$$

The velocity is $v = (-16t^2)' = -32t$ feet per second, and the acceleration is $a = (-32t)' = -32$ feet per second per second. The velocity changes at a constant rate. That is, the acceleration is constant. ◇

Finding Position from Velocity and Acceleration

To calculate the position of a moving object at any time it is enough to know the object's acceleration at all times, its initial position, and its initial velocity. This will be demonstrated in the next two examples in the special case that the acceleration is constant. In the first example, the acceleration is 0.

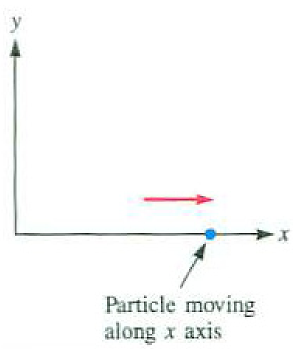


Figure 3.7.2:

EXAMPLE 3 In the simplest motion, no forces act on a moving particle, hence its acceleration is 0. Assume that a particle is moving on the x-axis and no forces act on it. Let its location at time t seconds be $x = f(t)$ feet. See Figure 3.7.2. If at time $t = 0$, $x = 3$ feet and the velocity is 5 feet per second, determine $f(t)$.

SOLUTION The assumption that no force operates on the particle tells us that there is no acceleration: $d^2x/dt^2 = 0$. Call the velocity v. Then

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = 0$$

Now, v is a function of time whose derivative is 0. At the end of Section 3.6 we saw that constant functions are the antiderivatives of 0. Thus, v must be constant:

$$v(t) = C \qquad \text{for some constant } C.$$

Since $v(0) = 5$, the constant C must be 5.

To find the position x as a function of time, note that its derivative is the velocity. Hence

$$\frac{dx}{dt} = 5$$

Similar reasoning tells us that $x = f(t)$ has the form

$$x = 5t + K \qquad \text{for some constant } K.$$

Now, when $t = 0$, $x = 3$. Thus $K = 3$. In short, at time t seconds, the particle is at $x = 5t + 3$ feet. ◇

The next example concerns the case in which the acceleration is constant, but not zero.

EXAMPLE 4 A ball is thrown straight up, with an initial speed of 64 feet per second, from a cliff 96 feet above a beach. Where is the ball t seconds later? When does it reach its maximum height? How high above the beach does the ball rise? When does the ball hit the beach? Assume that there is no air resistance and that the acceleration due to gravity is constant.

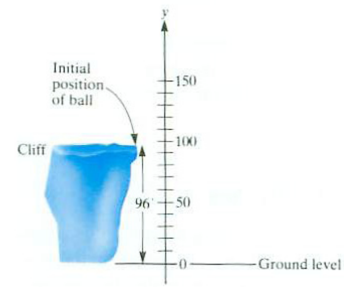


Figure 3.7.3:

SOLUTION Introduce a vertical coordinate axis to describe the position of the ball. It is more natural to call it the y-axis, and so the velocity is dy/dt and acceleration is d^2y/dt^2. Place the origin at ground level and let the positive part of the y-axis be above the ground, as in Figure 3.7.3. At time $t = 0$, the velocity dy/dt is 64, since the ball is thrown up at a speed of 64 feet per second. As time increases, dy/dt decreases from 64 to 0 (when the ball reaches the top of it path and begins its descent) and continues to decrease through larger and larger negative values as the ball falls to the ground. Since v is decreasing, the acceleration dv/dt is negative. The (constant) value of dv/dt, gravitational acceleration, is approximately -32 feet per second per second.

If it had been thrown down dy/dt would be -64.

From the equation

$$a = \frac{dv}{dt} = -32,$$

it follows that

$$v = -32t + C,$$

Velocity is an antiderivative of acceleration.

where C is some constant. To find C, recall that $v = 64$ when $t = 0$. Thus

$$64 = -32 \cdot 0 + C,$$

and $C = 64$. Hence $v = -32t + 64$ for any time t until the ball hits the beach.

So we have

$$\frac{dy}{dt} = v = -32t + 64.$$

Since the position function y is an antiderivative of the velocity, $-32t + 64$, we have

$$y(t) = -16t^2 + 64t + K,$$

where K is a constant. To find K, make use of the fact that $y = 96$ when $t = 0$. Thus

$$96 = -16 \cdot 0^2 + 64 \cdot 0 + K,$$

and $K = 96$.

We have obtained a complete description of the position of the ball at any time t while it is in the air:

$$y = -16t^2 + 64t + 96.$$

This, together with $v = -32t + 64$, provides answers to many questions about the ball's flight. (As a check, note that when $t = 0$, $y = 96$, the initial height.)

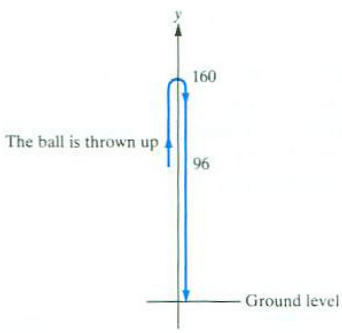


Figure 3.7.4:

When does it reach its maximum height? When it is neither rising nor falling. In other words, the velocity is neither positive nor negative, but must be 0. The velocity is zero when $-32t + 64 = 0$, which is when $t = 2$ seconds.

How high above the ground does the ball rise? Compute y when $t = 2$. This gives $-16 \cdot 2^2 + 64 \cdot 2 + 96 = 160$ feet. (See Figure 3.7.4.)

When does the ball hit the beach? When $y = 0$. Find t such that

$$y = -16t^2 + 64t + 96 = 0$$

Division by -16 yields the simpler equation $t^2 - 4t - 6 = 0$, which has the solutions

$$t = \frac{4 \pm \sqrt{16 + 24}}{2} = 2 \pm \sqrt{10}.$$

Since $2 - \sqrt{10}$ is negative and the ball cannot hit the beach before it is thrown, the only physically meaningful solution is $2 + \sqrt{10}$. The ball lands $2 + \sqrt{10}$ seconds after it is thrown; it is in the air for about 5.2 seconds.

The graphs of position, velocity, and acceleration as functions of time provide another perspective on the motion of the ball, as shown in Figure 3.7.4.

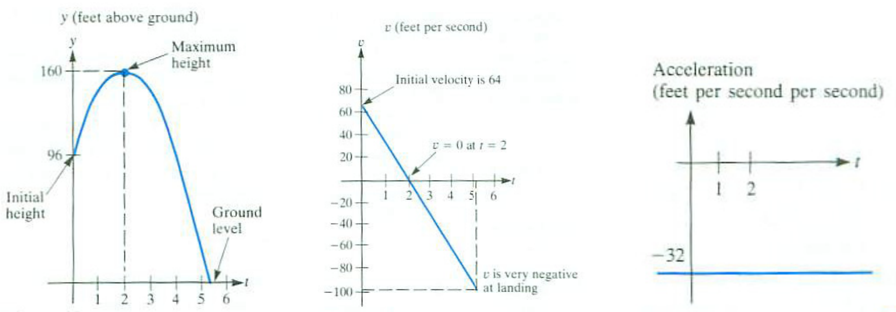
◇

Reasoning like that in Examples 3 and 4 establishes the following description of motion in all cases where the acceleration is constant.

OBSERVATION *Motion Under Constant Acceleration* Assume that a particle moving on the y-axis has a constant acceleration a at any time. Assume that at time $t = 0$ it has the initial change v_0 and has the initial y-coordinate y_0. Then at any time $t \geq 0$ its y-coordinate is

$$y = \frac{a}{2}t^2 + v_0 t + y_0.$$

In Example 3, $a = 0$, $v_0 = 5$, and $y_0 = 3$; in Example 4, $a = -32$ $v_0 = 64$,



(a) (b) (c)

Figure 3.7.5: (a) Position, (b) velocity, and (c) acceleration for the object in Example 4.

and $y_0 = 96$. Note that the data must be given in consistent units, for instance, all in meters or all in feet.

Summary

We defined the higher derivatives of a function. They are obtained by repeatedly differentiating. The second derivative is the derivative of the derivative, the third derivative being the derivative of the second derivative, and so on. The first and second derivatives, $D(f)$ and $D^2(f)$, are used in many applications. We used these two derivatives to analyze motion under constant acceleration. Higher-order derivatives will be used to estimate the error when approximating a function by a polynomial and when approximating an area of by the areas of rectangles or sections of parabolas.

EXERCISES for Section 3.7 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 16 find the first and second derivatives of the given functions.

1.[R] $y = 2x + 3$

2.[R] $y = e^{-x^3}$

3.[R] $y = x^5$

4.[R] $y = \ln(6x + 1)$

5.[R] $y = \sin(\pi x)$

6.[R] $y = 4x^3 - x^2 + x$

7.[R] $y = \frac{x}{x+1}$

8.[R] $y = \frac{x^2}{x-1}$

9.[R] $y = x\cos(x^2)$

10.[R] $y = \frac{x}{\tan(3x)}$

11.[R] $y = (x - 2)^4$

12.[R] $y = (x + 1)^3$

13.[R] $y = e^{3x}$

14.[R] $y = \tan(x^2)$

15.[R] $y = x^2 \arctan(3x)$

16.[R] $y = -\frac{\arcsin(2x)}{x^2}$

17.[R] Use calculus, specifically derivatives, to restate the following reports about the Leaning Tower of Pisa.

(a) "Until 2001, the tower's angle from the vertical was increasing more rapidly."

(b) "Since 2001, the tower's angle from the vertical has not changed."

HINT: Let $\theta = f(t)$ be the angle of deviation from the vertical at time t. NOTE: Incidently, the tower, begun in 1174 and completed in 1350, is 179 feet tall and leans about 14 feet from the vertical. Each day it leaned on the average, another $\frac{1}{5000}$ inch until the tower was propped up in 2001.

Exercises 18 to 20 concern Example 4.

18.[R]

(a) How long after the ball in Example 4 is thrown does it pass by the top of the hill?

(b) What are its speed and velocity at this instant?

19.[R] Suppose the ball in Example 4 had simply been dropped from the cliff. Find the position y as a function of time. How long would it take the ball to reach the beach?

20.[R] In view of the result of Exercise 19, provide a physical interpretation of the three terms on the right-hand side of the formula $y = -16t^2 + 64t + 96$.

21.[R] At time $t = 0$ a particle is at $y = 3$ feet and has a velocity of -3 feet per second; it has a constant acceleration of 6 feet per second per second. Find its position at any time t.

22.[R] At time $t = 0$ a particle is at $y = 10$ feet and has a velocity of 8 feet per second; it has a constant acceleration of -8 feet per second per second.

(a) Find its position at any time t.

(b) What is its maximum y coordinate.

23.[R] At time $t = 0$ a particle is at $y = 0$ feet and has a velocity of 0 feet per second. Find its position at any time t if its acceleration is always -32 feet per second per second.

24.[R] At time $t = 0$ a particle is at $y = -4$ feet and has a velocity of 6 feet per second; it has a constant acceleration of -32 feet per second per second.

(a) Find its position at any time t.

(b) What is its largest y coordinate.

In Exercises 25 to 34 find the given derivatives.

25.[R] $D^3\left(5x^2 - 2x + 7\right)$.

26.[R] $D^4\left(\sin(2x)\right)$.

27.[R] $D^n\left(e^x\right)$.

28.[R] $D(\sin(x))$, $D^2(\sin(x))$, $D^3(\sin(x))$, and $D^4(\sin(x))$.

29.[R] $D(\cos(x))$, $D^2(\cos(x))$, $D^3(\cos(x))$, and $D^4(\cos(x))$.

30.[R] $D(\ln(x))$, $D^2(\ln(x))$, $D^3(\ln(x))$, and $D^4(\ln(x))$.

31.[R] $D^4(x^4)$ and $D^5(x^4)$.

32.[M] $D^{200}(\sin(x))$

33.[M] $D^{200}(e^x)$

34.[M] $D^2(5^x)$

35.[M] Find all functions f such that $D^2(f) = 0$ for all x.

36.[M] Find all functions f such that $D^3(f) = 0$ for all x.

37.[M] A jetliner begins its descent 120 miles from the airport. Its velocity when the descent begins is 500 miles per hour and its landing velocity is 180 miles per hour. Assuming a constant deceleration, how long does the descent take?

38.[M] Let $y = f(t)$ describe the motion on the y-axis of an object whose acceleration has the constant value a. Show that

$$y = \frac{a}{2}t^2 + v_0 t + y_0$$

where v_0 is the velocity when $t = 0$ and y_0 is the position when $t = 0$.

39.[M] Which has the highest acceleration? Melanie Troxel's dragster, a 1997 Porsche 911 Turbo S, or an airplane being launched from an aircraft carrier? The plane reaches a velocity of 180 miles per hour in 2.5 seconds, within a distance of 300 feet. HINT: Assume each acceleration is constant.

40.[M] Why do engineers call the third derivative of position with respect to time the **jerk**?

41.[C] Give two functions f such that $D^2(f) = 9f$. Neither should be a constant multiple of the other.

42.[C] Give two functions f such that $D^2(f) = -4f$. Neither should be a constant multiple of the other.

43.[C] A car accelerates with constant acceleration from 0 (rest) to 60 miles per hour in 15 seconds. How far does it travel in this period? NOTE: Be sure to do your computations either all in seconds, or all in hours; for instance, 60 miles per hour is 88 feet per second.

44.[C] Show that a ball thrown straight up from the ground takes as long to rise as to fall back to its initial position. How does the velocity with which it strikes the ground compare with its initial velocity? How do the initial and landing speeds compare?

3.8 Precise Definition of Limits at Infinity: $\lim_{x\to\infty} f(x)$ L

One day a teacher drew on the board the graph of $y = x/2 + \sin(x)$, shown in Figure 3.8.1. Then the class was asked whether they thought that

$$\lim_{x\to\infty} f(x) = \infty.$$

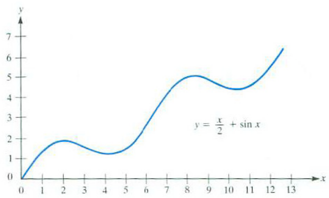


Figure 3.8.1:

A third of the class voted "No" because "it keeps going up and down." A third voted "Yes" because "the function tends to get very large as x increases." A third didn't vote. Such a variety of views on such a fundamental concept suggests that we need a more precise definition of a limit than the ones developed in Sections 2.2 and 2.3. (How would you vote?)

The definitions of the limits considered in Chapter 2 used such phrases as "x approaches a," "$f(x)$ approaches a specific number," "as x gets larger," and "$f(x)$ becomes and remains arbitrarily large." Such phrases, although appealing to the intuition and conveying the sense of a limit, are not precise. The definitions seem to suggest moving objects and call to mind the motion of a pencil point as it traces out the graph of a function.

This informal approach was adequate during the early development of calculus, from Leibniz and Newton in the seventeenth century through the Bernoullis, Euler, and Gauss in the eighteenth and early nineteenth centuries. But by the mid-nineteenth century, mathematicians, facing more complicated functions and more difficult theorems, no longer could depend solely on intuition. They realized that glancing at a graph was no longer adequate to understand the behavior of functions — especially if theorems covering a broad class of functions were needed.

It was Weierstrass who developed, over the period 1841–1856, a way to define limits without any hint of motion or pencils tracing out graphs. His approach, on which he lectured after joining the faculty at the University of Berlin in 1859, has since been followed by pure and applied mathematicians throughout the world. Even an undergraduate advanced calculus course depends on Weierstrass's approach.

In this section we examine how Weierstrass would define the "limits at infinity:"

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = L.$$

In the next section we consider limits at finite points:

$$\lim_{x\to a} f(x) = L.$$

The Precise Definition of $\lim_{x\to\infty} f(x) = \infty$

Recall the definition of $\lim_{x\to\infty} f(x) = \infty$ given in Section 2.2.

Informal definition of $\lim_{x\to\infty} f(x) = \infty$

1. $f(x)$ is defined for all x beyond some number
2. As x gets large through positive values, $f(x)$ becomes and remains arbitrarily large and positive.

To take us part way to the precise definition, let us reword the informal definition, paraphrasing it in the following definition, which is still informal.

Reworded informal definition of $\lim_{x\to\infty} f(x) = \infty$

1. Assume that $f(x)$ is defined for all x greater than the number c.
2. If x is sufficiently large and positive, then $f(x)$ is necessarily large and positive.

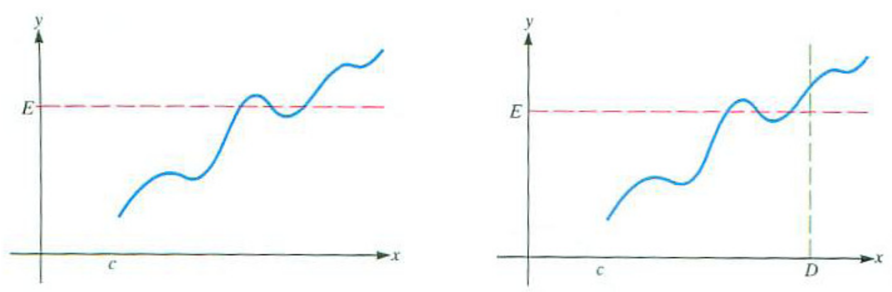
The precise definition parallels the reworded definition.

DEFINITION *(Precise definition of* $\lim_{x\to\infty} f(x) = \infty$*)*

1. Assume the $f(x)$ is defined for all x greater than some number c.
2. For each number E there is a number D such that for all $x > D$ it is true that $f(x) > E$.

The "challenge and reply" approach to limits. Think of E as the "enemy" and D as the "defense."

Think of the number E as a challenge and D as the reply. The *larger* E is, the *larger* D must usually be. Only if a number D (which depends on E) can he found for *every* number E can we make the claim that $\lim_{x\to\infty} f(x) = \infty$. In other words, D could be expressed as a function of E. To picture the idea behind the precise definition, consider the graph in Figure 3.8.2(a) of a function



(a) (b)

Figure 3.8.2:

f for which $\lim_{x\to\infty} f(x) = \infty$. For each possible choice of a horizontal line, say, at height E, if you are far enough to the right on the graph of f, you *stay above* that horizontal line. That is, there is a number D such that if $x > D$, then $f(x) > E$, as illustrated in Figure 3.8.2(b).

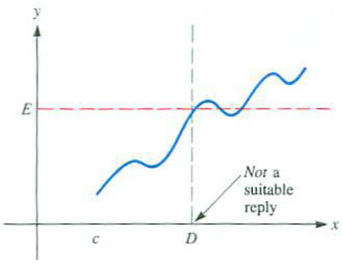


Figure 3.8.3:

The number D in Figure 3.8.3 is not a suitable reply. It is too small since there are some values of $x > D$ such that $f(x) \leq E$.

Examples 1 and 2 illustrate how the precise definition is used.

EXAMPLE 1 Using the precise definition, show that $\lim_{x\to\infty} 2x = \infty$.
SOLUTION Let E be any positive number. We must show that there is a number D such that whenever $x > D$ it follows that $2x > E$. (For example, if $E = 100$, then $D = 50$ would do because it is indeed the case that if $x > 50$, then $2x > 100$.) The number D will depend on E. Our goal is find a formula for D for any value of E.

Now, the inequality $2x > E$ is equivalent to

$$x > \frac{E}{2}.$$

D depends on E

In other words, if $x > E/2$, then $2x > E$. So choosing $D = E/2$ will suffice. To verify this: when $x > D$ $(= E/2)$, $2x > 2D = 2\frac{E}{2} = E$. This allows us to conclude that

$$\lim_{x\to\infty} 2x = \infty.$$

◇

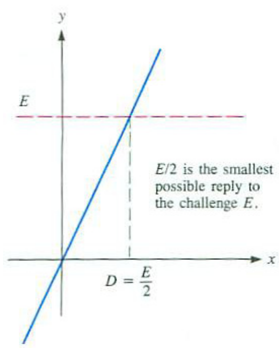


Figure 3.8.4:

In Example 1 a formula was provided for a suitable D in terms of E, namely, $D = E/2$ (see Figure 3.8.4. For instance, when challenged with $E = 1000$, the response $D = 500$ suffices. In fact, any larger value of D also is suitable. If $x > 600$, it is still the case that $2x > 1000$ (since $2x > 1200$). If one value of D is a satisfactory response to a given challenge E, then any larger value of D also is a satisfactory response.

Now that we have a precise definition of $\lim_{x\to\infty} f(x) = \infty$ we can settle the question, "Is $\lim_{x\to\infty}(x/2 + \sin(x)) = \infty$?"

EXAMPLE 2 Using the precise definition, show that $\lim\limits_{x\to\infty} \frac{x}{2} + \sin(x) = \infty$.
SOLUTION Let E be any number. We must exhibit a number D, depending on E, such that $x > D$ forces

$$\frac{x}{2} + \sin(x) > E. \qquad (3.8.1)$$

Now, $\sin(x) \geq -1$ for all x. So, if we can force

$$\frac{x}{2} + (-1) > E \qquad (3.8.2)$$

then it will follow that

$$\frac{x}{2} + \sin(x) > E.$$

The smallest value of x that satisfies inequality (3.8.1) can be found as follows:

$$\begin{array}{rcll} \frac{x}{2} & > & E+1 & \text{add 1 to both sides} \\ x & > & 2(E+1) & \text{multiply by a positive constant.} \end{array}$$

Thus $D = 2(E+1)$ will suffice. That is,

D depends on E

$$\text{If } x > 2(E+1), \text{ then } \frac{x}{2} + \sin(x) > E.$$

To verify this assertion we must check that $D = 2(E+1)$ is a satisfactory reply to E. Assume that $x > D = 2(E+1)$. Then

$$\begin{array}{lrcl} & \frac{x}{2} & > & E+1 \\ \text{and} & \sin(x) & \geq & -1. \end{array}$$

Adding these last two inequalities gives

If $a > b$ and $c \geq d$, then $a + c > b + d$.

$$\begin{array}{lrcl} & \frac{x}{2} + \sin(x) & > & (E+1) + (-1) \\ \text{or simply} & \frac{x}{2} + \sin(x) & > & E, \end{array}$$

which is inequality (3.8.1). Therefore we can conclude that

$$\lim_{x\to\infty} \left(\frac{x}{2} + \sin(x)\right) = \infty.$$

As x increases, the function does *become* and *remain* large, despite the small dips downward. ◇

The Precise Definition of $\lim_{x\to\infty} f(x) = L$

Next, recall the definition of $\lim_{x\to\infty} f(x) = L$ given in Section 2.2.

L is a finite number.

Informal definition of $\lim_{x\to\infty} f(x) = L$

1. $f(x)$ is defined for all x beyond some number
2. As x gets large through positive values, $f(x)$ approaches L.

Again we reword this definition before offering the precise definition.

Reworded informal definition of $\lim_{x\to\infty} f(x) = L$

1. Assume that $f(x)$ is defined for all x greater than some number c.
2. If x is sufficiently large, then $f(x)$ is necessarily near L.

Once again, the precise definition parallels the reworded definition. In order to make precise the phrase "$f(x)$ is necessarily near L," we shall use the absolute value of $f(x) - L$ to measure the distance from $f(x)$ to L. The following definition says that "if x is large enough, then $|f(x) - L|$ is as small as we please".

DEFINITION *(Precise definition of* $\lim_{x\to\infty} f(x) = L$*)*

1. Assume the $f(x)$ is defined for all x beyond some number c.
2. For each positive number ϵ there is a number D such that for all $x > D$ it is true that

$$|f(x) - L| < \epsilon.$$

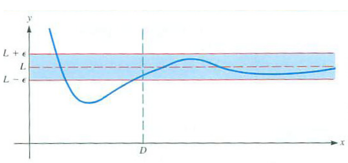


Figure 3.8.5:

Draw two lines parallel to the x-axis, one of height $L + \epsilon$ and one of height $L - \epsilon$. They are the two edges of an endless band of width 2ϵ and centered at $y = L$. Assume that for each positive ϵ, a number D can be found such that the part of the graph to the right of $x = D$ lies within the band. Then we say that "as x approaches ∞, $f(x)$ approaches L" and write

$$\lim_{x\to\infty} f(x) = L.$$

The positive number ϵ is the challenge, and D is a reply. The smaller ϵ is, the narrower the band is, and the larger D usually must be chosen. The geometric meaning of the precise definition of $\lim_{x\to\infty} f(x) = L$ is shown in Figure 3.8.5.

"ϵ" (epsilon) is the Greek letter corresponding to the English letter "e". Because mathematicians think of ϵ as being small, the number theorist, Paul Erdös, called children "epsilons."

EXAMPLE 3 Use the precise definition of "$\lim_{x\to\infty} f(x) = L$" to show that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right) = 1.$$

SOLUTION Here $f(x) = 1 + 1/x$, which is defined for all $x \neq 0$. The number L is 1. We must show that for each positive number ϵ, however small, there is a number D such that, for all $x > D$,

$$\left|\left(1 + \frac{1}{x}\right) - 1\right| < \epsilon. \tag{3.8.3}$$

Inequality (3.8.3) reduces to

$$\left|\frac{1}{x}\right| < \epsilon.$$

Since we may consider only $x > 0$, this inequality is equivalent to

$$\frac{1}{x} < \epsilon. \tag{3.8.4}$$

Multiplying inequality (3.8.4) by the positive number x yields the equivalent inequality

$$1 < x\epsilon. \tag{3.8.5}$$

Division of inequality (3.8.5) by the positive number ϵ yields

$$\frac{1}{\epsilon} < x \quad \text{or} \quad x > \frac{1}{\epsilon}.$$

D depends on ϵ.

These steps are reversible. This shows that $D = 1/\epsilon$ is a suitable reply to the challenge ϵ. If $x > 1/\epsilon$, then

$$\left|\left(1 + \frac{1}{x}\right) - 1\right| < \epsilon.$$

That is, inequality (3.8.3) is satisfied.

According to the precise definition of "$\lim_{x\to\infty} f(x) = L$", we conclude that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right) = 1.$$

◇

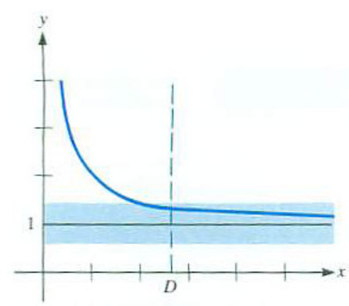


Figure 3.8.6:

The graph of $f(x) = 1+1/x$, shown in Figure 3.8.6, reinforces the argument. It seems plausible that no matter how narrow a band someone may place around the line $y = 1$, it will always be possible to find a number D such that the part of the graph to the right of $x = D$ stays within that band. In Figure 3.8.6 the typical band is shown shaded.

The precise definitions can also be used to show that some claim about an alleged limit is false. The next example illustrates how this is done.

EXAMPLE 4 Show that the claim that $\lim_{x\to\infty} \sin(x) = 0$ is false.

SOLUTION To show that the claim is *false*, we must exhibit a challenge $\epsilon > 0$ for which no response D can be found. That is, we must exhibit a positive number ϵ such that no D exists for which $|\sin(x) - 0| < \epsilon$ for all $x > D$.

Recall that $\sin(\pi/2) = 1$ and that $\sin(x) = 1$ whenever $x = \pi/2 + 2n\pi$ for any integer n. This means that there are arbitrarily large values of x for which

$\sin(x) = 1$. This suggests how to exhibit an $\epsilon > 0$ for which no response D can be found. Simply pick the challenge ϵ to be some positive number less than or equal to 1. For instance, $\epsilon = 0.7$ will do.

For any number D there is always a number $x^* > D$ such that we have $\sin(x^*) = 1$. This means that $|\sin(x^*) - 0| = 1 > 0.7$. Hence no response can he found for $\epsilon = 0.7$. Thus the claim that $\lim_{x\to\infty} \sin(x) = 0$ is false. $\diamond$

To conclude this section, we show how the precise definition of the limit can be used to obtain information about new limits.

EXAMPLE 5 Use the precise definition of "$\lim_{x\to\infty} f(x) = L$" to show that if f and g are defined everywhere and $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to\infty} g(x) = 3$, then $\lim_{x\to\infty}(f(x) + g(x)) = 5$.

SOLUTION The objective is to show that for each positive number ϵ, however small, there is a number D such that, for all $x > D$,

$$|(f(x) + g(x)) - 5| < \epsilon.$$

Observe that $|(f(x) + g(x)) - 5|$ can be written as $|(f(x) - 2) + (g(x) - 3))|$, and this is no larger than the sum $|f(x) - 2| + |g(x) - 3|$. If we can show that for all x sufficiently large that both $|f(x) - 2| < \epsilon/2$ and $|g(x) - 3| < \epsilon/2$, then there sum will be no larger than $\epsilon/2 + \epsilon/2 = \epsilon$.

Here is how this plan can be implemented.

The fact that $\lim_{x\to\infty} f(x) = 2$ implies for any given $\epsilon > 0$ there exists a number D_1 with the property that $|f(x) - 2| < \epsilon/2$ for all $x > D_1$. Likewise, the fact that $\lim_{c\to\infty} g(x) = 3$ implies for any given $\epsilon > 0$ there exists a number D_2 with the property that $|g(x) - 2| < \epsilon/2$ for all $x > D_2$.

Let D refer to the larger of D_1 and D_2. For any x greater than D we know that

$D = \max\{D_1, D_2\}$

$$|f(x) + g(x) - 5| < |f(x) - 2| + |g(x) - 3| < \epsilon/2 + \epsilon/2 = \epsilon.$$

According to the precise definition of a limit at infinity, we conclude that

$$\lim_{x\to\infty} (f(x) + g(x)) = 2 + 3 = 5.$$

$\diamond$

Summary

We developed a precise definition of the limit of a function as the argument becomes arbitrarily large: $\lim_{x\to\infty} f(x)$. The definition involves being able to respond to a challenge. In the case of an infinite limit, the challenge is a large number. In the case of a finite limit, the challenge is a small number used to describe a narrow horizontal band.

EXERCISES for Section 3.8 *Key:* R–routine, M–moderate, C–challenging

1.[R] Let $f(x) = 3x$.

(a) Find a number D such that, for $x > D$, it follows that $f(x) > 600$.

(b) Find another number D such that, for $x > D$, it follows that $f(x) > 600$.

(c) What is the smallest number D such that, for all $x > D$, it follows that $f(x) > 600$?

2.[R] Let $f(x) = 4x$.

(a) Find a number D such that, for $x > D$, it follows that $f(x) > 1000$.

(b) Find another number D such that, for $x > D$, it follows that $f(x) > 1000$.

(c) What is the smallest number D such that, for all $x > D$, it follows that $f(x) > 1000$?

3.[R] Let $f(x) = 5x$. Find a number D such that, for all $x > D$,

(a) $f(x) > 2000$,

(b) $f(x) > 10,000$.

4.[R] Let $f(x) = 6x$. Find a number D such that, for all $x > D$,

(a) $f(x) > 1200$,

(b) $f(x) > 1800$.

In Exercises 5 to 12 use the precise definition of the assertion "$\lim_{x\to\infty} f(x) = \infty$" to establish each limit.

5.[R] $\lim_{x\to\infty} 3x = \infty$

6.[R] $\lim_{x\to\infty} 4x = \infty$

7.[R] $\lim_{x\to\infty} (x + 5) = \infty$

8.[R] $\lim_{x\to\infty} (x - 600) = \infty$

9.[R] $\lim_{x\to\infty} (2x + 4) = \infty$

10.[R] $\lim_{x\to\infty} (3x - 1200) = \infty$

11.[R] $\lim_{x\to\infty} (4x + 100\cos(x)) = \infty$

12.[R] $\lim_{x\to\infty} (2x - 300\cos(x)) = \infty$

13.[R] Let $f(x) = x^2$.

(a) Find a number D such that, for all $x > D$, $f(x) > 100$.

(b) Let E be any nonnegative number. Find a number D such that, for all $x > D$, it follows that $f(x) > E$.

(c) Let E be any negative number. Find a number D such that, for all $x > D$, it follows that $f(x) > E$.

(d) Using the precise definition of "$\lim_{x\to\infty} f(x) = \infty$", show that $\lim_{x\to\infty} x^2 = \infty$.

14.[R] Using the precise definition of "$\lim_{x\to\infty} f(x) = \infty$", show that $\lim_{x\to\infty} x^3 = \infty$. HINT: See Exercise 13.

Exercises 15 to 22 concern the precise definition of "$\lim_{x\to\infty} f(x) = L$".

15.[R] Let $f(x) = 3 + 1/x$ if $x \neq 0$.

(a) Find a number D such that, for all $x > D$, it follows that $|f(x) - 3| < \frac{1}{10}$.

(b) Find another number D such that, for all $x > D$, it follows that $|f(x) - 3| < \frac{1}{10}$.

(c) What is the smallest number D such that, for all $x > D$, it follows that $|f(x) - 3| < \frac{1}{10}$?

(d) Using the precise definition of "$\lim_{x\to\infty} f(x) = L$", show that $\lim_{x\to\infty} (3 + 1/x) = 3$.

16.[R] Let $f(x) = 2/x$ if $x \neq 0$.

(a) Find a number D such that, for all $x > D$, it follows that $|f(x) - 0| < \frac{1}{100}$.

(b) Find another number D such that, for all $x > D$, it follows that $|f(x) - 0| < \frac{1}{100}$.

(c) What is the smallest number D such that, for all $x > D$, it follows that $|f(x) - 0| < \frac{1}{100}$?

(d) Using the precise definition of "$\lim_{x\to\infty} f(x) = L$", show that $\lim_{x\to\infty}(2/x) = 0$.

In Exercises 17 to 22 use the precise definition of "$\lim_{x\to\infty} f(x) = L$" to establish each limit.

17.[M] $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$ Hint: $|\sin(x)| \leq 1$ for all x.

18.[M] $\lim_{x\to\infty} \frac{x+\cos(x)}{x} = 1$

19.[M] $\lim_{x\to\infty} \frac{4}{x^2} = 0$

20.[M] $\lim_{x\to\infty} \frac{2x+3}{x} = 2$

21.[M] $\lim_{x\to\infty} \frac{1}{x-100} = 0$

22.[M] $\lim_{x\to\infty} \frac{2x+10}{3x-5} = \frac{2}{3}$

23.[M] Using the precise definition of "$\lim_{x\to\infty} f(x) = \infty$," show that the claim that $\lim_{x\to\infty} x/(x+1) = \infty$ is false.

24.[M] Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that the claim that $\lim_{x\to\infty} \sin(x) = \frac{1}{2}$ is false.

25.[M] Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that the claim that $\lim_{x\to\infty} 3x = 6$ is false.

26.[M] Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that for every number L the assertion that $\lim_{x\to\infty} 2x = L$ is false.

In Exercises 27 to 30 develop precise definitions of the given limits. Phrase your definitions in terms of a challenge number E or ϵ and a reply D. Show the geometric meaning of your definition on a graph.

27.[M] $\lim_{x\to\infty} f(x) = -\infty$

28.[M] $\lim_{x\to-\infty} f(x) = \infty$

29.[M] $\lim_{x\to-\infty} f(x) = -\infty$

30.[M] $\lim_{x\to-\infty} f(x) = L$

31.[M] Let $f(x) = 5$ for all x. (See Exercise 30

(a) Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that $\lim_{x\to\infty} f(x) = 5$.

(b) Using the precise definition of "$\lim_{x\to-\infty} f(x) = L$," show that $\lim_{x\to-\infty} f(x) = 5$.

32.[C] Is this argument correct? "I will prove that $\lim_{x\to\infty}(2x + \cos(x)) = \infty$. Let E be given. I want

$$\begin{array}{rrcl} & 2x+\cos(x) & > & E \\ \text{or} & 2x & > & E - \cos(x) \\ \text{so} & x & > & \frac{E-\cos(x)}{2}. \end{array}$$

Thus, if $D = \frac{E-\cos(x)}{2}$, then $2x + \cos(x) > E$."

33.[M] Use the precise definition of "$\lim_{x\to\infty} f(x) = L$," to prove this version of the sum law for limits: if $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty} g(x) = B$, then $\lim_{x\to\infty}(f(x) + g(x)) = A + B$. Hint: See Example 5.

34.[C] Use the precise definition of "$\lim_{x\to\infty} f(x) = L$," to prove this version of the product law for limits: if $\lim_{x\to\infty} f(x) = A$, then $\lim_{x\to\infty}(f(x)^2) = A^2$. Hint: $f(x)^2 - A^2 = (f(x) - A)(f(x) + A)$, and control the size of each factor.

35.[C] Use the precise definition of "$\lim_{x\to\infty} f(x) = L$," to prove this version of the product law for limits: if $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty} g(x) = B$, then $\lim_{x\to\infty}(f(x)g(x)) = AB$. Hint: To make use of the two given limits, write $f(x)$ as $A + (f(x) - A)$ and $g(x)$ as $B + (g(x) - B)$.

36.[C] Assume that $\lim_{x\to\infty} f(x) = 5$. Is there necessarily a number c such that for $x > c$, $f(x)$ stays in the closed interval $[4.5, 5]$? Explain in detail.

37.[C] Assume that $\lim_{x\to\infty} f(x) = 5$. Is there necessarily a number c such that for $x > c$, $f(x)$ stays in the open interval $(4, 5.5)$? Explain in detail.

38.[C]

Sam: I got lost in Example 5 when $\epsilon/2$ came out of nowhere.

Jane: It's just another ϵ.

Sam: Now I'm more confused.

Explain Jane's explanation for Sam's benefit.

3.9 Precise Definition of Limits at a Finite Point: $\lim_{x \to a} f(x) = L$

To conclude the discussion of limits, we extend the ideas developed in Section 3.8 to limits of a function at a number a.

Informal definition of $\lim_{x \to a} f(x) = L$

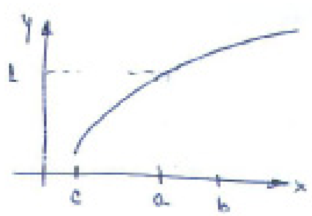


Figure 3.9.1:

Let f be a function and a some fixed number. (See Figure 3.9.1.)

1. Assume that the domain of f contains open intervals (c, a) and (a, b) for some number $c < a$ and some number $b > a$.
2. If, as x approaches a, either from the left or from the right, $f(x)$ approaches a specific number L, then L is called the **limit** of $f(x)$ as x approaches a. This is written

$$\lim_{x \to a} f(x) = L.$$

Keep in mind that a need not be in the domain of f. Even if it happens to be in the domain of f, the value of $f(a)$ plays no role in determining whether $\lim_{x \to a} f(x) = L$.

Reworded informal definition of $\lim_{x \to a} f(x) = L$

Let f be a function and a some fixed number.

1. Assume that the domain of f contains open intervals (c, a) and (a, b) for some number $c < a$ and some number $b > a$.
2. If x is is sufficiently close to a but not equal to a, then $f(x)$ is necessarily near L.

"δ" (delta) is the lower case version of the Greek letter "Δ"; it corresponds to the English letter "d."

The following precise definition parallels the reworded informal definition.

DEFINITION *(Precise definition of* $\lim_{x \to a} f(x) = L$*)* Let f be a function and a some fixed number.

1. Assume that the domain of f contains open intervals (c, a) and (a, b) for some number $c < a$ and some number $b > a$.

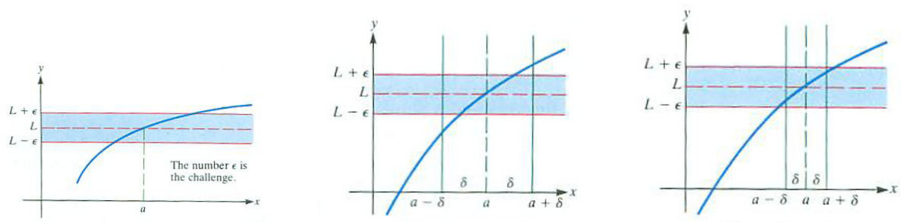
2. For each positive number ϵ there is a positive number δ such that

$$\begin{array}{rrcl} x \text{ that satisfy the inequality} & 0 < |x-a| & < & \delta \\ \text{it is true that} & |f(x)-L| & < & \epsilon. \end{array}$$

The meaning of $0 < |x-a| < \delta$

The inequality $0 < |x-a|$ that appears in the definition is just a fancy way of saying "x is not a." The inequality $|x-a| < \delta$ asserts that x is within a distance δ of a. The two inequalities may be combined as the single statement $0 < |x-a| < \delta$, which describes the open interval $(a-\delta, a+\delta)$ from which a is deleted. This deletion is made since the $f(a)$ pIays no role in the definition of $\lim_{x\to a} f(x)$.

Once again ϵ is the challenge. The reply is δ. Usually, the smaller ϵ is, the *smaller* δ will have to be.



(a) (b) (c)

Figure 3.9.2: (a) The number ϵ is the challenge. (b) δ is not small enough. (c) δ is small enough.

The geometric significance of the precise definition of "$\lim_{x\to a} f(x) = L$" is shown in Figure 3.9. The narrow horizontal band of width 2ϵ is again the challenge (see Figure 3.9(a)). The desired response is a sufficiently narrow vertical band, of width 2δ, such that the part of the graph within that vertical band (except perhaps at $x = a$) also lies in the horizontal band of width 2ϵ. In Figure 3.9(b) the vertical band shown is not narrow enough to meet the challenge of the horizontal band shown. But the vertical band shown in Figure 3.9(c) is sufficiently narrow.

Assume that for each positive number ϵ it is possible to find a positive number δ such that the parts of the graph between $x = a - \delta$ and $x = a$ and between $x = a$ and $x = a + \delta$ lie within the given horizontal band. Then we say that "as x approaches a, $f(x)$ approaches L". The narrower the horizontal band around the line $y = L$, the smaller δ usually must be.

EXAMPLE 1 Use the precise definition of "$\lim_{x\to a} f(x) = L$" to show that $\lim_{x\to 2}(3x+5) = 11$.

SOLUTION Here $f(x) = 3x+5$, $a = 2$, and $L = 11$. Let ϵ be a positive number. We wish to find a number $\delta > 0$ such that for $0 < |x-2| < \delta$ we have $|(3x+5) - 11| < \epsilon$.

So let us find out for which x it is true that $|(3x+5)-11| < \epsilon$. This inequality is equivalent to

$$\begin{array}{lrcl} & |3x-6| & < & \epsilon \\ \text{or} & 3|x-2| & < & \epsilon \\ \text{or} & |x-2| & < & \frac{\epsilon}{3}. \end{array}$$

Any positive number less than $\epsilon/3$ is also a suitable response.

Thus $\delta = \epsilon/3$ is a suitable response. If $0 < |x-2| < \epsilon/3$, then $|(3x+5)-11| < \epsilon$. ◇

The algebra of finding a response δ can be much more involved for other functions, such as $f(x) = x^2$. The precise definition of limit can actually be easier to apply in more general situations where f and a are not given explicitly. To illustrate, we present a proof of the Permanence Property.

When the Permanence Property was introduced in Section 2.5, the only justification we provided was a picture and an appeal to your intuition that a continuous function cannot jump instantaneously from a positive value to zero or a negative value — the function has to remain positive on some open interval. Mathematicians call this a "proof by handwaving".

EXAMPLE 2 Prove the Permanence Property: Assume that f is continuous in an open interval that contains a and that $f(a) = p > 0$. Then for any number $q < p$, there is an open interval I containing a such that $f(x) > q$ for all x in I.

The reason for this choice for ϵ will become clear in a moment.

SOLUTION Let $p = f(a) > 0$ and let q be any positive number less than p. Pick $\epsilon = p - q$. Because f is continuous at a there is a positive number δ such that

$$|f(a) - f(x)| < p - q \qquad \text{for } a - \delta < x < a + \delta.$$

Thus

$$-(p-q) < f(a) - f(x) < p - q.$$

In particular,

$$f(a) - f(x) < p - q \tag{3.9.1}$$

Because $f(a) = p$, (3.9.1) can be rewritten as

$$p - f(x) < p - q$$

or

$$f(x) > q.$$

Thus $f(x)$ is greater than q if x is in the interval $I = (a - \delta, a + \delta)$.

◇

One of the common uses of the Permanence Property is to say that if a continuous function is positive at a number, a, then there is an interval containing a on which the function is strictly positive. (This corresponds to $p = f(a) > 0$ and $q = 0$.)

Summary

This section developed a precise definition of the limit of a function as the argument approaches a fixed number: $\lim_{x\to a} f(x)$. This definition involves being able to respond to an arbitrary challenge number. In the case of a finite limit, the challenge is a small positive number. The smaller that number, the harder it is to meet the challenge.

In addition, it also gave a rigorous proof of the permanence principle.

EXERCISES for Section 3.9 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4 use the precise definition of "$\lim_{x\to a} f(x) = L$" to justify each statement.

1.[R] $\lim_{x\to 2} 3x = 6$

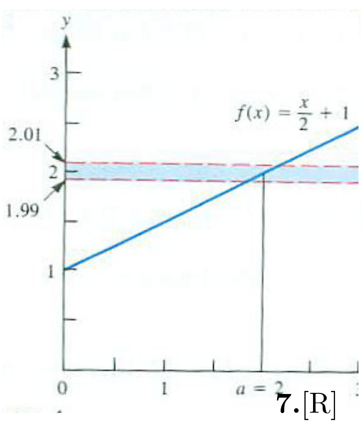
2.[R] $\lim_{x\to 3} (4x - 1) = 11$

3.[R] $\lim_{x\to 1} (x + 2) = 3$

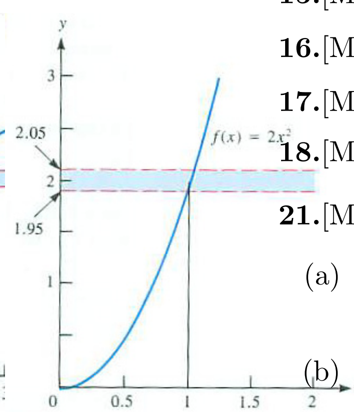
4.[R] $\lim_{x\to 5} (2x - 3) = 7$

In Exercises 5 and 8 find a number δ such that the point $(x, f(x))$ lies in the shaded band for all x in the interval $(a - \delta, a + \delta)$. HINT: Draw suitable vertical band for the given value of ϵ.

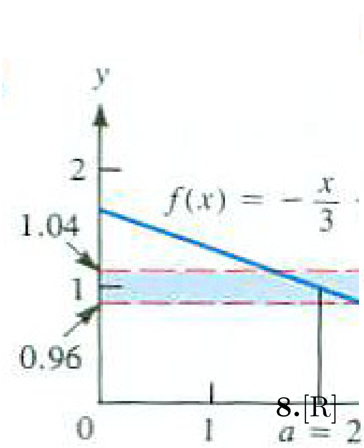
5.[R]



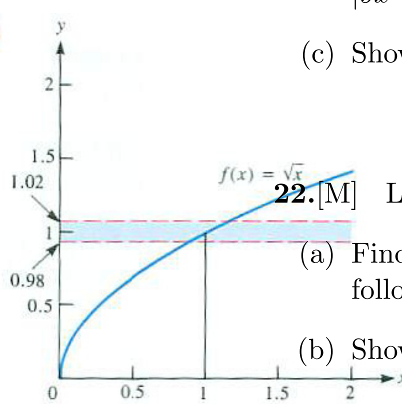
7.[R]



6.[R]



8.[R]



3 4 5

In Exercises 9 and 12 use the precise definition of "$\lim_{x\to a} f(x) = L$" to justify each statement.

9.[R] $\lim_{x\to 1} (3x + 5) = 8$

10.[R] $\lim_{x\to 1} \dfrac{5x + 3}{4} = 2$

11.[M] $\lim_{x\to 0} \dfrac{x^2}{4} = 0$

12.[M] $\lim_{x\to 0} 4x^2 = 0$

13.[M] Give an example of a number $\delta > 0$ such that $|x^2 - 4| < 1$ if $0 < |x - 2| < \delta$.

14.[M] Give an example of a number $\delta > 0$ such that $|x^2 + x - 2| < 0.5$ if $0 < |x - 1| < \delta$.

Develop precise definitions of the given limits in Exercises 15 to 20. Phrase your definitions in terms of a challenge, E or ϵ, and a response, δ.

15.[M] $\lim_{x\to a^+} f(x) = L$

16.[M] $\lim_{x\to a^-} f(x) = L$

17.[M] $\lim_{x\to a} f(x) = \infty$

18.[M] $\lim_{x\to a} f(x) = -\infty$

19.[M] $\lim_{x\to a^+} f(x) = \infty$

20.[M] $\lim_{x\to a^-} f(x) = \infty$

21.[M] Let $f(x) = 9x^2$.

(a) Find $\delta > 0$ such that, for $0 < |x - 0| < \delta$, it follows that $|9x^2 - 0| < \frac{1}{100}$.

(b) Let ϵ be any positive number. Find a positive number δ such that, for $0 < |x - 0| < \delta$ we have $|9x^2 - 0| < \epsilon$.

(c) Show that $\lim_{x\to 0} 9x^2 = 0$.

22.[M] Let $f(x) = x^3$.

(a) Find $\delta > 0$ such that, for $0 < |x - 0| < \delta$, it follows that $|x^3 - 0| < \frac{1}{1000}$.

(b) Show that $\lim_{x\to 0} x^3 = 0$.

23.[M] Show that the assertion "$\lim_{x\to 2} 3x = 5$" is false. To do this, it is necessary to exhibit a positive number ϵ such that there is no response number $\delta > 0$. HINT: Draw a picture.

24.[M] Show that the assertion "$\lim_{x\to 2} x^2 = 3$" is false.

25.[C] Review the proof of the Permanence Property given in Example 2. Recall that $p = f(a) > 0$ and q is chosen so that $p > q > 0$.

(a) Would the argument have worked if we had used $\epsilon = 2(p-q)$?

(b) Would the argument have worked if we had used $\epsilon = \frac{1}{2}(p-q)$?

(c) Would the argument have worked if we had used $\epsilon = q$?

(d) What is the largest value of ϵ for which the proof of the Permanence Property works?

26.[C] The Permanence Property discussed in Example 2 and Exercise 25 pertains to limits at a finite point a. State, and prove, a version of the Permanence Property that is valid "at ∞."

27.[M]

(a) Show that, if $0 < \delta < 1$ and $|x-3| < \delta$, then $|x^2-9| < 7\delta$. HINT: Factor x^2-9.

(b) Use (a) to deduce that $\lim_{x\to 3} x^2 = 9$.

28.[C]

(a) Show that, if $0 < \delta < 1$ and $|x-4| < \delta$, then

$$|\sqrt{x}-2| < \frac{\delta}{\sqrt{3}+2}.$$

(b) Use (a) to deduce that $\lim_{x\to 4} \sqrt{x} = 2$.

29.[C]

(a) Show that, if $0 < \delta < 1$ and $|x-3| < \delta$, then $|x^2+5x-24| < 12\delta$.
HINT: Factor $x^2+5x-24$.

(b) Use (a) to deduce that $\lim_{x\to 3}(x^2+5x) = 24$.

30.[C]

(a) Show that, if $0 < \delta < 1$ and $|x-2| < \delta$, then

$$\left|\frac{1}{x}-\frac{1}{2}\right| < \frac{\delta}{2}.$$

(b) Use (a) to deduce that $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

31.[C] Use the precise definitions of limits to prove: if f is defined in an open interval including a and f is continuous at a, so is $3f$.

32.[C] Use the precise definitions of limits to prove: if f and g are both defined in an open interval including a and both functions are continuous at a, so is $f+g$.

33.[C] Use the precise definitions of limits to prove: if f and g are both continuous at a, then their product, fg, is also continuous at a. NOTE: Assume that both functions are defined at least in an open interval around a.

34.[C] Assume that $f(x)$ is continuous at a and is defined at least on an open interval containing a. Assume that $f(x) = p > 0$. Using the precise definition of a limit, show that there is an open interval, I, containing a such that $f(x) > \frac{2}{11}p$ for all x in I.

3.S Chapter Summary

In this chapter we defined the derivative of a function, developed ways to compute derivatives, and applied them to graphs and motion.

The derivative of a function f at a number $x = a$ is defined as the limit of the slopes of secant lines through the points $(a, f(a))$ and $(b, f(b))$ as the input b is taken closer and closer to the input a.

Algebraically, the derivative is the limit of a quotient, "the change in the output divided by the change in the input". The limit is usually written in one of the following forms:

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a},\quad \lim_{h\to 0}\frac{f(a+h)-f(a)}{h},\quad \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}.G$$

The derivative is denoted in several ways, such as f', $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, and $D(f)$.

For the functions most frequently encountered in applications, this limit exists. Geometrically speaking, the derivative exists whenever the graph of the function on a very small interval looks almost like a straight line.

The derivative records how fast something changes. For instance, the velocity of a moving object is defined as the derivative of the object's position. Also, the derivative gives the slope of the tangent line to the graph of a function.

We then developed ways to compute the derivative of functions expressible in terms of the functions met in algebra and trigonometry, including exponentials with a fixed base and logarithms; the so-called "elementary functions". That development was based on three limits:

$$\begin{aligned}\lim_{x\to a}\frac{x^n-a^n}{x-a} &= na^{n-1}, \qquad n \text{ a positive integer}\\ \lim_{x\to 0}\frac{e^x-1}{x} &= 1\\ \lim_{x\to 0}\frac{\sin(x)}{x} &= 1.\end{aligned}$$

Using these limits, we obtained the derivatives of x^n, e^x, and $\sin(x)$. We showed, if we knew the derivatives of two functions, how to compute the derivatives of their sum, difference, product, and quotient. Naturally, this was based on the definition of the derivative as a limit.

The next step was the development of the most important computational tool: the Chain Rule. This enables us to differentiate a composite function, such as $\cos^3(x^2)$. It tells us that its derivative is $3\cos^2(x^2)(-\sin(x^2))(2x)$.

Differentiating inverse functions enabled us to show that the derivative of $\ln|x|$ is $\frac{1}{x}$ and the derivative of $\arcsin(x)$ is $\frac{1}{\sqrt{1-x^2}}$. The following list of derivatives of key functions should be memorized.

function	derivative
x^a (a constant)	ax^{a-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	e^x
a^x (a constant)	write $a^x = e^{x(\ln(a))}$
$\ln(x)$ $(x > 0)$	$1/x$
$\ln\|x\|$ $(x \neq 0)$	$1/x$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$
$\frac{1}{x}$	$-1/x^2$

Figure 3.S.1: Table of Common Functions and Derivatives.

As you work with derivatives you may begin to think of them as slope or velocity or rate of change, and forget their underlying definition as a limit. However, we will from time to time return to the definition in terms of limits as we develop more applications of the derivative.

We also introduced the antiderivative and, closely related to it, the slope field. While the derivative of an elementary function is again elementary, an antiderivative often is not. For instance, $\sqrt{1+x^3}$ does not have an elementary antiderivative. However, as we will see in Chapter 6, it does have an antiderivative. Chapter 8 will present a few ways to find antiderivatives.

The derivative of the derivative is the second derivative. In the case of motion, the second derivative describes acceleration. It is denoted several ways, such as $D^2 f$, $\frac{d^2 f}{dx^2}$, f'', and $f^{(2)}$. While the first and second derivatives suffice for most applications, higher derivatives of all orders are used in Chapter 5, where we estimate the error when approximating a function by a polynomial.

The final two sections returned to the notion of a limit, providing a precise definition of that concept.

EXERCISES for 3.S *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 19 find the derivative of the given function.

1.[R] $\exp(x^2)$

2.[R] 2^{x^2}

3.[R] $x^3\sin(4x)$

4.[R] $\frac{1+x^2}{1+x^3}$

5.[R] $\ln(x^3)$

6.[R] $\ln(x^3+1)$

7.[R] $\cos^4(x^2)\tan(2x)$

8.[R] $\sqrt{5x^2+x}$

9.[R] $\arcsin(\sqrt{3+2x})$

10.[R] $x^2\arctan(2x)e^{3x}$

11.[R] $\sec^2(3x)$

12.[R] $\sec^2(3x) - \tan^2(3x)$

13.[R] $\left(\frac{3+2x}{4+5x}\right)^3$

14.[R] $\frac{1}{1+2e^{-x}}$

15.[R] $\frac{x}{\sqrt{x^2+1}}$

16.[R] $(\arcsin(3x))^2$

17.[R] $x^2\arctan(3x)$

18.[R] $\sin^5(3x^2)$

19.[R] $\frac{1}{(2^x+3^x)^{20}}$

In Exercises 20 to 29 give an antiderivative for the given function. Use differentiation to check each answer. (Chapter 8 presents techniques for finding antiderivatives, but the ones below do not require these methods.)

20.[R] $4x^3$

21.[R] x^3

22.[R] $3/x^2$

23.[R] $\cos(x)$

24.[R] $\cos(2x)$

25.[R] $\sin^{100}(x)\cos(x)$

26.[R] $1/(x+1)$

27.[R] $5e^{4x}$

28.[R] $1/e^x$

29.[R] 2^x

In Exercises 30 to 51 carry out the differentiation to check each equation. The letters a, b, and c denote constants. NOTE: These problems provide good practice in differentiation and algebra. Each differentiation formula has a corresponding antiderivative formula. In fact, these exercises are based on several tables of antiderivatives.

30.[R] $\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{a^2+x^2}$

31.[R] $D\left(\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)\right) = \frac{1}{a^2-x^2}$

32.[R] $\left(\ln\left(x+\sqrt{a^2+x^2}\right)\right)' = \frac{1}{\sqrt{a^2+x^2}}$

33.[R] $\frac{d}{dx}\left(\frac{1}{a}\ln\left(\frac{x+\sqrt{a^2-x^2}}{x}\right)\right) = \frac{1}{x\sqrt{a^2+x^2}}$

34.[R] $D\left(\frac{-1}{b(a+bx)}\right) = \frac{1}{(a+bx)^2}$

35.[R] $\left(\frac{1}{b^2}\left(a+bx-a\ln(a+bx)\right)\right)' = \frac{x}{a+bx}$

36.[R] $\frac{d}{dx}\left(\frac{1}{b^2}\left(\frac{a}{2(a+bx)^2}-\frac{1}{a+bx}\right)\right) = \frac{x}{(a+bx)^3}$

37.[R] $D\left(\frac{1}{ab'-a'b}\ln\left(\frac{a'+b'x}{a+bx}\right)\right) = \frac{1}{(a+bx)(a'+b'x)}$ (a, b, a', b' constants)

38.[R] $\left(\frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2cx+b}{\sqrt{4ac-b^2}}\right)\right)' = \frac{1}{a+bx+cx^2}$ $(4ac > b^2)$

39.[R] $\frac{d}{dx}\left(\frac{-2}{\sqrt{b^2-4ac}}\ln\left(\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}}\right)\right) = \frac{1}{a+bx+cx^2}$ $(4ac < b^2)$

40.[R] $D\left(\frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right)\right) = \frac{1}{x\sqrt{x^2-a^2}}$

41.[R] $\left(\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\right.\right.$ ... $\sqrt{a^2-x^2}$ $(|x| < |a|)$

42.[R] $\frac{d}{dx}\left(\frac{-x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcs\right.$ ... $\frac{x^2}{\sqrt{a^2-x^2}}$ $(|x| < |a|)$

43.[R] $D\left(-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\left(\frac{x}{a}\right)\right.$ ... $\frac{\sqrt{a^2-x^2}}{x^2}$ $(|x| < |a|)$

44.[R] $\left(\arcsin(x)-\sqrt{1-x^2}\right)' =$ ... $\sqrt{\frac{1+x}{1-x}}$ $(|x| < 1)$

45.[R] $\frac{d}{dx}\left(\frac{x}{2}-\frac{1}{2}\cos(x)\sin(x)\right) =$ ... $\sin^2(x)$

46.[R] $D\left(x\arcsin x+\sqrt{1-x^2}\right)$ ... $\arcsin(x)$ $(|x| < 1)$

47.[R] $\left(x\tan^{-1}(x)-\frac{1}{2}\ln(1+x^2)\right.$ ... $\arctan(x)$

48.[R] $\frac{d}{dx}\left(\frac{e^{ax}}{a^2}\left(a^2-1\right)\right) = xe^{ax}$

49.[R] $D\left(x-\ln(1+e^x)\right) = \frac{1}{1+e^x}$

50.[R] $\left(\frac{x}{2}\left(\sin(\ln(ax))-\cos(\ln(a\right.\right.$ ... $\sin(\ln(ax))$

51.[R] $\left(\frac{e^{ax}(a\sin(bx)-b\cos(bx))}{a^2+b^2}\right)' =$ ... $e^{ax}\sin(bx)$

In Exercises 52 to 55 give two antiderivatives for each given function.

52.[M] xe^{x^2}

53.[M] $(x^2 + x)e^{x^3+3x}$

54.[M] $\cos^3(x)\sin(x)$

55.[M] $\sin(2x)$

56.[M] Verify that $2(\sqrt{x} - 1)e^{\sqrt{x}}$ is an antiderivative of $e^{\sqrt{x}}$.

In Exercises 57 to 60 (a) sketch the slope field and (b) draw the solution curve through the point $(0, 1)$.

57.[R] $dy/dx = 1/(x+1)$

58.[R] $dy/dx = e^{-x^2}$

59.[R] $dy/dx = -y$

60.[R] $dy/dx = y - x$

61.[R] Sam threw a baseball straight up and caught it 6 seconds later.

(a) How high above his head did it rise?

(b) How fast was it going as it left his hand?

(c) How fast was it going when he caught it?

(d) Translate the answers in (b) and (c) to miles per hour. (Recall: 60 mph = 88 fps.)

62.[M] Assuming that $D(x^4) = 4x^3$ and $D(x^7) = 7x^6$, you could find $D(x^3)$ directly by viewing x^3 as x^7/x^4 and using the formula for differentiating a quotient. Show how you could find directly $D(x^{11})$, $D(x^{-4})$, $D(x^{28})$, and $D(x^8)$.

63.[M] Let $y = x^{m/n}$, where $x > 0$ and m and $n \neq 0$ are integers. Assuming that y is differentiable, show that $\frac{dy}{dx} = \frac{m}{n}x^{\frac{m}{n}-1}$ by starting with $y^n = x^m$ and differentiating both y^n and x^m with respect to x. HINT: Think of y as $y(x)$ and remember to use the chain rule when differentiating y^n with respect to x.

64.[M] A spherical balloon is being filled with helium at the rate of 3 cubic feet per minute. At what rate is the radius increasing when the radius is (a) 2 feet? (b) 3 feet? HINT: The volume of a ball of radius r is $\frac{4}{3}\pi r^3$.

65.[M] An object at the end of a vertical spring is at rest. When you pull it down it goes up and down for a while. With the origin of the y-axis at the rest position, the position of the object t seconds later is $3e^{-2t}\cos(2\pi t)$ inches.

(a) What is the physical significance of 3 in the formula?

(b) What does e^{-2t} tell us?

(c) What does $\cos(2\pi t)$ tell us?

(d) How long does it take the object to complete a full cycle (go from its rest position, down, up, then down to its rest position)?

(e) What happens to the object after a long time?

66.[M] The motor on a moving motor boat is turned off. It then coasts along the x-axis. Its position, in meters, at time t (seconds) is $500 - 50e^{-3t}$.

(a) Where is it at time $t = 0$?

(b) What is its velocity at time t?

(c) What is its acceleration at time t?

(d) How far does it coast?

(e) Show that its acceleration is proportional to its velocity. NOTE: This means the force of the water slowing the boat is proportional to the velocity of the boat. (See also Exercise 78.)

67.[M] It is safe to switch the "sin" and "lim" in $\sin\left(\lim_{x\to 0}\frac{e^x-1}{x}\right) = \lim_{x\to 0}\left(\sin\left(\frac{e^x-1}{x}\right)\right)$. However, such a switch sometimes is not correct. Consider f defined by $f(x) = 2$ for $x \neq 1$ and $f(1) = 0$.

(a) Show that $f\left(\lim\limits_{x\to 0}\dfrac{e^x-1}{x}\right)$ is not equal to $\lim_{x\to 0} f\left(\frac{e^x-1}{x}\right)$.

(b) What property of the function $\sin(x)$ permits us to switch it with "lim"?

The preceding exercises offered an opportunity to practice computing derivatives. However, it is important to keep in mind the definition of a derivative as a limit. Exercises 68 to 72 will help to reinforce this definition.

68.[R] Define the derivative of the function $g(x)$ at $x = a$ in (a) the x and $x + h$ notation, (b) the x and a notation, and (c) the Δy and Δx notation.

69.[M] We obtained the derivative of $\sin(x)$ using the x and $x + h$ notation and the addition identity for $\sin(x+h)$. Instead, obtain the derivative of $\sin(x)$ using the x and a notation. That is, find

$$\lim_{x\to a} \frac{\sin(x) - \sin(a)}{x - a}.$$

(a) Show that $\sin(x) - \sin(a) = 2\sin\left(\frac{1}{2}(x - y)\right)\cos\left(\frac{1}{2}(x + y)\right)$.

(b) Use the identity in (a) to find the limit.

70.[M] We obtained the derivative for $\tan(x)$ by writing it as $\sin(x)/\cos(x)$. Instead, obtain the derivative directly by finding

$$\lim_{h\to 0} \frac{\tan(x + h) - \tan(x)}{h}.$$

HINT: The identity $\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$ will help.

71.[C] Show that $\frac{\tan(a)}{\tan(b)} > \frac{a}{b} > \frac{\sin(a)}{\sin(b)}$ for all angles a and b in the first quadrant with $a > b$. HINT: Be ready to make use of the two inequalities that squeezed $\sin(x)/x$ toward 1.

72.[C] We obtained the derivative of $\ln(x)$, $x > 0$, by viewing it as the inverse of $\exp(x)$. Instead, find the derivative directly from the definition. HINT: Use the x and h notation.

Exercises 73 and 74 show how we could have predicted that $\ln(x)$ would provide an antiderivative for $1/x$.

73.[C] The antiderivative of $1/x$ that passes through $(1,0)$ is $\ln(x)$. One would expect that for t near 1, the antiderivative of $1/x^t$ that passes through $(1,0)$ would look much like $\ln(x)$ when x is near 1. To verify that this is true

(a) graph the slope field for $1/x^t$ with $t = 1.1$

(b) graph the antiderivative of $1/x^t$ that passes through $(1,0)$ for $t = 1.1$

(c) repeat (a) and (b) for $t = 0.9$

(d) repeat (a) and (b) for $t = 1.01$

(e) repeat (a) and (b) for $t = 0.99$

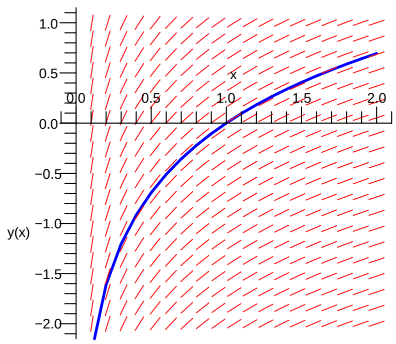


Figure 3.S.2:

The slope field for $1/x$ and the antiderivative of $1/x$ passing through $(1,0)$ are shown in Figure 3.S.2.

74.[C] (See Exercise 73.)

(a) Verify that for $t \neq 1$ the antiderivative of $1/x^t$ that passes through $(1,0)$ is $\frac{x^{1-t}-1}{1-t}$.

(b) Holding x fixed and letting t approach 1, show that

$$\lim_{t\to 1} \frac{x^{1-t}-1}{1-t} = \ln(x).$$

HINT: Recognize the limit as the derivative of a certain function at a certain input. Keep in mind that x is constant in this limit.

75.[C] Define f as follows:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$$

(a) What does the graph of f look like? NOTE: A dotted curve would indicate that points are missing.

(b) Does $\lim_{x\to 0} f(x)$ exist?

(c) Does $\lim_{x\to 1} f(x)$ exist?

(d) Does $\lim_{x\to\sqrt{2}} f(x)$ exist?

(e) For which numbers a does $\lim_{x\to a} f(x)$ exist?

76.[C] Define f as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ x^3 & \text{if } x \text{ is irrational.} \end{cases}$$

(a) What does the graph of f look like? NOTE: A dotted curve may be used to indicate that points are missing.

(b) Does $\lim_{x\to 0} f(x)$ exist?

(c) Does $\lim_{x\to 1} f(x)$ exist?

(d) Does $\lim_{x\to\sqrt{2}} f(x)$ exist?

(e) For which numbers a does $\lim_{x\to a} f(x)$ exist?

77.[C] A heavy block rests on a horizontal table covered with thick oil. The block, which is at the origin of the x-axis is given an initial velocity v_0 at time $t = 0$. It then coasts along the positive x-axis.
Assume that its acceleration is of the form $-k\sqrt{v(t)}$, where $v(t)$ is the velocity at time t and k is a constant. (That means it meets a resistance force proportional to the square root of its velocity.)

(a) Show that $\frac{dv}{dt} = -kv^{1/2}$.

(b) Is k positive or negative? Explain.

(c) Show that $2v^{1/2}$ and $-kt$ have the same derivative with respect to t.

(d) Show that $2v^{1/2} = -kt + 2v_0^{1/2}$.

(e) When does the block come to a rest? (Express that time in terms of v_0 and k.)

(f) How far does the block slide? (Express that distance in terms of v_0 and k.)

78.[C] A motorboat traveling along the x-axis at the speed v_0 stops its motor at time $t = 0$ when it is at the origin. It then coasts along the positive x-aixis.
Assuming the resistance force of the water is proportional to the velocity. That implies the acceleration of the boat is proportional to its velocity, $v(t)$. (See also Exercise 66.)

(a) Show that there is a constan
$-kv(t)$.

(b) Is k positive or negative? Ex

(c) Deduce that $\ln(v)$ and $-kt$ ha
tive with respect to t.

(d) Deduce that $\ln(v(t)) = -kt$ -

(e) Deduce that $v(t) = v_0 e^{-kt}$.

(f) According to (e), how long
tinue to move? (Express th
v_0 and k.)

(g) How far does it move during t
that distance in terms of v_0

79.[C] Archimedes used the follo
parabola in his study of the equ
bodies. Let P be any point on th
other than the origin. The normal
P meets the y-axis in a point Q.
and parallel to the x-axis meets the
Show that the length of QR is co
of the choice of P. NOTE: This
the **subnormal** of the graph; comp
cises 25 and 26 in Section 3.2.

Calculus is Everywhere # 3
Solar Cookers

A satellite dish is parabolic in shape. It is formed by rotating a parabola about its axis The reason is that all radio waves parallel to the axis of the parabola, after bouncing off the parabola, pass through a common point. This point is called the **focus** of the parabola. (See Figure C.3.1.) Similarly, the reflector behind a flashlight bulb is parabolic.

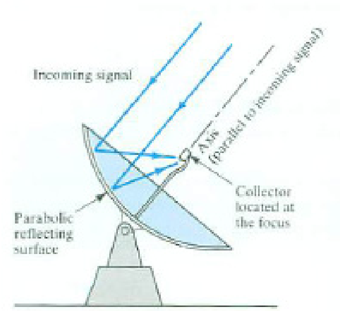


Figure C.3.1:

An ellipse also has a reflection property. Light, or sound, or heat radiating off one focus, after bouncing off the ellipse, goes through the other focus. This is applied, for instance, in the construction of computer chips where it is necessary to bake a photomask onto the surface of a silicon wafer. The heat is focused at the mask by placing a heat source at one focus of an ellipse and positioning the wafer at the other focus, as in Figure C.3.2.

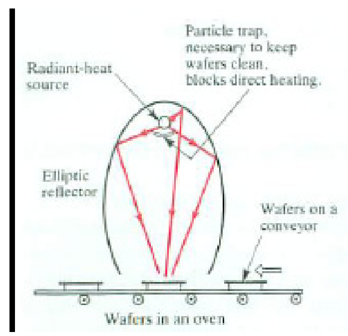


Figure C.3.2:

The reflection property is used in wind tunnel tests of aircraft noise. The test is run in an elliptical chamber, with the aircraft model at one focus and a microphone at the other.

Whispering rooms, such as the rotunda in the Capitol in Washington, D.C., are based on the same principle. A person talking quietly at one focus can be heard easily at the other focus and not at other points between the foci. (The whisper would be unintelligible except for the additional property that all the paths of the sound from one focus to the other have the same length.)

An ellipsoidal reflector cup is used for crushing kidney stones. (An ellipsoid is formed by rotating an ellipse about the line through its foci.) An electrode is placed at one focus and an ellipsoid positioned so that the stone is at the other focus. Shock waves generated at the electrode bounce off the ellipsoid, concentrate on the other focus, and pulverize the stones without damaging other parts of the body. The patient recovers in three to four days instead of the two to three weeks required after surgery. This advance also reduced the mortality rate from kidney stones from 1 in 50 to 1 in 10,000.

The reflecting property of the ellipse also is used in the study of air pollution. One way to detect air pollution is by light scattering. A laser is aimed through one focus of a shiny ellipsoid. When a particle passes through this focus, the light is reflected to the other focus where a light detector is located. The number of particles detected is used to determine the amount of pollution in the air.

The Angle Between Two Lines

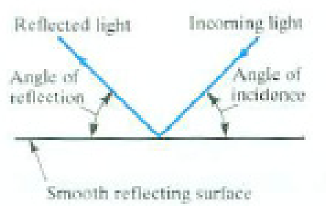
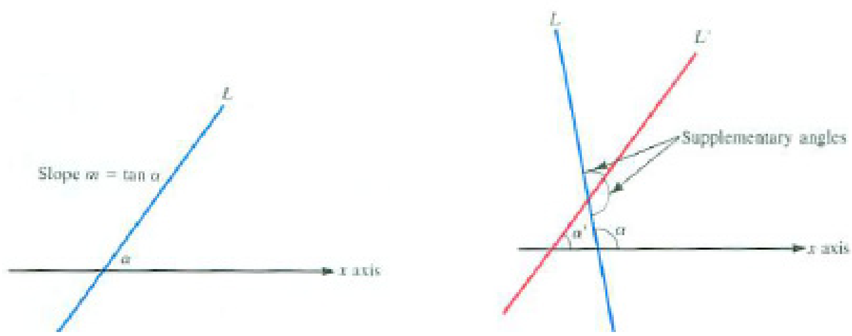


Figure C.3.3:

To establish the reflection properies just mentioned we will use the principle that the angle of reflection equals the angle of incidence, as in Figure C.3.3, and work with the angle between two lines, given their slopes.

Consider a line L in the xy-plane. It forms an **angle of inclination** α, $0 \le \alpha < \pi$, with the positive x-axis. The slope of L is $\tan(\alpha)$. (See Figure C.3.4(a).) If $\alpha = \pi/2$, the slope is not defined.



(a) (b)

Figure C.3.4:

Consider two lines L and L' with angles of inclination α and α' and slopes m and m', respectively, as in Figure C.3.4(b). There are two (supplementary) angles between the two lines. The following definition serves distinguishes one of these two angles as *the* angle between L and L'.

> **DEFINITION** *(Angle between two lines.)* Let L and L' be two lines in the xy-plane, named so that L has the larger angle of inclination, $\alpha > \alpha'$. The angle θ between L and L' is defined to be
>
> $$\theta = \alpha - \alpha'.$$
>
> If L and L' are parallel, define θ to be 0.

Note that θ is the counterclockwise angle from L' to L and that $0 \le \theta < \pi$. The tangent of θ is easily expressed in terms of the slopes m of L and m' of L'. We have

$$\begin{aligned}
\tan(\theta) &= \tan(\alpha - \alpha') && \text{definition of } \theta \\
&= \frac{\tan(\alpha)-\tan(\alpha')}{1+\tan(\alpha)\tan(\alpha')} && \text{by the identity for } \tan(A-B) \\
&= \frac{m-m'}{1+mm'}.
\end{aligned}$$

Thus

$$\tan(\theta) = \frac{m - m'}{1 + mm'}. \qquad \text{(C.3.1)}$$

The Reflection Property of a Parabola

Consider the parabola $y = x^2$. (The geometric description of this parabola is the set of all points whose distance from the point $(0, \frac{1}{4})$ equals its distance from the line $y = \frac{-1}{4}$, but this information is not needed here.)

In Figure C.3.5 we wish to show that angles A and B at the typical point (a, a^2) on the parabola are equal. We will do this by showing that $\tan(A) = \tan(B)$.

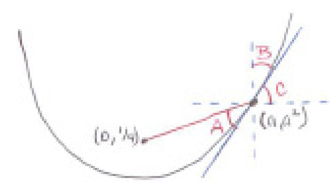


Figure C.3.5:

First of all, $\tan(C) = 2a$, the slope of the parabola at (a, a^2). Since A is the complement of C, $\tan(A) = 1/(2a)$.

The slope of the line through the focus $(0, \frac{1}{4})$ and a point on the parabola (a, a^2) is

$$\frac{a^2 - \frac{1}{4}}{a - 0} = \frac{4a^2 - 1}{4a}.$$

Therefore,

$$\tan(B) = \frac{2a - \frac{4a^2-1}{4a}}{1 + 2a\left(\frac{4a^2-1}{4a}\right)}.$$

Exercise 1 asks you to supply the algebraic steps to complete the proof that $\tan(B) = \tan(A)$.

The Reflection Property of an Ellipse

An ellipse consists of every point such that the sum of the distances from the point to two fixed points is constant. Let the two fixed points, called the **foci** of the ellipse, be a distance $2c$ apart, and the fixed sum of the distances be $2a$, where $a > c$. If the foci are at $(c, 0)$ and $(-c, 0)$ and $b^2 = a^2 - c^2$, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b^2 = a^2 - c^2$. (See Figure C.3.6.)

As in the case of the parabola, one shows $\tan(A) = \tan(B)$.

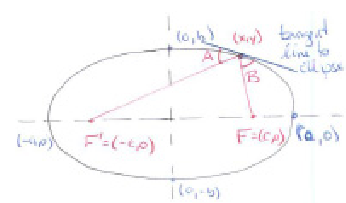


Figure C.3.6: Diocles, *On Burning Mirrors*, edited by G. J. Toomer, Springer, New York, 1976.

One reason to do Exercise 2 is to appreciate more fully the power of vector calculus, developed later in Chapter 14, for with that tool you can establish the reflection property of either the parabola or the ellipse in one line.

Diocles, in his book *On Burning Mirrors*, written around 190 B.C., studied spherical and parabolic reflectors, both of which had been considered by earlier scientists. Some had thought that a spherical reflector focuses incoming light at a single point. This is false, and Diocles showed that a spherical reflector subtending an angle of 60° reflects light that is parallel to its axis of symmetry to points on this axis that occupy about one-thirteenth of the radius. He proposed an experiment, "Perhaps you would like to make two examples of a burning-mirror, one spherical, one parabolic, so that you can measure the burning power of each." Though the reflection property of a parabola was already known, *On Burning Mirrors* contains the first known proof of this property.

Exercise 3 shows that a spherical oven is fairly effective. After all, a potato or hamburger is not a point.

EXERCISES

1.[R] Do the algebra to complete the proof that $\tan(A) = \tan(B)$.

2.[R] This exercise establishes the reflection property of an ellipse. Refer to Figure C.3.6 for a description of the notation.

(a) Find the slope of the tangent line at (x, y).

(b) Find the slope of the line through $F = (c, 0)$ and (x, y).

(c) Find $\tan(B)$.

(d) Find the slope of the line through $F' = (c', 0)$ and (x, y).

(e) Find $\tan(A)$.

(f) Check that $\tan(A) = \tan(B$

3.[M] Use trigonometry to show t
ror of radius r and subtending an
light parallel to its axis of symmetry
the axis in an interval of length $\Big($