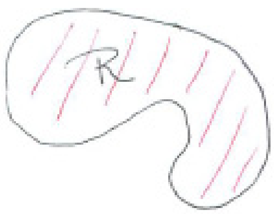
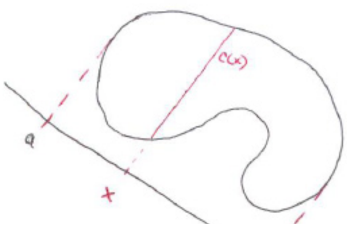
Chapter 7

Applications of the Definite Integral

7.1 Computing Area by Parallel Cross-Sections

In Section 6.1 we computed the area under $y = x^2$ and above the interval $[a, b]$, and later saw that it equals the definite integral $\int_a^b x^2\,dx$. Now we generalize the idea behind this example.





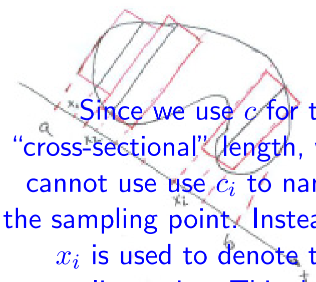


Figure 7.1.3:

Since we use c for the "cross-sectional" length, we cannot use c_i to name the sampling point. Instead, x_i is used to denote the sampling point. This does not cause any confusion since we are not using x_i to describe the endpoints of a partition.

Area as a Definite Integral of Cross Sections

How can we express the area of the region R shown in Figure 7.1.1 as a definite integral?

First, we introduce an "x-axis", as in Figure 7.1.2.

Assume that lines perpendicular to the axis for x in $[a, b]$, intersect the region R in an interval of length $c(x)$. The interval is called the **cross section** of R at x.

We approximate R by a collection of rectangles, just as we estimated the area of the region under $y = x^2$.

Pick an integer n, and divide the interval $[a, b]$ on the x-axis into n congruent sections. The total length of the interval $[a, b]$ is $b - a$; each section has width $\Delta x = \frac{b-a}{n}$. Then, in the i^{th} section, $i = 1, 2, \ldots, n$, we pick a "sampling number" x_i. For each of the n sections we form a rectangle of width Δx and height $c(x_i)$. These are indicated in Figure 7.1.3.

Since the i^{th} rectangle has area $c(x_i)\Delta x$, the total area of the n rectangles is $\sum_{i=1}^{n} c(x_i)\Delta x$. As n increases, the collection of rectangles provides a better approximation to the area of R. This suggests that:

$$\lim_{n\to\infty} \sum_{i=1}^{n} c(x_i)\Delta x = \text{area of region } R$$

But, by the definition of a definite integral,

$$\lim_{n\to\infty} \sum_{i=1}^{n} c(x_i)\Delta x = \int_a^b c(x)\,dx.$$

Thus,

$$\boxed{\text{area of } R = \int_a^b c(x)\,dx.}$$

Or, informally,

Area of a region equals the integral of its cross-sectional lengths.

Note that x need not refer to the x-axis of the xy-plane; it may refer to any conveniently chosen line in the plane. It may even refer to the y-axis; in this case the cross-sectional length would be denoted by $c(y)$.

To compute an area:

1. Find the endpoints a and b, and the cross-sectional length $c(x)$.

2. Evaluate $\int_a^b c(x)\ dx$ by the Fundamental Theorem of Calculus, if the antiderivative of $c(x)$ is elementary.

Chapter 6 showed how to accomplish Step 2. FTC I is used when the antiderivative is an elementary function, and other cases can be approximated numerically. The present section is concerned primarily with Step 1, how to find the cross-sectional length $c(x)$ and set up the definite integral.

If the region R happens to be the region under the graph of $f(x)$ and above the interval $[a, b]$, then the cross-sectional length is simply $f(x)$. We have already met this special case in Sections 6.2–6.4 with $f(x) = x^2$ and $f(x) = 2^x$.

EXAMPLE 1 Find the area of a disk of radius r.
SOLUTION Introduce an xy-coordinate system with its origin at the center of the disk, as in Figure 7.1.4.

The typical cross-section perpendicular to the x-axis is shown in Figure 7.1.5. The length of the cross-section, $\overline{AC}$, is twice $\overline{BC}$. By the Pythagorean Theorem,

$$x^2 + \overline{BC}^2 = r^2.$$

Then

$$\overline{BC}^2 = r^2 - x^2$$

and, because $|\overline{BC}|$, a length, is positive

$$\overline{BC} = \sqrt{r^2 - x^2}.$$

Because x is in $[-r, r]$,

$$\text{area of disk of radius } r = \int_{-r}^{r} 2\sqrt{r^2 - x^2}\ dx. \tag{7.1.1}$$

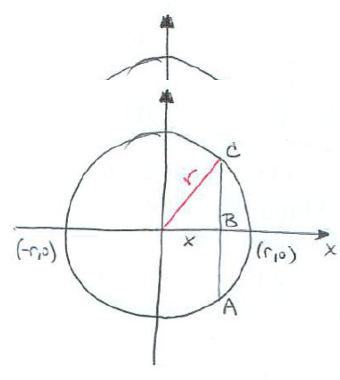


Figure 7.1.5:

Equation (7.1.2) is preferable because it reduces the chance of making an error when working with the subtraction of negative numbers.

By symmetry, we can also say that the total area is four times the area of a quadrant:

$$\text{area of disk of radius } r = 4\int_0^r \sqrt{r^2 - x^2}\, dx. \tag{7.1.2}$$

This completes the set up of the integral for the area of the region.

The next chapter presents a technique for finding an antiderivative of $\sqrt{r^2 - x^2}$. In the mean time, we use the table of integrals on the inside cover. According to formula 32,

$$\int \sqrt{r^2 - x^2}\, dx = \frac{r^2}{2}\left(\arcsin\left(\frac{x}{r}\right) + \frac{x}{r^2}\sqrt{r^2 - x^2}\right).$$

See Exercise 44

By FTC I,

$$\begin{aligned}
\int_0^r \sqrt{r^2 - x^2}\, dx &= \frac{r^2}{2}\left(\arcsin\left(\frac{x}{r}\right) + \frac{x}{r^2}\sqrt{r^2 - x^2}\right)\Big|_0^r \\
&= \frac{r^2}{2}\left(\arcsin\left(\frac{r}{r}\right) + \frac{r}{r^2}\sqrt{r^2 - r^2}\right) - \frac{r^2}{2}\left(\arcsin\left(\frac{0}{r}\right) + \frac{0}{r^2}\sqrt{r^2 - 0^2}\right) \\
&= \frac{r^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi r^2}{4}.
\end{aligned}$$

Reference: S. Stein: *Archimedes: What did he do besides cry Eureka?,*

Thus one quarter of the disk has area $\frac{\pi r^2}{4}$ and the whole disk has area πr^2. $\diamond$

Archimedes found the area in the next example, expressing it in terms of the area of a certain triangle (see Exercise 42). He used geometric properties of a parabola, since calculus was not invented until some 1900 years later.

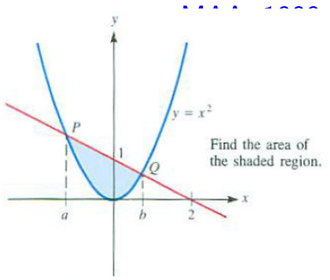


Figure 7.1.6:

EXAMPLE 2 Set up a definite integral for the area of a region above the parabola $y = x^2$ and below the line through $(2, 0)$ and $(0, 1)$ shown in Figure 7.1.6.

SOLUTION Since the x-intercept of the line is 2 and the y-intercept is 1, an equation for the line is

$$\frac{x}{2} + \frac{y}{1} = 1.$$

Hence $y = 1 - x/2$. The length $c(x)$ of a cross-section of the region taken parallel to the y-axis is, therefore

$$c(x) = \left(1 - \frac{x}{2}\right) - x^2 = 1 - \frac{x}{2} - x^2.$$

To find the interval $[a, b]$ of integration, we must find the x-coordinates of the points P and Q in Figure 7.1.5 where the line meets the parabola. For these values of x,

$$x^2 = 1 - \frac{x}{2},$$

so

$$2x^2 + x - 2 = 0. \tag{7.1.3}$$

The solutions to (7.1.3) are

$$x = \frac{-1 \pm \sqrt{17}}{4}.$$

Hence

$$\text{area} = \int_{(-1-\sqrt{17})/4}^{(-1+\sqrt{17})/4} \left(1 - \frac{x}{2} - x^2\right)\, dx.$$

The value of this definite integral is found in Exercise 33. ⋄

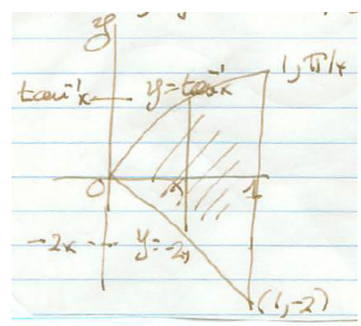


Figure 7.1.7:

EXAMPLE 3 Find the area of the region in Figure 7.1.7, bounded by $y = \arctan(x)$, $y = -2x$, and $x = 1$.

SOLUTION We will find the area two ways, first (a) with cross-sections parallel to the y-axis, then (b) with cross-sections parallel to the x-axis.

(a) The typical cross-section has length $\arctan(x) - (-2x) = \arctan(x) + 2x$. Thus the area is

$$\int_0^1 (\arctan(x) + 2x)\ dx.$$

It's easy to find $\int 2x\ dx$; it's just x^2. By the FTC,

Formula 71 in the cover of this book tells us that $\int \arctan(x)\ dx$ is $x\arctan(x) - \frac{1}{2}\ln(1+x^2)$. Use differentiation to check that this is correct.

$$\begin{aligned}
\int_0^1 (\arctan(x) + 2x)\ dx &= \left(x\arctan(x) - \frac{1}{2}\ln(1+x^2) + x^2\right)\Bigg|_0^1 \\
&= \left(1\arctan(1) - \frac{1}{2}\ln(1+1^2) + 1^2\right) \\
&\quad - \left(0\arctan(0) - \frac{1}{2}\ln(1+0^2) + 0^2\right) \\
&= \left(\frac{\pi}{4} - \frac{1}{2}\ln(2) + 1\right) - 0 \\
&= \frac{\pi}{4} + 1 - \frac{1}{2}\ln(2). \approx 1.4388 \qquad (7.1.4)
\end{aligned}$$

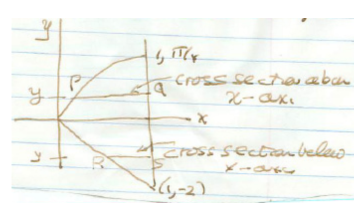


Figure 7.1.8:

(b) Now we use cross-sections parallel to the x-axis, as indicated in Figure 7.1.8.

Cross-sections above the x-axis involve the curved part of the boundary, while those below the x-axis involve the slanted line.

We must find the cross-sectional length as a function of y. That means we should first find the x-coordinates of P and Q, the ends of the typical cross-section above the x-axis. The x-coordinate of Q is 1. Let the x-coordinate of P be x, then $y = \arctan(x)$, so $x = \tan(y)$. Hence $c(y) = 1 - \tan(y)$ for $y \geq 0$. The length of RS, a typical cross-section below the x-axis, is $1 -$ (x-coordinate of R). Since R is on the line $y = -2x$, we have $x = -y/2$. Thus

$$c(y) = 1 - (-y/2) = 1 + y/2, \qquad \text{for } -2 \leq y \leq 0.$$

Note that the interval of integration is $[-2, \pi/4]$. Hence

$$\text{area of } R = \int_{-2}^{\pi/4} c(y)\, dy.$$

We have to break this integral into two separate ones:

$$\int_{-2}^{0} \left(1 + \frac{y}{2}\right)\, dy \text{ and } \int_{0}^{\pi/4} (1 - \tan(y))\, dy \tag{7.1.5}$$

Differentiate $\ln(\sec(y))$ to check this antiderivative. Because $\sec(y)$ is positive for $-\pi/2 < y < \pi/2$ it is not necessary to write the antiderivative as $\ln|\sec(y)|$; see Exercise 31.

It will be shown Example 3 in Section 8.5 that

$$\int \tan(y)\, dy = \ln(\sec(y)).$$

First,

$$\begin{aligned} \int_{-2}^{0} \left(1 + \frac{y}{2}\right)\, dy &= \left. \left(y + \frac{y^2}{4}\right) \right|_{-2}^{0} \\ &= \left(0 + \frac{0^2}{4}\right) - \left((-2) + \frac{(-2)^2}{4}\right) \\ &= 1 \end{aligned} \tag{7.1.6}$$

Second,

$$\begin{aligned}
\int_0^{\pi/4} (1-\tan(y))\ dy &= \left.(y-\ln\sec(y))\right|_0^{\pi/4} \\
&= \left(\frac{\pi}{4}-\ln(\sec(\frac{\pi}{4}))\right)-(0-\ln(\sec(0))) \\
&= \frac{\pi}{4}-\ln(\sqrt{2}) \qquad (7.1.7)
\end{aligned}$$

Adding (7.1.6) and (7.1.7) gives

$$\text{area of } R = \frac{\pi}{4}-\ln(\sqrt{2})+1 \qquad (7.1.8)$$

See Exercise 32.

The two answers (7.1.4) and (7.1.8) may look different but they agree, as you may show in Exercise 32. ⋄

In this example we could have simplified the solution by observing that the area below the x-axis is a triangle of area 1. But the purpose of Example 3 is to illustrate a general approach.

Summary

The key idea in this section, "area of a region equals integral of cross-sectional length," was already anticipated in Chapter 6. There we met the special case where the region is bounded by the graph of a function, the x-axis, and two lines perpendicular to the axis. In this section the concept was extended to more general regions.

EXERCISES for Section 7.1 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 6 (a) draw the region, (b) compute the lengths of vertical cross-sections ($c(x)$), and (c) compute the lengths of horizontal cross-sections ($c(y)$).

1.[R] The finite region bounded by $y = \sqrt{x}$ and $y = x^2$.

2.[R] The finite region bounded by $y = x^2$ and $y = x^3$.

3.[R] The finite region bounded by $y = 2x$, $y = 3x$, and $x = 1$.

4.[R] The finite region bounded by $y = x^2$, $y = 2x$, and $x = 1$.

5.[R] The triangle with vertices $(0,0)$, $(3,0)$, and $(0,4)$.

6.[R] The triangle with vertices $(1,0)$, $(3,0)$, and $(2,1)$.

In Exercises 7 to 12 find the indicated areas. Use the table of integrals provided inside the cover of this textbook to find antiderivatives, if necessary.

7.[R] Under $y = \sqrt{x}$ and above $[1,2]$

8.[R] Under $y = \sin(2x)$ and above $[\pi/6, \pi/3]$

9.[R] Under $y = e^{2x}$ and above $[0,1]$

10.[R] Under $y = 1/\sqrt{1-x^2}$ and above $[0, 1/2]$.

11.[R] Under $y = \ln(x)$ and above $[1, e]$

12.[R] Under $y = \cos(x)$ and above $[-\pi/2, \pi/2]$

In Exercises 13 to 20 find the indicated areas using cross-sections parallel to the x-axis.

13.[R] Between $y = x^2$ and $y = x^3$.

14.[R] Between $y = 2^x$ and $y = 2x$.

15.[R] Between $y = \arcsin(x)$ and $y = 2x/\pi$ (to the right of the y-axis).

16.[R] Between $y = 2^x$ and $y = 3^x$ (to the right of the y-axis).

17.[R] Between $y = \sin(x)$ and $y = \cos(x)$ (above $0, \pi/2]$.

18.[R] Between $y = x^3$ and $y = -x$ for x in $[1,2]$.

19.[R] Between $y = x^3$ and $y = \sqrt[3]{2x-1}$ for x in $[1,2]$.

20.[R] Between $y = 1 + x$ and $y = \ln(x)$ for x in $[1, e]$.

In Exercises 21 to 27 *set up a definite integral* for the area of the given region. These integrals will be evaluated in Exercises 36 to 42 in the Chapter 8 Summary.

21.[R] The region under the curve $y = \arctan(2x)$ and above the interval $[1/2, 1/\sqrt{3}]$.

22.[R] The region in the first quadrant below $y = -7x + 29$ and above the portion of $y = 8/(x^2 - 8)$ that lies in the first quadrant.

23.[R] The region below $y = 10^x$ and above $y = \log_{10}(x)$ for x in $[1, 10]$.

24.[R] The region under the curve $y = x/(x^2+5x+6)$ and above the interval $[1, 2]$.

25.[R] The region below $y = (2x + 1)/(x^2 + x)$ and above the interval $[2, 3]$.

26.[R] The region bounded by $y = \tan(x)$, $y = 0$, $x = 0$, and $x = \pi/2$ by (a) vertical cross-sections and (b) horizontal cross-sections.

27.[R] The region bounded by $y = \sin(x)$, $y = 0$, and $x = \pi/4$ (consider only $x \geq 0$) by (a) vertical cross-sections and (b) horizontal cross-sections.

28.[R]

(a) Draw the region inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(b) Find a definite integral for the area of the ellipse in (a) with horizontal cross-sections.

(c) Find a definite integral for the area of the ellipse in (a) with vertical cross-sections.

NOTE: See Exercise 43 in Chapter 8 Summary.

29.[R] Cross-sections in different directions lead to different definite integrals for the same area. While both integrals must give the same area, one of the two integrals can be easier to evaluate.

(a) Identify and evaluate the easier definite integral found in Exercise 26.

(b) Identify and evaluate the easier definite integral found in Exercise 27.

30.[R] Set up the definite integral for the area $A(b)$ of the region in the first quadrant under the curve $y = e^{-x}(\cos(x))^2$ and above the interval $[0, b]$.

31.[R] In Example 3 you are told that $\int \tan(y)\, dy = \ln(\sec(y))$. Verify this result, by differentiating.

32.[R] In Example 3 the area of the region bounded by $y = \arctan(x)$, $y = 2x$, and $x = 1$ is found to be both

$$\frac{\pi}{4} + 1 - \frac{1}{2}\ln(2) \qquad \text{and} \qquad \frac{\pi}{4} - \ln(\sqrt{2}) + 1.$$

Explain why these two answers are equal.

33.[M] In Example 2 the area of the region above the parabola $y = x^2$ and below the line through $(2, 0)$ and $(0, 1)$ is found to be

$$\text{area} = \int_{(-1-\sqrt{17})/4}^{(-1+\sqrt{17})/4} \left(1 - \frac{x}{2} - x^2\right)\, dx.$$

Find the value of this definite integral.

34.[M] Let R be the region bounded by $y = x^3$, $y = x + 2$, and the x-axis.

(a) Find a definite integral for the area of R. HINT: Define one or both of the endpoints as solutions to an equation.

(b) Use a graph or other method to approximate the endpoints.

(c) Use the estimates in (b) to obtain an estimate of the area of R.

35.[M] Let R be the region between $y = 3$ and $y = e^x/x$.

(a) Graph the region R.

(b) Find a definite integral for the area of R. HINT: You will encounter an equation that cannot be solved exactly. Identify the endpoints on the graph found in (a).

(c) Find approximate values for the endpoints of the definite integral for the area in (b).

(d) Because the antiderivative of e^x/x is not elementary, it is still not easy to estimate the area of R. What methods do we have for estimating this definite integral? Use one of these definite integrals to find an approximate value for the area of R.

36.[M] What fraction of the rectangle whose vertices are $(0, 0)$, $(a, 0)$, (a, a^4), and $(0, a^4)$, with a positive, is occupied by the region under the curve $y = x^4$ and above $[0, a]$?

37.[C]

(a) Draw the curve $y = e^x/x$ for $x > 0$, showing any asymptotes or critical points.

(b) Find the number t such that the area below $y = e^x/x$ and above the interval $[t, t+1]$ is a minimum.

HINT: Write $A(t) = \int_t^{t+1} f(x)\,dx = \int_0^{t+1} f(x)\,dx - \int_0^t f(x)\,dx$, then use FTC II.

38.[C] Let $A(t)$ be the area of the region in the first quadrant between $y = x^2$ and $y = 2x^2$ and inside the rectangle bounded by $x = t$, $y = t^2$, and the coordinate axes. (See the shaded region in Figure 7.1.9.) If $R(t)$ is the area of the rectangle, find

(a) $\lim_{t\to 0} \frac{A(t)}{R(t)}$

(b) $\lim_{t\to\infty} \frac{A(t)}{R(t)}$

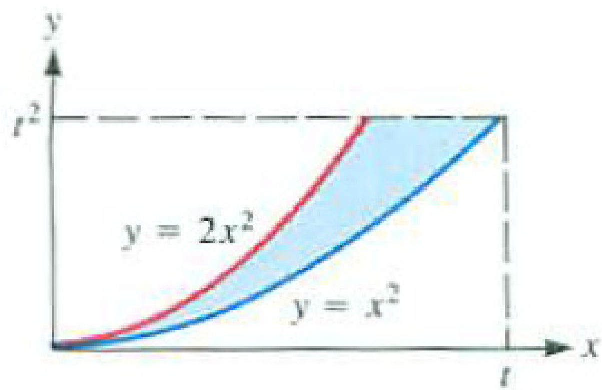


Figure 7.1.9:

39.[C] Figure 7.1.10 shows the graph of an increasing function $y = f(x)$ with $f(0) = 0$. Assume that $f'(x)$ is continuous and $f'(0) > 0$. Do not assume that $f''(x)$ exists. Our objective is to investigate

$$\frac{\text{area of shaded region under the curve}}{\text{area of triangle } ABC} \quad \text{as } t \text{ decreases toward 0.} \tag{7.1.9}$$

(a) Experiment with various functions, including some trigonometric functions and polynomials. NOTE: Make sure that $f'(0) > 0$.

(b) Make a conjecture about (7.1.9) and explain why it is true.

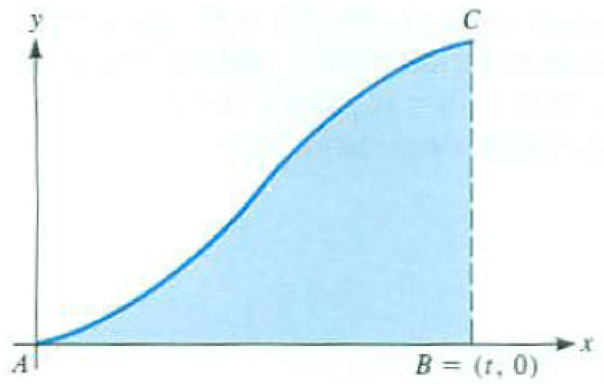
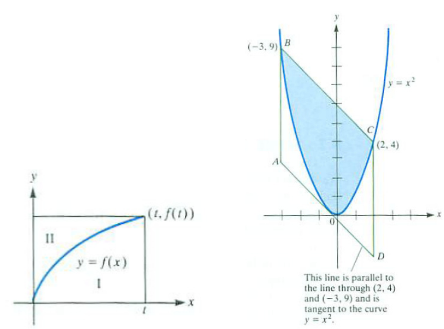


Figure 7.1.10:

40.[C] Repeat Exercise 39, but now assume that $f'(0) = 0$, f'' is continuous, and $f''(0) \neq 0$.

41.[C] Let f be an increasing function with $f(0) = 0$, and assume that it has an elementary antiderivative. Then f^{-1} is an increasing function, and $f^{-1}(0) = 0$. Prove that if f^{-1} is elementary, then it also has an elementary antiderivative. Hint: See Figure 7.1.11(a).



(a) (b)

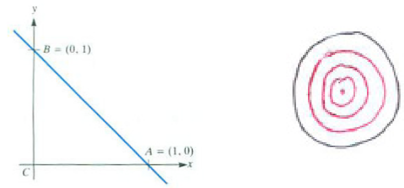
Figure 7.1.11:

42.[C] Show that the area of the shaded region in Figure 7.1.11(b) is two-thirds the area of the parallelogram $ABCD$. This is an illustration of a theorem of Archimedes concerning sectors of parabolas. He showed that the shaded area is 4/3 the area of triangle BOC. Note: See also Example 2.

43.[C] Figure 7.1.12(a) shows a right triangle ABC.

(a) Find equations for the lines parallel to each edge, AC, BC, and AB, that cut the triangle into two pieces of equal area.

(b) Are the three lines in (a) concurrent; that is, do they meet at a single point?



(a) (b)

Figure 7.1.12:

44.[C] Find the area of a disk of radius r by using concentric rings as suggested in Figure 7.1.12(b). The advantage of this approach is that it leads to an integral with a much simpler antiderivative than in Example 1. HINT: Approximate the area of each ring as the product of a circumference and the width of the ring.

7.2 Some Pointers on Drawing

None of us were born knowing how to draw solids. As we grew up, we lived in flatland: the surface of the Earth. Few high school math classes cover solid geometry, so calculus is often the first place where you have to think and sketch in terms of three dimensions. That is why we pause for a few words of advice on how to draw. Too often you cannot work a problem simply because your diagrams confuse even yourself. The following guidelines are not based on any profound artistic principles. Instead, they derive from years attempting to sketch diagrams that do more good than harm.

A Few Words of Advice

1. *Draw large.* Many students tend to draw diagrams that are so small that there is no room to place labels or to sketch cross-sections.

2. *Draw neatly.* Use a straightedge to make straight lines that are actually straight. Use a compass to make circles that look like circles. Draw each line or curve slowly.

A jar lid or soda can works just fine for drawing circles and circular arcs. Credit cards and ID badges make good straightedges.

3. *Avoid clutter.* If you end up with too many labels or the cross-section doesn't show up well, add separate diagrams for important parts of the figure.

4. *Practice.*

This example is continued in Example 1 in Section 7.4.

EXAMPLE 1 Draw a diagram of a ball of radius a that shows the circular cross-section made by a plane at a distance x from the center of the ball. Use the diagram to help find the radius of the cross-section as a function of x.

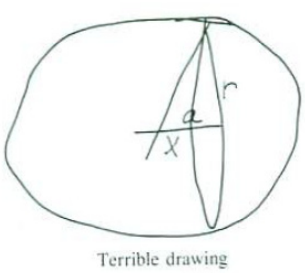


Figure 7.2.1:

TERRIBLE SOLUTION Is Figure 7.2.1 a potato or a ball? What segment has length r? What's x? What does the cross-section look like?

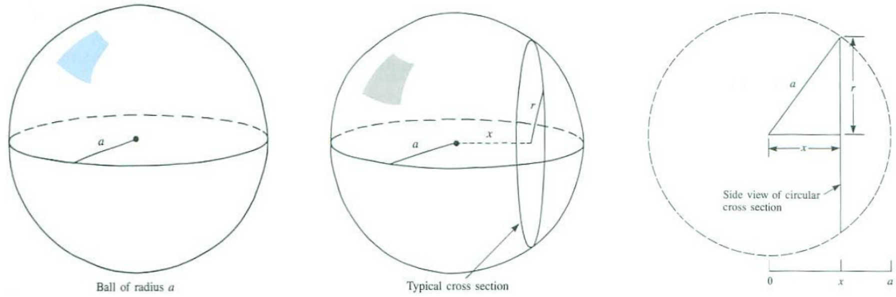
REASONABLE SOLUTION First, draw the ball carefully, as in Figure 7.2.2(a). The equator is drawn to give it perspective. Add a little shading.

Next show a typical cross-section at a distance x from the center, as in Figure 7.2.2(b). Shading the cross-section helps, too.

To find r, the radius of the cross-section, in terms of x, sketch a companion diagram. The radius we want is part of a right triangle. In order to avoid clutter, draw only the part of interest in a convenient side view, as in Figure 7.2.4(c).

Inspection of the right triangle in this figure shows that

$$r^2 + x^2 = a^2, \quad \text{hence that} \quad r = \sqrt{a^2 - x^2}.$$



(a) (b) (c)

Figure 7.2.2: NOTE: Add shading to cross-section in (b).

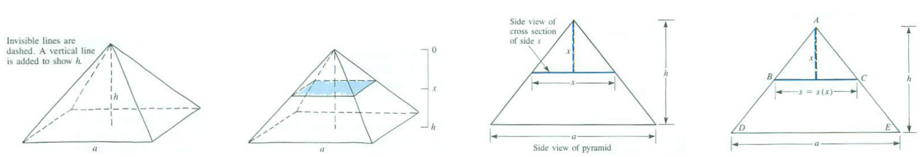
◇

This example is continued in Example 2 in Section 7.4.

EXAMPLE 2 A pyramid has a square base with a side of length a. The top of the pyramid is above the center of the base at a height h. Draw the pyramid and its cross-sections by planes parallel to the base. Then find the area of the cross-sections in terms of their distance x from the top.

Figure 7.2.3: Terrible drawing

TERRIBLE SOLUTION Figure 7.2.3 is too small; there's no room for the symbols. While it's pretty clear what side has length a, to what are the x and h attached? Also, without the hidden edges of the pyramid the shape of the base is not clear.



(a) (b) (c) (d)

Figure 7.2.4:

REASONABLE SOLUTION First draw a large pyramid with a square base, as in Figure 7.2.4(a). Note that the opposite edges of the base are drawn as parallel lines. While artists draw parallel lines as meeting in a point to enhance the sense of perspective, for our purposes it is more useful to use parallel lines to depict parallel lines. Then show a typical cross-section in perspective and side views, as in Figures 7.2.4(b) and (c). Note the x-axis, which is drawn separate from the pyramid.

The use of s is recommended because it suggests its meaning - side.

As x increases, so does s, the width of the square cross-section. Thus s is a function of x, which we could call $s(x)$ (or $f(x)$, if you prefer). A glance at Figure 7.2.4(b) shows that $s(0) = 0$ and $s(h) = 1$. To find $s(x)$ for all x in $[0, h]$, use the similar triangles ABC and ADE, shown in Figure 7.2.4(c). These triangles show that

$$\frac{x}{s} = \frac{h}{a}; \qquad \text{hence} \qquad s = \frac{ax}{h}. \tag{7.2.1}$$

Notice that $s = \frac{ax}{h}$ expresses s is a linear function of x. As a check on (7.2.1), replace x by 0 and by h; we get 0 and a for the respective values s, as expected. Finally, the area A of the cross-sections is given by

$$A = s^2 = \left(\frac{ax}{h}\right)^2.$$

$\diamond$

This example is continued in Exercise 18 in Section Section 7.4.

EXAMPLE 3 A cylindrical drinking glass of height h and radius a is full of water. It is tilted until the remaining water covers exactly half the base.

A. Draw a diagram of the glass and water.

B. Show a cross-section of the water that is a triangle.

C. Find the area of the triangle in terms of the distance x of the cross-section from the axis of the glass.

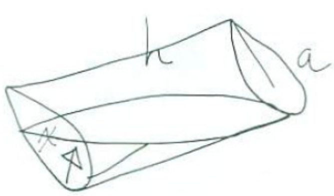
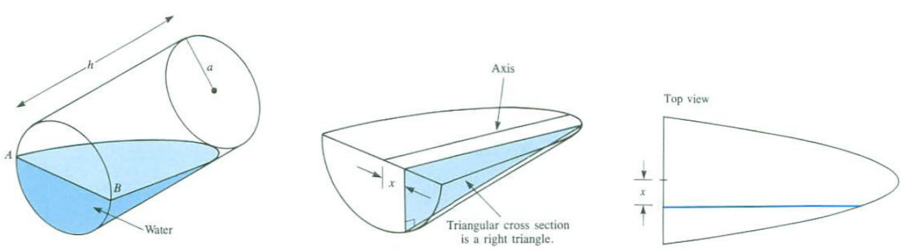


Figure 7.2.5:

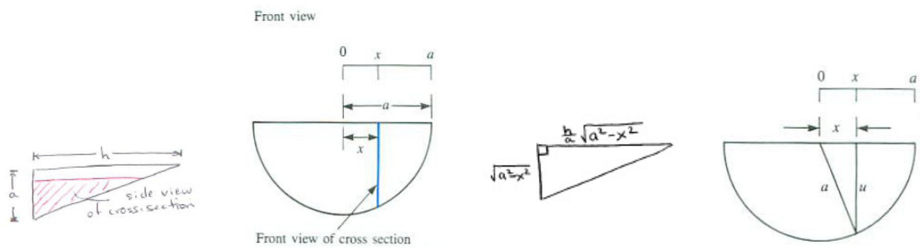
TERRIBLE SOLUTION The diagram in Figure 7.2.5 is too small. It is not clear what has length a. The cross-section is unclear. What does x refer to?



(a) (b) (c)

Figure 7.2.6:

REASONABLE SOLUTION First, draw a *neat, large* diagram of a slanted cylinder, as in Figure 7.2.6. Don't put in too much detail at first. When



(a) (b) (c) (d)

Figure 7.2.7:

showing the cross-section, draw only the water. Figures 7.2.6 and 7.2.7 show various views. Let u and v be the lengths of the two legs of the cross-section, as shown in Figure 7.2.7(d).

Comparing Figures 7.2.7(a) and (b), we have, by similar triangles, the relation

$$\frac{u}{a} = \frac{v}{h} \qquad \text{hence} \qquad v = \frac{h}{a}u.$$

Let $A(x)$ be the area of the cross-section at a distance x from the center of the base, as shown in Figure 7.2.6(b). If we can find u and v as functions of x, we will be able to write a formula for $A(x) = \frac{1}{2}uv$ in terms of x.

Figure 7.2.7(b) suggests how to find u. Copy it and draw in the necessary radius, as in Figure 7.2.7(d). By the Pythagorean Theorem,

$$u = \sqrt{a^2 - x^2}.$$

All told,

$$A(x) = \frac{1}{2}uv = \frac{1}{2}u\left(\frac{h}{a}u\right) = \frac{h}{2a}u^2 = \frac{h}{2a}(a^2 - x^2). \qquad (7.2.2)$$

As a check, note that

$$A(a) = \frac{h}{2a}(a^2 - a^2) = 0,$$

which makes sense. Also the formula (7.2.2) gives

$$A(0) = \frac{h}{2a}(a^2 - 0^2) = \frac{1}{2}ah,$$

again agreeing with the geometry of, say, Figure 7.2.6(b). ◇

Summary

When you look back at these three examples, you will see that most of the work is spent on making clear diagrams. If you can't draw a straight line free hand, use a straightedge. If you can't draw a circle, use a compass.

EXERCISES for Section 7.2 *Key:* R–routine, M–moderate, C–challenging

1.[R] Cross-sections of the pyramid in Example 2 are made by using planes perpendicular to the base and parallel to the edge of the base. What is the area of the cross-section made by a plane that is a distance x from the top of the pyramid?

(a) Draw a large perspective view of the pyramid.

(b) Copy the diagram in (a) and show the typical cross-section shaded.

(c) Draw a side view that shows the shape of the cross-section.

2.[R] Cross-sections of the water in Example 3 are made by using planes parallel to the plane that passes through the horizontal diameter of the base and the axis of the glass. What is the area of the cross-section made by a plane that is a distance x from the center of the base?

(a) Draw a large perspective view of the water and glass.

(b) Copy the diagram in (a) and show the typical cross-section shaded.

(c) Draw a side view that clearly shows the shape of the cross-section.

(d) Draw a different side view.

(e) Put necessary labels, such as x, a, and h, on the diagrams, where appropriate. (You will need to introduce more labels.)

(f) Find the area of the cross-section, $A(x)$, as a function of x.

3.[R] Cross-sections of the water in Example 3 are made by using planes perpendicular to the axis of the glass. Make clear diagrams, including perspective and side views, that show the typical cross-sections. Do not find its area.

4.[R] A lumberjack saws a wedge out of a cylindrical tree of radius a. His first cut is parallel to the ground and stops at the axis of the tree. His second cut makes an angle θ with the first cut and meets it along a diameter.

(a) Draw a typical cross-section that is a triangle.

(b) Find the area of the triangle as a function of x, the distance of the plane from the axis of the tree.

(c) Draw a typical cross-section that is a rectangle.

(d) Find the area of the rectangle as a function of x, the distance of the plane from the axis of the tree.

5.[R] A cylindrical glass is full of water. The glass is tilted until the remaining water just covers the base of the glass. (Try it!) The radius of the glass is a and its height is h. Consider parallel planes such that cross-sections of the water are rectangles.

(a) Make clear diagrams that show the situation. (Include a top view to show the cross-sections.)

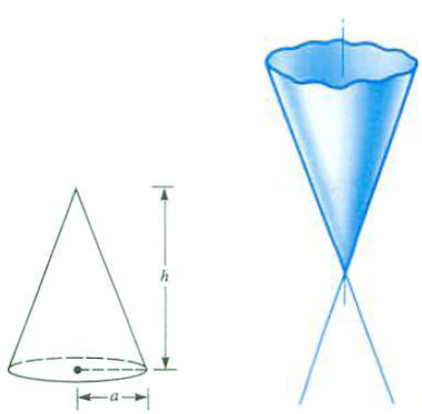
(b) Obtain a formula for the area of the cross-sections. *Advice:* The two planes at a distance x from the axis of the glass cut out cross-sections of different areas. So introduce an x-axis with 0 at the center of the base and extending from $-a$ to a in a convenient direction.

6.[R] Repeat Exercise 5, but this time consider parallel planes such that the cross-sections are trapezoids.

7.[R] A right circular cone has a radius a and height h as shown in Figure 7.2.8(a). Consider cross-sections made by planes parallel to the base of the cone.

(a) Draw perspective and side views of the situation.

(b) Drawing as many diagrams as necessary, find the area of the cross-section made by a plane at a distance x from the vertex of the cone.



(a) (b)

Figure 7.2.8:

8.[R] Draw the typical cross-section made by a plane parallel to the axis of the cone. Draw perspective and side views of the situation, but do not find a formula for the area of the cross-section. NOTE: See Exercise 7

9.[R] Figure 7.2.8(b) indicates an unbounded, solid right circular cone. Draw a cross-section that a bounded by (a) a circle, (b) an ellipse (but not a circle), (c) a parabola, and (d) a hyperbola.

10.[R] Draw a cross-section of a right circular cylinder that is (a) a circle, (b) an ellipse that is not a circle, and (c) a rectangle.

11.[R] Draw a cross-section of a solid cube that is (a) a square, (b) an equilateral triangle, (c) a five-sided polygon, and (d) a regular hexagon.

12.[R] The plane region between the curves $y = x$ and $y = x^2$ is spun around the x-axis to produce a solid resembling the bell of a trumpet.

(a) Draw the plane region.

(b) Draw the solid region produced by spinning this region around the x-axis.

(c) Draw the typical cross-section made by a plane perpendicular to the x-axis. Show this in both perspective and side views.

(d) Find the area of the cross-section in terms of the distance x of the plane from the origin to the x-axis.

13.[R] Obtain a circular stick such as a broom handle or a dowel. Saw off a piece, making one cut perpendicular to the axis and the second cut at an angle to the axis. Mark on the surface of the piece you cut out the borders of cross-sections that are (a) rectangles and (b) trapezoids.

7.3 Setting Up a Definite Integral

This section presents an informal shortcut for setting up a definite integral to evaluate some quantity. First, the formal and informal approaches are contrasted in the case of setting up the definite integral for area. Then the informal approach will be illustrated as commonly applied in a variety of fields.

The Complete Approach

Recall how the formula $A = \int_a^b f(x)\,dx$ was obtained (in Section 7.1). The interval $[a, b]$ was partitioned by the numbers $x_0 < x_1 < x_2 < \cdots < x_n$ with $x_0 = a$ and $x_n = b$. A sampling number c_i was chosen in each section $[x_{i-1}, x_i]$. For convenience, all the sections are of equal length, $\Delta x = (b - a)/n$. (See Figure 7.3.1.) We then form the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x \tag{7.3.1}$$

It equals the total area of the rectangular approximation in Figure 7.3.2.

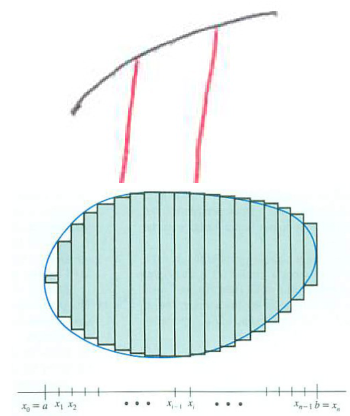


Figure 7.3.2: NOTE: Revise figure so not left-hand sum.

As Δx approaches 0, the sum (7.3.1) approaches the area of the region under consideration. But, by the definition of the definite integral, the sum (7.3.1) approaches

$$\int_a^b f(x)\,dx.$$

Thus

$$\text{Area} = \int_a^b f(x)\,dx. \tag{7.3.2}$$

That is the complete or "formal" approach to obtain formula (7.3.2). Now consider the "informal" approach, which is just a shorthand for the complete approach.

The Shorthand Approach

The heart of the complete approach is the *local estimate* $f(c_i)\Delta x$, the area of a rectangle of height $f(c_i)$ and width Δx, which is shown in Figure 7.3.4.

In the shorthand approach to setting up a definite integral attention is focused on the *local approximation*. No mention is made of the partition or the sampling numbers. We illustrate this shorthand approach by obtaining formula (7.3.2) informally. This is *not* a new method of integration, but just

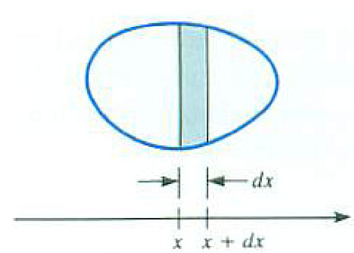


Figure 7.3.3:

a way to save time when setting up an integral - finding out the integrand and the interval of integration.

For example, consider a small positive number dx. What would be a good estimate of the area of the region corresponding to the short interval $[x, x+dx]$ of width dx shown in Figure 7.3.3? The area of the rectangle of width dx and height $f(x)$ shown in Figure 7.3.4 would seem to be a plausible estimate. The area of this thin rectangle is

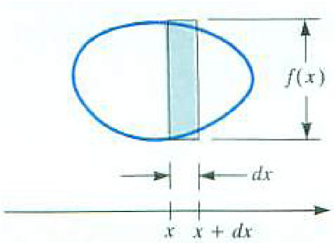


Figure 7.3.4:

$$f(x)\ dx. \tag{7.3.3}$$

Without further ado, we then write

$$\text{Area} = \int_a^b f(x)\ dx, \tag{7.3.4}$$

which is formula (7.3.2). The leap from the local approximation (7.3.3) to the definite integral (7.3.4) omits many steps of the complete approach. This informal approach is the shorthand commonly used in applications of calculus. It is the way engineers, physicists, biologists, economists, and mathematicians set up integrals.

It should be emphasized that it is only an abbreviation of the formal approach, which deals with approximating sums.

The Volume of a Ball

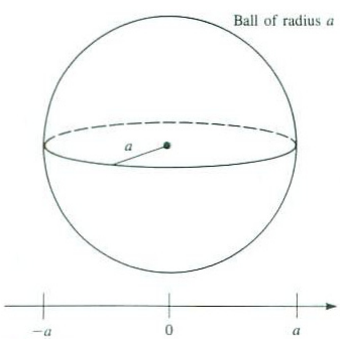


Figure 7.3.5:

EXAMPLE 1 Find the volume of a ball of radius a. First use the complete approach. Then use the shorthand approach.

SOLUTION Both approaches require good diagrams. In the complete approach we show an x-axis, a partition into sections of equal lengths, sampling numbers c_i, and the approximating disks. See Figures 7.3.5 and 7.3.6(a). The thickness of disk is Δx, as shown in the side view of Figure 7.3.6(b), while its radius is labeled r_i, as shown in the end view of Figure 7.3.6(c). The volume of this typical disk is

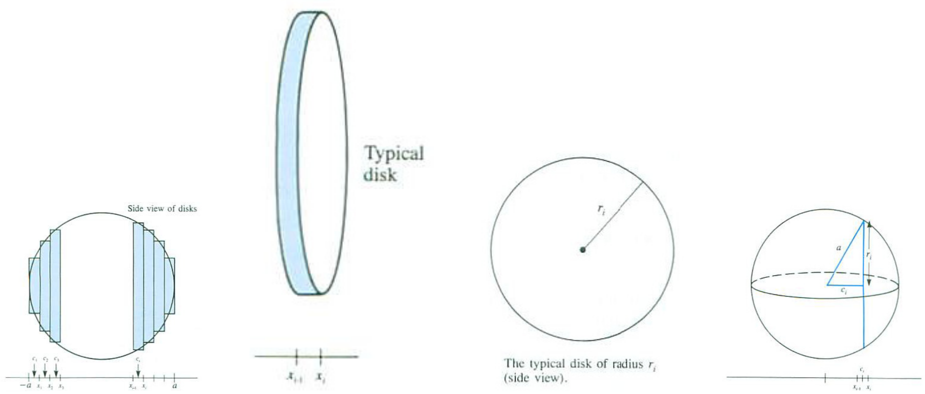
$$\pi r_i^2(\Delta x). \tag{7.3.5}$$

All that remains is to determine r_i. Figure 7.3.6(d) helps us do that. By the Pythagorean Theorem,

$$r_i^2 = a^2 - c_i^2. \tag{7.3.6}$$

Combining (7.3.1), (7.3.5), and (7.3.6) gives the typical estimate of the volume of a sphere of radius a:

$$\sum_{i=1}^{n} \pi(a^2 - c_i^2)\Delta x. \tag{7.3.7}$$



(a) (b) (c) (d)

Figure 7.3.6:

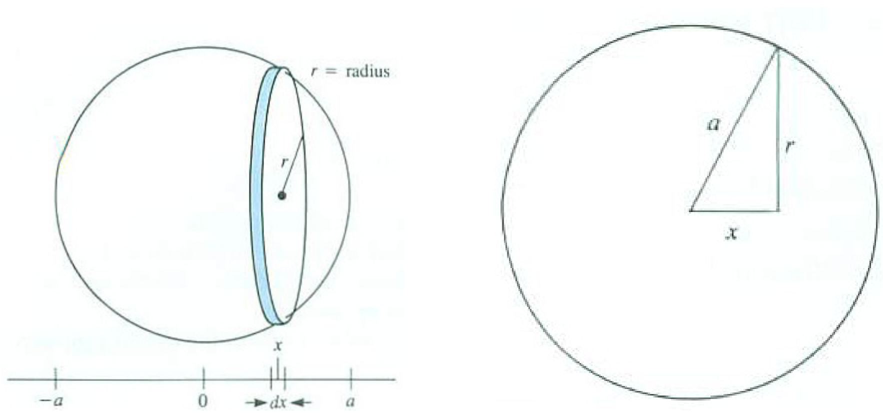
By the definition of the definite integral,

$$\lim_{\Delta x \to 0} \Sigma_{i=1}^{n} \pi(a^2 - c_i^2)\Delta x = \int_{-a}^{a} \pi(a^2 - x^2)\ dx.$$

Hence

$$\text{Volume of ball of radius } a = \int_{-a}^{a} \pi(a^2 - x^2)\ dx.$$

By the Fundamental Theorem of Calculus, the integral equals $4\pi a^3/3$.



(a) (b)

Figure 7.3.7:

Now for the shorthand approach. We draw only a short section of an x-axis and label its length dx. Then we draw an approximating disk, whose radius we label r, as in Figure 7.3.7(a). Since the disk has a base of area πr^2 and thickness dx, its volume is $\pi r^2\ dx$. Moreover, as Figure 7.3.7(b) shows, $r^2 = a^2 - x^2$. Hence the local approximation is

$$\pi(a^2 - x^2)\ dx. \tag{7.3.8}$$

Then, without further ado, without choosing any c_i or showing any approximating sum, we have

$$\text{Volume of ball of radius } a = \int_{-a}^{a} \pi(a^2 - x^2)\ dx.$$

The key to this bookkeeping is the local approximation (7.3.8) in differential form, which gives the necessary integrand. The limits of integration are determined separately. ◇

Volcanic Ash

EXAMPLE 2 After the explosion of a volcano, ash gradually settles from the atmosphere and falls on the ground. The depth diminishes with distance from the volcano. Assume that the depth of the ash at a distance x feet from the volcano is Ae^{-kx} feet, where A and k are positive constants. Set up a definite integral for the total volume of ash that falls within a distance b of the volcano.

SOLUTION First estimate the volume of ash that falls on a very narrow ring of width dx and inner radius x centered at the volcano. (See Figure 7.3.8(a).) This estimate can be made since the depth of the ash depends only on the distance from the volcano. On this ring the depth is almost constant.

The area of this ring is approximately that of a rectangle of length $2\pi x$ and width dx. (See Figure 7.3.8(b)) So the area of the ring is approximately

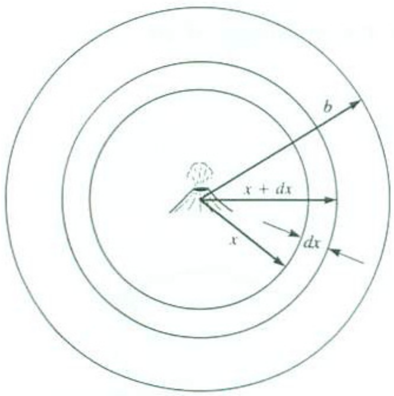
$$2\pi x\ dx.$$

Exercise 4 shows that its area is $2\pi x\ dx + \pi(dx)^2$.

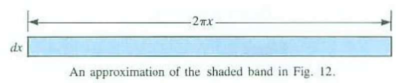
Although the depth of the ash on this narrow ring is not constant, it does not vary much. A good estimate of the depth throughout the ring is Ae^{-kx}. Thus the volume of the ash that falls on the typical ring of inner radius x and outer radius $x + dx$ is approximately

$$Ae^{-kx}(2\pi x)\ dx \text{ cubic feet.} \tag{7.3.9}$$

Once we have the key local estimate (7.3.9), we immediately write down the definite integral for the total volume of ash that falls within a distance b of the volcano:



(a)



(b)

Figure 7.3.8:

$$\text{Total volume} = \int_0^b Ae^{-kx} 2\pi x \, dx.$$

This completes the shorthand setting up the definite integral. (To evaluate this integral, use a formula from the inside front cover of this book or a technique in Chapter 8.) ◇

The limits of integration must be determined just as in the formal approach.

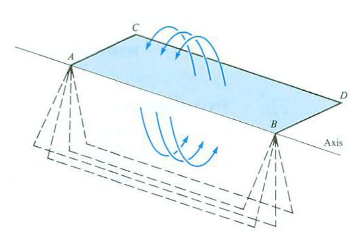
Kinetic Energy

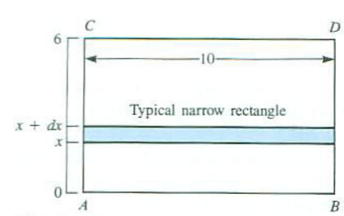
The next example of the informal approach to setting up definite integrals concerns kinetic energy. The kinetic energy associated with an object of mass m kilograms and velocity v meters per second is defined as

$$\text{Kinetic energy} = \frac{mv^2}{2} \text{joules.}$$

If the various parts of the objects are not all moving at the same speed, an integral is needed to express the total kinetic energy. We develop this integral in the next example.

EXAMPLE 3 A thin rectangular piece of sheet metal is spinning around one of its longer edges 3 times per second, as shown in Figure 7.3.9. The length of its shorter edge is 6 meters and the length of its longer edge is 10 meters. The density of the sheet metal is 4 kilograms per square meter. Find the kinetic energy of the spinning rectangle.





SOLUTION The farther a mass is from the axis, the faster it moves, and therefore the larger its kinetic energy. To find the total kinetic energy of the rotating piece of sheet metal, imagine it divided into narrow rectangles of length 10 meters and width dx meters parallel to the edge $\overline{AB}$; a typical one is shown in Figure 7.3.10. (Introduce an x-axis parallel to edge $\overline{AC}$ with the origin corresponding to A.) Since all points of the typical narrow rectangle move at roughly the same speed, we will be able to estimate its kinetic energy. That estimate will provide the key local appoximation in the informal approach to setting up a definite integral.

First of all, the mass of the typical rectangle is

$$4 \cdot 10 \; dx \text{ kilograms},$$

since its area is 10 dx square meters and the density is 4 kilograms per square meter.

Second, we must estimate its velocity. The narrow rectangle is spun 3 times per second around a circle of radius x. In 1 second each point in it covers a distance of about

$$3 \cdot 2\pi x = 6\pi x \text{ meters}.$$

Consequently, the velocity of the typical rectangle is

$$6\pi x \text{ meters per second}.$$

The local estimate of the kinetic energy associated with the typical rectangle is therefore

$$\frac{1}{2} \underbrace{40 \; dx}_{\text{mass}} \; \underbrace{(6\pi x)^2}_{\text{velocity squared}} \quad \text{joules}$$

The local approximation

or simply

$$720\pi^2 x^2 \; dx \text{ joules}. \qquad (7.3.10)$$

Having obtained the local estimate (7.3.10), we jump directly to the definite integral and conclude that

$$\text{Total energy of spinning rectangle} = \int_0^6 720\pi^2 x^2 \; dx \text{ joules}.$$

◇

Summary

This section presented a shorthand approach to setting up a definite integral for a quantity Q. In this method we estimate how much of the quantity Q corresponds to a very short section $[x, x+dx]$ of the x-axis, say $f(x)\;dx$. Then $Q = \int_a^b f(x)\;dx$, where a and b are determined by the particular situation.

EXERCISES for Section 7.3 *Key:* R–routine, M–moderate, C–challenging

1.[R] In Section 6.4 we showed that if $f(t)$ is the velocity at time t of an object moving along the x-axis, then $\int_a^b f(t)\ dt$ is the change in position during the time interval $[a, b]$. Develop this fact in the informal style of this section. Keep in mind that $f(t)$ may be positive or negative.

2.[R] The depth of rain at a distance r feet from the center of a storm is $g(r)$ feet.

(a) Estimate the total volume of rain that falls between a distance r feet and a distance $r+\ dr$ feet from the center of the storm. (Assume that dr is a small positive number.)

(b) Using (a), set up a definite integral for the total volume of rain that falls between $1,000$ and $2,000$ feet from the center of the storm.

3.[R] Consider a disk of radius a with the home base of production at the center. Let $G(r)$ denote the density of foodstuffs (in calories per square meter) at radius r meters from the home base. Then the total number of calories produced in the range is given by what definite integral?
NOTE: This analysis of primitive agriculture is taken from *Is There an Optimum Level of Population?*, edited by S. Fred Singer, McGraw-Hill, New York, 1971.

4.[R] In Example 2 the area of the ring with inner radius x and outer radius $x+dx$ was estimated to be about $2\pi x\ dx$.

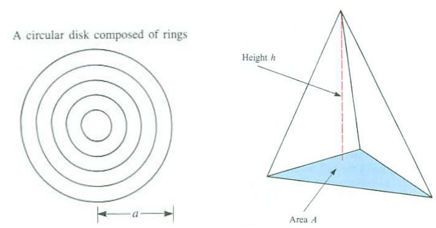
(a) Using the formula for the area of a circle, show that the area of the ring is $2\pi x\ dx + \pi(\ dx)^2$.

(b) Show that the ring has the same area as a trapezoid of height dx and bases of lengths $2\pi x$ and $2\pi(x+\ dx)$.

5.[R] Think of a circular disk of radius a as being composed of concentric circular rings, as in Figure 7.3.11(a).

(a) Using the shorthand approach, set up a definite integral for the area of the disk. (Draw a good picture of the local approximation.)

(b) Evaluate the integral in (a).



(a) (b)

Figure 7.3.11:

Exercises 6 to 8 concern the volumes of solids. In each case (a) draw a good picture of the local approximation of width dx, (b) set up the appropriate definite integral, and (c) evaluate the integral.

6.[R] A right circular cone of radius a and height h.

7.[R] A pyramid with a square base of side a and of height h. Its top vertex is above one corner of the base. (Use square cross sections.)

8.[R] A pyramid with a triangular base of area A and of height h. (The triangle can be any shape. See Figure 7.3.11(b).)

9.[M] At the time t hours, $0 \leq t \leq 24$, a firm uses electricity at the rate of $e(t)$ joules per hour. The rate schedule indicates that the cost per joule at time t is $c(t)$ dollars. Assume that both e and c are continuous functions.

(a) Estimate the cost of electricity consumed between times t and $t + dt$, where dt is a small positive number.

(b) Using (a), set up a definite integral for the total cost of electricity for the 24-hour period.

10.[M] The **present value** of a promise to pay one dollar t years from now is $g(t)$ dollars.

(a) What is $g(0)$?

(b) Why is it reasonable to assume that $g(t) \leq 1$ and that g is a decreasing function of t?

(c) What is the present value of a promise to pay q dollars t years from now?

(d) Assume that an investment made now will result in an income flow at the rate of $f(t)$ dollars per year t years from now. (Assume that f is a continuous function.) Estimate informally the present value of the income to be earned between time t and time $t + dt$, where dt is a small positive number.

(e) On the basis of the local estimate made in (d), set up a definite integral for the present value of all the income to be earned during the next b years.

11.[M] Let the number of females in a certain population in the age range from x years to $x + dx$ years, where dx is a small positive number, be approximately $f(x)\,dx$. Assume that, on average, women of age x produce $m(x)$ offspring during the year before they reach age $x + 1$. Assume that both f and m are continuous functions.

(a) What definite integral represents the number of women between ages a and b years?

(b) What definite integral represents the total number of offspring during the calendar year produced by women whose ages at the beginning of the calendar year were between a and b years?

Exercises 12 to 17 concern **kinetic energy**. They are all based on the concept that a particle of mass M moving with velocity V has the kinetic energy $MV^2/2$. (See Example 3.) An object whose density is the same at all its points is called **homogeneous**. If the object is planar, such as a square or disk, and has mass M kilograms and area A square meters, its density is M/A kilograms per square meter.

12.[M] The piece of sheet metal in Example 3 is rotated around the line midway between the edges AB and CD at the rate of 5 revolutions per second.

(a) Using the informal approach, obtain a local approximation for the kinetic energy of a narrow strip of the metal.

(b) Using (a), set up a definite integral for the kinetic energy of the piece of sheet metal.

(c) Evaluate the integral in (b).

13.[M] A circular piece of metal of radius 7 meters has a density of 3 kilograms per square meter. It rotates 5 times per second around an axis perpendicular to the circle and passing through the center of the circle.

(a) Devise a local approximation for the kinetic energy of a narrow ring in the circle.

(b) With the aid of (a), set up a definite integral for the kinetic energy of the rotating metal.

(c) Evaluate the integral in (b).

14.[M] The density of a rod x centimeters from its left end is $g(x)$ grams per centimeter. The rod has a length of b centimeters. The rod is spun around its left end 7 times per second.

(a) Estimate the mass of the rod in the section that is between x and $x + dx$ centimeters from the left end. (Assume that dx is small.)

(b) Estimate the kinetic energy of the mass in (a).

(c) Set up a definite integral for the kinetic energy of the rotating rod.

15.[M] A homogeneous square of mass M kilograms and side a meters rotates around an edge 5 times per second.

(a) Obtain a "local estimate" of the kinetic energy. What part of the square would you use? Why? Draw it.

(b) What is the local estimate?

(c) What definite integral represents the total kinetic energy of the square?

(d) Evaluate it.

16.[M] Repeat Exercise 15 for a square spun around a line through its center and parallel to an edge.

17.[M] Repeat Exercise 15 for a disk of radius a and mass M spinning around a line through its center and perpendicular to it. It is spinning at the rate of ω radians per second. (See Figure 7.3.12.)

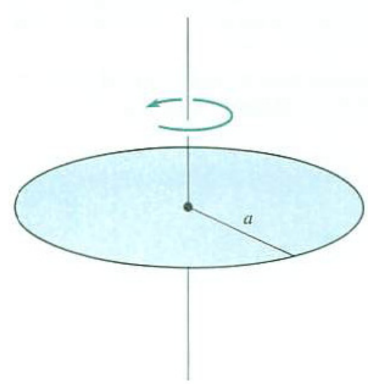


Figure 7.3.12:

In Exercises 18 and 19 you will meet definite integrals that cannot be evaluated by the Fundamental Theorem of Calculus (since the desired antiderivative is not elementary). Use (a) the trapezoidal and (b) Simpson's method with six sections to estimate the definite integrals.

18.[M] A homogeneous object of mass M occupies the region under $y = e^{x^2}$ and above $[0, 1]$. It is spun at the rate of ω radians per second around the y-axis. Estimate its kinetic energy.

19.[M] A homogeneous object of mass M occupies the region under $y = \sin(x)/x$ and above $[\pi/2, \pi]$. It is spun around the line $x = 1$ at the rate of ω radians per second. Estimate its kinetic energy.

In each of Exercises 20 to 23, find the kinetic energy of a planar homogeneous object that occupies the given region, has mass M, and is spun around the y-axis ω radians per second.

20.[M] The region under $y = e^x$ and above the interval $[1, 2]$.

21.[M] The region under $y = \arctan(x)$ and above the interval $[0, 1]$.

22.[M] The region under $y = 1/(1 + x)$ and above $[2, 4]$.

23.[M] The region under $y = \sqrt{1 + x^2}$ and above $[0, 2]$.

24.[M] A solid homogeneous right circular cylinder of radius a, height h, and mass M is spun at the rate of ω radians per second around its axis. Find its kinetic energy. (Include a good picture on which your local approximation is based.)

25.[M] A solid homogeneous ball of radius a and mass M is spun at the rate of ω radians per second around a diameter. Find its kinetic energy. (Include a good picture on which your local approximation is based.)

26.[C] Find the surface area of a sphere of radius a. Hint: Begin by estimating the area of the narrow band shown in Figure 7.3.13.

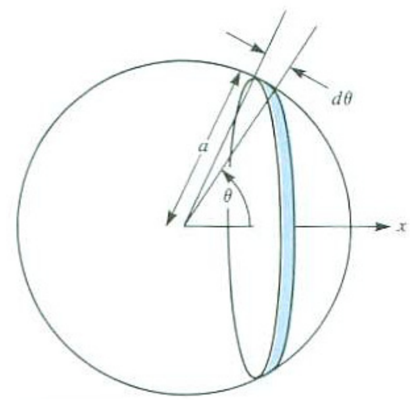


Figure 7.3.13:

27.[C] *[Actuarial tables]* Let $F(t)$ be the fraction of people born in 1900 who are alive t years later, $0 \leq F(t) \leq 1$.

(a) What is $F(150)$, probably?

(b) What is $F(0)$?

(c) Sketch the general shape of the graph of $y = F(t)$.

(d) Let $f(t) = F'(t)$. (Assume F is differentiable.) Is $f(t)$ positive or negative?

(e) What fraction of the people born in 1900 die during the time interval $[t, t+dt]$? (Express your answer in terms of F.)

(f) Answer (e), but express your answer in terms of f.

(g) Evaluate $\int_0^{150} f(t)\ dt$.

(h) What integral would you propose to call "the average life span of the people born in 1900"? Why?

28.[C] Let $F(t)$ be the fraction of ball bearings that wear out during the first t hours of use. Thus $F(0) = 0$ and $F(t) \leq 1$.

(a) As t increases, what would you think happens to $F(t)$?

(b) Show that during the short interval of time $[t, t + \ dt]$, the fraction of ball bearings that wear out is approximately $F'(t)\ dt$. (Assume F is differentiable.)

(c) Assume all wear out in at most $1,000$ hours. What is $F(1,000)$?

(d) Using the assumption in (b) and (c) devise a definite integral for the average life of the ball bearings.

29.[C] (*Poiseuille's law of blood flow*) A fluid flowing through a pipe does not all move at the same velocity. The velocity of any part of the fluid depends on its distance from the center of the pipe. The fluid at the center of the pipe moves fastest, whereas the fluid near the wall of the pipe moves slowest. Assume that the velocity of the fluid at a distance x centimeters from the axis of the pipe is $g(x)$ centimeters per second.

(a) Estimate the flow of fluid (in cubic centimeters per second) through a thin ring of inner radius r and outer radius $r + dr$ centimeters centered at the axis of the pipe and perpendicular to the axis.

(b) Using (a), set up a definite integral for the flow (in cubic centimeters per second) of fluid through the pipe. (Let the radius of the pipe be b centimeters.)

(c) Poiseuille (1797-1869), studying the flow of blood through arteries, used the function $g(r) = k(b^2 - r^2)$, where k is a constant. Show that in this case the flow of blood through an artery is proportional to the fourth power of the radius of the artery.

30.[C] The density of the earth at a distance of r miles from its center is $g(r)$ pounds per cubic mile. Set up a definite integral for the total mass of the earth. (Take the radius of the earth to be $4,000$ miles.)

7.4 Computing Volumes by Parallel Cross-Sections

See Problem 3 in Section 6.1.

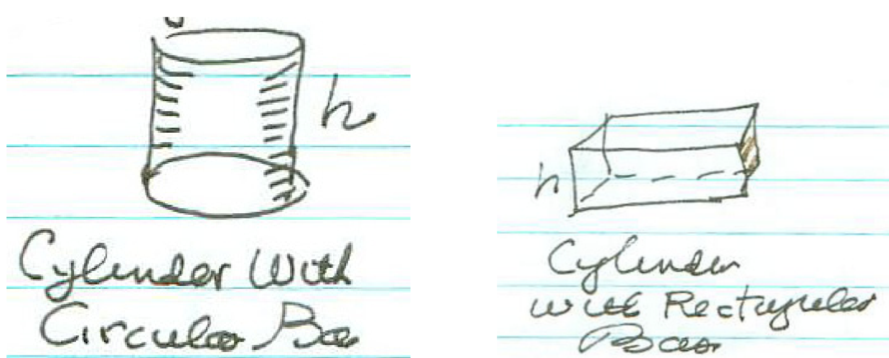
In Section 6.1 we computed areas by integrating lengths of cross-sections made by parallel lines. In this section we will use a similar approach, finding volumes by integrating areas of cross-sections made by parallel planes. We already saw an example of this method when we represented the volume of a tent as a definite integral.

Cylinders

(a) (b)

Figure 7.4.1:

Let $\mathcal{B}$ be a region in the plane (see Figure 7.4.1(a) and h a positive number. The **cylinder with base** $\mathcal{B}$ **and height** h consists of all line segments of length h perpendicular to $\mathcal{B}$, one end of which is in $\mathcal{B}$ and the other end is on a fixed side (above or below) of $\mathcal{B}$. This typical cylinder is shown in Figure 7.4.1(b). The top of the cylinder is congruent to $\mathcal{B}$. If $\mathcal{B}$ is a disk, the



(a) (b)

Figure 7.4.2: ARTIST: Final word in each caption is "Base"

cylinder is the customary circular cylinder of daily life (see Figure 7.4.2(a)). If $\mathcal{B}$ is a rectangle, the cylinder is a rectangular box (see Figure 7.4.2(b)).

We will make use of the formula for the volume of a cylinder:

The volume of a cylinder with base $\mathcal{B}$ and height h is

$$V = \text{Area of Base} \times \text{Height} = (\text{Area of } \mathcal{B}) \times h.$$

Volume as the Definite Integral of Cross-Sectional Area

Let's use the informal approach for setting up a definite integral to see how to use integration to calculate volumes of solids.

Consider the solid region $\mathcal{R}$ shown in Figure 7.4.3, which lies between the planes perpendicular to the x-axis at $x = a$ and at $x = b$. We use a cylinder to estimate the volume of the part of $\mathcal{R}$ that lies between two parallel planes a "small distance" dx apart, shown in perspective in Figure 7.4.4. This thin slab is not usually a cylinder. However, we can approximate it by a cylinder. To do this, let x be, say, the left endpoint of an interval of width dx. The plane perpendicular to the x-axis at x intersects $\mathcal{R}$ in a plane cross-section of area $A(x)$. The cylinder whose base is that cross-section and whose height is dx is a good approximation of the part of $\mathcal{R}$. It is the slab shown in Figure 7.4.5.

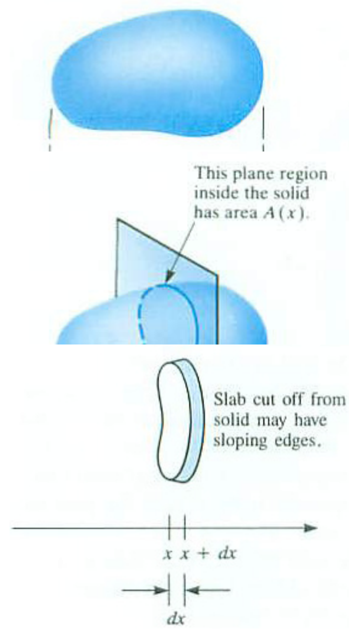


Figure 7.4.5:

We therefore have

$$\text{Local Approximation to Volume} = A(x)dx.$$

Then

$$\text{Volume of Solid} = \int_a^b A(x)\, dx.$$

In short, "volume equals the integral of cross-sectional area." To apply this idea, we compute $A(x)$. That is a where good drawings come in handy.

Given a particular solid, one just has to find a, b and the cross-sectional area $A(x)$ in order to construct a definite integral for its volume. These are the steps for finding the volume of a solid:

1. Choose a line to serve as an x-axis. See Figure 7.4.3.

2. For each plane perpendicular to that axis, find the area of the cross-section of the solid made by the plane. Call this area $A(x)$. See Figure 7.4.4.

3. Determine the limits of integration, a and b, for the region.

4. Evaluate the definite integral $\int_a^b A(x)\,dx$.

Formulas for the area of familiar plane regions are on the inside back cover.

Most of the effort is usually spent in finding the integrand $A(x)$.

In addition to the Pythagorean Theorem and properties of similar triangles, formulas for the areas of familiar plane figures may be needed. Also keep in mind that if corresponding dimensions of similar figures have a ratio k, then their areas have the ratio k^2; that is, area is proportional to the square of the ratios of the lengths of corresponding line segments.

Archimedes was the first person to find the volume of a ball. He did not express the volume as a number. Rather, in the style of mathematics of the 3$^{\text{rd}}$ century BC, he expressed the volume in terms of the volume of a simpler object: the volume of a ball is two-thirds the volume of the smallest cylinder that contains it. That he considered this one of his greatest accomplishments is evidenced by his request that his tomb be topped with a carving of a ball within a cylinder.

EXAMPLE 1 Find the volume of a ball of radius a.

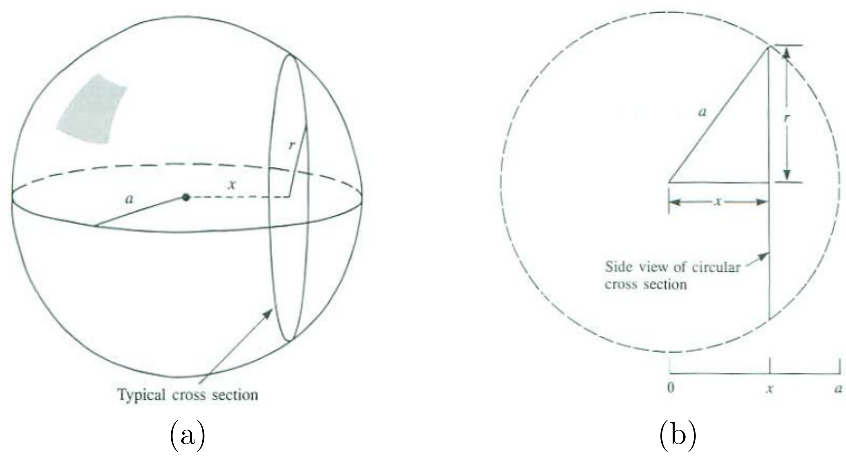


Figure 7.4.6: Cross-section (a) viewed in perspective and (b) from the side.

SOLUTION We sketch the typical cross-section in perspective and in side view (see Figure 7.4.6). The cross-section is a disk of radius r, which depends on x. The area of the cross-section is πr^2. To express this area in terms of x, use the Pythagorean Theorem, which tells us that $a^2 = x^2 + r^2$, hence $r^2 = a^2 - x^2$. So we have

$$\begin{aligned} \text{Volume} &= \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left(a^2x - \frac{x^3}{3}\right)\Big|_{-a}^{a} && \text{by FTC I} \\ &= \pi\left(\left(a^3 - \frac{a^3}{3}\right) - \left((-a)^3 - \frac{(-a)^3}{3}\right)\right) = \frac{4\pi}{3}a^3. \end{aligned}$$

◇

The next example concerns the solid region discussed in Example 3 of Section 7.2.

EXAMPLE 2 A cylindrical glass of height h and radius a is full of water. It is tilted until the remaining water covers exactly half the base. Find the volume of the remaining water.

SOLUTION We use the triangular cross-section shown in Figure 7.2.7.

Introduce the x-axis as in Figures 7.4.7 and 7.4.8. It was shown that the area of the cross-section at x is $\frac{1}{2}\frac{h}{a}(a^2-x^2)$. Thus,

$$\begin{aligned}\text{Volume} &= \int_{-a}^{a} \frac{h}{2a}(a^2-x^2)\,dx = \frac{h}{2a}\left(a^2x-\frac{x^3}{3}\right)\Big|_{-a}^{a} \qquad \text{by FTC I}\\ &= \frac{h}{2a}\left(\left(a^3-\frac{a^3}{3}\right)-\left(-a^3+\frac{a^3}{3}\right)\right) = \frac{h}{2a}\left(\frac{4}{3}a^3\right) = \frac{2}{3}ha^2.\end{aligned}$$

That's about 21% of the volume of the glass.

This calculation of the integral could be simplified by noting that the integrand is an even function (the volume to the right of 0 equals the volume to the left of 0). In this method we have

$$\begin{aligned}\text{Volume} &= 2\int_{0}^{a} \frac{h}{2a}\left(a^2-x^2\right)\,dx = \frac{h}{a}\left(a^2x-\frac{x^3}{3}\right)\Big|_{0}^{a}\\ &= \frac{h}{a}\left(\left(a^3-\frac{a^3}{3}\right)-(0-0)\right) = \frac{2}{3}ha^2\end{aligned}$$

The two solutions yield the same result. The second way avoids a lot of arithmetic with negative numbers, thus reducing the chance of making a mistake. ◇

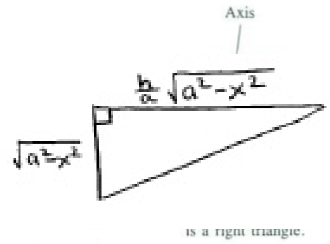
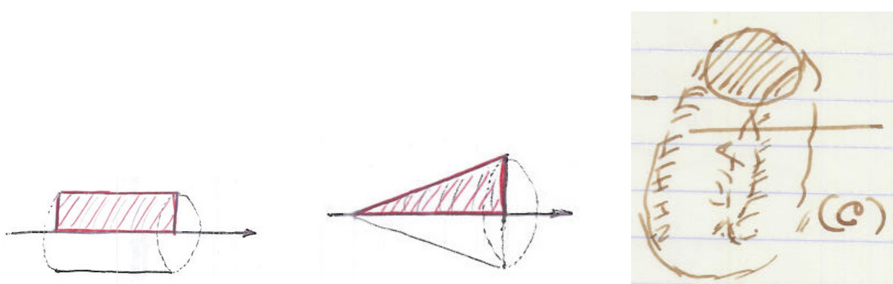


Figure 7.4.7:
Figure 7.4.8:

There is a much less chance for arithmetical error in this calculation.

Solids of Revolution

The solid formed by revolving a region $\mathcal{R}$ in the plane about a line in that plane that does not intersect the interior of $\mathcal{R}$ is called a **solid of revolution**.

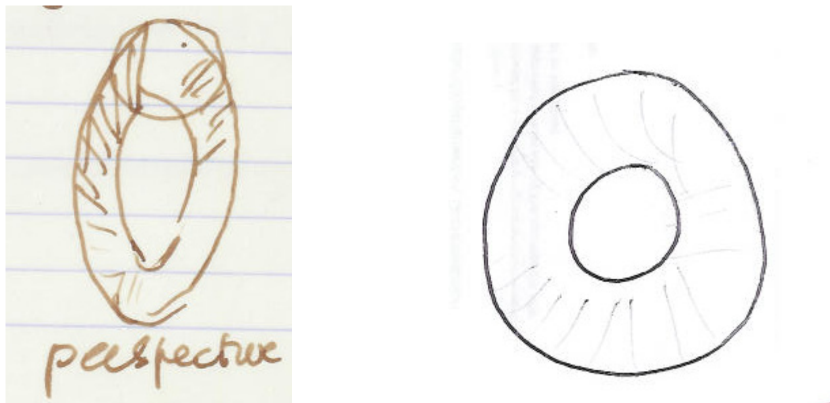


(a) (b) (c)

Figure 7.4.9:

Figure 7.4.9 shows three examples: (a) a circular cylinder obtained by revolving a rectangle about one of its edges, (b) a cone obtained by revolving a right triangle about one of its two legs, and (c) a torus ("doughnut" or "ring") formed by revolving a disk about a line outside the disk.

The cross-sections by planes perpendicular to the line around which the figure is revolved is either a disk or a "washer". The latter is a disk with a round hole. The cross-sections in Figure 7.4.9(a) and (b) are disks. In Figure 7.4.9(c) the cross-sections are washers. Figure 7.4.10 shows that the typical cross-section is a washer.

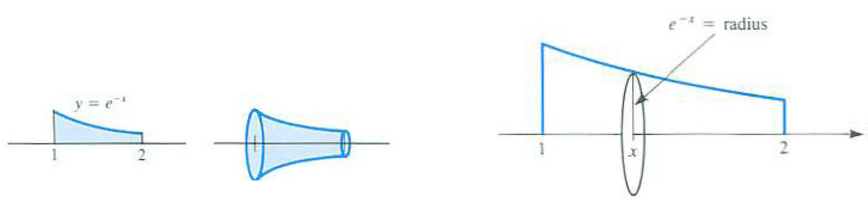


(a) (b)

Figure 7.4.10: (a) perspective (b) side view

EXAMPLE 3 The region under $y = e^{-x}$ and above $[1, 2]$ is revolved about the x-axis. Find the volume of the resulting solid of revolution. (See Figure 7.4.11(a).)

SOLUTION The typical cross-section by a plane perpendicular to the x-axis



(a) (b)

Figure 7.4.11:

is a disk of radius e^{-x}, as shown in Figure 7.4.11(b). The cross-sectional area is

$$\pi \left(e^{-x}\right)^2 = \pi e^{-2x}.$$

The volume of the solid is therefore

$$\int_1^2 \pi e^{-2x} \, dx.$$

Recall that $\frac{d}{dx}(e^{ax}) = ae^{ax}$, so that an antiderivative of e^{ax} is $\frac{1}{a}e^{ax}$. Hence,

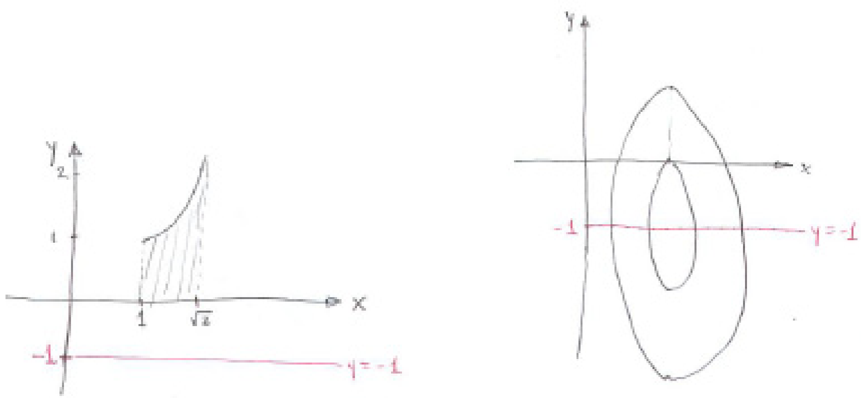
$$\int_1^2 \pi e^{-2x}\, dx \;=\; \frac{\pi}{-2}e^{-2x}\Big|_1^2 = \frac{\pi}{-2}\left(e^{-4}-e^{-2}\right) = \frac{\pi}{2}\left(e^{-2}-e^{-4}\right).$$

◇

The final two examples illustrate two themes: draw a good picture of the cross-section and integrate the cross-sectional area.

EXAMPLE 4 The region bounded by $y = x^2$, the lines $x = 1$ and $x = \sqrt{2}$, and the x-axis ($y = 0$). is revolved around the line $y = -1$. Find the volume of the resulting region $\mathcal{R}$.

SOLUTION Figure 7.4.12(a) shows the region being revolved and the line around which it is revolved. Figure 7.4.12(b) shows a perspective view of the typical cross-section.



(a) (b)

Figure 7.4.12:

The typical cross-section is a ring, with inner radius 1 and outer radius $1 + x^2$. Its area is therefore $\pi(1 + x^2)^2 - \pi(1)^2$.

Consequently, since "volume equals integral of cross-sectional area,"

$$\begin{aligned}
\text{Volume} &= \int_1^{\sqrt{2}} \left(\pi(1+x^2)^2 - \pi(1)^2\right)\, dx \\
&= \pi \int_1^{\sqrt{2}} (1+2x^2+x^4) - 1\, dx && \text{algebra} \\
&= \pi \int_1^{\sqrt{2}} \left(2x^2 + x^4\right)\, dx \\
&= \pi \left(\frac{2x^3}{3} + \frac{x^5}{5}\right)\Big|_1^{\sqrt{2}} && \text{FTC I} \\
&= \pi \left(\frac{32\sqrt{2}}{15} - \frac{13}{15}\right) && \text{arithmetic.}
\end{aligned}$$

◇

EXAMPLE 5 Find the volume of the solid formed by revolving the region in Figure 7.4.12(a) around the y-axis ($x = 0$).

SOLUTION The cross-sections by planes perpendicular to the y-axis are again rings (not disks). But something new enters the scene. For $0 \le y \le 1$ the cross-sections are between the vertical lines $x = 1$ and $x = \sqrt{2}$. For $1 \le y \le 2$ they are determined by the curve and the line $x = \sqrt{2}$.

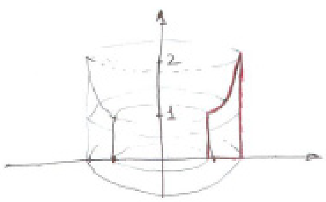


Figure 7.4.13:

The cross-sections for $0 \le y \le 1$, when rotated about the y-axis, fill out a cylinder whose height is 1 and whose base is a ring of area $\pi(\sqrt{2})^2 - \pi(1)^2 = \pi$. Thus, its volume (height times area of base) is $\pi(1) = \pi$. We did not need an integral for this.

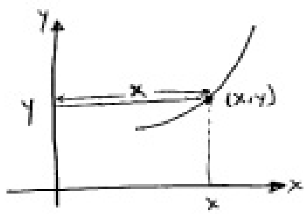


Figure 7.4.14:

The cross-sections for $1 \le y \le \sqrt{2}$ are rings whose outer radius is $\sqrt{2}$ and inner radius is determined by the curve $y = x^2$, as shown in Figure 7.4.14. Since $y = x^2$, the inner radius is $x = \sqrt{y}$. The area of these typical cross-sections is

$$\pi(\sqrt{2})^2 - \pi(\sqrt{y})^2.$$

Thus the typical local estimate of volume is

$$\left(\pi(\sqrt{2})^2 - \pi(\sqrt{y})^2\right)\ dy = (2\pi - \pi y)\ dy.$$

Therefore the volume swept out by these cross-sections is

$$\begin{aligned}
\int_1^{\sqrt{2}}(2\pi - \pi y)\ dy &= \left(2\pi y - \pi\tfrac{y^2}{2}\right)\Big|_1^{\sqrt{2}} && \text{FTC I}\\
&= \left(2\pi\sqrt{2} - \pi\right) - \left(2\pi - \tfrac{\pi}{2}\right)\\
&= 2\pi\sqrt{2} - \tfrac{5}{2}\pi.
\end{aligned}$$

Adding this to the volume obtained for the cylinder gives

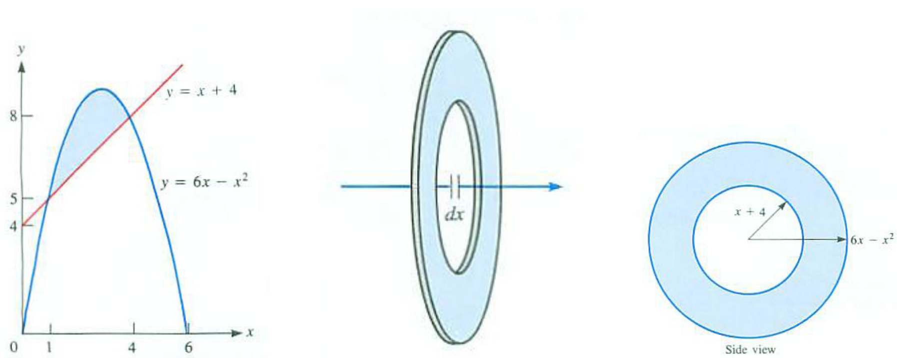
$$\begin{aligned}
\text{total volume} &= \left(2\pi\sqrt{2} - \frac{5}{2}\pi\right) + \pi\\
&= 2\pi\sqrt{2} - \frac{3}{2}\pi \approx 4.1734.
\end{aligned}$$

◇

EXAMPLE 6 The region bounded by the graphs of $y = x + 4$ and $y = 6x - x^2$, shown in Figure 7.4.15(a), is revolved about the x-axis to form a solid of revolution. Express the volume as a definite integral.

SOLUTION We first draw a local approximation to a thin slice of the solid (see Figure 7.4.15(b)). The side view in Figure 7.4.15(c) shows the area of the typical cross-section is

$$\pi\left(6x - x^2\right)^2 - \pi\left(x + 4\right)^2.$$



(a) (b) (c)

Figure 7.4.15:

This is the integrand. Next we find the interval of integration. The ends of the interval are determined by where the curves cross: $x + 4 = 6x - x^2$. Moving all terms to the left-hand side yields: $x^2 - 5x + 4 = 0$, or $(x - 1)(x - 4) = 0$. So the endpoints of the interval are $x = 1$ and $x = 4$. The volume of the solid is given by the definite integral

$$\int_1^4 \left(\pi\left(6x - x^2\right)^2 - \pi\left(x + 4\right)^2\right)\, dx.$$

◇

Summary

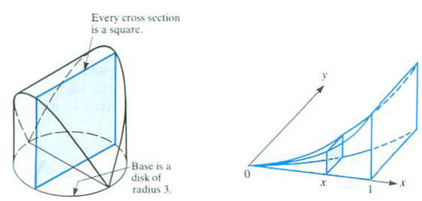
The key idea in this section is that "volume is the definite integral of cross-sectional area". To implement this idea we have to find that varying area and also the interval of integration. A solid of revolution, where the cross-section may be a disk or a ring, is just a special case.

EXERCISES for Section 7.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 8, (a) draw the solid, (b) draw the typical cross-section in perspective and side view, (c) find the area of the typical cross-section, (d) set up the definite integral for the volume, and (e) evaluate the definite integral (if possible).

1.[R] Find the volume of a cone of radius a and height h.

2.[R] The base of a solid is a disk of radius 3. Each plane perpendicular to a given diameter meets the solid in a square, one side of which is in the base of the solid. (See Figure 7.4.16(a).) Find its volume.



(a) (b)

Figure 7.4.16:

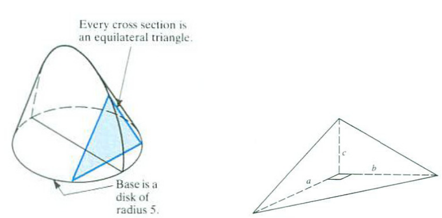
3.[R] The base of a solid is the region bounded by $y = x^2$, the line $x = 1$, and the x- and y-axes. Each cross-section perpendicular to the x-axis is a square. (See Figure 7.4.16(b).) Find the volume of the solid.

4.[R] Repeat Exercise 3 except that the cross-sections perpendicular to the base are equilateral triangles.

5.[R] Find the volume of a pyramid with a square base of side a and height h, using square cross-sections perpendicular to the base. The top of the pyramid is above the center of the base.

6.[R] Repeat Exercise 5, but using trapezoidal cross-sections perpendicular to the base.

7.[R] Find the volume of the solid whose base is the disk of radius 5 and whose cross-sections perpendicular to a diameter are equilateral triangles. (See Figure 7.4.17(a).)



(a) (b)

Figure 7.4.17:

8.[R] Find the volume of the pyramid shown in Figure 7.4.17(b) by using cross-sections perpendicular to the edge of length c.

In Exercises 9 to 14 set up a definite integral for the volume of the solid formed by revolving the given region R about the given axis.

9.[R] R is bounded by $y = \sqrt{x}$, $x = 1$, $x = 2$, and the x-axis, about the x-axis.

10.[R] R is bounded by $y = \frac{1}{\sqrt{1+x^2}}$, $x = 0$, $x = 1$, and the x-axis, about the x-axis.

11.[R] R is bounded by $y = x^{-1/2}$, $y = x^{-1}$, $x = 1$, and $x = 2$, about the x-axis.

12.[R] R is bounded by $y = x^2$ and $y = x^3$, about the y-axis.

13.[R] R is bounded by $y = \tan(x)$, $y = \sin(x)$, $x = 0$, and $x = \pi/4$, about the x-axis.

14.[R] R is bounded by $y = \sec(x)$, $y = \cos(x)$, $x = \pi/6$, and $x = \pi/3$, about the x-axis.

15.[R] A cylindrical drinking glass of height h and radius a, full of water, is tilted until the water just covers the base. Set up a definite integral that represents the amount of water left in the glass. Use rectangular cross-sections. Refer to Figure 7.4.18 and follow the directions preceding Exercise 1.

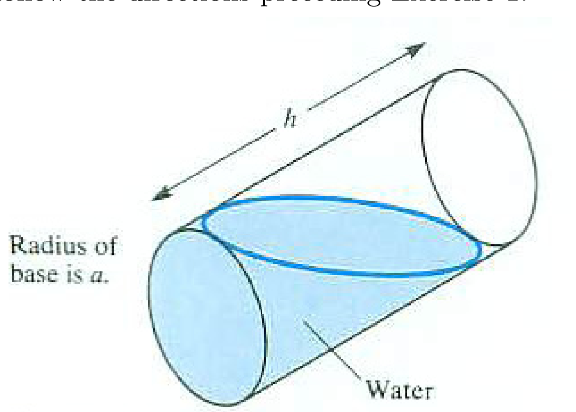


Figure 7.4.18:

16.[R] Repeat Exercise 15, but use trapezoidal cross-sections.

17.[R] Repeat Exercise 15 using only common sense. Don't use any calculus.

18.[M] A cylindrical drinking glass of height h and radius a, full of water, is tilted until the water remaining covers half the base.

(a) Set up a definite integral for the volume of water in the glass, using cross-sections that are parts of disks.

(b) Compare yours answer in (a) with the definite integral found in Example 2. Which definite integral looks easiest to evaluate?

19.[M] Repeat Exercise 18, but use rectangular cross-sections.

20.[M] A solid is formed in the following manner. A plane region R and a point P not in the plane are given. The solid consists of all line segments joining P to points in R. If R has area A and P is a distance h from the plane R, show that the volume of the solid is $Ah/3$. (See Figure 7.4.19.)

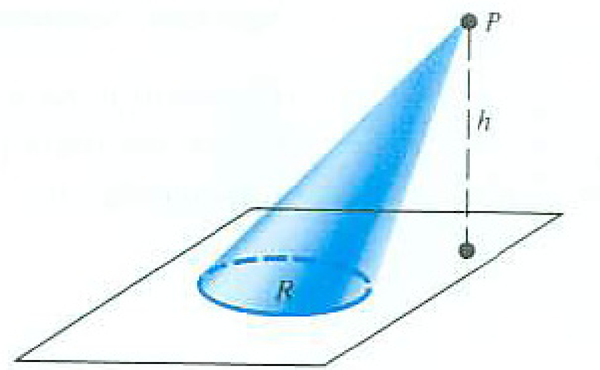


Figure 7.4.19:

21.[M] A drill of radius 4 inches bores a hole through a wooden sphere of radius 5 inches, passing symmetrically through the center of the sphere.

(a) Draw the part of the sphere removed by the drill.

(b) Find $A(x)$, the area of a cross-section of the region in (a) made by a plane perpendicular to the axis of the drill and at a distance x from the center of the sphere.

(c) Set up the definite integral for the volume of wood removed.

22.[M] What fraction of the volume of a sphere is contained between parallel planes that trisect the diameter to which they are perpendicular? (Leave your answer in terms of a definite integral.)

23.[M] The disk bounded by the circle $(x-b)^2+y^2=a^2$, where $0<a<b$, is revolved around the y-axis. Set up a definite integral for the volume of the doughnut (torus) produced.

In Exercises 24 to 27 set up definite integrals for (a) the area of R, (b) the volume formed when R is revolved around the x-axis, and (c) the volume formed when R is revolved around the y-axis.

24.[M] R is the region under $y = \tan(x)$ and above the interval $[0, \pi/4]$.

25.[M] R is the region under $y = e^x$ and above the interval $[-1, 1]$.

26.[M] R is the region under $y = 1/\sqrt{1 - x^2}$ and above the interval $[0, 1/2]$.

27.[M] R is the region under $y = \sin(x)$ and above the interval $[0, \pi]$.

28.[C] Set up a definite integral for the volume of one octant of the region common to two right circular cylinders of radius 1 whose axes intersect at right angles, as shown in Figure 7.4.20. NOTE: Contributed by Archimedes.

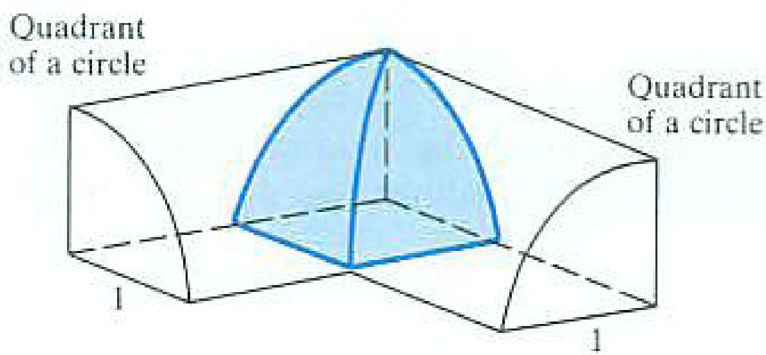


Figure 7.4.20:

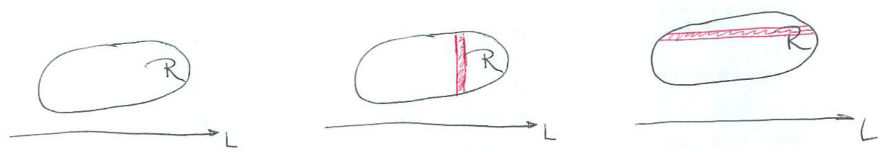
29.[C] When a convex region R of area A situated to the right of the y-axis is revolved around the y-axis, the resulting solid of revolution has volume V. When R is revolved around the line $x = -k$, the volume of the resulting solid is V^*. Express V^* in terms of k, A, and V. NOTE: The definition of convex can be found on page 134 in Section 2.5.

30.[C] Archimedes viewed a ball as a cone whose height is the radius of the ball and whose base is the surface of the ball. On that basis he computed that the volume of the ball is one third the product of the radius and the surface area. He then gave a rigorous proof of his conjecture.

Clever Sam, inspired by this, said "I'm going to get the volume of a circular cylinder in a new way. Say its radius is r and height is h. Then I'll view it as a cylinder made up of "r by h" rectangles, all of which have the axis as an edge. Then I pile them up to make a box whose base is an r by h rectangle and whose height is $2\pi r$ (the circumference of the cylinder's base). So the volume would be $2\pi r$ times rh, or $2\pi r^2 h$. That's twice the usual volume, so the standard formula is wrong." Is Sam right? (Explain.)

7.5 Computing Volumes by Shells

Imagine revolving the planar region $\mathcal{R}$ about the line L, as in Figure 7.5.1(a). We may think of $\mathcal{R}$ as being formed from narrow strips perpendicular to L, as in Figure 7.5.1(b). Revolving such a strip around L produces a washer (or disk). This is the approach used in the preceding section.



(a) (b) (c)

Figure 7.5.1:

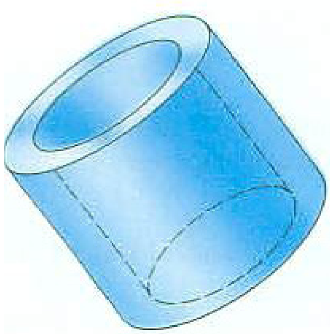
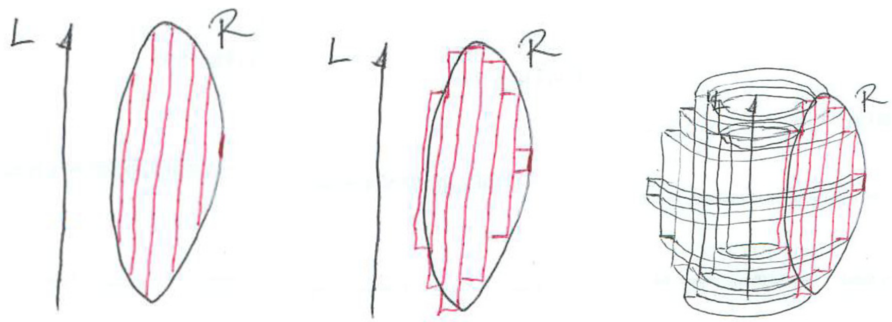


Figure 7.5.2:

However, we can also think of $\mathcal{R}$ as being formed from narrow strips *parallel* to L, as in Figure 7.5.1(c). Revolving such a strip around L produces a solid shaped like a bracelet or part of a drinking straw, as shown, in perspective, in Figure 7.5.2. We will call such a solid a **shell**. (Perhaps "tube" or "pipe" might be a better choice, but "shell" is standard in the world of calculus.)

This section describes how to find the volume of a solid of revolution using shells (instead of disks). Sometimes this approach provides an easier calculation.

The Shell Technique



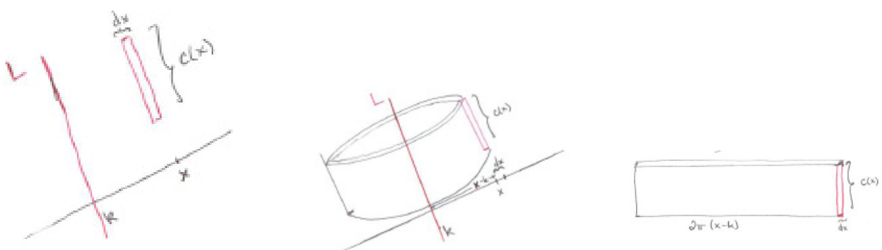
(a) (b) (c)

Figure 7.5.3:

To apply the shell technique we first imagine cutting the plane region $\mathcal{R}$ in Figure 7.5.3(a) into a finite number of narrow strips by lines parallel to L.

Each strip is then approximated by a rectangle as in Figure 7.5.3(b). Then we approximate the solid of revolution by a collection of tubes (like the parts of a collapsible telescope), as in Figure 7.5.3(c).

The key to this method is estimating the volume of each shell. Figure 7.5.4(a)



(a) (b) (c)

Figure 7.5.4:

shows the typical local approximation. Its height, $c(x)$, is the length of the cross-section of $\mathcal{R}$ corresponding to the value x on a line that we will call the x-axis. The radius of the shell, shown in Figure 7.5.4(b), is $x - k$, where k is the x-coordinate of the equation of the axis of rotation. Imagine cutting the shell along a direction parallel to L, unrolling it, and then laying it flat like a carpet. When laid flat, the shell resembles a thin slab of thickness dx, width $c(x)$, and length $2\pi(x - k)$, as shown in Figure 7.5.4(c). The volume of this shell, therefore, is about

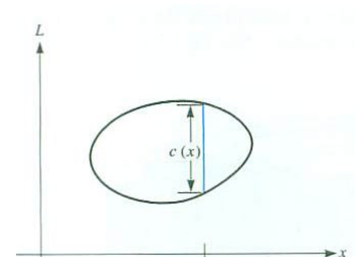
The exact volume of the shell is found in Exercise 23.

$$\text{Local Approximation to Volume of a Shell} = 2\pi(x-k)c(x)\ dx \qquad (7.5.1)$$

With the aid of the local approximation (7.5.1), we conclude that

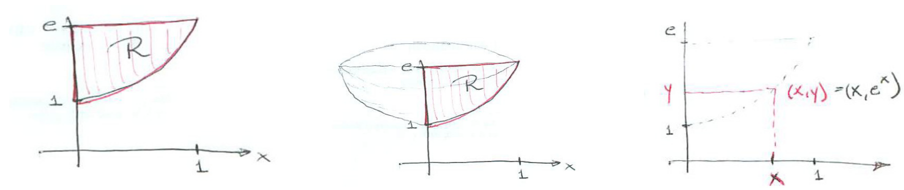
$$\text{Volume of Solid of Revolution} = \int_a^b 2\pi(x-k)c(x)\ dx. \qquad (7.5.2)$$

If $x - k$ is denoted $R(x)$, the "radius of the shell," as in Figure 7.5.5, then



$$\text{Volume of Solid of Revolution} = \int_a^b 2\pi R(x)c(x)\,dx.$$

EXAMPLE 1 The region $\mathcal{R}$ below the line $y = e$, above $y = e^x$, and to the right of the y-axis is revolved around the y-axis to produce a solid $\mathcal{S}$. Set up the definite integrals for the volume of $\mathcal{S}$ using (a) disks and (b) coaxial shells.



(a) (b) (c)

Figure 7.5.6:

SOLUTION Figure 7.5.6(a) shows the region $\mathcal{R}$ and Figure 7.5.6(b) shows the solid $\mathcal{S}$.

(a) If we use cross-sections perpendicular to the y-axis, as in the preceding section, we find that

$$\text{Volume} = \int_1^e \pi(\ln(y))^2\,dy.$$

This integrand has an elementary antiderivative, and we will learn how to find one in Chapter 8. Formula 66 (with $a = 1$) in the table on the inside cover of this book has $\int(\ln(x))^2\,dx = x((\ln(x))^2 - 2\ln(x) + 2)$, which you may check by differentiation. Thus

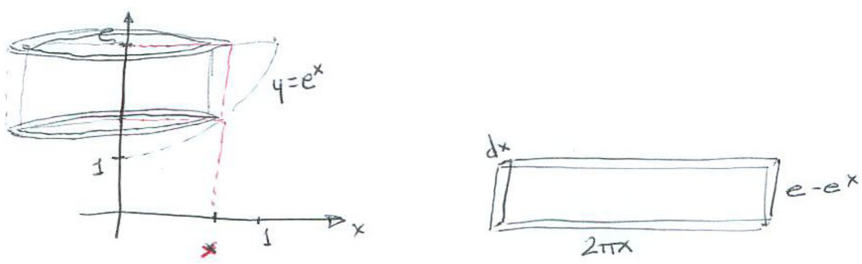
See Exercise 21

$$\text{Volume} = \pi(e - 2) \approx 2.2565.$$

(b) If we use cross-sections parallel to the x-axis, we meet a much simpler integration. The typical shell has radius x, height $e - e^x$, and thickness dx as shown in Figure 7.5.7(a).

The local approximation to the total volume of the shell is

$$\underbrace{2\pi x}_{\text{circumference}}\ \underbrace{(e - e^x)}_{\text{height}}\ \underbrace{dx}_{\text{thickness}},$$



(a) (b)

Figure 7.5.7:

so the volume of $\mathcal{S}$ is

$$\int_0^1 2\pi x\left(e - e^x\right)\,dx.$$

Now one needs an antiderivative of $2\pi x\left(e - e^x\right)$. In Chapter 8 we will learn how to do this, and we will find it is much easier to find than $\int \left(\ln(x)\right)\,dx$. The first part is trivial, $\int ex\,dx = \frac{e}{2}x^2$, and then formula 59 on the inside cover gives $\int xe^x\,dx = xe^x - e^x$. As expected, once again the volume is $\pi(e-2)$.

See Exercise 22

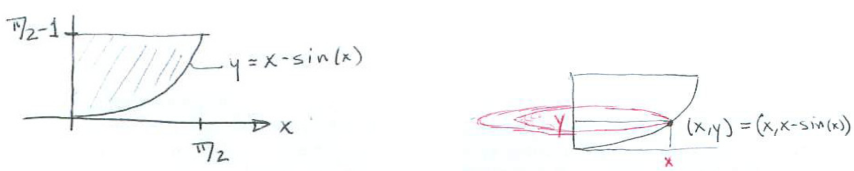
◇

In Example 1 both methods were feasible. In the next, the shell technique is clearly preferable.

It is not unusual to find one formulation much easier than the other.

EXAMPLE 2 The region $\mathcal{R}$ bounded by the line $y = \frac{\pi}{2} - 1$, the y-axis, and the curve $y = x - \sin(x)$ is revolved around the y-axis. Try to set up definite integrals for the volume of this solid using (a) disks and (b) coaxial shells.

The equation $y = x - \sin(x)$ is Kepler's equation, with $\mathrm{e} = 1$. See Exercise 23 on page 63.



(a) (b)

Figure 7.5.8:

SOLUTION The region $\mathcal{R}$ is displayed in Figure 7.5.8(a).

(a) To use the method of parallel cross-sections you would have to find the radius of the typical disk shown in Figure 7.5.8(b). The radius for each value of y is the value of x for which $x - \sin(x) = y$. In other words, we have to

For instance, when $y = 0$, then $x = 0$. When $y = \frac{\pi}{2} - 1$, then $x = \frac{\pi}{2}$.

express x as a function of y. This inverse function is not elementary, ending our hopes of using the FTC.

(b) On the other hand, the shell technique goes through smoothly. The typical shell, shown in Figure 7.5.9, has radius x and height $\frac{\pi}{2}-1-(x-\sin(x))$. The volume of the local approximation is

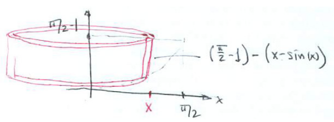


Figure 7.5.9:

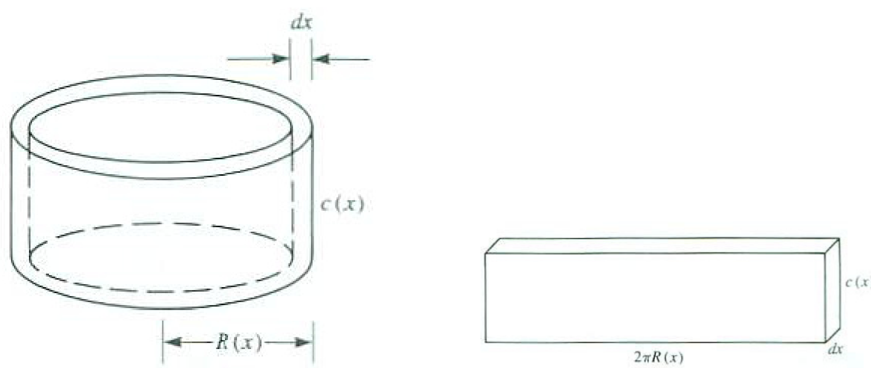
$$\underbrace{2\pi x}_{\text{circumference}}\ \underbrace{\left(\frac{\pi}{2}-1-(x-\sin(x))\right)}_{\text{height}}\ \underbrace{dx}_{\text{thickness}}.$$

The total volume of the bowl is then

$$\int_0^{\pi/2} 2\pi x\left(\frac{\pi}{2}-1-(x-\sin(x))\right)\,dx.$$

The value of this definite integral is found in Exercise 50 on page 774. ◇

Summary



(a) (b)

Figure 7.5.10:

The volume of a solid of revolution may be found by approximating the solid by concentric thin shells. The volume of such a shell is approximately $2\pi R(x)\ c(x)\ dx$. (See Figure 7.5.10.) The shell technique is often useful even when integration by cross-sections is difficult or impossible.

EXERCISES for Section 7.5 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4 draw a typical approximating cylindrical shell for the solid described, and set up a definite integral for the volume of the given solid. NOTE: When evaluating your definite integral, feel free to use the tables of antiderivatives in the inside covers of the text.

1.[R] The trapezoid bounded by $y = x$, $x = 1$, $x = 2$, and the x-axis is revolved around the x-axis.

2.[R] The trapezoid in Exercise 1 is revolved about the line $y = -3$.

(a) Repeat this problem when the trapezoid is revolved around the y-axis.

(b) Repeat this problem when the trapezoid is revolved around the $x = -3$.

3.[R] The triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ is revolved around the y-axis.

4.[R] The triangle in Exercise 3 is revolved about the line x-axis.

5.[R] Find a definite integral for the volume of the solid produced by revolving about the y-axis the finite region bounded by $y = x^2$ and $y = x^3$.

6.[R] Repeat Exercise 5, except the region is revolved around the x-axis.

7.[R] Set up a definite integral for the volume of the solid produced by revolving about the x-axis the finite region bounded by $y = \sqrt{x}$ and $y = \sqrt[3]{x}$.

8.[R] Repeat Exercise 7, except the region is revolved about the y-axis.

9.[R] Find a definite integral for the volume of the right circular cone of radius a and height h by the shell method.

10.[R] Set up a definite integral for the volume of the doughnut (ring, torus) produced by revolving the disk of radius a about a line L at a distance $b > a$ from its center. (See Figure 7.5.11.)

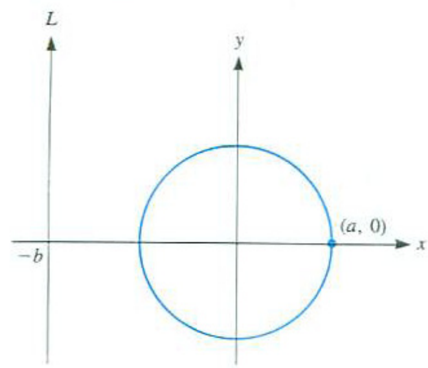


Figure 7.5.11:

11.[R] Let R be the region bounded by $y = x + x^3$, $x = 1$, $x = 2$, and the x-axis. Set up a definite integral for the volume of the solid produced by revolving R about (a) the x-axis and (b) the line $x = 3$.

12.[R] Set up a definite integral for the volume of the solid produced by revolving the region R in Exercise 11 about (a) the x-axis and (b) the line $y = -2$.

13.[R] Set up a definite integral for the volume of the solid of revolution formed by revolving the region bounded by $y = 2 + \cos(x)$, $x = \pi$, $x = 10\pi$, and the x-axis around (a) the y-axis and (b) the x-axis.

14.[R] The region below $y = \cos(x)$, above the x-axis, and between $x = 0$ and $x = \frac{\pi}{2}$ is revolved around the x-axis. Find a definite integral for the volume of the resulting solid of revolution by (a) parallel cross-sections and (b) concentric shells.

15.[R] Let R be the region below $y = 1/\left(1 + x^2\right)^2$ and above $[0, 1]$. Set up a definite integral for the volume of the solid produced by revolving R about the y-axis.

16.[R] The region between $y = e^{x^2}$, the x-axis, $x = 0$, and $x = 1$ is revolved about the y-axis.

(a) Set up a definite integral for the area of this region.

(b) Set up a definite integral for the volume of the solid produced.

NOTE: The FTC is of no use in evaluating the *area* of this region.

17.[R] The region R below $y = e^x\left(1 + \sin(x)\right)/x$ and above $[0, 10\pi]$ is revolved about the y-axis to produce a solid of revolution. (a) Find a definite integral for the volume of the solid by parallel cross-sections. (b) Find a definite integral for the volume of the solid by concentric shells. (c) Which definite integral do you think is easier to evaluate? Why?

18.[R] Let R be the region below $y = \ln(x)$ and above $[1, e]$. Find a definite integral for the volume of the solid produced by revolving R about (a) the x-axis and (b) the y-axis.

19.[R] Let R be the region below $y = 1/\left(x^2 + 4x + 1\right)$ and above $[0, 1]$. Find a definite integral for the volume of the solid produced by revolving R about the line $x = -2$.

20.[R] Let R be the region below $y = 1/\sqrt{2 + x^2}$ and above $[\sqrt{3}, \sqrt{8}]$. Set up a definite integral for the volume of the solid produced by revolving R about the (a) the x-axis and (b) the y-axis.

Exercises 21 and 22 complete Exercise 1. In that Example the region below $y = e$, above $y = e^x$, and to the right of the y-axis is revolved around the y-axis to form a solid $\mathcal{S}$.

21.[R] The volume of $\mathcal{S}$ using cross-sections perpendicular to the y-axis was found to be $\int_1^e \pi \left(\ln(y)\right)^2 \; dy$.

(a) Verify that $x\left((\ln(x))^2 - 2\ln(x) + 2\right)$ is an antiderivative of $(\ln(x))^2$.

(b) Find the volume of $\mathcal{S}$. Hint: Use FTC I.

22.[R] The volume of $\mathcal{S}$ using cross-sections parallel to the y-axis was found to be $\int_0^1 2\pi x \left(e - e^x\right) \; dx$.

(a) Verify that $xe^x - e^x$ is an antiderivative of xe^x.

(b) Find the volume of $\mathcal{S}$. Hint: Use FTC I.

23.[M] When we unrolled the shell as a carpet we pictured it as a rectangular solid whose faces meet at right angles. However, since the inner radius is x and the outer radius is $x + dx$ the circumference of the inside of the shell is less than the outer circumference.

(a) By viewing the shell as the difference between two circular cylinders, compute its exact volume.

(b) Show that this volume is $2\pi\left(x + \frac{dx}{2}\right)c(x)$.

This means that if we used $x + \frac{dx}{2}$ as our sampling number in the interval $[x, x + dx]$ instead of x, our local approximation to the volume of the shell would be exact.

The **kinetic energy** of a particle of mass m grams moving at a velocity of v centimeters per second is $mv^2/2$ ergs. Exercises 24 and 25 ask for the kinetic energy of rotating objects.

24.[M] A solid cylinder of radius r and height h centimeters has a uniform density of g grams per cubic centimeter. It is rotating at the rate of two revolutions per second around its axis.

(a) Find the speed of a particle at a distance x from the axis.

(b) Find a definite integral for the kinetic energy of the rotating cylinder.

25.[M] A solid ball of radius r centimeters has a uniform density of g grams per cubic centimeter. It is rotating around a diameter at the rate of three revolutions per second around its axis.

(a) Find the speed of a particle at a distance x from the diameter.

(b) Find a definite integral for the kinetic energy of the rotating ball.

26.[C] When a region $\mathcal{R}$ in the first quadrant is revolved around the y-axis, a solid of volume 24 is produced. When $\mathcal{R}$ is revolved around the line $x = -3$, a solid of volume 82 is produced. What is the area of $\mathcal{R}$?

27.[C] Let $\mathcal{R}$ be a region in the first quadrant. When it is revolved around the x-axis, a solid of revolution is produced. When it is revolved around the y-axis, another solid of revolution is produced. Give an example of such a region $\mathcal{R}$ with the property that the volume of the first solid *cannot* be evaluated by the FTC, but the volume of the second solid can be evaluated by the FTC.

7.6 Water Pressure Against a Flat Surface

This section shows how to use integration to compute the force of water against a submerged flat surface.

Introduction

Imagine the portion of the Earth's atmosphere directly above one square inch at sea level. That air forms a column some hundred miles high which weighs about 14.7 pounds. It exerts a pressure of 14.7 pounds per square inch (14.7 psi).

This pressure does not crush us because the cells in our body are at the same pressure. If we were to go into a vacuum, we would explode.

This is why astronauts wear pressurized suits.

The pressure inside a flat tire is 14.7 psi. When you pump up a bicycle tire so that the gauge reads 60 psi, the pressure is actually $60 + 14.7 = 74.7$ psi. The tire must be strong enough to avoid bursting.

Next imagine diving into a lake and descending 33 feet (10 meters). Extending that 100-mile-high column 33 feet into the water adds $(33)(12)(0.036227) = 14.7$ pounds of water. The pressure is now twice 14.7 psi. The pressure is now twice 14.7, or 29.4 psi. You cannot escape that pressure by turning your body, since at a given depth the pressure is the same in all directions.

One cubic foot of water weighs 62.6 pounds, so one cubic inch weighs $\frac{62.6}{1728} = 0.036227$ pounds and the density is 0.036227 pounds per cubic inch.

Pressure and force are closely related. If the force is the same throughout a region, then the pressure is simply "total force divided by area":

$$\text{pressure} = \frac{\text{force}}{\text{area}}.$$

Equivalently,

$$\text{force} = \text{pressure} \times \text{area}.$$

Thus, when the pressure is constant in a plane region it is easy to find the total force against it: multiply the pressure and the area of the region.

If the pressure varies in the region, we must make use of integration.

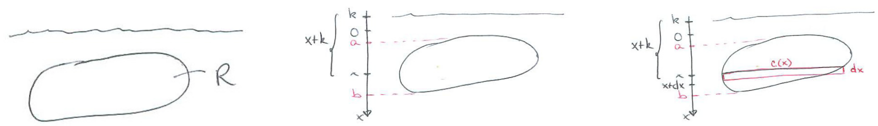
Using an Integral to Find the Force of Water

We will see how to find the total force on a flat submerged object due to the water. We will disregard the pressure due to the atmosphere. (See Figure 7.6.1(a).)

At a depth of h inches, water exerts a pressure of $0.036h$ psi. Therefore the water exerts a force on a flat <u>horizontal</u> object of area $\mathcal{A}$ square inches, at a depth of h inches equal to $0.036h\mathcal{A}$ pounds.

We approximate the density of water as 0.036 pounds per cubic inch.

To deal with, say, a vertical submerged surface takes more calculation, since the pressure is not constant over that surface. Imagine the surface $\mathcal{R}$, shown



(a) (b) (c)

Figure 7.6.1:

in Figure 7.6.1(b). Introduce a vertical x-axis, pointed down, with its origin $\mathcal{O}$, a distance k below the water's surface. $\mathcal{R}$ lies between lines corresponding to $x = a$ and $x = b$. The depth of the water corresponding to x is not x but $x + k$.

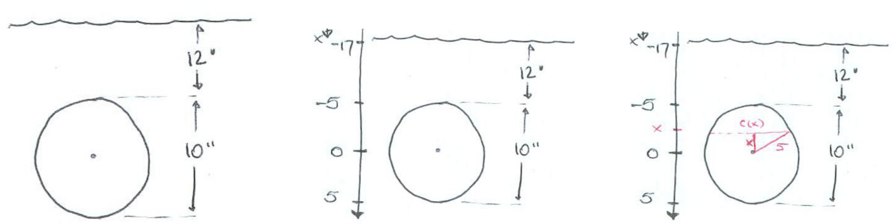
If the origin is at the water's surface, then $k = 0$.

As usual, we will find the local approximation of the force by considering a narrow horizontal strip corresponding to the interval $[x, x + dx]$ of the x-axis, as in Figure 7.6.1(c). Letting $c(x)$ denote the cross-sectional length, we see that the force of the water on this strip is approximately

$$\underbrace{(0.036)}_{\text{density of } H_2O} \quad \underbrace{(x + k)}_{\text{depth}} \quad \underbrace{c(x)\ dx}_{\text{area of strip}} \quad \text{pounds.}$$

Therefore

$$\boxed{\text{Force against } \mathcal{R} \text{ is } 0.036 \int_a^b (x + k)c(x)\ dx \text{ pounds.}}$$



(a) (b) (c)

Figure 7.6.2:

EXAMPLE 1 A circular tank is submerged in water. An end is a disk 10 inches in diameter. The top of the tank is 12 inches below the surface of the water. Find the force against one end.

SOLUTION The end of the tank is shown in Figure 7.6.2(a). Introduce

This placement of $\mathcal{O}$ will make it easier to compute the cross-section lengths.

a vertical x-axis with its origin $\mathcal{O}$ level with the center of the disk. (See Figure 7.6.2(b).) To find the cross-section $c(x)$ we use Figure 7.6.2(c).

By the Pythagorean Theorem applied to the right triangle in Figure 7.6.2(c) we have

For any number x, $|x|^2 = x^2$.

$$\left(\frac{c(x)}{2}\right)^2 + |x|^2 = 5^2.$$

Thus

$$(c(x))^2 + 4x^2 = 100.$$

So

$$c(x) = \sqrt{100 - 4x^2}.$$

Having found the cross-section as a function of x, we still must find the depth as a function of x. To do this, inspect Figure 7.6.3.

The depth $\overline{AC}$ equals $\overline{AB} + \overline{BC} = 12 + (x - (-5)) = 17 + x$.

As a check, let $x = 0$, when the depth is clearly 17 inches.

We have

$$\text{Local Estimate of Force} = \underbrace{(0.036)(x + 17)}_{\text{pressure}}\,\underbrace{\sqrt{100 - 4x^2}\,dx}_{\text{area}}.$$

From this we obtain

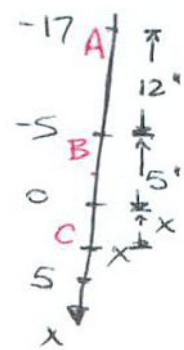


Figure 7.6.3:

$$\begin{aligned}\text{Total Force} &= \int_{-5}^{5} (0.036)(x+17)\sqrt{100-4x^2}\,dx \text{ pounds}\\ &= 0.036\int_{-5}^{5} x\sqrt{100-4x^2}\,dx + 0.036\int_{-5}^{5} 17\sqrt{100-4x^2}\,dx \text{ pounds.}\end{aligned}$$

The first integral is 0 because the integrand, $x\sqrt{100-4x^2}$, is an odd function and the interval of integration is symmetric about $x = 0$. The integrand in the second integral is even, so

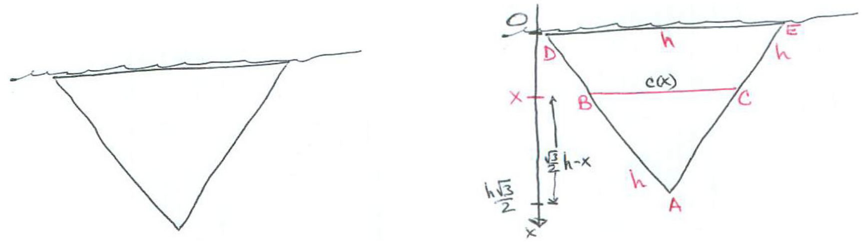
$$\int_{-5}^{5} \sqrt{100-4x^2}\,dx = 2\int_{0}^{5} \sqrt{100-4x^2}\,dx = 4\int_{0}^{5} \sqrt{25-x^2}\,dx = 4\,(\text{Area of one quarter of disk of radius } 5) = 4\left(\frac{1}{4}\pi\right.$$

Thus,

$$\text{Total Force} = (0.036)(17)(25\pi) \text{ pounds} \approx 48 \text{ pounds.}$$

◇

EXAMPLE 2 Figure 7.6.4(a) shows a submerged equilateral triangle of side h. Find the force of water against it.



(a) (b)

Figure 7.6.4:

SOLUTION In this case we place the origin of the vertical axis at the surface of the water (see Figure 7.6.4(b)). To set up an integral we must compute $c(x)$. Note $\frac{\sqrt{3}h}{2}$ is marked on the x-axis; it is the length of an altitude in the triangle.

The similar triangles ABC and ADE give us

$$\frac{c(x)}{h} = \frac{\frac{\sqrt{3}}{2}h - x}{\frac{\sqrt{3}}{2}h}.$$

Observe that $c(0) = h$ and $c(\frac{\sqrt{3}}{2}h) = 0$ and c is linear, which agree with Figure 7.6.4(b).

Thus,

$$c(x) = h - \frac{2x}{\sqrt{3}}.$$

The local estimate of force is therefore

$$\underbrace{0.036x}_{\text{pressure}} \underbrace{\left(h - \frac{2x}{\sqrt{3}}\right) dx}_{\text{area}}.$$

Hence

$$\begin{aligned}
\text{Total Force} &= \int\limits_0^{\frac{\sqrt{3}}{2}h} 0.036x\left(h - \frac{2x}{\sqrt{3}}\right) dx = 0.036 \int\limits_0^{\frac{\sqrt{3}}{2}h} \left(hx - \frac{2x^2}{\sqrt{3}}\right) dx \\
&= 0.036\left(h\frac{x^2}{2} - \frac{2}{\sqrt{3}}\frac{x^3}{3}\right)\Bigg|_0^{\frac{\sqrt{3}}{2}h} = 0.036\frac{h^3}{8} \text{ pounds.}
\end{aligned}$$

◇

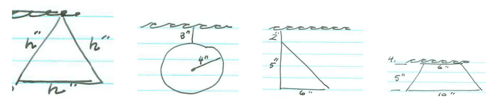
Summary

We introduced the notion of water pressure defined as "force divided by area" or "force per unit area." If the pressure is constant over a flat region of area

$\mathcal{A}$, the force is the product: pressure times area. When $p(x)$ is the pressure and $c(x)$ is the length of the typical horizontal cross-section, then $p(x)c(x)\ dx$ is a local approximation to the force. The water pressure $p(x)$ is 0.036 times the depth. The dimensions are in inches and the force is in pounds.

EXERCISES for Section 7.6 *Key:* R–routine, M–moderate, C–challenging

A cubic inch of water weighs 0.036 pounds. (All dimensions are in inches.)



(a) (b) (c) (d)

Figure 7.6.5:

In Exercises 1 to 4 find a definite integral for the force of water on the indicated surface.

1.[R] The triangular surface in Figure 7.6.5(a).

2.[R] The circular surface in Figure 7.6.5(b).

3.[R] The trapezoidal surface in Figure 7.6.5(c).

4.[R] The triangular surface in Figure 7.6.5(d).

In Exercises 5 and 6 the surfaces are tilted like the bottoms of many swimming pools. Find the force of the water against them.

5.[M] The surface is an a by b rectangle inclined at an angle of 30° ($\pi/6$ radians) to the horizontal. The top of the surface is at a depth k. (See Figure 7.6.6.)

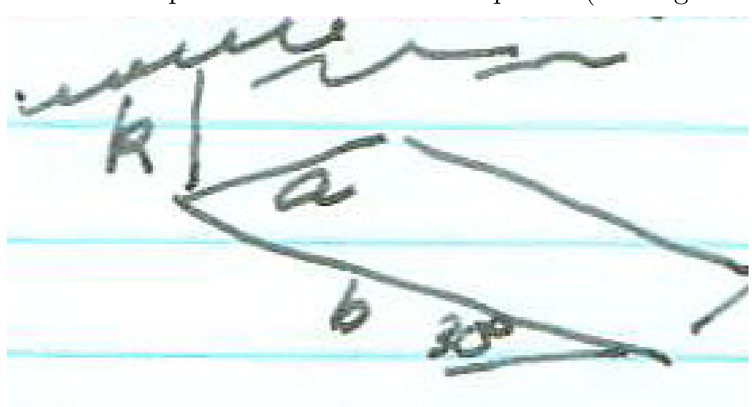


Figure 7.6.6:

6.[M] The surface is a disk of radius r tilted at an angle of 45° ($\pi/4$ radians) to the horizontal. Its top is at the surface of the water.

7.[M] A vertical disk is totally submerged. Show that the force of the water against it is the same as the product of its area and the pressure at its center.

8.[C] If the region in Exercise 7 is not vertical, is the same conclusion true?

9.[C] Let $\mathcal{R}$ be a convex planar region. $\mathcal{R}$ is called **centrally symmetric** if it contains a point P such that P is the midpoint of every chord of $\mathcal{R}$ that passes through P. For instance, a parallelogram is centrally symmetric. No triangle is. Now, assume that a centrally symmetric region is placed vertically in water and is completely submerged. Show that the force against it equals the product of its area and the pressure at P.

10.[C] Why is finding volume by shells essentially the same as finding the force against a submerged object?

7.7 Work

In this section we treat the work accomplished by a force operating along a line, for example the work done when you stretch a spring. If the force has the *constant* value F and it operates over a distance s in the *direction* of the force, then the work W accomplished is simply

$$\text{Work} = \text{Force} \cdot \text{Distance} \qquad \text{or} \qquad W = F \cdot s.$$

If force is measured in newtons and distance in meters, work is measured in newton-meters or joules. For example, the force needed to lift a mass of m kilograms at the surface of the earth is about $9.8m$ newtons.

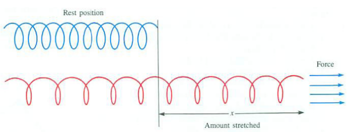


Figure 7.7.1:

A weightlifter who raises 100 kilograms a distance of 0.5 meter accomplishes $9.8(100)(0.5) = 490$ joules of work. On the other hand, the weightlifter who just carries the barbell from one place to another in the weightlifting room, without raising or lowering it, accomplishes no work because the barbell was moved a distance zero in the direction of the force.

The Stretched Spring

Hooke's law says a spring's force is proportional to the distance it is stretched.

As you stretch a spring (or rubber band) from its rest position, the further you stretch it the harder you have to pull. According to Hooke's law, the force you must exert is proportional to the distance that the spring is stretched, as shown in Figure 7.7.1. In symbols, $F = kx$, where F is the force and x is the distance from the rest position.

Because the force is *not* constant, we cannot compute the work accomplished just by multiplying force times distance. As usual, we need an integral, as the next example illustrates.

EXAMPLE 1 A spring is stretched 0.5 meter longer than its rest length. The force required to keep it at that length is 3 newtons. Find the total work accomplished in stretching the spring 0.5 meter from its rest position.

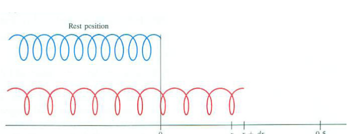


Figure 7.7.2:

SOLUTION Let us estimate the work involved in stretching the spring from x to $x + dx$. (See Figure 7.7.2.)

The distance dx is small. As the end of the spring is stretched from x to $x + dx$, the force is almost constant. Since the force is proportional to x, it is of the form kx for some constant k. We know that the force, F, is 3 when $x = 0.5$, so

$$F = kx \qquad \text{gives} \qquad 3 = k(0.5) \qquad \text{which implies} \qquad k = 6.$$

The work accomplished in stretching the spring from x to $x + dx$ is then approximately

$$\underbrace{kx}_{\text{force}} \cdot \underbrace{dx}_{\text{distance}} \text{ joule.}$$

Hence the total work is

$$\int_a^b kx\,dx = \int_0^{0.5} 6x\,dx = 3x^2\Big|_0^{0.5} = 0.75 \text{ joule.}$$

◇

Work in Launching a Rocket

The force of gravity that the earth exerts on an object diminishes as the object gets further away from the earth. The work required to lift an object 1 foot at sea level is greater than the work required to lift the same object the same distance at the top of Mt. Everest. However, the difference in altitudes is so small in comparison to the radius of the earth that the difference in work is negligible. On the other hand, when an object is rocketed into space, that the force of gravity diminishes with distance from the center of the earth is critical.

The earth's surface is about $4,000$ miles from its center.

According to Newton, the force of gravity on a given mass is proportional to the reciprocal of the square of the distance of that mass from the center of the earth. That is, there is a constant k such that the gravitational force at distance r from the center of the earth, $F(r)$, is given by

$$f(r) = \frac{k}{r^2}.$$

(See Figure 7.7.3.)

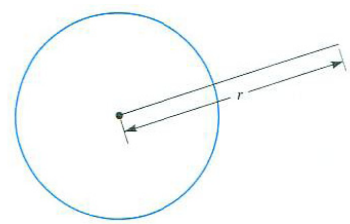


Figure 7.7.3:

WARNING It is important to remember that r is "distance to the center of the earth," not "distance to the surface."

EXAMPLE 2 How much work is required to lift a 1 pound payload from the surface of the earth to the moon, which is about $240,000$ miles away?

SOLUTION The work necessary to lift an object a distance x against a constant vertical force F is the product of force times distance:

$$\text{Work} = F \cdot x.$$

The unit for work is **joule**. 1 joule = 1 newton meter = 1 watt second = 0.7376 foot pound.

Since the gravitational pull of the earth on the payload *changes* with distance from the center of the earth, an integral will be needed to express the total work.

The payload weighs 1 pound at the surface of the earth. The farther it is from the center of the earth, the less it weighs, for the force of the earth on the mass is inversely proportional to the square of the distance of the mass from the center of the earth. Thus the force on the payload is given by k/r^2 pounds, where k is a constant, which will be determined in a moment, and r is the distance in miles form the payload to the center of the earth. When $r = 4,000$ (miles), the force is 1 pound; thus

$$1 \text{ pound} = \frac{k}{(4,000 \text{ miles})^2}.$$

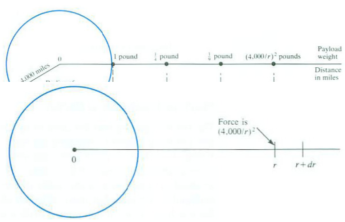


Figure 7.7.5:

From this it follows that $k = 4,000^2$, and therefore the gravitational force on a 1-pound mass is, in general, $(4,000/r)^2$ pounds. As the payload recedes from the earth, it loses weight (but not mass), as recorded in Figure 7.7.4. The work done in lifting the payload from a distance r to a distance $r + dr$ from the center of the earth is approximately

$$\underbrace{\left(\frac{4,000}{r}\right)^2}_{\text{force}} \underbrace{(\qquad\qquad dr)}_{\text{distance}} \text{ miles-pounds.}$$

(See Figure 7.7.5.)

Hence the work required to move the 1 pound mass from the surface of the earth to the moon is given by the integral

$$\begin{aligned}\int_{4,000}^{240,000} \left(\frac{4,000}{r}\right)^2 dr &= -\frac{4,000^2}{r}\Bigg|_{4,000}^{240,000} = -4,000^2\left(\frac{1}{240,000} - \frac{1}{4,000}\right) \\ &= -\frac{4,000}{60} + 4,000 \approx 3,933 \text{ miles-pounds}\end{aligned}$$

$= 2.8154 \times 10^7$ joules.

The work is just a little less than if the payload were lifted 4,000 miles against a constant gravitational force equal to that at the surface of the earth. ◇

Summary

The work accomplished by a constant force F that moves an object a distance x in the direction of the force is the product Fx, "force times distance." The work by a variable force, $F(x)$, moving an object over the interval $[a, b]$ is measured by an integral $\int_a^b F(x)\, dx$.

EXERCISES for Section 7.7 *Key:* R–routine, M–moderate, C–challenging

1.[R] A spring is stretched 0.20 meters from its rest length. The force required to keep it at that length is 5 newtons. Assuming that the force of the spring is proportional to the distance it is stretched, find the work accomplished in stretching the spring

(a) 0.20 meters from its rest length;

(b) 0.30 meters from its rest length.

2.[R] A spring is stretched 3 meters from its rest length. The force required to keep it at that length is 24 newtons. Assume that the force of the spring is proportional to the distance it is stretched. Find the work accomplished in stretching the spring

(a) 3 meters from its rest length;

(b) 4 meters from its rest length.

3.[R] Suppose a spring does not obey Hooke's law. Instead, the force it exerts when stretched x meters from its rest length is $F(x) = 3x^2$ Newtons. Find the work done in stretching the spring 0.80 meter from its rest length.

4.[R] Suppose a spring does not obey Hooke's law. Instead, the force it exerts when stretched x meters from its rest length is $F(x) = 2\sqrt{x}$ Newtons. Find the work done in stretching the spring 0.50 meter from its rest length.

5.[R] How much work is done in lifting the 1 pound payload the first $4,000$ miles of its journey to the moon? NOTE: See Example 2.

6.[R] If a mass that weighs 1 pound at the surface of the earth were launched from a position $20,000$ miles from the center of the earth, how much work would be required to send it to the moon ($240,000$ miles from the center of the earth)?

7.[R] Assume that the force of gravity obeys an inverse cube law, so that the force on a 1 pound payload a distance r miles from the center of the earth ($r \geq 4,000$) is $(4,000/r)^3$ pounds. How much work would be required to lift a 1 pound payload from the surface of the earth to the moon?

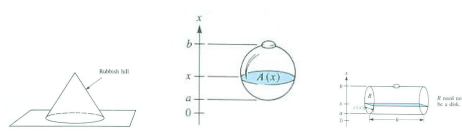
8.[R] Geologists, when considering the origin of mountain ranges, estimate the energy required to lift a mountain up from sea level. Assume that two mountains are composed of the same type of matter, which weighs k pounds per cubic foot. Both are right circular cones in which the height is equal to the radius. One mountain is twice as high as the other. The base of each is at sea level. If the work required to lift the matter in the smaller mountain above sea level is W, what is the corresponding work for the larger mountain?

9.[R] Assume that Mt. Everest has a shape of a right circular cone of height $30,000$ feet and radius $150,000$ feet, with unifrom density of 200 pounds per cubic foot.

(a) How much work was required to lift the material in Mt. Everest if it was initially all at sea level?

(b) How does this work compare with the energy of a 1 megaton hydrogen bomb? (One megaton is the energy in a million tons of TNT: about 3×10^{14} foot-pounds.)

10.[R] A town in a flat valley made a conical hill out of its rubbish, as shown in Figure 7.7.6(a). The work requireed to lift all the rubbish was W. Happy with the result, the town decided to make another hill with twice the volume, but of the same shape. How much work will be required to build this hill? Explain.



(a) (b) (c)

Figure 7.7.6:

11.[R] A container is full of water which weighs 64.2 pounds per cubic foot. All the water is pumped out of an opening at the top of the container. Develop a definite integral for the work accomplished. Hint: The integral involves only a, b, and $A(x)$, the cross-sectional area shown in Figure 7.7.6(b).

12.[R] A horizontal tank in the form of a cylinder with base R is full of water. The cylinder has height h feet. (See Figure 7.7.6(c).) Develop a definite integral for the total work accomplished when all the water is pumped out an opening at the top. Hint: Express the integral in terms of a, b, $c(x)$, and h.

Exercises 13 to 17 review differentiation. In each case compute the derivative of the given function.

13.[R] $\ln\left(x + \sqrt{a^2 + x^2}\right)$

14.[R] $\frac{1}{2ab} \ln\left|\frac{a+bx}{a-bx}\right|$

15.[R] $\frac{x^4}{8} - \left(\frac{x^3}{4} - \frac{3x}{8}\right)\sin(2x)$

16.[R] $x - \ln\left(1 + e^x\right)$

17.[R] $\frac{e^{ax}}{a^2+1}\left(a\sin(x) - \cos(x)\right)$

7.8 Improper Integrals

This section develops the analog of a definite integral when the interval of integration is infinite or the integrand becomes arbitrarily large in the interval of integration. The definition of a definite integral does not cover these cases.

Improper Integrals: Interval Unbounded

A question about areas will introduce the notion of an "improper integral." Figure 7.8.1 shows the region under $y = 1/x$ and above the interval $[1, \infty)$. Figure 7.8.2 shows the region under $y = 1/x^2$ and above the same interval.

Figure 7.8.1:

Let us compute the areas of the two regions. We might be tempted to say that the area in Figure 7.8.1 is $\int_1^\infty f(x)\ dx$. Unfortunately, the symbol $\int_1^\infty f(x)\ dx$ has not been given any meaning so far in this book. The definition of the definite integral $\int_a^b f(x)\ dx$ involves a limit of sums of the form

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{x-1}),$$

where each $x_i - x_{i-1}$ is the length of an interval $[x_{i-1}, x_i]$. If you cut the interval $[1, \infty)$ into a finite number of intervals, then at least one section has infinite length, and such a sum is meaningless.

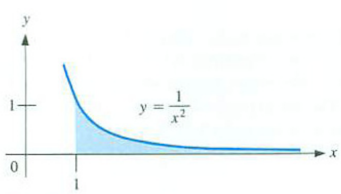


Figure 7.8.2:

It does make sense, however, to find the area of that part of the region in Figure 7.8.1 from $x = 1$ to $x = b$, where $b > 1$, and find what happens to that area as $b \to \infty$. To do this, first calculate $\int_1^b (1/x)\ dx$:

$$\int_1^b \frac{dx}{x} = \ln(x)\big|_1^b = \ln(b) - \ln(1) = \ln(b).$$

Then

$$\lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln(b) = \infty.$$

So the area of the region in Figure 7.8.1 is infinite.

Next, examine the area of the region in Figure 7.8.2. We first find

$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x}\bigg|_1^b = -\frac{1}{b} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{b}$$

Thus,

$$\lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = 1.$$

In this case the area is finite. Though the regions in Figures 7.8.1 and 7.8.2 look alike, one has an infinite area, and the other, a finite area. This contrast suggests the following definitions.

DEFINITION *(Convergent improper integral* $\int_a^\infty f(x)\ dx$.*)* Let f be continuous for $x \geq a$. If $\lim_{b\to\infty} \int_a^b f(x)\ dx$ exists, the function f is said to have a **convergent improper integral** from a to ∞. The value of the limit is denoted by $\int_a^\infty f(x)\ dx$:

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx.$$

We saw that $\int_1^\infty\ dx/x^2$ is a convergent improper integral with value 1.

DEFINITION *(Divergent improper integral* $\int_a^\infty f(x)\ dx$.*)* Let f be a continuous function for $x \geq a$. If $\lim_{b\to\infty} \int_a^b f(x)\ dx$ does not exist, the function f is said to have a **divergent improper integral** from a to ∞.

As we saw, $\int_1^\infty\ dx/x$ is a divergent improper integral.

The improper integral $\int_1^\infty\ dx/x$ is divergent because $\int_1^b\ dx/x \to \infty$ as $b \to \infty$. But an improper integral $\int_a^\infty f(x)\ dx$ can be divergent without being infinite. Consider, for instance, $\int_0^\infty \cos(x)\ dx$. We have

$$\int_0^b \cos(x)\ dx = \sin(x)|_0^b = \sin(b).$$

As $b \to \infty$, $\sin(b)$ does not approach a limit, nor does it become arbitrarily large. As $b \to \infty$, $\sin(b)$ just keeps going up and down in the range -1 to 1 infinitely often. Thus $\int_0^\infty \cos(x)\ dx$ is divergent.

The improper integral $\int_{-\infty}^b f(x)\ dx$.

The improper integral $\int_{-\infty}^b f(x)\ dx$ is defined similarly:

$$\int_{-\infty}^b f(x)\ dx = \lim_{a\to-\infty} \int_a^b f(x)\ dx.$$

If the limit exists, $\int_{-\infty}^b f(x)\ dx$ is a *convergent* improper integral. If the limit does not exist, it is a *divergent* improper integral.

The improper integral $\int_{-\infty}^\infty f(x)\ dx$

To deal with improper integrals over the entire x-axis, define

$$\int_{-\infty}^{\infty} f(x)\ dx$$

to be the sum

$$\int_{-\infty}^{0} f(x)\ dx + \int_{0}^{\infty} f(x)\ dx,$$

which will be called *convergent* if both

$$\int_{-\infty}^{0} f(x)\ dx \qquad \text{and} \qquad \int_{0}^{\infty} f(x)\ dx$$

are convergent. If at least one of the two is divergent, $\int_{-\infty}^{\infty} f(x)\ dx$ will be called divergent.

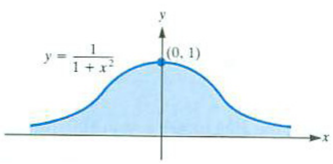


Figure 7.8.3:

EXAMPLE 1 Is the area of the region bounded by the curve $y = 1/(1+x^2)$ and the x-axis finite or infinite (see Figure 7.8.3).

SOLUTION The area in question equals $\int_{-\infty}^{\infty} dx/(1+x^2)$. Now,

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b\to\infty} (\tan^{-1}(b) - \tan^{-1}(0)) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Because $1/(1+x^2)$ is an even function, we deduce immediately that

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi,$$

the integral is convergent and the area in question is π. ◇

Shorthand Notation for $\int_a^{\infty} f(x)\ dx$

OBSERVATION *(Shorthand Notation for $\int_a^{\infty} f(x)\ dx$)* If $\int_a^{\infty} f(x)\ dx$ is convergent and $F(x)$ is an antiderivative of $f(x)$, then $\int_a^{\infty} f(x)\ dx = \lim_{b\to\infty} F(b) - F(a)$. In these situation we could write

$$\int_{a}^{\infty} f(x)\ dx = F(x)|_a^{\infty}$$

where it is understood that $F(\infty)$ is short for $\lim_{b\to\infty} F(b)$.

Comparison Test for Convergence of $\int_a^\infty f(x)\ dx$, $f(x) \geq 0$

The integral $\int_0^\infty e^{-x^2}\ dx$ is important in statistics. Is it convergent or divergent? We cannot evaluate $\int_0^b e^{-x^2}\ dx$ by the Fundamental Theorem since e^{-x^2} does not have an elementary antiderivative. Even so, there is a way of showing that $\int_0^\infty e^{-x^2}\ dx$ is convergent without finding its exact value. The method is described in Theorem 1.

Theorem. *Comparison test for convergence of improper integrals. Let $f(x)$ and $g(x)$ be continuous functions for $x \geq a$. Assume that $0 \leq f(x) \leq g(x)$ and that $\int_a^\infty g(x)\ dx$ is convergent. Then $\int_a^\infty f(x)\ dx$ is convergent and*

$$\int_a^\infty f(x)\ dx \leq \int_a^\infty g(x)\ dx.$$

In geometric terms, it asserts that if the area under $y = g(x)$ is finite, so is the area under $y = f(x)$. (See Figure 7.8.4.)

A similar convergence test holds for $g(x) \leq f(x) \leq 0$. If $\int_a^\infty g(x)\ dx$ converges, so does $\int_a^\infty f(x)\ dx$.

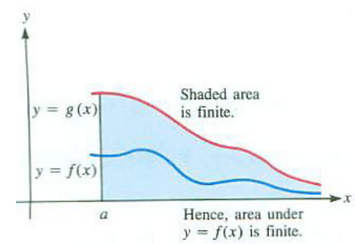


Figure 7.8.4:

EXAMPLE 2 Show that $\int_0^\infty e^{-x^2}\ dx$ is convergent and put a bound on its value.

SOLUTION Since e^{-x^2} does not have an elementary antiderivative, we compare $\int_0^\infty e^{-x^2}\ dx$ to an improper integral that we know converges.

For $x \geq 1, x^2 \geq x$; hence $e^{-x^2} \leq e^{-x}$. (See Figure 7.8.5.) Now,

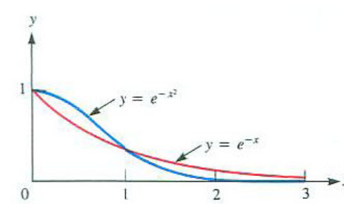


Figure 7.8.5:

$$\int_1^b e^{-x}\ dx = -e^{-x}\Big|_1^b = e^{-1} - e^{-b}.$$

Thus

$$\lim_{b\to\infty} \int_1^b e^{-x}\ dx = \frac{1}{e}$$

and the improper integral $\int_1^\infty e^{-x}\ dx$ is convergent.

The comparison test for convergence tells us that $\int_1^\infty e^{-x^2}\ dx$ is also convergent. Furthermore,

$$\int_1^\infty e^{-x^2}\ dx \leq \int_1^\infty e^{-x}\ dx = \frac{1}{e}.$$

In Exercise 34 of Section 17.3 we show that $\int_0^\infty e^{-x^2}\,dx$ equals $\sqrt{\pi/2} \approx 1.25331$.

Thus

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx \le \int_0^1 e^{-x^2}\,dx + \frac{1}{e}.$$

Since $e^{-x^2} \le 1$ for $0 < x \le 1$, we conclude that

$$\int_0^\infty e^{-x^2}\,dx \le 1 + \frac{1}{e}.$$

$\diamond$

Comparison Test for Divergence of $\int_a^\infty f(x)\,dx$.

Theorem. *Comparison test for divergence of improper integrals. Let $f(x)$ and $g(x)$ be continuous functions for $x \ge a$. Assume that $0 \le g(x) \le f(x)$ and that $\int_a^\infty g(x)\,dx$ is divergent. Then $\int_a^\infty f(x)\,dx$ is also divergent.*

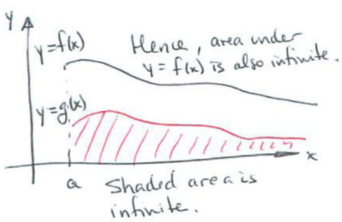


Figure 7.8.6:

A glance at Figure 7.8.6 suggests why this theorem is true. The area under $f(x)$ is larger than the area under $g(x)$. When the area under $g(x)$ is infinite, the area under f must also be infinite.

EXAMPLE 3 Show that $\int_1^\infty (x^2+1)/x^3\,dx$ is divergent.

SOLUTION For $x > 0$,

$$\frac{x^2+1}{x^3} > \frac{x^2}{x^3} = \frac{1}{x}.$$

Since $\int_1^\infty \frac{dx}{x} = \infty$, it follows that $\int_1^\infty (x^2+1)/x^3\,dx = \infty$. $\diamond$

Convergence of $\int_a^\infty f(x)\,dx$ When $\int_a^\infty |f(x)|\,dx$ Converges

Is $\int_0^\infty e^{-x}\sin(x)\,dx$ convergent or divergent? Because $\sin(x)$ takes on both positive and negative values, the integrand is not always positive, nor is it always negative. So we can't just compare it with $\int_0^\infty e^{-x}\,dx$.

The next theorem provides a way to establish the convergence of $\int_a^\infty f(x)\,dx$ when $f(x)$ is a function that takes on both positive and negative values. It says that if $\int_a^\infty |f(x)|\,dx$ converges, so does $\int_a^\infty f(x)\,dx$. The argument for this depends on showing that the "negative and positive parts of the function" both have convergent integrals.

Theorem 7.8.1. *Absolute-convergence test for improper integrals. If $f(x)$ is continuous for $x \ge a$ and $\int_a^\infty |f(x)|\,dx$ converges to the number L, then $\int_a^\infty f(x)\,dx$ is convergent and converges to a number between L and $-L$.*

Proof

We will break the function $f(x)$ into two continuous functions that do not change sign. That will enable us to use our comparison tests. Figure 7.8.7 shows the graphs of $y = f(x)$ and four functions closely related to $f(x)$.

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \text{ is positive} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad h(x) = \begin{cases} f(x) & \text{if } f(x) \text{ is negative} \\ 0 & \text{otherwise} \end{cases}$$

Note that $f(x) = g(x) + h(x)$, and that each of $g(x)$ and $h(x)$ is continuous for $x > a$. We will show that $\int_a^\infty g(x)\, dx$ and $\int_a^\infty h(x)\, dx$ both converge.

First, since $\int_a^\infty |f(x)|\, dx$ converges, has value L, and $0 \le g(x) \le |f(x)|$, we conclude that $\int_a^\infty g(x)\, dx$ converges, and the value of the integral is a nonnegative number A between 0 and L:

$$0 \le A \le \int_a^\infty |f(x)|\, dx = L.$$

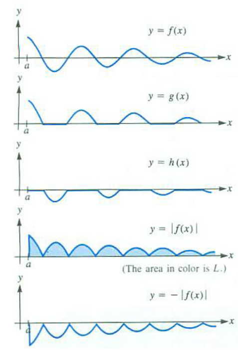


Figure 7.8.7:

Second, since $\int_a^\infty -|f(x)|\, dx$ converges, has value $-L$, and $0 \ge h(x) \ge -|f(x)|$, it follows that $\int_a^\infty h(x)\, dx$ converges to a nonpositive number B between $-L$ and 0:

$$0 \ge B \ge \int_a^\infty |f(x)|\, dx = -L.$$

Thus $\int_a^\infty f(x)\, dx = \int_a^\infty (g(x)+h(x))\, dx$ converges to $A+B$, which is a number somewhere in the interval $[-L, L]$. •

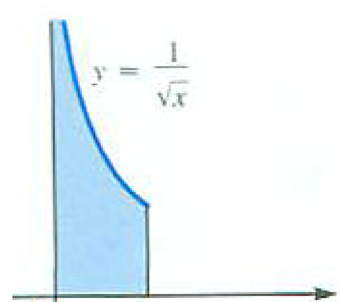
EXAMPLE 4 Show that $\int_0^\infty e^{-x}\sin(x)\, dx$ is convergent.

SOLUTION Since $|\sin(x)| \le 1$, we have $|e^{-x}\sin(x)| \le e^{-x}$. Now, $\int_0^\infty e^{-x}\, dx$ is convergent, as we saw in Example 2. Thus $\int_0^\infty e^{-x}\sin(x)\, dx$ is convergent. ◇

See Exercise 29.

Improper Integrals: Integrand Unbounded

The second type of improper integral is $\int_a^b f(x)\, dx$ in which $f(x)$ is unbounded in an interval $[a, b]$. For any partition of $[a, b]$, the approximating sum $\sum_{i=1}^n f(c_i)(x_i - x_{i-1})$ can be made arbitrarily large when c_i is chosen so that $f(c_i)$ is very large. The next example shows how to get around this difficulty.



See Exercise 38.

Figure 7.8.8:

EXAMPLE 5 Determine the area of the region bounded by $y = 1/\sqrt{x}$, $x = 1$, and the coordinate axes shown in Figure 7.8.8.

SOLUTION Resist for the moment the temptation to write "Area $= \int_0^1 1/\sqrt{x}\,dx$". The integral $\int_0^1 1/\sqrt{x}\,dx$ is not defined since its integrand is unbounded in $[0,1]$. Instead, consider the behavior of $\int_t^1 1/\sqrt{x}\,dx$ as t approaches 0 from the right. Since

$$\int_t^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_t^1 = 2\sqrt{1} - 2\sqrt{t} = 2(1-\sqrt{t}),$$

it follows that

$$\lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}} = 2.$$

The area in question is 2.

In Exercise 30 the same value for the area is obtained by taking horizontal cross-sections and evaluating an improper integral from 0 to ∞. $\diamond$

The reasoning in Example 5 motivates the definition of the second type of improper integral, in which the integrand rather than the interval is unbounded.

Convergent and Divergent Improper Integrals $\int_a^b f(x)\,dx$.

DEFINITION *(Convergent and Divergent Improper Integrals $\int_a^b f(x)\,dx$.)* Let f be continuous at every number in $[a,b]$ except at a. If $\lim_{t\to a^+}\int_t^b f(x)dx$ exists, the function f is said to have a **convergent improper integral** from a to b. The value of the limit is denoted $\int_a^b f(x)\,dx$.

If $\lim_{t\to a^+}\int_t^b f(x)dx$ does not exist, the function f is said to have a **divergent improper integral** from a to b; in brief, $\int_a^b f(x)\,dx$ does not exist.

In a similar manner, if f is not defined at b, define $\int_a^b f(x)\,dx$ as $\lim_{t\to b^-}\int_a^t f(x)dx$, if this limit exists.

As Example 5 showed, the improper integral $\int_0^1 1/\sqrt{x}\,dx$ is convergent and has the value 2.

A "proper" integral is a definite integral.

More generally, if a function $f(x)$ is not defined at certain isolated numbers, break the domain of $f(x)$ into intervals $[a,b]$ for which $\int_a^b f(x)\,dx$ is either improper or "proper"– that is, an ordinary definite integral.

For instance, the improper integral $\int_{-\infty}^{\infty} 1/x^2\,dx$ is troublesome for four reasons: $\lim_{x\to 0^-} 1/x^2 = \infty$, $\lim_{x\to 0^+} 1/x^2 = \infty$, and the range extends infinitely to the left and also to the right. (See Figure 7.8.9.) To treat the integral, write it as the sum of four improper integrals of the two basic types:

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\, dx = \int_{-\infty}^{-1} \frac{1}{x^2}\, dx + \int_{-1}^{0} \frac{1}{x^2}\, dx + \int_{0}^{1} \frac{1}{x^2}\, dx + \int_{1}^{\infty} \frac{1}{x^2}\, dx.$$

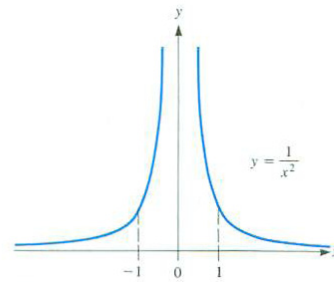


Figure 7.8.9:

Each of the four integrals on the right must be convergent in order for $\int_{-\infty}^{\infty} 1/x^2\, dx$ to be convergent. Only the first and last are, so $\int_{-\infty}^{\infty} 1/x^2\, dx$ is divergent.

Summary

We introduced two types of integrals that are not definite integrals, but are defined as limits of definite integrals. The "improper integral" $\int_a^{\infty} f(x)\, dx$ is defined as $\lim_{b\to\infty} \int_a^b f(x)\, dx$. If $f(x)$ is continuous in $[a, b]$ except at a, then $\int_a^b f(x)\, dx$ is defined as $\lim_{t\to a^+} \int_t^b f(x)\, dx$. The first type is far more common in applications. We also developed two comparison tests for convergence or divergence of $\int_a^{\infty} f(x)\, dx$, where the integrand keeps a constant sign. In the case where the integrand $f(x)$ may have both positive and negative values, we showed that if $\int_a^{\infty} |f(x)|\, dx$ converges, so does $\int_a^{\infty} f(x)\, dx$.

EXERCISES for Section 7.8 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 9 determine whether the improper integral is convergent or divergent. Evaluate the convergent ones if possible. Some exercises may require using the integral table in the back of the book.

1.[R] $\int_1^\infty \frac{dx}{x^3}$

2.[R] $\int_1^\infty \frac{dx}{\sqrt[3]{x}}$

3.[R] $\int_0^\infty e^{-x}\ dx$

4.[R] $\int_0^\infty \frac{dx}{x+100}$

5.[R] $\int_0^\infty \frac{x^3\ dx}{x^4+1}$

6.[R] $\int_1^\infty x^{-1.01}\ dx$

7.[R] $\int_0^\infty \frac{dx}{(x+2)^3}$

8.[R] $\int_0^\infty \sin(2x)\ dx$

9.[R] $\int_1^\infty x^{-0.99}\ dx$

10.[R] $\int_0^\infty \frac{e^{-x}\sin(x^2)}{x+1}\ dx$

11.[R] $\int_0^\infty \frac{dx}{x^2+4}$

12.[R] $\int_0^\infty \frac{x^2\ dx}{2x^3+5}$

13.[M] $\int_0^\infty \frac{dx}{(x+1)(x+2)(x+3)}$

14.[M] $\int_0^\infty \frac{\sin(x)}{x^2}\ dx$

15.[M] $\int_1^\infty \frac{\ln x\ dx}{x}$ NOTE: An antiderivative of $\ln(x)/x$ is $(\ln(x))^2/2$.

16.[M] $\int_0^\infty e^{-2x}\sin(3x)\ dx$

In Exercises 17 to 21 determine whether the improper integral is convergent or divergent. Evaluate the convergent ones if possible. Some exercises may require using the integral table in the back of the book.

17.[R] $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

18.[R] $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

19.[R] $\int_0^1 \frac{dx}{(x-1)^2}$

20.[M] $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\ dx$

21.[M] $\int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}}$ NOTE: This integrand is undefined at both endpoints, $x = 0$ and $x = 1$.

22.[R]

(a) For which values of k is $\int_0^1 x^k\ dx$ improper.

(b) For which values of k is $\int_0^1 x^k$ a convergent improper integral?

(c) For which values of k is $\int_0^1 x^k$ a divergent improper integral?

23.[R]

(a) For which values of k is $\int_1^\infty x^k\ dx$ convergent?

(b) For which values of k is $\int_1^\infty x^k\ dx$ divergent?

24.[R]

(a) For which positive constants p is $\int_0^1\ dx/x^p$ convergent? divergent?

(b) For which positive constants p is $\int_1^\infty\ dx/x^p$ convergent? divergent?

(c) For which positive constants p is $\int_0^\infty\ dx/x^p$ convergent? divergent?

25.[R] Let R be the region between the curves $y = 1/x$ and $y = 1/(x+1)$ to the right of the line $x = 1$. Is the area of R finite or infinite? If it is finite, evaluate it.

26.[R] Let R be the region between the curves $y = 1/x$ and $y = 1/x^2$ to the right of $x = 1$. Is the area of R finite or infinite? If it is finite, evaluate it.

27.[R] Describe how you would go about estimating $\int_0^\infty e^{-x^2}\ dx$ with an error less than 0.02. (Do not do the arithmetic.)

28.[R] Describe how you would go about estimating $\int_0^\infty \frac{dx}{\sqrt{1+x^4}}$ with an error less than 0.01. (Do not do the arithmetic.)

29.[M] Example 4 showed that $\int_0^\infty e^{-x}\sin(x)\ dx$ is convergent. Find its value. HINT: First find constants A and B such that $Ae^{-x}\sin(x) + Be^{-x}\cos(x)$ is an antiderivative of $e^{-x}\sin(x)$.

30.[M] In Example 5 the area of the region bounded by $y = 1/\sqrt{x}$, $x = 1$, and the coordinate axes was found to have area 2. Confirm this result by using horizontal cross sections and evaluating an improper integral from 0 to ∞.

31.[M] The function $f(x) = \frac{\sin(x)}{x}$ for $x \neq 0$ and $f(0) = 1$ occurs in communication theory. Show that the energy E of the signal represented by f is finite, where

$$E = \int_{-\infty}^{\infty} (f(x))^2 \; dx.$$

32.[M] Let $f(x)$ be a positive function and let R be the region under $y = f(x)$ and above $[1, \infty]$. Assume that the area of R is infinite. Does it follow that the volume of the solid of revolution formed by revolving R about the x-axis is infinite?

33.[M]

(a) Sketch the graph of $y = 1/x$, for $x > 0$.

(b) Is the part below the graph and above $(0, 1]$ congruent to the part below the graph and above $[1, \infty)$?

(c) What does this say about the convergence or divergence of $\int_0^1 \frac{dx}{x}$ and $\int_1^\infty \frac{dx}{x}$?

34.[M]

(a) Sketch the graph of $y = 1/x^2$ for $x > 0$.

(b) Is the part below the graph and above $(0, 1]$ congruent to the part below the graph and above $[1, \infty)$?

(c) What does this say about the convergence or divergence of $\int_0^1 \frac{dx}{x^2}$ and $\int_1^\infty \frac{dx}{x^2}$?

(d) What does this say about the convergence or divergence of $\int_0^1 \frac{dx}{\sqrt{x}}$ and $\int_1^\infty \frac{dx}{\sqrt{x}}$?

35.[M] In the study of the harmonic oscillator one meets the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + kx^2)^3},$$

where k is a positive constant. Show this improper integral is convergent.

36.[M] If $\int_0^\infty e^{-x^2} \; dx = \sqrt{\pi}/2$, show that $\int_0^\infty 2^{-x^2} \; dx = \sqrt{\pi}/\ln(4)$.

37.[M]

(a) Is the region under $y = 1/x^2$ and above $[1, \infty)$ congruent to the region under the same curve above $(0, 1]$?

(b) Is the region under $y = 1/x$ and above $[1, \infty)$ congruent to the region under the same curve above $(0, 1]$?

38.[C] Consider the improper integral $\int_0^1 \frac{dx}{x^2}$. Suppose the interval $[0, 1]$ is partitioned into n equal-width pieces. That is $x_i = i/n$ for all $i = 0, 1, \ldots, n$.

(a) Show that the approximating sum $S_n = \sum_{i=1}^{n} \frac{1}{c_i^2} \Delta x_i = \sum_{i=1}^{n} \frac{n}{i^2}$.

(b) Show that $\lim_{n\to\infty} S_n$ does not exist. HINT: Show that $S_n \geq n$ for all positive integers n.

39.[C] Plankton are small football-shaped organisms. The resistance they meet when falling through water is proportional to the integral

$$\int_0^{\infty} \frac{dx}{\sqrt{(a^2 + x)(b^2 + x)(c^2 + x)}},$$

where a, b, and c describe the dimensions of the plankton. Is this improper integral convergent or divergent? (Explain.)

40.[C] In R. P. Feynman, *Lectures on Physics*, Addison-Wesley, Reading, MA, 1963, appears this remark: "...the expression becomes

$$\frac{U}{V} = \frac{(kT)^4}{\hbar^3 \pi^2 c^3} \int_0^{\infty} \frac{x^3\, dx}{e^x - 1}.$$

This integral is just some number that we can get, approximately, by drawing a curve and taking the area by counting squares. It is roughly 6.5. The mathematicians among us can show that the integral is exactly $\pi^4/15$." Show at least that the integral is convergent.

41.[C]

(a) Assume that $f(x)$ is continuous and nonnegative and that $\int_1^{\infty} f(x)\, dx$ is convergent. Show by sketching a graph that $\lim_{x\to\infty} f(x)$ may not exist.

(b) Show that if we add the condition that f is a decreasing function, then $\lim_{x\to\infty} f(x) = 0$.

42.[C] Here is the standard proof of the absolute convergence test. Assume that $\int_0^{\infty} |f(x)|\, dx$ converges. Let $g(x) = f(x) + |f(x)|$. Note that $0 \leq g(x) \leq 2|f(x)|$. Thus $\int_0^{\infty} g(x)\, dx$ converges, that is, $\int_0^{\infty} (f(x) + |f(x)|)\, dx$ converges. It follows, since $f(x) = (f(x) + |f(x)|) - |f(x)|$, that $\int_0^{\infty} f(x)\, dx$ converges.

(a) Study this proof.

(b) State the advantages and disadvantages of each proof, the standard one and the proof in the text.

(c) Which proof do you prefer? Why?

43.[C] In the proof of the Absolute Convergence Test for Improper Integrals (Theorem 7.8.1), we assumed that the functions g and h are continuous. They are, as the following steps show:

(a) Show that $|f(x)| = \sqrt{(f(x))^2}$.

(b) Show that if $f(x)$ is continuous, so is $|f(x)|$.

(c) Show that $g(x) = \frac{1}{2}(f(x) + |f(x)|)$.

(d) Deduce that g is continuous.

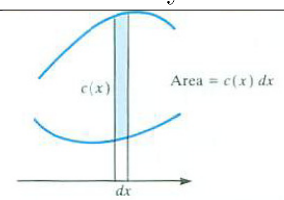
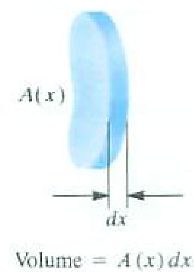
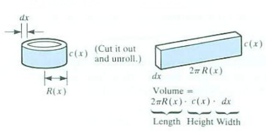
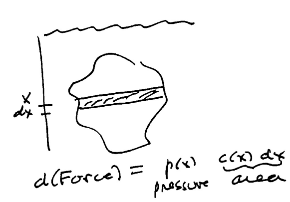
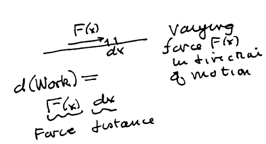
(e) Deduce that h is continuous.

44.[M] If A is in $[0, L]$ and B is in $[-L, 0]$, why is $A + B$ in $[-L, L]$?

7.S Chapter Summary

There are two ideas in this chapter. One is "make a large, clear drawing when setting up a definite integral." The other is "make a local estimate of the total quantity" — whether that quantity is area, volume, force of water, work, or something altogether different. If the local estimate is $f(x)\ dx$, the total quantity is represented by a definite integral $\int_a^b f(x)\ dx$ (or an improper integral).

The following table summarizes some of the applications of the definite integral.

Section	Concept	Memory Aid
7.1	Area $= \int_a^b c(x)\ dx$	
7.4	Volume $= \int_a^b A(x)\ dx$ Special Case: Solid of revolution (perpendicular cross sections)	
7.5	Volume $= \int_a^b 2\pi R(x)c(x)\ dx$ Special Case: Solid of revolution (parallel cross-sections)	
7.6	Force of water	
7.7	Work	

The final section, on improper integrals, shows how to deal with integrals over infinite intervals (that are surprisingly common) and integrands that become infinite (much less common).

EXERCISES for 7.S *Key:* R–routine, M–moderate, C–challenging

1.[M] Consider the parabola $y = x^2$ and two points on it, $P = (a, a^2)$ and $Q = (b, b^2)$.

(a) Show that the tangent to the parabola at the midpoint between P and Q, $R = \left(\frac{a+b}{2}, \left(\frac{a+b}{2}\right)^2\right)$ is parallel to the chord PQ.

(b) Show that the area of the parabola below the chord is $(b-a)^3/6$.

(c) Show that the area of triangle PQR is $(b-a)^3/4$.

Archimedes proved that the area of the parabolic section under PQ is 4/3 the area of triangle PQR. See S. Stein, *Archimedes: What did he do besides cry Eureka?*, MAA, Washington, DC, 1999 (pp. 51–60).

2.[M]

(a) The exponential function is an increasing function for all x. Use this fact to show that $e^x > 1$ for all $x > 0$.

(b) Suppose $f(t) > g(t)$ for all $t > a$. Explain why $\int_a^x f(t)\,dt > \int_a^x g(t)\,dt$ for all $x > a$.

(c) Use (b) to show that $e^x > 1 + x$ for all $x > 0$.

(d) Use (b) and (c) to show that $e^x > 1 + x + \frac{x^2}{2}$ for all $x > 0$.

3.[M] Extend the argument in Exercise 2 to show that $e^x > \sum_{i=0}^{n+1} \frac{x^i}{i!}$. Use this fact to show that $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$.

4.[M] The average distance of an electron from the nucleus of a hydrogen atom involves the integral

$$\int_0^\infty e^{-x}x^5\,dx.$$

Show that it is convergent. (Its value is $5! = 120$).

5.[M] If $\int_0^\infty f(x)\,dx$ is convergent, does it follow that

(a) $\lim_{x\to\infty} f(x) = 0$?

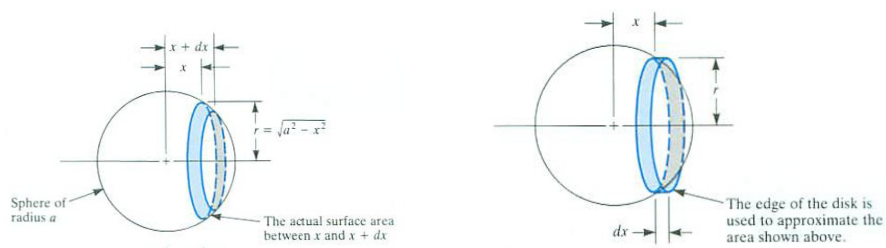
(b) $\lim_{x\to\infty} \int_x^{x+0.1} f(t)\ dt = 0$?

(c) $\lim_{x\to\infty} \int_x^{2x} f(t)\ dt = 0$?

(d) $\lim_{x\to\infty} \int_x^{\infty} f(t)\ dt = 0$?

NOTE: Compare with Exercise 18 in Chapter 11.

6.[C] Consider the following argument: "Approximate the surface area of the sphere of radius a shown in Figure 7.S.1(a) as follows. To approximate the surface area between x and $x + dx$, let us try using the area of the narrow curved part of the cylinder used to approximate the volume between x and $x+dx$. (This part is shaded in Figure 7.S.1(b).) This local approximation can be pictured (when unrolled and laid flat) as a rectangle of width dx and length $2\pi r$. The surface area of a sphere is $\int_{-a}^{a} 2\pi r\ dx = 4\pi \int_0^a \sqrt{a^2 - x^2}\ dx$. But $\int_0^a \sqrt{a^2 - x^2}\ dx = \pi a^2/4$, since it equals the area of a quadrant of a disk. Hence the area of the sphere is then $\pi^2 a^2$." This does not agree with the correct value, $4\pi a^2$, which was discovered by Archimedes in the third century B.C. What is wrong with this argument?



(a) (b)

Figure 7.S.1:

7.[C] Determine if the following improper integral converges or diverges: $\int_0^{\infty} \frac{x\ dx}{\sqrt{1+x^4}}$

8.[M] The probability that ball bearing A survives at least until time t will be denoted as $F(t)$. For ball bearing B let $G(t)$ be the probability that it survives at least until time t.

(a) Show that the probability that A lasts at least as long as B is $-\int_0^{\infty} F(t)G'(t)\ dt$.

(b) Similarly, the probability that B lasts at least as long as A is $-\int_0^{\infty} G(t)F'(t)\ dt$. Assume that the probability that A and B last exactly the same time is 0. Why should $-\int_0^{\infty} F(t)G'(t)\ dt - \int_0^{\infty} G(t)F'(t)\ dt = 1$? Show that it does equal 1.

In Exercise 9 assume $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$, which will be established in Section 17.3 (see Exercise 34 on 1424).
Let μ and σ be constants. The normal distribution, also called the Gaussian distribution and the bell curve, is given by the density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right).$$

9.[M]

(a) Show that the graph of f is symmetric with respect to the line $x = \mu$.

(b) Show that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

(c) Show that $\int_{-\infty}^{\infty} x f(x)\, dx = \mu$. NOTE: μ is the average value of x, and is called the **mean of the distribution**.

(d) Show that $\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \sigma^2$. NOTE: σ^2, called the **variance**, measures the deviation of x from the mean. The number σ is called the **standard deviation** of the distribution. Both measure the tendency of the data to spread out away from the mean.

(e) Show that $f(x)$ has two inflection points, which occur when $x = \mu + \sigma$ or $x = \mu - \sigma$.

(f) Sketch the graph of a typical $f(x)$.

The normal distribution, first introduced in Exercises 98 to 102 in Section 5.7, is defined for a variable that can take on both positive and negative values. However, such variables as incomes, life spans, amounts of rainfall, scores on examinations, and ages of first marriages, do not assume negative values. In these cases it may be more appropriate to use a **log-normal distribution**, which is defined only for $(0, \infty)$. (See, for instance, *The Lognormal Distribution*, by economists J. Atchison and J. A. C. Brown, 1957.)
Let $f(x)$ be the density in a normal distribution. The density, $g(x)$, of the log-normal distribution is defined, for $a > 0$, by the equation

$$\int_0^a \ell(x)\, dx = \int_{-\infty}^{\ln(a)} f(x)\, dx.$$

This says, "the probability that x is at most a is the probability that $\ln(x)$ is at most $\ln(a)$, as given by the normal distribution."
10.[C] In this problem $f(x)$ is the density of a normal distribution with mean μ and variance σ^2 and $g(x)$ is the density of the corresponding log-normal distribution.

(a) Show that $g(x) = \frac{1}{x} f(\ln(x))$ for $x > 0$.

(b) Show that $\int_0^\infty g(x)\ dx = 1$.

(c) Show that the mean value of the log-normal distribution, $\int_0^\infty xg(x)\ dx$, equals $e^{\mu+\frac{\sigma^2}{2}}$.

(d) Show that $\lim_{x\to\infty} g(x) = 0$.

(e) Show that $\lim_{x\to 0^+} g(x) = 0$.

(f) Show that the maximum of $g(x)$ occurs when x is $e^{\mu-\sigma^2}$.

(g) What is the maximum of $g(x)$?

(h) Show that $\int_0^{e^\mu} g(x)\ dx = \int_{e^\mu}^\infty g(x)\ dx$. Thus, half the area under the curve $y = g(x)$ lies to the left of e^μ.

(i) Sketch the general shape of the log-normal distribution. Remember that $g(x)$ is defined only for x in $(0, \infty)$.

Skill Drill: Derivatives

In Exercises 11 to 13 a, b, c, m, and p are constants. In each case verify that the derivative of the first function is the second function.

11.[R] $\frac{x}{a} - \frac{1}{ap}\ln(a + be^{px})$; $\frac{1}{a+be^{px}}$.

12.[R] $\frac{1}{\sqrt{-c}}\arcsin\left(\frac{-cx-b}{\sqrt{b^2-4ac}}\right)$; $\frac{1}{\sqrt{a+bx+cx^2}}$, for any negative number c.

13.[R] $\frac{1}{c}\ln\left(\sqrt{z + bx + cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right)$; $\frac{1}{\sqrt{a+bx+cx^2}}$, for any positive number c.

Calculus is Everywhere # 9

Escape Velocity

In Example 2 in Section 7.7 we saw that the total work required to lift a 1-pound payload from the surface of the earth to the moon is 3,933 mile-pounds. Since the radius of the earth is about 4,000 miles, the work required to launch a payload on an endless journey is given by the improper integral

$$\int_{4{,}000}^{\infty} \left(\frac{4,000}{r}\right)^2 dr = 4,000 \text{ mile-pounds.}$$

Because the integral is convergent, only a finite amount of energy is needed to send a payload on an endless journey — as the Voyager spacecraft has demonstrated. It takes only a little more energy than is required to lift the payload to the moon.

That the work required for the endless journey is finite raises the question "With what initial velocity must we launch the payload so that it never falls back, but continues to rise forever away from the earth?" If the initial velocity is too small, the payload will rise for a while, then fall back, as anyone who has thrown a ball straight up knows quite well.

The energy we supply the payload is kinetic energy. The force of gravity slows the payload and reduces its kinetic energy. We do not want the kinetic energy to shrink to zero. It it were ever zero, then the velocity of the payload would be zero. At that point the payload would start to fall back to earth.

As we will show, the kinetic energy of the payload is reduced by *exactly* the amount of work done on the payload by gravity. If v_{esc} is the minimal velocity needed for the payload to "escape" and not fall back, then

$$\frac{1}{2}mv_{\text{esc}}^2 = 4,000 \text{ mile-pounds,} \tag{C.9.1}$$

where m is the mass of the payload. Equation (C.9.1) can be solved for v_{esc}, the **escape velocity**.

In order to solve (C.9.1) for v_{esc}, we must calculate the mass of a payload that weighs 1 pound at the surface of the earth. The weight of 1 pound is the gravitational force of the earth pulling on the payload. Newton's equation

$$\text{Force} = \text{Mass} \times \text{Acceleration}, \tag{C.9.2}$$

known as his "second law of motion," provides the relationship among force, mass, and the acceleration of that mass that is needed.

The acceleration of an object at the surface of the earth is 32 feet per second per second, or 0.0061 miles per second per second. Then (C.9.2), for the 1-pound payload, becomes

$$1 = m(0.0061). \tag{C.9.3}$$

Combining (C.9.1) and (C.9.3) gives

$$\begin{aligned} \frac{1}{2}\frac{1}{0.0061}(v_{\text{esc}})^2 &= 4,000 \\ \text{or} \qquad (v_{\text{esc}})^2 = (8,000)(0.0061) &= 48.8. \end{aligned}$$

Hence $v_{\text{esc}} \approx 7$ miles per second, which is about 25,000 miles per hour, a speed first attained by human beings when Apollo 8 traveled to the moon in December 1968. All that remains is to justify the claim that the change in kinetic energy equals the work done by the force.

Let $v(r)$ be the velocity of the payload when it is r miles from the center of the earth. Let $F(r)$ be the force on the payload when it is r miles from the center of the earth. Since the force is in the opposite direction from the motion, we will define $F(r)$ to be negative.

Let a and b be numbers, $4,000 \leq a < b$. (See Figure C.9.1.) We wish to show that

$$\underbrace{\frac{1}{2}m(v(b))^2 - \frac{1}{2}m(v(a))^2}_{\text{change in kinetic energy}} = \underbrace{\int_a^b F(r)dr}_{\text{work done by gravity}} . \tag{C.9.4}$$

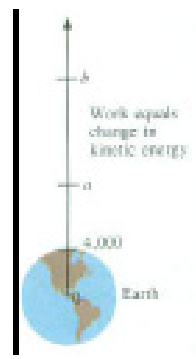


Figure C.9.1:

In this equation m is the payload mass. Note that both sides of (C.9.4) are negative.

Equation (C.9.4) resembles the Fundamental Theorem of Calculus. If we could show that $\frac{1}{2}m(v(r))^2$ is an antiderivative of $F(r)$, then (C.9.4) would follow immediately. Let us find the derivative of $\frac{1}{2}m(v(r))^2$ with respect to r and show that it equals $F(r)$:

$$\begin{aligned} \tfrac{d}{dr}\left(\tfrac{1}{2}m(v(r))^2\right) &= mv(r)\tfrac{dv}{dr} = mv(r)\tfrac{dv/dt}{dr/dt} && \text{(Chain Rule; } t \text{ is time)} \\ &= mv(r)\tfrac{d^2r/dt^2}{v(r)} = m\tfrac{d^2r}{dt^2} && (v(r) = \tfrac{dr}{dt}) \\ &= \text{mass} \times \text{acceleration} \\ &= F(r) && \text{(Newton's 2}^{\text{nd}} \text{ Law of Motion.} \end{aligned}$$

Hence (C.9.4) is valid and we have justified our calculation of escape velocity.

Incidentally, the escape velocity is $\sqrt{2}$ times the velocity required for a satellite to orbit the earth (and not fall into the atmosphere and burn up).

EXERCISES 1.[R] The earth is not a perfect sphere. The"mean radius" of the earth is about 3,959 miles. A more accurate value for the force of gravity is 32.174 feet per second per second. Repeat the derivation of the escape velocity using these values. References: `http://en.wikipedia.org/wiki/Earth_radius` and `http://en.wikipedia.org/wiki/Standard_gravity`.

2.[R] Repeat the derivation of the escape velocity using CGS units. That is, assume the radius of the earth is 6,371 kilometers and the force of gravity is 9.80665 meters per second per second.

3.[R] Determine the escape velocity from the moon. NOTE: What information do you need to complete this calculation?

4.[R] Determine the escape velocity from the sun.

Calculus is Everywhere # 10

Average Speed and Class Size

There are two ways to define your average speed when jogging or driving a car. You could jot down your speed at regular intervals of time, say, every second. Then you would just average those speeds. That average is called an average *with respect to time*. Or, you could jot down your velocity at regular intervals of distance, say, every hundred feet. The average of those velocities is called an average *with respect to distance*.

How do you think they would compare? If you kept a constant speed, c, the averages would both be c. Are they always equal, even if your speed varies? Would one of the averages always tend to be larger? Try to answer the question before we analyze it mathematically, with the aid of the **Cauchy-Schwartz inequality**.

pronounced: "ko-shee' shwartz"

There are several versions of the Cauchy-Schwartz inequality. The version we need here concerns two continuous functions, f and g, defined on an interval $[a, b]$. If $\int_a^b f(x)^2\ dx$ and $\int_a^b g(x)^2\ dx$ are small, then the absolute value of $\int_a^b f(x)g(x)\ dx$ ought to be small too. It is, as the following Cauchy-Schwartz inequality implies:

$$\left(\int_a^b f(x)g(x)\ dx\right)^2 \leq \int_a^b f(x)^2\ dx \int_a^b g(x)^2\ dx. \tag{C.10.1}$$

After showing some of its applications, we will use the quadratic formula to show that it is true.

First we use the inequality (C.10.1) to answer the question, "Which average of speed is larger, the one with respect to time or the one with respect to distance?"

Let the speed at time t be $v(t)$ and let $s(t)$ be the distance traveled up to time t. During the time interval from time a to time b the average of velocity with respect to time is

$$\frac{\int_a^b v(t)\ dt}{b-a} = \frac{s(b)-s(a)}{b-a}.$$

On the other hand, the average of velocity with respect to distance is defined as

$$\frac{\int_{s(a)}^{s(b)} v(s)\ ds}{s(b)-s(a)}, \tag{C.10.2}$$

where $v(s)$ denotes the velocity when the distance covered is s. Changing the independent variable in the numerator of (C.10.2) from s to t by the relation $ds = v(t)\ dt$, we obtain

$$\frac{\int_{s(a)}^{s(b)} v(s)\ ds}{s(b) - s(a)} = \frac{\int_a^b v(t)v(t)\ dt}{s(b) - s(a)}.$$

Noting that $s(b) - s(a) = \int_a^b v(t)\ dt$ and $b - a = \int_a^b 1\ dt$, we will show that the average with respect to time is less than or equal to the average with respect to distance, that is,

$$\frac{\int_a^b v(t)\ dt}{\int_a^b 1\ dt} \leq \frac{\int_a^b v(t)^2\ dt}{\int_a^b v(t)\ dt}.$$

Or, equivalently,

$$\left(\int_a^b v(t)\ dt\right)^2 \leq \int_a^b 1\ dt \int_a^b v(t)^2\ dt. \tag{C.10.3}$$

But, (C.10.3) is a special case of (C.10.1), with $f(t) = 1$ and $g(t) = v(t)$.

This implies that the average with respect to time is always less than or equal to the average with respect to distance. Exercise 1 shows a bit more: if the speed is not constant, then the average with respect to time is less than the average with respect to distance.

The way to show that inequality (C.10.1) holds is indirect but short. Introduce a new function, $h(t)$, defined by

$$h(t) = \int_a^b (f(x) - tg(x))^2\ dx = \int_a^b f(x)^2\ dx - 2t\int_a^b f(x)g(x)\ dx + t^2\int_a^b g(x)^2\ dx. \tag{C.10.4}$$

Because the first integrand in (C.10.4) is never negative, $h(t) \geq 0$. Now, $h(t) = pt^2 + qt + r$, where

$$p = \int_a^b g(x)^2\ dx, \qquad q = -2\int_a^b f(x)g(x)\ dx, \quad \text{and} \quad r = \int_a^b f(x)^2\ dx.$$

The parabola $y = h(t)$ never drops below the t-axis, and touches the t-axis at at most one point. Otherwise, if it touches the t-axis at two points, it would dip below that axis, forcing $h(t)$ to take on some negative values.

Because the equation $h(t) = 0$ has at most one solution, the discriminant $q^2 - 4pr$ must not be positive. Thus, $q^2 - 4pr \leq 0$, from which the Cauchy-Schwartz inequality follows.

EXERCISES

1.[M] Show that the only case when equality holds in (C.10.1) is when $g(x)$ is a constant times $f(x)$.

2.[M] The "discrete" form of the Cauchy-Schwartz inequality asserts that if a_1, a_2, a_3, ..., a_n and b_1, b_2, b_3, ..., b_n are real numbers, then

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} a_i^2.$$

(a) Prove this inequality.

(b) When does equality hold?

3.[M] Use the inequality in Exercise 2 to show that the average class size at a university as viewed by the registrar is usually smaller than the average class size as viewed by the students.

It is also the case that the average time between buses as viewed by the dispatcher is usually shorter than the average time between buses as viewed by passengers arriving randomly at a bus stop.
Reference: S. K. Stein, *An Inequality Between Two Average Speeds*, Transportation Research 22B (1988), pp. 469–471.

4.[C] A region R is bounded by the x-axis, the lines $x = 2$ and $x = 5$, and the curve $y = f(x)$, where f is a positive function. The area of R is A. When revolved around the x-axis it produces a solid of volume V.

(a) How large can V be?

(b) How small can V be?

Hint: In one of these two cases the Cauchy-Schwartz inequality on 683 may help.

5.[C] If the region R in the preceding exercise is revolved around the y-axis, what can be said about the maximum and minimum values for the volume of the resulting solid? Explain.

Chapter 8

Computing Antiderivatives

In Chapter 7 we saw several uses for definite integrals in geometry and physics. Similar applications of integration can be found in many other fields, including economics, engineering, biology, and statistics. Definite integrals are usually either evaluated using the Fundamental Theorem of Calculus or estimated numerically, as discussed in Section 6.5.

To evaluate $\int_a^b f(x)\,dx$ by the Fundamental Theorem of Calculus (FTC I) we need to find an antiderivative $F(x)$ of the integrand $f(x)$, then $\int_a^b f(x)\,dx$ is simply $F(b) - F(a)$. This chapter describes techniques for finding an antiderivative.

The problem of finding an antiderivative differs from that of finding a derivative in two important ways. First, the antiderivatives of some elementary functions, such as e^{x^2}, are not elementary. On the other hand, as we saw in Chapter 3, the derivatives of all elementary functions are elementary.

Second, a slight change in the form of a function can cause great change in the form of its antiderivative. For instance,

$$\int \frac{dx}{x^2+1} = \arctan(x) + C \qquad \text{while} \qquad \int \frac{x\,dx}{x^2+1} = \frac{1}{2}\ln(x^2+1) + C,$$

as you may check by differentiating $\arctan(x)$ and $\frac{1}{2}\ln(x^2+1)$. On the other hand, a slight change in the form of an elementary function produces only a slight change in the form of its derivative.

There are three ways to find an antiderivative:

- By hand, using techniques described in this chapter
- By an integral table
- By computer, calculator, or other automated integrator.

SHERMAN: I do not like the phrasing here, particularly the last part. Comments apply to computer and calculator. Call them automated integrators?

Section 8.1 illustrates a few shortcuts, describes how to use integral tables, and discusses the strengths and weaknesses of computer-based evaluation of integrals.

Section 8.2 presents "substitution," the most important technique for finding an antiderivative.

Section 8.3 describes"integration by parts," a technique that has many uses, such as in solving differential equations, besides finding antiderivatives.

Section 8.4 discusses the integration of rational functions.

Section 8.5 describes how to integrate some special integrands.

Section 8.6 offers an opportunity to practice the techniques when there is no clue as to which is the best to use.

8.1 Shortcuts, Tables, and Technology

In this section we list antiderivatives of some common functions and some shortcuts. Then we describe integral tables and the computation of antiderivatives by computers.

Some Common Integrands

Every formula for a derivative provides a corresponding formula for an antiderivative. For instance, since $(x^3/3)' = x^2$, it follows that

$$\int x^2\ dx = \frac{x^3}{3} + C.$$

The following miniature integral table lists a few formulas that should be memorized. Each can be checked by differentiating the right-hand side of the equation.

$\int x^a\ dx$	$=$	$\frac{x^{a+1}}{a+1} + C$	for $a \neq -1$
$\int \frac{1}{x}\ dx$	$=$	$\ln\|x\| + C$	This is $\int x^a\ dx$ for $a = -1$.
$\int \frac{f'(x)}{f(x)}\ dx$	$=$	$\ln\|f(x)\| + C$	if $f(x) > 0$, the absolute value can be omitted.
$\int (f(x))^n f'(x)\ dx$	$=$	$\frac{(f(x))^{n+1}}{n+1} + C$	for $n \neq -1$
$\int e^{ax}\ dx$	$=$	$\frac{e^{ax}}{a} + C$	
$\int \sin(ax)\ dx$	$=$	$\frac{-1}{a}\cos(ax) + C$	remember the negative sign
$\int \cos(ax)\ dx$	$=$	$\frac{1}{a}\sin(ax) + C$	
$\int \frac{1}{\sqrt{a^2-x^2}}\ dx$	$=$	$\arcsin\left(\frac{x}{a}\right) + C$	
$\int \frac{1}{a^2+x^2}\ dx$	$=$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$	
$\int \frac{1}{\|x\|\sqrt{x^2-a^2}}\ dx$	$=$	$\frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$	

Antiderivative of a polynomial

EXAMPLE 1 Find $\int (2x^4 - 3x + 2)\ dx$.

SOLUTION

$$\begin{aligned}\int (2x^4 - 3x + 2)\ dx &= \int 2x^4\ dx - \int 3x dx + \int 2\ dx \\ &= 2\int x^4\ dx - 3\int x\ dx + 2\int 1\ dx \\ &= 2\frac{x^5}{5} - 3\frac{x^2}{2} + 2x + C\end{aligned}$$

One constant of integration is enough

◇

EXAMPLE 2 Find $\int \frac{4x^3}{x^4+1}\ dx$

Antiderivative of f'/f

SOLUTION The numerator is precisely the derivative of the denominator. Hence

$$\int \frac{4x^3}{x^4+1}\ dx = \ln|x^4+1| + C$$

Since $x^4 + 1$ is always positive, the absolute-value sign is not needed, and $\int \frac{4x^3}{x^4+1}\ dx = \ln(x^4+1) + C$. ◇

Antiderivative of x^a

EXAMPLE 3 Find $\int \sqrt{x}\ dx$.

SOLUTION

$$\int \sqrt{x}\ dx = \int x^{1/2}\ dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} + C = \frac{2}{3}x^{3/2} + C$$

◇

EXAMPLE 4 Find $\int \frac{1}{x^3}\ dx$.

SOLUTION

$$\int \frac{1}{x^3}\ dx = \int x^{-3}\ dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C.$$

◇

EXAMPLE 5 Find $\int (3\cos(x) - 4\sin(2x) + \frac{1}{x^2})\ dx$.

SOLUTION

$$\begin{aligned}\int (3\cos(x) - 4\sin(2x) + \frac{1}{x^2})\ dx &= 3\int \cos(x)\ dx - 4\int \sin(2x)\ dx + \int \frac{1}{x^2} dx \\ &= 3\sin(x) + 2\cos(2x) - \frac{1}{x} + C.\end{aligned}$$

◇

EXAMPLE 6 Find $\int \frac{x}{1+x^2}\,dx$.
SOLUTION If the numerator was exactly $2x$, it would be the derivative of the denominator and we would have the case $\int (f'(x)/f(x))\,dx$: the antiderivative would be $\ln(1+x^2)$. But the numerator can be multiplied by 2 if we simultaneously divide by 2:

Multiplying the integrand by a constant

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx.$$

This step depends on the fact that a constant can be moved past the integral sign:

$$\frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}\cdot 2\int \frac{x}{1+x^2}\,dx = \int \frac{x}{1+x^2}\,dx.$$

Thus

Since $1+x^2>0$, the absolute value is not needed in $\ln(1+x^2)$.

$$\int \frac{x}{1+x^2}\,dx \quad = \quad \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)+C.$$

◇

Special Shortcuts

We present three shortcuts for evaluating some special but fairly common definite integrals. When one of these shortcuts can be used it saves a lot of work.
Shortcut 1 If f is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0. \tag{8.1.1}$$

Explanation. Recall that for an odd function $f(-x) = -f(x)$. Figure 8.1.1 suggests why (8.1.1) holds. The shaded area to the left of the y-axis equals the shaded area to the right. As integrals, however, these two areas represent quantities of opposite sign: $\int_{-a}^{0} f(x)\,dx = -\int_0^a f(x)\,dx$.

Therefore, the definite integral over the entire interval is 0.

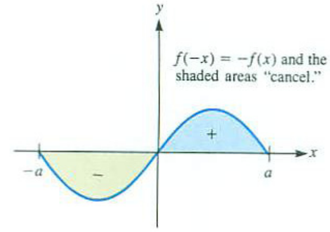


Figure 8.1.1:

EXAMPLE 7 Find $\int_{-2}^{2} x^3\sqrt{4-x^2}\,dx$.
SOLUTION The function $f(x) = x^3\sqrt{4-x^2}$ is odd. (Check it.) By the shortcut,

$$\int_{-2}^{2} x^3\sqrt{4-x^2} = 0.$$

◇

Shortcut 2 $\int_0^a \sqrt{a^2-x^2}\,dx = \frac{1}{4}\pi a^2$.

Note that this shortcut applies to a particular function over a particular interval.

Explanation The graph of $y = \sqrt{a^2-x^2}$ is part of a circle of radius a. The definite integral $\int_0^a \sqrt{a^2-x^2}\,dx$ is a quarter of the area of that circle. (See Figure 8.1.2.)

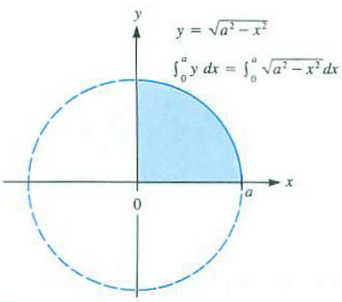


Figure 8.1.2:

EXAMPLE 8 Find $\int_0^1 \sqrt{1-x^2}\,dx$

SOLUTION Use Shortcut 2, with $a = 1$, to get

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}.$$

◇

Shortcut 3 If f is an even function,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

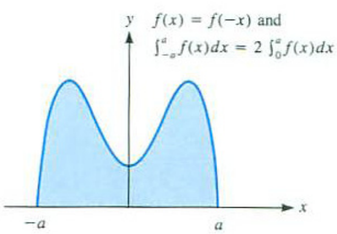


Figure 8.1.3:

Explanation A glance at Figure 8.1.3 suggests why this shortcut is valid.

EXAMPLE 9 Find $\int_{-1}^1 \sqrt{1-x^2}\,dx$.

SOLUTION Since $\sqrt{1-x^2}$ is an even function, by Shortcut 3:

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = 2\int_0^1 \sqrt{1-x^2}\,dx.$$

So, by Example 8, with $a = 1$,

$$\int_{-1}^{1} \sqrt{1-x^2}dx = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}.$$

◇

Using an Integral Table

An integral table lists antiderivatives. You will find a short integral table on the inside covers of this book. *Burington's Handbook of Mathematical Tables and Formulas*, 5th edition, McGraw-Hill, 1973, lists over 300 integrals in 33 pages. *CRC Standard Math Tables*, 30^{th} edition, CRC Press, 1996, lists more than 700 integrals in almost 60 pages. Two Wikipedia topics devoted to tables of integration are `http://en.wikipedia.org/wiki/List_of_integrals` and `http://en.wikipedia.org/wiki/Table_of_integrals`.

Often integral tables use "log" to denote "ln"; it is understood that e is the base. Most integral tables omit the constant of integration ($+C$).

The best way to use an integral table is to browse through one (buy one, check one out from the library, or navigate to an online table). Notice how the formulas are grouped. First might come the forms that everyone uses most frequently. Then may come "forms containing $ax+b$," then "forms containing $a^2 \pm x^2$," then "forms containing ax^2+bx+c," and so on, running through many different algebraic forms. There are separate sections with trigonometric forms, logarithmic, and exponential functions. The integral table on the inside front cover is similarly grouped.

EXAMPLE 10 Use the integral table to integrate

$$\int \frac{dx}{x\sqrt{3x+2}}.$$

SOLUTION Search until you find Formula 23,

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}} \right| \qquad b > 0,$$

and replace $ax+b$ by $3x+2$ and b by 2. Thus

$$\int \frac{dx}{x\sqrt{3x+2}} = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{3x+2}-\sqrt{2}}{\sqrt{3x+2}+\sqrt{2}} \right| + C.$$

◇

EXAMPLE 11 Use the integral table to integrate

$$\int \frac{dx}{x\sqrt{3x-2}}, \qquad x > 2/3.$$

SOLUTION This time we need Formula 24 with $b=-2$,

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}} \arctan\left(\sqrt{\frac{ax+b}{-b}}\right) \qquad b < 0.$$

Thus,

$$\int \frac{dx}{x\sqrt{3x-2}} = \frac{2}{\sqrt{2}} \arctan\left(\sqrt{\frac{3x-2}{2}}\right) + C$$

◇

Though the integrands in Examples 10 and 11 are similar, their antiderivatives are not.

There is no need to make a big fuss about integral tables. Be cautious and keep a cool head. Just match the patterns carefully, including any conditions on the variables and their coefficients. Note that some formulas are expressed in terms of an integral of a different integrand. In these cases you will have to search through the table more than once. (Exercises 35 and 36 illustrate this.)

Computers, Calculators, and Other Automated Integrators

Using an integral table is an exercise in "pattern matching", where you hunt for the formula that fits a particular integral. Computers are good at pattern matching, so it is not surprising that for many years computers have been used to find antiderivatives. MACSYMA is one of the earliest computer-based programs that perform the basic operations of calculus: limits, derivatives, integrals. Today, the most widely used computer algebra systems are Maple and Mathematica.

This technology is slowly creeping to handheld calculators. With such wide-ranging aids at our fingertips, calculus users do not need to rely as much on formal integration techniques or tables of integrals. What is essential is that they understand what an integral is, what it can represent, and how to utilize information obtained from an integral.

In addition to matching problems with formulas from large tables of integrals, these programs utilize various substitutions and computations to transform integrals into forms that can be evaluated.

In spite of the availability of integral tables, and computer programs, it is often simpler to use one of the techniques described later in this chapter.

EXERCISES for Section 8.1 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 14 find the integrals. Use the short list at the beginning of the section.

1.[R] $\int 5x^3\ dx$

2.[R] $\int (8+11x)\ dx$

3.[R] $\int x^{1/3}\ dx$

4.[R] $\int \sqrt[3]{x^2}\ dx$

5.[R] $\int \frac{6\ dx}{x^2}$

6.[R] $\int \frac{dx}{x^3}$

7.[R] $\int 5e^{-2x}\ dx$

8.[R] $\int \frac{5\ dx}{1+x^2}$

9.[R] $\int \frac{6\ dx}{|x|\sqrt{x^2-1}}$

10.[R] $\int \frac{5\ dx}{\sqrt{1-x^2}}$

11.[R] $\int \frac{4x^3\ dx}{1+x^4}$

12.[R] $\int \frac{e^x\ dx}{1+e^x}$

13.[R] $\int \frac{\sin(x)\ dx}{1+\cos(x)}$

14.[R] $\int \frac{dx}{1+3x}$

In Exercises 15 to 20, change the integrand into an easier one by algebra and find the antiderivative.

15.[R] $\int \frac{1+2x}{x^2}\ dx$ Hint: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

16.[R] $\int \frac{1+2x}{1+x^2}\ dx$

17.[R] $\int (x^2+3)^2\ dx$ Hint: First multiply out the integrand.

18.[R] $\int (1+e^x)^2\ dx$

19.[R] $\displaystyle\int (1+3x)x^2\ dx$

20.[R] $\displaystyle\int \frac{1+\sqrt{x}}{x}\ dx$

21.[R] A shortcut for $\displaystyle\int_0^{\pi/2} \sin^2(\theta)\ d\theta$.)

(a) Why would you expect $\displaystyle\int_0^{\pi/2} \cos^2(\theta)\ d\theta$ to equal $\displaystyle\int_0^{\pi/2} \sin^2(\theta)\ d\theta$?

(b) Why is $\displaystyle\int_0^{\pi/2} \sin^2(\theta)\ d\theta + \int_0^{\pi/2} \cos^2(\theta)\ d\theta = \pi/2$.

(c) Conclude that $\displaystyle\int_0^{\pi/2} \sin^2(\theta)\ d\theta = \pi/4$.

The integrals in Exercises 22 to 28 can be evaluated using one of the shortcuts. Hint: Is the integrand even or odd? Can you relate the integral to a known area? Recall the result of Exercise 21: $\displaystyle\int_0^{\pi/2} \cos^2(x)\ dx = \frac{\pi}{4} = \int_0^{\pi/2} \sin^2(x)\ dx$.

22.[R] $\displaystyle\int_{-1}^{1} x^5\sqrt{1+x^2}\ dx$

23.[R] $\displaystyle\int_{-\pi/2}^{\pi/2} \sin(3x)\cos(5x)\ dx$

24.[R] $\displaystyle\int_{-1}^{1} x^5\sqrt[4]{1-x^2}\ dx$

25.[R] $\displaystyle\int_{-\pi}^{\pi} \sin^3(x)\ dx$

26.[R] $\displaystyle\int_{0}^{5} \sqrt{25-x^2}\ dx$

27.[R] $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\ dx$

28.[R] $\displaystyle\int_{-3}^{3} (x^3\sqrt{9-x^2} + 10\sqrt{9-x^2})\ dx$

In Exercises 29 to 34 find the antiderivative with the aid of a table of integrals, such as the one inside the front cover.

29.[R]

(a) $\displaystyle\int \frac{dx}{(3x+2)^2}$

(b) $\displaystyle\int \frac{dx}{x(3x+2)}$

30.[R]

(a) $\displaystyle\int \frac{dx}{x\sqrt{3x+4}}$

(b) $\displaystyle\int \frac{dx}{x^2\sqrt{3x+4}}$

31.[R]

(a) $\displaystyle\int \frac{dx}{x\sqrt{3x-4}}$

(b) $\displaystyle\int \frac{dx}{x^2\sqrt{3x-4}}$

32.[R]

(a) $\displaystyle\int \frac{dx}{4x^2+9}$

(b) $\displaystyle\int \frac{dx}{4x^2-9}$

33.[R]

(a) $\displaystyle\int \frac{dx}{x^2+8x+7}$

(b) $\displaystyle\int \frac{dx}{x^2+2x+5}$

34.[R]

(a) $\displaystyle\int \frac{dx}{\sqrt{11-x^2}}$

(b) $\displaystyle\int \frac{dx}{\sqrt{11+x^2}}$

35.[M] Using the integral table on the inside front cover of the book, find $\displaystyle\int \frac{x\ dx}{\sqrt{2x^2+x+5}}$.
HINT: Use Formula 39 first, followed by Formula 38.

36.[M] Using the integral table in the front of the book, find

(a) $\displaystyle\int \frac{dx}{\sqrt{3x^2+x+2}}$

(b) $\displaystyle\int \frac{dx}{\sqrt{-3x^2+x+2}}$

8.2 The Substitution Method

This section describes the substitution method, which changes an integrand, preferably to one that we can integrate more easily. Several examples will illustrate the technique, which is the chain rule in disguise. Sometimes we can use a substitution to transform an integral not listed in an integral table to one that is listed. After the examples, the basis of the substitution method is provided.

The Substitution Method

EXAMPLE 1 Find $\int \sin(x^2)\ 2x\ dx$.
SOLUTION Note that $2x$ is the derivative of x^2. Make the substitution $u = x^2$. The differential of u is $du = \frac{d}{dx}(x^2)\ dx = 2x\ dx$ and so

$$\int (\sin(x^2))2x\ dx = \int \sin(u)\ du.$$

It is easy to find $\int \sin(u)\ du$:

$$\int \sin(u)\ du = -\cos(u) + C.$$

Replacing u by x^2 in $-\cos(u)$ yields $-\cos(x^2)$. Thus

Check the answer using the chain rule

$$\int \sin(x^2)2x\ dx = -\cos(x^2) + C.$$

◇

Contrast Example 1 with $\int \sin(x^2)\ dx$, which is not elementary. The presence of $2x$, the derivative of x^2, made it easy to find $\int(\sin(x^2))2x\ dx$.

Description of the Substitution Method

In Example 1, the integrand $f(x)$ could be written in the form

$$f(x) = \underbrace{g(h(x))}_{\text{function of } h(x)} \times \underbrace{h'(x)}_{\text{derivative of } h(x),} \tag{8.2.1}$$

for some function $h(x)$. To put it another way, the expression $f(x)\ dx$ could be written as

$$f(x)\ dx = \underbrace{g(h(x))}_{\text{function of } h(x)} \times \underbrace{h'(x)\ dx}_{\text{derivative of } h(x),} \tag{8.2.2}$$

Whenever this is the case, the substitution of u for $h(x)$ and du for $h'(x)\ dx$ transforms $\int f(x)\ dx$ to another integral, one involving u instead of x, $\int g(u)\ du$.

If you can find an antiderivative $G(u)$ of $g(u)$, replace u by $h(x)$. The resulting function, $G(h(x))$, is an antiderivative of $f(x)$. (This claim will be justified at the end of the section.)

The process of using substitution to evaluate an indefinite integral can be summarized as follows:

$$\int f(x)\ dx = \int g(h(x))\ h'(x)\ dx = \int g(u)\ du = G(u) + C = G(h(x)) + C.$$

EXAMPLE 2 Find $\int (1+x^3)^5\ x^2\ dx$.
SOLUTION The derivative of $1+x^3$ is $3x^2$, which differs from the x^2 in the integrand only by the constant factor 3. So let $u = 1 + x^3$. Hence

$$du = 3x^2\ dx \qquad \text{and} \qquad \frac{du}{3} = x^2\ dx. \tag{8.2.3}$$

Then

$$\int (1+x^3)^5 x^2\ dx = \int u^5 \frac{du}{3} = \frac{1}{3}\int u^5\ du = \frac{1}{3}\frac{u^6}{6} + C = \frac{(1+x^3)^6}{18} + C.$$

$\diamond$

If the factor x^2 were not present in the integrand in Example 2, you could still compute $\int (1+x^3)^5\ dx$. In this case you would have to multiply out $(1+x^3)^5$, which would be a polynomial of degree 15.

As Example 2 shows, you don't need exactly "derivative of $h(x)$" as a factor. Just "a constant times the derivative of $h(x)$" will do.

Similarly, $\int \frac{x^2}{\sqrt{1+x^3}}\ dx$ is easy (use $u = 1+x^3$), but $\int \frac{dx}{\sqrt{1+x^3}}$ is not elementary. The presence of x^2 makes a great difference.

Substitution in a Definite Integral

The substitution technique, or "change of variables," extends to definite integrals, $\int_a^b f(x)\ dx$, with one important proviso:

When making the substitution from x to u, be sure to replace the interval $[a,b]$ by the interval whose endpoints are $u(a)$ and $u(b)$.

An example will illustrate the necessary change in the limits of integration. The technique is justified in Theorem 8.2.

EXAMPLE 3 Evaluate $\int_1^2 3(1+x^3)^5\, x^2\, dx$.

SOLUTION Let $u = 1 + x^3$. Then $du = 3x^2\, dx$. Furthermore, as x goes from 1 to 2, $u = 1 + x^3$ goes from $1 + 1^3 = 2$ to $1 + 2^3 = 9$. Thus

This is the last you see of x.

$$\int_1^2 3(1+x^3)^5 x^2\, dx = \int_2^9 u^5\, du = \left.\frac{u^6}{6}\right|_2^9 = \frac{9^6 - 2^6}{6}.$$

Once you make the substitution in the integrand and the limits of integration, you work only with expressions involving u. There is no need to bring back x. ◇

The remaining examples present integrals needed in Section 8.4. They also show how some formulas in integral tables are obtained.

EXAMPLE 4 Integral tables include a formula for (a) $\int dx/(ax+b)$ and (b) $\int dx/(ax+b)^n$, $n \neq 1$. Obtain the formulas by using the substitution $u = ax + b$.

SOLUTION (a) Let $u = ax + b$. Hence $du = a\, dx$ and therefore $dx = du/a$. Thus

This is Formula 12 from the integral table.

$$\int \frac{dx}{ax+b} = \int \frac{du/a}{u} = \frac{1}{a}\int \frac{du}{u} = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C.$$

(b) The same substitution $u = ax + b$ gives

$$\begin{aligned}\int \frac{dx}{(ax+b)^n} &= \int \frac{du/a}{u^n} = \frac{1}{a}\int u^{-n}\, du = \frac{1}{a}\frac{u^{-n+1}}{(-n+1)} + C \\ &= \frac{(ax+b)^{-n+1}}{a(-n+1)} + C = \frac{1}{a(-n+1)(ax+b)^{n-1}} + C.\end{aligned}$$

◇

In the next Example we use u instead of x, to simplify Example 6.

EXAMPLE 5 Find $\displaystyle\int \frac{du}{4u^2+9}$.

SOLUTION $\displaystyle\int \frac{du}{4u^2+9}$ resembles $\displaystyle\int \frac{du}{u^2+1}$. This suggests rewriting $4u^2$ as $9t^2$, so we could then factor the 9 out of $9t^2 + 9$, getting $9(t^2+1)$. Here are the details.

Introduce t so $4u^2 = 9t^2$. To do this let $2u = 3t$, so $u = (3/2)\ t$. Then $du = (3/2)\ dt$. Also, $t = (2/3)u$. With this substitution we have

$$\begin{aligned}\int \frac{du}{4u^2+9} &= \int \frac{(3/2)\ dt}{9t^2+9} = \frac{3}{2}\cdot\frac{1}{9}\int\frac{dt}{t^2+1}\\ &= \frac{1}{6}\arctan(t) + C = \frac{1}{6}\arctan\left(\frac{2u}{3}\right) + C.\end{aligned}$$

◇

The next example uses a substitution together with "completing the square." To complete the square in the quadratic expression $x^2 + bx + c$ means adding and subtracting $(b/2)^2$ so that we get the simpler form "$v^2 + k$" where k is a constant:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}.$$

One squares half the coefficient of b: $(b/2)^2$. To complete the square in $ax^2 + bx + c$, where a is not 1, factor a out first:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right).$$

Then complete the square in $x^2 + (b/a)x + c/a$.

EXAMPLE 6 Find $\int \frac{dx}{4x^2+8x+13}$.

Note the subtraction of $4(1^2)$, not 1^2.

SOLUTION First complete the square in the denominator:

$$\begin{aligned}4x^2 + 8x + 13 &= 4(x^2 + 2x + \boxed{}) + 13 - 4\boxed{}\\ &= 4(x^2 + 2x + 1^2) + 13 - 4(1^2)\\ &= 4(x+1)^2 + 9.\end{aligned}$$

We now can rewrite the integral as

$$\int \frac{dx}{4(x+1)^2 + 9}$$

Let $u = x + 1$, hence $du = dx$ and we have

$$\int \frac{dx}{4(x+1)^2+9} = \int \frac{du}{4u^2+9}.$$

By a piece of good luck, we found in Example 5 that

$$\int \frac{du}{4u^2+9} = \frac{1}{6}\arctan\left(\frac{2u}{3}\right) + C$$

Putting all this together:

Check this by differentiating.

$$\begin{aligned}\int \frac{dx}{4x^2+8x+9} &= \int \frac{dx}{4(x+1)^2+9} = \int \frac{du}{4u^2+9} \\ &= \frac{1}{6}\tan^{-1}\left(\frac{2u}{3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{2(x+1)}{3}\right) + C.\end{aligned}$$

◇

The integral

$$\int \frac{2ax+b}{ax^2+bx+c}\,dx \tag{8.2.4}$$

is easy since it has the form $\int \frac{f'}{f}\,dx$. The integral is $\ln|az^2+b+c|+C$. This observation is the key to treating the integral in the next example.

EXAMPLE 7 Find $\displaystyle\int \frac{x}{4x^2+8x+13}\,dx$.

SOLUTION No substitution comes to mind. However, if $8x+8$, were in the numerator, we would have an easy integral, for $8x+8$ is the derivative of the denominator. So we will do a little algebra on x to get $8x+8$ into the numerator. We can write $x = \frac{1}{8}(8x+8) - \frac{8}{8} = \frac{1}{8}(8x+8) - 1$. Then we have

$$\begin{aligned}\int \frac{x\,dx}{4x^2+8x+13} &= \int \frac{\frac{1}{8}(8x+8)-1}{4x^2+8x+13}\,dx \\ &= \frac{1}{8}\int \frac{8x+8}{4x^2+8x+13} - \int \frac{dx}{4x^2+8x+13} \\ &= \frac{1}{8}\ln\left|4x^2+8x+13\right| - \frac{1}{6}\arctan\left(\frac{2(x+1)}{3}\right).\end{aligned}$$

◇

The techniques of completing the square, substitution, and rewriting x in the numerator, illustrated in Examples 6 and 7, show how to integrate any integrand of the form $\dfrac{1}{ax^2+bx+c}$ or $\dfrac{x}{ax^2+b+c}$.

Why Substitution Works

Theorem 8.2.1. *(Substitution in an indefinite integral) Assume that f and g are continuous functions and $u = h(x)$ is differentiable. Suppose that $f(x)$ can be written as $g(u)\frac{du}{dx}$ and that G is an antiderivative of g. Then $G(u(x))$ is an antiderivative of $f(x)$.*

Proof

We differentiate $G(u(x))$ and check that the result is $f(x)$:

$$\begin{array}{rcll} \dfrac{d}{dx}G(u(x)) & = & \frac{dG}{du}\frac{du}{dx} & \text{(Chain Rule)} \\ & = & g(u)\frac{du}{dx} & \text{(by definition of } G\text{)} \\ & = & f(x). & \text{(by assumption)} \end{array}$$

•

Theorem. *(Substitution in a definite integral) Under the same assumptions as in Theorem 8.2.1*

$$\int\limits_a^b f(x)\ dx = \int\limits_{u(a)}^{u(b)} g(u)\ du. \tag{8.2.5}$$

Warning: If x goes from a to b, $u(x)$ goes from $u(a)$ to $u(b)$. Be sure to change the limits of integration

Proof

Let $F(x) = G(u(x))$, where G is defined in the previous proof.

$$\begin{array}{rcll} \int_a^b f(x)\ dx & = & F(b) - F(a) & \text{(FTC I)} \\ & = & G(u(b)) - G(u(a)) & \text{(definition of } F\text{)} \\ & = & \int_{u(a)}^{u(b)} g(u)\ du & \text{(FTC, again)} \end{array}$$

•

Summary

This section introduced the most commonly used integration technique, "substitution:" If $f(x)\ dx$ can be written as $g(u(x))\ d(u(x))$ for a function $u(x)$ then $\int f(x)dx = \int g(u)\ du$ and $\int_a^b f(x)\ dx = \int_{u(a)}^{u(b)} g(u)\ du$.

It is to be hoped that finding $\int g(u)\ du$ is easier than finding $\int f(x)\ dx$. If it is not, try another substitution or a method presented in the rest of the chapter. There is no simple routine method for antidifferentiation of elementary functions. Practice in integration pays off in spotting which technique is most promising and also being able to transform an integral into one listed in an integral table.

EXERCISES for Section 8.2 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 14 use the given substitution to find the antiderivatives or definte integrals.

1.[R] $\int (1+3x)^5 3\ dx;\qquad u = 1+3x$

2.[R] $\int e^{\sin(\theta)} \cos(\theta)\ d\theta;\qquad u = \sin\theta$

3.[R] $\int\limits_0^1 \frac{x}{\sqrt{1+x^2}}\ dx;\qquad u = 1+x^2$

4.[R] $\int\limits_{\sqrt{8}}^{\sqrt{15}} x\sqrt{1+x^2}\ dx;\qquad u = 1+x^2$

5.[R] $\int \sin(2x)\ dx;\qquad u = 2x$

6.[R] $\int \frac{e^{2x}}{(1+e^{2x})^2}\ dx;\qquad u = 1+e^{2x}$

7.[R] $\int\limits_{-1}^2 e^{3x}\ dx;\qquad u = 3x$

8.[R] $\int\limits_2^3 \frac{e^{1/x}}{x^2}\ dx;\qquad u = \frac{1}{x}$

9.[R] $\int \frac{1}{\sqrt{1-9x^2}}\ dx;\qquad u = 3x$

10.[R] $\int \frac{t\ dt}{\sqrt{2-5t^2}};\qquad u = 2-5t^2$

11.[R] $\int\limits_{\pi/6}^{\pi/4} \tan(\theta)\sec^2(\theta)\ d\theta;\qquad u = \tan\theta$

12.[R] $\int\limits_{\pi^2/16}^{\pi^2/4} \frac{\sin(\sqrt{x})}{\sqrt{x}}\ dx;\qquad u = \sqrt{x}$

13.[R] $\int \frac{(\ln x)^4}{x}\ dx;\qquad u = \ln x$

14.[R] $\int \frac{\sin(\ln x)}{x}\ dx;\qquad u = \ln x$

Every antiderivative can be verified by checking that its derivative is the integrand. That is, if $\int f(x)\ dx = F(x)$, then $F'(x) = f(x)$. Exercises 15 to 21 ask you to verify an antiderivative found in one of the examples in this section.

15.[R] $\int (\sin(x^2))2x\ dx = -\cos(x^2) + C$ (Example 1)

16.[R] $\int (1+x^3)^5 x^2\ dx = \frac{(1+x^3)^6}{18} + C$ (Example 2)

17.[R] $\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$ (Example 4(a))

18.[R] $\int \frac{dx}{(ax+b)^n} = \frac{1}{a(-n+1)(ax+b)^{n-1}} + C$ (Example 4(b))

19.[R] $\int \frac{dx}{4x^2+9} = \frac{1}{6}\arctan\left(\frac{2x}{3}\right) + C$ (Example 5)

20.[R] $\int \frac{dx}{4x^2+8x+9} = \frac{1}{6}\tan^{-1}\left(\frac{2(x+1)}{3}\right) + C$ (Example 6)

21.[R] $\int \frac{x\ dx}{4x^2+8x+13} = \frac{1}{8}\ln\left|4x^2+8x+13\right| - \frac{1}{6}\arctan\left(\frac{2(x+1)}{3}\right)$ (Example 7)

In Exercises 22 to 47 use appropriate substitutions to find the antiderivatives.

22.[R] $\int (1-x^2)^5 x\ dx$

23.[R] $\int \frac{x\ dx}{(x^2+1)^3}$

24.[R] $\int x\sqrt[3]{1+x^2}\ dx$

25.[R] $\int \frac{\sin(\theta)}{\cos^2(\theta)}\ d\theta$

26.[R] $\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\ dt$

27.[R] $\int e^x \sin(e^x)\ dx$

28.[R] $\int \sin(3\theta)\ d\theta$

29.[R] $\int \frac{dx}{\sqrt{2x+5}}$

30.[R] $\int (x-3)^{5/2}\ dx$

31.[R] $\int \frac{dx}{(4x+3)^3}$

32.[R] $\int \frac{2x+3}{x^2+3x+2}\ dx$

33.[R] $\int \frac{2x+3}{(x^2+3x+5)^4}\ dx$

34.[R] $\displaystyle\int \frac{x^3}{\sqrt{1-x^8}}\ dx$

35.[R] $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^3}$

36.[R] $\displaystyle\int x^4 \sin(x^5)\ dx$

37.[R] $\displaystyle\int \frac{\cos(\ln(x))\ dx}{x}$

38.[R] $\displaystyle\int \frac{x}{1+x^4}\ dx$

39.[R] $\displaystyle\int \frac{x^3}{1+x^4}\ dx$

40.[R] $\displaystyle\int \frac{x\ dx}{(1+x)^3}$

41.[R] $\displaystyle\int \frac{x^2\ dx}{(1+x)^3}$

42.[R] $\displaystyle\int \frac{\ln(3x)\ dx}{x}$

43.[R] $\displaystyle\int \frac{\ln(x^2)\ dx}{x}$

44.[R] $\displaystyle\int \frac{(\arcsin(x))^2}{\sqrt{1-x^2}}\ dx$

45.[R] $\displaystyle\int \frac{dx}{\arctan(2x)(1+4x^2)}$

46.[R] $\displaystyle\int \frac{dx}{9x^2+1}$

47.[R] $\displaystyle\int \frac{dx}{9x^2+25}$

In Exercises 48 and 49 complete the square in each expression.

48.[R]

(a) $x^2+6x+10$

(b) $4x^2+6x+11$

49.[R]

(a) $x^2+\frac{5}{3}x+4$

(b) $3x^2+5x+12$

50.[R] Evaluate $\int \frac{dx}{x^2+2x+5}$

51.[R] Evaluate $\int \frac{dx}{2x^2+2x+5}$

52.[R] Evaluate $\int \frac{x}{x^2+2x+5}\,dx$

53.[R] Evaluate $\int \frac{x}{2x^2+2x+5}\,dx$

In Exercises 54 to 59 find the area of the region under the graph of the given function and above the given interval.

54.[R] $f(x) = x^2e^{x^3}$; $[1,2]$
55.[R] $f(x) = \sin^3(\theta)\cos(\theta)$; $[0,\pi/2]$
56.[R] $f(x) = \frac{x^2+3}{(x+1)^4}$; $[0,1]$ HINT: Let $u = x+1$.
57.[R] $f(x) = \frac{x^2-x}{(3x+1)^2}$; $[1,2]$
58.[R] $f(x) = \frac{(\ln(x))^3}{x}$; $[1,e]$
59.[R] $f(x) = \tan^5(\theta)\sec^2(\theta)$; $[0,\frac{\pi}{3}]$

In Exercises 60 to 63 use substitution to evaluate the integral.

60.[M] $\int \frac{x^2}{ax+b}\,dx$; $\qquad a \neq 0$

61.[M] $\int \frac{x}{(ax+b)^2}\,dx$; $\qquad a \neq 0$

62.[M] $\int \frac{x^2}{(ax+b)^2}\,dx$; $\qquad a \neq 0$

63.[M] $\int x(ax+b)^n\,dx$; for (a) $n=-1$, (b) $n=-2$

64.[M] Use a substitution to show that if f is an odd function then $\int_{-a}^{a} f(x)\,dx = 0$. HINT: First show that $\int_{-a}^{0} f(x)\,dx = -\int_0^a f(x)\,dx$ by using the substitution $u=-x$. (Do not refer to "areas".)

65.[M] Use a substitution to show that if f is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$. HINT: First show that $\int_{-a}^{0} f(x)\,dx = \int_0^a f(x)\,dx$ by using the substitution $u=-x$. (Do not refer to "areas".)

66.[M]

(a) Graph $y = \ln(x)/x$.

(b) Find the area under the curve in (a) and above the interval $[e, e^2]$

67.[C] Sam (using the substitution $u = \cos(\theta)$) claims that $\int 2\cos(\theta)\sin(\theta)\ d\theta = -\cos^2(\theta)$, while Jane (using the substitution $u = \sin(\theta)$) claims that the answer is $\sin^2(\theta)$. Who is right? Explain.

68.[C] Jane says, "$\int_0^\pi \cos^2(\theta)\ d\theta$ is obviously positive."
Sam claims, "No, its zero. Just make the substitution $u = \sin(\theta)$; hence $du = \cos(\theta)\ d\theta$. Then I get

$$\int_0^\pi \cos^2(\theta)\ d\theta \quad = \quad \int_0^\pi \cos(\theta)\cos(\theta)\ d\theta = \int_0^0 \sqrt{1-u^2}\ du = 0.$$

Simple."

(a) Who is right? What is the mistake?

(b) Use the identity $\cos^2(\theta) = (1 + \cos(2\theta))/2$ to evaluate the integral without substitution or the shortcut in Section 8.1.

69.[C] Jane asserts that $\int_{-2}^1 2x^2\ dx$ is obviously positive. "After all, the integrand is never negative and $-2 < 1$. It equals the area under $y = 2x^2$ and above $[-2, 1]$". "You're wrong again," Sam replies, "It's negative. Here are my computations. Let $u = x^2$; hence $du = 2x\ dx$. Then

$$\int_{-2}^1 2x^2\ dx \quad = \quad \int_{-2}^1 x \cdot 2x\ dx = \int_4^1 \sqrt{u}\ du = -\int_1^4 \sqrt{u}\ du,$$

which is obviously negative." Who is right? Explain.

8.3 Integration by Parts

Integration by substitution, described in the previous section, is based on the chain rule. The technique called "integration by parts," is based on the product rule for derivatives.

The Basis for "Integration by Parts"

It is a tradition to use u and v instead of the expected f and g.

If u and v are differentiable functions then

$$(uv)' = u'\ v + u\ v'.$$

This tells us that uv is an antiderivative of $u'\ v + u\ v'$:

$$uv = \int (u'\ v + u\ v')\ dx,$$

Then

$$uv = \int u'\ v\ dx + \int u\ v'\ dx,$$

which can be rearranged as

$$\boxed{\int u\ v'\ dx = u\ v - \int u'\ v\ dx \qquad (8.3.1)}$$

This equation tells us, "if you can integrate $u'\,v$, then you can integrate $u\,v'$." Now, $u'\,v$ may look quite different from $u\,v'$. Maybe $\int u'\,v\ dx$ is easier to find than $\int u\,v'\ dx$. The technique based on (8.3.1) is called "Integration by Parts".

Using the differentials $du = u'\ dx$ and $dv = v'\ dx$, we can replace (8.3.1) by the shorter version

$$\boxed{\int u\ dv = uv - \int v\ du \qquad (8.3.2)}$$

Typical Examples

EXAMPLE 1 Find $\int xe^{3x}\,dx$.

SOLUTION Let's see what happens if we let $u = x$. Because $u\,dv$ must equal $xe^{3x}\,dx$, we must choose $dv = e^{3x}\,dx$. That is, $v' = e^{3x}$. Then, differentiating u gives $du = dx$ and integrating v' gives $v = \int e^{3x}\,dx = e^{3x}/3$. The integration by parts formula, (8.3.2), tells us that:

$$\int \underbrace{x}_{u}\underbrace{e^{3x}\,dx}_{dv} = \underbrace{x}_{u}\underbrace{\frac{e^{3x}}{3}}_{v} - \int \underbrace{\frac{e^{3x}}{3}}_{v}\underbrace{dx}_{du} = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} = e^{3x}\left(\frac{x}{3} - \frac{1}{9}\right) + C$$

To check, differentiate $e^{3x}(\frac{x}{3} - \frac{1}{9}) + C$ and see that it's xe^{3x}. ◇

Look closely at Example 1 to see why it worked. The key is that the derivative of $u = x$ is simpler than u and also we could integrate $v' = e^{3x}$ to find v.

EXAMPLE 2 Find $\int x\ln(x)\,dx$.

SOLUTION Setting $dv = \ln(x)\,dx$ is not a wise move, since $v = \int \ln(x)\,dx$ is not immediately apparent. But setting $u = \ln(x)$ is promising because $du = d(\ln(x)) = \frac{1}{x}dx$ is much easier to handle than $\ln(x)$. This forces dv to be $x\,dx$. This second approach goes through smoothly:

$$\begin{aligned} u &= \ln(x) & dv &= x\,dx \\ du &= \tfrac{dx}{x} & v &= \tfrac{x^2}{2}. \end{aligned}$$

(Note that we needed to find $v = \int x\,dx$.) Thus

This antiderivative can be checked by differentiation.

$$\begin{aligned} \int x\ln(x)\,dx &= \int \underbrace{\ln(x)}_{u}\underbrace{x\,dx}_{dv} = \underbrace{\ln(x)}_{u}\underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v}\underbrace{\frac{dx}{x}}_{du} \\ &= \frac{x^2\ln(x)}{2} - \int \frac{x\,dx}{2} = \frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C. \end{aligned}$$

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General Guidelines for Applying Integration by Parts

General Guidelines for Applying Integration by Parts

The key to applying integration by parts is the selection of u and dv. The following three conditions should be met:

1. v can be found by integrating and should not be too messy.
2. du should not be messier than u.
3. $\int v\,du$ should be easier than the original $\int u\,dv$

The next example shows the general approach that can be used to integrate any inverse trigonometric function.

EXAMPLE 3 Find $\int \arctan(x)\ dx$.
SOLUTION Recall that the derivative of $\arctan(x)$ is $1/(1+x^2)$, a much simpler function than $\arctan(x)$. This suggests the following integration by parts:

$$\begin{array}{rclcrcl} u & = & \arctan(x) & \quad & dv & = & dx \\ du & = & \frac{dx}{1+x^2} & \quad & v & = & x \end{array}$$

Integrating an inverse trigonometric function by parts

$$\begin{aligned} \int \underbrace{\arctan(x)}_{u}\ \underbrace{dx}_{dv} &= \underbrace{\arctan(x)}_{u}\ \underbrace{x}_{v} - \int \underbrace{x}_{v}\ \underbrace{\frac{dx}{1+x^2}}_{du} \\ &= x\arctan(x) - \int \frac{x}{1+x^2}\ dx. \end{aligned}$$

It is easy to compute $\int \frac{x\ dx}{1+x^2}$, since the numerator is a constant times the derivative of the denominator:

$$\int \frac{x}{1+x^2}\ dx = \frac{1}{2}\int \frac{2x}{1+x^2}\ dx = \frac{1}{2}\ln(1+x^2).$$

Hence

$$\int \arctan(x)\ dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C.$$

You can check this by differentiation. ◇

Compare your answer with Formula 67 in the integral table in the front cover of the book.

To check that you understand the idea in Example 3, find $\int \arcsin(x)\ dx$ by the same method.

EXAMPLE 4 Find $\int x\sin(x)\ dx$.
SOLUTION There are two approaches. We could choose $u = \sin(x)$ and $dv = x\ dx$ or we could choose $u = x$ and $dv = \sin(x)\ dx$.

Approach 1: $u = \sin(x)$ and $dv = x\ dx$

$$\int x\sin(x)\ dx = \int \underbrace{\sin(x)}_{u}\ \underbrace{(x\ dx)}_{dv}.$$

Then $du = \cos(x)\ dx$, which is not any worse than $u = \sin(x)$. And, since $dv = x\ dx$, $v = x^2/2$. Thus,

$$\int \underbrace{\sin(x)}_{u}\ \underbrace{(x\ dx)}_{dv} = \underbrace{\sin(x)}_{u}\ \underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v}\ \underbrace{\cos(x)\ dx}_{du}.$$

We have replaced the problem of finding $\int x\sin(x)\ dx$ with the harder problem of finding $1/2\int x^2\cos(x)\ dx$. That is *not* progress: we have *raised* the exponent of x in the integrand from 1 to 2.

$\int u\ dv$ is more difficult than $\int v\ du$; Guideline 3 is not satisfied.

Approach 2: $u = x$ and $dv = \sin(x)\ dx$
With these choices for u and dv,

$$\begin{array}{rclcrcl} u & = & x & \qquad & dv & = & \sin(x)\ dx \\ du & = & dx & & v & = & -\cos(x). \end{array}$$

This time integration by parts goes through smoothly:

$$\begin{aligned} \int \underbrace{\sin(x)}_{u}\underbrace{(x\ dx)}_{dv} &= \underbrace{x}_{u}\underbrace{(-\cos(x))}_{v} - \int \underbrace{-\cos(x)}_{v}\underbrace{dx}_{du} \\ &= -x\cos(x) + \int \cos(x)dx = -x\cos(x) + \sin(x) + C. \end{aligned}$$

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All 3 Guidelines are satisfied by this choice of u and dv.

EXAMPLE 5 Find $\int x^2e^{3x}\ dx$.
SOLUTION If we let $u = x^2$, then $du = 2x\ dx$. This is good, for it *lowers* the exponent of x. Hence, try $u = x^2$ and therefore $dv = e^{3x}\ dx$:

$$\begin{array}{rclcrcl} u & = & x^2 & \qquad & dv & = & e^{3x}\ dx \\ du & = & 2x\ dx & & v & = & \frac{1}{3}e^{3x}. \end{array}$$

Thus

$$\begin{aligned} \int \underbrace{x^2}_{u}\underbrace{e^{3x}\ dx}_{dv} &= \underbrace{x^2}_{u}\underbrace{\frac{1}{3}e^{3x}}_{v} - \int \underbrace{\frac{1}{3}e^{3x}}_{v}\underbrace{2x\ dx}_{du} \\ &= \frac{x^2}{3}e^{3x} - \frac{2}{3}\int xe^{3x}\ dx \\ &= \frac{x^2}{3}e^{3x} - \frac{2}{3}\left(e^{3x}\left(\frac{x}{3}-\frac{1}{9}\right)+C\right) \qquad \text{by Example 1} \\ &= e^{3x}\left(\frac{x^2}{3}-\frac{2}{3}\left(\frac{x}{3}-\frac{1}{9}\right)\right)-\frac{2}{3}C \\ &= e^{3x}\left(\frac{x^2}{3}-\frac{2x}{9}+\frac{2}{27}\right)-\frac{2C}{3}. \end{aligned}$$

We may rename $-\frac{2C}{3}$, the arbitrary constant, as K, obtaining

$$\int x^2e^{3x}\ dx = e^{3x}\left(\frac{x^2}{3}-\frac{2x}{9}+\frac{2}{27}\right)+K.$$

Example 5 generalizes.

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The idea behind Example 5 applies to integrals of the form $\int P(x)g(x)\ dx$, where $P(x)$ is a polynomial and $g(x)$ is a function – such as $\sin(x)$, $\cos(x)$, or e^x – that can be repeatedly integrated. Let $u = P(x)$ and $dv = g(x)\ dx$. Then $du = P'(u)\ dx$ and $\int v\ du = \int P'(x)g(x)\ dx$ where $P'(x)$ has a lower degree that $P(x)$.

Definite Integrals and Integration by Parts

Integration by parts of a definite integral $\int_a^b f(x)\ dx$, where $f(x) = u(x)v'(x)$, takes the form

$$\begin{aligned}\int_a^b f(x)\ dx &= \int_a^b u\ dv = uv|_a^b - \int_a^b v\ du \\ &= u(v)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)\ dx.\end{aligned}$$

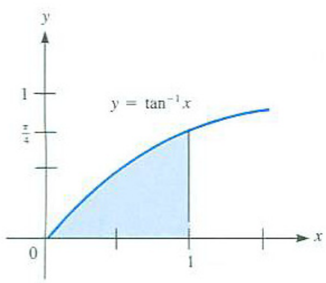


Figure 8.3.1:

EXAMPLE 6 Find the area under the curve $y = \arctan(x)$ and above [0,1]. (See Figure 8.3.1.)
SOLUTION The area is $\int_0^1 \arctan(x)\ dx$. By Example 3,

$$\int \arctan(x)\ dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C.$$

Since only one antiderivative is needed in order to apply the Fundamental Theorem of Calculus, we may choose $C = 0$. Then

$$\begin{aligned}\int_0^1 \arctan x\ dx &= x\arctan(x)|_0^1 - \frac{1}{2}\ln(1+x^2)\Big|_0^1 \\ &= 1\arctan(1) - 0\arctan(0) - \frac{1}{2}\ln(1+1^2) + \frac{1}{2}\ln(1+0^2) \\ &= \frac{\pi}{4} - \frac{1}{2}\ln(2) \approx 0.438824.\end{aligned}$$

◇

Reduction Formulas

Formulas 36, 43, and 46 in the table of integrals on the inside cover of this book express the integral of a function that involves the n^{th} power of some expressions in terms of the integral of a function that involves a lower power of the same expression. These are **reduction formulas** or **recursion formulas**. Usually they are obtained by integration by parts.

See Exercise 25 or Formula 43 in the table of integrals.

An example of a reduction formula is

$$\int \sin^n(x)\,dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx \qquad \text{for integer values of } n \geq 2 \tag{8.3.3}$$

EXAMPLE 7 Use the reduction formula (8.3.3) to evaluate $\int \sin^5(x)\,dx$.
SOLUTION In this case $n = 5$. By (8.3.3),

$$\int \sin^5(x)\,dx = -\frac{\sin^4(x)\cos(x)}{5} + \frac{4}{5}\int \sin^3(x)\,dx. \tag{8.3.4}$$

Use (8.3.3) again to dispose of $\int \sin^3(x)\,dx$. In this case $n = 3$:

$$\begin{aligned} \int \sin^3(x)\,dx &= -\frac{\sin^2(x)\cos(x)}{3} + \frac{2}{3}\int \sin(x)\,dx \\ &= -\frac{\sin^2(x)\cos(x)}{3} - \frac{2}{3}\cos(x) \end{aligned} \tag{8.3.5}$$

Combining (8.3.4) and (8.3.5) gives

$$\int \sin^5(x)\,dx = -\frac{\sin^4(x)\cos(x)}{5} + \frac{4}{5}\left(\frac{-\sin^2(x)\cos(x)}{3} - \frac{2}{3}\cos(x)\right) + C.$$

Every time (8.3.3) is used, the exponent of $\sin(x)$ decreases by 2. If you keep applying (8.3.3), you eventually run into the exponent 1 (as we did, because n is odd) or, if n is even, into the exponent 0. ◇

The next example shows how (8.3.3) can be obtained by integration by parts.

See Formula 43 in the inside cover of the text.

EXAMPLE 8 Obtain the reduction formula (8.3.3).
SOLUTION First write $\int \sin^n(x)\,dx$ as $\int \sin^{n-1}(x)\sin(x)\,dx$. Then let $u = \sin^{n-1}(x)$ and $dv = \sin(x)\,dx$. Thus

$$\begin{aligned} u &= \sin^{n-1}(x) & dv &= \sin(x)\,dx \\ du &= (n-1)\sin^{n-2}(x)\cos(x)\,dx & v &= -\cos(x). \end{aligned}$$

Integration by parts yields

$$\begin{aligned} &\int \underbrace{\sin^{n-1}(x)}_{u}\underbrace{\sin(x)\,dx}_{dv} \\ &= \underbrace{(\sin^{n-1}(x))}_{u}\underbrace{(-\cos(x))}_{v} - \int \underbrace{(-\cos(x))}_{v}\underbrace{(n-1)\sin^{n-2}(x)\cos(x)\,dx}_{du}. \end{aligned}$$

The integral on the right of this equation is

$$\begin{aligned} & -\int (n-1)\cos^2(x)\sin^{n-2}(x)\ dx \\ = & -(n-1)\int (1-\sin^2(x))\sin^{n-2}(x)\ dx \\ = & -(n-1)\int \sin^{n-2}(x)\ dx + (n-1)\int \sin^n(x)\ dx. \end{aligned}$$

Thus

$$\begin{aligned} & \int \sin^n(x)\ dx \\ = & -\sin^{n-1}(x)\cos(x) - \left(-(n-1)\int \sin^{n-2}(x)\ dx + (n-1)\int \sin^n(x)\ dx\right) \\ = & -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\ dx - (n-1)\int \sin^n(x)\ dx. \end{aligned}$$

Rather than being dismayed by the reappearance of $\int \sin^n(x)\ dx$, move that term to the left side to obtain:

$$n\int \sin^n(x)\ dx = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\ dx,$$

from which (8.3.3) follows. ◇

See Formula 46, with $a=1$, in the table on the front cover.

The reduction formula for $\int \cos^n x\ dx$ is obtained similarly.

EXAMPLE 9 Obtain the reduction formula for $\int \frac{dx}{(x^2+c)^n}$ where n is a positive integer.

SOLUTION The only choice that comes to mind for integration by parts is

$$\begin{aligned} u &= \tfrac{1}{(x^2+c)^n} & dv &= dx \\ du &= \tfrac{-2nx}{(x^2+c)^{n+1}} & v &= x. \end{aligned}$$

Integration by parts gives

$$\int \frac{dx}{(x^2+c)^n} = \frac{x}{(x^2+c)^{n+1}} + 2n\int \frac{x^2}{(x^2+c)^{n+1}}\ dx.$$

It looks as though we have just created a more compicated integrand. However, in the numerator of the integrand on the right-hand side, write x^2 as x^2+c-c. We then have

See Formula 28, with $a=1$, in the table on the front cover.

$$\int \frac{dx}{(x^2+c)^n} = \frac{x}{(x^2+c)^{n+1}} + 2n\int \frac{x^2+c}{(x^2+c)^{n+1}}\ dx - 2nc\int \frac{dx}{(x^2+c)^{n+1}} \quad (8.3.6)$$

Canceling out x^2+c in the first integrand on the right gives us an equation which could be rewritten to express $\int \frac{dx}{(x^2+c)^{n+1}}$ in terms of $\int \frac{dx}{(x^2+c)^n}$. ◇

See also Exercises 46 and 62 in this section.

An Unusual Example

In the next example one integration by parts appears at first to be useless, but two in succession lead to the successful evaluation of the integral.

EXAMPLE 10 Find $\int e^x \cos(x)\ dx$
SOLUTION There are two reasonable choices for applying integration by parts: $u = e^x$, $dv = \cos(x)\ dx$ or $u = \cos(x)$, $dv = e^x\ dx$. In neither case is du "simpler", but watch what happens when integration by parts is applied twice.

Following the first choice:

$$\begin{array}{rclcrcl} u & = & e^x & \qquad & dv & = & \cos(x)\ dx \\ du & = & e^x\ dx & & v & = & \sin(x) \end{array}$$

Then integration by parts proceeds as follows:

The second choice is explored in Exercise 57.

$$\int \underbrace{e^x}_{u} \underbrace{\cos(x)\ dx}_{dv} = \underbrace{e^x}_{u} \underbrace{\sin(x)}_{v} - \int \underbrace{\sin(x)}_{v} \underbrace{e^x\ dx}_{du}. \tag{8.3.7}$$

It may seem that nothing useful has been accomplished; $\cos(x)$ is replaced by $\sin(x)$. But watch closely as the new integral is also treated by an integration by parts. Capital letters U and V, instead of u and v, are used to distinguish this computation from the preceeding one.

Repeated integration by parts

$$\begin{array}{rclcrcl} U & = & e^x & \qquad & dV & = & \sin(x)\ dx \\ dU & = & e^x\ dx & & V & = & -\cos(x). \end{array}$$

So

$$\begin{aligned} \int \underbrace{e^x}_{U} \underbrace{\sin(x)\ dx}_{dV} &= \underbrace{e^x}_{U} \underbrace{(-\cos(x))}_{V} - \int \underbrace{(-\cos(x))}_{V} \underbrace{e^x\ dx}_{dU} \\ &= -e^x\cos(x) + \int e^x \cos(x)\ dx. \end{aligned} \tag{8.3.8}$$

Combining (8.3.7) and (8.3.8) yields

$$\begin{aligned} \int e^x \cos(x)\ dx &= e^x \sin(x) - \left(-e^x \cos(x) + \int e^x \cos(x)\ dx \right) \\ &= e^x(\sin(x) + \cos(x)) - \int e^x \cos(x)\ dx. \end{aligned}$$

Bringing $-\int e^x \cos x\ dx$ to the left side of the equation gives

$$2 \int e^x \cos(x)\ dx = e^x(\sin(x) + \cos(x)),$$

See Formula 63, with $a = 1$ and $b = 1$.

and we conclude that

$$\int e^x \cos(x)\ dx = \frac{1}{2} e^x (\sin(x) + \cos(x)) + C.$$

See Exercise 60.

◇

Summary

Integration by parts is described by the formula

$$\int u\ dv = uv - \int v\ du.$$

When you break up the original integral into the parts u and dv, try to choose them so that

1. You can find v and it is not too messy.
2. The derivative of u is nicer than u.
3. You can integrate $\int v\ du$.

Sometimes you have to apply integration by parts more than once, for instance, in finding $\int e^x \cos(x)\ dx$.

EXERCISES for Section 8.3 *Key:* R–routine, M–moderate, C–challenging

Use integration by parts to evaluate each of the integrals in Exercises 1 to 20.

1.[R] $\int xe^{2x}\ dx$

2.[R] $\int (x+3)e^{-x}\ dx$

3.[R] $\int x\sin(2x)\ dx$

4.[R] $\int (x+3)\cos(2x)\ dx$

5.[R] $\int x\ln(3x)\ dx$

6.[R] $\int (2x+1)\ln(x)\ dx$

7.[R] $\int\limits_1^2 x^2e^{-x}\ dx$

8.[R] $\int\limits_0^1 x^2e^{2x}\ dx$

9.[R] $\int\limits_0^1 \sin^{-1}(x)\ dx$

10.[R] $\int\limits_0^{1/2} \tan^{-1}(2x)\ dx$

11.[R] $\int x^2\ln(x)\ dx$

12.[R] $\int x^3\ln(x)dx$

13.[R] $\int\limits_2^3 (\ln(x))^2\ dx$

14.[R] $\int\limits_2^3 (\ln(x))^3\ dx$

15.[R] $\int\limits_1^e \frac{\ln(x)\ dx}{x^2}$

16.[R] $\displaystyle\int_e^{e^2} \frac{\ln(x)\ dx}{x^3}$

17.[R] $\displaystyle\int e^{3x}\cos(2x)\ dx$

18.[R] $\displaystyle\int e^{-2x}\sin(3x)\ dx$

19.[R] $\displaystyle\int \frac{\ln(1+x^2)\ dx}{x^2}$

20.[R] $\displaystyle\int x\ln(x^2)\ dx$

In Exercises 21 to 24 find the integrals two ways: (a) by substitution, (b) by integration by parts.

21.[R] $\displaystyle\int x\sqrt{3x+7}\ dx$

22.[R] $\displaystyle\int \frac{x\ dx}{\sqrt{2x+7}}$

23.[R] $\displaystyle\int x(ax+b)^3\ dx$

24.[R] $\displaystyle\int \frac{x\ dx}{\sqrt[3]{ax+b}}, \qquad a \neq 0$

25.[R] Use differentiation to verify (8.3.3).

26.[R] Use the recursion in Example 8 to find

(a) $\displaystyle\int \sin^2 x\ dx$

(b) $\displaystyle\int \sin^4 x\ dx$

(c) $\displaystyle\int \sin^6 x\ dx$

27.[R] Use the recursion in Example 8 to find

(a) $\displaystyle\int \sin^3 x\ dx$

(b) $\displaystyle\int \sin^5 x\ dx$

28.[R]

(a) Graph $y = e^x \sin x$ for x in $[0, \pi]$, showing extrema and inflection points.

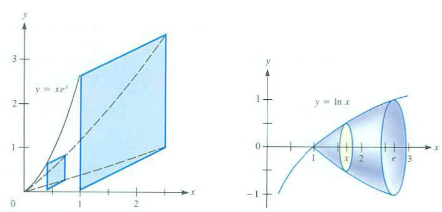
(b) Find the area of the region below the graph and above the interval $[0, \pi]$.

29.[R]

(a) Graph $y = e^{-x} \sin x$ for x in $[0, \pi]$, showing extrema and inflection points.

(b) Find the area of the region below the graph and above the interval $[0, \pi]$.

30.[R] Figure 8.3.2(a) shows a shaded region whose cross sections by planes perpendicular to the x-axis are squares. Find its volume.



(a) (b)

Figure 8.3.2:

31.[R] Figure 8.3.2(b) shows a solid whose cross sections by planes perpendicular to the x-axis are disks. The solid meets the x-axis in the interval $[y.e]$. Find its volume.

In Exercises 32 to 37 find the integrals. In each case a substitution is required before integration by parts can be used. In Exercises 36 and 37 the notation $\exp(u)$ is used for e^u. This notation is often used for clarity.

32.[M] $\displaystyle\int \sin(\sqrt{x})\, dx$

33.[R] In Exercise 67 in Section 6.4 it is claimed that $\frac{e^x}{x}$ does not have an elementary antiderivative. From this fact we can show other functions also lack elementary antiderivatives.

(a) Show that $\int \frac{e^x}{x}\, dx$ equals $\ln(x)e^x - \int \ln(x)e^x\, dx$ and also equals $\frac{e^x}{x} + \int \frac{e^x}{x^2}\, dx$ and $\int \frac{du}{\ln(u)}$ (where $u = e^x$). HINT: Each expression can be obtained from the first by an appropriate use of integration by parts or substitution.

(b) Deduce that $\displaystyle\int \ln(x)e^x\, ds$, $\int (e^x/x^2)\, dx$, and $\displaystyle\int 1/\ln(x)\, dx$ do not have elementary antiderivatives. NOTE: If one of these integrals has an elementary antiderivative, then they all do.

34.[M] Explain how you would go about finding

$$\int x^{10}(\ln x)^{18}\ dx$$

(Don't just say, "I'd use integral tables or a computer.") Explain why your approach would work, but include only enough calculation to convince a reader that it would succeed.

35.[M] Find $\int \sin(\sqrt[3]{x})\ dx$.

36.[M] Find $\int \exp(\sqrt{x})\ dx$. NOTE: Recall that $\exp(x) = e^x$.

37.[M] Find $\int \exp(\sqrt[3]{x})\ dx$

38.[M] Given that $\int \frac{\sin(x)}{x}\ dx$ is not elementary, deduce that $\int \cos(x)\ \ln(x)\ dx$ is not elementary.

39.[M] Given that $\int x\tan(x)\ dx$ is not elementary, deduce that $\int (x/\cos(x))^2\ dx$ is not elementary.

40.[M] Let I_n denote $\int_0^{\pi/2} \sin^n(\theta)\ d\theta$, where n is a nonnegative integer.

(a) Evaluate I_0 and I_1.

(b) Using the recursion in Example 8, show that

$$I_n = \frac{n-1}{n} I_{n-2}, \qquad \text{for } n \geq 2.$$

(c) Use (b) to evaluate I_2 and I_3.

(d) Use (c) to evaluate I_4 and I_5.

(e) Explain why $I_n = \frac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n}$ when n is odd.

(f) Explain why $I_n = \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} \cdot \frac{\pi}{2}$ when n is even.

(g) Explain why $\int_0^{\pi/2} \sin^n(\theta)\ d\theta = \int_0^{\pi/2} \cos^n(\theta)\ d\theta$. HINT: Use the substitution $u = \pi/2 - \theta$.

41.[M] Find $\int \ln(x+1)\,dx$ using

(a) $u = \ln(x+1)\,dx$, $dv = dx$, $v = x$

(b) $u = \ln(x+1)\,dx$, $dv = dx$, $v = x+1$

(c) Which is easier?

42.[M] Let n be a positive integer and a is a constants. Obtain a formula that expresses $\int x^n e^{-ax}\,dx$ in terms of $\int x^{n-1}e^{-ax}$.

43.[M] Find $\int x\sin(ax)\,dx$

44.[M] Let a be a constant and n a positive integer.

(a) Express $\int x^n \sin(ax)\,dx$ in terms of $\int x^{n-1}\cos(ax)\,dx$.

(b) Express $\int x^n \cos(ax)\,dx$ in terms of $\int x^{n-1}\sin(ax)\,dx$.

(c) Why do (a) and (b) enable us to find $\int x^n \sin(ax)\,dx$?

45.[M]

(a) Express $\int (\ln(x))^n\,dx$ in terms of $\int (\ln(x))^{n-1}\,dx$.

(b) Use (a) to find $\int (\ln(x))^3\,dx$

46.[M]

(a) Show how the integral $\int \frac{dx}{(ax^2+bx+c)^{n+1}}$ can be reduced to an integral of the form $\int \frac{du}{(u^2+p)^{n+1}}$.

(b) Use (a) and the recursion formula obtained in Exercise 62 to find a recursion formula for $\int \frac{dx}{(x^2+bx+c)^n}$. (How does your answer compare with Formula 35 in the integral table on the front cover of the text?)

In Exercises 47 to 50 obtain recursion formulas for the integrals.

47.[M] $\int x^n e^{ax}\,dx$, n a positive integer, a a nonzero constant

48.[M] $\int (\ln(x))^n\ dx$, n a positive integer

49.[M] $\int x^n \sin(x)\ dx$, n a positive integer

50.[R] $\int \cos^n(ax)\ dx$, n a positive integer.

Laplace Transform Let $f(t)$ be a continuous function defined for $t \geq 0$. Assume that, for certain fixed positive numbers r, $\int_0^\infty e^{-rt} f(t)\ dt$ converges and that $e^{-rt} f(t) \to 0$ as $t \to \infty$. Define $P(r)$ to be $\int_0^\infty e^{-rt} f(t)\ dt$. The function P is called the **Laplace transform** of the function f. It is an important tool for solving differential equations, and appears in the CIE on present value of future income (see page 786). In Exercises 51 to 55 find the Laplace transform of the given functions.

51.[M] $f(t) = t$

52.[M] $f(t) = t^2$

53.[M] $f(t) = e^t$ (assume $r > 1$)

54.[M] $f(t) = \sin(t)$

55.[M] $f(t) = \cos(t)$

56.[C] Let $P(x)$ be a polynomial.

(a) Check by differentiation that $(P(x) - P'(x) + P''(x) - \cdots)e^x$ is an antiderivative of $P(x)e^x$. (Note that the signs alternate and that the derivatives are taken to successively higher orders until they are constant, with value 0.)

(b) Use (a) to find $\int (3x^3 - 2x - 2)e^x\ dx$.

(c) Apply integration by parts to $\int P(x)e^x\ dx$ to show how the formula in (a) could be obtained.

57.[C] In Example 10, $\int e^x \cos(x)\ dx$ was evaluated by applying integration by parts twice, each time differentiating an exponential and antidifferentiating a trigonometric function. What happens when integration by parts is applied (twice, if necessary) when a trigonometric function is differentiated and an exponential is antidifferentiated. That is, to get started, apply integration by parts with $u = \cos(x)$ and $dv = e^x\ dx$.

58.[M] Find $\int_{-1}^{1} x^3 \sqrt{1 + x^{20}}\ dx$.

59.[M] Find $\int_{-\pi/4}^{\pi/4} \tan(x)(1 + \cos(x))^{3/2}\ dx$

60.[C] According to the reasoning in Example 10, it appears that $\int e^x \cos(x)\ dx$ must equal $\frac{1}{2}e^x(\sin(x) + \cos(x))$. This would contradict the fact that for any constant C, $\frac{1}{2}e^x(\sin(x) + \cos(x)) + C$ is also an antiderivative of $e^x \sin(x)$. Resolve the paradox.

61.[C]

(a) What does the graph of $y = \cos(ax)$ look like when $a = 1$? when $a = 2$? when $a = 3$? when a is a very large constant? Include graphs and a written description in your answers.

(b) Let $f(x)$ be a function with a continuous derivative. Assume that $f(x)$ is positive. What does the graph of $y = f(x)\cos(ax)$ look like when a is large? Express your response in terms of the graph of $y = f(x)$. Include a sketch of $y = f(x)\cos(ax)$ to give an idea of its shape.

(c) On the basis of (b), what do you think happens to

$$\int\limits_0^1 f(x)\cos(ax)\ dx$$

as $a \to \infty$? Give an intuitive explanation.

(d) Use integration by parts to justify your answer in (c).

62.[C] Solve (8.3.6) in Example 9 to obtain the reduction formula for $\int \frac{dx}{(ax^2+c)^n}$. To check your answer, compare it to Formula 28 in the integral table in the inside cover of this book with $a = 1$.

63.[C] If we have a recursion for $\int \frac{dx}{(ax^2+bx+c)^n}$ why don't we need one for $\int \frac{x\ dx}{(ax^2+bx+c)^n}$?

8.4 Integrating Rational Functions: The Algebra

Recall that a rational function is a polynomial or the quotient of two polynomials.

Every rational function, no matter how complicated, has an elementary integral which is the sum of some or all of these types of functions:

- rational functions (including polynomials),
- logarithms of linear or quadratic polynomials: $\ln(ax + b)$ or $\ln(ax^2 + bx + c)$, and
- arctangents of linear or quadratic polynomials: $\arctan(ax + b)$ or $\arctan(ax^2 + bx + c)$.

The reason is mainly algebraic. In an advanced algebra course it is proved that every rational function is the sum of much simpler rational functions, namely those of the forms:

$$\text{polynomials}, \frac{k}{(ax+b)^n}, \frac{d}{(ax^2+bx+c)^n}, \text{ and } \frac{ex}{(ax^2+bx+c)^n} \tag{8.4.1}$$

where a, b, c, d, e, k are constants and n is a positive integer. In Sections 8.2 and 8.3 we saw how to integrate each of these integrands. (See Examples 4 to 7 in Section 8.2 and Formulas 13, 14, 15, 35, 36, and 37.)

As this section is completely algebraic, our objective is to see how to express a rational function $f(x)$ as a sum of the functions in (8.4.1), that is, to find the **partial fraction decomposition** of $f(x)$. For instance, we will see how to find the decomposition

$$\frac{1}{2x^2 + 7x + 3} = \frac{2/5}{2x+1} - \frac{1/5}{x+3}.$$

Reducible and Irreducible Polynomials

A polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where a_n is not 0 is said to have degree n. The polynomials of degree one are called linear; those of degree two, quadratic. A polynomial of degree zero is a constant. If all the coefficients a_i are zero, we have the zero polynomial, which is not assigned a degree.

A polynomial of degree at least one is **reducible** if it is a product of nonconstant polynomials of lower degree. Otherwise, it is **irreducible**.

Every polynomial of degree one, $ax+b$, is clearly irreducible. A polynomial of degree two, ax^2+bx+c, is irreducible if its discriminant b^2-4ac is negative. (See Exercises 37 and 38.) However,

Recall: $a \neq 0$.

> FACT 1: Every polynomial of degree three or higher is reducible.

This is far from obvious. For instance, x^4+1 looks like it cannot be factored, but you can check that

$$x^4+1=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1).$$

On the other hand,

$$x^4-1=(x^2+1)(x^2-1)=(x^2+1)(x+1)(x-1).$$

The next non-obvious fact is that

> FACT 2: Every polynomial of degree at least one is either irreducible or the product of irreducible polynomials.

The factoring of x^4+1 and x^4-1, given above, illustrate both Facts 1 and 2.

Proper and Improper Rational Functions

In arithmetic, the rational number m/n is called proper if $|m|$ is less than $|n|$.

Let $A(x)$ and $B(x)$ be polynomials. The rational function $A(x)/B(x)$ is **proper** if the degree of $A(x)$ is less than the degree of $B(x)$. Otherwise, it is **improper**.

Every improper rational function is the sum of a polynomial and a proper rational function. The next example illustrates why this is true. It depends on long division.

EXAMPLE 1 Express $\frac{2x^3+1}{2x^2-x+1}$ as a polynomial plus a proper rational function.

SOLUTION We carry out long division

Keep dividing until the degree of the remainder is less than the degree of the divisor, or the remainder is 0.

$$\begin{array}{r|rrrrl}
 & & & x & +1/2 & \leftarrow \text{quotient} \\ \hline
2x^2-x+1 & 2x^3 & +0x^2 & +0x & +1 & \\
 & 2x^3 & -x^2 & +x & & \\ \cline{2-4}
 & & x^2 & -x & +1 & \\
 & & x^2 & -x/2 & +1/2 & \\ \cline{3-5}
 & & & -x/2 & +1/2 & \leftarrow \text{remainder}
\end{array}$$

Thus

$$2x^3+1=\left(2x^2-x+1)(x+\frac{1}{2}\right)+\left(-\frac{x}{2}+\frac{1}{2}\right).$$

Division by $2x^2 - x + 1$ gives us the representation

$$\underbrace{\frac{2x^3+1}{2x^2-x+1}}_{\text{improper}} = \underbrace{x + \frac{1}{2}}_{\text{polynomial}} + \underbrace{\frac{\left(\frac{-x}{2}+\frac{1}{2}\right)}{2x^2-x+1}}_{\text{proper}}.$$

To check this equation, just rewrite the right-hand side as a single fraction. ◇

To integrate a rational function we first check that it is proper. If it is improper, we carry out long division, and represent the function as the sum of a polynomial and a proper rational function. Since we already know how to integrate a polynomial we consider in the rest of this section only proper rational functions.

Partial Fractions

As mentioned in the introduction, every rational function is the sum of particularly simple rational functions, ones we know how to integrate. Here is a recipe for finding that representation for a proper rational function $A(x)/B(x)$.

1. Write $B(x)$ as a product of first-degree polynomials and irreducible second-degree polynomials.

Step 2: List summands of the form $\frac{k_i}{(px+q)^i}$.

2. If $px + q$ appears exactly n times in the factorizaiton of $B(x)$, form

$$\frac{k_1}{px+q} + \frac{k_2}{(px+q)^2} + \cdots + \frac{k_n}{(px+q)^n},$$

where the constants k_1, k_2, ..., k_n are to be determined later.

Step2: List summands of the form $\frac{r_jx+s_j}{(ax^2+bx+c)^j}$.

3. If ax^2+bx+c appears exactly m times in the factorization of $B(x)$, then form the sum

$$\frac{r_1x+s_1}{ax^2+bx+c} + \frac{r_2x+s_2}{(ax^2+bx+c)^2} + \cdots + \frac{r_mx+s_m}{(ax^2+bx+c)^m},$$

where the constants r_1, r_2, ..., r_m and s_1, s_2, ..., s_m are to be determined later.

4. Find all the constants (k_i's, r_j's, and s_j's) mentioned in Steps 2 and 3 so that the sum of the rational functions in Steps 2 and 3 equals $A(x)/B(x)$.

The rational functions in Steps 2 and 3 are called the **partial fractions** of $A(x)/B(x)$. This process deserves some comments.

Regarding Step 1

In practice the denominator $B(x)$ often already appears in factored form. If it does not, finding the factorization can be quite a challenge. To find first-degree factors, look for a root of $B(x) = 0$. If r is a root of $B(x)$, then $x - r$ is a factor. Divide $x-r$ into $B(x)$, getting a quotient $Q(x)$; so $B(x) = (x-r)Q(x)$. Repeat the process on $Q(x)$, continuing as long as you can find roots. Already you can see problems. Suppose you find a root numerically to several decimal places. Consequently your results of integration will be approximations. If you want $\int_a^b A(x)/B(x)\ dx$ it might be simpler just to approximate the definite integral.

After finding all the linear factors "what's left" has to be the product of second-degree polynomials. If the degree of "what's left" is just two, then you are happy: you have found the complete factorization. But, if that degree is 4 or 6 or higher, you face a task best to be avoided — or attacked with the assistance of a computer.

Regarding Steps 2 and 3

These steps refer to the number of times a factor occurs in the denominator. If you factor $2x^2+4x+2$, you may obtain $(x+1)(2x+2)$. Note that $2x+2$ is a constant times $x+1$. The factorization may be written as $2(x+1)^2$, where $x+1$ is a repeated factor. We say that "$x+1$ appears exactly two times in the factorization of $2x^2+4x+2$. Always collect factors that are constants times each other.

Regarding Step 4

Finding the unknown constants may take a lot of work. If there are only linear factors without repetition, the method illustrated in Example 3 is quick. Clearing denominators and comparing the corresponding coefficients of the polynomials on both sides of the resulting equation always works. The number of unknown constants always equals the degree of the denominator $B(x)$. If $B(x)$ has repeated linear or second-degree factors and the degree of $B(x)$ is "large," consider using a computing tool to find approximations to the coefficients.

EXAMPLE 2 What is the form of the partial fraction representation of

$$\frac{x^{10}+x+3}{(x+1)^2(2x+2)^3(x-1)^2(x^2+x+3)^2}? \tag{8.4.2}$$

SOLUTION The degree of the denominator is 11 and the degree of the numerator is 10. Thus (8.4.2) is proper. There is no need to divide the numerator by the denominator.

The factor $2x+2$ is $2(x+1)$. So $(x+1)^2(2x+2)^3$ should be written as $8(x+1)^5$. The discriminant of x^2+x+3 is $(1)^2-4(1)(3) = -11 < 0$; thus x^2+x+3 is irreducible. Therefore the partial fraction representation of (8.4.2) has the form

$$\frac{k_1}{x+1}+\frac{k_2}{(x+1)^2}+\frac{k_3}{(x+1)^3}+\frac{k_4}{(x+1)^4}+\frac{k_5}{(x+1)^5}$$
$$+\frac{k_6}{x-1}+\frac{k_7}{(x-1)^2}+\frac{r_1x+s_1}{x^2+x+3}+\frac{r_2x+s_2}{(x^2+x+3)^2}.$$

Note that the number of unknown constants equals the degree of the denominator in (8.4.2). ◇

Finding the constants in Example 2 would be a major task if done by hand. It would involve solving a system of 11 equations for the 11 unknown constants. Fortunately, this is an ideal problem for a computer to solve.

Denominator Has Only Linear Factors, Each Appearing Only Once

We illustrate this case, which can be done without a computer, by an example.

EXAMPLE 3 Express $\frac{1}{(2x+1)(x+3)}$ in the form $\frac{k_1}{2x+1}+\frac{k_2}{x+3}$ and then find $\int \frac{dx}{(2x+1)(x+3)}$.

SOLUTION

$$\frac{1}{(2x+1)(x+3)} = \frac{k_1}{2x+1} + \frac{k_2}{x+3}. \tag{8.4.3}$$

To find k_1, multiply both sides of (8.4.3) by the denominator of the term that contains k_1, $2x+1$, getting

$$\frac{1}{x+3} = k_1 + \frac{k_2(2x+1)}{x+3}. \tag{8.4.4}$$

Equation (8.4.4) is valid for all values of x except $x=-3$, in particular for the value of x that makes $2x+1=0$, namely $x=-1/2$. Evaluating (8.4.3) when $x=-1/2$ we get

$$\frac{1}{\left(\frac{-1}{2}\right)+3} = k_1 + 0.$$

We have found that k_1 is $\frac{2}{5}$.

The same idea can be used to solve for k_2: multiply both sides of (8.4.3) by $(x+3)$, obtaining

$$\frac{1}{2x+1} = \frac{k_1(x+3)}{2x+1} + k_2.$$

Replace x by -3, the solution to $x+3=0$, to obtain

$$\frac{1}{2(-3)+1} = 0 + k_2.$$

Thus $k_2 = \frac{-1}{5}$.

Since $k_1=\frac{2}{5}$ and $k_2=\frac{-1}{5}$, (8.4.3) takes the form

$$\frac{1}{(2x+1)(x+3)} = \frac{2/5}{2x+1} - \frac{1/5}{x+3}.$$

To verify this identity, check it by multiplying both sides by $(2x+1)(x+3)$, getting

$$1 = \frac{2}{5}(x+3) - \frac{1}{5}(2x+1) = \frac{2}{5}x + \frac{6}{5} - \frac{2}{5}x - \frac{1}{5} = \frac{5}{5}. \tag{8.4.5}$$

It checks.

For a quicker, but not complete, check replace x in (8.4.3) by a convenient number and see if the resulting equation is correct. Try it, with, say, $x = 0$.

Another way to solve for the unknown constants is to clear the denominator and equate coefficients of like powers of x. For instance, let us find k_1 and k_2 in (8.4.3). We obtain

$$1 = k_1(x+3) + k_2(x+3).$$

Collecting coefficients, we have

$$1 = (k_1 + 2k_2)x + (3k_1 + k_2). \tag{8.4.6}$$

Comparing coefficients on both sides of (8.4.6) we have

$$\begin{array}{rcll} 0 &=& k_1 + 2k_2 & \text{[equating coefficients of } x\text{]} \\ 1 &=& 3k_1 + k_2 & \text{[equating constant terms]} \end{array}$$

There are many ways to solve these simultaneous equations. One way is to use the first equation to express k_1 in terms of k_2: $k_1 = -2k_2$. Then replace k_1 by $-2k_2$ in the second, getting

$$1 = 3(-2k_2) + k_2 = -5k_2$$

from which it is seen that $k_2 = \frac{-1}{5}$. Then $k_1 = \frac{2}{5}$.

In general, in this method the number of equations always equals the number of unknowns, which is also equal to the degree of the denominator. If that degree is large, it is not realistic to do the calculations by hand. ◇

If the denominator is just a repeated linear factor, there are two options: "clearing the denominator and equate coefficients" or "substitution". For instance, the partial fraction representation of

$$\frac{7x+6}{(x+2)^2}$$

you could let $u = x+2$, hence $x = u-2$. Then

$$\begin{aligned} \frac{7x+6}{(x+2)^2} &= \frac{7(u-2)+6}{u^2} = \frac{7u}{u^2} - \frac{8}{u^2} \\ &= \frac{7u}{u^2} - \frac{8}{u^2} = \frac{7}{u} - \frac{8}{u^2} = \frac{7}{x+2} - \frac{8}{(x+2)^2}. \end{aligned}$$

This method for representing

$$\frac{A(x)}{(ax+b)^n}$$

Binomial Theorem: $(u+v)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} u^{n-k} v^k$

is practical if the degree of $A(x)$ is small. Here $u = ax + b$, hence $x = \frac{1}{a}(u - b)$. If the degree of $A(x)$ is not small, expressing a power of x, x^m, in terms of u would best be done by the Binomial Theorem, which is proved in Exercise 32 in Section 5.4.

The next example illustrates one way of dealing with a denominator that has both first and second degree factors.

EXAMPLE 4 Obtain the partial-fraction representation of $\frac{x^2}{x^4-1}$.

SOLUTION First factor the denominator: $x^4 - 1 = (x^2+1)(x+1)(x-1)$. There are constants c_1, c_2, c_3, and c_4 such that

As a check, note that there are 4 constraints to find and $x^4 - 1$ has degree 4.

$$\frac{x^2}{x^4-1} = \frac{c_1}{x+1} + \frac{c_2}{x-1} + \frac{c_3 x + c_4}{x^2+1}.$$

Clear the denominator, but *do not expand the right-hand side*:

$$x^2 = c_1(x-1)(x^2+1) + c_2(x+1)(x^2+1) + (c_3 x + c_4)(x-1)(x+1). \quad (8.4.7)$$

Instead, substitute $x = 1$ and $x = -1$ into (8.4.7) to obtain, respectively:

$$\begin{array}{rcll} 1 & = & 0 + 4c_2 + 0 & \text{[substitute } x = 1 \text{ in (8.4.7)]} \\ 1 & = & -4c_1 + 0 + 0 & \text{[substitute } x = -1 \text{ in (8.4.7)].} \end{array}$$

Setting $x = 0$ compares the constant terms on both sides of (8.4.7).

Already we see that $c_1 = \frac{-1}{4}$ and $c_2 = \frac{1}{4}$.

Next, substitute 0 for x in (8.4.7), obtaining

$$0 \;=\; -c_1 + c_2 - c_4 \qquad \text{[substituting } x = 0 \text{ in (8.4.7)].}$$

Hence $c_4 = \frac{1}{2}$.

We still have to find c_3. We could substitute another number, say $x = 2$, or compare coefficients in (8.4.7). Let us compare coefficients of just the highest degree, x^3. Without going to the bother of multiplying (8.4.7) out in full, we can read off the coefficient of x^3 on both sides by sight, getting

$$0 = c_1 + c_2 + c_3.$$

Since $c_1 = \frac{-1}{4}$, $c_2 = \frac{1}{4}$, if follows that $c_3 = 0$. Hence

$$\frac{x^2}{x^4-1} = \frac{\frac{-1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{2}}{x^2+1}.$$

◇

Example 4 used a combination of two methods: substituting convenient values of x and equating coefficients. We could have just compared coefficients. There would be an equation corresponding to each power of x up to x^3. That would give 4 equations in 4 unknowns. The Exercises suggest how to solve such equations, if you must solve them by hand.

The constant term corresponds to the power x^0.

Summary

We described ways to integrate rational functions. The key idea is algebraic: express the function as the sum of functions that are easier to integrate.

The first step is to check that the integrand is a proper rational function, that is, the degree of the numerator is less than the degree of the denominator. If it isn't, use long division to express the function as the sum of a polynomial and a proper rational function. A flowchart for this process is presented in Figure 8.4.1.

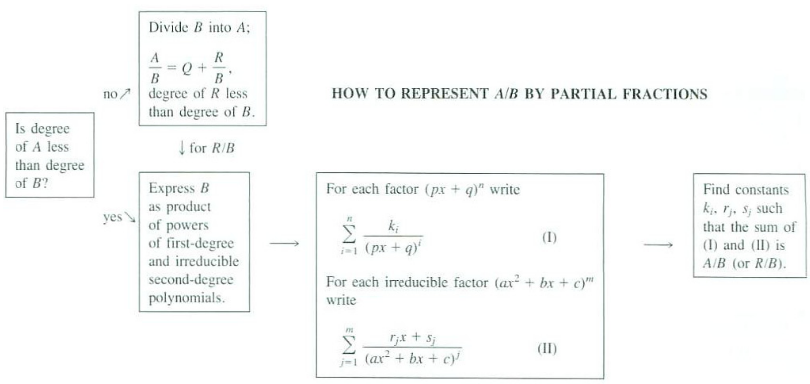


Figure 8.4.1:

THE REAL WORLD

Say that you wanted to compute the definite integral

$$\int_1^2 \frac{x+3}{x^3+x^2+x+2}\,dx.$$

One way is by partial fractions, but this can be tedious. You would probably prefer to estimate the definite integral by one of the approximation techniques in Section 6.5. Alternatively, computers and many scientific calculators, can be programmed to estimate a definite integral. On many graphing calculators you would enter the integrand, the variable of integration, and the limits of integration. In a matter of seconds the TI-89 provides 0.49353 as an approximation with an error less than 0.00001.

As noted in Chapter 6, in some cases computers and calculators can even give the exact (symbolic) value of a definite integral by first finding an antiderivative. In practical applications, however, formal antidifferentiation is not that important. The present example could theoretically be computed by partial fractions, but modern computational tools can evaluate it accurately to as many decimal places as we may want. For example, Simpson's rule with only 8 sections gives 0.514393 as an approximate value for this definite integral.

In other situations some of the coefficients in either the numerator or denominator of the integrand may be given only as decimal approximations. In these situations, too, it often is easier and more appropriate to use a computational method to obtain a numerical answer.

EXERCISES for Section 8.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 10 give the form of the partial fraction representation, but do not find the unknowns. NOTE: Each expression is already proper.

1.[R] $\dfrac{3x^3+5x+2}{(x-1)(x-2)(x-3)(x-4)}$

2.[R] $\dfrac{x^2-5x+3}{(x+1)^2(2x+3)}$

3.[R] $\dfrac{2x^2+x+1}{(x+1)^3}$

4.[R] $\dfrac{3x}{(x+1)(2x+2)}$

5.[R] $\dfrac{x^2-x+3}{(x+1)(x^2+1)}$

6.[R] $\dfrac{2x^2+3x+5}{(x-1)(x^2+x+1)}$

7.[R] $\dfrac{5x^3+x^2+1}{(x^2+x+1)^2}$

8.[R] $\dfrac{x^3+x+1}{(x^2+x+1)^3}$

9.[R] $\dfrac{x^2+x+2}{x^3-x}$

10.[R] $\dfrac{x^2+x+2}{x^4-1}$

11.[R] The rational function $1/(a^2-x^2)$, where a is constant, commonly appears in applications. Represent this function in partial fractions.

Exercises 12 to 15 concern improper rational functions. In each case express the given function as the sum of a polynomial and a proper rational function.

12.[R] $\dfrac{x^2}{x^2+x+1}$

13.[R] $\dfrac{x^3}{(x+1)(x+2)}$

14.[R] $\dfrac{x^5-2x+1}{(x+1)(x^2+1)}$

15.[R] $\dfrac{x^5+x}{(x+1)^2(x-2)}$

In Exercises 16 to 19 find the partial fraction representation.

16.[R] $\dfrac{5}{x^2-1}$

17.[R] $\dfrac{x+3}{(x+1)(x+2)}$

18.[R] $\dfrac{1}{(x-1)^2(x+2)}$

19.[R] $\dfrac{6x^2-2}{(x-1)(x-2)(2x-3)}$

20.[M] Show that $\dfrac{6+5e^{3x}+2e^{2x}+e^x}{5+e^{2x}+e^x}$ has an elementary antiderivative, but do not find it.

21.[M] Solve Example 3 by clearing the denominator in (8.4.3) to get

$$1 = k_1(x+3) + k_2(2x+1).$$

Replace x by any number you please. That gives an equation in k_1 and k_2. Then replace x by another number of your choice, to obtain a second equation in k_1 and k_2. Solve the equations. NOTE: Why are $x=-3$ and $x=-1/2$ the nicest choices?

22.[R] Express each of these polynomials as the product of first degree polynomials.

(a) x^2+2x+1

(b) x^2+5x-3

(c) x^2-4x-6

(d) $2x^2+3x-4$

23.[R] Which of these polynomials is irreducible:

(a) $3x^2+2x+1$

(b) $2x^2+4x+1$

In Exercises 24 to 33 express the rational function in terms of partial fractions.

24.[R] $\dfrac{5x^2-x-1}{x^2(x-1)}$

25.[R] $\dfrac{2x^2+3}{x(x+1)(x+2)}$

26.[R] $\dfrac{5x^2-2x-2}{x(x^2-1)}$

27.[R] $\dfrac{5x^2+9x+6}{(x+1)(x^2+2x+2)}$

28.[R] $\dfrac{5x^2+2x+3}{x(x^2+x+1)}$

29.[R] $\dfrac{x^3-3x^2+3x-3}{x^2-3x+2}$

30.[R] $\dfrac{3x^3+2x^2+3x+1}{x(x^2+1)}$

31.[R] $\dfrac{x^5+2x^4+4x^3+2x^2+x-2}{x^4-1}$

32.[R] $\dfrac{5x^2+6x+10}{(x-2)(x^2+3x+4)}$

33.[R] $\dfrac{3x^2-x-2}{(x+1)(2x^2+x+1)}$

34.[M]

(a) For which value of b is $3x^2+bx+2$ reducible? irreducible?

(b) For which value of b is $3x^2+bx-2$ reducible? irreducible?

35.[M]

(a) For which value of c is $3x^2+5x+c$ reducible? irreducible?

(b) For which value of c is $3x^2-5x+c$ reducible? irreducible?

36.[M] Sam was complaining to Jane, "I found this formula in my integral tables:

$$\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right| \qquad (a,\ b \text{ constants})$$

But my instructor said you won't get any logs other than logs of linear and quadratic polynomials."

Jane: "Maybe the table is wrong."

Sam: "I took the derivative. It's correct. Can I sue my instructor for misleading the young?"

Does Sam has a foundation for a case against his instructor? Explain.

We did not discuss the problem of factoring a polynomial $B(x)$ into linear and irreducible quadratic polynomials. Exercises 37 to 41 concern this problem when the degree of $B(x)$ is 2, 3, or 4.

37.[M]

(a) Show that if $b^2 - 4ac > 0$, then $ax^2 + bx + c = a(x - r_1)(x - r_2)$, where r_1 and r_2 are the distinct roots of $ax^2 + bx + c$.

(b) Show that if $b^2 - 4ac = 0$, then $ax^2 + bx + c = a(x - r)(x - r)$, with r the only root of $ax^2 + bx + c - 0$.

NOTE: The two parts show that if $b^2 - 4ac \geq 0$, then $ax^2 + bx + c$ is reducible. Compare with Exercise 38.

38.[M]

(a) Show that if $ax^2 + bx + c$ is reducible, then it can be written in the form $a(x - s_1)(x - s_2)$ for some real numbers s_1 and s_2.

(b) Deduce that s_1 and s_2 are the roots of $ax^2 + bx + c = 0$.

(c) Deduce that $b^2 - 4ac \geq 0$.

NOTE: From these three parts it follows that if $ax^2 + bx + c$ is reducible, then $b^2 - 4ac \geq 0$. Compare with Exercise 37.

39.[R] Factor each of these polynomials:

(a) $x^2 + 6x + 5$,

(b) $x^2 - 5$,

(c) $2x^2 + 6x + 3$.

40.[R]

(a) Show that $x^2 + 3x - 5$ is reducible.

(b) Using (a), find $\int dx/(x^2 + 3x - 5)$ by partial fractions.

(c) Find $\int dx/(x^2 + 3x - 5)$ by using an integral table.

41.[M] Compute as easily as possible.

(a) $\displaystyle\int \frac{x^3\,dx}{x^4 + 1}$

(b) $\displaystyle\int \frac{x\ dx}{x^4+1}$

(c) $\displaystyle\int \frac{dx}{x^4+1}$

42.[C] Show that any rational function of e^x has an elementary antiderivative. NOTE: That is, any function of the form $\frac{P(e^x)}{Q(e^x)}$ where P and Q are polynomials.

43.[C] If ax^2+bx+c is irreducible must ax^2-bx+c also be irreducible? Must ax^2+bx-c?

44.[C] Explain why every polynomial of odd degree has at least one linear factor. (Therefore, every polynomial of odd degree greater than one is reducible.)

45.[C] In arithmetic every fraction can be written as an integer plus a proper fraction. For instance, $\frac{25}{3}=8+\frac{1}{3}$. Why?

46.[C] In arithmetic, the analog of the partial fraction representation is this: Every fraction can be written as the sum of an integer and fractions of the form c/p^n, where p is a prime and $|c|$ is less than p. Check that this is true for 53/18.

47.[C] Let a be a solution of the equation $P(x)=0$, where $P(x)$ is a polynomial. Prove that $x-a$ must be a factor of $P(x)$. HINT: When you use long division to divide $P(x)$ by $x-a$, show why the remainder is 0. NOTE: This is the basis for the Factor Theorem (see Appendix B).

48.[C]

(a) Use the quadratic formula to find the roots of $x^2+7x+9=0$.

(b) With the aid of the Factor Theorem (Exercise 47), write x^2+7x+9 as the product of two linear polynomials.

(c) Check the factorization by multiplying it out.

8.5 Special Techniques

So far in this chapter you have met three techniques for computing integrals. The first, substitution, and the second, integration by parts, are used most often. Partial fractions applies to special rational functions and is used in solving some differential equations. In this section we compute some special integrals such as $\int \sin(mx)\cos(nx)\ dx$, $\int \sin^2(\theta)\ d\theta$, and $\int \sec(\theta)\ d\theta$, which you may meet in applications. Then we describe substitutions that deal with special classes of integrands.

Computing $\int \sin(mx)\sin(nx)\ dx$

m and n are integers

Fourier series are discussed in Section 12.7

The integrals $\int \sin(mx)\sin(nx)\ dx$, $\int \cos(mx)\sin(nx)\ dx$, and $\int \cos(mx)\cos(nx)\ dx$ are needed in the study of Fourier series, an important tool in the study of heat, sound, and signal processing. They can be computed with the aid of the identities:

$$\begin{aligned}
\sin(A)\sin(B) &= \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B);\\
\sin(A)\cos(B) &= \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B);\\
\cos(A)\cos(B) &= \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B).
\end{aligned}$$

These identities can be checked using the well-known identities for $\sin(A\pm B)$ and $\cos(A\pm B)$.

EXAMPLE 1 Find $\displaystyle\int_0^{\pi/4} \sin(3x)\sin(2x)\ dx$.

SOLUTION

$$\begin{aligned}
\int_0^{\pi/4} \sin(3x)\sin(2x)\ dx &= \int_0^{\pi/4} \left(\frac{1}{2}\cos(x) - \frac{1}{2}\cos(5x)\right)\ dx = \left(\frac{1}{2}\sin(x) - \frac{1}{10}\sin(5x)\right)\\
&= \left(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{20}\right) - \left(\frac{0}{2} - \frac{0}{10}\right) = \frac{3\sqrt{2}}{10} \approx 0.42426.
\end{aligned}$$

◇

Computing $\int \sin^2(x)\ dx$ and $\int \cos^2(x)\ dx$

These integrals can be computed with the aid of the identities

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \text{and} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}. \tag{8.5.1}$$

EXAMPLE 2 Find an antiderivative of $\sin^2(x)$:
SOLUTION

$$\int \sin^2(x)\ dx = \int \frac{1-\cos(2x)}{2}\ dx = \int \frac{dx}{2} - \int \frac{\cos(2x)}{2}\ dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C.$$

◇

Computing $\int \tan(\theta)\ d\theta$ and $\int \tan^2(\theta)\ d\theta$

Antiderivatives of $\tan(\theta)$ and $\sec(\theta)$ are found using similar methods.

EXAMPLE 3 Find $\displaystyle\int \tan(\theta)\ d\theta$.
SOLUTION The approach is to rewrite the integrand in a form where the trigonometric functions can be eliminated with a substitution. Here, this is accomplished by writing $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and using the substitution with $u = \cos(\theta)$ and $du = -\sin(\theta)$ as follows:

$$\int \tan(\theta)\ d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\ d\theta = \int \frac{-du}{u} = -\ln(u) + C = -\ln|\cos(\theta)| + C. \tag{8.5.2}$$

◇

Most integral tables have the formula

$$\int \tan(\theta)\ d\theta = \ln|\sec(\theta)| + C. \tag{8.5.3}$$

Exercise 49 shows that this formula agrees with (8.5.2).

Finding $\int \tan^2(\theta)\ d\theta$ is easier. Using the trigonometric identity $\tan^2(\theta) = \sec^2(\theta) - 1$, we obtain

$$\int \tan^2(\theta)\ d\theta = \int (\sec^2(\theta) - 1)\ d\theta = \tan(\theta) - \theta + C.$$

Computing $\int \sec(\theta)\ d\theta$

EXAMPLE 4 Find $\displaystyle\int \sec(\theta)\ d\theta$, assuming $0 \le \theta \le \pi/2$.
SOLUTION We begin by, once again, rewriting the integrand in a form where substitution can be used:

This integral is the key to Mercator maps, discussed in the CIE on page 861.

$$\int \sec(\theta)\ d\theta = \int \frac{1}{\cos(\theta)}\ d\theta = \int \frac{\cos(\theta)}{\cos^2(\theta)}\ d\theta = \int \frac{\cos(\theta)}{1-\sin^2(\theta)}\ d\theta.$$

The substitution $u = \sin(\theta)$ and $du = \cos(\theta)\ d\theta$ transforms this last integral into the integral of a rational function:

$$\begin{aligned}\int \frac{du}{1-u^2} &= \frac{1}{2}\int \left(\frac{1}{1+u} + \frac{1}{1-u}\right)\ du \\ &= \frac{1}{2}\left(\ln(1+u) - \ln(1-u)\right) + C \\ &= \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) + C.\end{aligned}$$

Because $\frac{1+u}{1-u}$ is positive for $-1 < u < 1$, absolute values are not needed.

Since $u = \sin(\theta)$,

$$\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) = \frac{1}{2}\ln\left(\frac{1+\sin(\theta)}{1-\sin(\theta)}\right).$$

Thus,

$$\int \sec(\theta)\ d\theta = \frac{1}{2}\ln\left(\frac{1+\sin(\theta)}{1-\sin(\theta)}\right) + C. \tag{8.5.4}$$

◇

Another formula for $\int \sec(\theta) d\theta$.

Most integral tables have the formula

$$\int \sec(\theta)\ d\theta = \ln|\sec(\theta) + \tan(\theta)| + C. \tag{8.5.5}$$

Exercise 48 shows that this formula agrees with (8.5.4).

In contrast to Example 4, $\int \sec^2(\theta)\ d\theta$ is easy, since it is simply $\tan(\theta) + C$.

The Substitution $u = \sqrt[n]{ax+b}$

The next example illustrates the use of the substitution $u = \sqrt[n]{ax+b}$. After the example we describe the integrands for which the substitution is appropriate.

EXAMPLE 5 Find $\displaystyle\int_4^7 x^2\sqrt{3x+4}\ dx$.

SOLUTION Let $u = \sqrt{3x+4}$, hence $u^2 = 3x+4$. Then $x = (u^2-4)/3$ and $dx = (2u/3)\ du$. Moreover, as x goes from 4 to 7, u goes from $\sqrt{16} = 4$ to $\sqrt{25} = 5$. Thus

$$\begin{aligned}\int_4^7 x^2\sqrt{3x+4}\ dx &= \int_4^5 \underbrace{\left(\frac{u^2-4}{3}\right)^2}_{x^2} \underbrace{u}_{\sqrt{3x+4}} \underbrace{\frac{2u}{3}\ du}_{dx} = \frac{2}{27}\int_4^5 (u^2-4)^2 u^2\ du \\ &= \frac{2}{27}\int_4^5 (u^6 - 8u^4 + 16u^2)\ du = \frac{1214614}{2835} \approx 428.43527.\end{aligned}$$

◇

Exercise 54 uses the substitution $u = \sqrt[n]{ax+b}$ to integrate any rational function of x and $u = \sqrt[n]{ax+b}$.

Three Trigonometric Substitutions

For the following substitutions we need the notion of a rational function in two variables, u and v. First, a polynomial in u and v is a sum of terms of the form $cu^m v^n$, where c is a number and m and n are nonnegative integers. The quotient of two such polynomials is called a **rational function in two variables**, and labeled $R(u,v)$. If one replaces u by x and v by $\sqrt{a^2-x^2}$ we obtain a rational function of x and $\sqrt{a^2-x^2}$, $R(x,\sqrt{a^2-x^2})$.

Any rational function of x and $\sqrt{a^2-x^2}$, where a is a constant, is transformed into a rational function of $\cos(\theta)$ and $\sin(\theta)$ by the substitution $x = a\sin(\theta)$. Similar substitutions are possible for integrands involving $\sqrt{a^2+x^2}$ or $\sqrt{x^2-a^2}$. In each case, one of the trigonometric identities $1-\sin^2(\theta) = \cos^2(\theta)$, $\tan^2(\theta)+1$, or $\sec^2(\theta)-1 = \tan^2(\theta)$ converts a sum or difference of squares into a perfect square.

How to integrate

If the integrand is a rational function of x and

Case 1 $\sqrt{a^2-x^2}$; let $x = a\sin(\theta)$ $\quad (a>0,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$. $\quad R(x,\sqrt{a^2-x^2})$

Case 2 $\sqrt{a^2+x^2}$; let $x = a\tan(\theta)$ $\quad (a>0,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2})$. $\quad R(x,\sqrt{a^2+x^2})$

Case 3 $\sqrt{x^2-a^2}$; let $x = a\sec(\theta)$ $\quad (a>0,\ 0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2})$. $\quad R(x,\sqrt{x^2-a^2})$

The motivation is simple. Consider Case 1, for instance. If you replace x in $\sqrt{a^2-x^2}$ by $a\sin(\theta)$, you obtain

How to make the square root sign in $\sqrt{a^2-x^2}$ disappear

$$\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin(\theta))^2} = \sqrt{a^2(1-\sin^2(\theta))} = \sqrt{a^2\cos^2(\theta)} = a\cos(\theta).$$

(Keep in mind that a and $\cos(\theta)$ are positive.) The important thing is that *the square root sign disappears.*

Case 3 raises a fine point. We have $a>0$. However, whenever x is negative, θ is an angle in the second-quadrant, so $\tan(\theta)$ is *negative*. In that case,

If $c<0$, $\sqrt{c^2} = -c$.

$$\sqrt{x^2-a^2} = \sqrt{(a\sec(\theta))^2-a^2} = a\sqrt{\sec^2(\theta)-1} = a\sqrt{\tan^2(\theta)} = a(-\tan(\theta))$$

In the Examples and Exercises involving Case 3 it will be assumed that x varies through nonnegative values, so that θ remains in the first quadrant and $\sqrt{\sec^2(\theta)-1} = \tan(\theta)$.

Note that for $\sqrt{a^2-x^2}$ to be meaningful, $|x|$ must be no larger than a. On the other hand, for $\sqrt{x^2-a^2}$ to be meaningful, $|x|$ must be at least as large as a. The quantity $\sqrt{a^2+x^2}$ is meaningful for all values of x.

EXAMPLE 6 Compute $\int \sqrt{1+x^2}\, dx$

SOLUTION The identity $\sqrt{1+\tan^2(\theta)} = \sec(\theta)$ suggests the substitution

$$\begin{aligned} x &= \tan(\theta) \\ \text{so that} \qquad dx &= \sec^2(\theta)\, d\theta. \end{aligned}$$

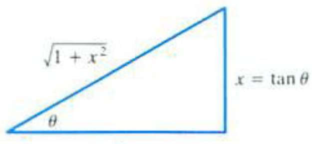


Figure 8.5.1:

(Figure 8.5.1 shows the geometry of this substitution.) Thus

$$\int \sqrt{1+x^2}\, dx = \int \sec(\theta)\sec^2(\theta)\, d\theta = \int \sec^3(\theta)\, d\theta.$$

By Formula 51 from the integral table on the front cover,

$$\int \sec^3(\theta)\, d\theta = \frac{\sec(\theta)\tan(\theta)}{2} + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| + C. \tag{8.5.6}$$

To express the antiderivative just obtained in terms of x rather than θ, it is necessary to express $\tan\theta$ and $\sec\theta$ in terms of x. Starting with the definition $x = \tan(\theta)$, find $\sec(\theta)$ by means of the relation $\sec(\theta) = \sqrt{1+\tan^2(\theta)} = \sqrt{1+x^2}$, as in Figure 8.5.1. Thus

$$\int \sqrt{1+x^2}\, dx = \frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\ln\left(\sqrt{1+x^2}+x\right) + C. \tag{8.5.7}$$

◇

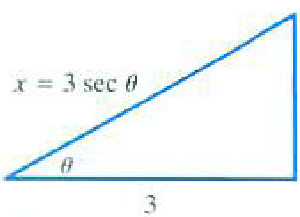


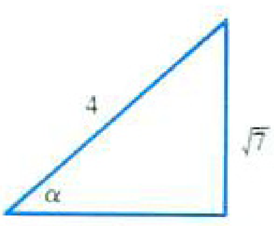
Figure 8.5.2:

EXAMPLE 7 Compute $\int_4^5 \frac{dx}{\sqrt{x^2-9}}$.

SOLUTION Let $x = 3\sec(\theta)$; hence $dx = 3\sec(\theta)\tan(\theta)\, d\theta$. (See Figure 8.5.2.) Thus, letting $\alpha = \operatorname{arcsec}(4/3)$ and $\beta = \operatorname{arcsec}(5/3)$, we obtain

$$\begin{aligned} \int_4^5 \frac{dx}{\sqrt{x^2-9}} &= \int_\alpha^\beta \frac{2\sec(\theta)\tan(\theta)\, d\theta}{\sqrt{9\sec^2(\theta)-9}} = \int_\alpha^\beta \frac{\sec(\theta)\tan(\theta)\, d\theta}{\tan(\theta)} \\ &= \int_\alpha^\beta \sec(\theta)\, d\theta = \ln|\sec(\theta)+\tan(\theta)|\Big|_\alpha^\beta \\ &= \ln\left(\tfrac{5}{3}+\tfrac{4}{3}\right) - \ln\left(\tfrac{4}{3}+\tfrac{\sqrt{7}}{3}\right) = \ln(3) - \ln\left(\tfrac{4+\sqrt{7}}{3}\right) \\ &= 2\ln(3) - \ln(4+\sqrt{7}) = \ln\left(\tfrac{9}{4+\sqrt{7}}\right) \approx 0.30325. \end{aligned}$$

Figures 8.5.3 and 8.5.4 were used to find $\tan(\alpha) = \frac{\sqrt{7}}{3}$ and $\tan(\beta) = \frac{4}{3}$. ◇



A Half-Angle Substitution for $R(\cos\theta, \sin\theta)$

Any rational function of $\cos(\theta)$ and $\sin(\theta)$ is transformed into a rational function of u by the substitution $u = \tan(\theta/2)$. This is sometimes useful after one of the three basic trigonometric substitutions has been used, leaving the integrand in terms of $\cos(\theta)$ and $\sin(\theta)$. The substitution $u = \tan(\theta/2)$ then yields an integral that can be treated by partial fractions. (See Exercises 56 and 57.)

Summary

We discussed some special integrals and integration techniques. First we saw how to evaluate the following common integrals:

$$\int \sin(mx)\sin(nx)\ dx, \quad \int \sin(mx)\cos(nx)\ dx, \quad \int \cos(mx)\cos(nx)\ dx,$$

$$\int \sin^2(x)\ dx, \quad \int \cos^2(x)\ dx,$$

$$\int \sec(\theta)\ d\theta, \quad \int \tan(\theta)\ d\theta, \quad \text{and} \int \tan^2(\theta)\ d\theta.$$

The integration of higher powers of the trigonometric functions is discussed in the exercises.

We also pointed out that the substitution $u = \sqrt[n]{ax+b}$ transforms a rational function in x and $\sqrt[n]{ax+b}$, $R(x, \sqrt[n]{ax+b})$ into a rational function of u. Similarly, $R(x, \sqrt[n]{a^2-x^2})$, $R(x, \sqrt[n]{x^2-a^2})$ and $R(x, \sqrt[n]{a^2+x^2})$ can be transformed into rational functions of $\cos(\theta)$ and $\sin(\theta)$ by trigonometric substitutions. $R(\cos(\theta), \sin(\theta))$ can be transformed into a rational function of u by the substitution $u = \tan(\theta/2)$, which can then be integrated by partial fractions.

EXERCISES for Section 8.5 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 16 are related to Examples 1 to 3. In Exercises 1 to 14 find the integrals.

1.[R] $\int \sin(5x)\sin(3x)\ dx$

2.[R] $\int \sin(5x)\cos(2x)\ dx$

3.[R] $\int \cos(3x)\sin(2x)\ dx$

4.[R] $\int \cos(2\pi x)\sin(5\pi x)\ dx$

5.[R] $\int \sin^2(3x)\ dx$

6.[R] $\int \cos^2(5x)\ dx$

7.[R] $\int (3\sin(2x) + 4\sin^2(5x))\ dx$

8.[R] $\int (5\cos(2x) + \cos^2(7x))\ dx$

9.[R] $\int (3\sin^2(\pi x) + 4\cos^2(\pi x))\ dx$

10.[R] $\int \sec(3\theta)\ d\theta$

11.[R] $\int \tan(2\theta)\ d\theta$

12.[R] $\int \sec^2(4x)\ dx$

13.[R] $\int \tan^2(5x)\ dx$

14.[R] $\int \frac{dx}{\cos^2(3x)}$

15.[R] Show that $\sin(A)\sin(B) = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$.

16.[R] Show that $\sin(A)\cos(B) = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$.

Exercises 17 to 19 develop the formulas that are the foundation for Fourier series, discussed in more detail in Section 12.7.

17.[M] Let m and k be positive integers. Show that

(a) $\int\limits_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\ dx = L.$

(b) $\displaystyle\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = 0.$

18.[M] Let m and k be positive integers. Show that

(a) $\displaystyle\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)\,dx = L.$

(b) $\displaystyle\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx = 0.$

19.[M] Let m and k be positive integers. Show that $\displaystyle\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx =$ 0.

Exercises 20 to 29 concern the substitution $u = \sqrt[n]{ax+b}$. In each case evaluate the integral.

20.[R] $\displaystyle\int x^2\sqrt{2x+1}\,dx$

21.[R] $\displaystyle\int \frac{x^2\,dx}{\sqrt[3]{x+1}}$

22.[R] $\displaystyle\int \frac{dx}{\sqrt{x+3}}$

23.[R] $\displaystyle\int \frac{\sqrt{2x+1}}{x}\,dx$

24.[R] $\displaystyle\int x\sqrt[3]{3x+2}\,dx$

25.[R] $\displaystyle\int \frac{\sqrt{x}+3}{\sqrt{x}-2}\,dx$

26.[R] $\displaystyle\int \frac{x\,dx}{\sqrt{x}+3}$

27.[R] $\displaystyle\int x(3x+2)^{5/3}\,dx$

28.[R] $\displaystyle\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}}$ Hint: Let $u = \sqrt[6]{x}$.

29.[R] $\displaystyle\int (x+2)\sqrt[5]{x-3}\,dx$

In Exercises 30 to 40 find the integrals using trigonometric substitution. (a is a positive constant.)

30.[R] $\int \sqrt{4-x^2}\ dx$

31.[R] $\int \frac{dx}{\sqrt{9+x^2}}$

32.[R] $\int \frac{x^2\ dx}{\sqrt{x^2-9}}$

33.[R] $\int x^3\sqrt{1-x^2}\ dx$

34.[R] $\int \frac{\sqrt{4+x^2}}{x}\ dx$

35.[R] $\int \sqrt{a^2-x^2}\ dx$

36.[R] $\int \frac{dx}{\sqrt{a^2-x^2}}$

37.[R] $\int \sqrt{a^2+x^2}\ dx$

38.[R] $\int \sqrt{a^2-x^2}\ dx$

39.[R] $\int \frac{dx}{\sqrt{25x^2-16}}$

40.[R] $\int_{\sqrt{2}}^{2} \sqrt{x^2-1}\ dx$

Exercises 41 and 42 concern the recursion formulas for $\int \tan^n(\theta)\ d\theta$ and $\int \sec^n(\theta)\ d\theta$.

41.[M] In Example 3 we found $\int \tan(\theta)\ d\theta$ and $\int \tan^2(\theta)\ d\theta$.

(a) Obtain the recursion

$$\int \tan^n(\theta)\ d\theta = \frac{\tan^{n-1}(\theta)}{n-1} - \int \tan^{n-2}(\theta)\ d\theta.$$

Begin by writing

$$\tan^n(\theta) \quad = \quad \tan^{n-2}(\theta)\tan^2(\theta) = \tan^{n-2}(\theta)(\sec^2(\theta)-1).$$

(b) Use the recursion formula to find $\displaystyle\int \tan^3(\theta)\ d\theta$.

(c) Find $\displaystyle\int \tan^4(\theta)\ d\theta$.

NOTE: See Example 3.

42.[R] In Example 4 we found $\int \sec(\theta)\ d\theta$ and $\int \sec^2(\theta)\ d\theta$.

(a) Obtain the recursion

$$\int \sec^n(\theta)\ d\theta = \frac{\sec^{n-2}(\theta)\tan(\theta)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(\theta)\ d\theta.$$

Begin by writing $\sec^n(\theta) = \sec^{n-2}(\theta)\sec^2(\theta)$, and integrating by parts. After the integration, $\tan^2(\theta)$ will appear in the integrand. Write it as $\sec^2(\theta)-1$.

(b) Evaluate $\displaystyle\int \sec^3(\theta)\ d\theta$.

(c) Evaluate $\int \frac{d\theta}{\cos^4(\theta)}$.

(d) Evaluate $\int \sec^2(2x)\ dx$.

NOTE: See Example 4.

43.[R] Find

(a) $\int \csc(\theta)\ d\theta$

(b) $\int \csc^2(\theta)\ d\theta$

44.[R] Find

(a) $\int \cot(\theta)\ d\theta$

(b) $\int \cot^2(\theta)\ d\theta$

45.[M] Consider $\int \sin^n(\theta)\cos^m(\theta)\ d\theta$, where m and n are nonnegative integers, and m is odd. To evaluate $\int \sin^n(\theta)\cos^m(\theta)\ d\theta$, write it as $\int \sin^n(\theta)\cos^{m-1}(\theta)\cos(\theta)\ d\theta$. Then, because $m-1$ is even, rewrite $\cos^{m-1}(\theta)$ as $(1-\sin^2(\theta))^{(m-1)/2}$ and use the substitution $u = \sin(\theta)$. Using this technique, find

(a) $\int \sin^3(\theta)\cos^3(\theta)\ d\theta$

(b) $\int \sin^4(\theta)\cos(\theta)\ d\theta$

(c) $\int_0^{\pi/2} \sin^4(\theta)\cos^3(\theta)\ d\theta$

(d) $\int \cos^5(\theta)\ d\theta$.

46.[M] How would you integrate $\int \sin^n(\theta)\cos^m(\theta)\ d\theta$, where m and n are nonnegative integers and n is odd? Illustrate your techniques by three examples. NOTE: See Exercise 45.

47.[M] The techniques in Exercises 45 and 46 apply to $\int \sin^n(\theta)\cos^m(\theta)\,d\theta$ only when at least one of m and n is odd. If both are even, first use the identities

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \qquad \text{and} \qquad \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}.$$

You will get a polynomial in $\cos(2\theta)$. If $\cos(2\theta)$ appears only to odd powers, the technique of Exercise 45 suffices. To treat an even power of $\cos(2\theta)$, use the identity $\cos^2(2\theta) = (1+\cos(4\theta))/2$ and continue. Using this method find

(a) $\int \cos^2(\theta)\sin^4(\theta)\,d\theta$

(b) $\int_0^{\pi/4} \cos^2(\theta)\sin^2(\theta)\,d\theta$

Antiderivatives of $\sec(\theta)$ and $\tan(\theta)$ were found in Examples 4 and 3. Exercises 48 to 50 explore some other antiderivatives of these functions.

48.[R] Let $0 \le \theta < \pi/2$.

(a) Show that $\displaystyle\int \sec(\theta)\,d\theta = \ln|\sec(\theta)+\tan(\theta)| + C$, by differentiating $\ln|\sec(\theta)+\tan(\theta)|$.

(b) Does (a) contradict the formula given in Example 4?

49.[R] Show that $-\ln(\cos(\theta))$ and $\ln(\sec(\theta))$ are both antiderivatives for $\tan(\theta)$.

50.[M] In 1645, Henry Bond conjectured from experimental data that $\int_0^{\theta} \sec(t)\,dt = \ln\left(\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)$ While Bond's conjecture was originally verified well before the advent of calculus, today we can verify Bond's conjecture by (i) checking that this formula holds for $\theta = 0$ and (ii) checking that the right-hand side is an antiderivative of $\sec(\theta)$. NOTE: Bond's conjecture is related to Mercator's projection (discussed in the CIE on page 861. Reference: `http://www.math.ubc.ca/~israel/m103/mercator/mercator.html` [permission needs to be requested from Robert Israel].

51.[R] The region R under $y = \sin(x)$ and above $[0,\pi]$ is revolved about the x-axis to produce a solid S.

(a) Draw R.

(b) Draw S.

(c) Set up a definite integral for the area of R.

(d) Set up a definite integral for the volume of S.

(e) Evaluate the integrals in (c) and (d).

52.[M] Transform the following integrals into integrals of rational functions of $\cos(\theta)$ and $\sin(\theta)$. Do *not* evaluate the integrals.

(a) $\int \frac{x+\sqrt{9-x^2}}{x^3}\, dx$

(b) $\int \frac{x^3\sqrt{5-x^2}}{1+\sqrt{5x^2}}\, dx$

53.[M] Transform the following integrals into integrals of rational functions of $\cos(\theta)$ and $\sin(\theta)$. Do *not* evaluate the integrals.

(a) $\int \frac{x^2+\sqrt{x^2-9}}{x}\, dx$

(b) $\int \frac{x^3\sqrt{5+x^2}}{x+2}]dx$

54.[M] Let $R(x,y)$ be a rational function of x and y. Let n be an integer greater than or equal to 2. Then $R(x, \sqrt[n]{ax+b})$ is a "rational function of x and $\sqrt[n]{ax+b}$." Let $R(x,y) = \frac{x+y^2}{2x-y}$.

(a) Evaluate $R(x, \sqrt[3]{4x+5})$.

(b) Use the substitution $u = \sqrt[3]{4x+5}$ to show that

$$\int \frac{x+(4x+5)^{2/3}}{2x-(4x+5)^{1/3}}\, dx = \frac{3}{8}\int \frac{(u^3+4u^2-5)u^2}{u^3-2u-5}\, du$$

NOTE: Do not attempt to evaluate this integral. The partial fraction decomposition of this integrand is very messy!

55.[M] Transform the following integrals into integrals of rational functions of u. Do *not* evaluate the integrals.

(a) $\displaystyle\int \frac{\sqrt[3]{x+2}}{x^2+(x+2)^{2/3}}\, dx$

(b) $\displaystyle\int \frac{\sqrt{x}+x+x^{3/2}}{\sqrt{x}+2}\, dx$

Exercises 56 to 58 concern $\int R(\cos(\theta), \sin(\theta))\, d\theta$.

56.[M] Let $-\pi < \theta < \pi$ and $u = \tan(\theta/2)$. (See Figure 8.5.5(a).) The following steps show that this substitution transforms $\int R(\cos\theta, \sin\theta)d\theta$ into the integral of a rational function of u (which can be integrated by partial fractions).

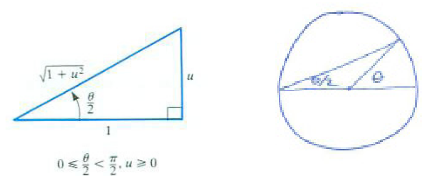
(a) Show that $\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{1+u^2}}$ and $\sin\left(\frac{\theta}{2}\right) = \frac{u}{\sqrt{1+u^2}}$.

(b) Using (a), show that $\cos(\theta) = \frac{1-u^2}{1+u^2}$.

(c) Show that $\sin(\theta) = \frac{2u}{1+u^2}$.

(d) Show that $d\theta = \frac{2\,du}{1+u^2}$. HINT: Note that $\theta = 2\arctan(u)$.

Combining (b), (c), and (d) shows that the substitution $u = \tan(\theta/2)$ transforms $\int R(\cos(\theta), \sin(\theta))\,d\theta$ into $\int R\left(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}\right)\,\frac{2}{1+u^2}\,du$, an integral of a rational function of u.



(a) (b)

Figure 8.5.5:

57.[M] Using the substitution $u = \tan(\theta/2)$, transform the following integrals into integrals of rational functions. HINT: Refer to Figure 8.5.5(b). (Do not evaluate them.)

(a) $\displaystyle\int \frac{1+\sin(\theta)}{1+\cos^2(\theta)}\,d\theta$

(b) $\displaystyle\int \frac{5+\cos(\theta)}{(\sin(\theta))^2+\cos(\theta)}\,d\theta$

(c) $\displaystyle\int_0^{\pi/2} \frac{5\,d\theta}{2\cos(\theta)+3\sin(\theta)}$ (Be sure to transform the limits of integraton also.)

58.[M] Compute $\displaystyle\int_0^{\pi/2} \frac{d\theta}{4\sin(\theta)+3\cos(\theta)}$.

59.[C] Explain why any rational function of $\tan(\theta)$ and $\sec(\theta)$ has an elementary antiderivative.

60.[C] Show that any rational function of x, $\sqrt{x+a}$, and $\sqrt{x+b}$ has an elementary antiderivative. HINT: Use the substitution $u = \sqrt{x+a}$.

However, it is not the case that every rational function of $\sqrt{x+a}$, $\sqrt{x+b}$, and $\sqrt{x+c}$ has an elementary antiderivative. For instance,

$$\int \frac{dx}{\sqrt{x}\sqrt{x+1}\sqrt{x-1}} = \int \frac{dx}{\sqrt{x^3-x}}$$

is not an elementary function.

61.[C] Every rational function of x and $\sqrt[n]{(ax+b)/(cx+d)}$ has an elementary antiderivative. Explain why.

62.[C] Assume $x-c$ is a factor of $Q(x)$ and not of $P(x)$. Also assume $(x-c)^2$ is not a factor of $Q(x)$. The term $A/(x-c)$ therefore appears in the partial fraction representation of $P(x)/Q(x)$. Show that $A = P(c)/Q'(c)$. HINT: First, multiply both sides of the partial fraction representation by $x-c$.

8.6 What to do When Confronted with an Integral

Since the exercises in each section of this chapter focus on the techniques of that section, it is usually clear what technique to use on a given integral. But what if an integral is met "in the wild," where there is no clue how to evaluate it? This section suggests what to do in this typical situation.

The more integrals you compute, the more quickly you will be able to choose an appropriate technique. Moreover, such practice will put you at ease in using integral tables or computer software. Besides, it may be quicker to find an integral by hand.

This table summarizes the techniques and shortcuts emphasized in this chapter.

General	Substitution	Section 8.2
	Integration by Parts	Section 8.3
	Partial Fractions	Sections 8.4 and 8.2
Special	if f is odd, then $\int_{-a}^{a} f(x)\ dx = 0$	Section 8.1
	if f is even, then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$	Section 8.1
	$\int_0^a \sqrt{a^2-x^2}\ dx = \frac{\pi a^2}{4}$	Section 8.1
	$\int \sin(mx)\sin(nx)\ dx$, etc.	Section 8.5
	$\int \sin^2(\theta)\ d\theta$, etc.	Section 8.5
	$\int \tan(\theta)\ d\theta$, $\int \sec(\theta)\ d\theta$, etc.	Section 8.5
	$\int R(x, \sqrt[n]{ax+b})\ dx$	Section 8.5
	$\int R(x, \sqrt{a^2-x^2})\ dx$, etc.	Section 8.5
	$\int R(\cos(\theta), \sin(\theta))\ dx$, etc.	Section 8.5

Table 8.6.1:

Exercises in Section 8.5 develop other specialized techniques, but they will not be required in this section.

A few examples will illustrate how to choose a method for computing an antiderivative.

EXAMPLE 1

$$\int \frac{x\ dx}{1+x^4}$$

See Exercise 57 in Section 7.5

SOLUTION DISCUSSION: Since the integrand is a rational function of x, partial fractions would work. This requires factoring x^4+1 and then representing $x/(1+x^4)$ as a sum of partial fractions. With some struggle it can be found that

$$x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$$

The constants A, B, C, and D will have to be found such that

$$\frac{x}{1+x^4} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$

The method would work but would certainly be tedious.

Try another attack. The numerator x is almost the derivative of x^2. The substitution $u = x^2$ is at least worth testing. With $u = x^2$ we find $du = 2x\ dx$ and so

$$\int \frac{x\ dx}{1+x^4} = \int \frac{du/2}{1+u^2},$$

which is easy:

Check by differentiating.

$$\int \frac{x\ dx}{1+x^4} = \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(x^2) + C.$$

◇

EXAMPLE 2

$$\int \frac{1+x}{1+x^2}\ dx.$$

SOLUTION DISCUSSION: This is a rational function of x, but partial fractions will not help, since the integrand is already in its partial-fraction form.

The numerator is not the derivative of the denominator, but it comes close enough to persuade us to break the integrand into two summands:

$$\int \frac{1+x}{1+x^2}\ dx = \int \frac{dx}{1+x^2} + \int \frac{x\ dx}{1+x^2}.$$

Both the latter integrals can be done in your head. The first is $\arctan(x)$, and the second is $(1/2)\ln(1+x^2)$. So

$$\int \frac{1+x}{1+x^2}\ dx = \arctan(x) + \frac{1}{2}\ln(1+x^2) + C.$$

◇

EXAMPLE 3

$$\int \frac{e^{2x}}{1+e^x}\ dx.$$

SOLUTION DISCUSSION: At first glance, this integral looks so peculiar that it may not even be elementary. However, e^x is a fairly simple function, with $d(e^x) = e^x\ dx$. This suggests trying the substitution $u = e^x$ and seeing what happens:

$$u = e^x \qquad\qquad du = e^x\ dx$$

It is essential to express dx completely in terms of u and du.

Thus

$$dx = \frac{du}{e^x} = \frac{du}{u}.$$

But what will be done to e^{2x}? Recalling that $e^{2x} = (e^x)^2 = u^2$, we anticipate there will be no difficulty:

$$\int \frac{e^{2x}}{1+e^x}\ dx = \int \frac{u^2}{1+u}\frac{du}{u} = \int \frac{u\ du}{1+u}.$$

Long division of $u/(u+1)$ also works.

which can be integrated quickly:

$$\begin{aligned}\int \frac{u\ du}{1+u} &= \int \frac{u+1-1}{1+u}\ du = \int \left(1 - \frac{1}{1+u}\right)\ du \\ &= u - \ln(|1+u|) + C = e^x - \ln(1+e^x) + C.\end{aligned}$$

The same substitution could have been done more elegantly:

$$\int \frac{e^{2x}}{1+e^x}\ dx = \int \frac{e^x(e^x\ dx)}{1+e^x} = \int \frac{u\ du}{1+u}.$$

◇

EXAMPLE 4

$$\int \frac{x^3\ dx}{(1-x^2)^5}.$$

SOLUTION DISCUSSION: Partial fractions would work, but the denominator, when factored, would be $(1+x)^5(1-x)^5$. There would be 10 unknown constants to find. Look for an easier approach.

Since the denominator is the obstacle, try $u = x^2$ or $u = 1 - x^2$ to see if the integrand gets simpler. Let us examine what happens in each case. Try $u = x^2$ first. Assume that we are interested only in getting an antiderivative for positive x, $x = \sqrt{u}$:

$$u = x^2 \qquad\qquad du = 2x\ dx \qquad\qquad dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}}.$$

Then

$$\int \frac{x^3\ dx}{(1-x^2)^5} = \int \frac{u^{3/2}}{(1-u)^5}\frac{du}{2\sqrt{u}} = \frac{1}{2}\int \frac{u\ du}{(1-u)^5}.$$

The same substitution could be carried out as follows:

$$\int \frac{x^3\ dx}{(1-x^2)^5} = \int \frac{x^2 x\ dx}{(1-x^2)^5} = \int \frac{u(du/2)}{(1-u)^5} = \frac{1}{2}\int \frac{u\ du}{(1-u)^5}.$$

The substitution $v = 1-u$ then results in an easy integral.

Verify this claim for yourself.

Observe that the two substitutions $u = x^2$ and $v = 1-u$ are equivalent to the single substitution $v = 1-x^2$. So, let us apply the substitution $u = 1-x^2$ to the original integral. Then $du = -2x\ dx$; thus

$$\int \frac{x^3\ dx}{(1-x^2)^5} = \int \frac{x^2(x\ dx)}{(1-x^2)^5} = \int \frac{(1-u)(-du/2)}{u^5} = \int \frac{1}{2}(u^{-4}-u^{-5})\ du,$$

an integral that can be computed without further substitution. So $u = 1-x^2$ is quicker than $u = x^2$. ◇

EXAMPLE 5

$$\int x^3 e^{x^2}\ dx.$$

SOLUTION DISCUSSION: Integration by parts may come to mind, since if $u = x^3$, then $du = 3x^2\ dx$ is simpler. However, dv must then be $e^{x^2}\ dx$ and force v to be non-elementary. This is a dead end.

So try integration by parts with $u = e^{x^2}$ and $dv = x^3\ dx$. What will $v\ du$ be? We have $v = x^4/4$ and $du = 2xe^{x^2}\ dx$, which is worse than the original $u\ dv$. The exponent of x has been raised by 2, from 3 to 5.

If we can raise an exponent, we should be able to lower it.

This time try $u = x^2$ and $dv = xe^{x^2}\ dx$; thus $du = 2x\ dx$ and $v = e^{x^2}/2$. Integration by parts yields

$$\begin{aligned}\int x^3 e^{x^2}\ dx &= \int \underbrace{x^2}_{u}\underbrace{xe^{x^2}\ dx}_{dv} = \underbrace{x^2}_{u}\underbrace{\frac{e^{x^2}}{2}}_{v} - \int \underbrace{\frac{e^{x^2}}{2}}_{v}\underbrace{2x\ dx}_{du} \\ &= \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C.\end{aligned}$$

See Exercise 71.

Another approach is to use the substitution $u = x^2$ followed by an integration by parts. ◇

EXAMPLE 6

$$\int \frac{1-\sin(\theta)}{\theta+\cos(\theta)} d\theta.$$

See Exercise 72.

SOLUTION DISCUSSION: The numerator is the derivative of the denominator, so the integral is simply $\ln|\theta + \cos\theta| + C$. ◇

EXAMPLE 7

$$\int \frac{1-\sin(\theta)}{\cos(\theta)}\, d\theta.$$

SOLUTION DISCUSSION: Break the integrand into two summands:

$$\begin{aligned}
\int \frac{1-\sin(\theta)}{\cos(\theta)}\, d\theta &= \int \left(\frac{1}{\cos(\theta)} - \frac{\sin(\theta)}{\cos(\theta)}\right)\, d\theta \\
&= \int \left(\sec(\theta) - \tan(\theta)\right)\, d\theta \\
&= \int \sec\theta\, d\theta - \int \tan(\theta)\, d\theta \\
&= \ln|\sec(\theta) + \tan(\theta)| + \ln|\cos(\theta)| + C.
\end{aligned}$$

Since $\ln(A) + \ln(B) = \ln(AB)$, the answer can be simplified to

$$\ln\left(|\sec(\theta) + \tan(\theta)|\,|\cos(\theta)|\right) + C.$$

But $\sec(\theta)\cos(\theta) = 1$ and $\tan(\theta)\cos(\theta) = \sin(\theta)$. The result becomes even simpler:

The absolute values are not needed because $1 + \sin(\theta) \geq 0$

$$\int \frac{1-\sin(\theta)}{\cos(\theta)}\, d\theta = \ln\left(1 + \sin(\theta)\right) + C.$$

◇

EXAMPLE 8

$$\int \frac{\ln x\, dx}{x}.$$

SOLUTION DISCUSSION: Integration by parts, with $u = \ln(x)$ and $dv = dx/x$, may come to mind. In that case, $du = dx/x$ and $v = \ln(x)$; thus

$$\int \underbrace{\ln(x)}_{u} \underbrace{\frac{dx}{x}}_{dv} = \underbrace{(\ln(x))}_{u}\underbrace{(\ln(x))}_{v} - \int \underbrace{\ln(x)}_{v} \underbrace{\frac{dx}{x}}_{du}.$$

Bringing $\int \ln(x)\, dx/x$ all to one side produces the equation

$$2\int \ln(x)\frac{dx}{x} = (\ln x)^2,$$

from which it follows that

$$\int \ln(x)\frac{dx}{x} = \frac{(\ln(x))^2}{2} + C.$$

The integration by parts approach worked, but is not the easiest one to use. Since $1/x$ is the derivative of $\ln(x)$, we could have used the substitution $u = \ln(x)$, which means $du = dx/x$. Thus

$$\int \frac{\ln(x)\ dx}{x} = \int u\ du = \frac{u^2}{2} + C = \frac{(\ln(x))^2}{2} + C.$$

◇

EXAMPLE 9

$$\int_0^{3/5} \sqrt{9 - 25x^2}\ dx.$$

SOLUTION DISCUSSION: This integral reminds us of $\int_0^a \sqrt{a^2 - x^2}\ dx = \pi a^2/4$, the area of a quadrant of a circle of radius a. This resemblance suggests a substitution u such that $25x^2 = 9u^2$ or $u = \frac{5}{3}x$, hence $dx = \frac{3}{5}\ du$. Then substitution gives

$$\begin{aligned} \int_0^{3/5} \sqrt{9 - 25x^2}\ dx &= \int_0^1 \sqrt{9 - 9u^2}\ \frac{3}{5}\ du = \frac{9}{5}\int_0^1 \sqrt{1 - u^2}\ du \\ &= \frac{9}{5} \cdot \frac{\pi}{4} = \frac{9\pi}{20} \approx 1.41372. \end{aligned}$$

◇

EXAMPLE 10

$$\int \sin^5(2x)\cos(2x)\ dx.$$

SOLUTION DISCUSSION: We could try integration by parts with $u = \sin^5(2x)$ and $dv = \cos(2x)\ dx$. (See Exercise 73.)

However, $\cos(2x)$ is almost the derivative of $\sin(2x)$. For this reason make the substitution

$$u = \sin(2x) \qquad\qquad du = 2\cos(2x)\ dx;$$

This means that

$$\cos(2x)\ dx = \frac{du}{2}.$$

and so

$$\int \sin^5(2x)\cos(2x)\ dx = \int u^5\ \frac{du}{2} = \frac{1}{2}\frac{u^6}{6} + C = \frac{\sin^6(2x)}{12} + C.$$

◇

EXAMPLE 11

$$\int_{-3}^{3} x^3 \cos(x)\ dx.$$

SOLUTION DISCUSSION: Since the integrand is of the form $P(x)\cos(x)$, where P is a polynomial, repeated integration by parts would work. On the other hand, x^3 is an odd function and $\cos(x)$ is an even function. The integrand is therefore an odd function and the integral over $[-3, 3]$ is 0. ◇

EXAMPLE 12

$$\int \sin^2(3x)\ dx.$$

SOLUTION DISCUSSION: You could rewrite this integral as $\int \sin(3x)\sin(3x)\ dx$ and use integration by parts. However, it is easier to use the trigonometric identity $\sin^2(\theta) = (1 - \cos 2(\theta))/2$:

$$\int \sin^2(3x)\ dx = \int \frac{1-\cos(6x)}{2}\ dx = \int \frac{dx}{2} - \int \frac{\cos(6x)}{2}\ dx = \frac{x}{2} - \frac{\sin(6x)}{12} + C.$$

◇

EXAMPLE 13

$$\int_{1}^{2} \frac{x^3 - 1}{(x+2)^2}\ dx.$$

SOLUTION DISCUSSION: Partial fractions would certainly work. (The first step would be division of $x^3 - 1$ by $x^2 + 4x + 4$.) However, the substitution $u = x + 2$ is easier because it makes the denominator simply u^2. We have

$$u = x + 2 \qquad du = dx \quad \text{and} \quad x = u - 2.$$

Thus

Note the new limits for u.

$$\begin{aligned}
\int_1^2 \frac{x^3-1}{(x+2)^2}\,dx &= \int_3^4 \frac{(u-2)^3-1}{u^2}\,du = \int_3^4 \frac{u^3-6u^2+12u-8-1}{u^2}\,du \\
&= \int_3^4 \left(u-6+\frac{12}{u}-\frac{9}{u^2}\right)\,du = \left(\frac{u^2}{2}-6u+12\ln|u|+\frac{9}{u}\right)\Big|_3^4 \\
&= \left(8-24+12\ln(4)+\frac{9}{4}\right) - \left(\frac{9}{2}-18+12\ln(3)+3\right) \\
&= -(\frac{13}{4})+12\ln(4)-12\ln(3) = 12\ln\left(\frac{4}{3}\right)-\frac{13}{4} \approx 0.20218.
\end{aligned}$$

◇

Summary

One word: PRACTICE.

Practice is the best way to improve your integration skills. Reading worked examples in a book is good, but doesn't provide practice making the necessary decisions and does not help you recognize when a particular approach will not be successful, or an error has been made.

Many integrals can be evaluated in several different ways, but one method is usually the easiest.

It is also important to learn to recognize integrals that can be evaluated without finding an antiderivative or are known to not have an elementary antiderivative.

EXERCISES for Section 8.6 *Key:* R–routine, M–moderate, C–challenging

All the integrals in Exercises 1 to 59 are elementary. In each case, list the technique or techniques that could be used to evaluate the integral. If there is a preferred technique, state what it is (and why). Do *not* evaluate the integrals.

1.[R] $\int \frac{1+x}{x^2}\ dx$

2.[R] $\int \frac{x^2}{1+x}\ dx$

3.[R] $\int \frac{dx}{x^2+x^3}$

4.[R] $\int \frac{x+1}{x^2+x^3}\ dx$

5.[R] $\int \arctan(2x)\ dx$

6.[R] $\int \arcsin(2x)\ dx$

7.[R] $\int x^{10}e^x\ dx$

8.[R] $\int \frac{\ln(x)}{x^2}\ dx$

9.[R] $\int \frac{\sec^2(\theta)\ d\theta}{\tan(\theta)}$

10.[R] $\int \frac{\tan(\theta)\ d\theta}{\sin^2(\theta)}$

11.[R] $\int \frac{x^3}{\sqrt[3]{x+2}}\ dx$

12.[R] $\int \frac{x^2}{\sqrt[3]{x^3+2}}\ dx$

13.[R] $\int \frac{2x+1}{(x^2+x+1)^5}\ dx$

14.[R] $\int \sqrt{\cos(\theta)}\sin(\theta)\ d\theta$

15.[R] $\int \tan^2(\theta)\ d\theta$

16.[R] $\int \frac{d\theta}{\sec^2(\theta)}$

17.[R] $\int e^{\sqrt{x}}\ dx$

18.[R] $\int \sin\sqrt{x}\ dx$

19.[R] $\int \frac{dx}{(x^2-4x+3)^2}$

20.[R] $\displaystyle\int \frac{x+1}{x^5}\ dx$

21.[R] $\displaystyle\int \frac{x^5}{x+1}\ dx$

22.[R] $\displaystyle\int \frac{\ln(x)}{x(1+\ln(x))}\ dx$

23.[R] $\displaystyle\int \frac{e^{3x}\ dx}{1+e^x+e^{2x}}$

24.[R] $\displaystyle\int \frac{\cos(x)\ dx}{(3+\sin(x))^2}$

25.[R] $\displaystyle\int \ln(e^x)\ dx$

26.[R] $\displaystyle\int \ln(\sqrt[3]{x})\ dx$

27.[R] $\displaystyle\int \frac{x^4-1}{x+2}\ dx$

28.[R] $\displaystyle\int \frac{x+2}{x^4-1}\ dx$

29.[R] $\displaystyle\int \frac{dx}{\sqrt{x}(3+\sqrt{x})^2}$

30.[R] $\displaystyle\int \frac{dx}{(3+\sqrt{x})^3}$

31.[R] $\displaystyle\int (1+\tan(\theta))^3 \sec^2(\theta)\ d\theta$

32.[R] $\displaystyle\int \frac{e^{2x}+1}{e^x-e^{-x}}\ dx$

33.[R] $\displaystyle\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\ dx$

34.[R] $\displaystyle\int \frac{(x+3)(\sqrt{x+2}+1)}{\sqrt{x+2}-1}\ dx$

35.[R] $\displaystyle\int \frac{(\sqrt[3]{x+2}-1\ dx}{\sqrt{x+2}+1}$

36.[R] $\displaystyle\int \frac{dx}{x^2-9}$

37.[R] $\displaystyle\int \frac{x+7}{(3x+2)^{10}}\ dx$

38.[R] $\displaystyle\int \frac{x^3\ dx}{(3x+2)^7}$

39.[R] $\displaystyle\int \frac{2^x+3^x}{4^x}\ dx$

40.[R] $\displaystyle\int \frac{2^x}{1+2^x}\ dx$

41.[R] $\displaystyle\int \frac{(x+\arcsin(x))\ dx}{\sqrt{1-x^2}}$

42.[R] $\displaystyle\int \frac{x+\arctan(x)}{1+x^2}\ dx$

43.[R] $\displaystyle\int x^3\sqrt{1+x^2}\ dx$

44.[R] $\displaystyle\int x(1+x^2)^{3/2}\ dx$

45.[R] $\displaystyle\int \frac{x\ dx}{\sqrt{x^2-1}}$

46.[R] $\displaystyle\int \frac{x^3}{\sqrt{x^2-1}}\ dx$

47.[R] $\displaystyle\int \frac{x\ dx}{(x^2-9)^{3/2}}$

48.[R] $\displaystyle\int \frac{\arctan(x)}{1+x^2}\ dx$

49.[R] $\displaystyle\int \frac{\arctan(x)}{x^2}\ dx$

50.[R] $\displaystyle\int \frac{\sin(\ln(x))}{x}\ dx$

51.[R] $\displaystyle\int \cos(x)\ln(\sin(x))\ dx$

52.[R] $\displaystyle\int \frac{x\ dx}{\sqrt{x^2+4}}$

53.[R] $\displaystyle\int \frac{dx}{x^2+x+5}$

54.[R] $\displaystyle\int \frac{x\ dx}{x^2+x+5}$

55.[R] $\displaystyle\int \frac{x+3}{(x+1)^5}\ dx$

56.[R] $\displaystyle\int \frac{x^5+x+\sqrt{x}}{x^3}\ dx$

57.[R] $\displaystyle\int (x^2+9)^{10}x\ dx$

58.[R] $\displaystyle\int (x^2+9)^{10}x^3\ dx$

59.[R] $\displaystyle\int \frac{x^4\ dx}{(x+1)^2(x-2)^3}$

In Exercises 60 to 62, (a) decide which positive integers n yield integrals you can evaluate and (b) evaluate them.

60.[M] $\displaystyle\int \sqrt{1+x^n}\ dx$

61.[M] $\int (1+x^2)^{1/n}\ dx$

62.[M] $\int (1+x)^{1/n}\sqrt{1-x}\ dx$

63.[M] Find $\int \frac{dx}{\sqrt{x+2}-\sqrt{x-2}}$.

64.[M] Find $\int \sqrt{1-\cos(x)}\ dx$.

In Exercises 65 to 70, evaluate the integrals.

65.[M] $\int \frac{x\ dx}{(\sqrt{9-x^2}^5}$

66.[M] $\int \frac{dx}{\sqrt{9-x^2}}$

67.[R] $\int \frac{dx}{x\sqrt{x^2+9}}$

68.[M] $\int \frac{x\ dx}{\sqrt{x^2+9}}$

69.[M] $\int \frac{dx}{x+\sqrt{x^2+25}}$

70.[M] $\int (x^3+x^2)\sqrt{x^2-5}\ dx$

71.[M]

(a) Evaluate $\int x^3 e^{x^2}$ using the substitution $u = x^2$ followed by an application of integration by parts.

(b) How does this approach compare with the one used in Example 5?

72.[M] In Example 6 it is found that

$$\int \frac{1-\sin(\theta)}{\theta+\cos(\theta)} d\theta = \ln|\theta+\cos\theta| + C.$$

Check this result by differentiation.

73.[M]

(a) Use integration parts to evaluate $\int \sin^5(2x)\cos(2x)\ dx$.

(b) How does this approach compare with the one used in Example 10?

8.S Chapter Summary

The previous section reviewed the techniques discussed in the chapter. Here we will offer some general comments on finding antiderivatives.

First of all, while the derivative of an elementary function is again elementary, that is not necessarily the case with antiderivatives. Moreover, it isn't easy to predict whether an antiderivative will be elementary. For instance $\ln(x)$ and $\frac{\ln(x)}{x}$ have elementary antiderivatives but $\frac{x}{\ln(x)}$ does not. Also, $x\sin(x)$ does, but $\frac{\sin(x)}{x}$ does not. Remembering that some elementary functions lack elementary antiderivatives can save you lots of time and frustration.

The substitution technique is the one that will come in handy most often, to reduce an integral to an easier one or to something listed in an integral table.

When an integrand involves a product or quotient, integration by parts may be of use.

The integrals of $\sin(mx)\sin(nx)$, $\sin(mx)\cos(nx)$, and $\cos(mx)\cos(nx)$ will be needed for the discussion of Fourier Series in Section 12.7.

A common partial fraction decomposition is

$$\frac{1}{a^2-x^2} = \frac{1}{2a}\left(\frac{1}{a-x} + \frac{1}{a+x}\right).$$

While it is comforting to know that every rational function has an elementary antiderivative, finding it can be a daunting task except for the simplest denominators. First, factoring the denominator into first and second degree polynomials may be a major hurdle. Second, finding the unknown coefficients in the representation could require lots of computation. In such cases, it may be simpler just to use Simpson's approximation (Section 6.5) — unless one absolutely needs to know the antiderivative. In such cases it might be best to take advantage of an automated integrators available through your calculator or computer.

As we will see in Chapter 12, approximating an integrand by a polynomial offers another way to estimate a definite or indefinite integral.

Some definite integrals over intervals of the form $[-a,a]$ can be simplified before evaluation. Other definite integrals can be evaluated using properties of even and odd functions. If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$; if f is an odd function, then $\int_{-a}^{a} f(x)\ dx = 0$. (For instance,

$\int_{-1}^{1} xe^{x^2}\ dx = 0$.)

Method	*Description*
Substitution (Section 8.2)	Introduce $u = h(x)$. If $f(x)\ dx = g(u)\ du$, then $\int f(x)\ dx = \int g(u)\ du$.
Substitution in a definite integral (Section 8.2)	If $u = h(x)$ with $f(x)\ dx = g(u)\ du$, then $\int_a^b f(x)\ dx = \int_{h(a)}^{h(b)} g(u)\ du$.
Table of Integrals (Section 8.1)	Obtain and become familiar with a good table of integrals. Remember to use substitution to put integrands into the proper form.
Integration by Parts (Section 8.3)	$\int u\ dv = uv - \int v\ du$. Choose u and dv so $u\ dv = f(x)\ dx$ and $\int v\ du$ is easier to integrate than $\int u\ dv$.
Partial Fractions (applies to any rational function of x) (Section 8.4 (and Section 8.2))	This is an algebraic method in which the integrand is written as a sum of a polynomial (which can be zero)) plus terms of the type $\frac{k_i}{(ax+b)^i}$ and $\frac{r_jx+s_j}{(ax^2+bx+c)^j}$.
Certain Trigonometric Products (Section 8.5)	$\int \sin(mx)\cos(nx)\ dx$, $\int \sin(mx)\sin(nx)\ dx$, $\int \cos(mx)\cos(nx)\ dx$ $\int \sin^2(x)\ dx$, $\int \cos^2(x)\ dx$ $\int \tan(x)\ dx$, $\int \tan^2(x)\ dx$ $\int \sec(x)\ dx$,
Rational Functions of x and one of $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ (Section 8.5)	For $\sqrt{a^2-x^2}$, let $x = a\sin(\theta)$. For $\sqrt{a^2+x^2}$, let $x = a\tan(\theta)$. For $\sqrt{x^2-a^2}$, let $x = a\sec(\theta)$.
Rational Functions of x and $\sqrt[n]{ax+b}$ (Section 8.5)	Let $u = \sqrt[n]{ax+b}$.
Rational Functions of $\cos(\theta)$ and $\sin(\theta)$ (Section 8.5)	Let $u = \tan\theta/2$.

Integrand	Method of Integration
$\frac{1}{(ax+b)^n}$	substitute $u = ax + b$
$\frac{1}{ax^2+c}$, a, $c > 0$	substitute $cu^2 = ax^2$: $u = \sqrt{\frac{a}{c}}x$
$\frac{1}{ax^2+bx+c}$, $b^2 - 4ac < 0$	factor out a, complete the square, then substitute
$\frac{x}{ax^2+bx+c}$, $b^2 - 4ac < 0$	first, write x in numerator as $\frac{1}{2a}(2ax+b) - \frac{b}{2a}$, then break into two parts. (That is, get $2ax+b$ into the numerator.)
$\frac{1}{(ax^2+bx+c)^n}$ $b^2 - 4ac < 0$, $n \geq 2$	use a recursive formula from the integral tables
$\frac{x}{(ax^2+bx+c)^n}$ $b^2 - 4ac < 0$, $n \geq 2$	express in terms of the previous type by the method in Example 7

Table 8.S.1: Antiderivatives of common forms that appear in partial fraction representations.

$f(t)$	$F(s) = L[f](s)$	Comments
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\cos(at)$	$\frac{s}{a^2+s^2}$	$s > 0$
$\sin(at)$	$\frac{a}{a^2+s^2}$	$s > 0$
te^{at}	$\frac{1}{(s-a)^2}$	$s > a$

Table 8.S.2: Brief table of Laplace Transforms

EXERCISES for 8.S *Key:* R–routine, M–moderate, C–challenging

1.[R]

(a) By an appropriate substitution, transform this definite integral into a simpler definite integral:

$$\int_0^{\pi/2} \sqrt{(1+\cos(\theta))^3}\sin(\theta)\ d\theta.$$

(b) Evaluate the new integral found in (a).

2.[R] Two of these antiderivatives are elementary functions; evaluate them.

(a) $\displaystyle\int \ln(x)\ dx$

(b) $\displaystyle\int \frac{\ln(x)}{x}\ dx$

(c) $\displaystyle\int \frac{dx}{\ln(x)}$

3.[R] Evaluate

(a) $\int_1^2 (1+x^3)^2\ dx$

(b) $\int_1^2 (1+x^3)^2 x^2\ dx$

4.[R] Use a table of integrals to compute

(a) $\displaystyle\int \frac{e^x\ dx}{5e^{2x}-3}$

(b) $\displaystyle\int \frac{dx}{\sqrt{x^2-3}}$

5.[R] Compute

(a) $\displaystyle\int \frac{dx}{x^3}$

(b) $\displaystyle\int \frac{dx}{\sqrt{x+1}}$

(c) $\displaystyle\int \frac{e^x}{1+5e^x}\ dx$

6.[R] Compute $\displaystyle\int \frac{5x^4-5x^3+10x^2-8x+4}{(x^2-1)(x-1)}\ dx.$

7.[R] Transform the definite integral

$$\int_0^3 \frac{x^3}{\sqrt{x+1}}\ dx$$

into another definite integral in the following ways (and evaluate each transformed integral).

(a) by the substitution $u = x+1$

(b) by the substitution $u = \sqrt{x+1}$.

(c) Which method was easier to apply?

8.[R]

(a) Transform the definite integral

$$\int_{-1}^4 \frac{x+2}{\sqrt{x+3}}\ dx$$

into an easier definite integral by a substitution.

(b) Evaluate the integral obtained in (a).

9.[R] Compute $\int x^2 \ln(1+x)\, dx$ (a) without an integral table, (b) with an integral table.

10.[R] Verify that the following factorizations into irreducible polynomials are correct.

(a) $x^3 - 1 = (x-1)(x^2+x+1)$

(b) $x^4 - 1 = (x-1)(x+1)(x^2+1)$

(c) $x^3 + 1 = (x+1)(x^2-x+1)$

Express each expression in Exercises 11 to 17 as a sum of partial fractions. (Do not integrate.) Exercise 10 may be helpful.

11.[R] $\dfrac{2x^2+3x+1}{x^3-1}$

12.[R] $\dfrac{x^4+2x^2-2x+2}{x^3-1}$

13.[R] $\dfrac{2x-1}{x^3+1}$

14.[R] $\dfrac{x^4+3x^3-2x62+3x-1}{x^4-1}$

15.[R] $\dfrac{2x+5}{x^2+3x+2}$

16.[R] $\dfrac{5x^3+11x^2+6x+1}{x^2+x}$

17.[R] $\dfrac{5x^3+6x^2+8x+5}{(x^2+1)(x+1)}$

18.[R] The Fundamental Theorem of Calculus can be used to evaluate one of these definite integrals, but not the other. Evaluate that integral using the FTC.

(a) $\displaystyle\int_0^1 \sqrt[3]{x}\sqrt{x}\, dx$

(b) $\displaystyle\int_0^1 \sqrt[3]{1-x}\sqrt{x}\, dx$

19.[R] Compute $\int \frac{x^3}{(x-1)^2}\,dx$

(a) using partial fractions

(b) using the substitution $u = x - 1$

(c) which method, (a) or (b), is easier in this case?

20.[R]

(a) Compute $\int \frac{x^{2/3}}{x+1}\,dx$.

(b) What does a table of integrals say about the indefinite integral in (a)?

21.[R] Compute $\int x\sqrt[3]{x+1}\,dx$ using

(a) the substitution $u = \sqrt[3]{x+1}$

(b) the substitution $u = x + 1$

In Exercises 22 to 25 evaluate the integrals.

22.[R] $\int\limits_0^1 (e^x+1)^3 e^x\,dx$

23.[R] $\int\limits_0^1 (x^4+1)^5 x^3\,dx$

24.[R] $\int\limits_1^e \frac{\sqrt{\ln(x)}}{x}\,dx$

25.[R] $\int\limits_9^{\pi/2} \frac{\cos(\theta)}{\sqrt{1+\sin(\theta)}}\,dx$

26.[R]

(a) Without an integral table, evaluate

$$\int \sin^5(\theta)\ d\theta \qquad \text{and} \qquad \int \tan^6(\theta)\ d\theta.$$

(b) Evaluate each integral with an integral table.

(c) Resolve any differences in the appearance of the antiderivatives found in (a) and (b).

27.[R] Two of these three antiderivatives are elementary. Find them, and explain why you know they are elementary (without necessarily evaluating the integral).

(a) $\int \sqrt{1 - 4\sin^2(\theta)}\ d\theta$

(b) $\int \sqrt{4 - 4\sin^2(\theta)}\ d\theta$

(c) $\int \sqrt{1 + \cos(\theta)}\ d\theta$

28.[R] Find $\int \cot(3\theta)\ d\theta$.

29.[R] Find $\int \csc(5\theta)\ d\theta$.

30.[R] Compute

(a) $\int \sec^5(x)\tan(x)\ dx$

(b) $\int \frac{\sin(x)}{\cos^3(x)}\ dx$

31.[R] Compute $\int \frac{x^3\ dx}{(1+x^2)^4}$ in two different ways:

(a) by the substitution $u = 1 + x^2$,

(b) by the substitution $x = \tan(\theta)$.

32.[R] Find $\int \frac{x\ dx}{\sqrt{9x^4+16}}$

(a) without an integral table,

(b) with an integral table.

33.[R] Transform $\int \frac{x^2\,dx}{\sqrt{1+x}}$ by each of the substitutions

(a) $u = \sqrt{1+x}$

(b) $y = 1 + x$

(c) $x = \tan^2(\theta)$

(d) Evaluate the easiest of the above three reformulations.

34.[R] Compute $\int x\sqrt{1+x}\,dx$ in three ways:

(a) $u = \sqrt{1+x}$,

(b) $u = 1 - \tan^2(\theta)$,

(c) by parts, with $u = x$ and $dv = \sqrt{1+x}\,dx$.

35.[R] Find $\int x\sqrt{(1-x^2)^5}\,dx$ using the substitutions

(a) $u = x^2$,

(b) $u = 1 - x^2$,

(c) $x = \sin(\theta)$.

In Exercises 36 to 48, evaluate the definite integral appearing in the given exercise.

36.[R] Exercise 21 in Section 7.1.
37.[R] Exercise 22 in Section 7.1.
38.[R] Exercise 23 in Section 7.1.
39.[R] Exercise 24 in Section 7.1.
40.[R] Exercise 25 in Section 7.1.
41.[R] Exercise 26 in Section 7.1.
42.[R] Exercise 27 in Section 7.1.

43.[R] Exercise 28 in Section 7.1.
44.[R] Exercise 30 in Section 7.1.
45.[M] Exercise 1 in Section 7.5.
46.[M] Exercise 2 in Section 7.5.
47.[M] Exercise 3 in Section 7.5.
48.[M] Exercise 4 in Section 7.5.

49.[M] The region $\mathcal{R}$ below the line $y = e$, above $y = e^x$, and to the right of the y-axis is revolved around the y-axis to form a solid $\mathcal{S}$. In Example 1 in Section 7.5 it is shown that the definite integral for the volume of $\mathcal{S}$ using disks is

$$\int_1^e \pi\left(\ln(y)\right)^2\ dy$$

and the volume of $\mathcal{S}$ using shells is

$$\int_0^1 2\pi x\left(e - e^x\right)\ dx.$$

Evaluate each integral. Which integral is easier to evaluate?

50.[M] The region $\mathcal{R}$ below the line $y = \frac{\pi}{2} - 1$, to the right of the y-axis, and above the curve $y = x - \sin(x)$ is revolved around the y-axis to form a solid $\mathcal{S}$. In Example 2 in Section 7.5 it is shown that the definite integral for the volume of $\mathcal{S}$ using disks cannot be evaluated in terms of elementary functions, and that the volume of $\mathcal{S}$ using shells is

$$\int_0^{\pi/2} 2\pi x\left(\frac{\pi}{2} - 1 - (x - \sin(x))\right)\ dx.$$

Evaluate the value of this integral.

51.[M]

(a) Evaluate $\int \frac{x+1}{x^2} e^{-x}\ dx$.

(b) Evaluate $\int \frac{ax-1}{ax^2} e^{ax}\ dx$, $a \neq 0$

52.[M] In Example 1 in Section 7.6 the total force on a submerged circular tank is found to be

$$\int_{-5}^{5} (0.036)(x+17)\sqrt{100-4x^2}\ dx = 0.036\int_{-5}^{5} x\sqrt{100-4x^2}\ dx + 0.036\int_{-5}^{5} 17\sqrt{100-4x^2}\ dx \text{ pou}$$

At that time, the value of this integral was found using the fact that the first integral has an odd integrand over an interval symmetric about the origin and by relating the second integral to the area of a quarter circle.

(a) Evaluate the first integral using the substitution $u = 100 - 4x^2$.

(b) Evaluate the second integral using the substitution $x^2 = 25\sin^2(\theta)$.

(c) Which approach is easier?

53.[M] Find $\int \frac{dx}{\sin(2x)}$ by first writing $\sin(2x)$ as $2\sin(x)\cos(x)$.

54.[M]

(a) Show that $\int_0^\infty \frac{\sin(kx)}{x}\,dx = \int_0^\infty \frac{\sin(x)}{x}\,dx$, where k is a positive constant.

(b) Show that $\int_0^\infty \frac{\sin(x)\cos(x)}{x}\,dx = \int_0^\infty \frac{\sin(x)}{x}\,dx$.

(c) If k is negative, what is the relation between $\int_0^\infty \frac{\sin kx}{x}\,dx$ and $\int_0^\infty \frac{\sin x}{x}\,dx$?

55.[M] Evaluate $\int_0^\infty e^{-x}\sin^2(x)\,dx$.

56.[M] Evaluate $\int_0^\infty e^{-x}\sin(x)\,dx$. NOTE: This integral was first encountered in Example 4 on page 667.

In statistics a function $F(x)$ defined on $[0,\infty)$ is called a **probability distribution** if $F(0) = 0$, $\lim_{x\to\infty} F(x) = 1$, and F has a nonnegative derivative f. The function f is called a **probability density**. The integral $\int_0^\infty xf(x)\,dx$ is called the **expected value** or **average value** of x. Exercises 57 and 58 show that if one of the integrals $\int_0^\infty xf(x)\,dx$ and $\int_0^\infty (1-F(x))\,dx$ is convergent, so is the other one and these two integrals are equal.

57.[M] Assume $\int_0^\infty xf(x)\,dx$ is finite.

(a) Show that $\int_k^\infty xf(x)\,dx$ approach zero as k approaches ∞.

(b) Using the fact that $\int_k^\infty xf(x)\,dx \geq \int_k^\infty kf(x)\,dx$, show that $\lim_{k\to\infty} k(1 - F(k)) = 0$.

(c) Show that

$$\int_0^k xf(x)\,dx = k(F(k)-1) + \int_0^k (1-F(x))\,dx.$$

HINT: Use integration by parts and $d(F(x)-1) = f(x)\,dx$.

(d) From (c) show that

$$\int_0^\infty x f(x)\ dx = \int_0^\infty (1 - F(x))\ dx.$$

58.[M] Assume that $\int_0^\infty (1 - F(x))\ dx$ is finite.

(a) Show that $\int_0^k f(x)\ dx = kF(k) - \int_0^k F(x)\ dx$. Hint: Use integration by parts with $dF(x) = f(x)\ dx$.

(b) Show $kF(k) - \int_0^k F(x)\ dx \le \int_0^k (1 - F(x))\ dx$.

(c) Show that $\int_0^\infty x f(x)\ dx$ is finite.

(d) Show that $\int_0^\infty x f(x)\ dx = \int_0^\infty (1 - F(x))\ dx$. Hint: Review Exercise 57.

Exercises 59 to 62 are related.

59.[M] Show that $\int_1^\infty (\cos(x))/x^2\ dx$ is convergent.

60.[M] Show that $\int_1^\infty (\sin(x))/x\ dx$ is convergent. Hint: Start with integration by parts.

61.[M] Show that $\int_0^\infty (\sin(x))/x\ dx$ is convergent.

62.[M] Show that $\int_0^\infty \sin(e^x)\ dx$ is convergent.

63.[M] In a statistics text it is asserted that for $\lambda > 0$ and n a positive integer

$$\int_0^\infty 1 - \left(1 - e^{-\lambda t}\right)^n\ dt = \frac{1}{\lambda}\sum_{k=1}^{n} \frac{1}{k}.$$

(a) Check this assertion for $n = 1$.

(b) Check this assertion for $n = 2$.

(c) Show that for all n the integral is convergent.

Hint: For (c), use the Binomial Theorem (see Exercise 32 in Section 5.4).

64.[M] Let $\int_{-\infty}^\infty f(x)\ dx$ be a convergent integral with value A.

(a) Express $\int_{-\infty}^\infty f(x + 2)\ dx$ in terms of A.

(b) Express $\int_{-\infty}^\infty f(2x)\ dx$ in terms of A.

65.[M] Find the error in the following computations: The substitution $x = y^2$, $dx = 2y\ dy$, yields

$$\begin{aligned}\int_0^1 \frac{1}{x}\ dx &= \int_0^1 \frac{2y}{y^2}\ dy = \int_0^1 \frac{2}{y}\ dy \\ &= 2\int_0^1 \frac{1}{y}\ dy = 2\int_0^1 \frac{1}{x}\ dx.\end{aligned}$$

Hence

$$\int_0^1 \frac{1}{x}\ dx = 2\int_0^1 \frac{1}{x}\ dx;$$

from which it follows that $\int_0^1 (1/x)\ dx = 0$.

Laplace Transforms were introduced in Exercises 51 to 55 in Section 8.3. Exercises 66 to 68 develop properties of Laplace Transforms.

66.[M] Let f and its derivative f' both have Laplace transforms. Let P be the Laplace transform of f, and let Q be the Laplace transform of f'. Show that

$$Q(r) = -f(0) + rP(r).$$

67.[M] Assume that $f(t) = 0$ for $t < 0$ and that f has a Laplace transform. Let a be a positive constant. Define $g(t)$ to be $f(t - a)$. Show that the Laplace transform of g is e^{-ar} times the Laplace transform of f. NOTE: The graph of g is the graph of f shifted to the right by a.

68.[C] Let P be the Laplace transform of f. Let a be a positive constant, and let $g(t) = f(at)$. Let P be the Laplace transform of f, and let Q be the Laplace transform of g. Show that $Q(r) = (1/a)P(r/a)$.

69.[M]

(a) Estimate $\int_0^1 \frac{\sin(x)}{x} dx$ by using the Maclaurin polynomial $P_6(x; 0)$ associated with $\sin(x)$ to approximate $\sin(x)$.

(b) Use the Lagrange form of the error to put an upper bound on the error in (a).

70.[M]

(a) Estimate $\int_{-1}^{1} \frac{e^x}{x+2} dx$ by using the Maclaurin polynomial $P_3(x; -2)$ associated with e^x to approximate e^x.

(b) Use the Lagrange form of the error to put an upper bound on the error in (a).

71.[M]

(a) Estimate $\int_{-1}^{1} \frac{e^x}{x-2} dx$ by using the Taylor polynomial $P_3(x; 2)$ associated with e^x to approximate e^x.

(b) Use the Lagrange form of the error to put an upper bound on the error in (a).

72.[M] Find $\displaystyle\int \frac{\ln(x^2)}{x^2} dx$.

73.[M] If a is a constant, show that $\int_{-\infty}^{\infty} e^{-(x-a)^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, dx = 2\int_0^{\infty} e^{-x^2}\, dx$.

74.[M] When studying the normal distribution in statistics one will meet an equation that amounts to

$$\frac{\int_{-\infty}^{\infty} x \exp(-(x-\mu)^2)\ dx}{\int_{-\infty}^{\infty} \exp(-(x-\mu)^2)\ dx} = \mu,$$

where μ is a constant. Show that the equation is correct. HINT: Make the substitution $t = x - \mu$.

75.[M] Show that $\int_1^{\infty} x\exp(-x^2)\ dx$ is less than $\int_0^1 x\exp(-x^2)\ dx$. This implies that the probability of a large disaster, compared to the long tail of the bell curve, is smaller than what must be planned for in spite of the growth of the coefficient x. As a result, economic predictions based on the bell curve may downplay the likelihood of rare events. This bias may have been one of the several factors that combined to produce the credit crisis and recession that began in 2007.

76.[C] For which values of the positive constant k is $\displaystyle\int_e^{\infty} \frac{dx}{x(\ln(x))^k}$ convergent? divergent?

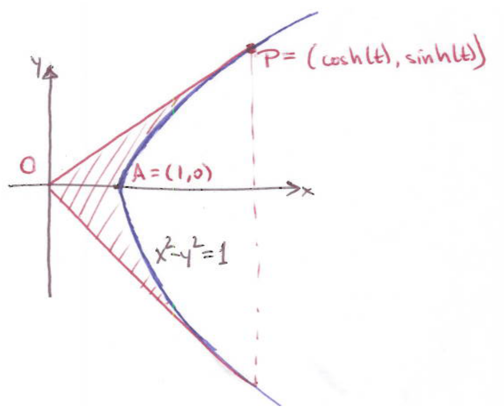


Figure 8.S.1:

77.[M] The formula for the area of region OAP in Figure 8.S.1 was found, in Exercise 64 in Section 6.5, to be

$$\frac{1}{2}\cosh(t)\sinh(t) - \int_1^{\cosh(t)} \sqrt{x^2-1}\,dx$$

Use the substitution $x = \cosh(u)$ to evaluate the definite integral. NOTE: See also Exercises 64 in Section 6.5 and 8 in Section 15.4.

The molecules in a gas move at various speeds. In 1859 James Maxwell developed a formula for the distribution of the speeds of a gas consisting of N molecules. The formula is

$$f(v) = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{\frac{-1}{2}\frac{mv^2}{kT}}$$

This means that for small values, dv, the number of molecules with speed between v and $v + dv$ is approximately $f(v)\ dv$. In the formula k is a physical constant, T is the absolute temperature, and m is the mass of a molecule. The only variable is v. Exercises 78 to 80 investigate Maxwell's model.

78.[C] Show that $\int_0^\infty f(v)\ dv = N$.

79.[C] (continuation of Exercise 78) The average speed of the molecules is

$$\frac{\int_0^\infty vf(v)\ dv}{N}.$$

Show that this equals $\sqrt{8kT/\pi m} \approx 1.5958\sqrt{kT/m}$.

80.[C] (continuation of Exercise 79) The "most probable speed" occurs where $f(v)$ has a maximum. Show that this speed is $\sqrt{2kT/m} \approx 1.4142\sqrt{kT/m}$. So the most likely speed is a bit less than the average speed.

81.[M] In the study of heat capacity of a crystal one meets

$$\int_0^b \frac{x^4 e^x}{(e^x - 1)^2}\ dx.$$

(a) Show that the integral is convergent.

(b) Is $\int_0^b \frac{xe^x}{(e^x-1)^2}\ dx$ convergent?

82.[M] Show that $\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{3/2}} = 2$.

83.[M]

(a) Show that $\int_0^{\infty} \frac{x^2}{(x^2+1)^{5/2}}$ is convergent.

(b) Show that the value of this improper integral is $1/3$.

84.[M] In the theory of probability one meets the equation

$$\int_0^{\infty} e^{-\lambda x} R(x)\ dx = \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} R'(x)\ dx + \frac{1}{\lambda} R(0)$$

Assuming the integrals are convergent, explain how the equation is obtained.

85.[M] The velocity of a particle at time t seconds is $e^{-t}\sin(\pi t)$ meters per second. Find how far it travels in the first second, from time $t = 0$ to $t = 1$,

(a) using the integral table in the front of the book,

(b) using Simpson's method with $n = 4$, expressing your answer to four decimal places.

HINT: Notice that the particle changes direction at $t = 1/2$ second.

86.[C] Assume that f is continuous on $[0, \infty)$ and has period one, that is, $f(x) = f(x + 1)$ for all x in $[0, \infty]$. Assume also that $\int_0^{\infty} e^{-x} f(x)\ dx$ is convergent. Show that

$$\int_0^{\infty} e^{-x} f(x)\ dx = \frac{e}{e-1} \int_0^1 e^{-x} f(x)\ dx.$$

87.[C] Assume that f is continuous on $[0, \infty)$ and has period $p > 0$. Let s be a positive number and assume $\int_0^\infty e^{-st} f(t)\ dt$ converges. Show that this improper integral equals

$$\frac{1}{1-e^{-sp}} \int\limits_0^p e^{-st} f(t)\ dt.$$

88.[C] The integral $\int_0^\infty x^{2n} e^{-kx^2}\ dx$ appears in the kinetic theory of gases. In Chapter 16, we will show that $\int_0^\infty e^{-x^2}\ dx = \sqrt{\pi}/2$. With the aid of this information, evaluate

(a) $\int_0^\infty e^{-kx^2}\ dx$,

(b) $\int_0^\infty x^2 e^{-kx^2}\ dx$.

89.[C] (continuation of Exercise 4) This exercise presents an alternate approach to evaluating the integral in Exercise 4. Express the integral as the Laplace transform of an appropriate function. Then, use a table of Laplace transforms to find the value of the integral.

90.[C] James Maxwell's "On the Geometric Mean Distance of Two Figures in a Plane," written in 1872, begins "There are several problems of great practical importance in electro-magnetic measurements, in which the value of the quantity has to be calculated by taking the sum of the logarithms of the distances of a system of parallel wires from a given point."
This leads him to several problems, of which this is the first.

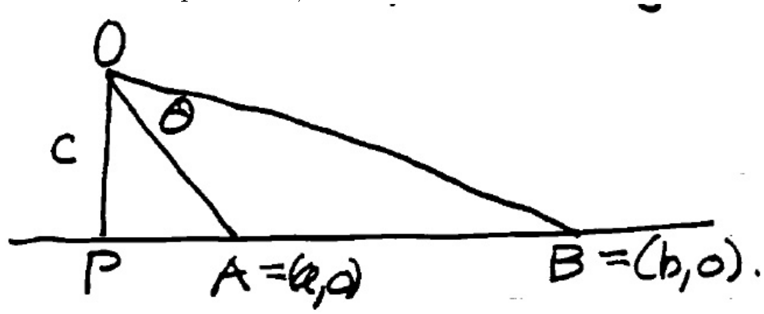


Figure 8.S.2:
A point $\mathcal{O}$ is a distance c from the line that contains the line segment AB. Let P be the point on that line nearest $\mathcal{O}$, as in Figure 8.S.2. Introduce a coordinate system in which P is the origin, AB lies on the x-axis, and OP lies on the y-axis.
Let $f(x)$ be the distance from $\mathcal{O}$ to $(x, 0)$.
Show that the average value of $\ln(f(x))$ for x in $[a, b]$ is

$$\frac{b\ln(b) - a\ln(a) - (b-a) + c\theta}{b-a},$$

where θ is the angle AOB in radians.

Exercises 91 to 92 are related to the CIE on Mercator maps (861).

91.[R] If the distance on a Mercator map is 3 inches from latitude 0° to latitude 20° how far is it on the map from (a) 60° to 80°, (b) 75° to 85°.

92.[M] Show that Bond's conjecture is correct. That is, that $\int_0^\alpha \sec(\theta)\ d\theta = \ln(\tan(\alpha/2+\pi/4))$

93.[M] Evaluate $\displaystyle\int \frac{\cos(\theta)}{(b^2+c^2\cos^2(\theta))^{1/2}}\ d\theta$. NOTE: This integral appears in Exercise 18. HINT: Let $u = c\cos(\theta)$.

94.[M] Show that $\int \sqrt{x}e^x\ dx$ is not elementary. HINT: Use the fact that $\int e^{x^2}\ dx$ is not elementary.

95.[C] We have seen that $\int e^{x^2}\ dx$ is not elementary.

(a) Show that for positive odd integers n, $\int x^n e^{x^2}\ dx$ is elementary.

(b) Show that for positive even integers n, $\int x^n e^{x^2}\ dx$ is not elementary.

96.[C] We have seen that $\int e^{x^2}\ dx$ and $\int \frac{e^x}{x}\ dx$ are not elementary.

(a) Show that $\int \frac{e^{x^2}}{x}\ dx$ is not elementary.

(b) Show that $\int \frac{e^{x^2}}{x^2}\ dx$ is not elementary.

(c) Show that for any positive integer n, $\int \frac{e^{x^2}}{x^n}\ dx$ is not elementary.

97.[C] We have seen that $\int \frac{e^x}{x}\ dx$ is not elementary.

(a) Show that for positive integers n, $\int x^n e^x\ dx$ is elementary.

(b) Show that for positive integers n, $\int \frac{e^x}{x^n}\ dx$ is not elementary

98.[C]

(a) Show that $\displaystyle\int x^2 e^{x^2}\ dx$ is not elementary.

(b) Show that $\displaystyle\int x^4 e^{x^2}\ dx$ is not elementary.

(c) Find non-zero values for a and b such that $\displaystyle\int (ax^4+bx^2)e^{x^2}\,dx$ is an elementary function.

99.[C] Show that $\int x^n e^{x^2}$ is elementary only when n is an odd positive integer.

100.[C] Let n be an integer. Show that $\int x^n e^x$ is elementary only when n is not negative.

101.[M]

Sam: I understand what a definite integral is — the limit of certain sums. I accept on faith that for a continuous function the limit exists. I agree that it is a handy idea, with many uses, but I don't see why I have to learn all those ways to compute it: antiderivatives, trapezoids, Simpson's method. My trusty calculator evaluates integrals to eight decimal places and a computer algebra system can often give me the exact expression.

Jane: What's your point?

Sam: I would make this text much shorter by omitting this chapter. This would allow us more time to spend on the stuff at the end.

Does Sam have a valid argument, for a change?

Exercises 102 to 107 all relate to the famous **bell curve** that arises in statistics.
102.[M] Use the fact that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ (see Exercise 34 in Section 17.3) to show that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}.$$

103.[M] Let σ (lower case Greek sigma corresponds to our letter s) be a positive constant. The famous **bell curve** is the graph of the function

$$f(x) = \frac{\exp\left(\frac{-x^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}.$$

Show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
104.[M] Show that f has inflection points at points where $x=\sigma$ and at $x=-\sigma$.

105.[M] Show that $\int_{-\infty}^{\infty} x^2 f(x)\,dx = \sigma^2$. Thus σ^2 measures the discrepancy from 0. It is called the **variance**.
106.[M] The mean value of x is defined as $\int_{-\infty}^{\infty} xf(x)\,dx$. Show that it is 0.

HINT: Avoid labor.

107.[M] Assume that $\int_{-\infty}^{\infty} g(x) = 1$ and $\int_{-\infty}^{\infty} xg(x)\ dx = k$. Let $h(x) = g(x-k)$. Show that $\int_{-\infty}^{\infty} h(x)\ dx = 1$, $\int_{-\infty}^{\infty} xh(x)\ dx = k$, and $\int_{-\infty}^{\infty} (x-k)^2 h(x)\ dx = \int_{-\infty}^{\infty} x^2 g(x)\ dx$.

108.[C] If $f(x)$ and $g(x)$ have elementary antiderivatives, which of the following necessarily do also? (a) $f(x)g(x)$, (b) $f(g(x))$, and (c) $f(x) + g(x)$. Justify each answer.

109.[C]

(a) Show that $e^{x^{1/2}}$ has an elementary antiderivative.

(b) Show that $e^{x^{1/3}}$ has an elementary antiderivative.

(c) Show that for every positive integer n, $e^{x^{1/n}}$ has an elementary antiderivative.

110.[C] When a curve situated above the x-axis is revolved around the x-axis, the area of the resulting surface of revolution is 31. When the curve is revolved around the line $y = -2$, the surface area of this solid is 75. How long is the curve?

111.[C] In a letter dated May 24, 1872 Maxwell wrote: "It is strange ... that W. Weber could not correctly integrate

$$\int_0^{\pi} \cos(\theta)\sin(\phi)\ d\phi \qquad \text{where} \qquad \tan(\theta) = \frac{A\sin(\phi)}{B + A\cos(\phi)},$$

but that everyone should have copied such a wild result as

$$\frac{B}{\sqrt{A^2+B^2}} \cdot \frac{B^4 + \frac{7}{6}A^2B^2 + \frac{2}{3}A^2}{B^4 + A^2B^2 + A^4}.$$

Of course there are two forms of the result according as A or B is greater." Assuming that A and B are positive, find the correct value of the integral. HINT: Begin by expressing $\cos(\theta)$ in terms of the constants ϕ, A, and B.

SHERMAN: Need specific reference to result in A&S.

112.[C] The following calculation appears in *Electromagnetic Fields*, 2nd ed., Roald K. Wangsness, Wiley, 1986. (See also Exercise 3 in the Chapter 12 Summary.)

(a) The substitution $\frac{\pi}{2}\cos(\theta) = \frac{1}{2}(\pi - v)$, turns $\int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}\ d\theta$ into

$$\frac{1}{4}\left(\int_0^{2\pi} \frac{1-\cos(v)}{v}\ dv + \int_0^{2\pi} \frac{1-\cos(v)}{2\pi - v}\ dv\right).$$

(b) Introducing $w = 2\pi - v$ shows that the two integrals with respect to v are equal.

(c) So we must find $\frac{1}{2}\int_0^{2\pi} \frac{1-\cos(v)}{v}\, dv$. The integrand does not have an elementary antiderivative. However, its value (2.438) is listed in integral tables. Reference: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed., Dover, 1964 (online version available at `http://www.math.sfu.ca/~cbm/aands/`.)

Skill Drill: Derivatives

In Exercises 113 and 114 a, b, c, m, and p are constants. In each case verify that the derivative of the first function is the second function.

113.[R] $\frac{e^{ax}(a\sin(px)-p\cos(px))}{a^2+p^2}$; $e^{ax}\sin(px)$.

114.[R] $\sec(x)+\ln\left(\tan\left(\frac{x}{2}\right)\right)$; $\frac{1}{\sin(x)\cos^2(x)}$.

Calculus is Everywhere # 11

An Improper Integral in Economics

Both business and government frequently face the question, "How much money do I need today to have one dollar t years in the future?"

Implicit in this question are such considerations as the present value of a business being dependent on its future profit and the cost of a toll road being weighed against its future revenue. We determine the present value of a business which depends on the future rate of profit.

To begin the analysis, assume that the annual interest rate r remains constant and that 1 dollar deposited today is worth e^{rt} dollars t years from now. This assumption corresponds to continuously compounded interest or to natural growth. Thus A dollars today will be worth Ae^{rt} dollars t years from now. What is the present value of the promise of 1 dollar t years from now? In other words, what amount A invested today will be worth 1 dollar t years from now? To find out, solve the equation $Ae^{rt} = 1$ for A. The solution is

t need not be an integer

The present value of \$1 t years from now is \$ e^{-rt}

$$A = e^{-rt}. \tag{C.11.1}$$

Now consider the present value of the future profit of a business (or future revenue of a toll road). Assume that the profit flow t years from now is at the rate $f(t)$. This rate may vary within the year; consider f to be a continuous function of time. The profit in the small interval of time dt, from time t to time $t + dt$, would be approximately $f(t)dt$. The total future profit, $F(T)$, from now, when $t = 0$, to some time T in the future is therefore

$$F(T) = \int_0^T f(t)dt. \tag{C.11.2}$$

But the **present value** of the future profit is *not* given by (C.11.2). It is necessary to consider the present value of the profit earned in a typical short interval of time from t to $t + dt$. According to (C.11.1), its present value is approximately

$$e^{-rt}f(t)dt.$$

Hence the present value of future profit from $t = 0$ to $t = T$ is given by

$$\int_0^T e^{-rt}f(t)dt. \tag{C.11.3}$$

The present value of all future profit is, therefore, the improper integral $\int_0^\infty e^{-rt} f(t)dt$.

To see what influence the interest rate r has, denote by $P(r)$ the present value of all future revenue when the interest rate is r; that is,

$$P(r) = \int_0^\infty e^{-rt} f(t)dt. \tag{C.11.4}$$

If the interest rate r is raised, then according to (C.11.4) the present value of a business declines. An investor choosing between investing in a business or placing the money in a bank account may find the bank account more attractive when r is raised.

A proponent of a project, such as a toll road, will argue that the interest rate r will be low in the future. An opponent will predict that it will be high. Of course, neither knows what the inscrutable future will do to the interest rate. Even so, the prediction is important in a cost-benefit analysis.

Equation (C.11.4) assigns to a profit function f (which is a function of time t) a present-value function P, which is a function of r, the interest rate. In the theory of differential equations, P is called the **Laplace transform** of f. This transform can replace a differential equation by a simpler equation that looks quite different.

The Laplace transform was first encountered in Exercises 51 to 55 in Section 8.3 and reappeared in Exercises 66 to 68 in Section 8.6.

EXERCISES

In Exercises 1 to 8 $f(t)$ is defined on $[0, \infty)$ and is continuous. Assume that for $r > 0$, $\int_0^\infty e^{-rt} f(t)dt$ converges and that $e^{-rt} f(t) \to 0$ as $t \to \infty$. Let $P(r) = \int_0^\infty e^{-rt} f(t)dt$. Find $P(r)$, the Laplace transform of $f(t)$, in Exercises 1 to 5.

1.[R] $f(t) = t$

2.[R] $f(t) = e^t$, assume $r > 1$

3.[R] $f(t) = t^2$

4.[R] $f(t) = \sin(t)$

5.[R] $f(t) = \cos(t)$

6.[M] Let P be the Laplace transform of f, and let Q be the Laplace transform of f'. Show that $Q(r) = -f(0) + rP(r)$.

7.[M] Let P be the Laplace transform of f, a a positive constant, and $g(t) = f(at)$. Let Q be the Laplace transform of g. Show that $Q(t) = \frac{1}{a} P\left(\frac{r}{a}\right)$.

8.[R] Which is worth more today, \$100, 8 years from now or \$80, five years from now?

(a) Assume $r = 4\%$.

(b) Assume $r = 8\%$.

(c) For which interest rate are the two equal?

Chapter 9

Polar Coordinates and Plane Curves

This chapter presents further applications of the derivative and integral. Section 9.1 describes polar coordinates. Section 9.2 shows how to compute the area of a flat region that has a convenient description in polar coordinates. Section 9.3 introduces a method of describing a curve that is especially useful in the study of motion.

The speed of an object moving along a curved path is developed in Section 9.4. It also shows how to express the length of a curve as a definite integral. The area of a surface of revolution as a definite integral is introduced in Section 9.5. The sphere is an instance of such a surface.

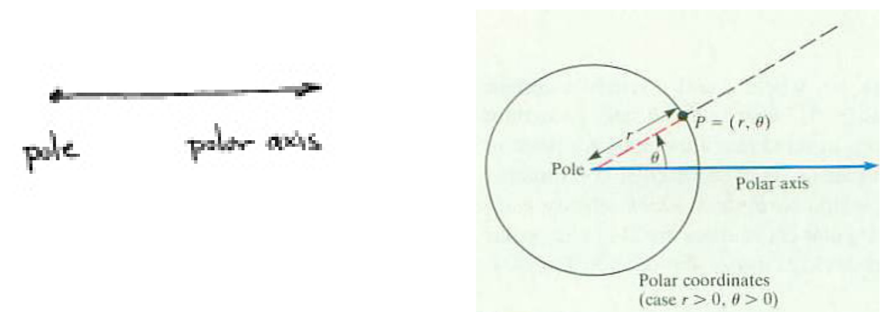
Section 9.6 shows how the derivative and second derivative provide tools for measuring how curvy a curve is at each of its points. This measure, called "curvature," will be needed in Chapter 15 in the study of motion along a curve.

9.1 Polar Coordinates

Rectangular coordinates provide only one of the ways to describe points in the plane by pairs of numbers. This section describes another coordinate system called "polar coordinates."

Polar Coordinates

The rectangular coordinates x and y describe a point P in the plane as the intersection of two perpendicular lines. Polar coordinates describe a point P as the intersection of a circle and a ray from the center of that circle. They are defined as follows.



(a) (b)

Figure 9.1.1:

When we say "The storm is 10 miles northeast," we are using polar coordinates: $r = 10$ and $\theta = \pi/4$.

Select a point in the plane and a ray emanating from this point. The point is called the **pole**, and the ray the **polar axis**. (See Figure 9.1.1(a).) Measure positive angles θ counterclockwise from the polar axis and negative angles clockwise. Now let r be a number. To plot the point P that corresponds to the pair of numbers r and θ, proceed as follows:

- If r is positive, P is the intersection of the circle of radius r whose center is at the pole and the ray of angle θ from the pole. (See Figure 9.1.1(b).)
- If r is 0, P is the pole, no matter what θ is.
- If r is negative, P is at a distance $|r|$ from the pole on the ray directly opposite the ray of angle θ, that is, on the ray of angle $\theta + \pi$.

In each case P is denoted (r, θ), and the pair r and θ are called the **polar coordinates** of P. The point (r, θ) is on the circle of radius $|r|$ whose center is the pole. The pole is the midpoint of the points (r, θ) and $(-r, \theta)$. Notice that the point $(-r, \theta + \pi)$ is the same as the point (r, θ). Moreover, changing the angle by 2π does not change the point; that is, $(r, \theta) = (r, \theta + 2\pi) = (r, \theta + 4\pi) = \cdots = (r, \theta + 2k\pi)$ for any integer k (positive or negative).

EXAMPLE 1 Plot the points $(3, \pi/4)$, $(2, -\pi/6)$, $(-3, \pi/3)$ in polar coordinates. See Figure 9.1.2.
SOLUTION

- To plot $(3, \pi/4)$, go out a distance 3 on the ray of angle $\pi/4$ (shown in Figure 9.1.2).
- To plot $(2, -\pi/6)$, go out a distance 2 on the ray of angle $-\pi/6$.
- To plot $(-3, \pi/3)$, draw the ray of angle $\pi/3$, and then go a distance 3 in the *opposite* direction from the pole.

$\diamond$

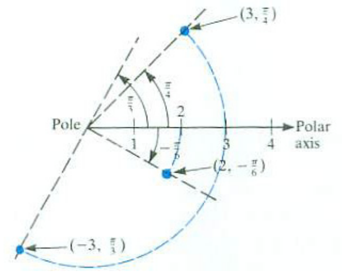


Figure 9.1.2:

It is customary to have the polar axis coincide with the positive x-axis as in Figure 9.1.3. In that case, inspection of the diagram shows the relation between the rectangular coordinates (x, y) and the polar coordinates of the point P:

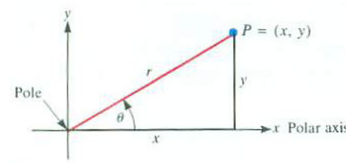


Figure 9.1.3:

The relation between polar and rectangular coordinates.

$$\boxed{\begin{array}{ll} x = r\cos(\theta) & y = r\sin(\theta) \\ r^2 = x^2 + y^2 & \tan(\theta) = \dfrac{y}{x} \end{array}}$$

These equations hold even if r is negative. If r is positive, then $r = \sqrt{x^2 + y^2}$. Furthermore, if $-\pi/2 < \theta < \pi/2$, then $\theta = \arctan(y/x)$.

Graphing $r = f(\theta)$

Just as we may graph the set of points (x, y), where x and y satisfy a certain equation, we may graph the set of points (r, θ), where r and θ satisfy a certain equation. Keep in mind that although each point in the plane is specified by a unique ordered pair (x, y) in rectangular coordinates, there are *many ordered pairs* (r, θ) *in polar coordinates that specify each point.* For instance, the point whose rectangular coordinates are $(1, 1)$ has polar coordinates $(\sqrt{2}, \pi/4)$ or $(\sqrt{2}, \pi/4 + 2\pi)$ or $(\sqrt{2}, \pi/4 + 4\pi)$ or $(-\sqrt{2}, \pi/4 + \pi)$ and so on.

The simplest equation in polar coordinates has the form $r = k$, where k is a positive constant. Its graph is the circle of radius k, centered at the pole. (See Figure 9.1.4(a).) The graph of $\theta = \alpha$, where α is a constant, is the line of inclination α. If we restrict r to be nonnegative, then $\theta = \alpha$ describes the ray ("half-line") of angle α. (See Figure 9.1.4(b).)

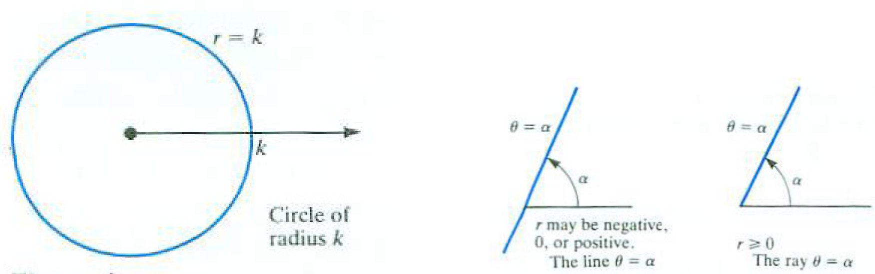


Figure 9.1.4:

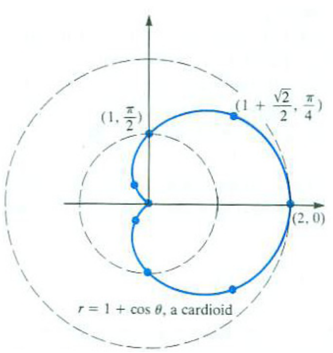


Figure 9.1.5: A cardioid is not shaped like a real heart, only like the conventional image of a heart.

EXAMPLE 2 Graph $r = 1 + \cos\theta$.

SOLUTION Begin by making a table: Since $\cos(\theta)$ has period 2π, we consider only θ in $[0, 2\pi]$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	2	$1 + \frac{\sqrt{2}}{2} \approx 1.7$	1	$1 - \frac{\sqrt{2}}{2} \approx 0.3$	0	$1 - \frac{\sqrt{2}}{2} \approx 0.3$	1	$1 + \frac{\sqrt{2}}{2} \approx 1.7$	2

Table 9.1.1:

As θ goes from 0 to π, r decreases; as θ goes from π to 2π, r increases. The last point is the same as the first. The graph begins to repeat itself. This heart-shaped curve, shown in Figure 9.1.5, is called a **cardioid**. ◇

Spirals turn out to be quite easy to describe in polar coordinates. This is illustrated by the graph of $r = 2\theta$ in the next example.

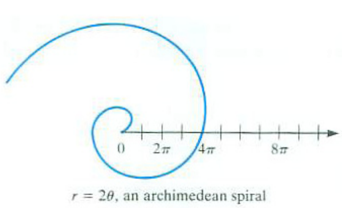


Figure 9.1.6:

EXAMPLE 3 Graph $r = 2\theta$ for $\theta \geq 0$.

SOLUTION First make a table:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	$\cdots$
r	0	π	2π	3π	4π	5π	$\cdots$

Increasing θ by 2π does *not* produce the same value of r. As θ increaes, r increases. The graph for $\theta \geq 0$ is an endless sprial, going infinitely often around the pole. It is indicated in Figure 9.1.6. ◇

If a is a nonzero constant, the graph of $r = a\theta$ is called an **Archimedean spiral** for a good reason: Archimedes was the first person to study the curve, finding the area within it up to any angle and also its tangent lines. The spiral with $a = 2$ is sketched in Example 3.

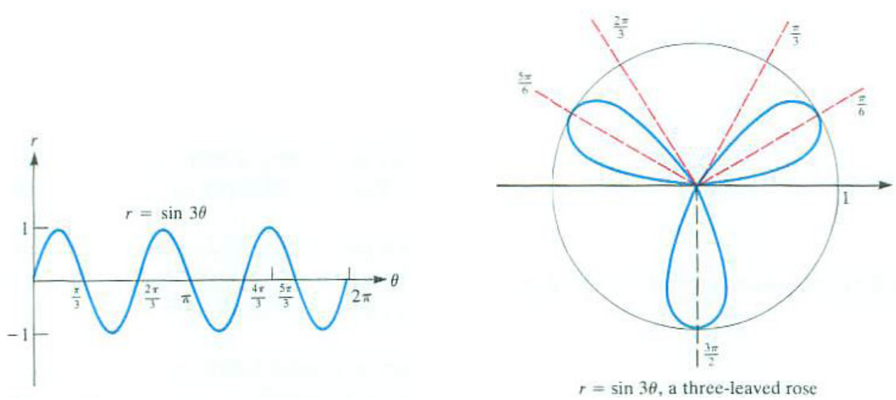
Polar coordinates are also convenient for describing loops arranged like the petals of a flower, as Example 4 shows.

EXAMPLE 4 Graph $r = \sin(3\theta)$.
SOLUTION Note that $\sin(3\theta)$ stays in the range -1 to 1. For instance, when $3\theta = \pi/2$, $\sin(3\theta) = \sin(\pi/2) = 1$. That tells us that when $\theta = \pi/6$, $r = \sin(3\theta) = \sin(3(\pi/6)) = \sin(\pi/2) = 1$. This case suggest that we calculate r at integer multiples of $\pi/6$, as in Table 9.1.2: The variation of r as a function

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
3θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{9\pi}{2}$	6π
$r = \sin(3\theta)$	0	1	0	-1	0	1	0	1	0

Table 9.1.2:

of θ is shown in Figure 9.1.7(a). Because $\sin(\theta)$ has period 2π, $\sin(3\theta)$ has period $2\pi/3$.

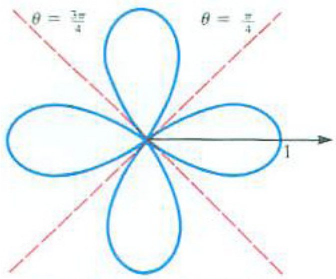


(a) (b)

Figure 9.1.7:

As θ increases from 0 up to $\pi/3$, 3θ increases from 0 up to π. Thus r, which is $\sin(3\theta)$, starts at 0 (for $\theta = 0$) up to 1 (for $\theta = \pi/6$) and then back to 0 (for $\theta = \pi/3$). This gives one of the three loops that make up the graph of $r = \sin(3\theta)$. For θ in $[\pi/3, 2\pi/3]$, $r = \sin(3\theta)$ is negative (or 0). This yields the lower loop in Figure 9.1.7(b). For θ in $[2\pi/3, \pi]$, r is again positive, and we obtain the upper left loop. Further choices of θ lead only to repetition of the loops already shown. ◇

The graph of $r = \sin(n\theta)$ or $r = \cos(n\theta)$ has n loops when n is an odd integer and $2n$ loops when n is an even integer. The next example illustrates the case when n is even.



$r = \cos 2\theta$, a four-leaved rose

Figure 9.1.8:

EXAMPLE 5 Graph the four-leaved rose, $r = \cos(2\theta)$.
SOLUTION To isolate one loop, find the two smallest nonnegative values of θ for which $\cos(2\theta) = 0$. These values are the θ that satisfy $2\theta = \pi/2$ and $2\theta = 3\pi/2$; thus $\theta = \pi/4$ and $\theta = 3\pi/4$. One leaf is described by letting θ go from $\pi/4$ to $3\pi/4$. For θ in $[\pi/4, 3\pi/4]$, 2θ is in $[\pi/2, 3\pi/2]$. Since 2θ is then a second- or third-quadrant angle, $r = \cos(2\theta)$ is *negative* or 0. In particular, when $\theta = \pi/2$, $\cos(2\theta)$ reaches its smallest value, -1. This loop is the bottom one in Figure 9.1.8. The other loops are obtained similarly. Of course, we could also sketch the graph by making a table of values. ◇

EXAMPLE 6 Transform the equation $y = 2$, which describes a horizontal straight line, into polar coordinates.
SOLUTION Since $y = r\sin\theta$, $r\sin\theta = 2$, or

$$r = \frac{2}{\sin(\theta)} = 2\csc(\theta).$$

This is more complicated than the Cartesian version of this equation, but is still sometimes useful. ◇

EXAMPLE 7 Transform the equation $r = 2\cos(\theta)$ into rectangular coordinates and graph it.
SOLUTION Since $r^2 = x^2 + y^2$ and $r\cos\theta = x$, first multiply the equation $r = 2\cos\theta$ by r, obtaining

$$r^2 = 2r\cos(\theta)$$

Hence

$$x^2 + y^2 = 2x.$$

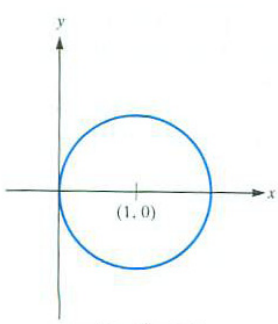
To graph this curve, rewrite the equation as

$$x^2 - 2x + y^2 = 0$$

and complete the square, obtaining

$$(x-1)^2 + y^2 = 1.$$

The graph is a circle of radius 1 and center at $(1, 0)$ in rectangular coordinates. It is graphed in Figure 9.1.9. ◇



$r = 2\cos\theta$, a circle

Figure 9.1.9:

> **Caution:** The step in Example 7 where we multiply by r deserves some attention. If $r = 2\cos(\theta)$, then certainly $r^2 = 2r\cos(\theta)$. However, if $r^2 = 2r\cos(\theta)$, it does not follow that $r = 2\cos(\theta)$. We can "cancel the r" only when r is not 0. If $r = 0$, it is true that $r^2 = 2r\cos(\theta)$, but it not necessarily true that $r = 2\cos(\theta)$. Since $r = 0$ satisfies the equation $r^2 = 2r\cos\theta$, the pole is on the curve $r^2 = 2r\cos\theta$. Luckily, it is also on the original curve $r = 2\cos(\theta)$, since $\theta = \pi/2$ makes $r = 0$. Hence the graphs of $r^2 = 2r\cos(\theta)$ and $r = 2\cos(\theta)$ are the same.
>
> However, as you may check, the graphs of $r = 2 + \cos(\theta)$ and $r^2 = r(2 + \cos(\theta))$ are *not* the same. The origin lies on the second curve, but not on the first.

The Intersection of Two Curves

Finding the intersection of two curves in polar coordinates is complicated by the fact that a given point has many descriptions in polar coordinates. Example 8 illustrates how to find the intersection.

EXAMPLE 8 Find the intersection of the curve $r = 1 - \cos(\theta)$ and the circle $r = \cos(\theta)$.

SOLUTION First graph the curves. The curve $r = \cos(\theta)$ is a circle half the size of the one in Example 7. Both curves are shown in Figure 9.1.10. (The curve $r = 1 - \cos(\theta)$ is a cardioid, being congruent to $r = 1 + \cos(\theta)$.) It appears that there are three points of intersection.

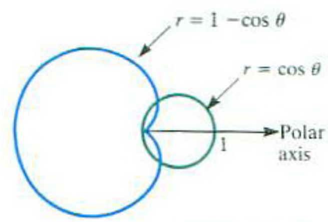


Figure 9.1.10:

A point of intersection is produced when one value of θ yields the same value of r in both equations, we would have

$$1 - \cos(\theta) = \cos(\theta).$$

Hence $\cos(\theta) = \frac{1}{2}$. Thus $\theta = \pi/3$ or $\theta = -\pi/3$ (or any angle differing from these by $2n\pi$, n an integer). This gives two of the three points, but it fails to give the origin. Why?

How does the origin get to be on the circle $r = \cos(\theta)$? Because, when $\theta = \pi/2$, $r = 0$. How does it get to be on the cardioid $r = 1 - \cos(\theta)$? Because, when $\theta = 0$, $r = 0$. The origin lies on both curves, but we would not learn this by simply equating $1 - \cos(\theta)$ and $\cos(\theta)$. ◇

When checking for the intersection of two curves, $r = f(\theta)$ and $r = g(\theta)$ in polar coordinates, examine the origin separately. The curves may also interect at other points not obtainable by setting $f(\theta) = g(\theta)$. This possibility is due to the fact the point (r, θ) is the same as the points $(r, \theta + 2n\pi)$ and $(-r, \theta + (2n + 1)\pi)$ for any integer n. The safest procedure is to graph the two curves first, identify the intersections in the graph, and then see why the curves intersect there.

Summary

We introduced polar coordinates and showed how to graph curves given in the form $r = f(\theta)$. Some of the more common polar curves are listed below.

Equation	Curve
$r = a$, $a > 0$	circle of radius a, center at pole
$r = 1 + \cos(\theta)$	cardioid
$r = a\theta$, $a > 0$	Archimedean spiral (traced clockwise)
$r = \sin(n\theta)$, n odd	n-leafed rose (one loop symmetric about $\theta = \pi/n$)
$r = \sin(n\theta)$, n even	$2n$-leafed rose
$r = \cos(n\theta)$, n odd	n-leafed rose (one loop symmetric about $\theta = 0$)
$r = \cos(n\theta)$, n even	$2n$-leafed rose
$r = a\csc(\theta)$	the line $y = a$
$r = a\sec(\theta)$	the line $x = a$
$r = a\cos(\theta)$, $a > 0$	circle of radius $a/2$ through pole and $(a/2, 0)$
$r = a\sin(\theta)$, $a > 0$	circle of radius $a/2$ through pole and $(0, a/2)$

Table 9.1.3:

To find the intersection of two curves in polar coordinates, first graph them.

EXERCISES for Section 9.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] Plot the points whose polar coordinates are

(a) $(1, \pi/6)$

(b) $(2, \pi/3)$

(c) $(2, -\pi/3)$

(d) $(-2, \pi/3)$

(e) $(2, 7\pi/3)$

(f) $(0, \pi/4)$

2.[R] Find the rectangular coordinates of the points in Exercise 1.

3.[R] Give at least three pairs of polar coordinates (r, θ) for the point $(3, \pi/4)$,

(a) with $r > 0$,

(b) with $r < 0$.

4.[R] Find polar coordinates (r, θ) with $0 \leq \theta < 2\pi$ and r positive, for the points whose rectangular coordinates are

(a) $(\sqrt{2}, \sqrt{2})$

(b) $(-1, \sqrt{3})$

(c) $(-5, 0)$

(d) $(-\sqrt{2}, -\sqrt{2})$

(e) $(0, -3)$

(f) $(1, 1)$

In Exercises 5 to 8 transform the equation into one in rectangular coordinates.

5.[R] $r = \sin(\theta)$

6.[R] $r = \csc(\theta)$

7.[R] $r = 4\cos(\theta) + 5\sin(\theta)$

8.[R] $r = 3/(4\cos(\theta) + 5\sin(\theta))$

In Exercises 9 to 12 transform the equation into one in polar coordinates.

9.[R] $x = -2$

10.[R] $y = x^2$

11.[R] $xy = 1$

12.[R] $x^2 + y^2 = 4x$

In Exercises 13 to 22 graph the given equations.

13.[R] $r = 1 + \sin\theta$

14.[R] $r = 3 + 2\cos(\theta)$

15.[R] $r = e^{-\theta/\pi}$

16.[R] $r = 4^{\theta/\pi}, \theta > 0$

17.[R] $r = \cos(3\theta)$

18.[R] $r = \sin(2\theta)$

19.[R] $r = 2$

20.[R] $r = 3$

21.[R] $r = 3\sin(\theta)$

22.[R] $r = -2\cos(\theta)$

23.[M] Suppose $r = 1/\theta$ for $\theta > 0$.

(a) What happens to the y coordinate of (r, θ) as $\theta \to \infty$?

(b) What happens to the x coordinate of (r, θ) as $\theta \to \infty$?

(c) Sketch the curve.

24.[R] Suppose $r = 1/\sqrt{\theta}$ for $\theta > 0$.

(a) What happens to the y coordinate of (r, θ) as $\theta \to \infty$?

(b) What happens to the x coordinate of (r, θ) as $\theta \to \infty$?

(c) Sketch the curve.

In Exercises 25 to 30, find the intersections of the curves after drawing them.

25.[R] $r = 1 + \cos(\theta)$ and $r = \cos(\theta) - 1$

26.[R] $r = \sin(2\theta)$ and $r = 1$

27.[R] $r = \sin(3\theta)$ and $r = \cos(3\theta)$

28.[R] $r = 2\sin(2\theta)$ and $r = 1$

29.[R] $r = \sin(\theta)$ and $r = \cos(2\theta)$

30.[R] $r = \cos(\theta)$ and $r = \cos(2\theta)$

A curve $r = 1 + a\cos(\theta)$ (or $r = 1 + a\sin(\theta)$) is called a **limaçon** (pronounced lee' · ma · son). Its shape depends on the choice of the constant a. For $a = 1$ we have the cardioid of Example 2. Exercises 31 to 33 concern other choices of a.

31.[R] Graph $r = 1 + 2\cos(\theta)$. (If $|a| > 1$, then the graph of $r = 1 + a\cos\theta$ crosses itself and forms two loops.)

32.[R] Graph $r = 1 + \frac{1}{2}\cos(\theta)$.

33.[C] Consider the curve $r = 1 + a\cos(\theta)$, where $0 \leq a \leq 1$.

(a) Relative to the same polar axis, graph the curves corresponding to $a = 0$, $1/4$, $1/2$, $3/4$, 1

(b) For $a = 1/4$ the graph in (a) is convex, but not for $a = 1$. Show that for $1/2 < a \leq 1$ the curve is not convex. NOTE: "Convex" is defined in Section 2.5 on page 134. HINT: Find the points on the curve farthest to the left and compare them to the point on the curve corresponding to $\theta = \pi$.

34.[M]

(a) Graph $r = 3 + \cos(\theta)$

(b) Find the point on the graph in (a) that has the maximum y coordinate.

35.[M] Find the y coordinate of the highest point on the right-hand leaf of the four-leaved rose $r = \cos(2\theta)$.

36.[M] Graph $r^2 = \cos(2\theta)$. Note that, if $\cos(2\theta)$ is negative, r is not defined and that, if $\cos(2\theta)$ is positive, there are two values of r, $\sqrt{\cos(2\theta)}$ and $-\sqrt{\cos(2\theta)}$. This curve is called a **lemniscate**.

In Appendix E it is shown that the graph of $r = 1/(1+e\cos(\theta))$ is a parabola if $e = 1$, an ellipse if $0 \leq e < 1$, and a hyperbola if $e > 1$. (e here denotes "eccentricity," not Euler's number.) Exercises 37 to 38 concern such graphs.

37.[M]

(a) Graph $r = \frac{1}{1+\cos(\theta)}$.

(b) Find an equation in rectangular coordinates for the curve in (a).

38.[M]

(a) Graph $r = \frac{1}{1-(1/2)\cos(\theta)}$.

(b) Find an equation in rectangular coordinates for the curve in (a).

39.[C] Where do the spirals $r = \theta$ and $r = 2\theta$, for $\theta \geq 0$, intersect?

9.2 Computing Area in Polar Coordinates

In Section 6.1 we saw how to compute the area of a region if the lengths of parallel cross sections are known. Sums based on rectangles led to the formula

$$\text{Area} = \int_a^b c(x)\ dx$$

where $c(x)$ denotes the cross-sectional length. Now we consider quite a different situation, in which sectors of circles, not rectangles, provide an estimate of the area.

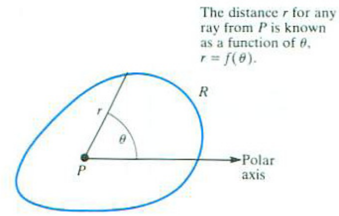


Figure 9.2.1:

Let R be a region in the plane and P a point inside it, that we take as the pole of a polar coordinate system. Assume that the distance r from P to any point on the boundary of R is known as a function $r = f(\theta)$. Also, assume that any ray from P meets the boundary of R just once, as in Figure 9.2.1.

The cross sections made by the rays from P are *not* parallel. Instead, like the spokes in a wheel, they all meet at the point P. It would be unnatural to use rectangles to estimate the area, but it is reasonable to use sectors of circles that have P as a common vertex.

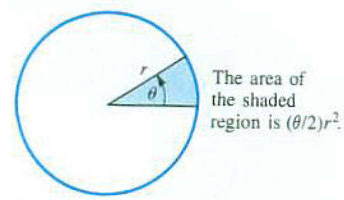


Figure 9.2.2:

Begin by recalling that in a circle of radius r a sector of central angle θ has area $(\theta/2)r^2$. (See Figure 9.2.2.) This formula plays the same role now as the formula for the area of a rectangle did in Section 6.1.

Area in Polar Coordinates

Let R be the region bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the curve $r = f(\theta)$, as shown in Figure 9.2.3. To obtain a **local estimate** for the area of R, consider the portion of R between the rays corresponding to the angles θ and $\theta + d\theta$, where $d\theta$ is a small positive number. (See Figure 9.2.4(a).) The area of the narrow wedge is shaded in Figure 9.2.4(a) is approximately that of a sector of a circle of radius $r = f(\theta)$ and angle $d\theta$, shown in Figure 9.2.4(b). The area of the sector in Figure 9.2.4(b) is

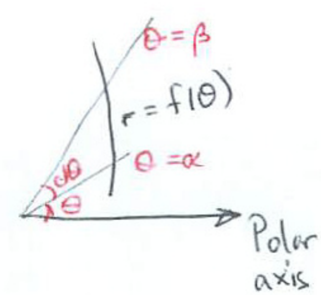


Figure 9.2.3:

$$\frac{f(\theta)^2}{2} d\theta. \tag{9.2.1}$$

Having found the local estimate of area (9.2.1), we conclude that the area of R is The area of the region bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the curve $r = f(\theta)$ is

How to find area in polar coordinates.

$$\int_\alpha^\beta \frac{f(\theta)^2}{2}\ d\theta \qquad \text{or simply} \qquad \int_\alpha^\beta \frac{r^2}{2} d\theta. \tag{9.2.2}$$

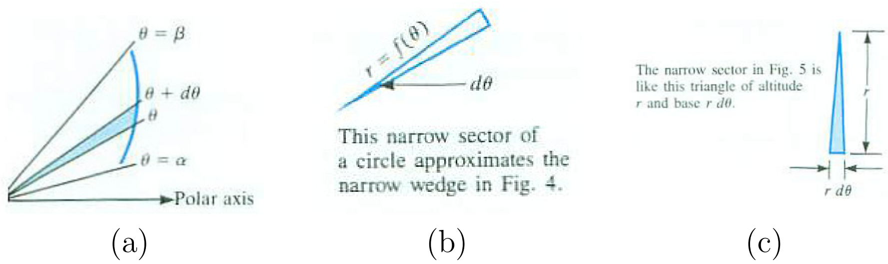


Figure 9.2.4:

Formula 9.2.2 is applied in Section 15.1 (and a CIE) to the motion of satellites and planets.

Area has dimensions of length squared.

Remark: It may seem surprising to find $(f(\theta))^2$, not just $f(\theta)$, in the integrand. But remember that area has the dimension "length times length." Since θ, given in radians, is dimensionless, being defined as "length of circular arc divided by length of radius", $d\theta$ is also dimensionless. Hence $f(\theta)\ d\theta$, having the dimension of length, not of area, could *not* be correct. But $\frac{1}{2}(f(\theta))^2\ d\theta$, having the dimension of area (length times length), is plausible. For rectangular coordinates, in the expressions $f(x)\ dx$, both $f(x)$ and dx have the dimension of length, one along the y-axis, the other along the x-axis; thus $f(x)\ dx$ has the dimension of area. As an aid in remembering the area of the narrow sector in Figure 9.2.4(b), note that it resembles a triangle of height r and base $r\ d\theta$, as shown in Figure 9.2.4(c). Its area is

Memory device

$$\frac{1}{2} \cdot \underbrace{r}_{\text{height}} \cdot \underbrace{rd\theta}_{\text{base}} = \frac{r^2 d\theta}{2}.$$

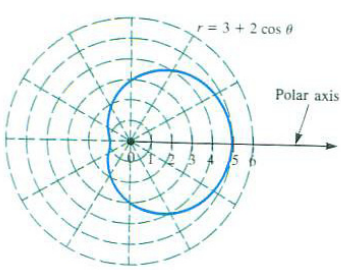


Figure 9.2.5:

EXAMPLE 1 Find the area of the region bounded by the polar curve $r = 3 + 2\cos(\theta)$, shown in Figure 9.2.5.
SOLUTION This cardiod is traced once for $0 \le \theta \le 2\pi$. By the formula just

obtained, this area is

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{2}(3+2\cos(\theta))^2 d\theta &= \frac{1}{2}\int_0^{2\pi} (9+12\cos(\theta)+4\cos^2(\theta))d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (9+12\cos(\theta)+2(1+\cos(2\theta))\ d\theta \\
&= \frac{1}{2}(9\theta+12\sin(\theta)+2\theta+\sin(2\theta))\Big|_0^{2\pi} = 11\pi.
\end{aligned}$$

◇

EXAMPLE 2 Find the area of the region inside one of the eight loops of the eight-leaved rose $r = \cos(4\theta)$.

SOLUTION To graph one of the loops, start with $\theta = 0$. For that angle, $r = \cos(4\cdot 0) = \cos 0 = 1$. The point $(r, \theta) = (1, 0)$ is the outer tip of a loop. As θ increases from 0 to $\pi/8$, $\cos(4\theta)$ decreases from $\cos(0) = 1$ to $\cos(\pi/2) = 0$. One of the eight loops is therefore bounded by the rays $\theta = \pi/8$ and $\theta = -\pi/8$. It is shown in Figure 9.2.6.

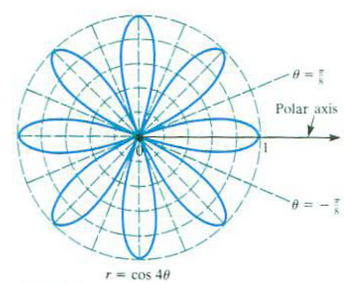


Figure 9.2.6:

The area of this loop, which is bisected by the polar axis, is

$$\begin{aligned}
\int_{-\pi/8}^{\pi/8} \frac{r^2}{2} d\theta &= \int_{-\pi/8}^{\pi/8} \frac{\cos^2(4\theta)}{2} d\theta = 2\cdot\frac{1}{4}\int_0^{\pi/8} (1+\cos(8\theta))\ d\theta \\
&= \frac{1}{2}\left(\theta+\frac{\sin(8\theta)}{4}\right)\Big|_0^{\pi/8} = \frac{1}{2}\left(\frac{\pi}{8}+\frac{\sin(\pi)}{8}\right) - 0 = \frac{\pi}{16} \approx 0.19635.
\end{aligned}$$

Notice how the fact that the integrand is an even function simplifies this calculation. ◇

The Area between Two Curves

Assume that $r = f(\theta)$ and $r = g(\theta)$ describe two curves in polar coordinates and that $f(\theta) \geq g(\theta) \geq 0$ for θ in $[\alpha, \beta]$. Let R be the region between these two curves and the rays $\theta = \alpha$ and $\theta = \beta$, as shown in Figure 9.2.7.

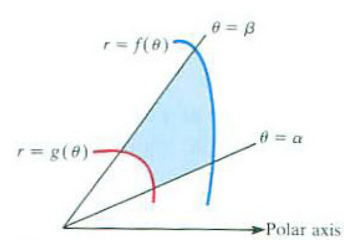


Figure 9.2.7:

The area of R is obtained by subtracting the area within the inner curve, $r = g(\theta)$, from the area within the outer curve, $r = f(\theta)$.

EXAMPLE 3 Find the area of the top half of the region inside the cardioid $r = 1 + \cos(\theta)$ and outside the circle $r = \cos(\theta)$.

We must integrate over two different intervals to find the two areas.

SOLUTION The region is shown in Figure 9.2.8. The top half of the circle $r = \cos(\theta)$ is swept out as θ goes from 0 to $\pi/2$. The area of this region is

$$\frac{1}{2}\int_0^{\pi/2} \cos^2(\theta)\, d\theta = \frac{\pi}{8}.$$

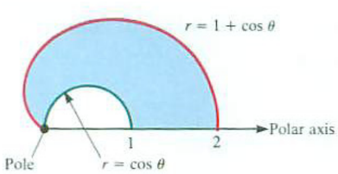


Figure 9.2.8: It's even easier to see this area as half the area of a circle of radius $1/2$: $\frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$.

The top half of the cardioid is swept out by $r = 1 + \cos(\theta)$ as θ goes from 0 to π; so its area is

$$\begin{aligned}
\frac{1}{2}\int_0^{\pi} (1+\cos(\theta))^2 d\theta &= \frac{1}{2}\int_0^{\pi} \left(1 + 2\cos(\theta) + \cos^2(\theta)\right) d\theta \\
&= \frac{1}{2}\int_0^{\pi} \left(1 + 2\cos(\theta) + \frac{1+\cos(2\theta)}{2}\right) d\theta \\
&= \frac{1}{2}\int_0^{\pi} \left(\frac{3}{2} + 2\cos(\theta) + \frac{\cos(2\theta)}{2}\right) d\theta \\
&= \frac{1}{2}\left(\frac{3\theta}{2} + 2\sin(\theta) + \frac{\sin(2\theta)}{4}\right)\Bigg|_0^{\pi} \\
&= \frac{3\pi}{4}.
\end{aligned}$$

Thus the area in question is

$$\frac{3\pi}{4} - \frac{\pi}{8} = \frac{5\pi}{8} \approx 1.96349.$$

◇

Summary

In this section we saw how to find the area within a curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$. The heart of the method is the local approximation by a narrow sector of radius r and angle $d\theta$, which has area $r^2 d\theta/2$. (It resembles a triangle of height r and base $r d\theta$.) This approximation leads to the formula,

$$\text{Area} = \int_\alpha^\beta \frac{r^2}{2} d\theta.$$

It is more prudent to remember the triangle than the area formula because you may otherwise forget the 2 in the denominator.

EXERCISES for Section 9.2 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 6, draw the bounded region enclosed by the indicated curve and rays and then find its area.

1.[R] $r = 2\theta$, $\alpha = 0$, $\beta = \frac{\pi}{2}$

2.[R] $r = \sqrt{\theta}$, $\alpha = 0$, $\beta = \pi$

3.[R] $r = \frac{1}{1+\theta}$, $\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{2}$

4.[R] $r = \sqrt{\sin(\theta)}$, $\alpha = 0$, $\beta = \frac{\pi}{2}$

5.[R] $r = \tan(\theta)$, $\alpha = 0$, $\beta = \frac{\pi}{4}$

6.[R] $r = \sec(\theta)$, $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{4}$

In each of Exercises 7 to 16 draw the region bounded by the indicated curve and then find its area.

7.[R] $r = 2\cos(\theta)$

8.[R] $r = e^{\theta}, 0 \leq \theta \leq 2\pi$

9.[R] Inside the cardioid $r = 3 + 3\sin(\theta)$ and outside the circle $r = 3$.

10.[R] $r = \sqrt{\cos(2\theta)}$

11.[R] One loop of $r = \sin(3\theta)$

12.[R] One loop of $r = \cos(2\theta)$

13.[R] Inside one loop of $r = 2\cos(2\theta)$ and outside $r = 1$

14.[R] Inside $r = 1 + \cos(\theta)$ and outside $r = \sin(\theta)$

15.[R] Inside $r = \sin(\theta)$ and outside $r = \cos(\theta)$

16.[R] Inside $r = 4 + \sin(\theta)$ and outside $r = 3 + \sin(\theta)$

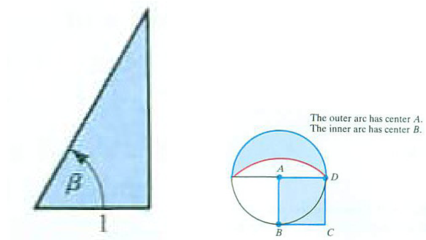
17.[M] Sketch the graph of $r = 4 + \cos(\theta)$. Is it a circle?

18.[M]

(a) Show that the area of the triangle in Figure 9.2.9(a) is $\int_0^{\beta} \frac{1}{2}\sec^2(\theta)d\theta$.

(b) From (a) and the fact that the area of a triangle is $\frac{1}{2}$(base)(height), show that $\tan(\beta) = \int_0^{\beta} \sec^2(\theta)d\theta$.

(c) With the aid of the equation in (b), obtain another proof that $(\tan(x))' = \sec^2(x)$.



(a) (b)

Figure 9.2.9:

19.[M] Show that the area of the shaded crescent between the two circular arcs is equal to the area of square $ABCD$. (See Figure 9.2.9(b).) This type of result encouraged mathematicians from the time of the Greeks to try to find a method using only straightedge and compass for constructing a square whose area equals that of a given circle. This was proved impossible at the end of the nineteenth century by showing that π is not the root of a non-zero polynomial with integer coefficients.

20.[M]

(a) Graph $r = 1/\theta$ for $0 < \theta \leq \pi/2$.

(b) Is the area of the region bounded by the curve drawn in (a) and the rays $\theta = 0$ and $\theta = \pi/2$ finite or infinite?

21.[M]

(a) Sketch the curve $r = 1/(1 + \cos(\theta))$.

(b) What is the equation of the curve in (a) in rectangular coordinates?

(c) Find the area of the region bounded by the curve in (a) and the rays $\theta = 0$ and $\theta = 3\pi/4$, using polar coordinates.

(d) Solve (c) using retangular coordinates and the equation in (b).

22.[M] Use Simpson's method to estimate the area of the bounded region between $r = \sqrt[3]{1 + \theta^2}$, $\theta = 0$, and $\theta = \pi/2$ that is correct to three decimal places.

23.[C] Estimate the area of the region bounded by $r = e^{\theta}$, $r = 2\cos(\theta)$ and $\theta = 0$. Hint: You may need to approximate a limit of integration.

24.[C] Figure 9.2.10 shows a point P inside a convex region $\mathcal{R}$.

(a) Assume that P cuts each chord through P into two intervals of equal length. Must each chord through P cut $\mathcal{R}$ into two regions of equal areas?

(b) Assume that each chord through P cuts $\mathcal{R}$ into two regions of equal areas. Must P cut each chord through P into two intervals of equal lengths?

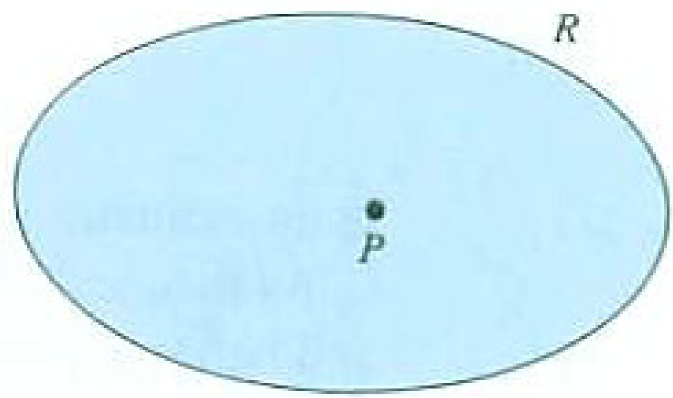


Figure 9.2.10:

25.[C] Let $\mathcal{R}$ be a convex region in the plane and P be a point on the boundary of $\mathcal{R}$. Assume that every chord of $\mathcal{R}$ that has an end at P has length at least 1.

(a) Draw several examples of such an $\mathcal{R}$.

(b) Make a general conjecture about the area $\mathcal{R}$.

(c) Prove it.

26.[C] Repeat Exercise 25, except that each chord through P has length not more than 1.

27.[C]

(a) Show that each line through the origin intersects the region bounded by the curve in Example 1 in a segment of length 6.

(b) Each line through the center of a disk of radius 3 also intersects the disk in a segment of length 6. Does it follow that the disk and the region in Example 1 have the same areas?

28.[C] Consider a convex region $\mathcal{R}$ in the plane and a point P inside it. If you know the length of each chord that passes through P. Can you then determine the area of $\mathcal{R}$

(a) if P is on the border of $\mathcal{R}$?

(b) if P is in the interior of $\mathcal{R}$?

Exercises 29 to 31, contributed by Rick West, are related.

29.[C] The graph of $r = \cos(n\theta)$ has $2n$ loops when n is even. Find the total area within those loops.

30.[C] The graph of $r = \cos(n\theta)$ has n loops when n is odd. Find the total area within those loops.

31.[C] Find the total area of all the petals within the curve $r = \sin(n\theta)$, where n is a positive integer. HINT: Take the cases n even or odd separately.

9.3 Parametric Equations

Up to this point we have considered curves described in three forms: "y is a function of x", "x and y are related implicitly", and "r is a function of θ". But a curve is often described by giving both x and y as functions of a third variable. We introduce this situation as it arises in the study of motion. It was the basis for the CIE on the Uniform Sprinkler in Chapter 5.

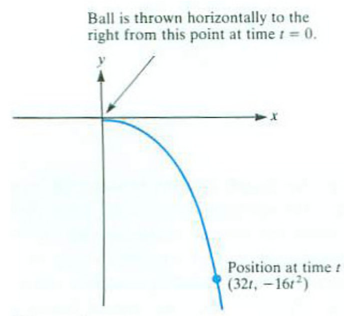


Figure 9.3.1:

Two Examples

EXAMPLE 1 If a ball is thrown horizontally out of a window with a speed of 32 feet per second, it falls in a curved path. Air resistance disregarded, its position t seconds later is given by $x = 32t$, $y = -16t^2$ relative to the coordinate system in Figure 9.3.1. Here the curve is completely described, not by expressing y as a function of x, but by expressing each of x and y as functions of a third variable t. The third variable is called a **parameter**. The equations $x = 32t$, $y = -16t^2$ are called **parametric equations** for the curve.

para meaning "together," *meter* meaning "measure".

In this example it is easy to eliminate t and so find a direct relation between x and y:

$$t = \frac{x}{32}.$$

Hence

$$y = -16\left(\frac{x}{32}\right)^2 = -\frac{16}{(32)^2}x^2 = -\frac{1}{64}x^2.$$

The path is part of the parabola $y = -\frac{1}{64}x^2$. ◇

In Example 2 elimination of the parameter would lead to a complicated equation involving x and y. One advantage of parametric equations is that they can provide a simple description of a curve, although it may be impossible to find an equation in x and y that describes the curve.

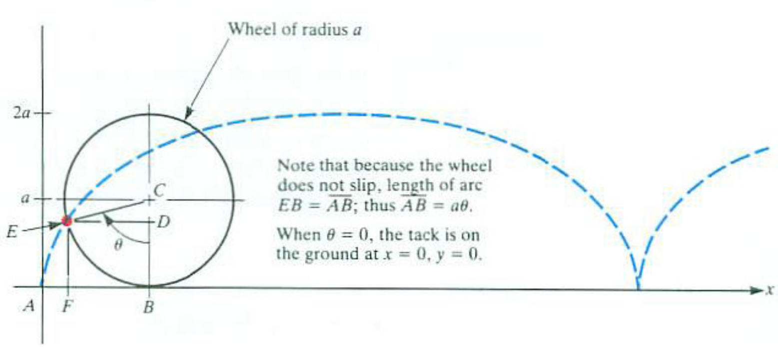


Figure 9.3.2:

EXAMPLE 2 As a bicycle wheel of radius a rolls along, a tack stuck in its circumference traces out a curve called a **cycloid**, which consists of a sequence of arches, one arch for each revolution of the wheel. (See Figure 9.3.2.) Find the position of the tack as a function of the angle θ through which the wheel turns.

SOLUTION Assume that the tack is initially at the bottom of the wheel. The x coordinate of the tack, corresponding to θ, is

$$|\overline{AF}| = |\overline{AB}| - |\overline{ED}| = a\theta - a\sin(\theta),$$

and the y coordinate is

$$|\overline{EF}| = |\overline{BC}| - |\overline{CD}| = a - a\cos(\theta).$$

Then the position of the tack, as a function of the parameter θ, is

$$x = a\theta - a\sin(\theta), \quad y = a - a\cos(\theta).$$

See Exercise 36.

In this case, eliminating θ leads to a complicated relation between x and y. ⋄

Any curve can be described parametrically. For instance, consider the curve $y = e^x + x$. It is perfectly legal to introduce a parameter t equal to x and write

$$x = t, \quad y = e^t + t.$$

This device may seem a bit artificial, but it will be useful in the next section in order to apply results for curves expressed by means of parametric equations to curves given in the form $y = f(x)$.

How to Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

How can we find the slope of a curve that is described parametrically by the equations

$$x = g(t), \quad y = h(t)?$$

An often difficult, perhaps impossible, approach is to solve the equation $x = g(t)$ for t as a function of x and substitute the result into the equation $y = h(t)$, thus expressing y explicitly in terms of x; then differentiate the result to find dy/dx. Fortunately, there is a very easy way, which we will now describe. Assume that y is a differentiable function of x. Then, by the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

from which it follows that

Slope of a parameterized curve

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{9.3.1}$$

It is assumed that in formula (9.3.1) dx/dt is not 0. To obtain d^2y/dx^2 just replace y in (9.3.1) by dy/dx, obtaining

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

EXAMPLE 3 At what angle does the arch of the cycloid shown in Example 2 meet the x-axis at the origin?
SOLUTION The parametric equations of the cycloid are

$$x = a\theta - a\sin(\theta) \qquad \text{and} \qquad y = a - a\cos(\theta).$$

Here θ is the parameter. Then

$$\frac{dx}{d\theta} = a - a\cos(\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = a\sin(\theta).$$

Consequently,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin(\theta)}{a - a\cos(\theta)} = \frac{\sin(\theta)}{1-\cos(\theta)}.$$

When $\theta = 0$, $(x, y) = (0, 0)$ and $\frac{dy}{dx}$ is not defined because $\frac{dx}{d\theta} = 0$. But, when θ is near 0, (x, y) is near the origin and the slope of the cycloid at $(0, 0)$ can be found by looking at the limit of the slope, which is $\sin\theta/(1 - \cos(\theta))$, as $\theta \to 0^+$. L'Hôpital's Rule applies, and we have

$$\lim_{\theta\to 0^+} \frac{\sin(\theta)}{1-\cos(\theta)} = \lim_{\theta\to 0^+} \frac{\cos(\theta)}{\sin(\theta)} = \infty.$$

Thus the cycloid comes in vertically at the origin, as shown in Figure 9.3.2. ⋄

EXAMPLE 4 Find d^2y/dx^2 for the cycloid of Example 2.
SOLUTION In Example 3 we found

$$\frac{dy}{dx} = \frac{\sin(\theta)}{1-\cos(\theta)}.$$

As shown in Example 3, $dx/d\theta = a - a\cos(\theta)$. To find $\frac{d^2y}{dx^2}$ we first compute

$$\frac{d}{d\theta}\left(\frac{dy}{dx}\right) = \frac{(1-\cos(\theta))\cos(\theta) - \sin(\theta)(\sin(\theta))}{(1-\cos(\theta))^2} = \frac{\cos(\theta)-1}{(1-\cos(\theta))^2} = \frac{-1}{1-\cos(\theta)}.$$

Thus

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{-1}{1-\cos(\theta)}}{a - a\cos(\theta)} = \frac{-1}{a(1-\cos(\theta))^2}.$$

Since the denominator is positive (or 0), the quotient, when defined, is negative. This agrees with Figure 9.3.2, which shows each arch of the cycloid as concave down. ◇

Summary

This section described parametric equations, where x and y are given as functions of a third variable, often time (t) or angle (θ). We also showed how to compute dy/dx and d^2y/dx^2:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

and replacing y by $\frac{dy}{dx}$,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

EXERCISES for Section 9.3 *Key:* R–routine, M–moderate, C–challenging

1.[R] Consider the parametric equations $x = 2t + 1$, $y = t - 1$.

(a) Fill in this table:

t	-2	-1	0	1	2
x					
y					

(b) Plot the five points (x, y) obtained in (a).

(c) Graph the curve given by the parametric equations $x = 2t + 1$, $y = t - 1$.

(d) Eliminate t to find an equation for the curve involving only x and y.

2.[R] Consider the parametric equations $x = t + 1$, $y = t^2$.

(a) Fill in this table:

t	-2	-1	0	1	2
x					
y					

(b) Plot the five points (x, y) obtained in (a).

(c) Graph the curve.

(d) Find an equation in x and y that describes the curve.

3.[R] Consider the parametric equations $x = t^2$, $y = t^2 + t$.

(a) Fill in this table:

t	-3	-2	-1	0	1	2	3
x							
y							

(b) Plot the seven points (x, y) obtained in (a).

(c) Graph the curve given by $x = t^2$, $y = t^2 + t$.

(d) Eliminate t and find an equation for the graph in terms of x and y.

4.[R] Consider the parametric equations $x = 2\cos(t)$, $y = 3\sin(t)$.

(a) Fill in this table, expressing the entries decimally:

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x									
y									

(b) Plot the eight distinct points in *(a)*.

(c) Graph the curve given by $x = 2\cos(t)$, $y = 3\sin(t)$.

(d) Using the identity $\cos^2(t) + \sin^2(t) = 1$, eliminate t.

In Exercises 5 to 8 express the curves parametrically with parameter t.

5.[R] $y = \sqrt{1 + x^3}$
6.[R] $y = \tan^{-1}(3x)$
7.[R] $r = \cos^2(\theta)$
8.[R] $r = 3 + \cos(\theta)$

In Exercises 9 to 14 find dy/dx and d^2y/dx^2 for the given curves.

9.[R] $x = t^3 + t$, $y = t^7 + t + 1$
10.[R] $x = \sin(3t)$, $y = \cos(4t)$
11.[R] $x = 1 + \ln(t)$, $y = t\ln(t)$
12.[R] $x = e^{t^2}$, $y = \tan(t)$
13.[R] $r = \cos(3\theta)$
14.[R] $r = 2 + 3\sin(\theta)$

In Exercises 15 to 16 find the equation of the tangent line to the given curve at the given point.

15.[R] $x = t^3 + t^2$, $y = t^5 + t$; $(2, 2)$
16.[R] $x = \frac{t^2+1}{t^3+t^2+1}$, $y = \sec 3t$; $(1, 1)$

In Exercises 17 and 18 find d^2y/dx^2.

17.[R] $x = t^3 + t + 1$, $y = t^2 + t + 2$
18.[R] $x = e^{3t} + \sin(2t)$, $y = e^{3t} + \cos(t^2)$

19.[R] For which values of t is the curve in Exercise 17 concave up? concave down?

20.[R] Let $x = t^3 + 1$ and $y = t^2 + t + 1$. For which values of t is the curve concave up? concave down?

21.[R] Find the slope of the three-leaved rose, $r = \sin(3\theta)$, at the point $(r, \theta) = (\sqrt{2}/2, \pi/12)$.

22.[R]

(a) Find the slope of the cardioid $r = 1 + \cos(\theta)$ at the point (r, θ).

(b) What happens to the slope in (a) as θ approaches π from the left?

(c) What does (b) tell us about the graph of the cardioid? (Show it on the graph.)

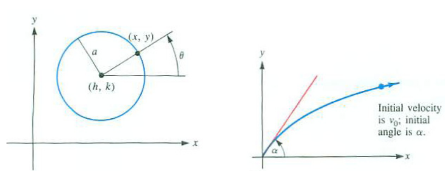
23.[R] Obtain parametric equations for the circle of radius a and center (h, k), using as parameter the angle θ shown in Figure 9.3.3(a).

24.[R] At time $t \geq 0$ a ball is at the point $(24t, -16t^2 + 5t + 3)$.

(a) Where is it at time $t = 0$?

(b) What is its horizontal speed at that time?

(c) What is its vertical speed at that time?



(a) (b)

Figure 9.3.3:

Exercises 25 to 27 analyze the trajectory of a ball thrown from the origin at an angle α and initial velocity v_0, as sketched in Figure 9.3.3(b). These results are used in the CIE on the Uniform Sprinkler in Chapter 5 (see page 472).

25.[R] It can be shown that if time is in seconds and distance in feet, then t seconds later the ball is at the point

$$x = (v_0 \cos(\alpha))t, \quad y = (v_0 \sin(\alpha))t - 16t^2.$$

(a) Express y as a function of x. HINT: Eliminate t.

(b) In view of (a), what type of curve does the ball follow?

(c) Find the coordinates of its highest point.

26.[R] Eventually the ball in Exercise 25 falls back to the ground.

(a) Show that, for a given v_0, the horizontal distance it travels is proportional to $\sin(2\theta)$.

(b) Use (a) to determine the angle that maximizes the horizontal distance traveled.

(c) Show that the horizontal distance traveled in (a) is the same when the ball is thrown at an angle of θ or at an angle of $\pi/2 - \theta$.

27.[R] Is it possible to extend the horizontal distance traveled by throwing the ball in Exercise 25 from the top of a hill? (Assume the hill has height d.) HINT: Work with the horizontal distance traveled, x, not the distance along the sloped ground.

28.[R] The spiral $r = e^{2\theta}$ meets the ray $\theta = \alpha$ at an infinite number of points.

(a) Graph the spiral.

(b) Find the slope of the spiral at each intersection with the ray.

(c) Show that at all of these points this spiral has the same slope.

(d) Show that the analog of (c) is not true for the spiral $r = \theta$.

29.[M] The spiral $r = \theta$, $\theta > 0$ meets the ray $\theta = \alpha$ at an infinite number of points (α, α), $(\alpha + 2\pi, \alpha)$, $(\alpha + 4\pi, \alpha)$, ... What happens to the angle between the spiral and the ray at the point $(\alpha + 2\pi n, \alpha)$ as $n \to \infty$?

30.[M] Let a and b be positive numbers. Consider the curve given parametrically by the equations

$$x = a\cos(t) \qquad y = b\sin(t).$$

(a) Show that the curve is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(b) Find the area of the region bounded by the ellipse in (a) by making a substitution that expresses $4\int_0^a y\,dx$ in terms of an integral in which the variable is t and the range of integration is $[0, \pi/2]$.

31.[M] Consider the curve given parametrically by

$$x = t^2 + e^t \qquad y = t + e^t$$

for t in $[0, 1]$.

(a) Plot the points corresponding to $t = 0$, $1/2$, and 1.

(b) Find the slope of the curve at the point $(1, 1)$.

(c) Find the area of the region under the curve and above the interval $[1, e + 1]$. [See Exercise 30(b).]

32.[M] What is the slope of the cycloid in Figure 9.3.2 at the first point on it to the right of the y-axis at the height a?

33.[M] The region under the arch of the cycloid

$$x = a\theta - a\sin(\theta), \quad y = a - a\cos(\theta) \qquad (0 \le \theta \le 2\pi)$$

and above the x-axis is revolved around the x-axis. Find the volume of the solid of revolution produced.

34.[M] Find the volume of the solid of revolution obtained by revolving the region in Exercise 33 about the y-axis.

35.[M] Let a be a positive constant. Consider the curve given parametrically by the equations $x = a\cos^3(t)$, $y = a\sin^3(t)$.

(a) Sketch the curve.

(b) Express the slope of the curve in terms of the parameter t.

36.[M] Solve the parametric equations for the cycloid, $x = a\theta - a\sin(\theta)$, $y = a - a\cos(\theta)$, for y as a function of x. NOTE: See Example 1.

37.[C] Consider a tangent line to the curve in Exercise 35 at a point P in the first quadrant. Show that the length of the segment of that line intercepted by the coordinate axes is a.

38.[C] L'Hôpital's rule in Section 5.5 asserts that if $\lim_{t\to 0} f(t) = 0$, $\lim_{t\to 0} g(t) = 0$, and $\lim_{t\to 0}(f'(t)/g'(t))$ exists, then $\lim_{t\to 0}(f(t)/g(t)) = \lim_{t\to 0}(f'(t)/g'(t))$. Interpret that rule in terms of the parameterized curve $x = g(t)$, $y = f(t)$. HINT: Make a sketch of the curve near $(0,0)$ and show on it the geometric meaning of the quotients $f(t)/g(t)$ and $f'(t)/g'(t)$.

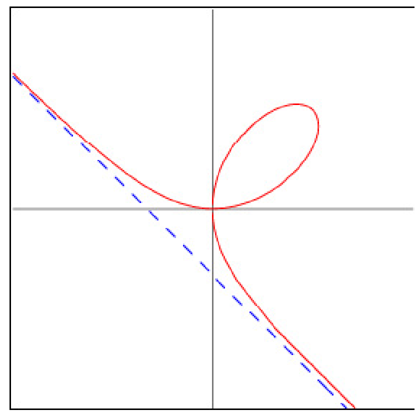


Figure 9.3.4:

39.[C] The **Folium of Descartes** is the graph of

$$x^3 + y^3 = 3xy.$$

The graph is shown in Figure 9.3.4. It consists of a loop and two infinite pieces both asymptotic to the line $x + y + 1 = 0$. Parameterize the curve by the slope t of the line joining the origin with (x, y). Thus for the point (x, y) on the curve, $y = xt$.

(a) Show that
$$x = \frac{3t}{1+t^3} \qquad \text{and} \qquad y = \frac{3t^2}{1+t^3}.$$

(b) Find the highest point on the loop.

(c) Find the point on the loop furthest to the right.

(d) The loop is parameterized by t in $[0, \infty)$. Which values of t parameterize the part in the fourth quadrant?

(e) Which values of t parameterize the part in the second quadrant?

(f) Show that the Folium of Descartes is symmetric with respect to the line $y = x$.

NOTE: Visit `http://en.wikipedia.org/wiki/Folium_of_Descartes` or do a Google search of "Folium Descartes" to see its long history that goes back to 1638.

9.4 Arc Length and Speed on a Curve

In Section 4.2 we studied the motion of an object moving on a line. If at time t its position is $x(t)$, then its velocity is the derivative $\frac{dx}{dt}$ and its speed is $\left|\frac{dx}{dt}\right|$. Now we will examine the velocity and speed of an object moving along a curved path.

Arc Length and Speed in Rectangular Coordinates

Consider an object moving on a path given parametrically by

$$\left\{\begin{array}{lcl} x & = & g(t) \\ y & = & h(t) \end{array}\right. ,$$

where g and h have continuous derivatives. Think of t as time, though the parameter could be anything, such as angle or even x itself.

First, let us find a formula for its speed.

Let $s(t)$ be the arc length covered from the initial time to an arbitrary time t. In a short interval of time, Δt, it travels a distance Δs along the path. We want to find

$$\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}.$$

We take an intuitive approach, and leave a more formal argument for Exercise 30.

During the time interval $[t, t+\Delta t]$ the object goes from P to Q on the path, covering a distance Δs, as shown in Figure 9.4.1. During this time its x-coordinate changes by Δx and its y-coordinate by Δy. The chord $\overline{PQ}$ has length $\sqrt{(\Delta x)^2+(\Delta y)^2}$.

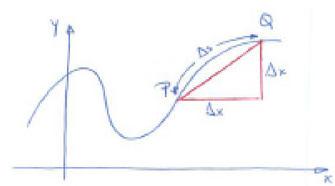


Figure 9.4.1:

We assume now that the curve is well behaved in the sense that $\lim_{\Delta t\to 0}\frac{\Delta s}{|\overline{PQ}|}=1$. In this case,

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} &= \lim_{\Delta t\to 0}\frac{|\overline{PQ}|}{\Delta t}=\lim_{\Delta t\to 0}\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}.\end{aligned}$$

We have just obtained the key result in this section:

$$\boxed{\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}$$

or, stated in terms of differentials,

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The rates at which x and y change determine how fast the arc length s changes, as recorded in Figure 9.4.2.

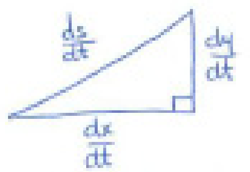


Figure 9.4.2:

Now that we have a formula for ds/dt, we simply integrate it to get the distance along the path covered during a time interval $[a, b]$:

$$\text{arc length} = \int_a^b ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \qquad (9.4.1)$$

If the curve is given in the form $y = f(x)$, one is free to use x as the parameter. Thus, a parametric representation of the curve is

$$x = x, \qquad y = f(x).$$

Then (9.4.1) becomes

$$\text{arc length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

WARNING *(Sign of $\frac{ds}{dt}$)* The arclength function is, by definition, an non-decreasing function. This means ds/dt is never negative. In fact, in most applications ds/dt will be strictly positive.

Three examples will show how these formulas are applied. The first goes back to the year 1657, when the 20-year old Englishman, William Neil, found the length of an arc on the graph of $y = x^{3/2}$. His method was much more complicated. Earlier in that century, Thomas Harriot had found the length of an arc of the spiral $r = e^\theta$, but his work was not widely published.

EXAMPLE 1 Find the arc length of the curve $y = x^{3/2}$ for x in $[0, 1]$. (See Figure 9.4.3.)

SOLUTION By formula (9.4.1),

$$\text{arc length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

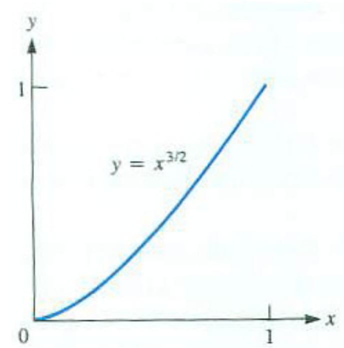


Figure 9.4.3:

Since $y = x^{3/2}$, we differentiate to find $dy/dx = \frac{3}{2}x^{1/2}$. Thus

$$\begin{aligned}
\text{arc length} &= \int_0^1 \sqrt{1 + \left(\tfrac{3}{2}x^{1/2}\right)^2}\, dx = \int_0^1 \sqrt{1 + \tfrac{9}{4}x}\, dx \\
&= \int_1^{13/4} \sqrt{u} \cdot \tfrac{4}{9}\, du \qquad \left(u = 1 + \tfrac{9}{4}x,\ du = \tfrac{9}{4}dx\right) \\
&= \tfrac{4}{9} \cdot \tfrac{2}{3} u^{3/2}\Big|_1^{13/4} = \tfrac{8}{27}\left(\left(\tfrac{13}{4}\right)^{3/2} - 1^{3/2}\right) \\
&= \tfrac{8}{27}\left(\tfrac{13^{3/2}}{8} - 1\right) = \tfrac{13^{3/2}-8}{27} \approx 1.43971.
\end{aligned}$$

◇

Incidentally, the arc length of the curve $y = x^a$ where a is a non-zero rational number, usually *cannot* be computed with the aid of the Fundamental Theorem of Calculus. The only cases in which it can be computed by the FTC are a=1 (the graph of $y = x$) and $a = 1 + \frac{1}{n}$ where n is an integer. Exercise 32 treats this question.

EXAMPLE 2 In Section 9.3 the parametric equations for the motion of a ball thrown horizontally with a speed of 32 feet per second (≈ 21.8 *mph*) were found to be $x = 32t$, $y = -16t^2$. (See Example 1 and Figure 9.3.1.) How fast is the ball moving at time t? Find the distance s which the ball travels during the first b seconds.

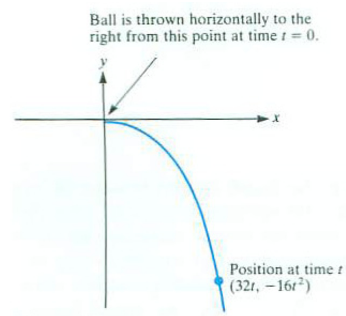


Figure 9.4.4:

SOLUTION From $x = 32t$ and $y = -16t^2$ we compute $\frac{dx}{dt} = 32$ and $\frac{dy}{dt} = -32t$. Its speed at time t is

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(32)^2 + (-32t)^2} = 32\sqrt{1+t^2} \text{ feet per second.}$$

The distance traveled is the arc length from $t = 0$ to $t = b$. By formula (9.4.1),

$$\text{arc length} = \int_0^b \sqrt{(32)^2 + (-32t)^2}\, dt = 32 \int_0^b \sqrt{1+t^2}\, dt.$$

This integral can be evaluated with an integration table or with the trigonometric substitution $x = \tan(\theta)$. An antiderivative is

See Formula 31 in the integral table.

$$\frac{1}{2}\left(t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right)$$

and the distance traveled is

$$16b\sqrt{1+b^2} + 16\ln\left(b+\sqrt{1+b^2}\right).$$

$\diamond$

EXAMPLE 3 Find the length of one arch of the cycloid found in Example 2 of Section 9.3.

SOLUTION Here the parameter is θ, $x = a\theta - a\sin(\theta)$, and $y = a - a\cos(\theta)$. To complete one arch of the cycloid, θ varies from 0 to 2π.

We compute

$$\frac{dx}{d\theta} = a - a\cos(\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = a\sin(\theta).$$

The square of the speed is

$$\begin{aligned}(a-a\cos(\theta))^2 + (a\sin(\theta))^2 &= a^2\left((1-\cos(\theta))^2 + (\sin(\theta))^2\right)\\ &= a^2\left(1 - 2\cos(\theta) + (\cos(\theta))^2 + (\sin(\theta))^2\right)\\ &= a^2\left(2 - 2\cos(\theta)\right)\\ &= 2a^2(1-\cos(\theta)).\end{aligned}$$

Using boxed formula (9.4.1) and the trigonometric identity $1 - \cos(\theta) = 2\sin^2(\theta/2)$, we have

$$\begin{aligned}\text{the length of one arch} &= \int_0^{2\pi} \sqrt{2a^2(1-\cos(\theta))}\ d\theta = a\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos(\theta)}\ d\theta\\ &= a\sqrt{2}\int_0^{2\pi}\sqrt{2}\sin\left(\frac{\theta}{2}\right)\ d\theta = 2a\int_0^{2\pi}\sin\left(\frac{\theta}{2}\right)\ d\theta\\ &= 2a\left(-2\cos\left(\frac{\theta}{2}\right)\Bigg|_0^{2\pi}\right) = 2a\left(-2(-1) - (-2)(1)\right) = 8a.\end{aligned}$$

This means that while θ varies from 0 to 2π, a bicycle travels a distance of $2\pi a \approx 6.28318a$ and a tack in the tread of the tire travels a distance $8a$. $\diamond$

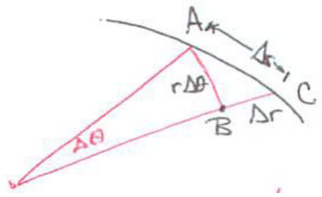


Figure 9.4.5:

Arc Length and Speed in Polar Coordinates

So far in this section curves have been described in rectangular coordinates. Next consider a curve given in polar coordinates by the equation $r = f(\theta)$.

We will estimate the length of arc Δs corresponding to small changes $\Delta\theta$ and Δr in polar coordinates, as shown in Figure 9.4.5. The region bounded by the circular arc AB, the straight segment BC, and AC, the part of the curve, resembles a right triangle whose two legs have lengths $r\Delta\theta$ and Δr. We assume Δs is well approximated by its hypotenuse, $\sqrt{(r\Delta\theta)^2 + (\Delta r)^2}$. Thus we expect

$$\begin{aligned}
\frac{ds}{d\theta} = \lim_{\Delta\theta\to 0} \frac{\Delta s}{\Delta\theta} &= \lim_{\Delta\theta\to 0} \frac{\sqrt{(r\Delta\theta)^2 + (\Delta r)^2}}{(\Delta\theta)} \\
&= \lim_{\Delta\theta\to 0} \sqrt{r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2} \\
&= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}
\end{aligned}$$

In short,

arc length for $r = f(\theta)$.

For a curve given in polar coordinates:

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \quad \text{or} \quad ds = \sqrt{(r\ d\theta)^2 + (dr)^2} = \sqrt{r^2 + (r')^2}\ d\theta.$$

This formula can also be obtained from the formula for the case of rectangular coordinates by using $x = r\cos(\theta)$ and $y = r\sin(\theta)$. (See Exercise 19.) However, we prefer the geometric approach because it is (i) more direct, (ii) more intuitive, and (iii) easier to remember.

See Exercise 19.

Arc Length of a Polar Curve $r = f(\theta)$

The length of the curve $r = f(\theta)$ for θ in $[\alpha, \beta]$ is $s = \int_\alpha^\beta ds$ where

$$ds = \sqrt{r^2 + (r')^2}\ d\theta = \sqrt{(f(\theta))^2 + (f'(\theta))^2}\ d\theta.$$

EXAMPLE 4 Find the length of the spiral $r = e^{-3\theta}$ for θ in $[0, 2\pi]$.

SOLUTION First compute

$$r' = \frac{dr}{d\theta} = -3e^{-3\theta},$$

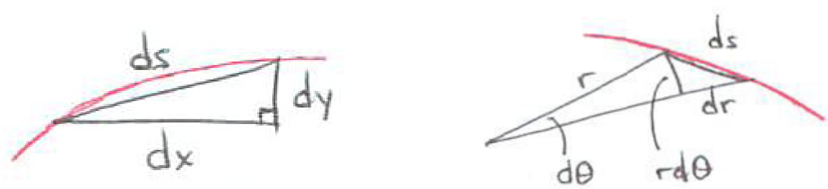
and then use the formula

$$\begin{aligned}
\text{Arc Length} &= \int_\alpha^\beta \sqrt{r^2+(r')^2}\,d\theta = \int_0^{2\pi} \sqrt{\left(e^{-3\theta}\right)^2+\left(-3e^{-3\theta}\right)^2}\,d\theta \\
&= \int_0^{2\pi} \sqrt{e^{-6\theta}+9e^{-6\theta}}\,d\theta = \sqrt{10}\int_0^{2\pi} \sqrt{e^{-6\theta}}\,d\theta \\
&= \sqrt{10}\int_0^{2\pi} e^{-3\theta}\,d\theta = \sqrt{10}\,\frac{e^{-3\theta}}{-3}\Bigg|_0^{2\pi} \\
&= \sqrt{10}\left(\frac{e^{-3\cdot 2\pi}}{-3}-\frac{e^{-3\cdot 0}}{-3}\right) = \sqrt{10}\left(\frac{e^{-6\pi}}{-3}+\frac{1}{3}\right) \\
&= \frac{\sqrt{10}}{3}\left(1-e^{-6\pi}\right) \approx 1.054093.
\end{aligned}$$

◇

Summary

This section concerns speed along a parametric path and the length of the path. If the path is described in rectangular coordinates, then Figure 9.4.6(a) conveys the key ideas. If in polar coordinates, Figure 9.4.6(b) is the key. It is much easier to recall these diagrams than the various formulas for speed and arc length. Everything depends on our old friend: the Pythagorean Theorem.



(a) (b)

Figure 9.4.6: (a) $ds = \sqrt{(dx)^2+(dy)^2}$ (b) $ds = \sqrt{(rd\theta)^2+(dr)^2}$

EXERCISES for Section 9.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 8 find the arc lengths of the given curves over the given intervals.

1.[R] $y = x^{3/2}$, x in $[1, 2]$
2.[R] $y = x^{2/3}$, x in $[0, 1]$
3.[R] $y = (e^x + e^{-x})/2$, x in $[0, b]$
4.[R] $y = x^2/2 - (\ln(x))/4$, x in $[2, 3]$
5.[R] $x = \cos^3(t)$, $y = \sin^3(t)$, t in $[0, \pi/2]$
6.[R] $r = e^\theta$, θ in $[0, 2\pi]$
7.[R] $r = 1 + \cos(\theta)$, θ in $[0, \pi]$
8.[R] $r = cos^2(\theta/2)$, θ in $[0, \pi]$

In each of Exercises 9 to 12 find the speed of the particle at time t, given the parametric description of its path.

9.[R] $x = 50t$, $y = -16t^2$
10.[R] $x = \sec(3t)$, $y = \sin^{-1}(4t)$
11.[R] $x = t + \cos(t)$, $y = 2t - \sin(t)$
12.[R] $\csc(\theta/2)$, $y = \tan^{-1}(\sqrt{t})$

13.[R]

(a) Graph $x = t^2$, $y = t$ for $0 \le t \le 3$.

(b) Estimate its arc length from $(0, 0)$ to $(9, 3)$ by an inscribed polygon whose vertices have x-coordinates 0, 1, 4, and 9.

(c) Set up a definite integral for the arc length of the curve in question.

(d) Estimate the definite integral in (c) by using a partition of $[0, 3]]$ into 3 sections, each of length 1, and the trapezoid method.

(e) Estimate the definite integral in (c) by Simpson's method with six sections.

(f) If your calculator has a program to evaluate definite integrals, use it to evaluate the definite integral in (c) to four decimal places.

14.[R]

(a) Graph $y = 1/x^2$ for x in $[1, 2]$.

(b) Estimate the length of the arc in (a) by using an inscribed polygon whose vertices at $(1, 1)$, $(\frac{5}{4}, \left(\frac{4}{5}\right)^2)$, $(\frac{3}{2}, \left(\frac{2}{3}\right)^2)$, and $(2, \frac{1}{4})$.

(c) Set up a definite integral for the arc length of the curve in question.

(d) Estimate the definite integral in (c) by the trapezoid method, using four equal length sections.

(e) Estimate the definite integral in (c) by Simpson's method with four sections.

(f) If your calculator has a program to evaluate definite integrals, use it to evaluate the definite integral in (c) to four decimal places.

15.[R] How long is the spiral $r = e^{-3\theta}$, $\theta \geq 0$?

16.[R] How long is the spiral $r = 1/\theta$, $\theta \geq 2\pi$?

17.[R] Assume that a curve is described in rectangular coordinates in the form $x = f(y)$. Show that

$$\text{Arc Length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

where y ranges in the interval $[c, d]$, using a little triangle whose sides have length dx, dy, and ds.

18.[R] Consider the arc length of the curve $y = x^{2/3}$ for x in the interval $[1, 8]$.

(a) Set up a definite integral for this arc length using x as the parameter.

(b) Set up a definite integral for this arc length using y as the parameter.

(c) Evaluate the easier of the two integrals found in parts (a) and (b).

NOTE: See Exercise 17.

19.[M] We obtained the formula $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$ geometrically.

(a) Obtain the same result by calculus, starting with $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$, and using the relations $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

(b) Which derivation do you prefer? Why?

20.[M] Let $P = (x, y)$ depend on θ as shown in Figure 9.4.7.

(a) Sketch the curve that P sweeps out.

(b) Show that $P = (2\cos(\theta), \sin(\theta))$.

(c) Set up a definite integral for the length of the curve described in P. (Do not evaluate it.)

(d) Eliminate θ and show that P is on the ellipse

$$\frac{x^2}{4} + \frac{y^2}{1} = 1.$$

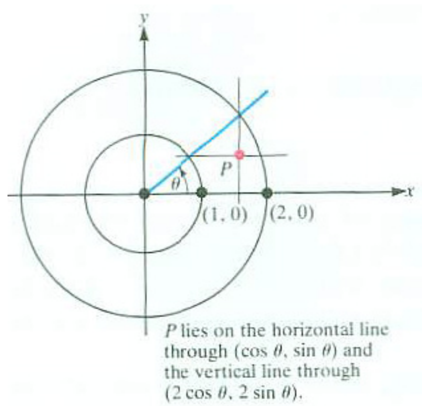


Figure 9.4.7:

21.[M]

(a) At time t a particle has polar coordinates $r = g(t)$, $\theta = h(t)$. How fast is it moving?

(b) Use the formula in (a) to find the speed of a particle which at time t is at the point $(r, \theta) = (e^t, 5t)$.

22.[M]

(a) How far does a bug travel from time $t = 1$ to time $t = 2$ if at time t it is at the point $(x, y) = (\cos \pi t, \sin \pi t)$?

(b) How fast is it moving at time t?

(c) Graph its path relative to an xy coordinate system. Where is it at time $t = 1$? At $t = 2$?

(d) Eliminate t to find a relation between x and y.

23.[M] Find the arc length of the Archimedean spiral $r = a\theta$ for θ in $[0, 2\pi]$ if a is a positive constant.

24.[M] Consider the cardioid $r = 1 + \cos\theta$ for θ in $[0, \pi]$. We may consider r as a function of θ or as a function of s, arc length along the curve, measured, say, from $(2, 0)$.

(a) Find the average of r with respect to θ in $[0, \pi]$.

(b) Find the average of r with respect to s. HINT: Express all quantities appearing in this average in terms of θ.

(See also Exercises 13 and 14 in the Chapter 9 Summary.)

25.[M] Let $r = f(\theta)$ describe a curve in polar coordinates. Assume that $df/d\theta$ is continuous. Let θ be a function of time t. Let $s(t)$ be the length of the curve corresponding to the time interval $[a, t]$.

(a) What definite integral is equal to $s(t)$?

(b) What is the speed ds/dt?

26.[M] The function $r = f(\theta)$ describes, for θ in $[0, 2\pi]$, a curve in polar coordinates. Assume r' is continuous and $f(\theta) > 0$. Prove that the average of r as a function of arc length is at least as large as the quotient $2A/s$, where A is the area swept out by the radius and s is the arc length of the curve. For which curve is the average equal to $2A/s$?

27.[M] The equations $x = \cos t$, $y = 2\sin t$, t in $[0, \pi/2]$ describe a quarter of an ellipse. Draw this arc. Describe at least two different ways of estimating the length of this arc. Compare the advantages and challenges each method presents. Use the method of your choice to estimate the length of this arc.

28.[M] When a curve is given in rectangular coordinates, its slope is $\frac{dy}{dx}$. To find the slope of the tangent line to the curve given in polar coordinates involves a bit more work.
Assume that $r = f(\theta)$. To begin use the relation

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta},$$

which is the Chain Rule in disguise ($\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$).

(a) Using the equations $y = r\sin(\theta)$ and $x = r\cos(\theta)$, find $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$.

(b) Show that the slope is

$$\frac{r\cos(\theta) + \frac{dr}{d\theta}\sin(\theta)}{-r\sin(\theta) + \frac{dr}{d\theta}\cos(\theta)}. \tag{9.4.2}$$

29.[M] Use (9.4.2) to find the slope of the cardioid $r = 1 + \sin(\theta)$ at $\theta = \frac{\pi}{3}$.

30.[M] Show that if $\lim_{\Delta t \to 0} \frac{\Delta s}{|PQ|} = 1$, then $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{|PQ|}{\Delta t}$.

31.[C] Let $y = f(x)$ for x in $[0, 1]$ describe a curve that starts at $(0, 0)$, ends at $(1, 1)$, and lies in the square with vertices $(0, 0)$,$(1, 0)$,$(1, 1)$, and $(0, 1)$. Assume f has a continuous derivative.

(a) What can be said about the arc length of the curve? How small and how large can it be?

(b) Answer (a) if it is assumed also that $f'(x) \geq 0$ for x in $[0, 1]$.

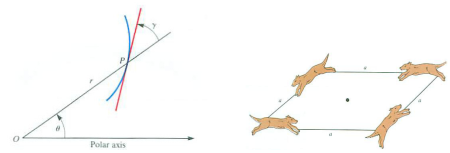
32.[C] Consider the length of the curve $y = x^m$, where m is a rational number. Show that the Fundamental Theorem of Calculus is of aid in computing this length only if $m = 1$ or if m is of the form $1 + 1/n$ for some integer n. *Hint:* Chebyshev proved that $\int x^p(1 + x)^q \, dx$ is elementary for rational numbers p and q only when at least one of p, q and $p + q$ is an integer.

33.[C] If one convex polygon P_1 lies inside another poligon P_2 is the perimeter of P_1 necessarily less than the perimeter of P_2? What if P_1 is not convex?

34.[C] One leaf of the cardioid $r = 1 + \sin(\theta)$ is traced as θ increases from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$. Find the highest point on that leaf in polar coordinates.

Exercises 35 and 36 form a unit. **35.**[C] Figure 9.4.8(a) shows the angle between the radius and tangent line to the curve $r = f(\theta)$. Using the fact that $\gamma = \alpha - \theta$ and that $\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$, show that $\tan(\gamma) = \frac{r}{r'}$. NOTE: See Exercise 36 for the derivation of $\tan(\gamma)$.

36.[C] The formula $\tan(\gamma) = r/r'$ in Exercise 35 is so simple one would expect a simple geometric explanation. Use the "triangle" in Figure 9.4.5 that we used to obtain the formula for $\frac{ds}{d\theta}$ to show that $\tan(\gamma)$ should be r/r'. NOTE: See Exercise 35.



(a) (b)

Figure 9.4.8: (a) ARTIST: (a) extend the (red) tangent line to the curve so it intersects the polar axis and label the angle made by the tangent to the curve with the polar axis as α

37.[C] Four dogs are chasing each other counterclockwise at the same speed. Initially they are at the four vertices of a square of side a. As they chase each other, each running directly toward the dog in front, they approach the center of the square in spiral paths. How far does each dog travel?

(a) Find the equation of the spiral path each dog follows and use calculus to answer this question.

(b) Answer the question without using calculus.

38.[C] We assumed that a chord $\overline{AB}$ of a smooth curve is a good approximation of the arc $\overset{\frown}{AB}$ when B is near to A. Show that the formula we obtained for arc length is consistent with this assumption. That is, if $y = f(x)$, $A = (a, f(a))$, $B = (x, f(x))$, then

$$\frac{\int_a^x \sqrt{1 + f'(t)^2}\, dt}{\sqrt{(x-a)^2 + (f(x) - f(a))^2}}$$

approaches 1 as x approaches a. Assume that $f'(x)$ is continuous. Hint: L'Hôpital's Rule is tempting but does not help. For simplicity, assume $a = 0 = f(0)$.

39.[C] In some approaches to arc length and speed on a curve the arc length is found first, then the speed. We outline this method in this Exercise.

Let $x = g(t)$, $y = h(t)$ where g and h have continuous derivatives. Let $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition of $[a, b]$ into n equal sections of length $\Delta t = (b-a)/n$. Let $P_i = (g(t_i), h(t_i))$, which we write as (x_i, y_i). Then the polygon $P_0 P_1 P_2 \cdots P_n$ is inscribed in the curve. We assume that as $n \to \infty$, the length of this polygon, $\sum_{i=1}^n |\overline{P_{i-1}P_i}|$ approaches the length of the curve from $(g(a), h(a))$ to $(g(b), h(b))$.

(a) Show that the length of the polygon is $\sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$.

(b) Show that the sum can be written as

$$\sum_{i=1}^{n} \sqrt{(g'(t_i^*))^2 + (h'(t_i^{**}))^2} \cdot \Delta t \tag{9.4.3}$$

for some t_i^* and t_i^{**} in $[t_{i-1}, t_i]$.

(c) Why would you expect the limit of (9.4.3) as $n \to \infty$ to be $\int_a^b \sqrt{(g'(t))^2 + h'(t))^2}\, dt$?
NOTE: This result is typically proved in Advanced Calculus, even though t_i^* and t_i^{**} may be different.

(d) From (c) deduce that the speed is $\sqrt{(g'(t))^2 + h'(t))^2}$.

9.5 The Area of a Surface of Revolution

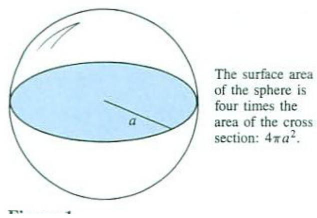
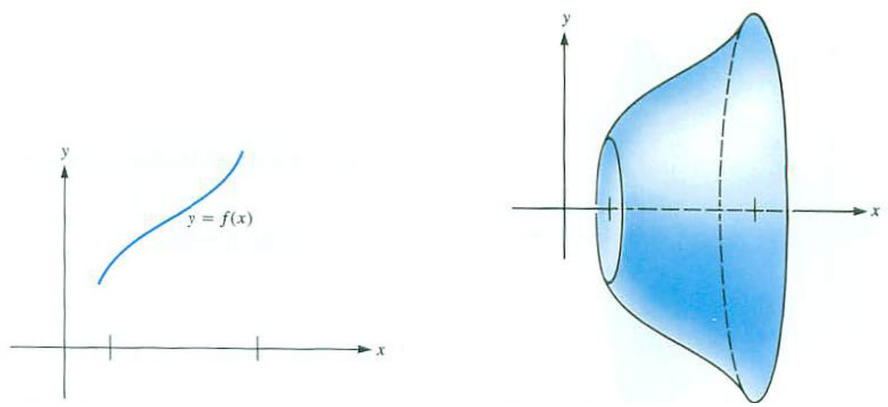


Figure 9.5.1:

In this section we develop a formula for expressing the surface area of a solid of revolution as a definite integral. In particular, we will show that the surface area of a sphere is four times the area of a cross section through its center. (See Figure 9.5.1.) This was one of the great discoveries of Archimedes in the third century B.C.

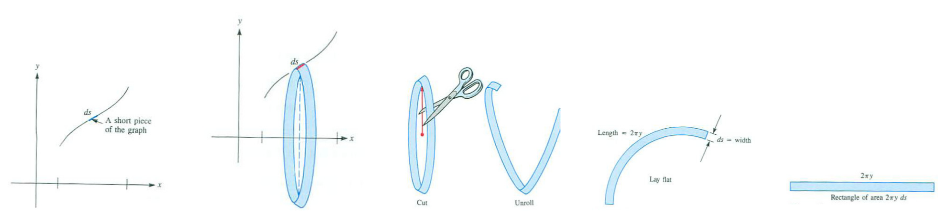
Let $y = f(x)$ have a continuous derivative for x in some interval. Assume that $f(x) \geq 0$ on this interval. When its graph is revolved about the x-axis it sweeps out a surface, as shown in Figures 9.5.2. To develop a definite integral



(a) (b)

Figure 9.5.2:

for this surface area, we use an informal approach.



(a) (b) (c) (d) (e)

Figure 9.5.3:

Consider a very short section of the graph $y = f(x)$. It is almost straight. Let us approximate it by a short line segment of length ds, a very small number. When this small line segment is revolved about the x-axis it sweeps out a narrow band. (See Figures 9.5.3(a) and (b).)

If we can estimate the area of this band, then we will have a local approximation of the surface area. From the local approximation we can set up a definite integral for the entire surface area.

Imagine cutting the band with scissors and laying it flat, as in Figures 9.5.3(c) and (d). It seems reasonable that the area of the flat band in Figure 9.5.3(d) is close to the area of a flat rectangle of length $2\pi y$ and width ds, as in Figure 9.5.3(e). (See Exercises 28 and 29.)

The gives us

$$\text{local approximation of the surface area of one slice} = 2\pi y\ ds.$$

which, in the usual way, leads to the formula

$$\boxed{\text{Surface area} = \int_{s_0}^{s_1} 2\pi y\ ds. \qquad (9.5.1)}$$

where $[s_0, s_1]$ describes the appropriate interval on the "s-axis". Since s is a clumsy parameter, for computations we will use one of the forms for ds to change (9.5.1) into more convenient integrals.

Assume that $y \geq 0$ and that dy/dx is continuous.

Say that the section of the graph $y = f(x)$ that was revolved corresponds to the interval $[a, b]$ on the x-axis, as in Figure 9.5.4. Then

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

and the surface area integral $\int_{s_0}^{s_1} 2\pi y\ ds$ becomes

$$\boxed{\text{Surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx. \qquad (9.5.2)}$$

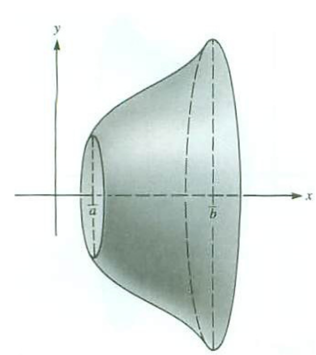


Figure 9.5.4:

EXAMPLE 1 Find the surface area of a sphere of radius a.

SOLUTION The circle of radius a has the equation $x^2 + y^2 = a^2$. The top

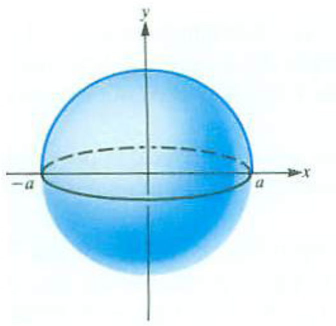


Figure 9.5.5:

half has the equation $y = \sqrt{a^2 - x^2}$. The sphere of radius a is formed by revolving this semi-circle about the x-axis. (See Figure 9.5.5.) We have

$$\text{surface area of sphere} = \int_{-a}^{a} 2\pi y\ ds.$$

Because $dy/dx = -x/\sqrt{a^2 - x^2}$ we find that

$$\begin{aligned} ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2}\ dx \\ &= \sqrt{1 + \frac{x^2}{a^2 - x^2}}\ dx = \sqrt{\frac{a^2}{a^2 - x^2}}\ dx = \frac{a}{\sqrt{a^2 - x^2}}\ dx. \end{aligned}$$

Thus,

$$\begin{aligned} \text{surface area of sphere} &= \int_{-a}^{a} 2\pi y\ ds = \int_{-a}^{a} 2\pi\sqrt{a^2 - x^2}\frac{a}{\sqrt{a^2 - x^2}}\ dx \\ &= \int_{-a}^{a} 2\pi a\ dx = 2\pi a x|_{-a}^{a} = 4\pi a^2. \end{aligned}$$

The surface area of a sphere is 4 times the area of its equatorial cross section. ◇

If the graph is given parametrically, $x = g(t)$, $y = h(t)$, where g and h have continuous derivatives and $h(t) \geq 0$, then it is natural to express the integral $\int_{s_0}^{s_1} 2\pi y\ ds$ as an integral over an interval on the t-axis. If t varies in the interval $[a, b]$, then

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

which leads to

$$\text{Surface area for a parametric curve} = \int_{a}^{b} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt. \qquad (9.5.3)$$

Formula 9.5.2 is just the special case of Formula 9.5.3 when the parameter is x.

As the formulas are stated, they seem to refer only to surfaces obtained by revolving a curve about the x-axis. In fact, they refer to revolution about any line. The factor y in the integrand, $2\pi y\ ds$, is the distance from the typical point on the curve to the axis of revolution. Replace y by R (for *radius*) to free ourselves from coordinate systems. (Use capital R to avoid confusion with polar coordinates.) The simplest way to write the formula for surface area of revolution is then

$$\boxed{\text{Surface area} = \int_c^d 2\pi R\ ds,}$$

where the interval $[c, d]$ refers to the parameter s. However, in practice arc length, s, is seldom a convenient parameter. Instead, x, y, t or θ is used and the interval of integration describes the interval through which the parameter varies.

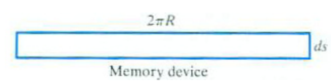


Figure 9.5.6: The key to this section.

To remember this formula, think of a narrow circular band of width ds and radius R as having an area close to the area of the rectangle shown in Figure 9.5.6.

EXAMPLE 2 Find the area of the surface obtained by revolving around the y-axis the part of the parabola $y = x^2$ that lies between $x = 1$ and $x = 2$. (See Figure 9.5.7.)

R is found by inspection of a diagram.

SOLUTION The surface area is $\int_a^b 2\pi R\ ds$. Since the curve is described as a function of x, choose x as the parameter. By inspection of Figure 9.5.7, $R = x$. Next, note that

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \sqrt{1 + 4x^2}\ dx.$$

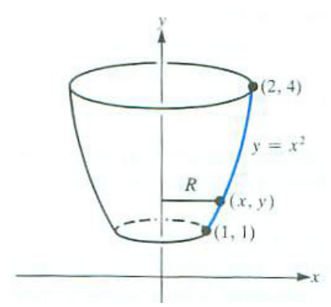


Figure 9.5.7:

The surface area is therefore

$$\int_1^2 2\pi x\sqrt{1 + 4x^2}\ dx.$$

To evaluate the integral, use the substitution

$$u = 1 + 4x^2 \qquad du = 8x\ dx.$$

Hence $x\,dx = du/8$. The new limits of integration are $u = 5$ and $u = 17$. Thus

$$\begin{aligned}\text{surface area} &= \int_5^{17} 2\pi\sqrt{u}\frac{du}{8} = \frac{\pi}{4}\int_5^{17}\sqrt{u}\,du \\ &= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_5^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.84649.\end{aligned}$$

◇

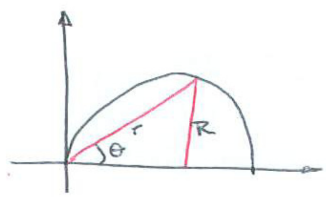


Figure 9.5.8:

EXAMPLE 3 Find the surface area when the curve $r = \cos(\theta)$, θ in $[0, \pi/2]$ is revolved around (a) the x-axis and (b) the y-axis.

SOLUTION The curve is shown in Figure 9.5.8. Note that it is the semicircle with radius $1/2$ and center $(1/2, 0)$. (a) We need to find both R and $ds/d\theta$. First, $R = r\sin(\theta) = \cos(\theta)\sin(\theta)$. And, using the formula for $\frac{ds}{d\theta}$ for a polar curve from Section 9.4 we have

$$\frac{ds}{d\theta} = \sqrt{r(\theta)^2 + r'(\theta)^2} = \sqrt{(\cos(\theta))^2 + (-\sin(\theta))^2} = 1.$$

Then

$$\begin{aligned}\text{surface area} &= \int_0^{\pi/2} 2\pi R\frac{ds}{d\theta}\,d\theta = \int_0^{\pi/2} 2\pi\cos(\theta)\sin(\theta)(1)\,d\theta \\ &= \int_0^{\pi/2} 2\pi\sin(\theta)\cos(\theta)\,d\theta = 2\pi\frac{\sin^2(\theta)}{2}\Big|_0^{\pi/2} = \pi.\end{aligned}$$

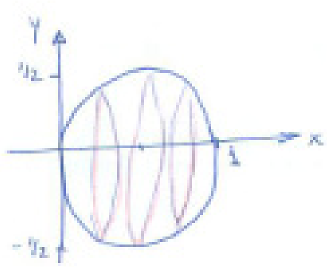
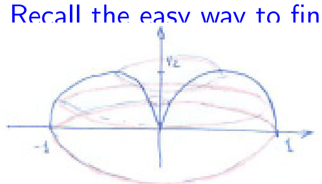


Figure 9.5.9:
Recall the easy way to find

This is expected since this surface of revolution is a sphere of radius $1/2$. See Figure 9.5.9.



$$\begin{aligned}\text{surface area} &= \int_0^{\pi/2} 2\pi R\frac{ds}{d\theta}\,d\theta = \int_0^{\pi/2} 2\pi\cos^2(\theta)(1)\,d\theta \\ &= 2\pi\int_0^{\pi/2}\cos^2(\theta)\,d\theta = 2\pi(\frac{\pi}{4}) = \frac{\pi^2}{2}.\end{aligned}$$

Figure 9.5.10:

This surface is the top half of a doughnut whose hole has just vanished. See Figure 9.5.10. ◇

Summary

This section developed a definite integral for the area of a surface of revolution. It rests on the local estimate of the area swept out by a short segment of length ds revolved around a line L at a distance R from the segment: $2\pi R\ ds$. (See Figure 9.5.11.) We gave an informal argument for this estimate; Exercises 28 and 29 develop it more formally.

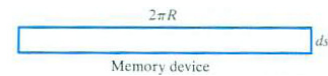


Figure 9.5.11:

EXERCISES for Section 9.5 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 4 set up a definite integral for the area of the indicated surface using the suggested parameter. Show the radius R on a diagram. Do *not* evaluate the definite integrals.

1.[R] The graph of $y = x^3$, x on the interval $[1, 2]$ revolved about the x-axis with parameter x.

2.[R] The graph of $y = x^3$, x on the interval $[1, 2]$ revolved about the line $y = -1$ with parameter x.

3.[R] The graph of $y = x^3$, x on the interval $[1, 2]$ revolved about the y-axis with parameter y.

4.[R] The graph of $y = x^3$, x on the interval $[1, 2]$ revolved about the y-axis with parameter x.

5.[R] Find the area of the surface obtained by rotating about the x-axis that part of the curve $y = e^x$ that lies above $[0, 1]$.

6.[R] Find the area of the surface formed by rotating one arch of the curve $y = \sin(x)$ about the x-axis.

7.[R] One arch of the cycloid given parametrically by the formula $x = \theta - \sin(\theta)$, $y = 1 - \cos(\theta)$ is revolved around the x-axis. Find the area of the surface produced.

8.[R] The curve given parametrically by $x = e^t \cos(t)$, $y = e^t \sin(t)$ $(0 \le t \le \pi/2)$ is revolved around the x-axis. Find the area of the surface produced.

In each of Exercises 9 to 16 find the area of the surface formed by revolving the indicated curve about the indicated axis. Leave the answer as a definite integral, but indicate how it could be evaluated by the Fundamental Theorem of Calculus.

9.[R] $y = 2x^3$ for x in $[0, 1]$; about the x-axis.

10.[R] $y = 1/x$ for x in $[1, 2]$; about the x-axis.

11.[R] $y = x^2$ for x in $[1, 2]$; about the x-axis.

12.[R] $y = x^{4/3}$ for x in $[1, 8]$; about the y-axis.

13.[R] $y = x^{2/3}$ for x in $[1, 8]$; about the line $y = 1$.

14.[R] $y = x^3/6 + 1/(2x)$ for x in $[1, 3]$; about the y-axis.

15.[R] $y = x^3/3 + 1/(4x)$ for x in $[1, 2]$; about the line $y = -1$.

16.[R] $y = \sqrt{1 - x^2}$ for x in $[-1, 1]$; about the line $y = -1$.

17.[M] Consider the smallest tin can that contains a given sphere.[1] (The height and diameter of the tin can equal the diameter of the sphere.)

(a) Compare the volume of the sphere with the volume of the tin can.

(b) Compare the surface area of the sphere with the total surface area of the can.

NOTE: See also Exercise 37.

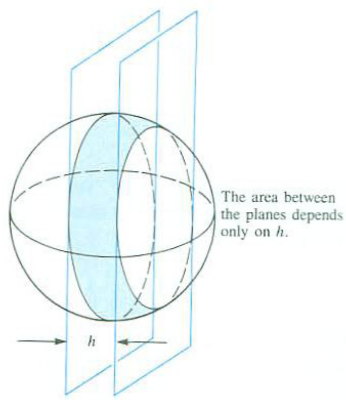


Figure 9.5.12:

18.[M]

(a) Compute the area of the portion of a sphere of radius a that lies between two parallel planes at distances c and $c+h$ from the center of the sphere ($0 \le c \le c+h \le a$).

[1] Archimedes, who obtained the solution about 2200 years ago, considered it his greatest accomplishment. Cicero wrote, about two centuries after Archimedes' death:

> I shall call up from the dust [the ancient equivalent of a blackboard] and his measuring-rod an obscure, insignificant person belonging to the same city [Syracuse], who lived many years after, Archimedes. When I was quaestor I tracked out his grave, which was unknown to the Syracusans (as they totally denied its existence), and found it enclosed all round and covered with brambles and thickets; for I remembered certain doggerel lines inscribed, as I had heard, upon his tomb, which stated that a sphere along with a cylinder had been set up on the top of his grave. Accordingly, after taking a good look around (for there are a great quantity of graves at the Agrigentine Gate), I noticed a small column rising a little above the bushes, on which there was the figure of a sphere and a cylinder. And so I at once said to the Syracusans (I had their leading men with me) that I believed it was the very thing of which I was in search. Slaves were sent in with sickles who cleared the ground of obstacles, and when a passage to the place was opened we approached the pedestal fronting us; the epigram was traceable with about half the lines legible, as the latter portion was worn away. [Cicero, *Tusculan Disputations*, vol. 23, translated by J. E. King, Loef Classical Library, Harvard Univeristy, Cambridge, 1950.]

Archimedes was killed by a Roman soldier in 212 B.C. Cicero was quaestor in 75 B.C.

(b) The result in (a) depends only on h, not on c. What does this mean geometrically? (See Figure 9.5.12.)

In Exercises 19 and 20 estimate the surface area obtained by revolving the specified arc about the given line. First, find a definite integral for the surface area. Then, use either Simpson's method with six sections or a programmable calculator or computer to approximate the value of the integral.

19.[M] $y = x^{1/4}$, x in $[1, 3]$, about the x-axis.

20.[M] $y = x^{1/5}$, x in $[1, 3]$, about the line $y = -1$.

Exercises 21 to 24 are concerned with the area of a surface obtained by revolving a curve given in polar coordinates.

21.[M] Show that the area of the surface obtained by revolving the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, around the polar axis is

$$\int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{r^2 + (r')^2}\, d\theta.$$

HINT: Use a local approximation informally.

22.[M] Use Exercise 21 to find the surface area of a sphere of radius a.

23.[M] Find the area of the surface formed by revolving the portion of the curve $r = 1 + \cos(\theta)$ in the first quadrant about (a) the x-axis, (b) the y-axis. HINT: The identity $1 + \cos(\theta) = 2\cos^2(\theta/2)$ may help in (b).

24.[M] The curve $r = \sin(2\theta)$, θ in $[0, \pi/2]$, is revolved around the polar axis. Set up an integral for the surface area.

25.[M] The portion of the curve $x^{2/3} + y^{2/3} = 1$ situated in the first quadrant is revolved around the x-axis. Find the area of the surface produced.

26.[M] Although the Fundamental Theorem of Calculus is of no use in computing the perimeter of the ellipse $x^2/a^2 + y^2/b^2 = 1$, it is useful in computing the surface area of the "football" formed when the ellipse is rotated about one of its axes.

(a) Assuming that $a > b$ and that the ellipse is revolved around the x-axis, find that area.

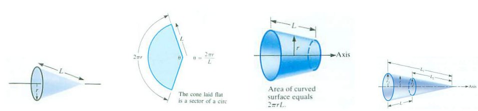
(b) Does your answer give the correct formula for the surface area of a sphere of radius a, $4\pi a^2$? HINT: Let b approach a from the left.

27.[M] The (unbounded) region bounded by $y = 1/x$ and the x-axis and situated to the right of $x = 1$ is revolved around the x-axis.

(a) Show that its volume is finite but its surface area is infinite.

(b) Does this mean that an infinite area can be painted by pouring a finite amount of paint into this solid?

Exercises 28 and 29 obtain the formula for the area of the surface obtained by revolving a line segment about a line that does not meet it. (This area was only estimated in the text.)



(a) (b) (c) (d)

Figure 9.5.13:

28.[M] A right circular cone has slant height L and radius r, as shown in Figure 9.5.13(a). If this cone is cut along a line through its vertex and laid flat, it becomes a sector of a circle of radius L, as shown in Figure 9.5.13(b). By comparing Figure 9.5.13(b) to a complete disk of radius L find the area of the sector and thus the area of the cone in Figure 9.5.13(a).

29.[M] Consider a line segment of length L in the plane which does not meet a certain line in the plane, called the axis. (See Figure 9.5.13(c).) When the line segment is revolved around the axis, it sweeps out a curved surface. Show that the area of this surface equals $2\pi rL$ where r is the distance from the midpoint of the line segment to the axis. The surface in Figure 9.5.3 is called a **frustum of a cone**. Follow these steps:

(a) Complete the cone by extending the frustum as shown in Figure 9.5.13(d). Label the radii and lengths as in that figure. Show that $\frac{r_1}{r_2} = \frac{L_1}{L_2}$, hence $r_1L_2 = r_2L_1$.

(b) Show that the surface area of the frustum is $\pi r_1L_1 - \pi r_2L_2$.

(c) Express L_1 as $L_2 + L$ and, using the result of (a), show that

$$\pi r_1L_1 - \pi r_2L_2 = \pi r_2(L_1 - L_2) + \pi r_1L = \pi r_2L + \pi r_1L.$$

(d) Show that the surface area of the frustum is $2\pi rL$, where $r = (r_1 + r_2)/2$. NOTE: This justifies our approximation $2\pi R\ ds$.

30.[C] The derivative (with respect to r) of the volume of a sphere is its surface area: $\frac{d}{dr}\left(4\pi r^3/3\right) = 4\pi r^2$. Is this simply a coincidence?

31.[C] Define the **moment of a curve around the x-axis** to be $\int_{s_1}^{s_2} y\ ds$, where s_1 and s_2 refer to the range of the arc length s. The **moment of the curve around the y-axis** is defined as $\int_{s_1}^{s_2} x\ ds$. The **centroid** of the curve, $(\overline{x}, \overline{y})$, is defined by setting

$$\overline{x} = \frac{\int_{s_1}^{s_2} x\ ds}{\text{length of curve}} \quad \overline{y} = \frac{\int_{s_1}^{s_2} y\ ds}{\text{length of curve}}$$

Find the centroid of the top half of the circle $x^2 + y^2 = a^2$.

32.[C] Show that the area of the surface obtained by revolving about the x-axis a curve that lies above it is equal to the length of the curve times the distance that the centroid of the curve moves. NOTE: See Exercise 31.

33.[C] Let a be a positive number and $\mathcal{R}$ the region bounded by $y = x^a$, the x-axis, and the line $x = 1$.

(a) Show that the centroid of $\mathcal{R}$ is $\left(\frac{a+1}{4a+2}, \left(\frac{a+1}{a+2}\right)^a\right)$.

(b) Show that the centroid of $\mathcal{R}$ lies in $\mathcal{R}$ for all large values of a.

NOTE: It is true that the centroid lies in $\mathcal{R}$ for all positive values of a, but this proof is more difficult.

34.[C] Use Exercise 32 to find the surface area of the doughnut formed by revolving a circle of radius a around a line a distance b from its center, $b \geq a$.

35.[C] Use Exercise 32 to find the area of the curved part of a cone of radius a and height h.

36.[C] For some continuous functions $f(x)$ the definite integral $\int_a^b f(x)\ dx$ depends only on the width of the interval $[a, b]$; namely, there is a function $g(x)$ such that

$$\int_a^b f(x)\ dx = g(b - a). \tag{9.5.4}$$

(a) Show that every constant function $f(x)$ satisfies (9.5.4).

(b) Prove that if $f(x)$ satisfies (9.5.4), then it must be constant.

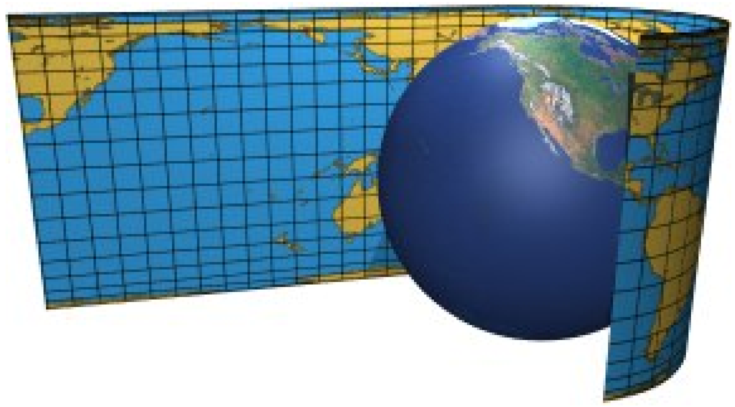


Figure 9.5.14: Source: `http://www.progonos.com/furuti/MapProj/Dither/ProjCyl/ProjCEA/projCEA.html`

NOTE: See Exercise 18.

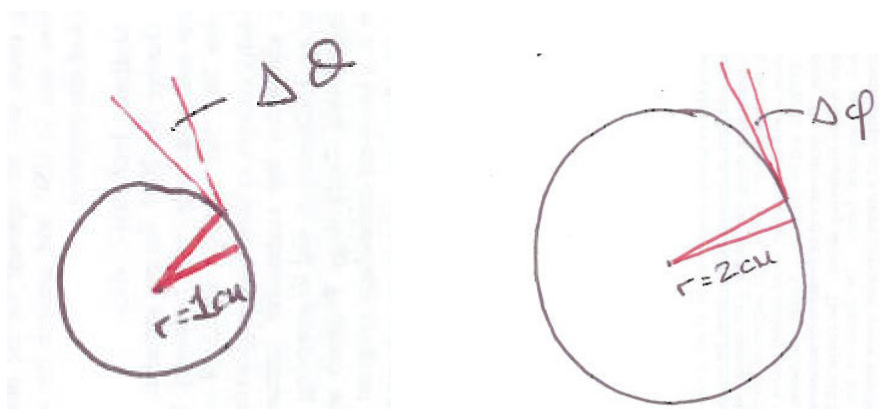
37.[C] The Mercator map discussed in the CIE of this chapter preserves angles. A **Lambert azimuthal equal-area projection** preserves areas, but not angles. It is made by projecting a sphere on a cylinder tangent at the equator by rays parallel to the equatorial plane and having one end on the diameter that joins the north and south poles, as shown in Figure 9.5.14.
Explain why a Lambert map preserves areas. HINT: See Exercise 17.

9.6 Curvature

In this section we use calculus to obtain a measure of the "curviness" or "curvature" at points on a curve. This concept will be generalized in Section 15.2 in the study of motion along a curved path in space.

Introduction

Imagine a bug crawling around a circle of radius one centimeter, as in Figure 9.6.1(a). As it walks a small distance, say 0.1 cm, it notices that its direction, measured by angle θ, changes. Another bug, walks around a larger circle, as in Figure 9.6.1(b). Whenever it goes 0.1 cm, its direction, measured by angle ϕ, changes by much less. The first bug feels that his circle is curvier than the circle of the second bug. We will provide a measure of "curviness" or **curvature**. A straight line will have "zero curvature" everywhere. A circle of radius a will turn out to have curvature $1/a$ everywhere. For other curves, the curvature varies from point to point.



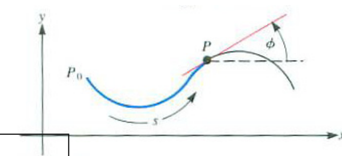
(a) (b)

Figure 9.6.1: The circle in (b) has twice the radius as the circle in (a). But, the change in $\Delta\phi$ in (b) is half that in (a).

Definition of Curvature

"Curvature" measures how rapidly the direction changes as we move a small distance along a curve. We have a way of assigning a numerical value to direction, namely, the angle of the tangent line. The *rate of change of this angle with respect to arc length* will be our measure of curvature.

DEFINITION *(Curvature)* Assume that a curve is given parametrically, with the parameter of the typical point P being s, the distance along the curve from a fixed P_0 to P. Let ϕ be the angle between the tangent line at P and the positive part of the x-axis. The **curvature** κ at P is the absolute value of the derivative, $\frac{d\phi}{ds}$:



$$\boxed{\text{curvature} = \kappa = \left|\frac{d\phi}{ds}\right|}$$

Figure 9.6.2:

κ is the Greek letter "kappa".

whenever the derivative exists. (See Figure 9.6.2.)

Observe that a straight line has zero curvature everywhere, since ϕ is constant.

The next theorem shows that curvature of a small circle is large and the curvature of a large circle is small, in agreement with the bugs' experience.

Theorem. *(Curvature of Circles) For a circle of radius a, the curvature $\left|\frac{d\phi}{ds}\right|$ is constant and equals $1/a$, the reciprocal of the radius.*

Proof

It is necessary to express ϕ as a function of arc length s on a circle of radius a. Refer to Figure 9.6.3. Arc length s is measured counterclockwise from the point P_0 on the x-axis. Then $\phi = \frac{\pi}{2} + \theta$, as Figure 9.6.3 shows. By definition of radian measure, $s = a\theta$, so that $\theta = s/a$. We can solve for ϕ, $\phi = \frac{\pi}{2} + \frac{s}{a}$. Differentiating with respect to arc length yields:

$$\frac{d\phi}{ds} = \frac{1}{a},$$

as claimed. •

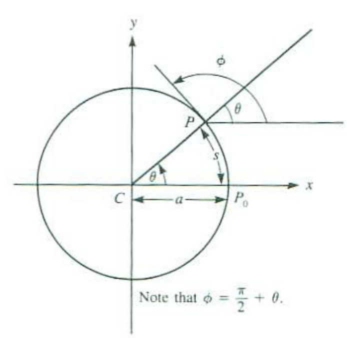


Figure 9.6.3:

Computing Curvature

When a curve is given in the form $y = f(x)$, the curvature can be expressed in terms of the first and second derivatives, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Theorem. *(Curvature of $y = f(x)$) Let arc length s be measured along the curve $y = f(x)$ from a fixed point P_0. Assume that x increases as s increases and that y' and y'' are continuous. Then*

The curvature of $y = f(x)$.

$$\text{curvature} = \kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}.$$

Proof

The Chain Rule, $\frac{d\phi}{dx} = \frac{d\phi}{ds}\frac{ds}{dx}$, implies

$$\frac{d\phi}{ds} = \frac{\frac{d\phi}{dx}}{\frac{ds}{dx}}.$$

As was shown in Section 9.3,

$$\frac{ds}{dx} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}.$$

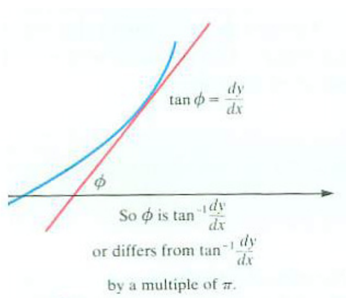


Figure 9.6.4:

All that remains is to express $\frac{d\phi}{dx}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Note that in Figure 9.6.4,

$$\frac{dy}{dx} = \text{slope of tangent line to the curve} = \tan(\phi). \qquad (9.6.1)$$

We find $\frac{d\phi}{dx}$ by differentiating both sides of (9.6.1) with respect to x, that is, both sides of the equation $\frac{dy}{dx} = \tan(\phi)$. Thus

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan(\phi)) = \sec^2(\phi)\cdot\frac{d\phi}{dx} = \left(1 + \tan^2(\phi)\right)\frac{d\phi}{dx} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)\frac{d\phi}{dx}.$$

Solving for $d\phi/dx$, we get

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}.$$

Consequently,

$$\frac{d\phi}{ds} = \frac{\frac{d\phi}{dx}}{\frac{ds}{dx}} = \frac{\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}},$$

and the theorem is proved. •

> **WARNING** *(Geometry of the Curvature)* One might have expected the curvature to depend only on the second derivative, $\frac{d^2y}{dx^2}$, since it records the rate at which the slope changes. This expectation is correct only when $\frac{dy}{dx} = 0$, that is, at critical points in the graph of $y = f(x)$. (See also Exercise 28.)

EXAMPLE 1 Find the curvature at a point (x, y) on the curve $y = x^2$.
SOLUTION In this case $\frac{dy}{dx} = 2x$ and $\frac{d^2y}{dx^2} = 2$. The curvature at (x, y) is

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}} = \frac{2}{(1 + (2x)^2)^{3/2}}.$$

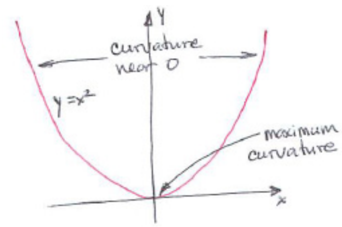


Figure 9.6.5:

The maximum curvature occurs when $x = 0$. The curvatures at (x, x^2) and at $(-x, x^2)$ are equal. As $|x|$ increases, the curve becomes straighter and the curvature approaches 0. (See Figure 9.6.5.) ◇

Curvature of a Parameterized Curve

Theorem 9.6 applies also to curves given parametrically.

Theorem 9.6 tells how to find the curvature if y is given as a function of x. But it holds as well when the curve is described parametrically, where x and y are functions of some parameter such as t or θ. Just use the fact that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}. \tag{9.6.2}$$

Both equations in (9.6.2) are special cases of

$$\frac{df}{dx} = \frac{\frac{df}{dt}}{\frac{dx}{dt}}.$$

And this equation is just the Chain Rule in disguise,

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}.$$

In the first equation in (9.6.2), the function f is y; in the second equation, f is $\frac{dy}{dx}$. Example 2 illustrates the procedure.

EXAMPLE 2 The cycloid determined by a wheel of radius 1 has the parametric equations

$$x = \theta - \sin(\theta) \qquad \text{and} \qquad y = 1 - \cos(\theta),$$

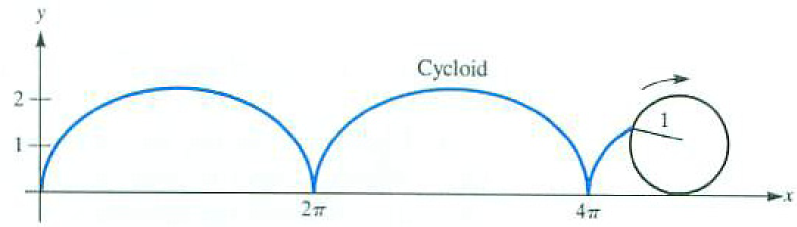


Figure 9.6.6:

as shown in Figure 9.6.6. Find the curvature at a typical point on this curve.
SOLUTION First find $\frac{dy}{dx}$ in terms of θ. We have

$$\frac{dx}{d\theta} = 1 - \cos(\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = \sin(\theta).$$

Thus

$$\frac{dy}{dx} = \frac{\sin(\theta)}{1 - \cos(\theta)}.$$

The parts of the cycloid near the x-axis are nearly vertical. See Exercise 29.

Similar direct calculations show that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left(\frac{\sin(\theta)}{1-\cos(\theta)}\right)}{1 - \cos(\theta)} = \frac{-1}{(1 - \cos(\theta))^2}.$$

Thus the curvature is

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}} = \frac{\left|\frac{-1}{(1-\cos(\theta))^2}\right|}{\left(\frac{2}{1-\cos(\theta)}\right)^{3/2}} = \frac{1}{2^{3/2}\sqrt{1 - \cos(\theta)}}.$$

Since $y = 1 - \cos(\theta)$ and $2^{3/2} = \sqrt{8}$, the curvature equals $1/\sqrt{8y}$. ◇

Radius of Curvature

A large radius of curvature implies a small curvature.

As Theorem 9.6 shows, a circle with curvature κ has radius $1/\kappa$. This suggests the following definition.

> **DEFINITION** *(Radius of Curvature)* The **radius of curvature** of a curve at a point is the reciprocal of the curvature:
>
> $$\text{radius of curvature} = \frac{1}{\text{curvature}} = \frac{1}{\kappa}.$$

As can be easily checked, the radius of curvature of a circle of radius a is, fortunately, a.

The cycloid in Example 2 has radius of curvature at the point (x, y) equal to $\sqrt{8y}$. The higher the point on the curve, the straighter the curve.

The line through a point P as a curve that looks most like the curve near P is the tangent line. The circle through P that looks most like the curve near P has the same slope at P as the curve and a radius equal to the radius of curvature at P. It is called the **osculating circle**, from the Latin "osculum = kiss." The tangent line is never called the "osculating line".

The Osculating Circle

At a given point P on a curve, the **osculating circle** at P is defined to be that circle which (a) passes through P, (b) has the same slope at P as the curve does, and (c) has the same curvature there.

For instance, consider the parabola $y = x^2$ of Example 1. When $x = 1$, the curvature is $2/5^{3/2}$ and the radius of curvature is $5^{3/2}/2 \approx 5.59017$. The osculating circle at $(1, 1)$ is shown in Figure 9.6.7.

Observe that the osculating circle in Figure 9.6.7 *crosses the parabola* as it passes through the point $(1, 1)$. Although this may be surprising, a little reflection will show why it is to be expected.

Think of driving along the parabola $y = x^2$. If you start at $(1, 1)$ and drive up along the parabola, the curvature diminishes. It is smaller than that of the circle of curvature at $(1, 1)$. Hence you would be turning your steering wheel to the left and would be traveling *outside* the osculating circle at $(1, 1)$. On the other hand, if you start at $(1, 1)$ and move toward the origin (to the left) on the parabola, the curvature increases and is greater than that of the osculating circle at $(1, 1)$, so you would be driving *inside* the osculating circle at $(1, 1)$. This informal argument shows why the osculating circle crosses the curve in general. In the case of $y = x^2$, the only osculating circle that does *not* cross the curve at its point of tangency is the one that is tangent at $(0, 0)$, where the curvature is a maximum.

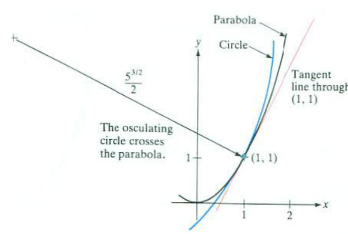


Figure 9.6.7:

Summary

We defined the curvature κ of a curve as the absolute value of the rate at which the angle between the tangent line and the x-axis changes as a function of arc length; curvature equals $\left|\frac{d\phi}{ds}\right|$. If the curve is the graph of $y = f(x)$, then

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}.$$

If the curve is given in terms of a parameter t then compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ with the aid of the relation

$$\frac{d(\;)}{dx} = \frac{\frac{d(\;)}{dt}}{\frac{dx}{dt}}, \tag{9.6.3}$$

Equation (9.6.3) is our old friend, the Chain Rule; just clear the denominator.

the empty parentheses enclosing first y, then $\frac{dy}{dx}$.

Radius of curvature is the reciprocal of curvature.

EXERCISES for Section 9.6 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 6 find the curvature and radius of curvature of the specified curve at the given point.

1.[R] $y = x^2$ at $(1, 1)$
2.[R] $y = \cos(x)$ at $(0, 1)$
3.[R] $y = e^{-x}$ at $(1, 1/e)$
4.[R] $y = \ln(x)$ at $(e, 1)$
5.[R] $y = \tan(x)$ at $(\frac{\pi}{4}, 1)$
6.[R] $y = \sec(2x)$ at $(\frac{\pi}{6}, 2)$

In Exercises 7 to 10 find the curvature of the given curves for the given value of the parameter.

7.[R] $\left\{\begin{array}{lcl} x &=& 2\cos(3t) \\ y &=& 2\sin(3t) \end{array}\right.$ at $t = 0$

8.[R] $\left\{\begin{array}{lcl} x &=& 1 + t^2 \\ y &=& t^3 + t^4 \end{array}\right.$ at $t = 2$

9.[R] $\left\{\begin{array}{lcl} x &=& e^{-t}\cos(t) \\ y &=& e^{-t}\sin(t) \end{array}\right.$ at $t = \frac{\pi}{6}$

10.[R] $\left\{\begin{array}{lcl} x &=& \cos^3(\theta) \\ y &=& \sin^3(\theta) \end{array}\right.$ at $\theta = \frac{\pi}{3}$

11.[R]

(a) Compute the curvature and radius of curvature for the curve $y = (e^x + e^{-x})/2$.

(b) Show that the radius of curvature at (x, y) is y^2.

12.[R] Find the radius of curvature along the curve $y = \sqrt{a^2 - x^2}$, where a is a constant. (Since the curve is part of a circle of radius a, the answer should be a.)

13.[R] For what value of x is the radius of curvature of $y = e^x$ smallest?
Hint: How does one find the minimum of a function?

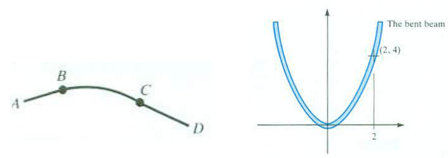
14.[R] For what value of x is the radius of curvature of $y = x^2$ smallest?

15.[M]

(a) Show that at a point where a curve has its tangent parallel to the x-axis its curvature is simply the absolute value of the second derivative d^2y/dx^2.

(b) Show that the curvature is never larger than the absolute value of d^2y/dx^2.

16.[M] An engineer lays out a railroad track as indicated in Figure 9.6.8(a). BC is part of a circle; AB and CD are straight and tangent to the circle. After the first train runs over the track, the engineer is fired because the curvature is not a continuous function. Why should the curvature be continuous?



(a) (b)

Figure 9.6.8:

17.[M] Railroad curves are banked to reduce wear on the rails and flanges. The greater the radius of curvature, the less the curve must be banked. The best bank angle A satisfies the equation $\tan(A) = v^2/(32R)$, where v is speed in feet per second and R is radius of curvature in feet. A train travels in the elliptical track

$$\frac{x^2}{1000^2} + \frac{y^2}{500^2} = 1$$

at 60 miles per hour. Find the best angle A at the points $(1000, 0)$ and $(0, 500)$. NOTE: x and y are measured in feet; 60 mph=88 fps.

18.[M] The flexure formula in the theory of beams asserts that the bending moment M required to bend a beam is proportional to the desired curvature, $M = c/R$, where c is a constant depending on the beam and R is the radius of curvature. A beam is bent to form the parabola $y = x^2$. What is the ratio between the moments required at (a) at $(0, 0)$ and (b) at $(2, 4)$? (See Figure 9.6.8(b).)

Exercises 19 to 21 are related.

19.[M] Find the radius of curvature at a typical point on the curve whose parametric equations are

$$x = a\cos\theta, \qquad y = b\sin\theta.$$

20.[M]

(a) Show, by eliminating θ, that the curve in Exercise 19 is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(b) What is the radius of curvature of this ellipse at $(a, 0)$? at $(0, b)$?

21.[M] An ellipse has a major diameter of length 6 and a minor diameter of length 4. Draw the circles that most closely approximate this ellipse at the four points that lie at the extremities of its diameters. (See Exercises 19 and 20.)

In each of Exercises 22 to 24 a curve is given in polar coordinates. To find its curvature write it in rectangular coordinates with parameter θ, using the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

22.[M] Find the curvature of $r = a\cos(\theta)$.

23.[M] Show that at the point (r, θ) the cardioid $r = 1 + \cos(\theta)$ has curvature $3\sqrt{2}/(4\sqrt{r})$.

24.[M] Find the curvature of $r = \cos(2\theta)$.

25.[M] If, on a curve, $dy/dx = y^3$, express the curvature in terms of y.

26.[M] As is shown in physics, the larger the radius of curvature of a turn, the faster a given car can travel around that turn. The required radius of curvature is proportional to the square of the maximum speed. Or, conversely, the maximum speed around a turn is proportional to the square root of the radius of curvature. If a car moving on the path $y = x^3$ (x and y measured in miles) can go 30 miles per hour at $(1, 1)$ without sliding off, how fast can it go at $(2, 8)$?

27.[M] Find the local extrema of the curvature of

(a) $y = x + e^x$

(b) $y = e^x$

(c) $y = \sin(x)$

(d) $y = x^3$

28.[M] Sam says, "I don't like the definition of curvature. It should be the rate at which the slope changes as a function of x. That is $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, which is the second derivative, $\frac{d^2y}{dx^2}$." Give an example of a curve which would have constant curvature according to Sam's definition, but whose changing curvature is obvious to the naked eye.

29.[M] In Example 2 some of the steps were omitted in finding that the cycloid given by $x = \theta - \sin(\theta)$, $y = 1 - \cos(\theta)$ has curvature $\kappa = 1/(2^{3/2}\sqrt{1 - \cos(\theta)}) = 1/\sqrt{8y}$. In this exercise you are asked to show all steps in this calculation.

(a) Verify that $\dfrac{dy}{dx} = \dfrac{\sin(\theta)}{1-\cos(\theta)}$.

(b) Show that $\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right) = \dfrac{-1}{1-\cos(\theta)}$

(c) Verify that $\dfrac{d^2y}{dx^2} = \dfrac{-1}{(1-\cos(\theta))^2}$.

(d) Show that $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{2}{1-\cos(\theta)}$.

(e) Compute the curvature, κ, in terms of θ.

(f) Express the curvature found in (e) in terms of x and y.

(g) At which points on the cycloid is the curvature largest?

(h) At which points on the cycloid is the curvature smallest?

30.[M] Assume that g and h are functions with continuous second derivatives. In addition, assume as we move along the parameterized curve $x = g(t)$, $y = h(t)$, the arc length s from a point P_0 increases as t increases. Show that

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

NOTE: Newton's dot notation for derivatives shortens the formula: $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $\dot{y} = \frac{dy}{dt}$, and $\ddot{y} = \frac{d^2y}{dt^2}$.

31.[M] Use the result of Exercise 30 to find the curvature of the cycloid of Example 2. NOTE: $x = \theta - \sin(\theta)$, $y = 1 - \cos(\theta)$

32.[C] (Contributed by G.D. Chakerian) If a planar curve has a constant radius of curvature must it be part of a circle? That the answer is "yes" is important in experiments conducted with a cyclotron: Physical assumptions imply that the path of an electron entering a uniform magnetic field at right angles to the field has constant curvature. Show that it follows that the path is part of a circle.

(a) Show that $\frac{ds}{d\phi} = R$, the radius of curvature.

(b) Show that $\frac{dx}{d\phi} = R\cos(\phi)$.

(c) Show that $\frac{dy}{d\phi} = R\sin(\phi)$.

(d) With the aid of (b) and (c), find an equation for the curvature involving x and y.

HINT: For (b) and (c) draw the little triangle whose hypotenuse is like a short piece of arc length ds on the curve and whose legs are parallel to the axes. For (d), think about antiderivatives. NOTE: Physicists show why the radius of curvature is constant, leaving it to the mathematicians to show that therefore the path is a circle.

33.[C] At the top of the cycloid in Example 2 the radius of curvature is twice the diameter of the rolling circle. What would you have guessed the radius of curvature to be at this point? Why is it not simply the diameter of the wheel, since the wheel at each moment is rotating about its point of contact with the ground?

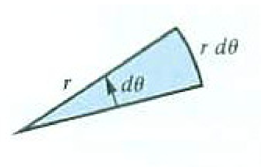
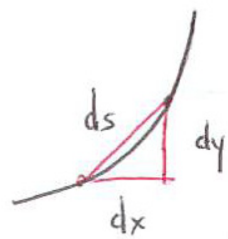
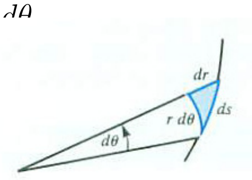
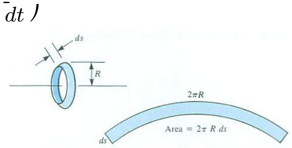
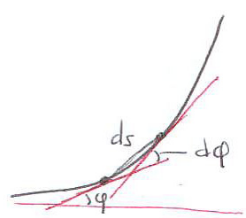
34.[C] A smooth convex curve has length L.

(a) Show that the average of its curvature, as a function of arc length, is $2\pi/L$.

(b) Check that the formula in (a) is correct for a circle of radius a.

9.S Chapter Summary

This chapter deals mostly with curves described in polar coordinates and curves given parametrically. The following table is a list of reminders for most of the ideas in the chapter.

Concept	*Memory Aid*	*Comment*
Area $= \int_\alpha^\beta \frac{r^2}{2}\,d\theta$		The narrow sector resembl gle of base $r\ d\theta$ and height $\frac{1}{2}(r\ d\theta)(r) = \frac{1}{2}r^2\ d\theta$.
Arc length $= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$ Arc length $= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$		A short part of the curve straight, suggesting $(ds)^2$ $(dy)^2$.
Arc length $= \int_\alpha^\beta \sqrt{r^2 + (r')^2}\ d\theta$ $= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right.}$ Speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right.}$ $= \sqrt{\left(r\frac{d\theta}{dt}\right)^2 + \left(\frac{}{dt}\right)}$		The shaded area with two c looks like a right triangle, $(ds)^2 = (rd\theta)^2 + (dr)^2$.
Area of surface of revolution $= \int_a^b 2\pi R\ ds$		
Curvature $= \kappa = \left\lvert\frac{d\phi}{ds}\right\rvert$		Using the chain rule to wr $\left\lvert\frac{(d\phi/dx)}{(ds/dx)}\right\rvert$ one gets the forn $\frac{\lvert y''\rvert}{(1+(y')^2)^{3/2}}$

If a curve is given parametrically, its curvature can be found by replacing $\frac{dy}{dx}$ by $\frac{dy/dt}{dx/dt}$, and, similarly, $\frac{d^2y}{dx^2} = \frac{\frac{d}{dx}}{\left(\frac{dy}{dx}\right)}$ by $\frac{\frac{d}{dy}\left(\frac{dy}{dx}\right)}{dx/dt}$.

Section 15.2 defines curvature of a curve in space, using vectors. It is consistent with the definition given here for curves that happen to lie in a plane.

EXERCISES for 9.S *Key:* R–routine, M–moderate, C–challenging

1.[R] When driving along a curvy road, which is more important in avoiding car sickness, $d\phi/ds$ or $d\phi/dt$, where t is time.

2.[R] Some definite integrals can be evaluated by interpretting them as the area of an appropriate region. Consider $\int_0^{\pi/2} \cos^2(\theta)\, d\theta$.

(a) Evaluate $\int_0^{\pi/2} \cos^2(\theta)\, d\theta$ by identifying it as the area of an appropriate region.

(b) Evaluate $\int_0^{\pi/2} \cos^2(\theta)\, d\theta$ with the use of a double angle formula.

(c) Repeat (a) and (b) for $\int_0^{\pi} \sin^2(\theta)\, d\theta$.

(d) Repeat (a) and (b) for $\int_{\pi}^{2\pi} \sin^2(\theta)\, d\theta$.

3.[R] The solution to Example 3 (Section 9.2) requires the evaluation of the definite integrals $\int_0^{\pi/2} \cos^2(\theta)\, d\theta$ and $\int_0^{\pi} (1+\cos(\theta))^2\, d\theta$. Evaluate these definite integrals making use of the ideas in Exercise 2 as much as possible.

4.[M] A physics midterm includes the following information: For $r = \sqrt{x^2+y^2}$ and y constant,

$$\text{(a)} \int \frac{dx}{r} = \ln(x+r), \qquad \text{(b)} \int \frac{x\ dx}{r} = r, \qquad \text{(c)} \int \frac{dx}{r^3} = \frac{x}{y^2 r}.$$

Show by differentiating that these equations are correct.

5.[M] (Contributed by Jeff Lichtman.) Let f and g be two continuous functions such that $f(x) \geq g(x) \geq 0$ for x in $[0,1]$. Let R be the region under $y = f(x)$ and above $[0,1]$; let R^* be the region under $y = g(x)$ and above $[0,1]$.

(a) Do you think the center of mass of R is at least as high as the center of mass of R^*? (Give your opinion, without any supporting calculations.)

(b) Let $g(x) = x$. Define $f(x)$ to be $\frac{1}{3}$ for $0 \leq x \leq \frac{1}{3}$ and let $f(x)$ be x if $\frac{1}{3} \leq x \leq 1$. (Note that f is continuous.) Find $\bar{y}$ for R and also for R^*. (Which is larger?)

(c) Let a be a constant, $0 \leq a \leq 1$. Let $f(x) = a$ for $0 \leq x \leq a$, and let $f(x) = x$ for $a \leq x \leq 1$. Find $\bar{y}$ for R.

(d) Show that the number a for which $\bar{y}$ defined in (c) is a minimum is a root of the equation $x^3 + 3x - 1 = 0$.

(e) Show that the equation in (d) has only one real root q.

(f) Find q to four decimal places.

(g) Show that $\bar{y} = q$

Exercises 6 and 7 require an integral version of the Cauchy-Schwarz inequality (see Exercise 29):

$$\int_0^{2\pi} f(\theta)g(\theta)\ d\theta \leq \left(\int_0^{2\pi} f(\theta)^2\ d\theta\right)^{1/2} \left(\int_0^{2\pi} g(\theta)^2\ d\theta\right)^{1/2}.$$

6.[C] Let P be a point inside a region in the plane bounded by a smooth convex curve. ("Smooth" means it has a continuously defined tangent line.) Place the pole of a polar coordinate system at P. Let $d(\theta)$ be the length of the chord of angle θ through P. Show that $\int_0^{2\pi} d(\theta)^2\ d\theta \leq 8A$, where A is the area of the region.

7.[C] Show that if $\int_0^{2\pi} d(\theta)^2\ d\theta = 8A$ then P is the midpoint of each chord through P.

8.[C] Let $r = f(\theta)$ describe a convex curve surrounding the origin.

(a) Show that $\int_0^{2\pi} r\ d\theta \leq$ arc length of the boundary.

(b) Show that if equality holds in (a), the curve is a circle.

9.[C] Let $r(\theta)$, $0 \leq \theta \leq 2\pi$, describe a closed convex curve of length L.

(a) Show that the average value of $r(\theta)$, as a function of θ, is at most $L/(2\pi)$.

(b) Show that the if average is $L/(2\pi)$, then the curve is a circle.

10.[C]

Sam: I've discovered an easy formula for the length of a closed curve that encloses the origin.

Jane: Well?

Sam: First of all, $\int_0^{2\pi} \sqrt{r^2 + (r')^2}\ d\theta$ is obviously greater than or equal to $\int_0^{2\pi} \mathbf{r}\ d\theta$.

Jane: I'll give you this much, because $(r')^2$ is never negative.

Sam: Now, if a and b are not negative, $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.

Jane: Why?

Sam: Just square both sides. So $\sqrt{r^2+(r')^2} \leq \sqrt{r^2} + \sqrt{(r')^2} = r + r'$.

Jane: Looks all right.

Sam: Thus

$$\int_0^{2\pi} \sqrt{r^2+(r')^2}\ d\theta \leq \int_0^{2\pi} (r+r')\ d\theta = \int_0^{2\pi} r\ d\theta + \int_0^{2\pi} r'\ d\theta.$$

But $\int_0^{2\pi} r'\ d\theta$ equals $r(2\pi) - r(0)$, which is 0. So, putting all this together, I get

$$\int_0^{2\pi} r\ d\theta \leq \int_0^{2\pi} \sqrt{r^2+(r')^2}\ d\theta \leq \int_0^{2\pi} r\ d\theta.$$

So the arc length is simply $\int_0^{2\pi} r\ d\theta$.

Jane: That couldn't be right. If it were, it would be an Exercise.

Sam: They like to keep a few things secret to surprise us on a mid-term.

Who is right, Sam or Jane? Explain.

Skill Drill: Derivatives

In Exercises 11 and 12 a, b, c, m, and p are constants. In each case verify that the derivative of the first function is the second function.

11.[R] $\frac{1}{\sqrt{c}} \arcsin\left(\frac{cx-b}{\sqrt{b^2+ac}}\right)$; $\sqrt{\frac{c}{a+2bx-cx^2}}$.

12.[R] $\frac{1}{c}\sqrt{a+2bx+cx^2} - \frac{b}{\sqrt{c}} \ln\left(b + cx + \sqrt{c}\sqrt{a+2bx+cx^2}\right)$; $\frac{x}{a+bx+cx^2}$ (assume c is positive).

In Exercises 13 and 14, L is the length of a smooth curve C and P is a point within the region A bounded by C.

13.[M]

(a) Show that the average distance from P to points on the curve, averaged with respect to arc length is greater than or equal to $2A/L$.

(b) Give an example when equalify holds.

14.[M]

(a) Show that the average distance from P to points on the curve, averaged with respect to the polar angle is greater than or equal to $L/(2\pi)$.

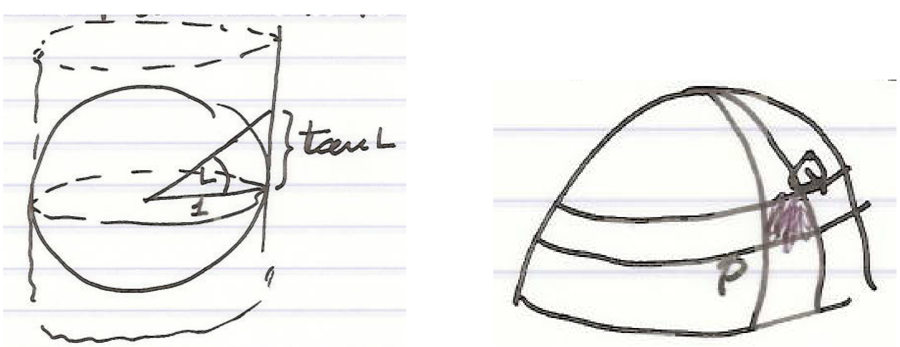
(b) Give an example when equalify holds.

(See also Exercise 24 in Section 9.4.)

Calculus is Everywhere # 12
The Mercator Map

One way to make a map of a sphere is to wrap a paper cylinder around the sphere and project points on the sphere onto the cylinder by rays from the center of the sphere. This **central cylindrical projection** is illustrated in Figure C.12.1(a).

A web search for "map projection" leads to detailed information about this and other projections. The US Geological Society has some particularly good resources.



(a) (b)

Figure C.12.1:

Points at latitude L project onto points at height $\tan(L)$ from the equatorial plane.

A **meridean** is a great circle passing through the north and south poles. It corresponds to a fixed longitude. A short segment on a meridian at latitude L of length dL projects onto the cylinder in a segment of length approximately $d(\tan(L)) = \sec(L)^2\, dL$. This tells us that the map magnifies short vertical segments at latitude L by the factor $\sec^2(L)$.

Points on the sphere at latitude L form a circle of radius $\cos(L)$. The image of this circle on the cylinder is a circle of radius 1. That means the projection magnifies horizontal distances at latitude L by the factor $\sec(L)$.

Consider the effect on a small "almost rectangular" patch bordered by two meridians and two latitude lines. The patch is shaded in Figure C.12.1(b). The vertical edges are magnified by $\sec^2(L)$, but the horizontal edges by only $\sec(L)$. The image on the cylinder will not resemble the patch, for it is stretched more vertically than horizontally.

The path a ship sailing from P to Q makes a certain angle with the latitude line through P. The map just described distorts that angle.

The ship's caption would prefer a map without such a distortion, one that preserves direction. That way, to chart a voyage from point A to point B on the sphere of the Earth at a fixed compass heading, he would simply draw a straight line from A to B on the map to determine the compass setting.

Gerhardus Mercator, in 1569, designed a map that does not distort small patches hence preserves direction. He figured that since the horizontal magnification factor is $\sec(L)$, the vertical magnification should also be $\sec(L)$, not $\sec^2(L)$.

This condition can be stated in the language of calculus. Let y be the height on the map that represents latitude L_0. Then Δy should be approximately $\sec(L)\Delta L$. Taking the limit of $\Delta y/\Delta L$ and ΔL approaches 0, we see that $dy/dL = \sec(L)$. Thus

$$y = \int_0^{L_0} \sec(L)\, dL. \tag{C.12.1}$$

Mercator, working a century before the invention of calculus, did not have the concept of the integral or the Fundamental Theorem of Calculus. Instead, he had to break the interval $[0, L_0]$ into several short sections of length ΔL, compute $(\sec(L))\Delta L$ for each one, and sum these numbers to estimate y in (C.12.1).

We, coming after Newton and Leibniz, can write

$$y = \int_0^{L_0} \sec(L)\, dL = \ln|\sec(L) + \tan(L)|\,\Big|_0^{L_0} = \ln(\sec(L_0) + \tan(L_0)) \qquad \text{for } 0 \leq L_0 \leq \pi$$

In 1645, Henry Bond conjectured that, on the basis of numerical evidence, $\int_0^\alpha \sec(\theta)\, d\theta = \ln(\tan(\alpha/2 + \pi/4))$ but offered no proof. In 1666, Nicolaus Mercator (no relation to Gerhardus) offered the royalties on one of his inventions to the mathematician who could prove Bond's conjecture was right. Within two years James Gregory provided the missing proof.

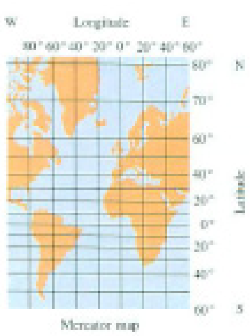


Figure 12 shows a Mercator map. Such a map, though it preserves angles, greatly distorts areas: Greenland looks bigger than South America even though it is only one eighth its size. The first map we described distorts areas even more than does a Mercator map.

EXERCISES

1.[R] Draw a clear diagram showing why segments at latitude L are magnified vertically by the factor $\sec(L)$.

2.[R] Explain why a short arc of length dL in Figure C.12.1(a) projects onto a short interval of length approximately $\sec^2(L)\, dL$.

3.[R] On a Mercator map, what is the ratio between the distance between the lines representing latitudes $60°$ and $50°$ to the distance between the lines representing latitudes $40°$ amd $30°$?

4.[M] What magnifying effect does a Mercator map have on areas?

Chapter 10

Sequences and Their Applications

When trying to write $1/3$ as a decimal, we meet the following sequence of numbers:

$$0.3, 0.33, 0.333, 0.3333, \ldots$$

The more 3s we write, the closer the numbers are to $1/3$.

When estimating a definite integral $\int_a^b f(x)\,dx$, we pick a positive integer n, divide the interval $[a,b]$ into n equal pieces of length $\Delta x = (b-a)/n$, pick a number c_i in the i^{th} interval and form the sum $E_n = \sum_{i=1}^n f(c_i)\Delta x$. In this way we obtain a sequence of estimates,

$$E_1, E_2, E_3, \ldots, E_n, \ldots.$$

As n increases the estimates approach $\int_a^b f(x)\,dx$, if $f(x)$ is continuous.

In the analysis of APY (annual percentage yield on an account at a bank), in CIE #3 in Chapter 2 (page 161) we encounter the sequence

$$(1+1/1)^1, (1+1/2)^2, (1+1/3)^3, \ldots, (1+1/n)^n, \ldots.$$

As n increases, these numbers approach e.

What happens to the numbers

$$S_1 = 1, S_2 = 1+\frac{1}{2}, S_3 = 1+\frac{1}{2}+\frac{1}{3}, S_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, \ldots, S_n = \sum_{k=1}^n \frac{1}{k}, \ldots$$

as we add more and more reciprocals of integers? Do the S_n get arbitrarily large or do they approach some finite number? When students, neither author guessed right.

Chapters 10, 11, and 12 concern the behavior of endless sequences of numbers. Such sequences arise in estimating a solution of an equation. They also provide a way to estimate such important functions as e^x, $\sin(x)$, and $\ln(x)$, and therefore a way to estimate such integrals as $\int_0^1 e^{x^2}\, dx$, for which the fundamental theorem of calculus is of no help. They also offer another way to evaluate indeterminate limits.

10.1 Introduction to Sequences

A **sequence** of numbers,

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

is a function that assigns to each positive integer n a number a_n. The number a_n is called the n^{th} **term** of the sequence. For example, the sequence

$$\left(1+\frac{1}{1}\right)^1, \left(1+\frac{1}{2}\right)^2, \left(1+\frac{1}{3}\right)^3, \ldots, \left(1+\frac{1}{n}\right)^n, \ldots$$

was first seen in Section 2.2 and was later shown to be related to the number e. In this case, the n^{th} term of the sequence is

$$a_n = \left(1+\frac{1}{n}\right)^n.$$

n	a_n
1	2.0000
2	2.2500
3	2.3704
4	2.4414
5	2.4883
10	2.5937
100	2.7048
1000	2.7169
10000	2.7181

For example, $a_1 = (1+1)^1 = 2$, $a_2 = \left(1+\frac{1}{2}\right)^2 = \frac{9}{4} = 2.25$, $a_{10} = \left(1+\frac{1}{10}\right)^{10} \approx 2.5937$, and $a_{100} = \left(1+\frac{1}{100}\right)^{10} \approx 2.7048$.

The notation $\{a_n\}$ is an abbreviation for the sequence $a_1, a_2, \ldots, a_n, \ldots$. Read a_1 as "a sub 1" and a_n as "a sub n."

The "sub" stands for "subscript."

If, as n gets larger, a_n approaches a number L, then L is called the **limit** of the sequence $\{a_n\}$. When the sequence $a_1, a_2, \ldots, a_n, \ldots$ has a limit L we say it is convergent and write

$$\lim_{n\to\infty} a_n = L.$$

For instance, we write

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

A sequence need not begin with the term a_1. Later, sequences of the form $a_0, a_1, a_2, \ldots$ will be considered. In such a case, a_0 is called the **zeroth term**. In other instances we consider sequences $a_k, a_{k+1}, a_{k+2}, \ldots$ that begin with a_k for $k > 1$. These sequences are called a "tail" of the sequence $a_1, a_2, a_3, \ldots$. Two important features of any sequence are i) the terms of a sequence are defined only for integers and ii) the sequence never ends.

The Sequence $\{r^n\}$

The next example introduces a simple but important type of sequence called a **geometric sequence**.

EXAMPLE 1 A certain (small) piece of equipment depreciates in value over the years. In fact, at the end of any year it has only 80% of the value it had at the beginning of the year. What happens to its value in the long run if its value when new is \$1?

SOLUTION Let a_n be the value of the equipment at the end of the n^{th} year. Call the initial value $a_0 = 1$. At the end of year 1 the value is $a_1 = 0.8(1)$. Similarly, $a_2 = 0.8(0.8) = 0.8^2 = 0.64$ and $a_3 = 0.8(0.8^2) = 0.8^3$. After n years the value is $a_n = 0.8^n$. This question is asking about the limit of the sequence $\{0.8^n\}$. After 5 years, the value is just under \$0.33. In another five years the value is reduced to about \$0.11, and at the end of year 20, the value is roughly \$0.01. This is strong evidence that

n	0.8^n
0	$0.8^0 = 1$
1	$0.8^1 = 0.8$
2	$0.8^2 = 0.64$
3	$0.8^3 = 0.512$
4	$0.8^4 = 0.4096$
5	$0.8^5 = 0.3277$
10	$0.8^{10} = 0.1074$
20	$0.8^{20} = 0.0115$

$$\lim_{n\to\infty} 0.8^n = 0.$$

◇

Even if the piece of equipment in Example 1 retained 99% of its value each year, in the long run it would still be worth less than a dime, then less than a penny, etc. The data in Table 10.1.1 indicates that 0.99^n approaches 0 as $n \to \infty$, but much more slowly than 0.8^n does.

n	0	1	2	3	4	5	10	20	100	200
0.99^n	1	0.99	0.9801	0.9703	0.9606	0.9510	0.9044	0.8179	0.3660	0.1340

Table 10.1.1:

On the basis of Example 1, it is plausible that if $0 \le r < 1$, then $\lim_{n\to\infty} r^n = 0$. Moreover, the closer r is to 1, the slower r^n approaches 0. To show that this is the case, we introduce a property of the real numbers which we will use often. It concerns **monotone sequences**. A sequence is **monotone** either if it is nondecreasing ($a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \dots$) or nonincreasing ($a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \dots$).

Every bounded and monotone sequence converges.

Theorem 10.1.1. *Let $\{a_n\}$ be a nondecreasing sequence with the property that there is a number B such that $a_n \le B$ for all n. That is, $a_1 \le a_2 \le a_3 \le a_4 \le \cdots \le a_n \le a_{n+1} \le \dots$ and $a_n \le B$ for all n. Then the sequence $\{a_n\}$ is convergent and a_n approaches a number L less than or equal to B.*

Similarly, if $\{a_n\}$ is a nonincreasing sequence and there is a number B such that $a_n \ge B$ for all n, then the sequence $\{a_n\}$ is convergent and its limit is greater than or equal to B.

Figure 10.1.1 suggests the first part of Theorem 10.1.1 is plausible. The monotonicity prevents the terms from backtracking or entering a cycle. Without the bound on the terms, the sequence could continue to approach ∞. Any sequence that is both bounded and monotone has to converge to a limit.

Theorem 10.1.1 is proved in advanced calculus.

The next theorem shows the power of Theorem 10.1.1.

Figure 10.1.1:

Theorem 10.1.2. *If* $0 < r < 1$ *then* $\{r^n\}$ *converges to* 0*.*

Proof

Let r be a number between 0 and 1. The sequence r^1, r^2, r^3, $\ldots r^n$, $\ldots$ is decreasing and each term is greater than 0. By Theorem 10.1.1, the sequence has a limit, L, and $L \geq 0$.

The sequence r^2, r^3, $\ldots$, r^{n+1}, $\ldots$ also approaches L. We then have

$$L = \lim_{n\to\infty} r^{n+1} = \lim_{n\to\infty} rr^n = r \lim_{n\to\infty} r^n = rL.$$

In short,

$$L = rL.$$

Thus $(1-r)L = 0$. So either $1 - r = 0$ or $L = 0$. Because $0 < r < 1$, $1 - r$ is not zero, L has to be 0, which shows that $\lim_{n\to\infty} r^n = 0$. •

The behavior of $\{r^n\}$ for other values of r is much more easily obtained:

1. If $r = 1$, then $r^n = 1$ for all n. So $\lim_{n\to\infty} r^n = 1$.

2. If $r > 1$, then r^n gets arbitrarily large as $n \to \infty$. Hence is divergent.

3. If $r < -1$, then $|r|^n$ gets arbitrarily large. Thus $\lim_{n\to\infty} r^n$ does not converge.

4. If $r = -1$, then the sequence is -1, 1, -1, 1, $\ldots$. which is divergent.

5. If $-1 < r < 0$, then $\lim_{n\to\infty} r^n = 0$.

6. If $r = 0$, then $r^n = 0$ for all n. So $\lim_{n\to\infty} r^n = 0$.

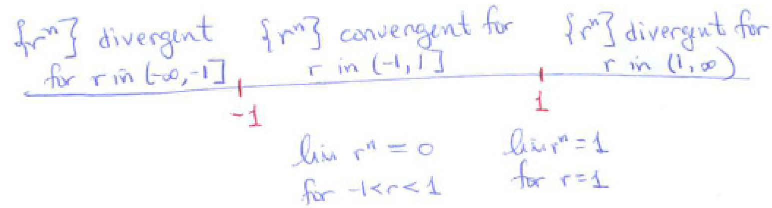


Figure 10.1.2:

Figure 10.1.2 records this information.

We prove (2) and (5). First, (2). If $r > 1$, the sequence r, r^2, r^3, r^4, $\dots$, r^n, $\dots$ is monotone increasing. The terms either approach a limit, L, or they get arbitrarily large. In the first case we would have, as before, $(1-r)L = 0$, which implies $L = 0$ (because $1 - r0$ is not zero). That's impossible since every term is greater than or equal to r.

To prove (5), let $-1 < r < 0$ and note that $|r^n| = |r|^n$ approaches zero as $n \to \infty$ (by Theorem 10.1.2). Since the absolute value of r^n approaches 0, so must r^n.

The terms of a convergent sequence usually never equal their limit, L, but merely get closer to it as the index, n, increases.

Informal definition of $\lim_{n\to\infty} a_n = \infty$.

If a_n becomes and remains arbitrarily large and positive as n gets larger, the sequence diverges and we write $\lim_{n\to\infty} a_n = \infty$. For instance, $\lim_{n\to\infty} 2^n = \infty$. Similarly, $\lim_{n\to\infty}(-2^n) = -\infty$. But, for $\lim_{n\to\infty}(-2)^n$ all we can say is that the sequence diverges because the values alternate between positive and negative values and $\lim_{n\to\infty} |2^n| = \lim_{n\to\infty} 2^n = \infty$.

The Sequence $\{k^n/n!\}$

Example 2 introduces a type of sequence that occurs in the study of $\sin(x)$, $\cos(x)$, and e^x.

EXAMPLE 2 Does the sequence defined by $a_n = 3^n/n!$ converge or diverge?

SOLUTION The first terms of this sequence are recorded (to four decimal places) in Table 10.1.2. Although a_2 is larger than a_1 and a_3 is equal to a_2, from a_4 through a_8, as Table 10.1.2 shows, the terms decrease.

The numerator 3^n becomes large as $n \to \infty$, influencing a_n to grow large. But the denominator $n!$ also becomes large as $n \to \infty$, influencing the quotient a_n to shrink toward 0. For $n = 1$ and $n = 2$ the first influence dominates, but then, as the table shows, the denominator $n!$ grows faster than the numerator 3^n, forcing a_n toward 0.

n	1	2	3	4	5	6	7	8
3^n	3	9	27	81	243	729	$2,187$	$6,561$
$n!$	1	2	6	24	120	720	$5,040$	$40,320$
$a_n = 3^n/n!$	3.0000	4.5000	4.500	3.3750	2.0250	1.0125	0.4339	0.1627

Table 10.1.2:

To see why the denominator grows so fast that the quotient $3^n/n!$ approaches 0, consider a_{10}. This term can be expressed as the product of 10 fractions:

$$a_{10} = \frac{3^{10}}{10!} = \frac{3}{1}\frac{3}{2}\frac{3}{3}\frac{3}{4}\frac{3}{5}\frac{3}{6}\frac{3}{7}\frac{3}{8}\frac{3}{9}\frac{3}{10}.$$

The first three fractions are greater than or equal to 1, but the seven remaining fractions are all less than or equal to $\frac{3}{4}$. Thus

$$a_{10} < \frac{3}{1}\frac{3}{2}\frac{3}{3}\left(\frac{3}{4}\right)^7.$$

Similarly,

$$a_{100} < \frac{3}{1}\frac{3}{2}\frac{3}{3}\left(\frac{3}{4}\right)^{97}.$$

More generally, for $n > 4$,

$$a_n < \frac{3}{1}\frac{3}{2}\frac{3}{3}\left(\frac{3}{4}\right)^{n-3}.$$

By Theorem 10.1.2,

$$\lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0,$$

from which it follows that $\lim_{n\to\infty} a_n = 0$. ◇

Reasoning like that in Example 2 shows that for any fixed number k,

This limit will be used often.

$$\boxed{\lim_{n\to\infty}\frac{k^n}{n!} = 0.}$$

This means that the factorial grows faster than any exponential k^n.

Properties of Limits of Sequences

The limits of sequences $\{a_n\}$ behave like the limits of functions $f(x)$, as discussed in Section 2.4. The most important properties are summarized in Theorem 10.1.3 without proof.

Remember that A and B are numbers (not $\pm\infty$).

Theorem 10.1.3. *If* $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$*, then*

- $\lim_{n\to\infty}(a_n + b_n) = A + B$.
- $\lim_{n\to\infty}(a_n - b_n) = A - B$.
- $\lim_{n\to\infty}(a_n b_n) = AB$.
- $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{A}{B}$ *(*$B \neq 0$*)*.
- *If* k *is a constant,* $\lim_{n\to\infty} ka_n = kA$*. In particular,* $\lim_{n\to\infty}(-a_n) = -\lim_{n\to\infty} a_n$.
- *If* f *is continuous on an open interval that contains* A*, then* $\lim_{n\to\infty} f(a_n) = f(A)$.

For instance,

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{3}{n} + \left(\frac{1}{2}\right)^n\right) &= 3\lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\left(\frac{1}{2}\right)^2 \\ &= 3\cdot 0 + 0 \\ &= 0.\end{aligned}$$

Techniques for dealing with $\lim_{x\to\infty} f(x)$ can often help to determine the limit of a sequence. The essential point is

$$\text{if } \lim_{x\to\infty} f(x) = L \quad \text{then} \quad \lim_{n\to\infty} f(n) = L.$$

EXAMPLE 3 Find $\lim_{n\to\infty}\frac{n}{2^n}$.
SOLUTION Consider the function $f(x) = \frac{x}{2^x}$. By l'Hôpital's Rule (∞-over-∞ case),

$$\lim_{x\to\infty}\frac{x}{2^x} = \lim_{x\to\infty}\frac{1}{2^x\ln(2)} = 0.$$

Thus

$$\lim_{n\to\infty}\frac{n}{2^n} = 0.$$

◇

WARNING *(On Limits of Sequences and Limits of Functions)*
The converse of the statement "if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$" is not true. For example, take $f(x) = \sin(\pi x)$. Then $\lim_{n\to\infty} f(n) = 0$, but $\lim_{x\to\infty} f(x)$ does not exist.

The Precise Definition of $\lim_{n\to\infty} a_n = L$

In Sections 3.8 and 3.9 limit concepts were given precise (as opposed to informal) definitions. The following definition is in the same spirit.

Precise definition of limit of a sequence.

DEFINITION *(Limit of a sequence.)* The number L is the **limit of the sequence** $\{a_n\}$ if for each $\epsilon > 0$ there is an integer N such that

$$|a_n - L| < \epsilon \quad \text{for all integers } n > N.$$

EXAMPLE 4 Use the precise definition of the limit of a sequence to show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

SOLUTION Given $\epsilon > 0$ we want to show that there is an integer N such that

$$\left|\frac{1}{n} - 0\right| < \epsilon \quad \text{for all integers } n > N.$$

For instance, if $\epsilon = 0.01$, we want

$$\begin{aligned} \left|\frac{1}{n} - 0\right| &< 0.01 \\ \text{or simply} \qquad \frac{1}{n} &< 0.01 = \frac{1}{100}. \end{aligned}$$

This inequality holds for $n > 100$. Hence $N = 100$ suffices. (So does any integer greater than 100.)

The general case is similar. We wish to have

$$\begin{aligned} \left|\frac{1}{n} - 0\right| &< \epsilon \\ \text{or} \qquad \frac{1}{n} &< \epsilon \\ \text{Hence,} \qquad 1 &< n\epsilon \\ \text{and finally} \qquad n &> \frac{1}{\epsilon}. \end{aligned}$$

Any integer $N > 1/\epsilon$ suffices. ◇

k^n and Energy from the Atom

In a particular nuclear chain reaction, when a neutron strikes the nucleus of an atom of uranium or plutonium, on the average a certain number of neutrons split off. Call this number k. These k neutrons then strike further atoms. Since each splits off k neutrons, in this second generation there are k^2 neutrons. In the third generation there are k^3 neutrons, and so on. Each generation is born in a fraction of a second and produces energy.

If k is less than 1, then the chain reaction dies out, since $k^n \to 0$ as $n \to \infty$. A successful chain reaction — whether in a nuclear reactor or an atomic bomb — requires that k be greater than 1, since $k^n \to \infty$ as $n \to \infty$.

In September 1941, Enrico Fermi and Leo Szilard achieved $k = 0.87$ with a uranium pile at Columbia University. In 1942, they obtained an encouraging $k = 0.918$. Iin the meantime, Samuel Allison at the Univeristy of Chicago, Fermi and Szilard attained $k = 1.0006$. With this k the neutron intensity doubled every 2 minutes. They had achieved the first controlled, sustained, chain reaction, producing energy from the atom. Fermi let the pile run for 4.5 minutes. Had he let it go on much longer, the atomic pile, the squash court, the university, and part of Chicago might have disappeared.

Eugene Wigner, one of the scientists present, wrote, "We felt as, I presume, everyone feels who has done something that he knows will have very far-reaching consequences which he cannot foresee."

Szilard had a different reaction: "There was a crowd there and then Fermi and I stayed there alone. I shook hands with Fermi and I said I thought this day would go down as a black day in the history of mankind."

However it may be regarded, December 2, 1942, is a historic date. Before that date k was less than 1, and $\lim_{n\to\infty} k^n = 0$. After that date, k was larger than 1 and $\lim_{n\to\infty} k^n = 0$.

Based on Richard Rhodes, *The Making of the Atomic Bomb*, Simon and Schuster, New York, 1986.

Summary

We defined convergent sequences and their limits and divergent sequences, which have no limit. The sequences $\{r^n\}$ and $\{k^n/n!\}$ will be used often in Chapters 10, 11, and 12. We have

$$\lim_{n\to\infty} r^n = 0 \quad (|r| < 1) \qquad \lim_{n\to\infty} \frac{k^n}{n!} = 0 \quad (k \text{ any constant}).$$

Also, a bounded monotone sequence converges, even though we may not be able to find its limit exactly.

EXERCISES for Section 10.1 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 18 write out the first three terms of the given sequence and state whether it converges or diverges. If it converges, give its limit.

1.[R] $\{0.999^n\}$

2.[R] $\{1.001^n\}$

3.[R] $\{1^n\}$

4.[R] $\{(-0.8)^n\}$

5.[R] $\{(-1)^n\}$

6.[R] $\{(-1.1)^n\}$

7.[R] $\{n!\}$

8.[R] $\left\{\frac{10^n}{n!}\right\}$

9.[R] $\left\{\frac{3n+5}{5n-3}\right\}$

10.[R] $\left\{\frac{(-1)^n}{n}\right\}$

11.[R] $\left\{\frac{\cos(n)}{n}\right\}$

12.[R] $\{n\sin(1/n)\}$ HINT: A limit in Section 2.2 will help.

13.[R] $\{n(a^{1/n}-1)\}$ HINT: A limit in Section 2.2 will help.

14.[R] $\left\{\frac{n}{2^n}+\frac{3n+1}{4n+2}\right\}$

15.[M] $\left\{\left(1+\frac{2}{n}\right)^n\right\}$

16.[M] $\left\{\left(\frac{n-1}{n}\right)^n\right\}$

17.[M] $\left\{\left(1+\frac{1}{n^2}\right)^n\right\}$ HINT: Write $f(n)^{g(n)}$ as $e^{g(n)\ln(f(n))}$.

18.[M] $\left\{\left(1+\frac{1}{n}\right)^{n^2}\right\}$

19.[R] Assume that each year inflation eats away 2 percent of the value of a dollar. Let a_n be the value of one dollar after n years.

(a) Find a_4.

(b) Find $\lim_{n\to\infty} a_n$.

20.[R] Let $a_n = 6^n/n!$.

(a) Fill in this table:

n	1	2	3	4	5	6	7	8
a_n								

(b) Plot the points (n, a_n) corresponding to each column in the table above. NOTE: Let the n-axis be the horizontal axis.

(c) What is the largest value of a_n? What is the corresponding n?

(d) What is $\lim_{n\to\infty} a_n$?

21.[R] What is the largest value of $(11.8)^n/n!$? Explain.

22.[M] Find an index n such that 0.999^n is less than 0.0001

(a) by experimenting with the aid of your calculator

(b) by solving the equation $0.999^x = 0.0001$

23.[M] Find the first index n such that 1.0006^n is larger than 2

(a) by experimenting with the aid of your calculator

(b) by solving the equation $1.0006^x = 2$.

In Exercises 24 and 25 determine the limits of the given sequences by first identifying each limit as a definite integral, $\int_a^b f(x)\,dx$, for a suitable interval $[a, b]$ and function $f(x)$. HINT: Review Section 6.2

24.[M]

$$a_n = \sum_{k=1}^{n} \left(\frac{1}{n}\right)^2 \frac{1}{n}$$

25.[M]

$$a_n = \sum_{k=1}^{n} \frac{n}{n^2 + i^2}$$

26.[M] For each integer $n \geq 1$, let

$$a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} = \sum_{k=n}^{2n} \frac{1}{k}.$$

For example, $a_3 = 1/3 + 1/4 + 1/5 + 1/6 = 0.95$.

(a) Compute decimal approximations to a_n for $n = 1, 2, 3, 4,$ and 5.

(b) Show that $\{a_n\}$ is a monotone and bounded sequence.

(c) Show that it has a limit and that the limit is at least 1/2.

27.[C] We showed that for $-1 < r < 0$, $\lim_{n\to\infty} r^n = 0$ by considering $|r^n|$. Here is a more direct argument.

(a) Let $r = -s$, $0 < s < 1$. Show that for even n, $r^n = s^n$ and for odd n, $r^n = -(s^n)$.

(b) Show that the sequence $\{r^{2n}\}$ converges to 0.

(c) Show that the sequence $\{r^{2n-1}\}$ converges to 0.

(d) Conclude that $\lim_{n\to\infty} f^n = 0$.

28.[C] The binomial theorem asserts that if n is a positive integer, then $(1 + x)^n$ is equal to $1 + nx$ plus other terms that are positive if $x > 0$. Use this to show that if $r > 1$, then $\lim_{n\to\infty} r^n = \infty$.

29.[C] Exercise 28 makes use of the binomial theorem. It was not necessary to use the binomial theorem, as this exercise shows. Assume that $x > 0$.

(a) Show that $(1 + x)^n \geq 1 + nx$ for $n = 1$.

(b) Assume that you know that $(1 + x)^n \geq 1 + nx$ when n is 100. Show that it follows that $(1 + x)^n \geq 1 + nx$ when n is 101.

(c) Explain why $(1 + x)^n \geq 1 + nx$ for all positive integers n.

30.[C] The sequence $\{a_n\}$ with $a_n = \sum_{k=n}^{2n} \frac{1}{k}$ was shown to be convergent in Exercise 26. Show that the limit of this sequences is $\ln(2)$ by expressing it as a certain definite integral and evaluating that integral.

31.[C] Let $a_n = \sum_{k=2n}^{3n} \frac{1}{k}$. Does $\{a_n\}$ converge or diverge? If it converges, find its limit.

32.[C] Using the precise definition of $\lim_{a_n} = L$, prove that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

33.[C] Use the precise definition of $\lim_{n\to\infty} a_n = L$ to prove $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.

34.[C] Use the precise definition of $\lim_{n\to\infty} a_n = L$ to prove $\lim_{n\to\infty} \frac{3}{n^2} = 0$.

35.[C] Use the precise definition of $\lim_{n\to\infty} a_n = L$ to prove that the statement $\lim_{n\to\infty}(-1)^n = 0$ is false.

36.[C]

(a) What would be the precise definition of $\lim_{n\to\infty} a_n = \infty$?

(b) Use the precise definition in (a) and the precise definition of $\lim_{n\to\infty} a_n = L$ to show that:
$$\text{if } \lim_{n\to\infty} a_n = \infty, \text{ then } \lim_{n\to\infty} 1/a_n = 0.$$

SHERMAN: Move this, and others?, to Chapter Summary?

37.[C]

Sam: I'm going to prove, using the precise definition, that if $0 < r < 1$, then $\lim_{n\to\infty} r^n = 0$.

Jane: I'll listen.

Sam: I want to show that there is an integer N such that $|r^n - 0| < \epsilon$ if $n > N$, in other words, $r^n < \epsilon$, if n is big enough. To get hold of n, I take logarithms, obtaining $n\ln(r) < \ln(\epsilon)$. Then I'll divide by $\ln(r)$.

Jane: How do you know r has a log?

Sam: Well, $r = e^{\ln(r)}$.

Jane: You mean the equation $r = e^x$ has a solution?

Sam: Sure, that's what a log is all about.

Jane: Since r is less than 1, x would be negative. May I write it as $-p$ where p is positive?

Sam: If you want to, why not?

Jane: So you're saying that r can be written as $(1/e)^p$ for some positive number p. You're assuming that no matter how small r is, there is a positive number p so that $(1/e)^p$ will equal it. Right?

Sam: Right. But why all this fuss?

Jane: To say that $(1/e)^p$ gets as small as you please is just a special case of what you're trying to prove. You're wandering in circles.

Who's right, Jane or Sam? If Sam is right, finish his proof.

10.2 Recursively-Defined Sequences and Fixed Points

The terms in each sequence considered in Section 10.1 were given by an explicit formula$a_n = f(n)$. Often a sequence is not given explicitly. Instead, each term (after the first) may be expressed in terms of earlier terms. For instance, the sequence of powers $a_0 = r^0 = 1$, $a_1 = r^1 = r$, $a_2 = r^2$, ..., $a_n = r^n$, ... can be described this way:

> The first term, a_0, is 1.
> For $n \geq 1$, $a_n = ra_{n-1}$.

That is, each term after a_0 is r times the preceding term. We will describe a technique for finding the limit of such sequences, defined indirectly, if they are convergent.

Sequences Defined Recursively

A sequence given by a formula that describes the n^{th} term in terms of previous terms is said to be given **recursively**. If a_n depends only on its immediate predessor, we would have $a_n = f(a_{n-1})$, for some function f. If a_n depends on both a_{n-1} and a_{n-2}, then there would be a function f such that $a_n = f(a_{n-1}, a_{n-2})$.

EXAMPLE 1 Let $a_0 = 1$ and $a_n = na_{n-1}$ for $n \geq 1$. Give an explicit definition of $\{a_n\}$.
SOLUTION $a_1 = 1a_0 = 1$; $a_2 = 2a_1 = 2 \cdot 1$; $a_3 = 3a_2 = 3 \cdot 2 \cdot 1$; $a_4 = 4a_3 = 4 \cdot 3 \cdot 2 \cdot 1$. Evidently, a_n is $n!$, "n factorial," the product of all integers from 1 to n. ◇

EXAMPLE 2 Let $b_0 = 1$ and $b_1 = 1$ and $b_n = b_{n-1} + b_{n-2}$ for $n \geq 2$. Compute b_2, b_3, b_4, and b_5.
SOLUTION $b_2 = b_1 + b_0 = 1 + 1 = 2$; $b_3 = b_2 + b_1 = 2 + 1 = 3$; $b_4 = b_3 + b_2 = 3 + 2 = 5$; $b_5 = b_4 + b_3 = 5 + 3 = 8$. This sequence, which appears often in both pure and applied mathematics, is called the **Fibonacci sequence**.

The terms in the Fibonacci sequence are positive and become arbitrarily large as n gets larger. The Fibonacci sequence diverges (to ∞). ◇

The Fibonacci sequence appears in the following problem from Chapter XII of the *Liber abaci* of Leonard Fibonacci. This book appeared in 1202 (hand copied) and was revised in 1228.

> A man put a pair of rabbits in a place surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year if every month each pair produces a new pair which from the second month on can produce another pair?

For a discussion of the Fibonacci sequence and the Golden Ratio and the myths that surround it, see S. Stein, "Strength in Numbers," John Wiley and Sons, New York, 1996 (p. 39).

Finding the Limit of a Recursive Sequence

Assume that a sequence satisfies the relation $a_n = f(a_{n-1})$ and has a limit L. Since $a_n \to L$ as $n \to \infty$, we also have $a_{n-1} \to L$ and $n \to \infty$. Now assume also that f is continuous. Then we have, because f is continuous,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(a_{n-1}) = f\left(\lim_{n\to\infty} a_{n-1}\right).$$

Hence,

$$L = f(L) \qquad (10.2.1)$$

There could be other solutions.

According to (10.2.1), L is a solution to the equation $x = f(x)$. A number L such that $f(L) = L$ is called a **fixed point** of f.

EXAMPLE 3 Let $f(n) = rf(n-1)$ where $0 < r < 1$. Let $a_1 = 1$. Use (10.2.1) to find $\lim_{n\to\infty} a_n$.

This is the same argument as in Section 10.1, but obtained now directly from (10.2.1).

SOLUTION We recognize this recursion as giving the sequence $1, r, r^2, \ldots$. Since this is a monotonic sequence bounded below by 0, it has a limit L. Thus

$$L = f(L) = rL.$$

Since r is not 1, $L = 0$. ◇

Exercises 42 to 45 provide a proof that the sequence of ratios of the Fibonacci sequence converges.

EXAMPLE 4 Define c_n to be the ratio of successive terms in the Fibonacci sequence $\{b_n\}$: $c_n = \frac{b_n}{b_{n-1}}$ for all $n \geq 2$. Assuming this sequence converges, find its limit.

SOLUTION $c_2 = \frac{b_2}{b_1} = \frac{1}{1} = 1$. For $n \geq 3$ the definition of the Fibonacci sequence can be used to obtain a formula relating c_n to c_{n-1}:

$$c_n = \frac{b_n}{b_{n-1}} = \frac{b_{n-1} + b_{n-2}}{b_{n-1}} = 1 + \frac{b_{n-2}}{b_{n-1}} = 1 + \frac{1}{c_{n-1}}.$$

So

$$c_n = 1 + \frac{1}{c_{n-1}} \qquad \text{for all } n \geq 3. \tag{10.2.2}$$

Thus, $c_n = f(c_{n-1})$ where $f(x) = 1 + \frac{1}{x}$.

The table showing the first few terms of this sequence suggests that this sequence converges. Note that the sequence is neither increasing nor decreasing, so Theorem 10.1.1 does not apply.

n	c_n
2	1.000000
3	2.000000
4	1.500000
5	1.666667
6	1.600000
7	1.625000
8	1.615385
9	1.619048
10	1.617647
15	1.618037
25	1.618034

Assume that $\lim_{n\to\infty} c_n$ exists and call that limit L. Then, by (10.2.2),

$$\begin{aligned} \lim_{n\to\infty} c_n &= \lim_{n\to\infty}\left(1 + \frac{1}{c_{n-1}}\right) = 1 + \frac{1}{\lim_{n\to\infty} c_{n-1}} \\ L &= 1 + \frac{1}{L}. \\ L^2 - L - 1 &= 0. \end{aligned}$$

So,

e,

The two solutions to $L^2 - L - 1 = 0$ are

$$L = \frac{1}{2}\left(1 + \sqrt{5}\right) \qquad L = \frac{1}{2}\left(1 - \sqrt{5}\right).$$

Since every term in this sequence is positive, the limit cannot be negative. The only possible limit is

$$L = \frac{1}{2}\left(1 + \sqrt{5}\right) \approx 1.61803.$$

$\frac{1}{2}\left(1 + \sqrt{5}\right) \approx 1.618034$ is known as the **Golden Ratio**.

◇

A Famous Recursion

The recursion $p_{n+1} = kp_n$ describes a population growing at a rate proportional to the amount present. If the initial population is p_1, then $p_2 = kp_1$, $p_3 = k^2p_1$, $p_4 = k^3p_1$, ... For $k > 1$, the population would increase without bound. But a population cannot do that. Instead, let us assume it approaches a limiting population, which we will say is 1. As it approaches this size, the struggle to find food slows its growth. Taking this into consideration, we assume that $\{p_n\}$ satisfies the **logistic equation**:

$$p_{n+1} = kp_n(1 - p_n).$$

The behavior of this equation, consideredon its own is surprising. For instance, if k is near 3.5699456 the behavior of the sequence changes a great deal even when k is changed only a little.

EXAMPLE 5 Examine the sequence given by $p_{n+1} = kp_n(1 - p_n)$ for $0 \leq k \leq 1$.

SOLUTION For $p_0 = 0$ or 1, $p_n = 0$ for all $n \geq 1$. For $0 < p_0 < 1$, $p_1 = kp_0(1 - p_0)$ is at most $p_0(1 - p_0)$, which is less than p_0. Similarly, p_2 is less than p_1, and, in general we have $p_{n+1} < p_n$. The sequence $\{p_n\}$ decreases but stays above 0. Therefore it has a limit L, and $L \geq 0$. Taking limits on both sides of (10.2) shows that $L = kL(1 - L).Either$L=0or1=k(1-L)$, hence$L=0orL=1-1/k.But1-1/k$iseithernegative(if$0¡k¡1$)or$0(if $k = 1$). So $L = 0$. ◇

Summary

A recursive sequence is one whose n^{th} term is given in terms of previous terms. If a_n depends only on its immediate predecessor, then $a_n = f(a_{n-1})$. If a_1, a_2, $\ldots$, a_{n-1}, a_n, $\ldots$ converges to L, then $f(L) = L$. Thus L is a root of the equation $f(x) = x$. It is called a **fixed point** of F.

EXERCISES for Section 10.2 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 6 give an explicit formula for a_n as a function of n.

1.[R] $a_0 = 1$, $a_n = -a_{n-1}$ for $n \geq 1$

2.[R] $a_0 = 3$, $a_n = a_{n-1}/n$ for $n \geq 1$

3.[R] $a_0 = 2$, $a_n = 3 + a_{n-1}$ for $n \geq 1$

4.[R] $a_0 = 5$, $a_n = -a_{n-1}/2$ for $n \geq 1$

5.[R] $a_1 = 1$, $a_n = a_{n-1} + 1/n$ for $n \geq 2$

6.[R] $a_1 = 1$, $a_n = -a_{n-1} + (-1)^n/n$ for $n \geq 2$

In Exercises 7 to 12 describe a_n in terms of a_{n-1} and an initial term a_0.

7.[R] $a_n = 3^n$, $n = 0, 1, 2, \ldots$

8.[R] $a_n = 5/n!$, $n = 0, 1, 2, \ldots$

9.[R] $a_n = 3n!$, $n = 0, 1, 2, \ldots$

10.[R] $a_n = 2n + 5$, $n = 0, 1, 2, \ldots$

11.[R] $a_n = 1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2$, $n = 1, 2, 3, \ldots$

12.[R] $a_n = 1/2 + 1/4 + 1/8 + \cdots + 1/2^{n-1}$, $n = 0, 1, 2, \ldots$

13.[R] Define $\{b_n\}$ by $b_0 = 2$ and $b_1 = 1/b_{n-1}$ for $n \geq 1$.

(a) Find $b_1, b_2, \ldots, b_5$.

(b) Show that if $\{b_n\}$ converges, its limit is 1 or -1.

(c) Does $\{b_n\}$ converge?

(d) For which choices of b_0 does $\{b_n\}$ converge to 1?

(e) For which choices of b_0 does $\{b_n\}$ converge to -1?

(f) For which choices of b_0 does $\{b_n\}$ diverge?

14.[R] Consider the logistic recursion (10.2) with $k = 2$, that is $p_{n+1} = 2p_n(1-p_n)$.

(a) Choose p_0 between 0 and 1/2. Find enough p_n to be able to conjecture if the sequences converge.

(b) Repeat (a) for another value of p_0 between 0 and 1/2.

(c) Repeat (a) with p_0 between 1/2 and 1.

(d) Repeat (a) for another value of p_0 between 1/2 and 1.

(e) What happens to the sequence $\{p_n\}$ if p_0 is 0 or 1?

(f) What happens to the sequence $\{p_n\}$ if p_0 is 1/2?

(g) For which values of p_0 does $\{p_n\}$ converge? And, in those cases, to what limit?

15.[R] For which values of x does $\left\{\frac{x^n}{n!}\right\}$ converge?

16.[R] For which values of x does $\left\{\frac{x^n}{2^n}\right\}$ converge?

17.[R] For which values of x does $\left\{\frac{x^n}{n^2}\right\}$ converge?

18.[R] For which values of x does $\left\{\frac{x^n}{\sqrt{n}}\right\}$ converge?

19.[R] Let $a_{n+2} = a_n + 2a_{n+1}$ with $a_0 = 1 = a_1$ and $c_n = a_n/a_{n-1}$. Examine $\{c_n\}$ numerically, deciding whether it converges and, if so, what it's limit might be.

20.[R] Explore the sequence $\{a_n\}$ where $a_{n+1} = a_n - a_{n-1}$ for $n \geq 2$ if

(a) $a_0 = 3$ and $a_1 = 4$,

(b) $a_0 = 1$ and $a_1 = 0$,

(c) the general case, $a_0 = a$, $a_1 = b$.

21.[R] Consider the logistic recursion (10.2) with $0 < k \leq 4$. Show that if p_0 is in the interval $[0, 1]$, then p_n is also in $[0, 1]$ for all $n \geq 0$.

22.[R] Let $a_{n+2} = (a_n + 3a_{n+1})/4$, with $a_0 = 0$ and $a_1 = 1$.

(a) Compute enough terms of $\{a_n\}$ to guess the limit, L.

(b) When you take limits of both sides of the recursion equation, what equation do you get for L?

23.[M] Consider the recursion $a_{n+2} = (1 + a_{n+1})/a_n$.

(a) Starting with $a_1 = 1$ and $a_2 = 2$, compute a_3, a_4, a_5, a_6, a_7, and a_8.

(b) Repeat (a) with $a_1 = 3$ and $a_2 = 3$.

(c) Repeat (a) with a_1 and a_2 of your choice.

(d) Explain what is going on.

24.[M] Let k and p be positive numbers and define the sequence $\{f_n\}$ as follows: given f_1, define $f_{n+1} = k(f_n)^p$ for $n \geq 1$.

(a) Assuming this sequence converges, find its limit.

(b) Explain how to choose k so that the sequence converges to 2.

25.[M] Show that if $0 \leq k \leq 4$, $0 \leq p_0 \leq 1$, and $p_{n+1} = kp_n(1 - p_n)$, then $0 \leq p_n \leq 1$.

26.[M]

(a) Investigate the logistic sequence $\{p_n\}$ for $k = 2$.

(b) Make a conjecture based on (a).

(c) Let $p_n = \frac{1}{2} + q_n$. Show that $q_{n+1} = -2q_n^2$.

(d) Use (c) to discuss your conjecture in (b).

27.[M] A path that is $1'$ by n' is to be tiled with $1' \times 1'$ tiles and $1' \times 2'$ tiles. Let a_n be the number of ways this can be done.

(a) Obtain a recursive formula for a_n.

(b) Use your formula found in (a) to find a_{10}.

28.[M] Repeat Exercise 27 with $1' \times 1'$ and $1' \times 3'$ tiles.

29.[M] Repeat Exercise 27 with $1' \times 2'$ and $1' \times 3'$ tiles.

30.[M] Let $u(n)$ be the number of ways of tiling a 2 by n rectangle with 1 by 2 dominoes.

(a) Find $u(1)$, $u(2)$, and $u(3)$.

(b) Find a recursive definition of the function u.

Exercises 31 to 34 illustrate some of the characteristics that make the logistic recursion $p_{n+1} = kp_n(1 - p_n)$ so interesting. In each case, create two sequences corresponding to two values of k in the indicated range and with different values for the initial value, p_0.

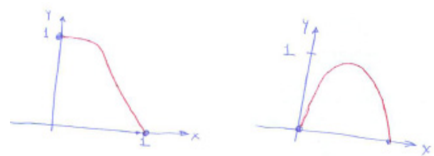
31.[M] $1 < k < 3$

32.[M] $3 < k < 3.4$

33.[M] $3.4 < k < 3.5$

34.[M] $3.6 < k < 4$

35.[M] Figure 10.2.1(a) shows the graph of a decreasing continuous function f such that $f(0) = 1$ and $f(1) = 0$.



(a) (b)

Figure 10.2.1:

(a) Show that f has exactly one fixed point in the interval $[0, 1]$. That is, show that there is one number a with $0 \leq a \leq 1$ that satisfies $f(a) = a$. Hint: Draw the line $y = x$ on the graph of $y = f(x)$.

(b) If $0 < x < a$, in what interval does $f(x)$ lie?

(c) If $a < x < 1$, in what interval does $f(x)$ lie?

(d) Use the graphs of $y = f(x)$ and $y = x$ to find all values of x for which $f(f(x)) > x$ and all values of x for which $f(f(x)) < x$.

36.[M] Let f be a decreasing function such that $f(0) = 1$ and $f(1) = 0$ and the graph of f is symmetric with respect to the line $y = x$. Examine the sequence x, $f(x)$, $f(f(x))$, ... for x in $[0, 1]$. What can you say about the convergence of this sequence?

37.[M] Let k, c_1, and c_2 be positive numbers. Define the sequence $\{c_n\}$ as follows: given c_1, c_2, define $c_n = (1 + kc_{n-1})/c_{n-2}$ for $n \geq 3$. Assuming this sequence converges, find the possible limits.

38.[C] Examine the sequence $\{x_n\}$ determined by $x_{n+1} = f(x_n)$ with $f(x) = 1-x^2$ for various inputs in $[0,1]$. Does f have a fixed point?

39.[C] Let $f(x) = 1 - x$, $g(x) = 1 - 1.1x$, and $h(x) = 1 - 0.9x$. Let $x_0 = 0.4$. Examine what happens to the sequences determined by each function.

40.[C] Assume that f is a decreasing function for x in $[0,1]$, $f(1) = 0$, and $-1 < f'(x) < 0$.

(a) What can be said about $f(0)$?

(b) Show that f has a unique fixed point.

(c) Assume $f(a) = a$. Show that if $1 \geq x > a$, then $f(x) < a$ and if $0 \leq x < a$, then $f(x) > a$.

(d) Let $g(x) = f(f(x))$. Examine the sequence x, $g(x)$, $g(g(x))$, $\ldots$ for x in $[0,1]$.

(e) Show that for all x in $[0,1]$ the sequence x, $f(x)$, $f(f(x))$, $\ldots$, approaches a.

41.[C] Figure 10.2.1(b) is the graph of a function for which $f(0) = 0$, $f(1) = 0$, $f''(x) \leq 0$, and $0 \leq f(x) \leq 1$.

(a) Show that f has at least one fixed point.

(b) Show that if $f'(0) \geq 1$, then f has only one fixed point.

(c) Show that if $f'(0) < 1$, it has exactly two fixed points.

Exercises 42 to 45 show all of the steps in the proof that the sequence introduced in Example 4 converges. Recall that $c_2 = 1$ and $c_n = 1 + \frac{1}{c_{n-1}}$ for all $n \geq 3$.

42.[C] Let $\{d_n\}$ be the sequence formed from the terms of $\{c_n\}$ with an odd index. That is, $d_n = c_{2n-1}$ for all $n \geq 2$.

(a) Show that $d_n \leq 2$ for all $n \geq 2$.

(b) Show that $\{d_n\}$ is a decreasing sequence.

(c) Explain why you know $\{d_n\}$ converges.

(d) What is $\lim_{n\to\infty} d_n$?

43.[C] Let $\{e_n\}$ be the sequence formed from the terms of $\{c_n\}$ with an even index. That is, $e_n = c_{2n}$ for all $n \geq 1$.

(a) Show that $e_n \geq 1$ for all $n \geq 1$.

(b) Show that $\{e_n\}$ is a increasing sequence.

(c) Explain why you know $\{e_n\}$ converges.

(d) What is $\lim_{n\to\infty} e_n$?

44.[C] Let $\{x_n\}$ be a sequence with the property that the (sub)sequence of odd terms converges to L, $\lim_{n\to\infty} x_{2n-1} = L$, and the (sub)sequence of even terms converges to M, $\lim_{x\to\infty} x_{2n} = M$. Show:

(a) if $L \neq M$ then $\{x_n\}$ diverges

(b) if $L = M$ then $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = L$.

45.[C] Use Exercises 42 to 44 to prove that $\{c_n\}$ converges. Hence its limit is the Golden Ratio.

46.[C] Let k be a number and define the sequence $\{d_n\}$ as follows: given d_0, define $d_n = kd_{n-1}^2$ for $n \geq 1$.

(a) Assuming the sequence converges, find its limit.

(b) Explain how to choose k so that this sequence converges to $3/2$.

10.3 Bisection Method for Solving $f(x) = 0$

One way to estimate the solution of an equation $f(x) = 0$, called a **root**, is to zoom in on it with a graphing calculator. However, precision is limited by the resolution of the display. This section and the next describe techniques for estimating a root to as many decimal places as you may need. The technique in this section is based on the fact that a continuous function that is positive at one input and negative at another input has a root between the two inputs.

Bisection Method for Solving $f(x) = 0$

A **root** of f is a solution to $f(x) = 0$.

Let $f(x)$ be a function. A *solution* or *root* of the equation $f(x) = 0$ is a number r such that $f(r) = 0$. The graph of $y = f(x)$ passes through the point $(r, 0)$, as shown in Figure 10.3.1.

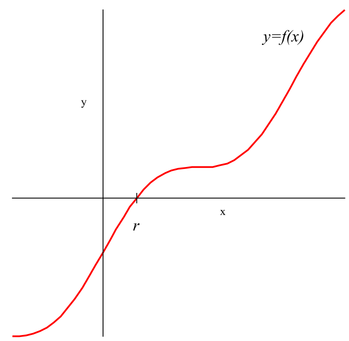


Figure 10.3.1:

Let $f(x)$ be a continuous function defined at least on an interval $[a_0, b_0]$, with $a_0 < b_0$. Assume that $f(a_0)$ and $f(b_0)$ have opposite signs, one negative, the other positive. By the Intermediate Value Theorem, $f(x)$ has at least one root in $[a_0, b_0]$.

Not knowing where in $[a_0, b_0]$ a root lies, evaluate f at the midpoint, $m_0 = (a_0 + b_0)/2$. If, by chance, $f(m_0) = 0$, one has found a root and the search is over. Otherwise, the sign of $f(m_0)$ is opposite the sign of one (and only one) of $f(a_0)$ and $f(b_0)$.

If $f(a_0)$ and $f(m_0)$ have opposite signs, then a root must be in the interval $[a_0, m_0]$, which is half the width of $[a_0, b_0]$. On the other hand, if $f(m_0)$ and $f(b_0)$ have opposite signs, a root lies in $[m_0, b_0]$, again half the width of $[a_0, b_0]$.

In either case we have trapped a root in an interval half the width of $[a_0, b_0]$. Call this shorter interval $[a_1, b_1]$. Figure 10.3.2 shows the two possibilities for $[a_1, b_1]$ in the case when $f(a_0) > 0$ and $f(b_0) < 0$.

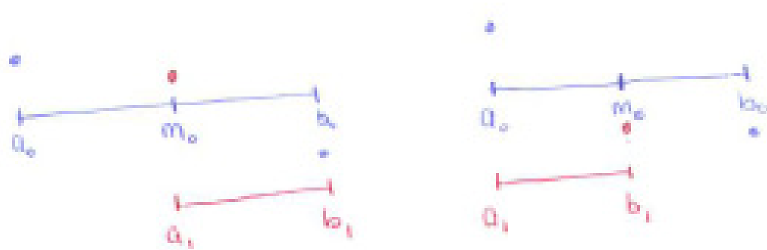


Figure 10.3.2:

The Bisection Method is a recursive algorithm.

Then repeat the process, starting at $[a_1, b_1]$. In this way you obtain a sequence of shorter and shorter intervals $[a_0, b_0]$, $[a_1, b_1]$, $[a_2, b_2]$, ..., each half as long as its predecessor. Thus, the length of $[a_n, b_n]$ is $(b_0 - a_0)/2^n$.

An Illustration of the Bisection Method

The bisection method is so named because at each step an interval is bisected.

When x is large and positive $f(x) = x + \sin(x) - 2$ is positive. When x is large and negative, $f(x)$ is negative. Therefore $f(x) = 0$ has at least one solution. The derivative of $f(x)$ is $1+\cos(x)$, which is positive except at odd multiples of π, when it is zero. Thus, $f(x)$ is an increasing function, which implies that it cannot have more than one root. Let r be the unique root of $x+\sin(x)-2 = 0$.

Begin the search for the root by finding an interval on which we can be certain the root will lie.

The larger $[a_o, b_0]$ is the longer this process will take.

Since $f(0) = -2$, the root must be positive. Using $\sin(x) \geq -1$ we know $f(x) = x + \sin(x) - 2 \geq x - 1 - 2 = x - 3$ and so $f(4)$ must be positive. Let $a_0 = 0$ and $b_0 = 4$. The root will be found in the interval $[a, b] = [0, 4]$.

The middle of this interval is $m_0 = (a_0 + b_0)/2 = 2$. Evaluate $y_0 = f(m_0) = f(2) \approx 0.909297$. Because $y_0 > 0$ we now know the root is in the interval $[a_1, b_1] = [0, 2]$.

The middle of the new interval is $m_1 = (a_1 + b_1)/2 = 1$. Then $y_1 = f(m_1) = f(1) \approx -0.15829$. Now $y_1 < 0$ so the root is trapped in the interval $[a_2, b_2] = [1, 2]$.

The third iteration of this process yields $m_2 = 1.5$ and $y_2 = f(1.5) \approx 0.497495$. Then, $[a_3, b_3] = [1, 1.5]$.

An additional ten iterations for the above problem are shown in Table 10.3.1. After 13 iterations the root is known to exist on the interval $[a_{13}, b_{13}] = [1.105957, 1.106445]$. The midpoint of this interval, $m_{13} = 1.106201$, differs from r by at most half the width of $[a_{13}, b_{13}]$, that is, by at most 0.000244.

Rounding to three decimal digits, every number in $[1.105957, 1.106445]$ rounds to 1.106.

If the iterations were continued without end, this process defines sequences $\{a_n\}$ and $\{b_n\}$. Of course, one stops when the length of the interval containing r is short enough.

EXAMPLE 1 Use the bisection method to estimate the square root of 3 to three decimal places.

SOLUTION The square root of 3 is the positive number whose square is 3: $x^2 = 3$ or $x^2 - 3 = 0$. We are looking for the positive root of $f(x) = x^2 - 3$.

The function f is continuous. We know $\sqrt{3}$ is between 1 and 2. This suggests using bisection method with initial interval $[1, 2]$.

The first 11 iterations of the bisection method are displayed in Table 10.3.2. After 7 iterations the approximation $\sqrt{3} \approx m_7 = 1.730469$ is accurate to two decimal places: $\sqrt{3} \approx 1.73$. After another 4 iterations the approximation is accurate to three decimal places: $\sqrt{3} \approx 1.732$. ◇

n	a_n	b_n	m_n	y_n	$b_n - a_n$
0	0.000000	4.000000	2.000000	0.909297	4.000000
1	0.000000	2.000000	1.000000	−0.158529	2.000000
2	1.000000	2.000000	1.500000	0.497495	1.000000
3	1.000000	1.500000	1.250000	0.198985	0.500000
4	1.000000	1.250000	1.125000	0.027268	0.250000
5	1.000000	1.125000	1.062500	−0.063925	0.125000
6	1.062500	1.125000	1.093750	−0.017895	0.062500
7	1.093750	1.125000	1.109375	0.004796	0.031250
8	1.093750	1.109375	1.101562	−0.006522	0.015625
9	1.101562	1.109375	1.105469	−0.000857	0.007812
10	1.105469	1.109375	1.107422	0.001971	0.003906
11	1.105469	1.107422	1.106445	0.000558	0.001953
12	1.105469	1.106445	1.105957	−0.000149	0.000977
13	1.105957	1.106445	1.106201	0.000204	0.000488

Table 10.3.1:

n	a_n	b_n	m_n	y_n	$b_n - a_n$
0	1.000000	2.000000	1.500000	−0.750000	1.000000
1	1.500000	2.000000	1.750000	0.062500	0.500000
2	1.500000	1.750000	1.625000	−0.359375	0.250000
3	1.625000	1.750000	1.687500	−0.152344	0.125000
4	1.687500	1.750000	1.718750	−0.045898	0.062500
5	1.718750	1.750000	1.734375	0.008057	0.031250
6	1.718750	1.734375	1.726562	−0.018982	0.015625
7	1.726562	1.734375	1.730469	−0.005478	0.007812
8	1.730469	1.734375	1.732422	0.001286	0.003906
9	1.730469	1.732422	1.731445	−0.002097	0.001953
10	1.731445	1.732422	1.731934	−0.000406	0.000977
11	1.731934	1.732422	1.732178	0.000440	0.000488

Table 10.3.2:

The bisection method is known as a "bracketing method" because the two sequences bracket the solution.

Why the Bisection Method Works

The bisection method applied to $f(x)$ produces two sequences, $a_0 \le a_1 \le a_2 \le \cdots$ and $b_0 \ge b_1 \ge b_2 \ge \cdots$. If no a_n or b_n is a root of f, the sequences do not end. The sequence of left endpoints, $\{a_n\}$, is monotone increasing and the sequence of right endpoints is monotone decreasing. Moreover, since every a_n is less than or equal to b_0, $\{a_n\}$ is bounded. Thus $\{a_n\}$, being bounded and monotone, has a limit, $A \le b_0$. Similarly, $\{b_n\}$ also has a limit, $B \ge a_0$.

Recall that the length of the interval $[a_n, b_n]$ is $b_n - a_n = (b_0 - a_))/2^n$. This means that $\{b_n - a_n\}$ is a geometric sequence with ratio $1/2$, which is less than 1. Thus, $\lim_{n\to\infty}(b_n - a_n) = 0$, and we have

$$0 = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = B - A.$$

Consequently, $A = B$.

But, why is A a root of f?

Consider the sequence

$$f(a_0), f(b_0), f(a_1), f(b_1), f(a_2), f(b_2), \cdots f(a_n), f(b_n), \cdots . \tag{10.3.1}$$

Since f is continuous, (10.3.1) has a limit, $f(A)$. Moreover, the fact that one of $f(a_n)$ and $f(b_n)$ is positive means the limit, $f(A)$, cannot be negative. Similarly, because one of each pair of entries in (10.3.1) is negative, $f(A)$ cannot be positive. Thus, $f(A) = 0$ and A is a root of f.

Summary

In the bisection method for finding a root of a function f, one first finds two inputs a_0 and b_0 for which $f(a_0)$ and $f(b_0)$ have opposite signs. Then one evaluates f at the midpoint m_0. The function f will have opposite signs at the endpoints of exactly one of the intervals: $[a_0, m_0]$ or $[m_0, b_0]$. Call this new interval $[a_1, b_1]$, then repeat the process on this new interval. Continue to repeat the process until the interval is short enough to assure an estimate of the root that meets the desired accuracy.

EXERCISES for Section 10.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 and 2, use the bisection method to find a_1 and b_1.

1.[R] $a_0 = 2$, $b_0 = 6$, $f(2) = 0.3$, $f(4) = 1.5$, $f(6) = -1.2$

2.[R] $a_0 = 1$, $b_0 = 3$, $f(1) = -4$, $f(2) = -1.5$, $f(3) = 1$

3.[R] In this exercise use the bisection method to approximate $\sqrt{2}$. Let $a_0 = 1$, $b_0 = 2$, and $f(x) = x^2 - 2$. Fill in the following table as you carry out the first five steps of the bisection method.

n	a_n	b_n
0	0	2
1		
2		
3		
4		
5		

4.[R] In this exercise the ideas in Exercise 3 are used to estimate $\sqrt{5}$ with the bisection method.

(a) Use $f(x) = x^2 - 5$ and start with $a_0 = 2$, and $b_0 = 3$. Continue until the interval $[a_n, b_n]$ is shorter than 0.01, that is, $b_n - a_n < 0.01$.

(b) How many more steps of the bisection method are needed to reduce the interval by another factor of 10, that is, $b_n - a_n < 0.001$? HINT: This can be answered *without* computing every a_n and b_n.

5.[R] In this exercise the ideas in Exercise 3 are used to estimate $\sqrt[3]{2}$ with the bisection method.

(a) Use $f(x) = x^3 - 2$ and start with $a_0 = 1$, and $b_0 = 2$. Continue until the interval $[a_n, b_n]$ is shorter than 0.01, that is, $b_n - a_n < 0.01$.

(b) How many more steps of the bisection method are needed to reduce the interval by another factor of 10, that is, $b_n - a_n < 0.001$?

In Exercises 6 to 9 use the ideas in Exercise 3 to estimate the given numbers to the indicated number of decimal places.

6.[R] $\sqrt{15}$ to 3 decimal places HINT: Use $f(x) = x^2 - 15$ with $a_0 = 3$ and $b_0 = 4$.

7.[R] $\sqrt{19}$ to 2 decimal places

8.[R] $\sqrt[3]{7}$ to 4 decimal places

9.[R] $\sqrt[3]{25}$ to 3 decimal places

10.[R] Let $f(x) = x^5 + x - 1$.

(a) Show that there is a root of the function $f(x)$ in the interval $[0, 1]$.

(b) Apply five steps of the bisection method with $a_0 = 0$ and $b_0 = 1$.

(c) Why is the root unique?

11.[R] Let $f(x) = x^4 + x - 19$.

(a) Show that $f(2) < 0 < f(3)$. What additional property of f assures that there is exactly one root r between 2 and 3?

(b) Using the bisection method with $[a_0, b_0] = [2, 3]$, find an interval of length no more than 0.01 where this root must be found.

(c) The second real root of $f(x)$ is negative. Find an interval of length one in which this root must exist.

(d) Repeat (b) using the interval found in (c) as the initial interval.

12.[R] In estimating $\sqrt{3}$ with the bisection method, Sam imprudently chooses the initial interval to be $[0, 10]$.

(a) How many steps of the bisection method will Sam have to execute before he has an interval shorter than 0.005?

(b) Jane started with $[1, 2]$. How many steps of the bisection method will she need to execute before she has an interval shorter than 0.0005?

13.[R] Let $f(x) = 2x^3 - x^2 - 2$.

(a) Show that there is exactly one root of the equation $f(x) = 0$ in the interval $[1, 2]$.

(b) Using $[a_0, b_0] = [1, 2]$ as a first interval, apply two steps of the bisection method..

14.[R]

(a) Graph $y = x$ and $y = \cos(x)$ relative to the same axes.

(b) Using the graph in (a), find an interval of length no more than 0.25 that contains the positive solution of the equation $x = \cos(x)$. Is there a negative solution?

(c) Using your estimate in (b) as $[a_0, b_0]$, apply the bisection method until the interval is shorter than 0.001.

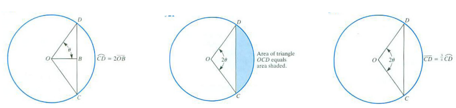
15.[R]

(a) Graph $y = \cos(x)$ and $y = 2\sin(x)$ relative to the same axes.

(b) Without using the graph in (a), explain how you know there is exactly one solution in $[0, \pi/2]$.

(c) Using $[a_0, b_0] = [0, \pi/2]$, apply the bisection method until the length of the interval is no more than 0.001.

In Exercises 16 to 18 (Figure 10.3.3) use the bisection method to estimate θ (to two decimal places). Angles are in radians. Also show that there is only one answer if $0 < \theta < \pi/2$.



(a) (b) (c)

Figure 10.3.3:

16.[R] Figure 10.3.3(a)

17.[R] Figure 10.3.3(b)

18.[R] Figure 10.3.3(c)

19.[R]

(a) Graph $y = x\sin(x)$ for x in $[0, \pi]$.

(b) Using the first and second derivatives, show that the fuction has a unique relative maximum in the interval $[0, \pi]$.

(c) Show that the maximum value of $x\sin(x)$ occurs when $x\cos(x) + \sin(x) = 0$.

(d) Use bisection method, with $[a_0, b_0] = [0, \pi/2]$, to find an estimate for a root of $x\cos(x) + \sin(x) = 0$ that is accurate to at least two decimal digits.

20.[R]

(a) Graph $y = x \cos x$ for x in $[0, \pi]$.

(b) Using the first and second derivatives, show that there is a unique relative maximum in the interval $[0, \pi/2]$.

(c) Show that the maximum value of $x \cos x$ occurs when $\cos x - x \sin x = 0$.

(d) Use the bisection method, with $[0, \pi/2]$, to find an interval $[a_n, b_n]$ with length no more than 0.01 that contains a solution of $\cos x - x \sin x = 0$.

21.[R] Use the bisection method to estimate the maximum value of $y = 2\sin(x) - x^2$ over the interval $[0, \pi/2]$.

22.[R] Use the bisection method to estimate the maximum value of $y = x^3 + \cos(x)$ over the interval $[0, \pi/2]$.

23.[R] We can show that the error in the bisection method diminishes rather slowly. Let $[a_0, b_0]$ be the initial interval containing the root r and let $[a_1, b_1]$ be the next estimate, obtained by the bisection method.

(a) Show that $b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$.

(b) Let $[a_2, b_2]$ be the next interval obtained by the bisection method. Show that $b_2 - a_2 = \frac{1}{2}(b_1 - a_1) = \frac{1}{4}(b_0 - a_0)$.

(c) Explain why, in general, $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \frac{1}{2^n}(b_0 - a_0)$.

(d) How many steps of the bisection method are needed to obtain an interval no longer than L ($L > 0$) containing the given root.

24.[M] The equation $x \tan(x) = 1$ occurs in the theory of vibrations.

(a) How many roots does it have in $[0, \pi/2]$?

(b) Use the bisection method to estimate each root to two decimal places.

25.[M] Use the bisection method to approximate all local extrema of $g(x) = 2x - (x + 1)e^{-x}$ to three decimal places. How do you know you have found all extrema? Note: See also Example 3 in Section 10.4.

26.[M]

(a) Show that a critical number of the function $f(x) = (\sin x)/x$ for $x \neq 0$ and $f(0) = 1$ satisfies the equation $\tan x = x$.

(b) Show that $(\sin(x))/x$ is an even function. Thus we will consider only positive x.

(c) Graph the function $\tan(x)$ and x relative to the same axes. How often do they cross for x in $[\pi/2, 3\pi/2]$? for x in $[3\pi/2, 5\pi/2]$? Base your answer on your graphs.

(d) Show that $\tan(x) - x$ is an increasing function for x in $[\pi/2, 3\pi/2]$. What does that tell us about the number of solutions of the equation $\tan(x) = x$ for x in $[\pi/2, 3\pi/2]$?

(e) How many critical numbers does the function $f(x)$ have?

(f) Use the bisection method with $[a_0, b_0] = [\pi/2, 3\pi/2]$ to estimate the critical number in $[\pi/2, 3\pi/2]$ to at least two decimal places.

27.[M] Examine the solutions of the equation $2x + \sin(x) = 2$. How many are there? Use the bisection method with appropriate initial intervals to evaluate each solution to two decimal places. Explain the steps in your solution in complete sentences.

28.[M] How many solutions does the equation $\sin(x) = x$ have? Explain how you could use the bisection method to estimate each solution.

29.[M] Explain how you could use the bisection method to estimate $\sqrt[5]{a}$.

30.[M]

Sam: I have a better way than the bisection method.

Jane: What do you propose?

Sam: I trisect the interval into three equal intervals using two points.

Jane: What's so good about that?

Sam: I cut the error by a factor of 3 each step.

Jane: But you have to compute two points and evaluate the function there. That's four calculations instead of two.

Sam: But my method cuts the error so fast, it's still better, so the gain outweighs the cost.

Is Sam right?
Assume the initial interval is $[0, 1]$ and estimate the "cost" to reduce the length of the interval containing the root go the small number E.

31.[M]

Sam: I have a better way than the bisection method.

Jane: What is it?

Sam: I break the interval into four equal intervals by three points.

Jane: Then?

Sam: I find on which of the four intervals the root must lie. I do two of the bisection steps in one step. So it must be more efficient.

Jane: That all depends. I'll think about it.

Think about it.

10.4 Newton's Method for Solving $f(x) = 0$

This section presents another way to find a sequence of approximations to a solution of $f(x) = 0$. **Newton's Method** uses information about f and its derivative to produce estimates that usually converge much faster than the sequences obtained by the bisection method.

The Idea Behind Newton's Method

Figure 10.4.1 shows the graph of a function f which has a root r and initial estimate x_0. (You may make the initial estimate by looking at a graph, or doing some calculations on your calculator.)

To get a (hopefully) better estimate of r, find where the tangent at $P = (x_0, f(x_0))$ crosses the x-axis. Call the new estimate x_1, as shown in Figure 10.4.1.

Then repeat the process using x_1, instead of x_0, as the estimate of the root r. This produces an estimate x_2. Repeating the process produces a sequence x_0, x_1, x_2, …, x_n, …. However, in practice, you stop Newton's Method when two successive estimates are sufficiently close together.

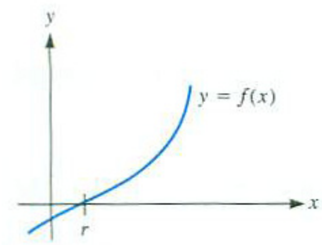


Figure 10.4.1: ARTIST: Please add labels for x_0 and $f(x_0)$.

The Key Formula

What difficulties arise if $f'(x_0) = 0$?

To obtain a formula for x_1 in terms of x_0, observe that the slope of the tangent at P in Figure 10.4.1 is $f'(x_0)$ and also $(f(x_0) - 0)/(x_0 - x_1)$. We assume $f'(x_0)$ is not zero, that is, the tangent at P is not parallel to the x-axis. Thus

$$f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1}$$

or

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}.$$

Consequently, we have the key formula for applying Newton's Method:

Newton's Recursion

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{10.4.1}$$

The same idea gives $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ and so on for x_3, x_4, …. In general, we have the recursive definition,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{10.4.2}$$

Before we examine whether the sequence converges, we illustrate the technique with some examples.

EXAMPLE 1 In the previous section, 13 iterations of the bisection method were needed to estimate the unique solution to $f(x) = x + \sin(x) - 2 = 0$ to 3 decimal places. Let's see how Newton's Method deals with the same problem.
SOLUTION A reasonable initial estimate is $x_0 = 2$, because it cancels the -2 in $x+\sin(x)-2$. The derivative of $x+\sin(x)-2$ is $1+\cos(x)$. The Newton recursion formula, (10.4.1), reads

$$x_{n+1} = x_n - \frac{x_n + \sin(x_n) - 2}{1 + \cos(x_n)}.$$

The first six iterations of Newton's Method are shown in Table 10.4.1.

Note that $f(x_5) = 0$. As a result, all subsequent estimates will be identical to x_5. We conclude that $r \approx x_5 = 1.106060$ and that this estimate is accurate to six decimal places.

n	x_n	$f(x_n)$	$f'(x_n)$
0	2.000000	0.909297	0.583853
1	0.442592	−1.129124	1.903644
2	1.035731	−0.104034	1.509898
3	1.104632	−0.002069	1.449463
4	1.106060	−0.000001	1.448188
5	1.106060	0.000000	1.448187
6	1.106060	0.000000	1.448187

Table 10.4.1:

Each iteration of the bisection method is much easier to implement than Newton's method. However, Newton's Method needs only 5 steps to obtain an approximation of the root to f accurate to (at least) six decimal places while after 13 iterations the bisection method yields an approximation, $p_{13} \approx 1.106201$, accurate to only three decimal places. ◇

EXAMPLE 2 Use Newton's method to estimate the square root of 3, that is, the positive root of the equation $x^2 - 3 = 0$.
SOLUTION Here $f(x) = x^2 - 3$ and $f'(x) = 2x$. According to (10.4.1), if the initial estimate is x_0, then the next estimate x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 3}{2x_0} = \frac{1}{2}\left(x_0 + \frac{3}{x_0}\right).$$

For our initial estimate, let us use $x_0 = 2$. Its square is 4, which isn't far from 3. Then

$$x_1 = \frac{1}{2}\left(x_0 + \frac{3}{x_0}\right) = \frac{1}{2}\left(2 + \frac{3}{2}\right) = 1.75.$$

Repeat, using $x_1 = 1.75$ to obtain the next estimate:

$$x_2 = \frac{1}{2}\left(x_1 + \frac{3}{x_1}\right) = \frac{1}{2}\left(1.75 + \frac{3}{1.75}\right) \approx 1.73214.$$

One more step of the process yields (to five decimals) $x_3 \approx 1.73205$, which is close to $\sqrt{3}$. The decimal expansion of $\sqrt{3}$ begins 1.7320508. See Figure 10.4.2, which shows x_0, x_1 and the graph of $f(x) = x^2 - 3$, and Table 10.4.2 the numerical values used in these computations.

In fact, x_3 agrees with $\sqrt{3}$ to seven decimals.

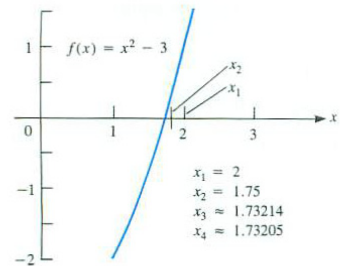


Figure 10.4.2: NOTE: Renumber indices.

n	x_n	$f(x_n)$	$f'(x_n)$
0	2.000000	1.000000	4.000000
1	1.750000	0.062500	3.500000
2	1.732143	0.000319	3.464286
3	1.732051	0.000000	3.464102
4	1.732051	0.000000	3.464102

Table 10.4.2:

When the same problem was solved using the bisection method in Example 1, after 11 iterations the best approximation to r is $p_{11} = 1.732178$. This approximation to $\sqrt{3}$ is accurate to only three decimal places. ◇

Compare with Table 10.3.2.

In practice, stop the process when either $|f(x_n)|$ or the difference between successive estimates, $|x_n - x_{n-1}|$, become sufficiently small.

Compare with Exercise 25.

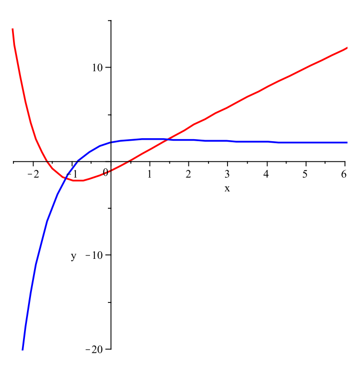
EXAMPLE 3 Use Newton's method to approximate the location of the local extrema of $g(x) = 2x - (x + 1)e^{-x}$.

SOLUTION This problem, which was first solved in Exercise 25 in Section 10.3 is equivalent to asking for all roots of $f(x) = g'(x) = 2 + xe^{-x}$.

To find an initial guess to start Newton's method, notice that $f(0) = 2$ and $f(x) > 0$ for all positive numbers x. Looking for a negative value of x that makes $f(x)$ negative, we see that $f(-2) = 2 + (-2)e^2 = 2 - 2e^2 < 0$ because $e > 1$.

The first few iterations of Newton's method with $x_0 = -1$ are shown in Table 10.4.3. After four steps the process is stopped because $f(x_3) = 0$. The critical number of g is approximately $x^* \approx x_3 = -0.852606$. This is correct to all six decimal places shown.

Because $g'(x)$ is negative to the immediate left of x^* and is positive to the immediate right of x^* we conclude that x^* is a local minimum of $g(x) =$



n	x_n	$f(x_n)$	$f'(x_n)$	$\lvert x_n - x_{n-1}\rvert$
0	-1.000000	-0.718282	5.436564	
1	-0.867879	-0.067163	4.449017	0.132121
2	-0.852783	-0.000773	4.346941	0.015096
3	-0.852606	0.000000	4.345751	0.000177
4	-0.852606	0.000000	4.345751	0.000000

Table 10.4.3:

$2x-(x+1)e^{-x}$. The graphs of g and $g' = f$ are shown in Figure 10.4.3. Observe the only local extremum is the local minimum near $x = -0.85$. ◇

Remarks on Newton's Method

The assumption that f'' exists implies f' (and f) are continuous.

In an interval where $f''(x)$ is positive, the graph of $y = f(x)$ is concave up, and lies above its tangents, as shown in Figure 10.4.4. If x_1 is to the right of r, the sequence x_1, x_2, x_3, ... is monotone and is bounded below by r. Thus, the sequence converges to a limit $L \geq r$. To show that L is r, take limits of both sides of the Newton recursion formula, (10.4.2):

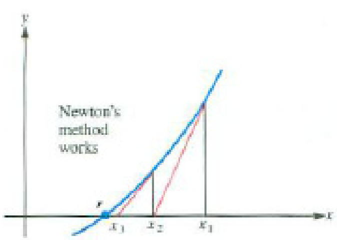


Figure 10.4.4:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(x_n - \frac{f(x_n)}{f'(x_n)}\right) \tag{10.4.3}$$

obtaining (10.4.4)

$$L = L - \frac{f(L)}{f'(L)} \tag{10.4.5}$$

Thus, $0 = -f(L)/f'(L)$, so $f(L) = 0$, and L is a root of f.

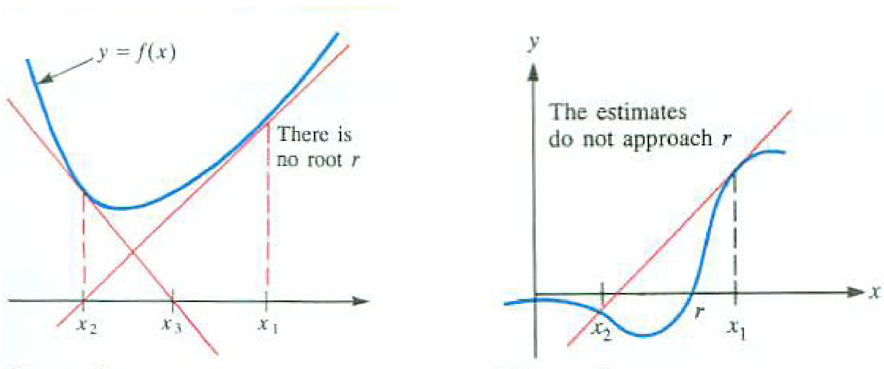
The reasoning that obtained (10.4.5) from (10.4.3) shows, more generally, if the sequence produced by Newton's Method converges, its limit is a root.

The equation $f(x) = 0$ may not have a solution. In that case the sequence of estimates produced by Newton's method does not approach a specific number but may wander all over the place, as in Figure 10.4.5(a).

It is also possible that there is a root r, but your initial guess x_0 is so far from r that the sequence of estimates does not approach r. See Figure 10.4.5(b).

The tangent at $(x_n, f(x_n))$ is horizontal and does not intersect the x-axis.

Of course, if x_n is a number where $f'(x_n) = 0$, then the Newton recursion, which has $f'(x_n)$ in the denominator, makes no sense.



(a) (b)

Figure 10.4.5:

How Good is Newton's Method

Newton's method for solving $x^2 - 3 = 0$ revisited from a different point of view.

When you use Newton's method, you produce a sequence of estimates x_0, x_1, x_2, ... of a root r. How quickly does the sequence approach r? In other words, how rapidly does the difference between the estimate x_n and the root r, $|x_n - r|$, approach 0?

To get a feel for the rate at which $|x_n - r|$ shrinks as we keep using Newton's method, take the case in Example 2, where we were estimating $\sqrt{3}$ using the recursion

$$x_{n+1} = \frac{1}{2}\left(x_1 + \frac{3}{x_n}\right).$$

In the following table, we list, x_1, x_2, x_3, x_4 to seven decimal places and compare to $\sqrt{3} \approx 1.7320508$:

Estimate	Value	Agreement with $\sqrt{3}$
x_1	2.000000000	Initial guess
x_2	1.750000000	First two digits
x_3	1.732142857	First four digits
x_4	1.732050810	First eight digits

At each stage the number of correct digits tends to *double*. This means the error at one step is roughly the square of the error of the previous guess,

$$|x_n - r| \le M\,|x_{n-1} - r|^2$$

for an appropriate constant M. This constant depends on the maximum of the absolute values of the first and second derivatives. By contrast, the iterates for the bisection method tend to cut the error $|x_n - r|$ in half at each step. Because $2^3 < 10 < 2^4$, it generally takes 3 or 4 steps to gain one more decimal place accuracy.

This difference is evident in the number of iterations needed in each algorithm to achieve the same accuracy.

Summary

This section developed Newton's method for estimating a root of an equation, $f(x) = 0$. You start with an estimate x_0 of the root, then compute

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Then repeat the process, obtaining the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{for all } n = 1,\ 2,\ 3,\ \ldots.$$

When $f'(r) \neq 0$ and f' is continuous, the iterates in Newton's Method converge to r provided the initial guess is sufficiently close to r.

The Newton iterates converge quickly to the root: there is a constant M such that

$$|x_n - r| \leq M\,|x_{n-1} - r|^2$$

while the iterates computed by the bisection method converge slowly:

$$|x_n - r| \leq \frac{1}{2}\,|x_{n-1} - r|\,.$$

While, in general, Newton's method converges faster than the bisection method the actual performance depends on f and the initial estimates.

Iterative methods for finding a root generally stop when either $|f(x_n)|$ or $|x_{n+1} - x_n|$ become small enough.

EXERCISES for Section 10.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises1 and 2, use Newton's method to find x_1.

1.[R] $x_0 = 2,\ f(2) = 0.3,\ f'(2) = 1.5$

2.[R] $x_0 = 3,\ f(3) = 0.06,\ f'(3) = 0.3$

3.[R] Let a be a positive number. Show that the Newton recursion formula for estimating $\sqrt{a}$ is given by

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{a}{x_i}\right).$$

NOTE: The sequence defined in Exercise 3 was the Babylonian method for estimating $\sqrt{a}$. If the guess x_0 is smaller than $\sqrt{a}$, then a/x_0 is larger than $\sqrt{a}$. So x_1 is the average of two numbers between which $\sqrt{a}$ lies.

4.[R] Use the formula of Exercise 3 to estimate $\sqrt{15}$. Choose $x_0 = 4$ and compute x_1 and x_2 to three decimals.

5.[R] Use the formula of Exercise 3 to estimate $\sqrt{19}$. Choose $x_0 = 4$ and compute x_1 and x_2 to three decimals.

6.[R] Use Newton's Method to estimate $\sqrt[3]{7}$. Choose $x_0 = 2$ and compute x_1 and x_2 to three decimals.

7.[R] Use Newton's Method to estimate $\sqrt[3]{25}$. Choose $x_0 = 3$ and compute x_1 and x_2 to three decimals.

8.[R] In this exercise the ideas in Exercise 3 are used to estimate $\sqrt{5}$ with Newton's method.

(a) Use $f(x) = x^2 - 5$ and start with $x_0 = 2$. Continue until the consecutive estimates differ by at most 0.01, that is, $x_{n+1} - x_n < 0.01$.

(b) How many more steps of Newton's method are needed to reduce the interval by another factor of 10, that is, $x_{n+1} - x_n < 0.001$?

9.[R] Estimate $\sqrt[3]{2}$ with Newton's method.

(a) Use $f(x) = x^3 - 2$ and start with $x_0 = 1$. Continue until the consecutive estimates differ by at most 0.01, that is, $x_{n+1} - x_n < 0.01$.

(b) How many more steps of Newton's method are needed to reduce the interval by another factor of 10, that is, $x_{n+1} - x_n < 0.001$?

10.[R] Let $f(x) = x^5 + x - 1$.

(a) Using $x_0 = \frac{1}{2}$ as a first estimate, apply Newton's method to find a second estimate x_1.

(b) Show that there is a root of the function $f(x)$ in the interval $[0, 1]$.

(c) Why is the root unique?

11.[R] Let $f(x) = x^4 + x - 19$.

(a) Apply Newton's method, starting with $x_0 = 2$. Compute x_1 and x_2.

(b) Show that $f(2) < 0 < f(3)$. What additional property of f assures that there is exactly one root r between 2 and 3?

(c) The second real root of $f(x)$ is negative. Find an interval of length one on which this root must exist.

(d) Use the left endpoint of the interval in (c) as the initial guess for Newton's method. Compute x_1 and x_2.

12.[R] In estimating $\sqrt{3}$ with Newton's method, Sam imprudently chooses $x_0 = 10$. What does Newton's method give for x_1, x_2, and x_3?

13.[R] Let $f(x) = 2x^3 - x^2 - 2$.

(a) Show that there is exactly one root of the equation $f(x) = 0$ in the interval $[1, 2]$.

(b) Using $x_0 = \frac{3}{2}$ as a first estimate, apply Newton's method to find x_2 and x_3.

14.[R]

(a) Graph $y = x$ and $y = \cos(x)$ relative to the same axes.

(b) Using the graph in (a), estimate the positive solution of the equation $x = \cos(x)$. Is there a negative solution?

(c) Using your estimate in (b) as x_0, apply Newton's method until consecutive estimates agree to four decimal places.

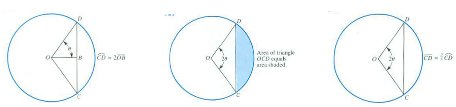
15.[R]

(a) Graph $y = \cos(x)$ and $y = 2\sin(x)$ relative to the same axes.

(b) Using the graph in (a), estimate the solution that lies in $[0, \pi/2]$.

(c) Using your estimate in (b) as x_0, apply Newton's method until consecutive estimates agree to four decimal places.

In Exercises 16 to 18 (Figure 10.4.6) use Newton's method to estimate θ (to two decimal places). Angles are in radians. Also show that there is only one answer if $0 < \theta < \pi/2$.



(a) (b) (c)

Figure 10.4.6:

16.[R] Figure 10.4.6(a)

17.[R] Figure 10.4.6(b)

18.[R] Figure 10.4.6(c)

19.[R] The equation $x\tan(x) = 1$ occurs in the theory of vibrations.

(a) How many roots does it have in $[0, \pi/2]$?

(b) Use Newton's method to estimate each root to two decimal places.

20.[R]

(a) Show that a critical number of the function $f(x) = (\sin x)/x$ for $x \neq 0$ and $f(0) = 1$ satisfies the equation $\tan x = x$.

(b) Show that $(\sin(x))/x$ is an even function. Thus we will consider only positive x.

(c) Graph the function $\tan(x)$ and x relative to the same axes. How often do they cross for x in $[\pi/2, 3\pi/2]$? for x in $[3\pi/2, 5\pi/2]$? Base your answer on your graphs.

(d) Show that $\tan(x) - x$ is an increasing function for x in $[\pi/2, 3\pi/2]$. What does that tell us about the number of solutions of the equation $\tan(x) = x$ for x in $[\pi/2, 3\pi/2]$?

(e) How many critical numbers does the function $f(x)$ have?

(f) Use Newton's method to estimate the critical number in $[\pi/2, 3\pi/2]$ to at least two decimal places.

21.[R] Examine the solutions of the equation $2x + \sin(x) = 2$. How many are there? Use Newton's method to evaluate each solution to two decimal places. Explain the steps in your solution in complete sentences.

22.[R] How many solutions does the equation $\sin(x) = x$ have? Explain how you could use Newton's method to estimate each solution.

23.[R] Explain how you could use Newton's method to obtain a formula for estimating $\sqrt[5]{a}$.

Exercises 24 and 25 show that care should be taken in applying Newton's method.
24.[R] Let $f(x) = 2x^3 - 4x + 1$.

(a) Show that there must be a root r of $f(x) = 0$ in $[0, 1]$.

(b) Take $x_0 = 1$, and apply Newton's method to obtain x_1 and x_2.

(c) Graph f, and show what is happening in the sequence of estimates.

25.[R] Apply Newton's method to the function $f(x) = x^3 - x$, starting with $x_0 = 1/\sqrt{5}$.

(a) Compute x_1 and x_2 exactly (not as decimal approximations).

(b) Graph $x^3 - x$ and explain why Newton's method fails in this case.

26.[R] Let $f(x) = x^2 + 1$

(a) Using Newton's method with $x_0 = 2$, compute x_1, x_2, x_3, and x_4 to two decimal places.

(b) Using the graph of f, show geometrically what is happening in (a).

(c) Using Newton's method with $x_1 = \sqrt{3}/3$, compute x_2 and x_3. What happens to x_n as $n \to \infty$?

(d) What happens when you use Newton's method, startng with $x_1 = 1$?

27.[R] Assume that $f'(x) > 0$, $f"(x) < 0$ for all x, and $f(r) = 0$.

(a) Sketch a possible graph of $y = f(x)$.

(b) Describe the behavior of the sequence of Newton's estimates x_0, x_1,..., x_n, ... when you choose $x_0 > r$. Include a sketch.

(c) Describe the behavior of the sequence if you choose $x_0 < r$. Include a sketch.

28.[M] Let $f(x) = 1/x + 5$

(a) Graph $f(x)$ showing its x-intercepts.

(b) For which x_0 does Newton's Method sequence converge to a solution to $f(x) = 0$?

(c) For which x does Newton Method sequence not converge?

29.[M] Let $f(x) = \frac{1}{x^2} - 5$ and assume the same questions as in the preceding exercise.

30.[M]

(a) Graph $y = x\sin(x)$ for x in $[0, \pi]$.

(b) Using the first and second derivatives, show that it has a unique relative maximum in the interval $[0, \pi]$.

(c) Show that the maximum value of $x\sin(x)$ occurs when $x\cos(x) + \sin(x) = 0$.

(d) Use Newton's method, with $x_0 = \pi/2$, to find an estimate x_1 for a root of $x\cos(x) + \sin(x) = 0$.

(e) Use Newton's method again to find x_2.

31.[M]

(a) Graph $y = x\cos x$ for x in $[0, \pi]$.

(b) Using the first and second derivatives, show that it has a unique relative maximum in the interval $[0, \pi/2]$.

(c) Show that the maximum value of $x\cos x$ occurs when $\cos x - x\sin x = 0$.

(d) Use Newton's method, with $x_0 = \pi/4$, to find an estimate x_1 for a root of $\cos x - x\sin x = 0$.

(e) Use Newton's method again to find x_2.

32.[M] Use Newton's method to estimate the maximum value of $y = 2\sin(x) - x^2$ over the interval $[0, \pi/2]$.

33.[M] Use Newton's method to estimate the maximum value of $y = x^3 + \cos(x)$ over the interval $[0, \pi/2]$.

34.[M] We can show that the error in Newton's method diminishes rapidly (compared to the bisection method). Let x_0 be an estimate of the root r and let x_1 be the second estimate, obtained by Newton's method. Assume $f'(x_0) \neq 0$.
Using the first-order Taylor polynomial with remainder, centered at $a = x_0$, we may write

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(c)}{2}(x - x_1)^2 \qquad (10.4.6)$$

where c is a number between x and x_1. (See Section 5.4 on page 398.)

(a) In (10.4.6), replace x by r and use the definition of x_1 to show that

$$x_1 - r = \frac{f^{(2)}(c)}{2f'(x_0)}(r - x_0)^2,$$

where c is between x_1 and r.

(b) Assume that $x_0 > r$ and that $f'(x)$ and $f''(x)$ are positive for x in $[r, x_0]$. Indicate on a diagram where the numbers x_1, x_2 ... are situated. Then use (a) to discuss how the error, $r - x_n$, behaves as n increases.

35.[C] Let p be a positive number.

(a) Graph $f(x) = 1/x - p$.

(b) For which choices of the initial estimate of a root of f will Newton's Method converge to r?

36.[C] Throughout this section we have assumed we knew the derivative $f'(x)$.

However, the derivative may be too complicated, or perhaps you just know the values of $f(x)$ at certain points. When you make an initial guess of a root of f, how would you calculate a plausible "better approximation"? HINT: What could you use instead of the tangent line?

10.S Chapter Summary

Infinite sequences of numbers a_k, a_{k+1}, ... arise in many contexts. (The initial index, k, can be any non-negative integer.) For instance, they arise when estimating a root of an equation of the form $f(x) = 0$. Any equation, $g(x) = h(x)$ can be transformed to that form, for it is equivalent to $g(x) - h(x) = 0$.

One way to estimate a root of $f(x) = x$ is to pick an estimate, a, of a root and compute $f(a)$, $f(f(a))$, $f(f(f(a)))$, ... If this sequence has a limit, r, then $f(r) = r$.

The bisection method provides estimates of the roots of $f(x) = 0$. One looks for numbers a and b at which $f(x)$ has opposite signs. If f is continuous, it has a root in the interval (a, b). Let m be the midpoint of that interval. Then either m is a root or its sign is opposite the sign of one of $f(a)$ and $f(b)$. Repeat, using either (a, m) or (m, b) depending on which interval has ends of opposite signs (when plugged into f). This process continues until the intervals are short enough. Usually, the midpoint of the final interval is the final approximation to the root and the error estimate is half the length of the interval.

Newton's method for solving $f(x) = 0$ depends on using a tangent to approximate the graph of $f(x)$. It yields the recursion $x_2 = x_1 - f(x_1)/f'(x_1)$. Repeat the process until one has the desired accuracy.

EXERCISES for 10.S *Key:* R–routine, M–moderate, C–challenging

1.[R] Let $a_0 = 0$ and $a_n = a_{n-1} + 2n - 1$ for $n \geq 1$.

(a) Compute a few values of a_n (at least through a_5) and conjecture an explicit formula for a_n.

(b) Show that if your formula is correct for $n = k$, then it is correct for $n = k + 1$.

2.[C]

(a) Graph $f(x) = \cos(\pi/2\ x)$ for x in $[0, 1]$.

(b) Let a be the unique fixed point of f on $[0, 1]$. Estimate a by looking at your graph in (a).

(c) Use Newton's Method to estimate a to 2 decimal places.

(d) Use the bisection method to estimate a to 2 decimal places.

(e) Does the sequence $\cos(\pi/2\ x)$, $\cos(\pi/2\ \cos(\pi/2\ x))$, ... converge for every x in $[0, 1]$.

3.[R] Sketch the graph of a concave-down function f with the properties that $f(1/2) = 1/2$, $f(0) = 0$, $f(1) = 0$, and $f'(0) > 1$.

4.[R] Like Exercise 3 but with $0 < f(0) < 1$.

5.[R] In Example 1 in Section 4.1 it was shown that $f(t) = (t^2 - 1)\ln\left(\frac{t}{\pi}\right)$ has one critical number on $[1, \pi]$. Use Newton's Method to estimate this critical number to three decimal digits.

6.[R] In Example 2 in Section 4.1 it was shown that $f(x) = x^3 - 6x^2 + 15x + 3$ has exactly one real root. Use Newton's method to approximate this root to three decimal places.

7.[M]

(a) Graph $y = xe^{-x^2}$.

(b) Estimate the area of the region bounded by $y = xe^{-x^2}$, the line $x + y = 1$, and the x-axis.

NOTE: You will need Newton's method of estimating a solution of an equation.

8.[M] The spiral $r = \theta$ meets the circle $r = 2\sin(\theta)$ at a point other than the origin. Use Newton's method to estimate the coordinates of that point. (Give both the polar and rectangular coordinates of the point of intersection.)

9.[M] The equation $M = E - e\sin(E)$, known as **Kepler's equation**, occurs in the study of planetary motion. (M involves E, position, and e, the eccentricity of the orbit, a number between 0 and 1.)

(a) Sketch what the graph of M as a function of E looks like.

(b) Show that $E - \sin(E)$ is an increasing function of E.

(c) In view of (b), E is a function of M, $E = g(M)$. Use Newton's method to find $g(E)$ if $e = 0.2$.

(d) Which x_0 lead to convergent sequences? HINT: A graphing calculator or computer can be used to simplify the calculations.

Consider the problem of finding a solution to $g(x) = 0$. There are usually several ways to rewrite this equation as $f(x) = x$. The challenge is to choose the function

f so that the sequence with $a_n = f(a_{n-1})$ converges. Then $L = \lim_{n\to\infty} a_n$ is a solution to $g(x) = 0$. In Exercises 10 to 13 we develop and apply a general result known as the **Fixed Point Theorem**.

10.[M] In this exercise we develop a version of the Fixed Point Theorem that will explain what is happening in Exercises 11 and 13. Basically, if r is a fixed point of f, that is, a number such that $f(r) = r$, then the errors $e_n = r - a_n$ satisfy $r - e_n = f(r - e_{n-1})$.

(a) Fill in the details to show why $r - e_n = f(r - e_{n-1})$.

(b) Replace $f(r - e_{n-1})$ with the linear approximation to f at r and derive the (approximate) result: $e_n \approx f'(r)e_{n-1}$ for all $n \geq 0$.

(c) Show that if $e_n \approx f'(r)e_{n-1}$ for all $n \geq 0$, then $e_n \approx (f'(r))^{n+1}e_0$.

(d) Explain why $e_n \to 0$ if $|f'(r)| < 1$ and $\{e_n\}$ diverges if $|f'(r)| > 1$. That is, a_n converges to r if $|f'(r)| < 1$, and $\{a_n\}$ does not converge to r if $|f'(r)| > 1$.

Consider the question of finding a solution to $g(x) = x + \ln(x) = 0$. There are several ways to reformulate this problem as a fixed point problem, that is to solve an equation of the form $f(x) = x$. Exercises 11 and 12 show that the Fixed Point Theorem can be used to explain why some reformulations are more useful than others for finding a root of $g(x) = 0$.

11.[M]

(a) Let $f_1(x) = -\ln(x)$. Verify that $g(x) = 0$ and $f_1(x) = x$ have the same solution.

(b) Compute $|f_1'(r)|$ where r is close to the solution to $g(x) = 0$. What does this tell you about the sequence with $a_n = f_1(a_{n-1})$?

(c) Let $x_0 = 0.5$ and compute x_1, x_2, x_3, and x_4 using $x_n = f_1(x_{n-1})$. Why can't you compute x_5?

12.[M]

(a) Let $f_2(x) = e^{-x}$. Verify that $g(x) = 0$ and $f_2(x) = x$ have the same solution.

(b) Compute $|f_2'(r)|$ where r is close to the solution to $g(x) = 0$. What does this tell you about the sequence with $a_n = f_2(a_{n-1})$?

(c) Let $x_0 = 0.5$ and compute x_1, x_2, x_3, and x_4 using $x_n = f_2(x_{n-1})$. What happens as $n \to \infty$?

The function $g(x) = x^2 - 2x - 3$ has two roots: $x = 3$ and $x = -1$. In Exercises 13 to 15 we will explore three different ways to use fixed-point iterations to find these roots.

13.[M]

(a) Show that solving $g(x) = 0$ is equivalent to finding a fixed point of $f_1(x) = \sqrt{2x+3}$.

(b) Compute $|f_1'(r)|$, where r is close to either root of $g(x) = 0$. What does this tell you about the sequence $a_n = f_1(a_{n-1})$?

(c) Let $x_0 = 0.5$ and compute x_1, x_2, x_3, and x_4 using $x_n = f_1(x_{n-1})$. What happens $\lim_{n\to\infty} x_n$?

14.[M]

(a) Show that solving $g(x) = 0$ is equivalent to finding a fixed point of $f_2(x) = 3/(x-2)$.

(b) Compute $|f_2'(r)|$, where r is close to either root of $g(x)$. What does this tell you about the sequence $a_n = f_2(a_{n-1})$?

(c) Let $x_0 = 0.5$ and compute x_1, x_2, x_3, and x_4 using $x_n = f_2(x_{n-1})$. What happens $\lim_{n\to\infty} x_n$?

15.[M]

(a) Show that solving $g(x) = 0$ is equivalent to finding a fixed point of $f_3(x) = \frac{1}{2}(x^2 - 3)$.

(b) Compute $|f_3'(r)|$, where r is close to the solutions to $g(x) = 0$. What does this tell you about the sequence $a_n = f_3(a_{n-1})$?

(c) Let $x_0 = 0.5$ and compute x_1, x_2, x_3, and x_4 using $x_n = f_3(x_{n-1})$. What happens $\lim_{n\to\infty} x_n$?

(d) Which of these three methods is the best way to find the solutions to $g(x) = 0$?

Exercises 16 and 17 will be used in Exercises 18 and 19.

16.[M] Find $\lim_{x\to 0} \frac{\tan(x)-x}{2x-\sin(2x)}$.

17.[M] Find $\lim_{x\to 0} \frac{\tan(x)-x}{x-\sin(x)}$.

18.[M] Let P_n be the perimeter of a regular polygon with n sides that circumscribes a circle of radius 1. Similarly, let p_n be the perimeter of an inscribed regular polygon of n sides. When n is large, which is the better estimate of the perimeter of the circle? To decide, examine the limit of $\frac{P_n-2\pi}{2\pi-p_n}$. (Form an opinion before you calculate.) HINT: See Exercise 16.

19.[M] Let A_n be the perimeter of a regular polygon with n sides that circumscribes a circle of radius 1. Similarly, let a_n be the perimeter of an inscribed regular polygon of n sides. When n is large, which is the better estimate of the perimeter of the circle? To decide, examine the limit of $\frac{A_n-\pi}{\pi-a_n}$. (Form an opinion before you calculate.) HINT: See Exercise 17.

20.[M] (Contributed by Frank Saminiego.) Assume that a_i and b_i, $0 \leq i \leq n$, are positive and the ratios a_i/b_i increase as a function of the index i. (That is, $a_0/b_0 < a_1/b_1 < \cdots < a_n/b_n$.) Then it is known that

$$f(x) = \frac{\sum_{i=0}^n a_i x^i}{\sum_{i=0}^n b_i x^i}$$

is an increasing function for $x > 0$. This fact is used in the statistical theory of reliability.
Verify the assertion for (a) $n = 1$ and (b) $n = 2$. HINT: Show that $f'(x) > 0$.

21.[C] Let $u(n)$ be the number of ways of tiling a 3 by n rectangle with 1 by 3 dominoes.

(a) Find $u(1)$, $u(2)$, and $u(3)$.

(b) Find a recursive definition of the function u.

(c) Use (b) to find $u(10)$.

22.[C] In the study of the hydrogen atom, one meets the integral

$$\int_0^\infty r^n e^{-kr}\, dr$$

Here n is a non-negative integer and k a positive constant. Show that it equals $n!/k^{n+1}$. HINT: First find the value for $n = 0$. Then use integration by parts. NOTE: $n!$ is the factorial of n, $n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$

In Exercise 23 the binomial distribution for the case when the number of successes is small leads to the Poisson distribution in CIE 15 in Chapter 12. Exercise 24 shows

how the bell curve arises from the situation when the number of successes is large, near the most likely outcome. In both cases, one determines certain limits.
23.[C] The following limit occurs in the elementary theory of probability:

$$\lim_{N\to\infty} \frac{N!}{n!(N-n)!}\left(\frac{k}{N}\right)^n\left(1-\frac{k}{N}\right)^{N-n},$$

where n is a fixed positive integer and k is a positive constant. Show that the limit is

$$\frac{k^n e^{-k}}{n!}.$$

24.[C]

DOUG/SHERMAN: Need to be sure to find (or re-write) this exercise.

25.[C] Let the mass of a bacteria culture at the end of n intervals of time be C_n. If there is adequate nutrients, it doubles each interval, that is, $C_{n+1} = 2C_n$. When the population is large it does not reproduce as quickly. In that case, according to the Verhulst model (1848) there is a constant K such that

$$C_{n+1} = \frac{2}{1+\frac{C_n}{K}}C_n.$$

Show that $\lim_{n\to\infty} C_n = K$. HINT: Set $R_n = 1/C_n$.

26.[C] The recursion $P_{n+1} = re^{\frac{-P_n}{K}}P_n$ was introduced by W. E. Ricker in 1954 in the study of fish populations. P_n denotes the fish population at the n^{th} time interval, while r and K are constants, with r being the maximum reproduction rate. Examine the recursion when $K = 10,000$, $P_0 = 5,000$ and (a) $r = 20$ and (b0 $r = 10$. As you will see, the highly unpredictable sequence $\{P_n\}$ depends dramatically on r. Such sensitivity to r is an early example of "chaos."
References: F. C. Hoppensteadt and C. S. Peskin, *Mathematics in Medicine and the Life Sciences*, Springer, NY 1991 (p. 21)
W. E. Ricker, *Stock and Prerecruitment*, J. Fish Res. Bd., Canada, 11 (1954), pp. 559–623.

Calculus is Everywhere # 13
Hubbert's Peak

SHERMAN: This CIE needs an ending. Did you have something in mind?

In the CIE for Chapter 6, Hubbert combined calculus concepts with counting squares. Later he developed specific functions and used more techniques of calculus in "Oil and Gas Supply Modeling", NBS Special Publication 631, U.S. Department of Commerce, National Bureau of Standards, May, 1982. (NOTE: NBS is now the National Institute of Standards and Technology (NIST).)

In his approach, Q_∞ denotes the total amount of oil reserves a the time oil is first extracted and t, time. The derivative dQ/dt is the rate at which oil is extracted. $Q(t)$ denotes the amount extracted up to time t. Hubbert assumes $Q(0) = 0$ and $(dQ/dt)(0) = 0$. He wants to obtain a formula for $Q(t)$.

"The curve of dQ/dt versus Q between 0 and Q_∞ can be represented by the Maclaurin series

$$\frac{dQ}{dt} = c_0 + c_1 Q + c_2 Q^2 + c_3 Q^3 + \cdots .$$

Since, when $Q = 0$, $dQ/dt = 0$, it follows that $c_0 = 0$.

"Since the curve must return to 0 when $Q = Q_\infty$, the minimum number of terms that permit this, and the simplest form of the equation, becomes the second degree equation

$$\frac{dQ}{dt} = c_1 Q + c_2 Q^2.$$

By letting $a = c_1$ and $b = -c_2$, this can be rewritten as

$$\frac{dQ}{dt} = aQ - bQ^2.$$

"Since when $Q = Q_\infty$, $dQ/dt = 0$,

$$aQ_\infty - bQ_\infty^2 = 0$$

or

$$b = \frac{a}{Q_\infty}$$

and

$$\frac{dQ}{dt} = a\left(Q - \frac{Q^2}{Q_\infty}\right). \tag{C.13.1}$$

"This is the equation of a parabola .... The maximum value occurs when the slope is 0, or when

$$a - \frac{2a}{Q_\infty} Q = 0,$$

or

$$Q = \frac{Q_\infty}{2}.$$

"It is to be emphasized that the curve of dQ/dt versus Q does not have to be a parabola, but that a parabola is the simplest mathematical form that this curve can assume. We may regard the parabolic form as a sort of idealization for all such actual data curves."

He then points out that

$$\frac{dQ/dt}{Q} = a - \frac{a}{Q_\infty}.$$

"This is the equation of a straight line. The plotting of this straight line gives the values for its constraints Q_∞ and a."

Because the rate of production, dQ/dt, and the total amount produced up to time t, namely, $Q(t)$ and observable, the line can be drawn and its intercepts read off the graph. (The two intercepts are $(0, a)$ and $(Q_\infty, 0)$.)

Hubbert then compares this with actual data, which it approximates fairly well.

Equation (C.13.1) can be written as

$$\frac{dQ}{dt} = \frac{a}{Q_\infty} Q\left(Q_\infty - Q\right),$$

which says, "The rate of production is proportional both to the amount already produced and to the reserves $Q_\infty - Q$." This is related to the logistic equation describing bounded growth. (See Exercises 35 to 37 in Section 5.6.)

This approach, which is more formal than the one in CIE 8 at the end of Chapter 6, concludes that as Q approaches Q_∞, the rate of production will decline, approaching 0. This means the Age of Oil will end.

Chapter 11

Series

How is $\sin(\theta)$ computed? One approach might be to draw a right triangle with one angle θ, as in Figure 11.0.1. Then measure the lengths of the opposite side b and the length of the hypotenuse c and calculate b/c ("opposite over hypotenuse"). (Try it!) You are lucky if you get even two decimal places correct. Clearly this method cannot give the many decimal places a calculator displays for $\sin(\theta)$, even if you draw a gigantic triangle.

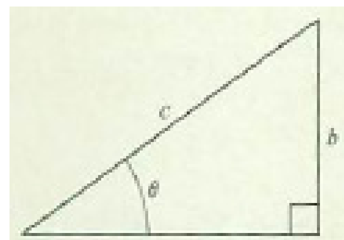


Figure 11.0.1:

One way to obtain this accuracy will be described in Chapter 12. The idea is to use polynomials to evaluate important functions like $sin(x)$, $\arctan(x)$, e^x, and $\ln(x)$ to as many decimal places as we please. For instance, when $|x| \leq 1$, the polynomial

$$x - \frac{x^3}{6} + \frac{x^5}{120}$$

approximates $\sin(x)$ with an error less than 0.0002 (provided angle x is given in radians). This means the estimate will be correct to at least three decimal places for angles less than about 57°.

1 radian $= \frac{180°}{\pi} \approx 57.29578°$

Such an estimate has other uses than simply evaluating a function. Consider the definite integral

$$\int_0^1 \frac{\sin(x)}{x}\, dx.$$

The Fundamental Theorem of Calculus is useless here since $\sin(x)/x$ does not have an elementary antiderivative. But, we can evaluate

$$\int_0^1 \frac{x - \frac{x^3}{6} + \frac{x^5}{120}}{x}\, dx = \int_0^1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right)\, dx.$$

Since the integrand is now a polynomial, the Fundamental Theorem of Calculus can be used to obtain the estimate

$$\left(x - \frac{x^3}{18} + \frac{x^5}{600}\right)\Big|_0^1 = 1 - \frac{1}{18} + \frac{1}{600} \approx 0.94611$$

which gives $\int_0^1 \sin(x)/x \; dx$ to three decimal places.

An overview of this chapter, and Chapter 12, is given at the end of Section 11.1.

11.1 Informal Introduction to Series

The main goal of this chapter and the next is to show how polynomials can be used to approximate functions that are not polynomials. Table 11.1.1 shows some of the formulas we will obtain.

The larger n is, the better the approximation, as long as we keep x in the appropriate interval.

Function	Approximating Polynomial	Interval
$\frac{1}{1-x}$	$1+x+x^2+x^3+\cdots\vert x^n$	$\vert x\vert < 1*$
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}$	all x
$\ln(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}$	$\vert x\vert \leq 1$
$\sin(x)$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	all x

Table 11.1.1:

Example 1 illustrates the use of such polynomials.

EXAMPLE 1 Use the approximations in Table 11.1.1 to estimate $\sqrt{e} = e^{1/2}$.

SOLUTION By the first row of the table, for each positive integer n,

$$1+\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^2}{2!}+\frac{\left(\frac{1}{2}\right)^3}{3!}+\cdots+\frac{\left(\frac{1}{2}\right)^n}{n!}$$

is an estimate of $e^{1/2}$. Let us compare some of these estimates, keeping in mind that as n increases we expect the estimates to improve. The sums in the

n	$1+\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^2}{2!}+\frac{\left(\frac{1}{2}\right)^3}{3!}+\cdots+\frac{\left(\frac{1}{2}\right)^n}{n!}$	Decimal Form	Sum
1	$1+\frac{1}{2}$	$1+0.5$	1.5
1	$1+\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^2}{2!}$	$1+0.5+0.125$	1.625
1	$1+\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^2}{2!}+\frac{\left(\frac{1}{2}\right)^3}{3!}$	$1+0.5+0.125+0.02083\ldots$	$1.64583\ldots$
4	$1+\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^2}{2!}+\frac{\left(\frac{1}{2}\right)^3}{3!}+\frac{\left(\frac{1}{2}\right)^4}{4!}$	$1+0.5+0.125+0.02083+0.00260\ldots$	$1.6484375\ldots$

Table 11.1.2:

rightmost column form a sequence that converges to $e^{1/2} \approx 1.64872$, to five decimal places. The estimate with $n = 4$ is correct to three decimal places. $\diamond$

There is little point in making an estimate if we have no idea about the size of its "error" — the difference between an estimate and the number we are estimating. We will focus on two closely related questions.

1. How can we estimate the "error"?

2. How can we choose n to achieve a prescribed accuracy, say, to 10 decimal places?

Calculus delights in resolving such battles.

Example 1 depicts a battle between two forces. On the one hand, the individual summands are getting very small — shrinking toward 0; so their sums may not get very large. On the other hand, there are more and more of summands in each estimate; so their sums might become arbitrarily large.

In Example 1 the first force is stronger, and the sums — no matter how many summands we take — stay less than $\sqrt{e} \approx 1.64872$. But, in Example 2 the sums behave quite differently.

EXAMPLE 2 What happens to sums of the form

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \tag{11.1.1}$$

as the integer n gets larger and larger? Will they stay less than some fixed number or will they get arbitrarily large, eventually passing 100, then 1,000, and so on?

SOLUTION Table 11.1.3 lists values of (11.1.1) for n up through 5.

n	$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$	*Decimal Form (7 places)*
1	$\frac{1}{\sqrt{1}}$	1.0000000
2	$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$	1.7071068
3	$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}$	2.2844571
4	$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}$	2.7844571
5	$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}}$	3.2316706

Table 11.1.3:

As of November 2010, the fastest computer could perform 2.57×10^{15} floating-point computations per second. Source: `http://Top500.org/`.

These computations do not answer the question: What will happen to the sums as n becomes arbitrarily large? In fact, even if we calculated the values of $1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n}$ all the way to $n = 1,000,000$, we still would not know the answer. Why? Because we can't be sure what happens to the sums when n is a billion or a quadrillion or larger. Do the sums get arbitrarily large or do they stay below some fixed number? No computer, even the world's fastest supercomputer, can answer that question.

However, an algebraic insight helps us answer the question. Observe that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$$

has n summands and that the smallest of them is $1/\sqrt{n}$. Therefore (11.1.1) is at least as large as

$$\underbrace{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}}_{n \text{ summands}} = n\left(\frac{1}{\sqrt{n}}\right) = \sqrt{n}.$$

Thus $1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n}$ is at least as large as $\sqrt{n}$. (In fact, when $n \geq 2$, the sum is larger than $\sqrt{n}$.)

As n gets larger and larger, $\sqrt{n}$ grows arbitrarily large. For $n = 1,000,000$, for instance, we have

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{1,000,000}} \geq \sqrt{1,000,000} = 1,000.$$

So the sums of the form $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$ also become arbitrarily large. They do *not* stay less than some fixed number. $\diamond$

WARNING *(Traveler's Advisory)* In both Examples 1 and 2, the individual summands form sequences that converge to 0:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n}{n!} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Yet in the first case, the sums stay less than $\sqrt{e}$, while in the second the sums grow arbitrarily large. This contrast shows that we must be careful when dealing with such sums, especially since they may play a role in approximating important functions.

Summary

THINGS TO COME

In most of this chapter the summands are constants. In Chapter 12 the summands involve a variable.

§11.2 introduces the notion of a "series" as a sequence formed by adding up more and more terms from a sequence of numbers.

§§11.3–11.6 develop methods for determining when these sums converge to a number and, if they do, how big the error is when you use a particular finite sum to estimate that number.

§§12.1 and 12.2 build on Section 5.4 and apply series in various ways. Review Taylor polynomials (5.4) before reading this section.

§§12.3–12.4 shows how a series approximating one function can be used to find a series approximating a related function

§§12.5-12.6 develops complex numbers and uses thems to show that the functions $\sin(x)$ and $\cos(x)$ are intimately related to the exponential function e^x. This relation is used in physics, engineering, and mathematics.

§12.7 introduces series that are the sum of terms of the form $a_n \sin(nx)$ and $b_n \cos(nx)$ for $n = 1, 2, 3, \ldots$.

As you work through Chapters 11 and 12, check back to this outline from time to time. It will help you keep track of what you are doing, and why.

EXERCISES for Section 11.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] Estimate $\sqrt[3]{e} = e^{1/3}$ by using the following approximations with $x = \frac{1}{3}$.

(a) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

(b) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

2.[R] Estimate $1/e = e^{-1}$ using the following approximations with $x = -1$.

(a) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

(b) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!}$

3.[R] As shown in Section 5.4 the polynomial $x - x^3/6$ is an excellent approximation to $\sin(x)$ (angle measured in radians) for $|x| \le \frac{1}{2}$. Using a calculator or computer, fill in Table 11.1.4 to seven decimal places.

x	$\sin(x)$	$x - \frac{x^3}{6}$	$\sin(x) - \left(x - \frac{x^3}{6}\right)$
0.1			
0.2			
0.3			
0.4			
0.5			

Table 11.1.4:

NOTE: The results should illustrate that this estimate is accurate to at least three decimal places, for these values of x.

4.[R] The polynomial $x - x^3/3! + x^5/5!$ is an excellent approximation to $\sin(x)$ (angle in radians) for $|x| \le 1$. Using a calculator or computer, compute the approximation to at least seven decimal places:

(a) $\sin(1)$,

(b) $x - x^3/3! + x^5/5!$ when $x = 1$.

(c) To how many decimal places do these results agree?

5.[R] Estimate $\int_{1/2}^{1}(e^x - 1)/x\, dx$ by approximating e^x by the polynomial

(a) $1 + x + \frac{x^2}{2!}$,

(b) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

(c) The exact value of this definite integral, to seven decimal places, is 0.7477507. To how many decimal places do each of these results agree with the exact value?

6.[R] Estimate $\int_{1/4}^{1/2} \sin(x)/x\, dx$ by approximating $\sin(x)$ by the polynomial

(a) x.

(b) $x - \frac{x^3}{3!}$.

(c) $x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

(d) The exact value of this definite integral, to seven decimal places, is 0.2439738. To how many decimal places do each of these results agree with the exact value?

7.[R]

(a) The polynomial $x - x^2/2 + x^3/3 - \cdots + (-1)^{n-1}x^n/n$, $|x| \leq 1$, is a good estimate of $\ln(1 + x)$ when n is large. So, to estimate $\ln(1.5)$, which is $\ln(1 + 0.5)$, we use the polynomial with x replaced by $\frac{1}{2}$. Use a calculator or computer to fill in Table 11.1.5.

n	$\frac{1}{2} - \left(\frac{1}{2}\right)^2/2 + \left(\frac{1}{2}\right)^3/3 - \cdots + (-1)^{n-1}\left(\frac{1}{2}\right)^n/n$	Decimal Form
1		
2		
3		
4		
5		

Table 11.1.5:

(b) Use your calculator or a computer to compute $\ln(1.5)$.

(c) What is the error between this approximation and the result for $n = 5$ in Table 11.1.5?

8.[R] (See Exercise 7.)

(a) To estimate $\ln(0.5)$, write it as $\ln(1 + (\frac{-1}{2}))$. Fill in Table 11.1.6.

n	$-\left(\frac{-1}{2}\right) - \left(\frac{-1}{2}\right)^2/2 - \left(\frac{-1}{2}\right)^3/3 - \cdots - \left(\frac{-1}{2}\right)^n/n$	Decimal Form
1		
2		
3		
4		
5		

Table 11.1.6:

(b) Use your calculator or a computer to compute $\ln(1.5)$.

(c) What is the error between this approximation and the result for $n = 5$ in Table 11.1.5?

9.[M] One way to approximate $\ln(2)$ is to write it as $\ln(1+1)$ and use a polynomial in Exercise 7 that approximates $\ln(1 + x)$ with $x = 1$. Another way is to note that $\ln(2) = -\ln(0.5)$ and use the approach of Exercise 8. Using the polynomial approximation of degree 5 ($n = 5$) in both cases, decide which gives the better estimate.

10.[C] What happens to sums of the form

$$\frac{1}{\sqrt[3]{1}} + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{n}}$$

as n gets larger? Explore and explain.

11.[M]

(a) Using results from Section 1.4, show that, for $x \neq 1$,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1}{1-x} - \frac{x^n}{1-x}. \qquad (11.1.2)$$

(b) Now assume that $|x| < 1$. Then x^n approaches 0 as n increases (as was shown in Section 10.1). Thus, for $|x| < 1$ and large n, $1 + x + x^2 + \cdots + x^{n-1}$ is a polynomial approximation for the function $1/(1 - x)$.

(c) Compute $1 + x + x^2 + \cdots + x^{n-1}$ for $n = 6$ and $x = 0.3$. How much does this differ from $1/(1 - x)$ for $x = 0.3$?

(d) The same as (c), with $x = -0.9$.

Exercises 12 and 13 use (11.1.2) to derive polynomial approximations to $\ln(1+x)$ and $\arctan(x)$. These two problems both start from the same idea. We begin by expressing (11.1.2) in the form

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^{n-1} + \frac{t^n}{1-t} \qquad (t \neq 1).$$

Replace t with $-t$, getting

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^{n-1}t^{n-1} + \frac{(-1)^n t^n}{1+t} \qquad (t \neq -1). \qquad (11.1.3)$$

12.[C] This exercise derives the sequence of polynomial approximations to $\ln(1+x)$ listed in Table 11.1.1 on page 923.

(a) Integrate both sides of (11.1.3) over the interval from 0 to x, $x > 0$, to show that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n} + (-1)^n \int_0^x \frac{t^n}{1+t}\,dt.$$

(b) Show that for $0 \leq x \leq 1$, $\int_0^x (t^n/(1+t))\,dt$ approaches 0 as n increases. Hint: $1/(1+t) \leq 1$ for $t \geq 0$.

13.[C] This exercise obtains a sequence of polynomials that approximate $\arctan(x)$ for $|x| \leq 1$ and shows one way of computing π. The key is that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$. To begin, replace t by $-t^2$ in (11.1.2) to obtain

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^{n-1}t^{2n-2} + \frac{(-1)^n t^{2n}}{1+t^2} \qquad \text{(for all } t\text{)}. \quad (11.1.4)$$

(a) Consider only $0 \leq x \leq 1$. Integrate both sides of (11.1.4) over $[0, x]$ to show that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt. \qquad (11.1.5)$$

(b) Show that for fixed x, $0 < x < 1$, the integral in (11.1.3) approaches 0 as $n \to \infty$.

(c) Use the polynomial in (a), with $n = 5$ (so its degree is 9) to estimate arctan(1).

(d) Use the result in (c) to estimate π. HINT: $\arctan(1) = \frac{\pi}{4}$

14.[C] In this exercise we will see what happens to sums of the form

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

as n increases. Do these sums get arbitrarily large or do they approach some number?

n	$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$	Sum, as fraction	Sum, as decimal
1			
2			
3			
4			
5			

Table 11.1.7:

(a) Fill in at least 5 rows of Table 11.1.7. Add more rows if you wish.

(b) On the basis of your computations, what do you think happens to the sums as n increases. HINT: If you don't see a pattern, go up to $n = 10$.

(c) Justify your opinion in (b).

15.[C]

(a) Use the polynomial in (11.1.5), with $n = 5$, to estimate $\arctan\left(\frac{1}{2}\right)$ in radians. Then, translate the answer into degrees.

(b) Use the result in (a) to estimate arctan(2) in radians. HINT: For positive x, what is the relation between $\arctan(1/x)$ and $\arctan(x)$?

(c) Draw a right triangle with one leg 20 cm long and the other 10 cm; use it and a protractor to estimate arctan(2).

(d) What does your calculator or computer give as an estimate of arctan(2)?

(e) To how many decimal places does the estimate in (b) agree with the value found in (d)? To how many decimal places does the measurement in (c) agree with the value found in (d)?

11.2 Series

The goal of this section is to introduce sequences formed by adding up more and more terms of a given sequence.

Series

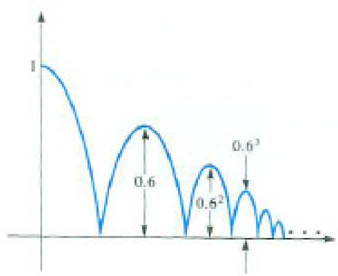


Figure 11.2.1:

Consider a tennis ball that is dropped from a height of 1 meter. It rebounds 0.6 meter. It continues to bounce, and each fall is 60% as high as the previous fall. (See Figure 11.2.1.) What is the total distance the ball falls?

The third fall is $(0.6)^2$ meter, the next is $(0.6)^3$ meter, and so on. In general, the n^{th} time the ball falls, it falls a distance $(0.6)^{n-1}$ meter. While it is clear this geometric sequence converges to zero, we are more interested in the question:

"What happens to the sum $\quad 1+0.6+(0.6)^2+\cdots+(0.6)^n \quad$ as $n \to \infty$?"

Example 1 explores this question.

Similar sums arise in many applications. Exercise 30 is an application to medicine and Exercise 31 presents an example from economics.

EXAMPLE 1 Given the geometric progression 1, 0.6, $(0.6)^2$, $(0.6)^3$, $\ldots$, form a new sequence $\{S_n\}$ as follows:

$$\begin{aligned} S_1 &= 1, \\ S_2 &= 1+0.6, \\ S_3 &= 1+0.6+(0.6)^2, \end{aligned}$$

and, in general,

$$S_n = 1+0.6+(0.6)^2+\cdots+(0.6)^{n-1}.$$

Each S_n is the sum of n terms of the sequence $\{a_n\}$ with $a_n = 0.6^n$ for $n = 0$, 1, 2, $\ldots$. Does the sequence $\{S_n\}$ converge or diverge? If it converges, what is the limit?

SOLUTION To examine the behavior of S_n as $n \to \infty$, note that each S_n is the sum of the first n terms in a geometric sequence. So

$$S_n = \frac{1-(0.6)^n}{1-0.6}$$

and so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-(0.6)^n}{1-0.6} = \frac{1}{1-0.6} = 2.5.$$

◇

The rest of this section expands upon the ideas introduced in Example 1.

Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a sequence. From this sequence a new sequence $S_1, S_2, S_3, \ldots, S_n, \ldots$ can be formed:

$$\begin{aligned} S_1 &= a_1 = \sum_{k=1}^{1} a_k, \\ S_2 &= a_1 + a_2 = \sum_{k=1}^{2} a_k, \\ S_3 &= a_1 + a_2 + a_3 = \sum_{k=1}^{3} a_k, \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{\infty} a_k. \end{aligned}$$

The sequence of sums, $S_1, S_2, S_3, \ldots, S_n, \ldots$, is called the **series** obtained from the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$. It can also be defined by the recursion, $S_{n+1} = S_n + a_{n+1}$.

Traditionally, $\{S_n\}$ is referred to as "the series whose n^{th} term is a_n." Common notations for the sequence $\{S_n\}$ are $\sum_{k=1}^{\infty} a_k$ and $a_1 + a_2 + a_3 + \cdots + a_k \cdots$. The sum

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is called a **partial sum** or the n^{th} **partial sum**. If the sequence of partial sums of a series converges to L, then L is called the **sum** of the series and the series is said to be **convergent**. We write

$$\lim_{n\to\infty} S_n = L.$$

Frequently one writes $L = a_1 + a_2 + \cdots + a_n + \cdots$. Remember, however, that we do *not* add an infinite number of terms; we take the limit of finite sums. A series that is not convergent is called **divergent**.

Only finitely many summands are ever added up.

A Note on Notation Starting with the sequence $a_1, a_2, \ldots, a_n, \ldots$, we form a new sequence, $S_1, S_2, \ldots, S_n, \ldots$, whose terms are the partial sums $S_1 = a_1$, $S_2 = a_1 + a_2, \ldots, S_n = a_1 + a_2 + \cdots + a_n$. The symbol

$$\sum_{k=1}^{\infty} a_k$$

is short for this sequence $S_1, S_2, \ldots, S_n, \ldots$. If the sequence of partial sums converges to a number L, we also write

$$\sum_{k=1}^{\infty} a_k = L.$$

The symbol $\sum_{k=1}^{\infty} a_k$ has two meanings.

So the symbol $\sum_{k=1}^{\infty} a_k$ stands for two different concepts: a sequence of partial sums and also, if that sequence converges, for its limit. This limit is called the "sum" of the series.

So, in Example 1, we investigated the *series*

$$\sum_{k=1}^{\infty} 0.6^{k-1},$$

namely, the sequence of partial sums 1, $1+0.6$, $1+0.6+0.6^2$, …, $1+0.6+0.6^2+\cdots+(0.6)^{n-1}$. This sequences converges to 2.5. That permits us to write

$$\sum_{k=1}^{\infty}(0.6)^{k-1} = 2.5,$$

which says, "The series $\sum_{k=1}^{\infty}(0.6)^{k-1} = 2.5$ converges to the number 2.5." We also say, for the sake of brevity, "Its sum is 2.5."

Just as a sequence need not start with a_1, a series can start with any term, such as a_0 or a_k, and we would write $\sum_{k=0}^{\infty} a_k$ or $\sum_{i=1}^{\infty} a_i$ or $\sum_{j=k}^{\infty} a_j$. Notice that there is nothing special about the index for a series. The most common indices are n, k, j, and i.

Geometric Series

Example 1 concerns the series whose n^{th} term is $(0.6)^{n-1}$:

$$S_n = 1 + 0.6 + 0.6^2 + \cdots + 0.6^{n-1}.$$

Geometric sums with a finite number of terms are discussed in Section 5.4.

It is a special case of a geometric series, which will now be defined.

DEFINITION *(Geometric Series)* Let a and r be real numbers. The series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

is called the 'geometric series with initial term a and ratio r.

The series in Example 1 is a geometric series with initial term 1 and ratio 0.6.

Theorem 11.2.1. *If $-1 < r < 1$, the geometric series*

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \qquad \textit{converges to } \frac{a}{(1-r}.$$

Proof

See Exercise 11 in Section 11.1.

Let S_n be the sum of the first n terms: $S_n = a + ar + \cdots + ar^{n-1}$. The formula for the finite geometric sum is $S_n = \frac{a(1-r^n)}{1-r}$. Since $-1 < r < 1$, the individual terms converge to zero: $\lim_{n\to\infty} ar^n = 0$. Thus

See also Exercise 28.

$$\lim_{n\to\infty} S_n = \frac{a}{1-r},$$

proving the theorem. •

The series in Example 1 is a geometric series with first term a and ratio $r = 0.6$. It converges and has the sum

$$\frac{1}{1-0.6} = \frac{1}{0.4} = 2.5.$$

The n^{th} Term Test for Divergence

Theorem 11.2.1 says nothing about geometric series in which $r \geq 1$ or $r \leq -1$. The next theorem, which concerns series in general, not just geometric series, will be useful in settling this case.

Theorem (n^{th}-Term Test for Divergence.)**.** *If* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $a_1 + a_2 + \cdots + a_n + \cdots$ *diverges. (The same conclusion holds if* $\{a_n\}$ *has no limit.)*

We take an indirect approach.

Proof

Assume that the series $a_1 + a_2 + \cdots + a_n + \cdots$ converges. Since S_n is the sum of $a_1 + a_2 + \cdots + a_n$, while S_{n-1} is the sum of the first $n-1$ terms, it follows that $S_n = S_{n-1} + a_n$, or

$$a_n = S_n - S_{n-1}.$$

Because we have assumed the series converges, let $S = \lim_{n\to\infty} S_n$. Then we also have $S = \lim_{n\to\infty} S_{n-1}$, since S_{n-1} runs through the same numbers as S_n. Thus

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} (S_n - S_{n-1}) \\ &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\ &= S - S \\ &= 0. \end{aligned}$$

This proves the theorem. •

If a series converges, its n^{th}-term must approach 0.

The n^{th}-Term Test for Divergence implies that if $a \neq 0$ and $r \geq 1$, the geometric series

$$a + ar + \cdots + ar^{n-1} + \cdots$$

diverges. For instance, if $r = 1$,

$$\lim_{n\to\infty} ar^n = \lim_{n\to\infty} a1^n = a,$$

which is not 0. If $r > 1$, then r^n gets arbitrarily large as n increases; hence $\lim_{n\to\infty} ar^n$ does not exist. Similarly, if $r \le -1$, $\lim_{n\to\infty} ar^n$ does not exist. The above results and Theorem 11.2.1 can be summarized by this statement: The geometric series

$$\sum_{i=1}^{\infty} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots .$$

Warning: Even if the n^{th} term approaches 0, the series still can diverge.

for $a \neq 0$, converges if and only if $|r| < 1$.

The n^{th}-Term Test for Divergence tells us that *if* the series $a_1 + a_2 + a_3 + \cdots$ converges, *then* a_n approaches 0 as $n \to \infty$. The converse of this statement is *not true*. If a_n approaches 0 as $n \to \infty$, it does *not* follow that the series $a_1 + a_2 + a_3 + \cdots$ converges. Be careful to make this distinction.

Recall the series

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$$

discussed in Example 2 in Section 11.1. Even though its n^{th} term approaches 0 as $n \to \infty$, the sums get arbitrarily large. The n^{th} term approaches 0 so "slowly" that the sums S_n get arbitrarily large.

The harmonic series was so named by the Greeks because of the role of $1/n$ in musical harmony.

Nicole Oresme, 1323–1382, one of the most influential philosophers of the Middle Ages, `http://en.wikipedia.org/wiki/Nicole_Oresme`

In the next example, the n^{th} term approaches 0 much faster than $1/\sqrt{n}$ does. Still, the series diverges. The series in this example is called the **harmonic series**. The argument that it diverges is due to the French mathematician Nicolas of Oresme, who presented it about the year 1360.

EXAMPLE 2 Show that the harmonic series $1/1 + 1/2 + \cdots + 1/n + \cdots$ diverges.

SOLUTION Collect the summands in longer and longer groups. Except for the first two terms, each group contains twice the number of summands as it predecessor:

$$1 + \underbrace{\frac{1}{2}}_{\text{1 term}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\text{2 terms}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\text{4 terms}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}}_{\text{8 terms}} + \cdots .$$

The sum of the terms in each group is at least $\frac{1}{2}$. For instance,

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

and

$$\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16} = \frac{8}{16} = \frac{1}{2}$$

Since the repeated addition of $\frac{1}{2}$'s produces sums as large as we please, the series diverges. ◇

An important moral: The n^{th}-term test is only a test for divergence.

If the series $a_1 + a_2 + \cdots + a_n + \cdots$ converges, it *follows* that $a_n \to 0$. However, if $a_n \to 0$, it *does not follow* that $a_1 + a_2 + \cdots + a_n + \cdots$ converges. Indeed, there is no general, practical rule for determining whether a series converges or diverges. Fortunately, a few rules suffice to decide on the convergence or divergence of the most common series. They will be presented in this chapter.

Because convergence or divergence of a series is decided by looking at the convergence or divergence of the sequence of partial sums, the basic properties for sequences are also true for series.

Exercise 36 asks for the proof.

Theorem 11.2.2. *A. If $\sum_{i=1}^{\infty} a_i$ is a convergent series with sum L and if c is a number, then $\sum_{i=1}^{\infty} ca_i$ is convergent and has the sum cL.*

B. If $\sum_{k=1}^{\infty} b_i$ is a convergent series with sum M, then $\sum_{k=1}^{\infty}(a_n + b_n)$ is a convergent series with sum $L + M$.

Front ends do not affect convergence.

Keep in mind that you can disregard any finite number of terms when deciding whether a series is convergent or divergent. If you delete a finite number of terms from a series and what is left converges, then the series you started with converges. Another way to look at this is to note that a "front end," $a_1 + a_2 + \cdots + a_n$. does not influence convergence or divergence. It is rather a "tail end," $a_{n+1} + a_{n+2} + \cdots$ that matters. The sum of the series is the sum of any tail end plus the sum of the corresponding front end; that is, for any positive integer m,

$$\sum_{k=1}^{\infty} a_k = \underbrace{\sum_{k=1}^{m} a_k}_{\text{front end}} + \underbrace{\sum_{k=m+1}^{\infty} a_k}_{\text{tail end}}.$$

Suppose that $\sum_{i=1}^{\infty} p_i$ is a series with positive terms and you can show that there is a number B such that every partial sum $S_1 = p_1$, $S_2 = p_1 + p_2$, $\ldots$, $S_n = p_1 + p_2 + \cdots + p_n$, is less than or equal to B. By Theorem 10.1.1 of Section 10.1, they have a limit L, which is less than or equal to B. (See Figure 11.2.2.) This means that $\sum_{k=1}^{\infty} p_i$ is convergent (and its sum is less than or equal to B). This observation will be useful in establishing the convergence of a series of non-negative terms, even though it does not tell us the exact sum of the series.

Figure 11.2.2:

A similar statement holds for the series $\sum_{k=1}^{\infty} a_i$ in which $a_i \leq 0$ for all n. If there is a number A such that each partial sum is greater than or equal to A, then the series converges and its sum is greater than or equal to A.

Example 3 introduces a series that is representative of many series that arise in the study of $\sin(x)$, $\cos(x)$, and e^x.

EXAMPLE 3 Does the series defined by $\sum_{k=0}^{\infty} \frac{2^k}{k!}$ converge or diverge?
SOLUTION First, note that the first index is $k = 0$, not $k = 1$. This has no bearing on the convergence or divergence of this series (it's part of the front end), but it does affect the value of the series (assuming it converges).

Define $a_k = 2^k/k!$ for $k = 0, 1, 2, \ldots$. The partial sums of the series are $S_n = \sum_{k=0}^{n} a_k$ for $n = 0, 1, 2, \ldots$. From the relation $S_{n+1} = S_n + a_{n+1}$ and the fact that a_{n+1} is positive, we see that $S_{n+1} - S_n = a_{n+1} > 0$ and so $\{S_n\}$ is an increasing sequence.

By the same reasoning used in Section 5.4, we can conclude that for $k > 3$,

$$a_k = \frac{2}{1}\frac{2}{2}\left(\frac{2}{3}\right)^{k-2}$$

This observation that the terms of the series are bounded by the terms of a convergent geometric series is the key to concluding that the partial sums of this series are bounded. For $n \geq 2$:

$$S_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \sum_{k=2}^{n} a_k < 1 + 2 + \sum_{k=2}^{n} 2\left(\frac{2}{3}\right)^{k-2}.$$

Add the rest of the terms of the geometric series with first term 2 and ratio 2/3 are added into the above bound, we conclude that

$$S_n < 1 + 2 + \sum_{k=2}^{\infty} 2\left(\frac{2}{3}\right)^{k-2} = 1 + 2 + \frac{2}{1 - \frac{2}{3}} = 1 + 2 + 6 = 9.$$

Thus, the series $\sum k = 0^{\infty} \frac{2^k}{k!}$ converges because the sequence of partial sums for the series is monotone and bounded above (by 9). The actual value of this limit will be found later. ◇

The same ideas can be used to prove that $\sum_{K+!}^{\infty} \frac{k^n}{k!}$, for any positive number k, converges.

Summary

Given any sequence $\{a_k\}$ we can form a new sequence $\{S_n\}$,where S_n is the sum of the first n terms of $\{a_k\}$, $S_n = a_1 + a_2 + \cdots + a_n$. The new sequence is called the "series" derived from the original sequence $\{a_k\}$. If the series converges, then a_k must approach 0 as $k \to \infty$. (The converse is not true.) It follows that if a_k does not approach 0 as $k \to \infty$, then the series $a_1 + a_2 + \cdots + a_n + \cdots$ diverges.

If $a_k = ar^{k-1}$, where $|r| < 1$, we obtain the geometric series or simply as $\sum_{k=0}^{\infty} ar^k$, which converges to $a/(1-r)$.

If, for each index, a_k is non-negative and $a_1 + a_2 + \cdots + a_k \leq B$ for some fixed number B for all k, then $\sum_{k=1}^{\infty} a_k$ is convergent and approaches a number no larger than B. This principle was used in this section to show that $\sum_{k=0}^{\infty} \frac{2^k}{k!}$ converges.

EXERCISES for Section 11.2 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 4 each concern a series $\sum_{k=1}^{\infty} a_k$ and the sequence of its partial sums $\{S_n\}$. (Based on suggestions by James T. Vance Jr.)

1.[R] Suppose you know that $a_n \to 0$ as $n \to \infty$. Which of the following statements are true. (More than one may be true.)

(a) The series definitely converges.

(b) The series definitely diverges.

(c) There is not enough information to decide whether the series diverges or converges.

(d) More information is needed to determine the sum of the series.

(e) $S_n \to 0$ as $n \to \infty$.

(f) $\sum_{k=1}^{\infty} a_k = 0$.

2.[R] Suppose you know that $a_n \to 6$ as $n \to \infty$. Which of the following statements are true. (More than one may be true.)

(a) The series definitely converges.

(b) The series definitely diverges.

(c) There is not enough information to decide whether the series diverges or converges.

(d) More information is needed to determine the sum of the series.

(e) $S_n \to 0$ as $n \to \infty$.

(f) $\displaystyle\sum_{k=1}^{\infty} a_k = 6$.

3.[R] Suppose you know that $S_n \to 3$ as $n \to \infty$. Which of the following statements are true. (More than one may be true.)

(a) The series definitely converges.

(b) The series definitely diverges.

(c) There is not enough information to decide whether the series diverges or converges.

(d) More information is needed to determine the sum of the series.

(e) The sum of the series is 3.

(f) $\sum_{k=1}^{\infty} a_k = 3$.

4.[R] Suppose you know that $S_n = n/(n+1)$. Which of the following statements are true. (More than one may be true.)

(a) The series definitely converges.

(b) The k^{th} term of the series diverges.

(c) The k^{th} term of the series converges.

(d) The k^{th} term of the series is $1/(k(k+1))$.

(e) The series is a geometric series.

5.[R] This exercise concerns the series $\sum_{k=1}^{\infty} 5(-1/2)^k$.

(a) Express the fourth term of this series as a decimal.

(b) Express the fourth partial sum of this series as a decimal.

(c) Find the limit as $k \to \infty$ of the k^{th} term of the series.

(d) Find the limit as $n \to \infty$ of the n^{th} partial sum of the series.

(e) Does the series converge? If so, what is its sum?

6.[R] This exercise concerns the series $\sum_{k=1}^{\infty} 3(1/10)^k$.

(a) Express the third term of this series as a decimal.

(b) Express the third partial sum of this series as a decimal.

(c) Find the limit as $k \to \infty$ of the k^{th} term of the series.

(d) Find the limit as $n \to \infty$ of the n^{th} partial sum of the series.

(e) Does the series converge? If so, what is its sum?

In Exercises 7 to 14 determine whether the given geometric series converges. If it does, find its sum.

7.[R] $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^{k-1} + \cdots$

8.[R] $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + \left(\frac{-1}{3}\right)^{k-1} + \cdots$

9.[R] $\displaystyle\sum_{k=1}^{\infty} 10^{-k}$

10.[R] $\displaystyle\sum_{k=1}^{\infty} 10^{k}$

11.[R] $\displaystyle\sum_{k=1}^{\infty} 5(0.99)^{k}$

12.[R] $\displaystyle\sum_{k=1}^{\infty} 7(-1.01)^{k}$

13.[R] $\displaystyle\sum_{k=1}^{\infty} 4\left(\frac{2}{3}\right)^{k}$

14.[R] $\frac{-3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots + \frac{3}{(-2)^k} + \cdots$

In Exercises 15 to 22 determine whether the given series converge or diverge. Find the sums of the convergent series.

15.[R] $-5 + 5 - 5 + 5 - \cdots + (-1)^k 5 + \cdots$

16.[R] $\sum_{k=1}^{\infty} \frac{1}{(1+(1/k))^k}$

17.[R] $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k}$

18.[R] $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2k+1}$

19.[R] $\displaystyle\sum_{k=1}^{\infty} 6\left(\frac{4}{5}\right)^{k}$

20.[R] $\displaystyle\sum_{k=1}^{\infty} 100\left(\frac{-8}{9}\right)$

21.[R] $\displaystyle\sum_{k=1}^{\infty} \left(2^{-k} + 3^{-k}\right)$

22.[R] $\displaystyle\sum_{k=1}^{\infty} \left(4^{-k} + k^{-1}\right)$

23.[R] What is the total distance traveled — both up and down — by the ball described in the opening paragraph of this section?

24.[R] A rubber ball, when dropped on concrete, rebounds 90 percent of the distance it falls. If it is dropped from a height of 6 feet, how far does it travel — both up and down — before coming to rest?

25.[M] The repeating decimal

$$3.171717\ldots,$$

where the 17's continue forever, can be viewed as 3 plus a geometric series:

$$3 + \frac{17}{100} + \frac{17}{100^2} + \frac{17}{100^3} + \cdots .$$

Using the formula for the sum of a geometric series, write the decimal as a fraction.

26.[M] (See Exercise 25.) Evaluate the repeating decimal $0.3333\cdots$.

27.[M] (See Exercise 25.) Evaluate the repeating decimal $4.1256256256\ldots$ (with 256 repeating).

28.[M] Show that if $|r| < 1$, the sum of the geometric series $a + ar + ar^2 + \cdots$ differs from S_n by $ar^n/(1-r)$.

29.[M] This is a quote from an economics text: "The present value of the land, if a new crop is planted at time t, $2t$, $3t$, etc., is

$$P = g(t)e^{-rt} + g(t)e^{-2rt} + g(t)e^{-3rt} + \cdots .$$

By the formula for the sum of a geometric series,

$$P = \frac{g(t)e^{-rt}}{1-e^{-rt}}."$$

Check that the missing step, which simplified the formula for P, was correct.

30.[M] A patient takes A grams of a certain medicine every 6 hours. The amount of each dose active in the body t hours later is Ae^{-kt} grams, where k is a positive constant and time is measured in hours.

(a) Show how immediately after taking the medicine for the n^{th} time, the amount active in the body is

$$S_n = A + Ae^{-6k} + Ae^{-12k} + \cdots + Ae^{-6(n-1)k}.$$

(b) If, as $n \to \infty$, $S_n \to \infty$, the patient would be in danger. Does $S_n \to \infty$? If not, what is $\lim_{n\to\infty} S_n$?

(See also Exercise 115 in the Chapter 5 Summary.)

31.[M] Deficit spending by the federal government inflates the nation's money supply. However, much of the money paid out by the government is spent in turn by those who receive it, thereby producing additional spending. This produces a chain reaction, called by economists the *multiplier effect*. It results in much greater total spending than the government's original expenditure. To be specific, suppose the government spends 1 billion dollars and that the recipients of that expenditure in turn spend 80 percent while retaining 20 percent. Let S_n be the *total* spending generated after n transactions in the chain, 80 percent of receipts being expended at each step.

(a) Show that $S_n = 1 + 0.8 + 0.8^2 + \cdots + 0.8^{n-1}$ billion dollars.

(b) Show that as n increases, the total spending approaches 5 billion dollars. (In this case the multiplier is 5.)

(c) What would the total spending be if 90 percent of receipts is spent at each step instead of 80 percent?

NOTE: The subprime mortgage foreclosures in 2008 caused a similar ripple effect, threatening a recession.

32.[M] Assume a man falls $16t^2$ feet in t seconds and twice as long to bounce as it took to fall a given distance. How long does the ball in Exercise 24 bounce?

Exercises 33 to 35 are related to the following question: A gambler tosses a coin until a head appears. On the average, how many times does she toss it to get a head?

33.[M]

(a) Repeat this experiment 10 times. Each run consists of tossing a coin until a head appears. Average the lengths of the 10 trials.

(b) The probability of a run of length one is $\frac{1}{2}$, since a head must appear on the first toss. The probability of a run of length two is $\left(\frac{1}{2}\right)^2$. The probability of having a head appear for the first time on toss k is $\left(\frac{1}{2}\right)^k$. It is shown in probability theory that the average number of tosses to get a head is $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$. NOTE: This is a theoretical average approached as the experiment is repeated many times. Compute $\displaystyle\sum_{k=1}^{8} \frac{k}{2^k}$.

34.[C] Oresme, around the year 1360, summed the series $\sum_{k=1}^{\infty} \frac{k}{2^k}$ by drawing the endless staircase shown in Figure 11.2.3, in which each stair after the first has width 1 and is half as high as the stair immediately to its left.

(a) By looking at the staircase in two ways, show that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots.$$

(b) Use (a) to sum $\sum_{k=1}^{\infty}\frac{k}{2^k}$.

(c) Use the same idea to find $\sum_{k=1}^{\infty}kp^k$, when $0<p<1$.

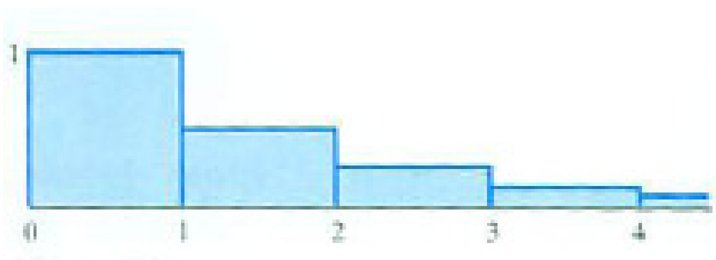


Figure 11.2.3:

35.[C]

(a) Using your calculator compute enough partial sums of the series $\sum_{k=1}^{\infty}k3^{-k}$ to offer an opinion as to whether it converges or diverges.

(b) Show that it converges. HINT: The coefficient k is less than 2^k.

(c) On the basis of (a), what do you think its sum is?

36.[C] Use the precise definition of convergence from Section 10.2 to prove each of the following statements:

(a) If c is a number and $\sum_{k=1}^{\infty}a_k$ is a convergent series with sum L, then $\sum_{k=1}^{\infty}ca_k$ is a convergent series with sum cL.

(b) If $\sum_{k=1}^{\infty}a_k$ and $\sum_{k=1}^{\infty}b_k$ are convergent series with sums L and M, respectively, then $\sum_{k=1}^{\infty}(a_k+b_k)$ is a convergent series with sum $L+M$.

11.3 The Integral Test

In this section we use integrals of the form $\int_a^\infty f(x)\ dx$ to establish convergence or divergence of series whose terms are positive and decreasing. Furthermore, we obtain a way of analyzing the error when we use a partial sum to estimate the sum of the series.

The Integral Test

Let $f(x)$ be a decreasing positive function. We obtain a sequence from $f(x)$ by defining a_n to be $f(n)$. For instance, the sequence $1/1$, $1/2$, $1/3$, $\dots$, $1/n$, $\dots$ is obtained from the function $f(x) = 1/x$. It turns out that the convergence (or divergence) of the series $\sum_{i=1}^{\infty} a_i$ is closely connected with the convergence (or divergence) of the improper integral $\int_1^\infty f(x)\ dx$. This connection is expressed in the following theorem:

Theorem (Integral Test)**.** *Let $f(x)$ be a continuous decreasing function such that $f(x) > 0$ for $x \geq 1$. Let $a_n = f(n)$ for each positive integer n. Then*

A. If $\int_1^\infty f(x)\ dx$ is convergent, then so is the series $\sum_{k=1}^{\infty} a_k$.

B. If $\int_1^\infty f(x)\ dx$ is divergent, then so is the series $\sum_{k=1}^{\infty} a_k$.

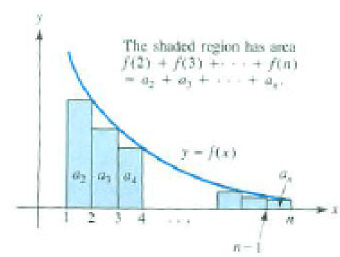


Figure 11.3.1:

Proof

Figures 11.3.1 and 11.3.2 are the key to the proof. Note how the rectangles are constructed in each case.

In Figure 11.3.1 the rectangles lie *below* the curve $y = f(x)$. Each rectangle has width 1. Comparing the staircase area with the area under the curve gives the inequality

$$a_2 + a_3 + \cdots + a_n < \int_1^n f(x)\ dx,$$

and therefore

$$a_1 + a_2 + a_3 + \cdots + a_n < a_1 + \int_1^n f(x)\ dx. \qquad (11.3.1)$$

If $\int_1^\infty f(x)\ dx$ is convergent, with value I, then

$$a_1 + a_2 + \cdots + a_n < a_1 + I.$$

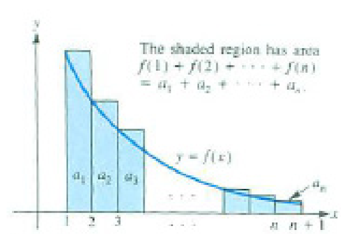


Figure 11.3.2:

Since the partial sums of the series $\sum_{k=1}^{\infty} a_k$ are all bounded by the number $a_1 + I$, the series $\sum_{k=1}^{\infty} a_k$ converges and its sum is less than or equal to $a_1 + I$.

Now, Figure 11.3.2 shows that

$$a_1 + a_2 + \cdots + a_n > \int_1^{n+1} f(x)\,dx. \tag{11.3.2}$$

If follows that if $\int_1^\infty f(x)\,dx$ diverges, then so must the series $\sum_{k=1}^\infty a_k$. •

EXAMPLE 1 Use the integral test to determine the convergence or divergence of

(a) $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots = \sum_{k=1}^\infty \frac{1}{k}$

(b) $\frac{1}{1^{1.01}} + \frac{1}{2^{1.01}} + \cdots + \frac{1}{n^{1.01}} + \cdots = \sum_{k=1}^\infty \frac{1}{k^{1.01}}$

SOLUTION

(a) Observe that this is the harmonic series, which was shown in Example 2 in Section 11.2 to diverge. To apply the Integral Test to this series, let $f(x) = 1/x$. This is a decreasing positive function for $x > 0$. Then $a_k = f(k) = 1/k$. We have

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} (\ln(b) - \ln(1)) = \infty$$

Since $\int_1^\infty \frac{dx}{x}$ is divergent, so is the series $\sum_{i=1}^\infty \frac{1}{n}$.

Even though the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^{1.01}}$ are near each other, the integrals $\int \frac{dx}{x}$ and $\int \frac{dx}{x^{1.01}}$ behave very differently.

(b) Let $f(x) = 1/x^{1.01}$, which is a decreasing positive function. Then $a_k = f(k) = 1/k^{1.01}$. We have

$$\begin{aligned}\int_1^\infty \frac{dx}{x^{1.01}} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^{1.01}} = \lim_{b\to\infty} \left.\frac{x^{-1.01+1}}{-1.01+1}\right|_1^b = \lim_{b\to\infty} \left.\frac{x^{-0.01}}{-0.01}\right|_1^b \\ &= \lim_{b\to\infty} \left(\frac{b^{-0.01}}{-0.01} - \frac{1^{-0.01}}{-0.01}\right) = 0 - (-100) = 100.\end{aligned}$$

Since $\int_1^\infty dx/x^{1.01}$ is convergent, so is $\sum_{k=1}^\infty 1/k^{0.01}$. By (11.3.1), its sum is less than $a_1 + 100 = 101$.

◇

The argument in Example 1 extends to a family of series known as p-series.

DEFINITION *([)p*-series] For a positive number p, the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

is called a p-**series**.

For example, when $p = 1$ we obtain the harmonic series $\sum_{k=1}^{\infty} 1/k$ and for $p = 1.01$, the series $\sum_{k=1}^{\infty} 1/k^{1.01}$.

An argument similar to those in Example 1 establishes the following theorem.

Theorem 11.3.1. *If $0 < p \leq 1$, the p-series $\sum_{i=1}^{\infty} 1/i^p$ diverges. If $p > 1$, the p-series $\sum_{i=1}^{\infty} 1/i^p$ converges.*

Note that there is a p-series for each positive number p. A negative exponent p would not give a series of interest. For instance, when $p = -1$, we obtain $\sum_{k=1}^{\infty} 1/k^{-1} = \sum_{k=1}^{\infty} k$, which is clearly divergent since its n^{th} term does not approach 0 as $n \to \infty$. (For any negative p, $\lim_{i\to\infty} 1/n^p = \infty$.)

Controlling the Error

When we use a front end of a series (a partial sum) to estimate the sum of the whole series, there will be an error, namely, the sum of the corresponding tail end. For the sum of a front end to be a good estimate of the sum of the whole series, we must be sure that the sum of the corresponding tail end is small. Otherwise, we would be like the carpenter who measures a board as "5 feet long with an error of perhaps as much as 5 feet." That is why we wish to be sure that the sum of the tail end is small.

Partial sum = front end;
Error = tail end.

Let S_n be the sum of the first n terms of a convergent series $\sum_{k=1}^{\infty} a_k$ whose sum is S. The difference

$$R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

$R_n = \sum_{k=n+1}^{\infty} a_k$

is called the **remainder** or **error** in using the sum of the first n terms to approximate the sum of the series. That is,

$$\underbrace{a_1 + a_2 + \cdots + a_n}_{\text{partial sum } S_n} + \underbrace{a_{n+1} + a_{n+2} + \cdots}_{\text{tail end } R_n} = \underbrace{a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+2} + \cdots}_{\text{sum of series } S}$$

so

$$S_n + R_n = S.$$

For a series whose terms are positive and decreasing, use an improper integral to estimate the error. The reasoning depends again on comparing a staircase of rectangles with the area under a curve.

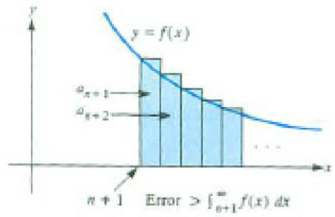


Figure 11.3.3:

Recall that $f(x)$ is a continuous decreasing positive function. The error in using $S_n = f(1) + f(2) + \cdots + f(n) = \sum_{i=1}^{n} f(i)$ to approximate $\sum_{i=1}^{\infty} f(i)$ is the sum $\sum_{i=n+1}^{\infty} f(i)$. This sum is the area of the endless staircase of rectangles shown in Figure 11.3.3. Comparing the rectangles with the region under the curve $y = f(x)$, we conclude that

$$R_n = a_{n+1} + a_{n+2} + \cdots = f(n+1) + f(n+2) + \cdots > \int_{n+1}^{\infty} f(x)\, dx. \quad (11.3.3)$$

Inequality (11.3.3) gives a *lower* estimate of the error.

The staircase in Figure 11.3.4, which lies below the curve, gives an *upper* estimate of the error. Inspection of Figure 11.3.4 shows that

$$R_n = a_{n+1} + a)n + 2 + \cdots = f(n+1) + f(n+2) + \cdots < \int_{n}^{\infty} f(x)\, dx.$$

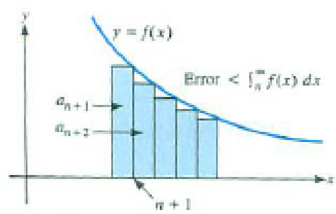


Figure 11.3.4:
Estimating the error

Putting these observations together yields the following estimate of the error.

Theorem 11.3.2 (A bound on the error)**.** *Let $f(x)$ be a continuous decreasing positive function such that $\int_1^{\infty} f(x)\, dx$ is convergent. Then the error R_n in using $f(1) + f(2) + \cdots + f(n)$ to estimate $\sum_{i=1}^{\infty} f(i)$ satisfies the inequality*

$$\int_{n+1}^{\infty} f(x)\, dx \le R_n \le \int_{n}^{\infty} f(x)\, dx. \quad (11.3.4)$$

EXAMPLE 2 The first five terms of the series $1/1^2 + 1/2^2 + \cdots + 1/n^2 + \cdots$ are used to estimate the sum of the series.

(a) Put upper and lower bounds on the error in using just those terms.

(b) Use the bounds in (a) to estimate $\sum_{k=1}^{\infty} 1/k^2$.

SOLUTION First, observe that the series with terms $a_k = 1/k^2$ is the p-series with $p = 2$. Since $p > 1$, this series converges. Also, the function $f(x) = 1/x^2$ is continuous, decreasing, and positive for $x \ge 1$.

(a) By inequality (11.3.4) of Theorem 11.3.2, the error R_5 satisfies the inequality

$$\int_6^{\infty} \frac{dx}{x^2} < R_5 < \int_5^{\infty} \frac{dx}{x^2}.$$

Now, $$\int_5^\infty \frac{dx}{x^2} = \left.\frac{-1}{x}\right|_5^\infty = 0 - \left(\frac{-1}{5}\right) = \frac{1}{5}.$$

Similarly, $$\int_6^\infty \frac{dx}{x^2} = \frac{1}{6}.$$

Thus $$\frac{1}{6} < R_5 < \frac{1}{6}.$$

(b) The sum of the first five terms of the series is

$$S_5 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \approx 1.463611.$$

Since the sum of the remaining terms (the "tail end") is between $\frac{1}{6}$ and $\frac{1}{5}$, the sum of the series is between $1.463611+0.166666$ and $1.463611+0.2$, hence between 1.6302 and 1.6636. (In the 17th century Euler proved that this sum is $\pi^2/6 \approx 1.644934068$.

Keep more digits than you need until all calculations have been done. Then, "round down" lower bounds and "round up" upper bounds.

◇

Estimating a Partial Sum S_n

We still restrict our attention to series that satisfy the hypotheses of the integral test in Theorem 11.3. That is, there is a continuous, positive, and decreasing function $f(x)$ such that $f(n) = a_n$.

Just as we can use an (improper) integral to estimate the sum of a tail end of such a series, we can also use a (definite) integral to estimate a partial sum $S_n = a_1 + a_2 + \cdots + a_n$.

In the course of proving Theorem 11.3, we obtained equations (11.3.1) and (11.3.2). Taken together, they give us the inequalities

$$\int_1^{n+1} f(x)\ dx < a_1 + a_2 + \cdots + a_n < a_1 + \int_1^{n} f(x)\ dx. \tag{11.3.5}$$

If we can evaluate $\int_1^{n+1} f(x)\ dx$ and $\int_1^{n} f(x)\ dx$ by the Fundamental Theorem of Calculus, we may use (11.3.5) to put upper and lower bounds on $S_n =$

$\sum_{k=1}^{n} a_k$. These estimates are valid whether the series $\sum_{k=1}^{\infty} a_k$ converges or diverges.

EXAMPLE 3 Use (11.3.5) to estimate the sum of the first million terms of the harmonic series.

SOLUTION By (11.3.5)

$$\int_1^{1{,}000{,}001} \frac{dx}{x} < \sum_{k=1}^{1{,}000{,}000} \frac{1}{k} < 1 + \int_1^{1{,}000{,}000} \frac{dx}{x}.$$

hence

$$\ln(1,000,001) < \sum_{k=1}^{1{,}000{,}000} \frac{1}{k} < 1 + \ln(1,000,000).$$

Evaluating the logarithm with a calculator, we conclude that

$$13.8155 < \sum_{i=1}^{1{,}000{,}000} \frac{1}{i} < 14.8156.$$

◇

Summary

We developed a test for convergence or divergence for series whose terms a_k are of the form $f(k)$ for a continuous, positive, decreasing function $f(x)$. The series converges if $\int_1^{\infty} f(x)\, dx$ converges, and diverges if $\int_1^{\infty} f(x)\, dx$ diverges.

We also used integrals to analyze the error in using a partial sum S_n of such a series as an estimate of the sum of the series. (Rather than memorizing the formulas, just draw the appropriate staircase diagrams.)

We assumed $f(x)$ is decreasing for $x \geq 1$. Actually, Theorem 11.3 holds if we assume that $f(x)$ is decreasing from some point on, that is, there is some number a such that $f(x)$ is decreasing for $x \geq a$. (The argument for this type of integral involves similar staircase diagrams.)

EXERCISES for Section 11.3 *Key:* R–routine, M–moderate, C–challenging

Use the integral test in Exercises 1 to 8 to determine whether each series diverges or converges.

1.[R] $\sum_{k=1}^{\infty} \frac{1}{k^{1.1}}$

2.[R] $\sum_{k=1}^{\infty} \frac{1}{k^{0.9}}$

3.[R] $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

4.[R] $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

5.[R] $\sum_{k=1}^{\infty} \frac{1}{k\ln(k)}$

6.[R] $\sum_{k=1}^{\infty} \frac{1}{k+1,000}$

7.[R] $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$

8.[R] $\sum_{k=1}^{\infty} \frac{k^3}{e^k}$

Use Theorem 11.3.1 in Exercises 9 to 12 to determine whether each series diverges or converges.

9.[R] $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}}$

10.[R] $\sum_{k=1}^{\infty} \frac{1}{k^3}$

11.[R] $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

12.[R] $\sum_{k=1}^{\infty} \frac{1}{k^{0.999}}$

13.[R]

(a) Prove that if $p > 1$, the p-series converges.

(b) Give two numbers between which its sum lies.

14.[R]

(a) If you used S_{100} to estimate $\sum_{k=1}^{\infty} 1/k^2$, what could you say about the error R_{100}?

(b) How large should you choose k to be sure that the error R_k is less than 0.0001?

15.[R]

(a) If you used S_{1000} to estimate $\sum_{k=1}^{\infty} 1/k^3$, what could you say about the error R_{1000}?

(b) How large should you choose k to be sure that the error R_k is less than 0.0001?

16.[R]

(a) How many terms of the series $\sum_{k=1}^{\infty} 1/k^4$ should you use to be sure that the remainder is less than 0.0001?

(b) Estimate $\sum_{k=1}^{\infty} 1/k^4$ to three decimal places.

17.[R] Repeat Exercise 16 for the series $\sum_{k=1}^{\infty} 1/k^5$.

In each of Exercises 18 to 21 (a) compute the sum of the first four terms of the series to four decimal places, (b) give upper and lower bound on the error R_4, (c) combine (a) and (b) to estimate the sum of the series.

18.[M] $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$

19.[M] $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$

20.[M] $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

21.[M] $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k}$

22.[M] Prove that if $p \leq 1$, the p-series diverges.

23.[M] What does the integral test say about the geometric series $\sum_{k=1}^{\infty} p^k$, when

$0 < p < 1$?

24.[M] Let $f(x)$ be a positive continuous function that is decreasing for $x \geq a$. Let $a_k = f(k)$. Show in detail (with appropriate diagrams and exposition) why $\int_a^\infty f(x)\,dx$ and $\sum_{k=1}^\infty a_k$ both converge or both diverge. Use your own words. Don't just mimic the book's treatment of the case $a = 1$.

25.[M] (See Exercise 24.) Show that $\sum_{k=1}^\infty k^3 e^{-k}$ converges.

26.[M] Show that for $n \geq 2$,

$$2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1.$$

27.[M]

(a) By comparing the sum with integrals, show that

$$\ln\left(\frac{201}{100}\right) < \frac{1}{100} + \frac{1}{101} + \frac{1}{102} + \cdots + \frac{1}{200} < \ln\left(\frac{200}{99}\right).$$

(b) Find $\lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}\right)$.

28.[M]

(a) Let $f(x)$ be a decreasing continuous positive function for $x \geq 1$ such that $\int_1^\infty f(x)\,dx$ is convergent. Show that

$$\int_1^\infty f(x)\,dx < \sum_{k=1}^{\infty} f(k) < f(1) + \int_1^\infty f(x)\,dx.$$

(b) Use (a) to estimate $\sum_{k=1}^\infty 1/k^2$.

29.[M] In Example 1 we showed that the p-series for $p = 1$ diverges but the p-series for $p = 1.01$ converges. This contrast occurs even though the corresponding terms of the two series seem to resembe each other so closely. (For instance, $1/7^{1.01} \approx 0.140104$, $1/7^1 \approx 0.142857$.) What happens to the ratio $(1/k^{1.01})/(1/k)$ as $k \to \infty$.

In Exercises 30 and 31 concern products, rather than sums, of numbers.

30.[C] Let $\{a_n\}$ be a sequence of positive numbers. Denote the product $(1+a_1)(1+a_2)\cdots(1+a_n)$ by $\prod_{k=1}^n (1+a_k)$.

(a) Show that $\sum_{k=1}^{\infty} a_k \leq \prod_{k=1}^{n}(1+a_k)$.

(b) Show that if $\lim_{k\to\infty} \prod_{k=1}^{n}(1+a_k)$ exists, then $\sum_{k=1}^{\infty} a_k$ is convergent.

31.[C] (This continues Exercise 30.)

(a) Show that $1+a_k \leq e^{a_k}$. HINT: Show that $1+x \leq e^x$ for $x>0$.

(b) Show that if the series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{n\to\infty} \prod_{k=1}^{n}(1+a_k)$ exists.

32.[C] Here is an argument that there is an infinite number of primes. Assume that there is only a finite number of primes, p_1, p_2, ..., p_m.

(a) Show that

$$\frac{1}{1-1/p_k} = 1 + \frac{1}{p_k} + \frac{1}{p_k^2} + \frac{1}{p_k^3} + \cdots .$$

(b) Show then that

$$\frac{1}{1-1/p_1}\frac{1}{1-1/p_2}\cdots\frac{1}{1-1/p_m} = \sum_{k=1}^{\infty}\frac{1}{k}.$$

NOTE: Assume the series can be multiplied term-by-term.

(c) From (b) obtain a contradiction.

11.4 The Comparison Tests

So far in this chapter three tests for the convergence (or divergence) of a series have been presented. The first concerned a special type of series, a geometric series. The second, the n^{th}-term test for divergence, asserts that if the n^{th} term of a series does *not* approach 0, the series diverges. The third, the integral test, applies to certain series of positive terms. In this section two further tests are developed; the comparison and limit-comparison tests. We still consider only tests for series with positive terms.

Comparison Tests

The first test is similar to the comparison test for improper integrals in Section 7.8.

Comparison Tests

Theorem (Comparison Tests for Convergence and Divergence)**.**

(a) If $0 \le p_k \le c_k$ for each k and $\sum_{k=1}^{\infty} c_k$ converges, so does $\sum_{k=1}^{\infty} p_k$.

(b) If $0 \le d_k \le p_k$ for each k and $\sum_{k=1}^{\infty} d_k$ diverges, so does $\sum_{k=1}^{\infty} p_k$.

Proof

(a) Let the sum of the series $c_1 + c_2 + \cdots$ be C. Let S_n denote the partial sum $p_1 + p_2 + \cdots + p_n$. Then, for each n,

$$S_n = p_1 + p_2 + \cdots + p_n \le c_1 + c_2 + \cdots + c_n \le C.$$

Since the p_n's are non-negative,

$$S_1 \le S_2 \le \cdots \le S_n \le \cdots .$$

Since each S_n is less than or equal to C, Theorem 10.1.1 of Section 10.1 assures us that the sequence $\{S_n\}$ converges to a number L (less than or equal to C). In other words, the series $p_1 + p_2 + \cdots$ converges (and its sum is less than or equal to the sum $c_1 + c_2 + \cdots$).

$S_1 \le S_2 \le \cdots \le S_n \le \cdots \le C$

Logically, (b) is the contrapositive of (a).

(b) The divergence test follows immediately from the convergence test. If the series $p_1 + p_2 + \cdots$ converged, so would the series $d_1 + d_2 + \cdots$, which is assumed to diverge.

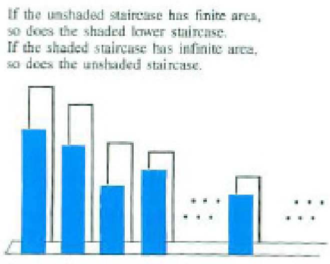


Figure 11.4.1:

•

Figure 11.4.1 present the two comparison tests in Theorem 11.4 in terms of endless staircases.

In order to apply the comparison test to a series of positive terms you have to compare it to a series whose convergence or divergence you already know. What series can you use for comparison? You know the p-series converges for $p > 1$ and diverges for $p \leq 1$. Also a geometric series $\sum_{k=1}^{\infty} r^k$ with positive terms converges for $0 \leq r < 1$ but diverges for $r \geq 1$. Moreover, when we multiply one of theses series by a non-zero constant, we don't affect its convergence or divergence.

EXAMPLE 1 Does the series

$$\sum_{k=1}^{\infty} \frac{k+1}{k+2}\frac{1}{k^2} = \frac{2}{3} \cdot \frac{1}{1^2} + \frac{3}{4} \cdot \frac{1}{2^2} + \frac{4}{5} \cdot \frac{1}{3^2} + \cdots$$

converge or diverge?

SOLUTION The coefficients $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$, ... approach 1 as $k \to \infty$, so they are a minor influence. The series resembles the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \cdots ,$$

which was shown by the integral test to be convergent. Since the fraction $(k+1)/(k+2)$ is less than 1,

$$\frac{k+1}{k+2}\frac{1}{k^2} < \frac{1}{k^2}.$$

Thus, by the comparison test for convergence, the series

$$\frac{2}{3} \cdot \frac{1}{1^2} + \frac{3}{4} \cdot \frac{1}{2^2} + \frac{4}{5} \cdot \frac{1}{3^2} + \cdots$$

also converges. However, the test does not tell us the sum of the series. ◇

EXAMPLE 2 Does the series

$$\sum_{k=1}^{\infty} \frac{k+1}{k+2} \cdot \frac{1}{k} = \frac{2}{3} \cdot \frac{1}{1} + \frac{3}{4} \cdot \frac{1}{2} + \cdots + \frac{k+1}{k+2} \cdot \frac{1}{k} + \cdots$$

converge or diverge?

SOLUTION Again the coefficient $(k+1)/(k+2)$ is a minor influence. We suspect that $1/k$ is the main influence and that the series diverges.

Unfortunately, the terms in this series are *less* than the terms of the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. So the divergence test does not directly apply. However, $(k+1)/(k+2)$ is greater than $1/2$. Now, the series

$$\frac{1}{2} \cdot \frac{1}{1} + \frac{1}{2} \cdot \frac{1}{2} + \cdots + \frac{1}{2} \cdot \frac{1}{k} + \cdots$$

is also divergent, since it's just a multiple of a divergent series. The divergence part of the comparison test applies: the series

$$\sum_{k=1}^{\infty} \frac{k+1}{k+2} \frac{1}{k}$$

is, term by term, larger than the terms of the divergent series

$$\sum_{k=1}^{\infty} \frac{1}{2} \frac{1}{k}.$$

Hence, $\sum_{k=1}^{\infty} \frac{k+1}{k+2} \frac{1}{k}$ is divergent. ◇

Limit-Comparison Tests

There is a variation of the comparison test that produces a much quicker solution of Example 2. It is the **limit-comparison test**.

Limit-Comparison Tests

Theorem (Limit-Comparison Tests for Convergence and Divergence)**.** *Let $\sum_{k=1}^{\infty} p_k$ be a series of positive terms to be tested for convergence or divergence.*

A. Let $\sum_{k=1}^{\infty} c_k$ be a convergent series of positive terms. If $\lim_{k\to\infty} \frac{p_k}{c_k}$ exists, then $\sum_{k=1}^{\infty} c_k$ also converges.

B. Let $\sum_{k=1}^{\infty} d_k$ be a divergent series of positive terms. If $\lim_{k\to\infty} \frac{p_k}{d_k}$ exists and is not 0 or if the limit is infinite, then $\sum_{k=1}^{\infty} p_k$ also diverges.

Proof

We shall prove part (a). Let $a = \lim_{k\to\infty} \frac{p_k}{c_k}$. Since as $k \to \infty$, $p_k/c_k \to a$, there must be an integer N such that, for all $n \geq N$, p_k/c_k remains less than, say, $a+1$. Thus

$$p_k < (a+1)c_k \qquad \text{for all } n \geq N.$$

Now the series

$$(a+1)c_N + (a+1)c_{N+1} + \cdots + (a+1)c_k + \cdots ,$$

being $a+1$ times the tail end of a convergent series, is itself convergent. By the comparison test,

$$p_N + p_{N+1} + \cdots + p_k + \cdots$$

is convergent. Hence $p_1 + p_2 + \cdots + p_k + \cdots$ is convergent.

Part (B) can be proved in a similar manner. •

Note that in part B of the Limit-Comparison Test nothing is said about the case $\lim_{k\to\infty} p_k/d_k = 0$. In this circumstance the series $\sum_{k=1}^{\infty} p_k$ can either converge or diverge. For instance, take $\sum_{k=1}^{\infty} d_k$ to be the divergent series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}}$. The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent and $\lim_{k\to\infty} \frac{1/k^2}{1/\sqrt{k}} = 0$. Contrarily, the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent and again $\lim_{k\to\infty} \frac{1/k}{1/\sqrt{k}} = 0$.

The next example shows how convenient the limit-comparison test is. Contrast the solution in Example 3 with that in Example 2.

EXAMPLE 3 Does the series

$$\sum_{k=1}^{\infty} \frac{k+1}{k+2} \cdot \frac{1}{k} = \frac{2}{3} \cdot \frac{1}{1} + \frac{3}{4} \cdot \frac{1}{2} + \cdots + \frac{k+1}{k+2} \cdot \frac{1}{k} + \cdots$$

converge or diverge?

SOLUTION As with Example 2, we expect this series to behave like the harmonic series. For this reason we examine the ratio between corresponding terms:

$$\lim_{k\to\infty} \frac{\frac{k+1}{k+2} \cdot \frac{1}{k}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k+1}{k+2} = 1.$$

Since the limit is not 0, and the harmonic series diverges, the Limit-Comparison Test tells us that $\sum_{k=1}^{\infty} \frac{k+1}{k+2} \cdot \frac{1}{k}$ diverges. ◇

EXAMPLE 4 Does

$$\sum_{k=1}^{\infty} \frac{(1+1/k)^k \left(1+(-1/2)^k\right)}{2^k}$$

converge or diverge?

See Section 2.2.

SOLUTION Note that as $k \to \infty$, $(1+1/k)^k \to e$ and $1+(-1/2)^k \to 1$. The major influence is the 2^k in the denominator. So use the Limit-Comparison Test. The given series resembles the convergent geometric series with first term $\frac{1}{2}$ and ratio also $\frac{1}{2}$: $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \cdots$. Then

$$\lim_{k\to\infty} \frac{\frac{\left(1+\frac{1}{k}\right)^k \left(1+\left(\frac{1}{2^k}\right)^k\right)}{2^k}}{\frac{1}{2^k}} = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k \left(1+\left(\frac{-1}{2^k}\right)^k\right) = e \cdot 1 = e.$$

Since $\sum_{k\to\infty} 2^{-k}$ is convergent, so is the given series. ◇

EXAMPLE 5 Does $\sum_{k=1}^{\infty} k^3 3^{-k}$ converge or diverge?
SOLUTION The typical term $k^3 3^{-k}$ is dominated by the exponential factor, $1/3^k$. For this reason we suspect that the series $\sum_{k=1}^{\infty} k^3 3^{-k}$ might also converge. We try the Limit-Comparison Test, obtaining

$$\lim_{k\to\infty} \frac{\frac{k^3}{3^k}}{\frac{1}{3^k}} = \lim_{k\to\infty} k^3 = \infty.$$

Since the limit is not finite, the test gives no information. So we start over and look at $k^3/3^k$ a little closer.

See also Section 5.5.

The numerator k^3 approaches ∞ much more slowly than 3^k, so we still suspect that $\sum_{k=1}^{\infty} k^3/3^k$ converges. Now, k^3 approaches ∞ *more slowly than any exponential* b^k with $b > 1$. For example, for large k, k^3 is less than $(1.5)^k$. This means that for large k

$$\frac{k^3}{3^k} < \frac{(1.5)^k}{3^k} = (0.5)^k.$$

The geometric series $\sum_{k=1}^{\infty}(0.5)^k$ converges. Since $k^3/3^k < (0.5)^k$ for all but a finite number of values of k, the Comparison Test tells us that $\sum_{k=1}^{\infty} k^3/3^k$ converges. ◇

Summary

We developed two tests for convergence or divergence of a series with positive terms, $\sum_{k=1}^{\infty} p_k$. If, for each k, p_k is less than the corresponding term of a convergent series, then $\sum_{k=1}^{\infty} p_k$ converges. If p_k is larger than the corresponding term of a divergent series of positive terms, then $\sum_{k=1}^{\infty} p_k$ diverges. This Comparison Test is the basis for the Limit-Comparison Test, which is often easier to apply. This test depends only on the limit of the ratio of p_k to the corresponding term of a series of positive terms known to converge or diverge.

EXERCISES for Section 11.4 *Key:* R–routine, M–moderate, C–challenging

Use the comparison test in Exercises 1 to 4 to determine whether each series converges or diverges.

1.[R] $\sum_{k=1}^{\infty} \frac{1}{k^2+3}$

2.[R] $\sum_{k=1}^{\infty} \frac{k+2}{(k+1)\sqrt{k}}$

3.[R] $\sum_{k=1}^{\infty} \frac{\sin^2(k)}{k^2}$

4.[R] $\sum_{k=1}^{\infty} \frac{1}{k2^k}$

Use the limit-comparison test in Exercises 5 to 8 to determine whether each series converges or diverges.

5.[R] $\sum_{k=1}^{\infty} \frac{5k+1}{(k+2)k^2}$

6.[R] $\sum_{k=1}^{\infty} \frac{2^k+k}{3^k}$

7.[R] $\sum_{k=1}^{\infty} \frac{k+1}{(5k+2)\sqrt{k}}$

8.[R] $\sum_{k=1}^{\infty} \frac{(1+1/k)^k}{k^2}$

In Exercises 9 to 28 use any test discussed so far in this chapter to determine whether each series converges or diverges.

9.[R] $\sum_{k=1}^{\infty} \frac{k^2k}{3^k}$

10.[R] $\sum_{k=1}^{\infty} \frac{2^k}{k^2}$

11.[R] $\sum_{k=1}^{\infty} \frac{1}{k^k}$

12.[R] $\sum_{k=1}^{\infty} \frac{1}{k!}$

13.[R] $\sum_{k=1}^{\infty} \frac{4k+1}{(2k+3)k^2}$

14.[R] $\sum_{k=1}^{\infty} \frac{k^2(2^k+1)}{3^k+1}$

15.[R] $\sum_{k=1}^{\infty} \frac{1+\cos(k)}{k^2}$

16.[R] $\sum_{k=1}^{\infty} \frac{ln(k)}{k}$

17.[R] $\sum_{k=1}^{\infty} \frac{ln(k)}{k^2}$

18.[R] $\sum_{k=1}^{\infty} \frac{5^k}{k^k}$

19.[R] $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

20.[R] $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\ln(k)}$

21.[R] $\sum_{k=1}^{\infty} \frac{e^{2k}}{\pi^k}$

22.[R] $\sum_{k=1}^{\infty} \frac{k^2 e^k}{\pi^k}$

23.[R] $\sum_{k=1}^{\infty} \frac{3k+1}{2k+10}$

24.[R] $\sum_{k=1}^{\infty} \frac{4}{2k^2-k}$

25.[R] $\sum_{k=1}^{\infty} \frac{1}{\ln(k)}$

26.[R] $\sum_{k=1}^{\infty} \frac{1}{\sin(1/k)}$

27.[R] $\sum_{k=1}^{\infty} \left(\frac{k+1}{k+3}\right)^k$

28.[R] $\sum_{k=1}^{\infty} \left(\frac{k}{2k-1}\right)^k$

In Exercises 29 to 34, assume that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series with positive terms. What, if anything, can we conclude about the convergence or divergence of $\sum_{k=1}^{\infty} a_k$ if:

29.[M] If $\sum_{k=1}^{\infty} b_k$ is divergent and $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$?

30.[M] If $\sum_{k=1}^{\infty} b_k$ is convergent and $\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$?

31.[M] If $\sum_{k=1}^{\infty} b_k$ is convergent and $3b_k \leq a_k \leq 5b_k$?

32.[M] If $\sum_{k=1}^{\infty} b_k$ is divergent and $3b_k \leq a_k \leq 5b_k$?

33.[M] If $\sum_{k=1}^{\infty} b_k$ is convergent and $a_k < b_k^2$?

34.[M] If $\sum_{k=1}^{\infty} b_k$ is divergent and $b_k \to 0$ as $k \to \infty$, and $a_k < b_k^2$?

35.[M] For which values of the positive number x does the series $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k2^k}$ converge? diverge?

36.[M] For which values of the positive exponent m does the series $\sum_{k=1}^{\infty} \frac{1}{k^m \ln(k)}$ converge? diverge?

37.[C] Prove part B of the Limit-Comparison Test for Convergence and Divergence.

38.[C] For which constants p does $\sum_{k=1}^{\infty} k^p e^{-k}$ converge?

39.[C]

(a) Show that $\sum_{k=1}^{\infty} 1/(1+2^k)$ converges.

(b) Show that the sum of the series in (a) is between 0.64 and 0.77. HINT: Use the first three terms and control the sum of the rest of the series by comparing it to the sum of a geometric series.

40.[C]

(a) Show that $\sum_{k=n+1}^{\infty} 1/k!$ is less than the sum of the geometric series whose first term is $1/(n+1)!$ and whose ratio is $1/(n+2)$.

(b) Use (a) with $n = 4$ to show that

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} < \sum_{k=0}^{\infty} \frac{1}{k!} < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \cdot \frac{1}{1 - \frac{1}{6}}.$$

(c) From (b) deduce that

$$2.71 < \sum_{k=0}^{\infty} \frac{1}{k!} < 2.72.$$

(d) Find a value of n such that $\sum_{k=n+1}^{\infty} 1/k! < 0.0005$.

(e) Use (d) to estimate $\sum_{k=0}^{\infty} 1/k!$ to three decimal places.

41.[C] Prove the following result, which is used in the statistical theory of stochastic processes: Let $\{a_n\}$ and $\{c_n\}$ be two sequences of non-negative numbers such that $\sum_{k=1}^{\infty} a_k c_k$ converges and $\lim_{n\to\infty} c_n = 0$. Then $\sum_{k=1}^{\infty} a_k c_k^2$ converges.

42.[C] Find a specific number B, expressed as a decimal, such that

$$\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2} < B.$$

43.[C] Find a specific number B, expressed as a decimal, such that

$$\sum_{k=1}^{\infty} \frac{k+2}{k+1} \cdot \frac{1}{n^3} < B.$$

44.[C] Estimate $\sum_{k=1}^{\infty} \frac{1}{k2^k}$ to three decimal places.

45.[C] Let $\sum_{k=1}^{\infty} a_k$ be a convergent series with only positive terms. Must $\sum_{k=1}^{\infty} (a_k)^2$ also converge?

46.[C] Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be convergent series with only positive terms. Must $\sum_{k=1}^{\infty} a_k b_k$ converge? HINT: Review the Cauchy-Schwarz inequality in CIE 10 in Chapter 7.

11.5 Ratio Tests

The next test is suggested by the test for the convergence of a geometric series. In a geometric series the ratio between consecutive terms is constant. The "Ratio Test" concerns series where this ratio is "almost constant".

The Ratio Test

Ratio Test

Theorem 11.5.1 (Ratio Test). *Let $p_1 + p_2 + \cdots + p_n + \cdots$ be a series of positive terms. Assume $\lim_{k\to\infty} p_{k+1}/p_k$ exists and call it r.*

(a) If r is less than 1, the series converges.

(b) If r is greater than 1 or r is infinite, the series diverges.

(c) If r is equal to 1 or r does not exist, no conclusion can be drawn (the series may converge or may diverge).

Proof

The idea behind the Ratio Test is to compare the original series to a geometric series. Here is how that works.

(a) Assume $r = \lim_{k\to\infty} \frac{p_{k+1}}{p_k} < 1$. Select a number s such that $r < s < 1$. Then there is an integer N such that for all $k \geq N$,

$$\begin{aligned} \frac{p_{k+1}}{p_k} &< s \\ \text{and, therefore,} \qquad p_{k+1} &< sp_k. \end{aligned}$$

Using this inequality, we deduce that

$$\begin{aligned} p_{N+1} &< p_N \\ p_{N+2} < sp_{N+1} &< s(sp_N) < s^2 p_N \\ p_{N+3} < sp_{N+2} &< s(s^2 p_N) < s^3 p_N, \end{aligned}$$

and so on.

Thus the terms of the series

$$p_N + p_{N+1} + p_{N+2} + \cdots$$

are less than the corresponding terms of the geometric series

$$p_N + sp_N + s^2 p_N + \cdots$$

(except for the first term, p_N, which equals the first term of the geometric series). Since $s < 1$, the latter series converges. By the comparison test, $p_N + p_{N+1} + p_{N+2} + \cdots$ converges. Adding in the front end,

$$p_1 + p_2 + \cdots + p_{N-1},$$

still results in a convergent series.

(b) If $r > 1$ or is infinite, then for all k from some point on p_{k+1} is larger than p_k. Thus the n^{th} term of the series $p_1 + p_2 + \cdots$ cannot approach 0. By the n^{th}-term test for divergence the series diverges.

•

No information if r is 1 or does not exist.

When $r = 1$ or r does not exist, anything can happen; the series may diverge or it may converge. (Exercise 21 illustrates these possibilities.) In these cases, one must look to other tests to determine whether the series diverges or converges.

The Ratio Test is a natural test to try if the k^{th} term of a series involves powers of a fixed number, or factorials, as the next two examples show.

EXAMPLE 1 Show that the series $p + 2p^2 + 3p^3 + \cdots + kp^k + \cdots$ converges for any fixed number p for which $0 < p < 1$.

SOLUTION Let a_k denote the k^{th} term of the series. Then

$$a_k = kp^k \qquad \text{and} \qquad a_{k+1} = (k+1)p^{k+1}.$$

The ratio between consecutive terms is

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)p^{k+1}}{kp^k} = \frac{k+1}{k}p.$$

Thus

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = p < 1,$$

and the series converges. ◇

The value of this series is found in Exercise 34.

EXAMPLE 2 Determine the positive values of x for which the series

$$\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} \cdots$$

converges and for which values of x it diverges. (Each choice of x determines a specific series with constant terms.)

SOLUTION If we start the series with $k = 0$, then the n^{th} term, a_k is $x^k/k!$. Thus

$$a_{k+1} = \frac{x^{k+1}}{(k+1)!},$$

and therefore

$$\frac{a_{k+1}}{a_k} = \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}} = x\frac{k!}{(k+1)!} = \frac{x}{k+1}.$$

In the next section, it will be shown that this series converges for all negative values of x, too.

Since x is fixed,

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{x}{k+1} = 0.$$

By the Ratio Test, the series converges for all positive x. ◇

The next example uses the Ratio Test to establish divergence.

EXAMPLE 3 Show that the series $2/1 + 2^2/2 + \cdots + 2^k/k + \cdots$ diverges.
SOLUTION In this case, $a_k = 2^k/k$ and

$$\frac{a_{k+1}}{a_k} = \frac{\frac{2^{k+1}}{k+1}}{\frac{2^k}{k}} = \frac{2^{k+1}}{k+1}\frac{k}{2^k} = 2\frac{k}{k+1}.$$

The series is like a geometric series with ratio 2.

Thus

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 2,$$

which is larger than 1. By the Ratio Test, this series diverges. ◇

It is not really necessary to call on the powerful Ratio Test to establish the divergence of the series in Example 3. Since $\lim_{k\to\infty} 2^k/k = \infty$, its k^{th} term gets arbitrarily large; by the k^{th}-term test, the series diverges. (Comparison with the harmonic series also demonstrates divergence.)

The Root Test

The next test, closely related to the Ratio Test, is of use when the k^{th} term contains only k^{th} powers, such as k^k or 3^k. It is not useful if factorials such as $k!$ are present.

Root Test

Theorem 11.5.2 (Root Test)**.** *Let $\sum_{k=1}^{\infty} p_k$ be a series of positive terms. Assume $\lim_{k\to\infty} \sqrt[k]{p_k}$ exists and call it r. Then*

A. *If r is less than 1, the series converges.*

B. *If r is greater than 1 or r is infinite, the series diverges.*

C. If r is equal to 1 or r does not exist, no conclusion can be drawn (the series may converge or may diverge).

The proof of the Root Test is outlined in Exercises 22 and 23.

EXAMPLE 4 Use the Root Test to determine whether $\sum_{k=1}^{\infty} 3^k/k^{k/2}$ converges or diverges.
SOLUTION We have

$$r = \lim_{k\to\infty} \sqrt[k]{\frac{3^k}{k^{k/2}}} = \lim_{k\to\infty} \frac{3}{\sqrt{k}} = 0.$$

By the Root Test, the series converges. ◇

Summary

We developed two tests for convergence or divergence of a series $\sum_{k=1}^{\infty} p_k$ with positive terms, both motivated by geometric series. In the Ratio Test, we examine $\lim_{k\to\infty} p_{k+1}/p_k$ and in the Root Test, $\lim_{k\to\infty} \sqrt[k]{p_k}$. The Root Test is convenient when only powers appear. The Ratio Test is convenient to use when the terms involve powers and factorials.

EXERCISES for Section 11.5 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 6 apply the Ratio Test to decide whether the series converges or diverges. If that test gives no information, use another test to decide.

1.[R] $\sum_{k=1}^{\infty} \frac{k^2}{3^k}$

2.[R] $\sum_{k=1}^{\infty} \frac{(k+1)^2}{k2^k}$

3.[R] $\sum_{k=1}^{\infty} \frac{k\ln(k)}{3^k}$

4.[R] $\sum_{k=1}^{\infty} \frac{k!}{3^k}$

5.[R] $\sum_{k=1}^{\infty} \frac{(2k+1)(2^k+1)}{3^k+1}$

6.[R] $\sum_{k=1}^{\infty} \frac{k!}{k^k}$

In Exercises 7 and 8 use the Root Test to determine whether the series converge or diverge.

7.[R] $\sum_{k=1}^{\infty} \frac{k^k}{3^{k^2}}$

8.[R] $\sum_{k=1}^{\infty} \frac{(1+1/k)^k(2k+1)^k}{(3k+1)^k}$

Each series found in Exercises 9 to 14 converges. Use any legal means to find a number B in decimal form that is larger than the sum of the series.

9.[R] $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

10.[R] $\sum_{k=1}^{\infty} \frac{k}{3^k}$

11.[R] $\sum_{k=1}^{\infty} \frac{1}{k^3}$

12.[R] $\sum_{k=1}^{\infty} \frac{\sin^2(k)}{k^2}$

13.[R] $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$

14.[R] $\sum_{k=1}^{\infty} \frac{\left(1+\frac{2}{k}\right)^k}{1.1^k}$

Each series in Exercises 15 to 18 diverges. Use any legal means to find a number m such that the m^{th} partial sum of the series exceeds 1,000.

15.[R] $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$

16.[R] $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

17.[R] $\displaystyle\sum_{k=1}^{\infty} (1.01)^k$

18.[R] $\displaystyle\sum_{k=1}^{\infty} \frac{(k+2)^2}{k+1} \cdot \frac{1}{\sqrt{k}}$

19.[M] Use the result of Example 2 to show that, for $x > 0$, $\lim_{k\to\infty} x^k/k! = 0$. NOTE: This was established directly in Section 11.2.

20.[M] Solve Example 3 using the Root Test.

21.[M] This exercise shows that the Ratio Test gives no information if $\displaystyle\lim_{k\to\infty} \frac{p_{k+1}}{p_k} = 1$.

(a) Show that for $p_k = 1/k$, $\sum_{k=1}^{\infty} p_k$ diverges and $\lim_{k\to\infty} \frac{p_{k+1}}{p_k} = 1$.

(b) Show that for $p_k = 1/k^2$, $\sum_{k=1}^{\infty} p_k$ converges and $\lim_{k\to\infty} \frac{p_{k+1}}{p_k} = 1$.

22.[M] This exercise shows that the Root Test gives no information if $\displaystyle\lim_{k\to\infty} \sqrt[k]{p_k} = 1$.

(a) Show that for $p_k = 1/k$, $\sum_{k=1}^{\infty} p_k$ diverges and $\lim_{k\to\infty} \sqrt[k]{p_k} = 1$.

(b) Show that for $p_k = 1/k^2$, $\sum_{k=1}^{\infty} p_k$ converges and $\lim_{k\to\infty} \sqrt[k]{p_k} = 1$.

23.[C] (Proof of the Root Test, Theorem 11.5.2.)

(a) Assume that $r = \lim_{k\to\infty} \sqrt[k]{p_k} < 1$. Pick any s with $r < s < 1$, and then pick N such that $\sqrt[k]{p_k} < s$ for all $k > N$. Show that $p_k < s^k$ for all $k > N$ and compare a tail end of $\sum_{k=1}^{\infty} p_k$ to a geometric series.

(b) Assume that $r = \lim_{k\to\infty} \sqrt[k]{p_k} > 1$. Pick any s with $1 < s < r$, and then pick N such that $\sqrt[k]{p_k} > s$ for all $k > N$. Show that $p_k > s^k$ for all $k > N$. From this conclude that $\sum_{k=1}^{\infty} p_k$ diverges.

Skill Drill

In Exercises 24 to 26 a, b, c, m, and p are constants. In each case verify that the derivative of the first function is the second function.

24.[R] $a^2 x\sin(ax)$; $\sin(ax) - ax\cos(ax)$

25.[R] $\ln\left|ax^2+bx+c\right|$; $\dfrac{2ax+b}{ax^2+bx+c}$

26.[R] $x\arctan(ax) - \dfrac{1}{2a}\ln\left(1+a^2x^2\right)$; $\arctan(ax)$

In Exercises 27 to 32 a, b, c, and n are constants and n is positive. Use integration techniques to obtain each of the following reduction formulas.

27.[R] $\displaystyle\int x^n\sin(ax)\ dx = -\frac{1}{a}\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)\ dx$

28.[R] $\displaystyle\int x^n\cos(ax)\ dx = \frac{1}{a}\cos(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\ dx$

29.[R] $\displaystyle\int \frac{dx}{x^2\sqrt{ax+b}} = \frac{-\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$

30.[R] $\displaystyle\int \frac{dx}{(ax^2+c)^(n+1)} = \frac{1}{2nc}\frac{x}{(ax^2+c)^n} + \frac{2n-3}{2nc}\int \frac{dx}{(ax^2+c)^n}$

31.[R] $\displaystyle\int \frac{dx}{(ax^2+bx+c)^{n+1}} = \frac{2ax+b}{n(4ac-b^2)(ax^2+bx+c)^n} + \frac{2(2n-1)a}{n(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^n}$

32.[R] $\displaystyle\int (\ln(ax))^2\ dx = x^2\left((\ln(ax))^2 - 2\ln(ax) + 2\right)$

11.6 Tests for Series with Both Positive and Negative Terms

The tests for convergence or divergence in Sections 11.3 to 11.5 concern series whose terms are positive. This section examines series that have both positive and negative terms. Two tests for the convergence of such a series are presented. The alternating-series test applies to series whose terms alternate in sign (+, -, +, -, ...) and decrease in absolute value. In the absolute-convergence test, the signs may vary in any way.

Alternating Series

DEFINITION *(Alternating Series)* If $p_1, p_2, \ldots, p_n, \ldots$ is a sequence of positive numbers, then the series

$$\sum_{k=1}^{\infty}(-1)^{k+1}p_k = p_1 - p_2 + p_3 - p_4 + \cdots + (-1)^{k+1}p_k + \cdots$$

and the series

$$\sum_{k=1}^{\infty}(-1)^{k}p_k = -p_1 + p_2 - p_3 + p_4 - \cdots + (-1)^{k}p_k + \cdots$$

are called **alternating series**.

For instance,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{k+1}\frac{1}{2k-1} + \cdots$$

and

$$1 - 1 + 1 - 1 + \cdots + (-1)^k + \cdots$$

are alternating series.

By the n^{th}-term test, the second series diverges. The following theorem implies that the first series converges.

Alternating-Series Test

Theorem. *(Alternating-Series Test) If $p_1, p_2, \ldots, p_k, \ldots$ is a decreasing sequence of positive numbers such that $\lim_{k\to\infty} p_k = 0$, then the series whose k^{th} term is $(-1)^{k+1}p_k$,*

$$\sum_{k=1}^{\infty}(-1)^{k+1}p_k = p_1 - p_2 + p_3 - \cdots + (-1)^{k+1}p_k + \cdots ,$$

converges.

Proof

We will prove the theorem in the special case when $p_k = 1/k$, that is, the **alternating harmonic series**

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1}\frac{1}{k} + \cdots .$$

The argument easily generalizes to prove the general theorem. (See Exercise 33.)

Consider first the partial sums of an *even* number of terms, S_2, S_4, S_6, ... For clarity, group the summands in pairs:

$$\begin{array}{lcll} S_2 & = & (1-\frac{1}{2}) & \\ S_4 & = & (1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4}) & = S_2+(\frac{1}{3}-\frac{1}{4}) \\ S_6 & = & (1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6}) & = S_4+(\frac{1}{5}-\frac{1}{6}) \\ \vdots & & & \end{array}$$

Since $\frac{1}{3}$ is larger than $\frac{1}{4}$, the difference $\frac{1}{3}-\frac{1}{4}$ is positive. Therefore, S_4, which equals $S_2 + (\frac{1}{3}-\frac{1}{4})$, is larger than S_2. Similarly, $S_6 > S_4$. More generally:

$$S_2 < S_4 < S_6 < S_8 < \cdots .$$

Figure 11.6.1:

The sequence of even partial sums, $\{S_{2n}\}$ is increasing. (See Figure 11.6.1.)

Next, it will be shown that S_{2n} is less than 1, the first term of the sequence. First of all,

$$S_2 = 1 - \frac{1}{2} < 1.$$

Next, consider S_4:

$$\begin{array}{lcll} S_4 & = & 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4} & \\ & = & 1-(\frac{1}{2}-\frac{1}{3})-\frac{1}{4} & \\ & < & 1-(\frac{1}{2}-\frac{1}{3}) & \text{because } \frac{1}{4} \text{ is positive} \\ & < & 1 & \text{because } \frac{1}{2}-\frac{1}{3} \text{ is positive.} \end{array}$$

Similarly,

$$\begin{array}{lcll} S_6 & = & 1-(\frac{1}{2}-\frac{1}{3})-(\frac{1}{4}-\frac{1}{5})-\frac{1}{6} & \\ & < & 1-(\frac{1}{2}-\frac{1}{3})-(\frac{1}{4}-\frac{1}{5}) & \text{because } \frac{1}{6} \text{ is positive} \\ & < & 1-(\frac{1}{2}-\frac{1}{3}) & \text{because } \frac{1}{4}-\frac{1}{5} \text{ is positive} \\ & < & 1 & \text{because } \frac{1}{2}-\frac{1}{3} \text{ is positive.} \end{array}$$

In general then,

$$S_{2n} < 1 \qquad \text{for all } n.$$

The sequence

$$S_2, S_4, S_6, \ldots$$

is therefore increasing and yet bounded by the number 1, as indicated in Figure 11.6.2. By Theorem 10.1.1 of Section 10.1, $\lim_{n\to\infty} S_{2n}$ exists. Call this limit S, which is less than or equal to 1. (See Figure 11.6.2.)

All that remains is to show that the *odd* partial sums

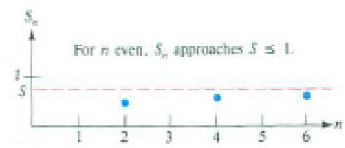


Figure 11.6.2:

$$S_1, S_3, S_5, \ldots$$

also converge to S.

Note that

$$\begin{array}{lclcl} S_3 & = & 1 - \frac{1}{2} + \frac{1}{3} & = & S_2 + \frac{1}{3} \\ S_5 & = & 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} & = & S_4 + \frac{1}{5}. \end{array}$$

In general,

$$S_{2k+1} = S_{2k} + \frac{1}{2k+1}.$$

In the general case, the term $1/(2k+1)$ will be replaced by p_{2k+1}.

Thus

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty}\left(S_{2k} + \frac{1}{2k+1}\right) = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty}\frac{1}{2k+1} = S + 0 = S.$$

Since the sequence of even partial sums, S_2, S_4, S_6, ..., S_{2k}, ..., and the sequence of odd partial sums, S_1, S_3, S_5, ..., S_{2k+1}, ..., both have the same limit, S, it follows that

$$\lim_{k\to\infty} S_k = S.$$

Thus the alternating harmonic series

$$\sum_{k=1}^{\infty}(-1)^k\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges. In Chapter 12 it will be shown that this sum is $\ln(2)$.

See Exercise 28.

A similar argument applies to any alternating series whose k^{th} term approaches 0 and whose terms decrease in absolute value. •

Decreasing alternating series

An alternating series, such as the alternating harmonic series, whose terms decrease in absolute value as k increases will be called a **decreasing alternating series**. Theorem 11.6 shows that a decreasing alternating series whose

k^{th} term approaches zero as $k \to \infty$ converges.

EXAMPLE 1 Estimate the sum S of the alternating harmonic series.
SOLUTION These are the first five partial sums:

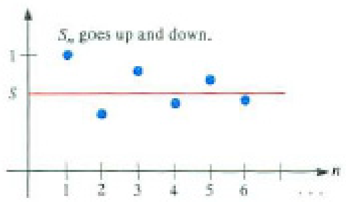


Figure 11.6.3:

$$\begin{array}{lcll}
S_1 & = & 1 & = 1.00000 \\
S_2 & = & 1 - \frac{1}{2} & = 0.50000 \\
S_3 & = & 1 - \frac{1}{2} + \frac{1}{3} \approx 0.5 + 0.33333 & = 0.83333 \\
S_4 & = & S_3 - \frac{1}{4} \approx 0.83333 - 0.25 & = 0.58333 \\
S_5 & = & S_4 + \frac{1}{5} \approx 0.58333 + 0.2 & = 0.78333
\end{array}$$

Figure 11.6.3 is a graph of S_n as a function of n. The odd partial sums S_1, S_3, ... approach S from above. The even partial sums S_2, S_4, ... approach S from below. For instance,

$$S_4 < S < S_5$$

gives the information that $0.58333 < S < 0.8334$. (See Figure 11.6.4.) $\diamond$

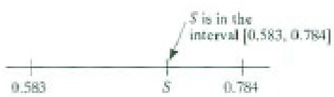


Figure 11.6.4:

As Figure 11.6.3 suggests, any partial sum of a series satisfying the hypothesis of the alternating-series test differs from the sum of the series by less than the absolute value of the first omitted term. That is, if S_n is the sum of the first n terms of the series and S is the sum of the series, then the error

$$R_n = S - S_n$$

The error in estimating the sum of a decreasing alternating series.

has absolute value at most p_{n+1}, which is the absolute value of the first omitted term. Moreoveer, S is between S_n and S_{n+1} for every n.

EXAMPLE 2 Does the series

$$\frac{3}{1!} - \frac{3^2}{2!} + \frac{3^3}{3!} - \frac{3^4}{4!} + \frac{3^5}{5!} - \cdots + (-1)^{k+1}\frac{3^k}{k!} + \cdots$$

converge or diverge?
SOLUTION This is an alternating series. By Example 2 of Section 11.2, its k^{th} term approaches 0. Let us see whether the absolute values of the terms decrease in size, term-by-term. The first few absolute values are

$$\begin{array}{lcl}
\frac{3}{1!} & = & 3 \\
\frac{3^2}{2!} & = & \frac{9}{2} = 4.5 \\
\frac{3^3}{3!} & = & \frac{27}{6} = 4.5 \\
\frac{3^4}{4!} & = & \frac{81}{24} = 3.375.
\end{array}$$

At first, they increase. However, the fourth term is less than the third. Let us show that the rest of the terms decrease in size. For instance,

At first the terms increase, but then they decrease.

$$\frac{3^5}{5!} = \frac{3}{4}\frac{3^4}{4!} < \frac{3^4}{4!},$$

and, for $n \geq 3$,

$$\frac{3^{k+1}}{(k+1)!} = \frac{3}{n+1}\frac{3^k}{k!} < \frac{3^k}{k!}.$$

By the alternating-series test, the tail end that begins

$$\frac{3^3}{3!} - \frac{3^4}{4!} + \frac{3^5}{5!} - \frac{3^6}{6!} - \cdots$$

converges. Call its sum S. If the front end

$$\frac{3}{1!} - \frac{3^2}{2!}$$

is added on, we obtain the original series, which therefore converges and has the sum

$$\frac{3}{1!} - \frac{3^2}{2!} + S.$$

◇

As Example 2 illustrates, the alternating-series test works as long as the k^{th} term approaches 0 and the terms decrease in size from some point on.

It may seem that any alternating series whose k^{th} term approaches 0 converges. *This is not the case*, as shown by this series:

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{4} + \cdots, \tag{11.6.1}$$

whose terms alternate $2/k$ and $-1/k$.

Let S_n be the sum of the first n terms of (11.6.1). Then

$$\begin{aligned} S_2 &= \tfrac{2}{1} - \tfrac{1}{1} &&= \tfrac{1}{1}, \\ S_4 &= \left(\tfrac{2}{1} - \tfrac{1}{1}\right) + \left(\tfrac{2}{2} - \tfrac{1}{2}\right) &&= \tfrac{1}{1} + \tfrac{1}{2}, \\ S_6 &= \left(\tfrac{2}{1} - \tfrac{1}{1}\right) + \left(\tfrac{2}{2} - \tfrac{1}{2}\right) + \left(\tfrac{2}{3} - \tfrac{1}{3}\right) &&= \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3}, \end{aligned}$$

and, more generally,

$$S_{2n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Since S_{2n} gets arbitrarily large as $n \to \infty$, the series (11.6.1) diverges.

Recall that the harmonic series diverges.

Absolute Convergence

Consider the series

$$a_1 + a_2 + \cdots + a_n \cdots ,$$

whose terms may be positive, negative, or zero. It is reasonable to expect it to behave at least as "nicely" as the corresponding series with non-negative terms

$$|a_1| + |a_2| + \cdots + |a_n| + \cdots ,$$

since by making all the terms positive we give the series more chance to diverge. This is similar to the case with improper integrals in Section 7.8, where it was shown that if $\int_a^\infty |f(x)|\ dx$ converges, then so does $\int_a^\infty f(x)\ dx$. The next theorem (and its proof) is similar to the Absolute-Convergence Test for Improper Integrals in Section 7.8. (Re-read it. It's on page 666.)

Absolute-Convergence Test

Theorem 11.6.1. *(Absolute-Convergence Test) If the series $\sum_{k=1}^\infty |a_k|$ converges, then so does the series $\sum_{k=1}^\infty a_k$. Furthermore, if $\sum_{k=1}^\infty |a_k| = S$, then $\sum_{k=1}^\infty a_k$ is between $-S$ and S.*

Proof

We introduce two series in order to record the behavior of the positive and negative terms in $\sum_{k=1}^\infty a_k$ separately. Let

$$b_k = \begin{cases} a_k & \text{if } a_k \text{ is positive} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad c_k = \begin{cases} a_k & \text{if } a_k \text{ is negative} \\ 0 & \text{otherwise} \end{cases}.$$

Note that $a_k = b_k + c_k$. To establish the convergence of $\sum_{k=1}^\infty a_k$ we show that both $\sum_{k=1}^\infty b_k$ and $\sum_{k=1}^\infty c_k$ converge. First of all, since b_k is non-negative and $b_k \le |a_k|$, the series of positive terms, $\sum_{k=1}^\infty b_k$, converges by the comparison test. In fact, it converges to a number $P \le S$.

Since c_k is non-positive, and $c_k \ge -|a_k|$, the series of negative terms, $\sum_{k=1}^\infty c_k$, converges to a number $N \ge -S$. Thus $\sum_{k=1}^\infty a_k = \sum_{k=1}^\infty (b_k + c_k)$ converges to $P + N$, which is between $-S$ and S. •

EXAMPLE 3 Examine the series

$$\frac{\cos(x)}{1^2} + \frac{\cos(2x)}{2^2} + \frac{\cos(3x)}{3^2} + \cdots + \frac{\cos(kx)}{k^2} + \cdots \qquad (11.6.2)$$

for convergence or divergence.

SOLUTION The number x is fixed. The numbers $\cos(kx)$ may be positive, negative, or zero, in an irregular manner. However, for all k, $|\cos(kx)| \le 1$.

The series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2}$$

is the p-series with $p = 2$, which converges (by the integral test). Since $\left|\frac{\cos(kx)}{k^2}\right| \leq \frac{1}{k^2}$, the series

When you study Fourier series, in Section 12.7, you will learn that for $0 \leq x \leq 2\pi$, series (11.6.2) sums to $\frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)$.

$$\frac{|\cos(x)|}{1^2} + \frac{|\cos(2x)|}{2^2} + \frac{|\cos(3x)|}{3^2} + \cdots + \frac{|\cos(kx)|}{k^2} + \cdots \qquad (11.6.3)$$

converges by the comparison test. Theorem 11.6.1 then tells us that (11.6.2) converges. ◇

WARNING *(Converse of Theorem 11.6.1 is false)* If $\sum_{k\to\infty} a_k$ converges, then $\sum_{k\to\infty} |a_k|$ may converge or diverge. The standard counterexample to the converse of Theorem 11.6.1 is the alternating harmonic series, $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots$. This series converges, as was shown by the alternating-series test (Theorem 11.6). But, when all of the terms are replaced by their absolute values, the resulting serise is the harmonic series, $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$, which diverges (it is a p-series with $p = 1$).

The following definitions are frequently used in describing these various cases of convergence or divergence.

DEFINITION *(Absolute Convergence)* A series $a_1 + a_2 + \cdots$ is said to **converge absolutely** if the series $|a_1| + |a_2| + \cdots$ converges.

Theorem 11.6.1 can then be stated simply: "If a series converges absolutely, then it converges."

DEFINITION *(Conditional Convergence)* A series $a_1 + a_2 + \cdots$ is said to **converge conditionally** if it converges but does *not* converge absolutely.

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges conditionally.

For instance, the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

Absolute-Limit-Comparison Test

When you combine the limit-comparison test for positive series with the absolute-convergence test, you obtain a single test, described in Theorem 11.6.2.

Absolute-Limit-Comparison Test

Theorem 11.6.2. *(Absolute-Limit-Comparison Test) Let* $\sum_{k=1}^{\infty} a_k$ *be a series whose terms may be negative or positive. Let* $\sum_{k=1}^{\infty} c_k$ *be a convergent series of positive terms. If*

$$\lim_{k\to\infty} \left| \frac{a_k}{c_k} \right|$$

exists, then $\sum_{k=1}^{\infty} a_k$ *is absolutely convergent, hence convergent.*

Proof

Note that $|a_k/c_k| = |a_k|/c_k$, since c_k is positive. The limit-comparison test tells us that $\sum_{k=1}^{\infty} |a_k|$ converges. Then the absolute-convergence test assures us that $\sum_{k=1}^{\infty} a_k$ converges. •

One advantage of the absolute-convergence test over the limit-comparison test is that we don't have to follow it by the absolute-convergence test. Another is that we don't have to worry about the arithmetic of negative numbers.

EXAMPLE 4 Show that

$$\frac{3}{2}\left(\frac{1}{2}\right) - \frac{5}{2}\left(\frac{1}{2}\right)^2 + \frac{7}{3}\left(\frac{1}{2}\right)^3 - \cdots + (-1)^{k+1}\frac{2k+1}{k}\left(\frac{1}{2}\right)^k + \cdots \qquad (11.6.4)$$

converges.

SOLUTION Consider the series with positive terms

$$\frac{3}{2}\left(\frac{1}{2}\right) + \frac{5}{2}\left(\frac{1}{2}\right)^2 + \frac{7}{3}\left(\frac{1}{2}\right)^3 + \cdots + \frac{2k+1}{k}\left(\frac{1}{2}\right)^k + \cdots .$$

The fact that $(2k+1)/k \to 2$ as $k \to \infty$ suggests use of the limit-comparison test, comparing the second series to the convergent geometric series $\sum_{k=1}^{\infty}(1/2)^k$. We have

$$\lim_{k\to\infty} \frac{\frac{2k+1}{k}\left(\frac{1}{2}\right)^k}{\left(\frac{1}{2}\right)^k} = 2.$$

Thus $\sum_{k=1}^{\infty}((2k+1)/k)(1/2)^k$ converges. Consequently, the first series (11.6.4), with both positive and negative terms, converges absolutely. Thus it converges. ◇

Absolute-Ratio Test

Absolute-Ratio Test

The ratio test of Section 11.5 also has an analog that applies to series with both negative and positive terms.

Theorem 11.6.3 (Absolute-Ratio Test). *Let $\sum_{k=1}^{\infty} a_k$ be a series such that*

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = r < 1.$$

Then $\sum_{k=1}^{\infty} a_k$ converges. If $r > 1$ or if $\lim_{k\to\infty}|a_{k+1}/a_k| = \infty$, then $\sum_{k=1}^{\infty} a_k$ diverges. If $r = 1$, then the Absolute-Ratio Test gives no information.

Proof

Take the case $r < 1$. By the Ratio Test, $\sum_{k=1}^{\infty}|a_k|$ converges. Since $\sum_{k=1}^{\infty}|a_k|$ converges, it follows that $\sum_{k=1}^{\infty} a_k$ converges also.

The case $r > 1$ is treated in Exercise 34.

The case $r = \infty$ can be treated as follows. If $\lim_{k\to\infty}|a_{k=1}/a_k| = \infty$, the ratio $|a_{k+1}|/|a_k|$ gets arbitrarily large as $k \to \infty$. So from some point on the positive numbers $|a_k|$ increase. By the k^{th}-Term Test for Divergence, $\sum_{k=1}^{\infty} a_k$ is divergent. •

The Absolute-Ratio Test avoids work with minus signs.

Theorem 11.6.3 establishes the convergence of the series in Example 4 as follows. Let $a_k = (-1)^{k+1}\frac{(2k+1)}{k2^k}$. Then

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k+2}\frac{(2k+3)}{(k+1)2^{k+1}}}{(-1)^{k+1}\frac{(2k+1)}{k2^k}}\right| = \frac{2k+3}{2k+1}\cdot\frac{k}{k+1}\cdot\frac{1}{2},$$

which approaches $r = \frac{1}{2}$ as $k \to \infty$. Thus $\sum_{k=1}^{\infty} a_k$ converges (in fact, absolutely).

Rearrangements

$1 + 13 + 15 + 27 = 13 + 27 + 15 + 1$

The sum of a finite collection of numbers does not depend on the order in which they are added. A series that converges absolutely is similar: no matter how the terms of an absolutely convergent series are rearranged, the new series converges and has the same sum as the original series. It might be expected that any convergent series has this property, but this is not the case. For instance, the alternating harmonic series

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \tag{11.6.5}$$

does not. To show this, rearrange the terms so that two positive terms alternate with one negative term, as follows:

$$\frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots. \tag{11.6.6}$$

Rearranging the terms in a conditionally convergent series is dangerous.

The positive summands in (11.6.6) have much more influence than the negative summands. In the battle between the positives and the negatives, the positives will win by a bigger margin in (11.6.6) than in (11.6.5). In fact, the sum of (11.6.6) is $\frac{3}{2}\ln(2)$, while Exercise 27 shows that the sum of (11.6.5) is $\ln(2)$.

Conditionally convergent series are so sensitive that they can be made to sum to any number that you choose. To be precise, Riemann proved: if $\sum_{k=1}^{\infty} a_k$ is a conditionally convergent series and s is any real number, then there is a rearrangement of $\sum_{k=1}^{\infty} a_k$ whose sum is s. This is proved in Exercise 40.

Summary

Earlier in this chapter we described ways to test for the convergence or divergence of series whose terms are all positive. This section describes several tests for series that may be a mix of positive and negative terms.

- If the signs alternate and the absolute value of the terms decreases and approach 0, the series converges. [Alternating-Series Test]
- If the series converges when "all the terms are made positive," then it converges. [Absolute-Convergence Test]
- This Absolute-Convergence Test in combination with the Limit-Comparison Test gives us a single test, called the Absolute-Limit-Comparison Test.
- The Absolute-Convergence Test in combination with the Ratio Test gives us the Absolute-Ratio Test. (This will be the most important test in Chapter 12.)

EXERCISES for Section 11.6 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 8 concern alternating series. Determine which series converge and which diverge. Explain your reasoning.

1.[R] $\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\cdots+(-1)^{k+1}\frac{k}{k+1}+\cdots$

2.[R] $-\frac{1}{1+\frac{1}{2}}+\frac{1}{1+\frac{1}{4}}-\frac{1}{1+\frac{1}{8}}+\cdots+(-1)^{k}\frac{1}{1+2^{-k}}+\cdots$

3.[R] $\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots+(-1)^{k+1}\frac{1}{\sqrt{k}}+\cdots$

4.[R] $\frac{5}{1!}-\frac{5^2}{2!}+\frac{5^3}{3!}-\frac{5^4}{4!}+\cdots+(-1)^{k+1}\frac{5^k}{k!}+\cdots$

5.[R] $\frac{3}{\sqrt{1}}-\frac{2}{\sqrt{1}}+\frac{3}{\sqrt{2}}-\frac{2}{\sqrt{2}}+\frac{3}{\sqrt{3}}-\frac{2}{\sqrt{3}}+\cdots$

6.[R] $\sqrt{1}-\sqrt{2}+\sqrt{3}-\sqrt{4}+\cdots+(-1)^{k+1}\sqrt{k}+\cdots$

7.[R] $\frac{1}{3}-\frac{2}{5}+\frac{3}{7}-\frac{4}{9}+\frac{5}{11}-\cdots+(-1)^{k+1}\frac{k}{2k+1}+\cdots$

8.[R] $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots+(-1)^{k+1}\frac{1}{k^2}+\cdots$

9.[R] Consider the alternating harmonic series

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}.$$

(a) Compute S_5 and S_6 to five decimal places.

(b) Is the estimate S_5 smaller or larger than the sum of the series?

(c) Use (a) and (b) to find two numbers between which the sum of the series must lie.

10.[R] Consider the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{-k}}{k}$.

(a) Estimate the sum of the series using S_6.

(b) Estimate the error R_6.

11.[R] Does the series

$$\frac{2}{1}-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\cdots+(-1)^{k+1}\left(\frac{n+1}{n}\right)+\cdots.$$

converge or diverge?

In Exercises 12 to 26 determine which series diverge, converge absolutely, or converge conditionally. Explain your answers.

12.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt[3]{k^2}}$

13.[R] $\displaystyle\sum_{k=1}^{\infty}\ln\left(\frac{1}{k}\right)$

14.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k\ln(k)}$

15.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{\sin(k)}{k^{1.01}}$

16.[R] $\displaystyle\sum_{k=1}^{\infty}\left(1-\cos\left(\frac{\pi}{k}\right)\right)$

17.[R] $\displaystyle\sum_{k=1}^{\infty}(-1)^k\cos\left(\frac{\pi}{k^2}\right)$

18.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{(-2)^k}{k!}$

19.[R] $\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}-\cdots$ NOTE: There are two +'s alternating with two −'s.

20.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{(-3)^k(1+k^2)}{k!}$

21.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{\cos(k\pi)}{2k+1}$

22.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k(k+5)}{k^2}$

23.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{(-9)^k}{10^k+k}$

24.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt[3]{k}}$

25.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{(-1.01)^k}{k!}$

26.[R] $\displaystyle\sum_{k=1}^{\infty}\frac{(-\pi)^{2k+1}}{(2k+1)!}$

27.[R] For which values of x does $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k!}$ converge?

28.[R] The series $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} 2^{-k}$ is both a geometric series and a decreasing alternating series whose k^{th} term approaches 0.

(a) Compute S_6 to three decimal places.

(b) Using the fact that the series is a decreasing alternating series, put a bound on R_6.

(c) Using the fact that the series is a geometric series, compute R_6 exactly.

29.[M]

(a) How many terms of the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ must you take to be sure the error is less than 0.005? Explain.

(b) Estimate $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ to two decimal places.

30.[M] Estimate $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$ to two decimal places. Explain your reasoning.

31.[M]

(a) Show $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$ converges.

(b) Estimate the sum of the series in (a) to two decimal places.

32.[C] Let $P(x)$ and $Q(x)$ be two polynomials of degree at least one. Assume that for $n \geq 1$, $Q(n) \neq 0$. What relation must there be between the degrees of $P(x)$ and $Q(x)$ if

(a) $\frac{P(k)}{Q(k)} \to 0$ as $k \to \infty$?

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{P(k)}{Q(k)}$ converges absolutely?

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{P(k)}{Q(k)}$ converges absolutely?

33.[C] The Alternating-Series Test was proved only for the alternating harmonic series. Prove it in general. HINT: The only difference is that the k^{th} term is $(-1)^{k+1}p_k$ instead of $(-1)^{k+1}/k$.

34.[C] This exercise treats the second half of the absolute-ratio test.

(a) Show that if

$$\rho = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| > 1,$$

then $|a_k| \to \infty$ as $k \to \infty$. HINT: First show that there is a number r, $r > 1$, such that for some integer N, $|a_{k+1}| > r|a_k|$ for all $k \geq N$.

(b) From (a) deduce that a_k does not approach 0 as $k \to \infty$.

35.[M] Consider . For which values of x does the series $\displaystyle\sum_{k=1}^{\infty} \frac{kx^k}{2k+1}$ diverge? converge conditionally? converge absolutely? Record your conclusions in a diagram on the x-axis.

36.[M] Repeat Exercise 35 for the series *(a)* $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k!}$ and *(b)* $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$.

37.[C] Is this argument okay? Add the alternating harmonic series to half of itself:

$$\begin{array}{ccccccccccccccc}
1 & -\frac{1}{2} & +\frac{1}{3} & -\frac{1}{4} & +\frac{1}{5} & -\frac{1}{6} & +\frac{1}{7} & -\frac{1}{8} & +\frac{1}{9} & -\frac{1}{10} & -\frac{1}{11} & +\frac{1}{12} & +\cdots & = & S \\
 & \frac{1}{2} & & -\frac{1}{4} & & +\frac{1}{6} & & -\frac{1}{8} & & +\frac{1}{10} & & -\frac{1}{12} & +\cdots & = & \frac{1}{2}S \\
\hline
1 & & +\frac{1}{3} & -\frac{1}{2} & +\frac{1}{5} & & +\frac{1}{7} & -\frac{1}{4} & +\frac{1}{9} & & -\frac{1}{11} & & +\cdots & = & \frac{3}{2}S
\end{array}$$

Rearranging the last line produces the alternating harmonic series, whose sum is S. Thus $S = \frac{3}{2}S$, from which it follows that $S = 0$.

38.[C]

Sam: I have a neat proof that absolute convergence implies convergence. First of all,

$$a_n = a_n + |a_n| - |a_n|.$$

Jane: True, but why do that?

Sam: Don't interrupt me. Just wait. Now $a_n + |a_n|$ is 0 if a_n is negative and it's $2|a_n|$ if a_n is positive. Right?

Jane: If you say so.

Sam: Just think.

Jane: Yes, I agree.

Sam: So $0 \le a_n + |a_n| \le 2|a_n|$. Right? So $\sum(a_n + |a_n|)$ converges.

Jane: Yes.

Sam: You can fill in the rest, yes?

Jane: Oh, neat.

Sam: Yeh, mathematicians really like this proof.

Is the proof correct? (Explain your answer.) Which proof do you prefer, this one or the one on page 978?

39.[C] If $\displaystyle\sum_{k=1}^{\infty} a_k$ converges and $a_k > 0$ for all k, what, if anything, can we say about the convergence or divergence of (a) $\displaystyle\sum_{k=1}^{\infty} \sin(a_k)$ and (b) $\displaystyle\sum_{k=1}^{\infty} \cos(a_k)$?

40.[C] Prove that if $\sum_{k=1}^{\infty} a_k$ is a conditionally convergent series and s is any real number, then there is a rearrangement of $\sum_{k=1}^{\infty} a_k$ whose sum is s. Hint: A conditionally convergent series must have an endless supply of both positive and negative numbers. And, the series of positive terms and the series of negative terms, separately, diverge. Use these facts to explain how to construct a rearrangement that converges to s.

41.[C] In the proof of the Absolute-Convergence Theorem, why does $\sum_{k=1}^{\infty} c_k$ converge and have a sum greater than or equal to $-S$?

42.[C] The Absolute-Convergence Test asserts that $\sum_{k=1}^{\infty} a_k$ is between $-S$ and S. Why is that?

11.S Chapter Summary

This chapter concerns sequences formed by adding a finite number of terms from another sequence: $S_n = a_1 + a + 2 + \cdots + a_n$. Two questions motivate the sections:

- Does $\lim_{n\to\infty} S_n$ exist?
- If the limit exists, how do we estimate it?

If the limit exists, it is denoded $\displaystyle\sum_{k=1}^{\infty} a_k$, though we never add an infinite number of summands.

Some of the tests for convergene of divergence apply only to series whose terms are positive (or all are negative): the Integral Test, the Comparison Tests, and the Ratio Tests.

For series whose terms a_n may be both positive and negative, the key is that if $\sum_{k=1}^{\infty} |a_k|$ converges so must $\sum_{k=1}^{\infty} a_n$. Moreover, if $\sum_{k=1}^{\infty} |a_k| = L$, then $-L \le a_k \le L$.

If the series alternates, $a_1 - a_2 + a_3 - a_4 + \cdots$ and $a_k \to 0$ monotonically, then $\sum_{k=1}^{\infty} a_k$ converges.

The Integral Test, the Comparison Tests, and the formula for the sum of a geometric series also provide ways to estimate the error in using a particular S_n to approximate the sum of the series.

EXERCISES for 11.S *Key:* R–routine, M–moderate, C–challenging

1.[R] Explain in your own words.

(a) Why the Comparison Test for convergence works.

(b) Why the Ratio Test for convergence works.

(c) Why the Alternating-Series Test works.

(d) Why the Absolute-Convergence Test works.

2.[R] How many terms of the series $\sum_{k=1}^{\infty}(-1)^{n+1}(1/n^2)$ should be used to estimate its sum to three-decimal place accuracy?

3.[R] For which type of series does each of these tests imply convergence:

(a) Alternating-Series Test

(b) Integral Test

(c) Comparison Test

(d) Absolute-Convergence Test

(e) Absolute-Ratio Test.

4.[R] Assume that $|a_k| \leq 1/2^n$ for $n \geq 1$.

(a) Must $\sum_{k=1}^{\infty} |a_k|$ converge? If so, what can you say about its sum?

(b) Must $\sum_{k=1}^{\infty} a_k$ converge? If so, what can you say about its sum?

Sometimes convergence or divergence of a series can be established by more than one of the tests developed in this chapter. In Exercises 5 to 10 determine the convergence or divergence of the given series by as many tests can be applied in each case.

5.[R] $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

6.[R] $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i}{3^i}$

7.[R] $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+1}$

8.[R] $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2-2}$

9.[R] $\displaystyle\sum_{i=1}^{\infty} \left. \frac{3+1/n}{2+1/n} \right)^n$

10.[R] $\displaystyle\sum_{n=1}^{\infty} \left. \frac{2}{3+1/n} \right)^n$

11.[R] What is the Comparison Test and how can it be used to estimate the error when using part of a series to approximate the sum of the series.

12.[R] What do the three expressions "convergent," "conditionally convergent," and "absolutely convergent" mean.

13.[R] What tests could be used to to test a series for convergence if you know that $\lim_{k\to\infty} a_{k+1}/a_k = -1/3$? Explain.

14.[R] For what values of s does $\sum_{k=1}^{\infty} a_n s^n$ converge?

15.[R] Fpr what values of q does $\sum_{k=1}^{\infty} 1/n^p$ converge?

16.[R] if $\lim_{k\to\infty} a_{k+1}/a_k = 1$, what can we conclude about the series $\sum_{k\to\infty} a_k$?

17.[R] For what values of q does $\sum_{k=1}^{\infty}(-1)^n n^q$ (a) converge? (b) converge absolutely?

18.[M] If $\sum_{k=0}^{\infty} a_k$ is convergent, does it follow that

(a) $\lim_{n\to\infty} a_n = 0$?

(b) $\lim_{n\to\infty}(a_n + a_{n+1}) = 0$?

(c) $\lim_{n\to\infty} \sum_{k=n}^{2n} a_k = 0$?

(d) $\lim_{n\to\infty} \sum_{k=n}^{\infty} a_k = 0$?

NOTE: Compare with Exercise 5 in Chapter 7.

19.[M] In an energy problem one meets the integral

$$\int_0^{\pi/2} \frac{\sin x}{e^x - 1}\, dx.$$

Note that the integrand is not defined at $x = 0$. Is that a big obstacle? Is this integral convergent or divergent? NOTE: Do not try to evaluate the integral.

20.[M] Give an example of a convergent series of positive terms $\{a_k\}$ such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist but $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ is not ∞.

21.[C] Assume that f is continuous on $[0,\infty)$ and has period one, that is, $f(x) = f(x+1)$ for all x in $[0,\infty)$. Assume also that $\int_0^{\infty} e^{-x} f(x)\, dx$ is convergent. Show that

$$\int_0^{\infty} e^{-x} f(x)\, dx = \frac{e}{e-1} \int_0^1 e^{-x} f(x)\, dx.$$

In Exercises 22 to 27 a short formula for estimating $n!$ is obtained. **22.**[C] Let f have the properties that for $x \geq 1$, $f(x) \geq 0$, $f'(x) > 0$, and $f''(x) > 0$. Let a_n be the area of the region below the graph of $y = f(x)$ and above the line segment that joins $(n, f(n))$ with $(n+1, f(n+1))$.

(a) Draw a large-scale version of Figure 11.S.1. The individual regions of area a_1, a_2, a_3, and a_4 should be clear and not too narrow.

(b) Using geometry, show that the series $a_1 + a_2 + a_3 + \cdots$ converges and has a sum no larger than the area of the triangle with vertices $(1, f(1))$, $(2, f(2))$, $(1, f(2))$.

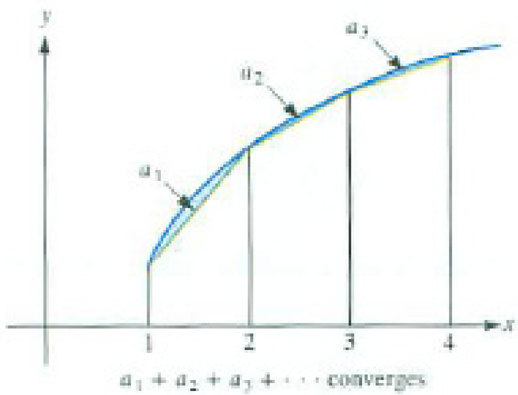


Figure 11.S.1:

23.[C] Let $y = \ln(x)$.

(a) Using Exercise 22, show that as $n \to \infty$,

$$\int_1^n \ln(x)\, dx - \left(\frac{\ln(1)+\ln(2)}{2} + \frac{\ln(2)+\ln(3)}{2} + \cdots + \frac{\ln(n-1)+\ln(n)}{2}\right)$$

has a limit; denote this limit by C.

(b) Show that (a) is equivalent to the assertion

$$\lim_{n\to\infty} \left(n\ln(n) - n + 1 - \ln(n!) + \ln(\sqrt{n})\right) = C.$$

24.[C] From Exercise 23(b), deduce that there is a constant k such that

$$\lim_{n\to\infty} \frac{n!}{k(n/e)^n\sqrt{(n)}} = 1.$$

Exercises 25 and 26 are related. Review Example 8 of Section 8.3 first.

25.[C] Let $I_n = \int_0^{\pi/2} \sin^n(\theta)\, d\theta$, where n is is a nonnegative integer.

(a) Evaluate I_0 and I_p.

(b) Show that

$$I_{2n} = \frac{2n-1}{2n}\frac{2n-3}{2n-2}\cdots\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \qquad \text{and} \qquad I_{2n+1} = \frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{4}{5}\frac{2}{3}.$$

(c) Show that
$$\frac{I_7}{I_6} = \frac{6}{7}\frac{6}{5}\frac{4}{5}\frac{4}{3}\frac{2}{3}\frac{2}{1}\frac{2}{\pi}.$$

(d) Show that
$$\frac{I_{2n+1}}{I_{2n}} = \frac{2n}{2n+1}\frac{2n}{2n-1}\frac{2n-2}{2n-1}\cdots\frac{2}{3}\frac{2}{1}\frac{2}{\pi}.$$

(e) Show that
$$\frac{2n}{2n+1}I_{2n} < \frac{2n}{2n+1}I_{2n-1} = I_{2n+1} < I_{2n},$$
and thus $\lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = 1$.

(f) From (d) and (e), deduce that
$$\lim_{n\to\infty}\frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5}\frac{6\cdot 6}{5\cdot 7}\cdots\frac{(2n)(2n)}{(2n-1)(2n+1)} = \frac{\pi}{2}.$$

This is **Wallis's formula**, usually written in shorthand as
$$\frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5}\frac{6\cdot 6}{5\cdot 7}\cdots = \frac{\pi}{2}$$

26.[C]

(a) Show that $2\cdot 4\cdot 6\cdot 8\cdots 2n = 2^n n!$.

(b) Show that $1\cdot 3\cdot 5\cdot 7\cdots(2n-1) = \frac{(2n)!}{2^n n!}$¿

(c) From Exercise 25 deduce that
$$\lim_{n\to\infty}\frac{(n!)^2 4^n}{(2n)!\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}.$$

27.[C]

(a) Using Exercise 26(c), show that k in Exercise 24 equals $\sqrt{2\pi}$. Thus a good estimate of $n!$ is provided by the formula
$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

This is known as **Stirling's formula**.

(b) Using the factorial key on a calculator, compute $(20)!$. Then compute the ratio $\sqrt{2\pi n}(n/e)^n/n!$ for $n = 20$.

28.[C] Let $\{a_k\}$ and $\{b_k\}$ be sequences of positive terms. Assume that for all k

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}.$$

(a) Prove that if $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$. HINT: Rewrite the inequality as $a_{k+1}/b_{k+1} \leq a_k/b_k$,

(b) Use the result in (a) to prove that if $\lim_{k\to infty} a_{k+1}a_k = r < 1$, then $\sum_{k=1}^{\infty} a_k$ converges.

Calculus is Everywhere # 14

$E = mc^2$

This could also appear as a boxed item in Chapter 12.

The equation $E = mc^2$ relates the energy associated with an object to its mass and the speed of light. Where does it come from?

Newton's second law of motion reads: "Force is the rate at which the momentum of an object changes." The momentum of an object of mass m and velocity v is the product mv. Denoting the force by F, we have

$$F = \frac{d}{dt}(mv).$$

If the mass is constant, this reduces to the familiar "force equals mass times acceleration." But what if the mass m is not constant? What if the mass of an object changes as its velocity changes?

According to Einstein's Special Theory of Relativity, announced in 1905, the mass does change, in a manner described by the equation:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{C.14.1}$$

For a satellite circling the Earth at 17,000 miles per hour, v/c is less than $1/2500$.

Here m_0 is the mass at rest, v is the velocity, and c is the velocity of light. If v is not zero, m is larger than m_0. When v is small (compared to the velocity of light) then m is only slightly larger than m_0. However, as v approaches the velocity of light, the mass becomes arbitrarily large.

Consider moving an object, initially at rest, in a straight line. If the velocity at time t is $v(t)$, then the displacement is $x(t) = \int_0^t v(s)\ ds$. Assuming the object is initially at rest $v(0) = 0$, the work done by a varying force F in moving the object during the time interval $[0, T]$ is

$$\begin{aligned}
W &= \int_0^T F(t)v(t)\ dt = \int_0^T (mv)'v\ dt \\
&= (mv)v\Big|_0^T - \int_0^T mv(v')\ dt && \text{integration by parts} \\
&= m(v(T))^2 - \int_0^T \frac{m_0 v v'}{\sqrt{1-\frac{v^2}{c^2}}}\ dt \\
&= m(v(T))^2 - \left(-c^2 m_0 \sqrt{1 - \frac{v^2}{c^2}}\right)\Bigg|_0^T && \text{FTC} \\
&= m(v(T))^2 - \left(-c^2 m_0 \sqrt{1 - \frac{(v(T))^2}{c^2}} + c^2 m_0 \sqrt{1 - \frac{0^2}{c^2}}\right) && \text{since } v(0) = 0 \\
&= m(v(T))^2 + c^2 m_0 \sqrt{1 - \frac{(v(T))^2}{c^2}} - m_0 c^2 \\
&= m(v(T))^2 + mc^2 \left(1 - \frac{(v(T))^2}{c^2}\right) - m_0 c^2 && \text{using (C.14.1)} \\
&= m(v(T))^2 + mc^2 - m(v(T))^2 - m_0 c^2 \\
&= mc^2 - m_0 c^2.
\end{aligned}$$

We can interpret this as saying that the "total energy associated with the object" increases from m_0c^2 to mc^2. The energy of the object at rest is then m_0c^2, called its **rest energy**.

That is the mathematics behind the equation $E = mc^2$. It suggests that mass may be turned into energy, as Einstein predicted. For instance, in a nuclear reactor some of the mass of the uranium is indeed turned into energy in the fission process. Also, the mass of the sun decreases as it emits radiant energy.

What about the equation that states kinetic energy is half the product of the mass and the square of the velocity? That is what (C.14.2) resembles when v is small (compared to c). In this case the first two terms of the binomial series for $(1-x^2)^{-1/2}$, with $x = v^2/c^2$, give

$$\begin{aligned} mc^2 - m_0c^2 = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}c^2 - mc^2 &\approx m_0c^2\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) - m_0c^2 \\ &= m_0c^2 + \frac{1}{2}\frac{m_0c^2v^2}{c^2} - m_0c^2 \\ &= \frac{m_0v^2}{2}. \end{aligned}$$

So the increase in energy is well approximated by the familiar kinetic energy, $\frac{1}{2}m_0v^2$.

Chapter 12

Applications of Series

The preceding chapter developed several tests for determining the convergence or divergence of infinite series. This chapter applies infinite series to approximate functions such as e^x and $\sin(\sqrt{x})$, evaluate integrals, and calculate limits of the indeterminate form "zero-over-zero". After asection devoted to complex numbers, we will use them to show that there is a close link between trigonometric and exponential functions.

12.1 Taylor Series

Section 5.4 introduced the n^{th}-order Taylor polynomial of a function f centered at a as the polynomial P_n that agrees with f and its first n derivatives at $x = a$:

$$\begin{aligned} P_n(x;a) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}$$

The sequence of Taylor polynomials $P_0(x;a)$, $P_1(x;a)$, $\ldots$, $P_n(x;a)$, $\ldots$ can now be viewed as the sequence of partial sums of the infinite series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This series is called the **Taylor series at a associated with the function f**. When $a = 0$, the series is also called the **Maclaurin series associated with f**.

EXAMPLE 1 Find the Maclaurin series associated with $f(x) = e^x$.
SOLUTION In Example 1 of Section 5.4 the third-order Maclaurin polynomial of $f(x) = e^x$ was found to be $P_3(x;0) = 1 + x + x^2/2! + x^3/3!$. This is the front end of the Maclaurin series associated with the exponential function. The calculation of the coefficient for the general term in the Maclaurin series is summarized in Table 12.1.1.

k	$f^{k)}(x)$	$f^{(k)}(0)$	$f^{(k)}/k!$
0	e^x	$e^0 = 1$	$1/0! = 1$
1	e^x	1	$1/1! = 1$
2	e^x	1	$1/2! = 1/2$
3	e^x	1	$1/3! = 1/6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	e^x	1	$1/k!$

Table 12.1.1:

The Maclaurin series associated with the exponential function is

$$\sum_{k=0}^{\infty} \frac{1}{k!}x^k = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^k}{k!} + \cdots.$$

◇

EXAMPLE 2 Find the Taylor series at $x = 1$ associated with $f(x) = 1/x$.
SOLUTION The calculation of the coefficient for the general term in the Taylor series at $x = 1$ associated with $f(x) = 1/x$ is summarized in Table 12.1.2. Notice that the terms alternate in sign, with the terms for even values of k

k	$f^{k)}(x)$	$f^{(k)}(0)$	$f^{(k)}/k!$
0	$1/x = x^{-1}$	1	$1/0! = 1$
1	$-x^{-2}$	-1	$-1/1! = -1$
2	$2x^{-3}$	2	$2/2! = 1$
3	$-6x^{-4}$	-6	$-6/3! = -1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$(-1)^k k! x^{-(k+1)}$	$(-1)^k k!$	$(-1)^k$

Table 12.1.2:

being positive. The Taylor series at $x = 1$ associated with $f(x) = 1/x$ is

$$\sum_{k=0}^{\infty}(-1)^k(x-1)^k = 1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^k(x-1)^k+\cdots.$$

◇

We have been careful to say the Taylor series is *associated with* the original function $f(x)$. We did not say that it converges, nor did we say, if it converges, that it converges to $f(x)$. To understand the relation between a function and its Taylor series, observe that the Taylor series found in Example 2 is a geometric series with first term 1 and ratio $(1-x)$. We know this series converges whenever $|(1-x)| < 1$, and, for any x in $(0, 2)$, its limit is the given function $1/x$:

$$\sum_{k=0}^{\infty}(1-x)^k = \frac{1}{1-(1-x)} = \frac{1}{x}.$$

$(-1)^n(x-1)^n = (1-x)^n$

See Theorem 11.2.1 in Section 11.2 on page 934.

For all other values of x, this geometric series diverges.

In this case the Taylor series represents the original function in the sense that the Taylor polynomials converge to the function on $(0, 2)$ as $n \to \infty$. However, we did not use the Lagrange formula for the error to establish this fact. To see why not, go to Exercises 27 and 29.

See Exercise 31 for a function whose Maclaurin series does not represent the function.

To determine when the Taylor series converges to the function from which it was obtained, recall that the remainder of the n^{th}-order Taylor polynomial is the difference between the function and the n^{th}-order Taylor polynomial:

$$R_n(x; a) = f(x) - P_n(x; a).$$

If $\lim_{n\to\infty} R_n(x,a) = 0$, the Taylor series represents the original function and we will write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

see Theorem 5.4.1

The Lagrange Form of the Remainder, described in Section 5.4, tells us that the remainder can be written in the form

$$R_n(x,a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some number } c_n \text{ between } x \text{ and } a.$$

EXAMPLE 3 Show that the Maclaurin series for the exponential function, e^x, represents e^x for all x.

SOLUTION In Example 1 the Maclaurin series associated with the exponential function was found to be

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Recall that all derivatives of e^x are e^x.

To determine if this series represents e^x for all x we must show that the remainder $R_n(x,0) \to 0$ as $n \to \infty$ for all numbers x. According to the Lagrange form of the remainder (see Theorem 5.4.1),

$$R_n(x,0) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-0)^{n+1} = \frac{e^{c_n}}{(n+1)!}x^{n+1}$$

where c_n is a number between 0 and x and depends on n.

First, consider the case where x is positive. Since $0 \le c_n \le x$ and the exponential function is increasing, we have $1 \le e^{c_n} \le e^x$, hence

$$\frac{1}{(n+1)!}x^{n+1} \le R_n(x,0) \le \frac{e^x}{(n+1)!}x^{n+1}.$$

See Section 11.2, Example 3.

Since x is a fixed positive number, we know that both $\lim_{n\to\infty} \frac{1}{(n+1)!}x^{n+1} = 0$ and $\lim_{n\to\infty} \frac{e^x}{(n+1)!}x^{n+1} = 0$. Thus, $\lim_{n\to\infty} R_n(x,0) = 0$.

When x is negative the analysis is similar, except that $x \le c_n \le 0$. The reader may carry out the details leading again to the conclusion that $\lim_{n\to\infty} R_n(x,0) = 0$.

See Exercise 10

The final case is $x = 0$. Notice that, for each $n \ge 0$, $P_n(0;0) = 1$ and so $R_n(0;0) = 0$.

Because we have shown that $\lim_{n\to\infty} R_n(x,0) = 0$ for all numbers x, we can write

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This provides a way to estimate e^x using only addition, multiplication, and division. In particular, when $x = 1$, it gives a series representation of e:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots .$$

Euler used this formula to evaluate e to 23 decimal places (without the aid of a calculator). ◇

EXAMPLE 4 Find the Maclaurin series for $f(x) = \cos(x)$ and show that it represents $f(x)$ for all x.

SOLUTION We must show that $R_n(x, 0) \to 0$ as $n \to \infty$. By the Lagrange form of the remainder,

$$R_n(x, 0) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x - 0)^{n+1},$$

where c_n is between 0 and x.

Since $f(x) = \cos(x)$, we have $f^{(1)}(x) = -\sin(x)$, $f^{(2)}(x) = -\cos(x)$, $f^{(3)}(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, and so on. All derivatives are either $\pm\cos(x)$ or $\pm\sin(x)$. Thus, for any nonnegative integer n and any real number c,

$$\left| f^{(n+1)}(c) \right| \leq 1.$$

Consequently,

$$|R_n(x, 0)| = \left| \frac{f^{(n+1)}(c_n)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!},$$

which approaches 0 as $n \to \infty$.

Hence the Maclaurin series associated with $f(x) = \cos(x)$ represents $\cos(x)$ for all numbers x.

The information in Table 12.1.3 helps to determine the Maclaurin series for $\cos(x)$. Notice that every odd-power term is zero and the signs alternate between successive even-power terms. Thus, for any number x,

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

◇

k	$f^{k)}(x)$	$f^{(k)}(0)$	$f^{(k)}/k!$
0	$\cos(x)$	1	$1/0! = 1$
1	$-\sin(x)$	0	$0/1! = 0$
2	$-\cos(x)$	-1	$-1/2! = -1/2$
3	$\sin(x)$	0	$0/3! = 0$
4	$\cos(x)$	1	$1/4! = 1/24$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 12.1.3:

Summary

The Taylor series at a associated with a function is the series whose partial sums are the n^{th}-order Taylor polynomials of the function. This series represents the original function only for inputs such that the remainder of the n^{th}-order Taylor polynomial approaches zero as $n \to \infty$: $\lim_{n\to\infty} R_n(x,a) = 0$. The Lagrange form of the remainder, Theorem 5.4.1 from Section 5.4, helps to show that the remainder converges to zero, though, as Exercise 27 illustrates, in some cases it may not be strong enough to do that.

Function	Series	Interval of Convergence
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	all x: $(-\infty,\infty)$
$\cos(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	all x: $(-\infty,\infty)$
$\frac{1}{x}$	$\sum_{k=0}^{\infty}(-1)^k(x-1)^k = \sum_{k=0}^{\infty}(1-x)^k$	$0 < x < 2 : (0,2)$

Table 12.1.4:

EXERCISES for Section 12.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] State without using any mathematical symbols the formula for the terms of a Taylor series of a function around a number that may not be zero.

2.[R] State without using any mathematical symbols the formula for the terms of a Maclaurin series of a function.

In Exercises 3 to 8 compute the Maclaurin series associated with the given function
3.[R] $1/(1+x)$
4.[R] $1/(1-x)$
5.[R] $\ln(1+x)$
6.[R] $\ln(1-x)$
7.[R] $\sin(x)$
8.[R] e^{-x}

9.[R] Let $f(x) = 2 + 3x - 4x^2$.

(a) Find $P_1(x; 0)$, $P_2(x; 0)$, and $P_3(x; 0)$.

(b) What is the Maclaurin series associated with $f(x)$?

(c) Find $P_1(x; 1)$, $P_2(x; 1)$, and $P_3(x; 1)$.

(d) What is the Taylor series at $x = 1$ associated with $f(x)$?

10.[R] Let $f(x) = e^x$. Show that $\lim_{n\to\infty} R_n(x; 0) = 0$ for any negative number x. This completes the proof that the exponential function is represented by its Maclaurin series for all numbers x (see Example 3).

11.[R] Show that the Maclaurin series associated with $\sin(x)$ represents $\sin(x)$ for all x.

12.[R] Show that the Maclaurin series associated with e^{-x} represents e^{-x} for all x.

13.[R] Show that the Maclaurin series associated with any polynomial $f(x)$ represents the polynomial for all x. Hint: Examine $R_n(x; 0)$.

14.[R] Show that the Maclaurin series associated with $1/(1+x)$ represents $1/(1+x)$ for all $-1/2 < x < 1$. Hint: Examine $R_n(x; 0)$. Note: Actually, the representation holds for $-1 < x < 1$. See also Exercise 28.

15.[R] Show that the Taylor series in powers of $x-a$ for e^x represents e^x for all x.

16.[R] Show that the Taylor series in powers of $x-a$ for $\cos(x)$ represents $\cos(x)$ for all x.

17.[M]

(a) Which polynomials are even functions?

(b) If f is an even function, is $P_n(x;0)$ necessarily an even function? Explain.

18.[M]

(a) Which polynomials are odd functions?

(b) If f is an odd function, is $P_n(x;0)$ necessarily an odd function? Explain.

19.[M]

(a) Is $f(x) = \arctan(x)$ an odd function? an even function? neither?

(b) Find $P_3(x;0)$.

(c) What powers of x in the Maclaurin series associated with $\arctan(x)$ will have coefficients of 0?

20.[M] Obtain the first two non-zero terms of the Maclaurin series associated with $\arctan(x)$. NOTE: In Section 12.4 we describe a shortcut for finding all the terms of the series for $\arctan(x)$.

21.[M]

(a) Find the Maclaurin series associated with $(1+x)^4$.

(b) Find the Maclaurin series associated with $(1+x)^n$, where n is a positive integer.

22.[C] Do there exist any polynomials $p(x)$ such that $\sin(x) = p(x)$ for all x in the interval $[1, 1.0001]$? Explain.

23.[C] Do there exist any polynomials $p(x)$ such that $\ln(x) = p(x)$ for all x in the interval $[1, 1.0001]$? Explain.

24.[M] Let f be a function that has derivatives of all orders for all x. Assume that $\left|f^{(n)}(x)\right| \leq n$ for all n. Show why $f(x)$ is represented by its Maclaurin series for all x.

25.[M] Let f be a function that has derivatives of all orders for all x. Assume that $\left|f^{(n)}(x)\right| \leq 2^n$ for all n. Does $R_n(x;0)$ necessarily approach 0 for all x as $n \to \infty$? If not, for which x does it necessarily approach 0?

26.[M] Since $e^x e^y = e^{x+y}$, the product of the Maclaurin series for e^x and e^y should be the Maclaurin series for e^{x+y}. Check that for terms up to degree 3 in the series for e^{x+y}, this is the case.

27.[C] We know that for $|x| < 1$ the geometric series $\sum_{k=0}^{\infty} x^k$ converges to $1/(1-x)$, because we had a formula for the error, namely, $\left|x^{n+1}/(1-x)\right|$. This exercise shows that the Lagrange form for the error is not strong enough to show that the series converges to $1/(1-x)$ in $(-1, 1)$.

(a) Show that the Maclaurin series for $1/(1-x)$ is $\sum_{k=0}^{\infty} x^k$.

(b) Show that the Lagrange form of the error gives $R_n(x;0) = \frac{1}{1-c_n}\left(\frac{x}{1-c_n}\right)^{n+1}$ for some c_n between 0 and x.

(c) Show that for $x = 1/3$, the formula for $R_n(x;0)$ implies that it approaches 0 as $n \to \infty$.

(d) Show that for $x = 2/3$, the formula for $R_n(x;0)$ does not imply that it approaches 0 as $n \to \infty$.

(e) Show that for $x = -2/3$, the formula for $R_n(x;0)$ implies that it approaches 0 as $n \to \infty$.

(f) For which x in $(-1, 1)$ is the formula for $R_n(x;0)$ useful in showing that $R_n(x;0)$ approaches 0 as $n \to \infty$?

28.[C] In Example 3(b) the Taylor series at $x = 1$ for $1/x$ is found to be

$$1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^k (x-1)^k + \cdots.$$

(a) This is a geometric series. What is the first term? What is the ratio?

(b) Show that this geometric series converges to $1/x$ for all $0 < x < 2$, and diverges for all other values of x.

(c) The Lagrange form for the remainder is

$$R_n(x;1) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-1)^{n+1}.$$

Show that

$$f^{(n+1)}(c_n) = \frac{(-1)^{n+1}(n+1)!}{c_n^{n+2}}$$

and

$$|R_n(x;1)| = \frac{1}{c_n}\left(\frac{|x-1|}{c_n}\right)^{n+1}.$$

(Assume $0 < x < 2$ and c_n is between x and 1.)

(d) Show that if $1/2 < x < 2$ then $|x-1|/c_n < 1$ and so $R_n(x;1) \to 0$ as $n \to \infty$ for these values of x.

(e) Show that if $x = 1/2$ then $1/2 \le |x-1|/c_n \le 1$ and so the Lagrange formula fails to show that $R_n(1/2;1) \to 0$ as $n \to \infty$.

(f) Show that if $0 < x < 1/2$ then the Lagrange formula fails to show that $R_n(1/2;1) \to 0$ as $n \to \infty$.

HINT: First, complete Exercise 27. NOTE: Another form of the remainder expresses $R_n(x;a)$ as an integral instead of a derivative. That form, which shows that $R_n(x;1) \to 0$ as $n \to \infty$ for all x with $|x-1| < 1$, is found in any advanced calculus text. See also Exercise 14.

29.[C] This problem examines three ways to estimate the error in using a front-end of $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ to estimate e^{-1}.

(a) Use the Lagrange formula to obtain an estimate of the error in using the front-end up through $(-1)^m/m!$ to estimate e^{-1}

(b) Estimate the error by noticing the series is alternating and the terms decrease in absolute value

(c) Estimate the error by comparing $\sum_{k=m+1}^{\infty} \left|\frac{(-1)^k}{k!}\right|$ to a geometric series, which is easy to sum.

(d) Which of the three methods provides the smallest (best) estimate of the error?

Exercises 30 and 31 present a non-zero function whose Maclaurin series has the value 0 for all x, and therefore does not represent the function. This function is so "flat" at the origin that all its derivatives are zero there.

30.[C] The following steps show that $\lim_{x\to 0} \frac{e^{1/x^2}}{x^n} = 0$ for all positive numbers n:

(a) Why does it suffice to consider only $x > 0$?

(b) Let $v = 1/x^2$ and translate the limit to

$$\lim_{v\to\infty} v^{n/2}e^{-v}.$$

(c) This limit is similar to a limit treated in Section 5.5. Show that it equals 0.

(d) Show that $\lim\limits_{n\to\infty} \dfrac{p(x)e^{-1/x^2}}{x^n} = 0$ for any polynomial $p(x)$.

31.[C] Let $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$.

(a) Show f is continuous at 0.

(b) Show f is differentiable at 0.

(c) Show that $f'(0) = 0$.

(d) Show that $f''(0) = 0$.

(e) Explain why $f^{(n)}(0) = 0$ for all integers $n \geq 0$.

(f) What is the Maclaurin series associated with f?

(g) Why does the example use e^{-1/x^2} instead of the simpler $e^{-1/x}$.

Q: Why bother with Tayor series when my calculator can give these answers to more decimal places faster? A: Read the section.

12.2 Two Applications of Taylor Series

The front-end of the Taylor series representation of a function $f(x)$ approximates the function. After an example to remind us how to estimate a value of a function, the remaining examples in this section illustrate how to evaluate certain limits and to approximate the value of definite integrals.

Using a Front-End to Estimate $f(x)$

EXAMPLE 1 Estimate $\sqrt{\sqrt{e}+3}$ using the first four terms of the Maclaurin series for $\sqrt{e^x+3}$. Discuss the error.
SOLUTION To find the third-degree Maclaurin polynomial we need the first three derivatives of $f(x)$:

$$f'(x) = \frac{e^x}{2\sqrt{e^x+3}}, \qquad f''(x) = \frac{e^x(e^x+6)}{4(e^x+3)^{3/2}}, \qquad f'''(x) = \frac{e^x(e^{2x}+6e^x+36)}{8(e^x+3)^{5/2}}$$

From these, and the observation that $e^0 = 1$ and $\sqrt{e^0+3} = 2$, give us the desired approximating Maclaurin polynomial:

$$f(x) \approx P_3(x;0) = 2 + \frac{1}{4}x + \frac{7}{64}x^2 + \frac{43}{1536}x^3.$$

Then $\sqrt{\sqrt{e}+3} = f\left(\frac{1}{2}\right) \approx P_3(\frac{1}{2};0) = \frac{26491}{12288} \approx 2.1558431.$

By Lagrange's form of the remainder, the error can be written as

$$\frac{f^{(4)}(c)\left(\frac{1}{2}\right)^4}{4!} \qquad \text{for some number } c \text{ in } [0,1/2].$$

To obtain an upper bound on $\left|f^{(4)}(x)\right| = \left|\frac{e^x(e^{3x}+12e^{2x}-36e^x+216)}{16(e^x+3)^{7/2}}\right|$ for x in $[0,1/2]$, notice that $1 = e^0 < e^x < 4^{1/2} = 2$ for all x in $[0,1/2]$. An upper bound on the numerator is found by using $e^x < 2$:

$$\left|e^x\left(e^{3x}+12e^{2x}-36e^x+216\right)\right| \leq 2(2^3+12(2)^3+36(2)+216) = 648.$$

Likewise, using $e^x > e^0 = 1$ provides a lower bound on the denominator:

$$16(e^x+3)^{7/2} \geq 16(1+3)^{7/2} = 16(128).$$

Combined, the error is bounded above by

$$\frac{\left|f^{(4)}(c)\right|\left(\frac{1}{2}\right)^4}{4!} \leq \frac{648}{16(128)}\frac{1}{16}\frac{1}{24} = \frac{27}{32768} \approx 0.000823 \qquad \text{for all } c \text{ in } [0,1/2].$$

Thus, to three decimal places, $\sqrt{\sqrt{e}+3} \approx 2.156.$ ◇

To obtain more accurate estimates use more terms in the Maclaurin series for $f(x) = \sqrt{e^x+3}$.

Using a Taylor Series to Find a Limit

The next example shows how series can be used to evaluate the limit of an indeterminate form.

EXAMPLE 2 Use the Maclaurin series for $\sin(x)$ and e^x to evaluate

$$\lim_{x\to 0} \frac{\left(e^{2x}-1\right)^4}{\sin\left(x^3\right)}.$$

SOLUTION We know the Maclaurin series for e^x represents the exponential function for all numbers x. This allows us to write the numerator of the limit as

$$\begin{aligned}\left(e^{2x}-1\right)^4 &= \left(\left(1+(2x)+\frac{(2x)^2}{2}+\cdots\right)-1\right)^4 \\ &= \left((2x)+\frac{(2x)^2}{2}+\cdots\right)^4 = 16x^4+64x^5+\frac{416}{3}x^6+\cdots.\end{aligned}$$

Likewise, because the Maclaurin series for $\sin(x)$ represents the sine function for all numbers x, the denominator of the limit can be written as

$$\sin\left(x^3\right) = (x^4) - \frac{(x^4)^3}{6} + \cdots = x^4 - \frac{1}{6}x^1 2 + \cdots.$$

These two results can be combined to find the limit:

$$\begin{aligned}\lim_{x\to 0} \frac{\left(e^{2x}-1\right)^4}{\sin\left(x^3\right)} &= \lim_{x\to 0} \frac{16x^4+64x^5+\frac{416}{3}x^6+\cdots}{x^4-\frac{1}{6}x^{12}+\cdots} \\ &= \lim_{x\to 0} \frac{x^4\left(16+64x+\cdots\right)}{x^4\left(1+\frac{1}{6}x^8+\cdots\right)} = \lim_{x\to 0} \frac{16+64x+\cdots}{1+\frac{1}{6}x^8+\cdots} = \frac{16}{1} = 16.\end{aligned}$$

◇

The limit in Example 2 has the indeterminate form $\frac{0}{0}$. It could also have been obtained using l'Hôpital's rule — applied four times.

Using a Taylor Series to Estimate an Integral

The integral describes the "bell curve."

In statistics, the integral $\int_{-\infty}^{b}(1/\sqrt{2\pi})e^{-x^2/2}\,dx$ is of major importance. Since $e^{-x^2/2}$ does not have an elementary antiderivative, the integral must be estimated by other means. Tables of values of this function can be found in almost any mathematical handbook. For example, Abramowitz, M. and Stegun, I. A. (Eds.). *Handbook of Mathematical Functions with Formulas,* *Graphs, and Mathematical Tables*, (9^{th} printing, New York: Dover, pp. 931-933, 1972). lists values of this function for b in the interval $[0, 4]$ at intervals of 0.01.

The next example shows how to estimate $\int_a^b f(x)\, dx$ when $f(x)$ is represented by a Taylor series.

EXAMPLE 3 Use the Maclaurin series for e^x to estimate $\int_0^1 e^{-x^2}\, dx$.

SOLUTION The first step is to obtain the Maclaurin series for the integrand: e^{-x^2}. Because

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

we can replace x with $-x^2$ to obtain

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots . \qquad (12.2.1)$$

For $0 \leq |x| \leq 1$, (12.2.1) is a convergent alternating series. Every partial sum that ends with a negative term is smaller than e^{-x^2}; every partial sum that ends with a positive term is larger than e^{-x^2}. For example,

$$1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} < e^{-x^2} < 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!}.$$

Hence

$$\int_0^1 \left(1 - x^2 + \tfrac{x^4}{2!} - \tfrac{x^6}{3!}\right) dx \;<\; \int_0^1 e^{-x^2}\, dx \;<\; \int_0^1 \left(1 - x^2 + \tfrac{x^4}{2!} - \tfrac{x^6}{3!} + \tfrac{x^8}{4!}\right) dx,$$

or

$$1 - \tfrac{1}{3} + \tfrac{1}{5\cdot 2!} - \tfrac{1}{7\cdot 3!} \;<\; \int_0^1 e^{-x^2}\, dx \;<\; 1 - \tfrac{1}{3} + \tfrac{1}{5\cdot 2!} - \tfrac{1}{7\cdot 3!} + \tfrac{1}{9\cdot 4!}.$$

From this it follows that

$$0.742 < \int_0^1 e^{-x^2}\, dx < 0.748.$$

◇

The approach used in Example 3 is best for small values of b.

The final example shows how a Maclaurin series can be used to estimate a number to a prescribed number of decimal places.

EXAMPLE 4 Use the Taylor series at $x = 1$ for $1/x$ to estimate $\ln(3/2)$ to four decimal places.

SOLUTION The starting point is the definition of the equation $\ln(t) = \int_1^t \frac{dx}{x}$. In particular, we are looking at $\ln(3/2) = \int_1^{3/2} \frac{dx}{x}$.

The integrand can be replaced by a series representation provided the series represents the function throughout the interval. By Example 2, the Taylor series at 1 represents $1/x$ for $0 < x < 2$:

$$1/x = 1 + (1-x) + (1-x)^2 + \cdots = \sum_{k=0}^{\infty} (1-x)^k.$$

Since we are integrating over the interval $[1, 3/2]$, we obtain a series representation for $\ln(3/2)$ in the following manner:

$$\begin{aligned} \ln(3/2) &= \int_1^{3/2} \frac{dx}{x} = \int_1^{3/2} \sum_{k=0}^{\infty} (1-x)^k \, dx = \sum_{k=0}^{\infty} \int_1^{3/2} (1-x)^k \, dx \\ &= \sum_{k=0}^{\infty} \frac{-1}{k+1} (1-x)^{k+1} \Big|_1^{3/2} = \sum_{k=0}^{\infty} \frac{-1}{k+1} \left(\frac{-1}{2}\right)^{k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)2^{k+1}}. \end{aligned}$$

So

$$\ln(3/2) = \frac{1}{1 \cdot 2^1} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \cdot 2^k}.$$

This is a convergent series.

Why is it convergent?

Approximations of $\ln(3/2)$ can be found by truncating this series after any finite number of terms, such as this three-term sum:

$$\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 8}.$$

We learned in Section 11.6 that the error in truncating a convergent alternating series is bounded by the absolute value of the first omitted term, the "next term." Table 12.2.1 shows the first 10 partial sums and the next term. The next term (in absolute value) drops below 0.00005 when $N = 9$. This means t0.4054 is an estimate of $\ln(3/2)$ that is accurate to four decimal places. of $\ln(3/2)$ is 0.4054. ◇

N	$\sum_{k=0}^{N} \frac{(-1)^k}{(k+1)2^{k+1}}$	$\frac{(-1)^{N+1}}{(N+2)2^{N+2}}$
0	0.5	−0.125000
1	0.375	0.041666
2	0.416666	−0.015625
3	0.401042	0.006250
4	0.407292	−0.002604
5	0.404687	0.001116
6	0.405804	−0.000488
7	0.405315	0.000217
8	0.405532	−0.000097
9	0.405434	0.000044

Table 12.2.1:

The critical step was the switching of the sum and the integral:

$$\int_1^{3/2} \sum_{k=0}^{\infty} (1-x)^k \, dx = \sum_{k=0}^{\infty} \int_1^{3/2} (1-x)^k \, dx$$

This will be discussed in Section 12.4.

Summary

The Taylor series associated with a function can be used to evaluate limits and to approximate function values and definite integrals. These ideas will be studied in greater detail in Sections 12.5 and 12.6.

EXERCISES for Section 12.2 *Key:* R–routine, M–moderate, C–challenging

1.[R] Use the Maclaurin series for e^x to estimate $1/e = e^{-1}$ to three decimal places.

2.[R] Use the Maclaurin series for e^x to estimate e^2 to three decimal places.

3.[R] Use the Maclaurin series for $\cos(x)$ to estimate $\cos(20^\circ)$ to three decimal places. HINT: First, convert 20° to radians. Use $\pi \approx 3.14159$ to keep round-off errors under control.

4.[R]

(a) Estimate $\cos(40^\circ)$ to three decimal places using the Maclaurin series for $\cos(x)$.

(b) Estimate $\cos(40^\circ)$ to three decimal places using the Taylor series for $\cos(x)$ in powers of $x - \pi/4$.

(c) Which estimate requires fewer terms to obtain the desired accuracy? Why?

HINT: Use $\pi \approx 3.14159$ to keep round-off errors under control.

5.[R] Evaluate $\displaystyle\lim_{x\to 0} \frac{\cos(x) - (1 - x^2/2 + x^4/24 - \cdots}{e^x}$.

6.[M]

(a) Show $\displaystyle\int_0^1 (e^x - 1)/x \, dx$ is finite, even though the integrand is not defined 0.

(b) Show that $1 + \dfrac{1}{2\cdot 2!} + \dfrac{1}{3\cdot 3!} + \dfrac{1}{4\cdot 4!} + \dfrac{1}{5\cdot 5!}$ is an estimate of the integral in (a).

(c) The error in using the sum in (b) is $\frac{1}{6\cdot 6!} + \frac{1}{7\cdot 7!} + \frac{1}{8\cdot 8!} + \frac{1}{9\cdot 9!} + \cdots$. Show that this is less than $\frac{1}{6\cdot 6!}\left(1 + \frac{1}{7}\left(\frac{1}{7}\right) + \frac{1}{7}\left(\frac{1}{7}\right)^2 + \frac{1}{7}\left(\frac{1}{7}\right)^3 + \cdots\right)$.

(d) From (c) deduce that the error is less than 0.000237.

7.[M] Find $\displaystyle\lim_{x\to 0} \frac{\cos(x)(e^x)^2 - 1}{\sin^2(x)}$ using (a) Maclaurin series and (b) l'Hôpital's rule.

8.[M]

(a) Show that for x in $[0,2]$

$$x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}\leq e^x-1\leq x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\frac{e^2x^{n+1}}{(n+1)!}.$$

(b) Use (a) to find $\int_0^2 \frac{e^x-1}{x}\,dx$ to three decimal places.

9.[M] Find $\int_0^1 \frac{1-\cos(x)}{x}\,dx$ to three decimal places, using an approach like that in Exercise 8.

10.[M] Estimate $\int_0^\infty e^{-5x^2}\,dx$ following these steps:

(a) Find a number b such that

$$\int_b^\infty e^{-5x^2}\,dx < 0.0005.$$

(Use the fact that $e^{-5x^2} < e^{-5x}$ for $x > 1$.)

(b) Let b be the number you found in (a). Estimate $\int_0^b e^{-5x^2}\,dx$ with an error of less than 0.0005. (Use the Maclaurin series for e^{-5x^2}.)

(c) Combine (a) and (b) to get a two decimal place estimate of $\int_0^\infty e^{-5x^2}\,dx$.

11.[M] Estimate $\int_0^\infty \frac{\cos(x^6/100)-1}{x^6}\,dx$, following these steps:

(a) Find a number b such that

$$\left|\int_b^\infty \frac{\cos(x^6/100)-1}{x^6}\,dx\right| < 0.001.$$

(Use the fact that $|\cos(x)|\leq 1$.)

(b) Let b be the number you found in (a). Estimate

$$\int_0^b \frac{\cos(x^6/100)-1}{x^6}\,dx,$$

with an error less that 0.001. (Use the Maclaurin series for $\cos(x)$.)

(c) Combine (a) and (b) to get a two decimal place estimate for

$$\int_0^\infty \frac{\cos(x^6/100) - 1}{x^6}\, dx.$$

12.[C] Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by

(a) the Fundamental Theorem of Calculus (approximate π to 3 decimal places),

(b) Simpson's method (six sections),

(c) trapezoid method (six sections),

(d) using the first six non-zero terms of the series $1 - x^2 + x^4 - \cdots$ for $1/(1+x^2)$.

13.[C] Repeat Exercise 12 for $\int_0^1 \frac{dx}{1+x^3}$.

14.[C] Assume that $f(x)$ has a continuous fourth derivative. Let M_4 be the maximum of $\left|f^{(4)}(x)\right|$ for x in $[-1,1]$. Show that

$$\left| \int_{-1}^{1} f(x)\, dx - f\left(\frac{1}{\sqrt{3}}\right) - f\left(\frac{-1}{\sqrt{3}}\right) \right| \leq \frac{7M_4}{270}.$$

Hint: Use the representation $f(x) = f(0) + f'(0)x + f''(0)x^2/2 + f^{(3)}(0)x^3/6 + f^{(4)}(c)x^4/24$, where c depends on x.

12.3 Power Series and Their Interval of Convergence

Our use of Taylor polynomials to approximate a function led us to consider series of the form

$$\sum_{k=0}^{\infty} b_k(x-a)^k = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_k(x-a)^k + \cdots .$$

Such a series is called a **power series** in $x - a$. If $a = 0$, we obtain a series in powers of x:

$$\sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + b_2 x^2 + \cdots + b_k x^k + \cdots .$$

We will now look at some properties of power series and see that they behave very much like polynomials.

The Radius of Convergence of a Power Series

The power series $b_0 + b_1x + b_2x^2 + \cdots$ certainly converges when $x = 0$. It may or may not converge for other choices of x. However, as Theorem 12.3.1 will show, if the series converges at a certain value c, it converges at any number x whose absolute value is less than $|c|$, that is, throughout the interval $(-|c|, |c|)$. Since the proof of Theorem 12.3.1 uses the comparison test and the absolute-convergence test, it offers a nice review of important concepts from Chapter 11.

For each fixed choice of x, a power series becomes a series with constant terms.

Theorem 12.3.1. *Let c be a nonzero number such that Assume that $\sum_{k=0}^{\infty} b_k c^k$ converges. Then, if $|x| < |c|$, $\sum_{k=0}^{\infty} b_k x^k$ converges. In fact, it converges absolutely.*

The proof is at the end of this section.

By Theorem 12.3.1, the set of numbers x such that $\sum_{k=0}^{\infty} b_k x^k$ converges has no holes. In other words, it is one unbroken piece, which includes the number 0. Moreover, if r is in the set, so is the entire open interval $(-|r|, |r|)$.

The x's for which the series converges form an interval with 0 at its midpoint.

There are two possibilities. In the first case, there are arbitrarily large r's such that the series converges for x in $(-r, r)$. This means that the series converges for all x. In the second case, there is an upper bound on the numbers r such that the series converges for x in $(-r, r)$. It is shown in advanced calculus that there is then a smallest upper bound on the r's; call it R. Consequently, either

See Figure 12.3.1.
Note that convergence or divergence at R and $-R$ is not mentioned.

1. $b_0 + b_1x + b_2x^2 + \cdots$ converges for all x

or

2. there is a number R such that $b_0 + b_1x + b_2x^2 + \cdots$ converges for all x such that $|x| < R$ but diverges for $|x| > R$.

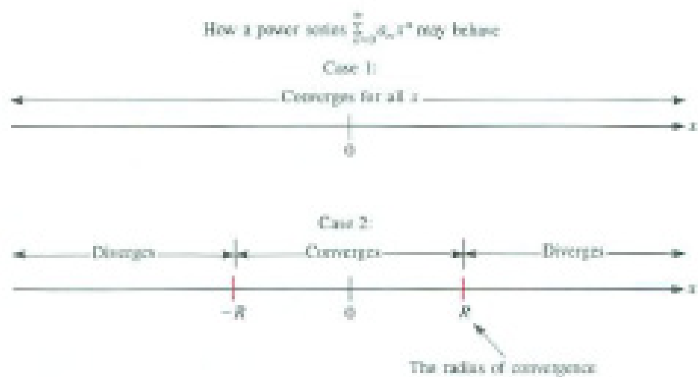


Figure 12.3.1:

In the second case, R is called the **radius of convergence** of the series. In the first case, the radius of convergence is said to be infinite, $R = \infty$. For the geometric series $1+x+x^2+\cdots+x^k+\cdots$, $R = 1$, since the series converges when $|x| < 1$ and diverges when $|x| > 1$. (It also diverges when $x = 1$ and $x = -1$.) A power series with radius of convergence R may or may not converge at $x = R$ and at $x = -R$. These observations are summarized as Theorem 12.3.2.

Theorem 12.3.2. *Let R be the radius of convergence for the power series $\sum_{k=0}^{\infty} b_k x^k$. If $R = 0$, the series converges only for $x = 0$. If R is a positive real number, the series converges for $|x| < R$ and diverges for $|x| > R$. If R is ∞, the series converges for all x.*

EXAMPLE 1 Find all value of x for which $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{k+1}x^k}{k} + \cdots$ converges.

SOLUTION Because of the presence of x^k and the fact that x may be negative, use the absolute-ratio test. The absolute value of the ratio of successive terms is

$$\left| \frac{\frac{(-1)^{k+2}x^{k+1}}{k+1}}{\frac{(-1)^{k+1}x^k}{k}} \right| = \frac{k}{k+1}|x|.$$

As $k \to \infty$, $k/(k+1) \to 1$. Thus,

$$\begin{array}{ll} \text{if } |x| < 1, & \lim_{k\to\infty} \frac{k}{k+1}|x| = |x| < 1; \\ \text{if } |x| > 1, & \lim_{k\to\infty} \frac{k}{k+1}|x| = |x| > 1. \end{array}$$

The absolute-ratio test takes care of $|x| < 1$ and $|x| > 1$.

By the absolute-ratio test, the series converges for $|x| < 1$ and diverges for $|x| > 1$. The radius of convergence is $R = 1$. It remains to see what happens at the endpoints, 1 and -1.

Checking convergence at $x = 1$

For $x = 1$, we obtain the alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

This series converges, by the alternating-series test. Thus, $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{k+1}x^k}{k} + \cdots$ converges for $x = 1$.

Checking convergence at $x = -1$

What about $x = -1$? The series becomes

$$(-1) - \frac{(-1)^2}{2} + \frac{(-1)^3}{3} - \frac{(-1)^4}{4} + \cdots + \frac{(-1)^{k+1}(-1)^k}{k} + \cdots$$

or

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{k} + \cdots,$$

which, being the negative of the harmonic series, diverges.

The series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{k+1}x^k}{k} + \cdots$ converges only for $-1 < x \leq 1$. Figure 12.3.2 records what we found about this series.

Figure 12.3.2:

◇

EXAMPLE 2 Find the radius of convergence of

$$\sum_{k=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots,$$

the Maclaurin series for e^x.

SOLUTION Because of the presence of the powers x^k, the factorial $k!$, and that x may be positive or negative, the Absolute-Ratio Test is the logical test to use to determine the radius of convergence. The absolute value of the ratio between successive terms is

$$\left| \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}} \right| = \frac{k!}{(k+1)!}|x| = \frac{|x|}{k+1}.$$

Since

$$\lim_{k \to \infty} \frac{|x|}{k+1} = 0,$$

A case where $R = \infty$

the limit of the ratio between successive terms is 0. Since 0 is less than 1, the series converges for all x. The series converges for all x. The radius of convergence R is infinite. ◇

The next example represents the opposite extreme, $R = 0$.

EXAMPLE 3 Find the radius of convergence of the series

$$\sum_{k=1}^{\infty} k^k x^k = 1x + 2^2x^2 + 3^3x^3 + \cdots + k^k x^k + \cdots .$$

Every power series converges for at least one value of x.

SOLUTION The series converges for $x = 0$.

If $x \neq 0$, consider the k^{th} term $k^k x^k$, which can be written as $(kx)^k$. As $k \to \infty$, $|kx| \to \infty$. By the n^{th} term test, this series diverges. In short, the series converges only when $x = 0$. The radius of convergence in this case is $R = 0$. ◇

A case where $R = 0$

The Radius of Convergence of $\sum_{k=0}^{\infty} b_k(x-a)^k$

Just as a power series in x has an associated radius of convergence, so does a power series in $x - a$. To see this, consider any such power series,

$$\sum_{k=0}^{\infty} b_k(x-a)^k = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots . \tag{12.3.1}$$

Let $u = x - a$. Then series (12.3.1) becomes

$$\sum_{k=0}^{\infty} b_k u^k = b_0 + b_1 u + b_2 u^2 + \cdots . \tag{12.3.2}$$

Series (12.3.2) has a certain radius of convergence R. That is, (12.3.2) converges for $|u| < R$ and diverges for $|u| > R$. Consequently (12.3.1) converges

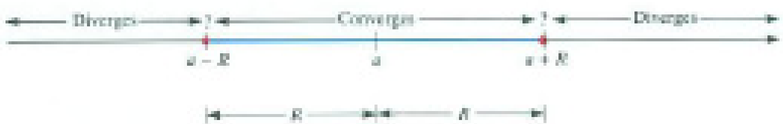


Figure 12.3.3:

for $|x-a| < R$ and diverges for $|x-a| > R$. The number R is called the radius of convergence of the series (12.3.1).

R may be zero, positive, or infinite.

As Figure 12.3.3 suggests, the series $\sum_{k=0}^{\infty} b_k(x-a)^k$ converges in an interval $(a-R, a+R)$, whose midpoint is a. The question marks in Figure 12.3.3 indicate that the series may converge or may diverge at the ends of teh interval, $a-R$ and $a+R$. These cases must be decided separately.

These observations are summarized in the following theorem.

Theorem 12.3.3. *Let R be the radius of convergence for the power series $\sum_{k=0}^{\infty} b_k(x-a)^k$. If $R = 0$, the series converges only for $x = a$. If R is a positive real number, the series converges for $|x-a| < R$ and diverges for $|x-a| > R$. If $R = \infty$, the series converges for all numbers x.*

EXAMPLE 4 Find all values of x for which

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(x-1)^k}{k} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \quad (12.3.3)$$

converges.

SOLUTION Note that this is Example 1 with x replaced by $x-1$. Thus $x-1$ plays the role that x played in Example 1. Consequently, series (12.3.3) converges for $-1 < x-1 \le 1$, that is, for $0\mathcal{L}\,[x] \le 2$, and diverges for all other values of x. Its radius of convergence is $R = 1$. The set of values where the series converges is an interval whose midpoint is 1.

The convergence of (12.3.3) is recorded in Figure 12.3.4. ◇

The General Binomial Theorem

Appendix C reviews the binomial theorem.

Figure 12.3.4:

$\begin{pmatrix} r \\ k \end{pmatrix} = \frac{r!}{k!(r-k)!}$

If r is 0 or a positive integer, $(1+x)^r$, when multiplied out, is a polynomial of degree r. Its Maclaurin series has only a finite number of nonzero terms, the one of highest degree being x^r. The formula

$$(1+x)^r = \sum_{k=0}^{r} \frac{r!}{k!(r-k)!} x^k$$

To remember it, recall taht the coefficient of x^k has k factors in both the numerator and denominator. The factors in the numerator start from r and decrease by one. The factors in the denominator start from 1 and increase by 1.

is known as the **binomial theorem**. It can also be written as

$$(1+x)^r = \sum_{k=0}^{r} \frac{r(r-1)\cdots(r-(k-1))}{1\cdot 2\cdots k} x^k.$$

Newton generalized the binomial theorem to all exponents, as illustrated in Example 5.

EXAMPLE 5 Find the Maclaurin series associated with $f(x) = (1+x)^r$, where r is not 0 or a positive integer and determine its radius of convergence.
SOLUTION The Maclaurin series for $f(x)$ is $\sum_{k=0}^{\infty} b_k x^k$ where $b_k = \frac{f^{(k)}(0)}{k!}$. The following table will help in computing $f^{(k)}(0)$:

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$(1+x)^r$	1
1	$r(1+x)^{r-1}$	r
2	$r(r-1)(1+x)^{r-2}$	$r(r-1)$
3	$r(r-1)(r-2)(1+x)^{r-2}$	$r(r-1)(r-2)$
$\vdots$	$\vdots$	$\vdots$
k	$r(r-1)\cdots(r-k+1)(1+x)^{r-k}$	$r(r-1)(r-2)\cdots(r-k+1)$

Table 12.3.1:

Consequently, the Maclaurin series associated with $(1+x)^r$ is

$$1 + rx + \frac{r(r-1)}{1\cdot 2}x^2 + \frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}x^3 + \cdots. \qquad (12.3.4)$$

Note that the series has an infinite number of terms (it does not stop) because r is not a positive integer or 0.

For $x = 0$, the series clearly converges. So consider $x \neq 0$. The presence of x^k, which could be positive or negative, and of $k!$ in the denominator of the general term in the series suggests using the absolute-ratio test to determine the radius of convergence. Let a_k be the term in the Maclaurin series for $(1+x)^r$ that contains the power x^k. Then

$$a_k = \frac{r(r-1)(r-2)\cdots(r-k+1)}{1\cdot 2\cdot 3\cdots k}x^k,$$

and

$$a_{k+1} = \frac{r(r-1)(r-2)\cdots(r-k+1)(r-k)}{1\cdot 2\cdot 3\cdots k(k+1)}x^k.$$

Thus

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{r(r-1)(r-2)\cdots(r-k+1)(r-k)}{1\cdot 2\cdot 3\cdots k(k+1)}x^k}{\frac{r(r-1)(r-2)\cdots(r-k+1)}{1\cdot 2\cdot 3\cdots k}x^k}\right| = \left|\frac{r-k}{k+1}x\right|.$$

Since r is fixed,

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = |x|.$$

By the absolute-ratio test, series (12.3.4) converges when $|x| < 1$ and diverges when $|x| > 1$. ◇

In Example 5 it was shown that for $|x| < 1$ the Maclaurin series associated with $(1+x)^r$ converges to something, but does it converge to $(1+x)^r$?

Let us check the case $r = -1$. When $r = -1$, series (12.3.4) becomes

$$1 + (-1)x + \frac{(-1)(-2)}{1\cdot 2}x^2 + \frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3}x^3 + \cdots,$$

or

$$1 - x + x^2 - x^3 + \cdots.$$

This series is a geometric series with first term 1 and ratio $-x$. It, therefore, converges for $|x| < 1$. Moreover, it does represent the function $(1+x)^r = (1+x)^{-1}$.

But, as was pointed out at the end of Section 12.2, Lagrange's formula for the remainder $R_k(x,0)$ does not show that $R_k(x,0) \to 0$ as $k \to \infty$. It is true that for $|x| < 1$ series (12.3.4) does converge to $(1+x)^r$. The fact that $(1+x)^r$ is equal to the series (12.3.4) is known as the **general binomial theorem** or, simply, the **binomial theorem**. Series (12.3.4) is called the **binomial expansion** of $(1+x)^r$.

See Exercises 34 to 37 in the next section for a proof that (12.3.4) represents $(1+x)^r$ for $|x| < 1$.

Proofs of Theorem 12.3.1

Proof (of Theorem 12.3.1)
Since $\sum_{k=0}^{\infty} b_k c^k$ converges, the k^{th} term $a_k c^k$ approaches 0 as $k \to \infty$. Thus there is an integer N such that for $k \geq N$, $|b_k c^k| \leq 1$, say. From here on in the proof, consider only $k \geq N$. Now,

$$b_k x^k = b_k c^k \left(\frac{x}{c}\right)^k.$$

Since
$$\left|b_k x^k\right| = \left|b_k c^k\right| \left|\frac{x}{c}\right|^k,$$

it follows that for $k \geq N$,

$$\left|b_k x^k\right| \leq \left|\frac{x}{c}\right|^k \qquad (\text{since } |b_k c^k| \leq 1 \text{ for } k \geq N).$$

The series $\sum_{k=0}^{\infty} \left|\frac{x}{c}\right|^k$ is a geometric series with the ratio $|x/c| < 1$. Hence it converges.

Since $|b_k x^k| \leq \left|\frac{x}{c}\right|^k$ for $k \geq N$, the series $\sum_{k=0}^{\infty} \left|b_k x^k\right|$ converges (by the comparison test). Thus $\sum_{k=N}^{\infty} b_k x^k$ converges (in fact, absolutely). Putting in the front end $\sum_{k=0}^{N-1} b_k x^k$, we conclude that the series $\sum_{k=0}^{\infty} b_k x^k$ converges absolutely if $|x| < |c|$. •

Summary

Motivated by Taylor series, we investigated series of the form $\sum_{k=0}^{\infty} b_k x^k$ and, more generally, $\sum_{k=0}^{\infty} b_k (x-a)^k$. Associated with each such series is a radius of convergence R. (If the series converges for all x, we take R to be infinite.) If $\sum_{k=0}^{\infty} b_k x^k$ has radius of convergence R, then it converges (absolutely) for all x in $(-R, R)$, but diverges for all x such that $|x| > R$. Similarly, if $\sum_{k=0}^{\infty} b_k (x-a)^k$ has radius of convergence R, it converges for all x such that x is in $(a-R, a+R)$ but diverges for $|x-a| > R$. Convergence or divergence at the endpoints of the interval of convergence must be checked separately.

EXERCISES for Section 12.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 12 draw the appropriate diagrams (like Figure 12.3.4) showing where the series converge or diverge. Explain your work.

1.[R] $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$

2.[R] $\sum_{k=1}^{\infty} \frac{x^k}{\sqrt{k}}$

3.[R] $\sum_{k=0}^{\infty} \frac{x^k}{3^k}$

4.[R] $\sum_{k=1}^{\infty} k^2 e^{-k} x^k$

5.[R] $\sum_{k=0}^{\infty} \frac{2k^2+1}{k^2-5} x^k$

6.[R] $\sum_{k=1}^{\infty} \frac{x^k}{k}$

7.[R] $\sum_{k=0}^{\infty} \frac{x^k}{(2k)!}$

8.[R] $\sum_{k=0}^{\infty} \frac{2^k x^k}{k!}$

9.[R] $\sum_{k=0}^{\infty} \frac{x^k}{(2k+1)!}$

10.[R] $\sum_{k=0}^{\infty} k! x^k$

11.[R] $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$

12.[R] $\sum_{k=1}^{\infty} \frac{2^k x^k}{n}$

13.[R] Assume that $\sum_{k=0}^{\infty} b_k x^k$ converges for $x = 9$ and diverges when $x = -12$. What, if anything, can be said about

(a) convergence when $x = 7$?

(b) absolute convergence when $x = -7$?

(c) absolute convergence when $x = 9$?

(d) convergence when $x = -9$?

(e) divergence when $x = 10$?

(f) divergence when $x = -15$?

(g) divergence when $x = 15$?

14.[R] Assume that $\sum_{k=0}^{\infty} b_k x^k$ converges for $x = -5$ and diverges when $x = 8$. What, if anything, can be said about

(a) convergence when $x = 4$?

(b) absolute convergence when $x = 4$?

(c) convergence when $x = 7$?

(d) absolute convergence when $x = -5$?

(e) convergence when $x = -9$?

(f) convergence when $x = -9$?

15.[R] If $\sum_{k=0}^{\infty} b_k x^k$ converges whenever x is positive, must it converge whenever x is negative?

16.[R] If $\sum_{k=0}^{\infty} b_k 6^k$ converges, what can be said abou the convergence of

(a) $\sum_{k=0}^{\infty} b_k (-6)^k$?

(b) $\sum_{k=0}^{\infty} b_k 5^k$?

(c) $\sum_{k=0}^{\infty} b_k (-5)^k$?

In Exercises 17 to 28 draw the appropriate diagrams showing where the series converge and diverge.

17.[R] $\sum_{k=0}^{\infty} \frac{(x-2)^k}{k!}$

18.[R] $\sum_{k=0}^{\infty} \frac{(x-1)^k}{k3^k}$

19.[R] $\sum_{k=0}^{\infty} \frac{(x-1)^k}{k+3}$

20.[R] $\sum_{k=0}^{\infty} \frac{(x-4)^k}{2k+1}$

21.[R] $\sum_{k=0}^{\infty} \frac{k(x-2)^k}{2k+3}$

22.[R] $\sum_{k=0}^{\infty} \frac{(x-5)^k}{k\ln(k)}$

23.[R] $\sum_{k=0}^{\infty} \frac{(x+3)^k}{5^k}$

24.[R] $\sum_{k=0}^{\infty} k(x+1)^k$

25.[R] $\sum_{k=0}^{\infty} \frac{(x-5)^k}{k^2}$

26.[R] $\sum_{k=0}^{\infty} (-1)^k frac(x+4)^k k+2$

27.[R] $\sum_{k=0}^{\infty} k!(x-1)^k$

28.[R] $\sum_{k=0}^{\infty} \frac{k^2+1}{k^3+1}(x+2)^k$

In Exercises 29 to 34 write out the first five (non-zero) terms of the binomial expansion of the given functions.

29.[R] $(1+x)^{1/2}$

30.[R] $(1+x)^{1/3}$

31.[R] $(1+x)^{3/2}$

32.[R] $(1+x)^{-2}$

33.[R] $(1+x)^{-3}$

34.[R] $(1+x)^{-4}$

35.[R]

(a) If a power series $\sum_{k=0}^{\infty} b_k x^k$ diverges when $x = 3$, at which x must it diverge?

(b) If a power series $\sum_{k=0}^{\infty} b_k (x+5)^k$ diverges when $x = -3$, at which x must it diverge?

36.[R] If $\sum_{k=0}^{\infty} b_k (x-3)^k$ converges for $x = 7$, at what other values of x must the series necessarily converge?

37.[M] Find the radius of convergence of $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$.

38.[M] If $\sum_{k=0}^{\infty} b_k x^k$ has a radius of convergence 3 and $\sum_{k=0}^{\infty} c_k x^k$ has a radius of convergence 5, what can be said about the radius of convergence of $\sum_{k=0}^{\infty}(b_k+c_k)x^k$?

39.[M]

(a) Using the first four nonzero terms of the Maclaurin series for $\sqrt{1+x^3}$, estimate $\int_0^1 \sqrt{1+x^3}\,dx$. NOTE: This integral cannot be evaluated by the Fundamental Theorem of Calculus.

(b) Evaluate the integral in (a) to three decimal places by Simpson's method.

40.[M]

(a) Write the first four terms of the Maclaurin series associated with $f(x) = (1+x)^{-2}$.

(b) Find a formula for the general term in the Maclaurin series associated with $f(x)$.

(c) Replace x by $-x$ to obtain the Maclaurin series for $(1-x)^{-2}$. NOTE: Give the first four nonzero terms.

41.[M] What is the radius of convergence for the Maclaurin series associated with

(a) e^x

(b) $\sin(x)$

(c) $\cos(x)$

(d) $\ln(1+x)$

(e) $\arctan(x)$?

SHERMAN/DOUG: Revise, or remove!

42.[C] Show that the general binomial expansion of $(1+x)^r$ represents $(1+x)^r$.

43.[C] In R. P. Feynman, *Lectures on Physics*, Addison-Wesley, Reading, MA 1963, this statement appears in Section 15.8 of Volume 1:

> An approximate formula to express the increase of mas, for the case when the velocity is small, can be found by expanding $m_0/\sqrt{1-v^2/c^2} = m_0(1-v^2/c^2)^{-1/2}$ in a power series, using the binomial theorem. We get
>
> $$m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} = m_0\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\cdots\right).$$
>
> We see clearly from the formula that the series converges rapidly when v is small and the terms after the first two or three are neglible.

Check the expansion and justify the equation.

44.[C] In *Introduction to Fluid Mechanics*, by Stephan Whitaker, Krieger, New York, 1981, the following argument appears in the discussion of flow through a nozzle:

> The pressure p equals
>
> $$\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/(1-\gamma)}.$$
>
> By the binomial theorem and the fact that $v^2 = M^2\gamma RT$:
>
> $$p = 1-\frac{1}{2}\frac{v^2}{RT}+\frac{\gamma(2\gamma-1)}{8}M^4+\cdots.$$

Fill in the steps. NOTE: γ is the specific heat, which is about 1.4, and M is a Mach number, which is in the range 1 to 2.

45.[C]

(a) The ellipse $x^2/a^2 + y^2/b^2 = 1$ for $a \le b$ has the parameterization

$$x = a\cos(t), \qquad y = b\sin(t).$$

Show that the arc length of one quadrant of an ellipse is

$$\int_0^{\pi/2} b\sqrt{1 - \left(1 - \left(\frac{a}{b}\right)^2\right)\sin(t)^2}\ dt.$$

NOTE: The integrand does not have an elementary antiderivative.

(b) Assume that in (a), $a < b$. Then the arc length integral has the form $\int_0^{\pi/2} b\sqrt{1 - k^2\sin(t)^2}\ dt$, where $0 < k < 1$.

The "elliptic integral"

$$E = \int_0^{\pi/2} b\sqrt{1 - k^2\sin(t)^2}\ dt$$

is tabulated in mathematical handbooks for many values of k in $[0, 1]$. Using the binomial theorem and the formula for $\int_0^{\pi/2} \sin^2(\theta)\ d\theta$ (Formula 73 in the table of integrals), obtain the first four non-zero terms of E as a series of powers of k^2.

12.4 Manipulating Power Series

Where they converge, power series behave like polynomials. You can differentiate or integrate them term-by-term. You can add, subtract, multiply, and divide them. We will state these properties precisely, and apply them. While most of the discussion will be spent on power series in x, the same ideas apply to power series in $(x-a)$. Proofs can be found in any advanced calculus text.

Differentiating a Power Series

See the Sum and Difference Rules in Section 3.3

In Section 3.3 we showed that you can differentiate the sum of a finite number of functions by adding their derivatives. Theorem 12.4.1 generalizes this to power series in x.

Theorem 12.4.1 (Differentiating a power series)**.** *Assume $R > 0$ and that $\sum_{k=0}^{\infty} b_k x^k$ converges to $f(x)$ for $|x| < R$. Then for $|x| < R$, f is differentiable, $\sum_{k=1}^{\infty} k b_k x^{k-1}$ converges, and*

$$f'(x) = a_1 + 2a_2x^2 + 3a_3x^3 + \cdots .$$

This theorem is *not* covered by the fact that the derivative of the sum of a *finite* number of functions is the sum of their derivatives.

EXAMPLE 1 Obtain a power series for the function $1/(1-x)^2$ from the power series for $1/(1-x)$.

SOLUTION From the formula for the sum of a geometric series, we know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \text{for } |x| < 1.$$

According to Theorem 12.4.1, if we differentiate both sides of this equation, we obtain a true equation, namely,

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots \qquad \text{for } |x| < 1.$$

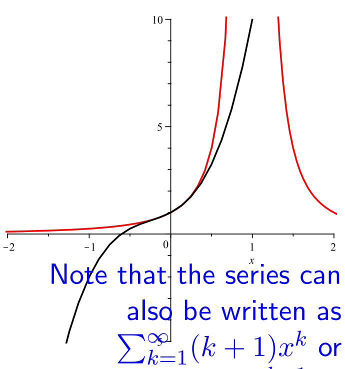


Figure 12.4.1:

Note that the series can also be written as $\sum_{k=1}^{\infty}(k+1)x^k$ or $\sum_{k=1}^{\infty} kx^{k-1}$ or $\sum_{k=0}^{\infty}(k+1)x^k$.

This can be expressed in summation notation. The geometric series is $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. When we differentiate both sides of this equation, we obtain $\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1}$. (See Figure 12.4.1.)

Theorem 12.4.1 does not say anything about convergence at the endpoints of the interval of convergence. When $x = 1$ the series is $\sum_{k=1}^{\infty} k$ which diverges (because the terms of this series do not approach 0). This is not surprising, because the derivative (and, in fact, the original function) are not defined when $x = 1$. When $x = -1$, $\frac{1}{(1-x)^2} = \frac{1}{4}$, so the derivative of the function is well-defined. But, when the series for the derivative is evaluated at $x = -1$

we get the series $\sum_{k=0}^{\infty}(-1)^{k-1}k$. As when $x = 1$, the terms of this series do not converge to zero and the series diverges. $\diamond$

Suppose that $f(x)$ has a power-series representation $b_0 + b_1x + b_2x^2 + \cdots$; Theorem 12.4.1 enables us to find the coefficients b_0, b_1, b_2, $\ldots$.

Theorem 12.4.2. *(Formula for b_k) Let R be a positive number and suppose that $f(x)$ is represented by the power series $\sum_{k=0}^{\infty} b_kx^k$ for $|x| < R$; that is,*

$$f(x) = b_0 + b_1x + b_2x^2 + \cdots + b_kx^k + \cdots \qquad \text{for } |x| < R.$$

Then

$$b_k = \frac{f^{(k)}(0)}{k!}. \tag{12.4.1}$$

The proof is practically the same as the derivation of the formulas for the coefficients of Taylor polynomials in Section 5.4. It consists of repeated differentiation and evaluation of the higher derivatives at 0.

Theorem 12.4.2 also tells us that there can be at most one series of the form $\sum_{k=0}^{\infty} b_kx^k$ that represents $f(x)$, for the coefficients b_k are completely determined by $f(x)$ and its derivatives. That series must be the Maclaurin series we obtained in Section 12.1. For instance, the series $1+x+x^2+x^3+\cdots$, which sums to $1/(1-x)$ for $|x| < 1$ must be the Maclaurin series for $1/(1-x)^2$.

Integrating a Power Series

Just as we may differentiate a power series term by term, we can integrate it term by term.

Theorem 12.4.3. *(Integrating a power series) Assume that $R > 0$ and*

$$f(x) = b_0 + b_1x + b_2x^2 + \cdots + b_kx^k + \cdots \qquad \text{for } |x| < R.$$

Then

$$b_0x + b_1\frac{x^2}{2} + b_2\frac{x^3}{3} + \cdots + b_k\frac{x^{k+1}}{k+1} + \cdots$$

converges for $|x| < R$, and

$$\int_0^x f(t)\ dt = b_0x + b_1\frac{x^2}{2} + b_2\frac{x^3}{3} + \cdots + b_k\frac{x^{k+1}}{k+1} + \cdots .$$

WARNING *(Choosing Variables of Integration)* Note that t is used as the variable of integration. This is done to avoid writing $\int_0^x f(x)\ dx$, an expression in which x describes both the interval $[0, x]$ and the independent variable of the integrand.

The next example shows the power of Theorem 12.4.3.

EXAMPLE 2 Integrate the power series for $1/(1+x)$ to obtain the power series in x for $\ln(1+x)$.

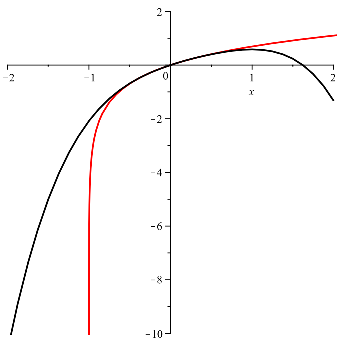
SOLUTION Start with the geometric series $1/(1-x) = 1 + x + x^2 + \cdots$ for $|x| < 1$. Replace x by $-x$ and obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots \qquad \text{for } |x| < 1.$$

By Theorem 12.4.3, $\int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$ for $|x| < 1$.

Now,

$$\begin{aligned} \int_0^x \frac{dt}{1+t} &= \ln(1+t)|_0^x \\ &= \ln(1+x) - \ln(1+0) \\ &= \ln(1+x). \end{aligned}$$



Power series for $\ln(1+x)$
Figure 12.4.2:

Thus

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \text{ for } |x| < 1.$$

◇

The power series for $\ln(1+x)$ could also have been obtained using Theorem 12.4.2. But this would have required finding a general form for the derivatives of $\ln(1+x)$, and their values at $x = 0$.

The derivation in Example 2 is more straightforward, and it gives the radius of convergence without additional work.

The Algebra of Power Series

In addition to differentiating and integrating power series, we may also add, subtract, multiply, and divide them just like polynomials. Theorem 12.4.4 states the rules for these operations.

Theorem 12.4.4. *The algebra of power series. Assume that*

$$f(x) = \sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + b_2 x^2 + \cdots \quad \textit{for } |x| < R_1$$

and

$$g(x) = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots \quad \textit{for } |x| < R_2.$$

Let R be the smaller of R_1 and R_2. Then, for $|x| < R$, *A.* $f(x)+g(x)=\sum_{k=0}^{\infty}(b_k+c_k)x^k=(b_0+c_0)+(b_1+c_1)x+(b_2+c_2)x^2+\cdots$

B. $f(x)-g(x)=\sum_{k=0}^{\infty}(b_k-c_k)x^k=(b_0-c_0)+(b_1-c_1)x+(b_2-c_2)x^2+\cdots$

C. $f(x)g(x)=(b_0c_0)+(b_0c_1+b_1c_0)x+(b_0c_2+b_1c_1+b_2c_0)x^2+\cdots$

D. *$f(x)/g(x)$ is obtainable by long division, provided $g(x)\neq 0$ for all $|x|<R$.*

This says "multiply two power series the way you multiply polynomials — term by term: start with the constant terms and work up."

EXAMPLE 3 Find the first four terms of the Maclaurin series for $e^x/(1-x)$.

SOLUTION There are at least three ways to approach this problem. The direct approach is to use Theorem 12.4.2; this requires finding the first three derivatives of $e^x/(1-x)$ evaluated at $x=0$. A second idea is to divide the power series for e^x by $1-x$. The third idea is to multiply the power series for e^x and the power series for $1/(1-x)$.

See Exercise 6

As multiplication is generally easier to carry out than division, that is the option we choose. The power series for e^x is $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ for all x (radius of convergence is ∞) and the power series for $1/(1-x)$ is $1+x+x^2+x^3+\cdots$ for $|x|<1$ (radius of convergence is 1):

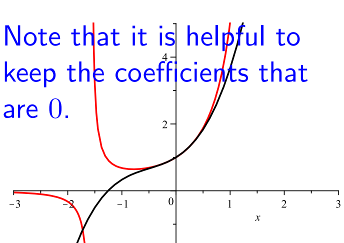
$$\begin{aligned}
e^x\frac{1}{1-x} &= \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3}+\cdots\right)\left(1+x+x^2+x^3+\cdots\right)\\
&= (1\cdot 1)+(1\cdot 1+1\cdot 1)x+\left(1\cdot 1+1\cdot 1+\frac{1}{2!}\cdots\right)x^2\\
&\quad+\left(1\cdot 1+1\cdot 1+\frac{1}{2!}\cdots+\frac{1}{3!}\cdot 1\right)x^3+\cdots\\
&= 1+2x+\frac{5}{2}x^2+\frac{8}{3}x^3+\cdots.
\end{aligned}$$

The power series for e^x converges for all x, $R_1=\infty$ and the power series for $1/(1-x)$ converges for $|x|<1$. According to Theorem 12.4.2, the power series for $e^x/(1-x)$, whose first four terms we just found, has radius of convergence $R=1$. The power series for $e^x/(1-x)$ is valid for $|x|<1$. ◇

EXAMPLE 4 Find the first four terms of the Maclaurin series for $e^x/\cos(x)$.

SOLUTION Write the Maclaurin series for e^x and $\cos(x)$ up through the

Note that it is helpful to keep the coefficients that are 0.



terms of degree 3 and arrange the long division as follows:

$$
\begin{array}{r|ccccccccc}
 & 1 & + & x & + & x^2 & + & \frac{2}{3}x^3 & + & \cdots \\
\cline{2-10}
1 + 0x - \frac{x^2}{2} + 0x^3 + \cdots & 1 & + & x & + & \frac{x^2}{2} & + & \frac{x^3}{6} & + & \cdots \\
 & 1 & + & 0x & + & -\frac{x^2}{2} & + & 0x^3 & + & \cdots \\
\cline{2-10}
 & & & x & + & x^2 & + & \frac{x^3}{6} & + & \cdots \\
 & & & x & + & 0x^2 & - & \frac{x^3}{2} & + & \cdots \\
\cline{2-10}
 & & & & & x^2 & + & \frac{2x^3}{3} & + & \cdots \\
 & & & & & x^2 & + & 0x^3 & - & \cdots \\
\cline{2-10}
 & & & & & & & \frac{2x^3}{3} & + & \cdots \\
 & & & & & & & \frac{2x^3}{3} & + & \cdots \\
\cline{2-10}
 & & & & & & & & & \cdots
\end{array}
$$

Thus, the Maclaurin series for $e^x/\cos(x)$ begins

$$\frac{e^x}{\cos(x)} = 1 + x + x^2 + \frac{2x^3}{3} + \cdots$$

What happens when $|x| = \pi/2$?

Even though the power series for e^x and $\cos(x)$ both have infinite radius of convergence, the power series for $e^x/\cos(x)$ converges only for $|x| < \pi/2$. Why would you expect "trouble" at $\pi/2$?

We could have found the front-end of this Maclaurin series using Theorem 12.4.2. Of course, the results would be the same. But, this approach does not give any information about the radius of convergence of this power series. $\diamond$

Power series in $x - a$

Power Series Around a

The various theorems and methods of this section were stated for power series in x. Analogous theorems hold for power series in $x-a$. Such series may be differentiated and integrated term by term inside the interval in which they converge. For instance, Theorem 12.4.2 generalizes to the following result.

Theorem 12.4.5. *(Formula for b_k) Let R be a positive number and suppose that $f(x)$ is represented by the power series $\sum_{k=0}^{\infty} b_k(x-a)^k$ for $|x-a| < R$; that is,*

$$f(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_k(x-a)^k + \cdots \qquad \text{for } |x-a| < R.$$

Then

$$b_k = \frac{f^{(k)}(a)}{k!}.$$

The proof of Theorem 12.4.5 is similar to that of Theorem 12.4.2. To determine b_k: differentiate k times, evaluate the result when $x = a$, and divide by $k!$.

Endpoints

Each theorem in this section includes information on the radius of convergence of a power series obtained from another power series. Convergence at the endpoints is never guaranteed; it must be checked separately in every case.

In Example 1 we found the power series in x for $1/(1-x)^2$ is

$$1 + 2x + 3x^2 + \cdots = \sum_{k=1}^{\infty} kx^{k-1} \tag{12.4.2}$$

for $|x| < 1$. When $x = 1$ this series becomes $\sum_{k=1}^{\infty} k$, and, when $x = -1$ it is $\sum_{k=1}^{\infty} k(-1)^{k-1}$. Each of these series diverges because its terms do not approach 0 as $k \to \infty$. Thus, (12.4.2) converges only on the open interval $(-1, 1)$.

In Example 2 the power series for $\ln(1 + x)$ was found to be

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \tag{12.4.3}$$

again for $|x| < 1$.

When $x = 1$ the series becomes $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. This is the alternating harmonic series, which converges. When $x = -1$ the series becomes $\sum_{k=1}^{\infty} \frac{-1}{k}$ which diverges because it is the negative of the harmonic series. This means the interval of convergence for (12.4.3) is $(-1, 1]$.

We still do not know this series, with $x = 1$, converges to $\ln(2)$. This is shown in Exercise 28.

Some series converge at both endpoints. You can never tell what will happen until you check each endpoint.

How Some Calculators Find e^x

The power series in x for e^x is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots .$$

For $x = 10$, this would give

$$e^{10} = 1 + 10 + \frac{10^2}{2!} + \frac{10^3}{3!} + \cdots + \frac{10^k}{k!} + \cdots .$$

Although the terms eventually become very small, the first few terms are quite large. (For instance, the fifth term, $10^4/4!$, is about 417.) So when x is large, the series for e^x provides a time-consuming procedure for calculating e^x.
Some calculators use the following method instead.
The values of e^x at certain inputs are built into the memory:

$$\begin{aligned}
e^1 &\approx 2.718281828459 \\
e^{10} &\approx 22,026.46579 \\
e^{100} &\approx 2.6881171 \times 10^{43} \\
e^{0.1} &\approx 1.1051709181 \\
e^{0.01} &\approx 1.0100501671 \\
e^{0.001} &\approx 1.0010005002.
\end{aligned}$$

To find $e^{315.425}$, say, the calculator makes use of the identities $e^{x+y} = e^x e^y$ and $(e^x)^y = e^{xy}$ and computes

$$\left(e^{100}\right)^3 \left(e^{10}\right)^1 \left(e^1\right)^5 \left(e^{0.1}\right)^4 \left(e^{0.01}\right)^2 \left(e^{0.001}\right)^5 \approx 9.71263198 \times 10^{136}.$$

This result is accurate to six decimal places.

Summary

We showed how to operate with power series to obtain new power series — by differentiation, integration, or an algebraic operation, such as multiplying or dividing two series. For instance, from the geometric series for $1/(1+x)$, you can obtain the series for $\ln(1+x)$ by integration, or the series for $-1/(1+x)^2$ by differentiation.

The radius of convergence for a derived power series is determined directly from the radius of convergence of the original series and the operation performed. However, convergence at the endpoints must be checked for each series.

EXERCISES for Section 12.4 *Key:* R–routine, M–moderate, C–challenging

1.[R] Differentiate the Maclaurin series for $\sin(x)$ to obtain the Maclaurin series for $\cos(x)$.

2.[R] Differentiate the Maclaurin series for e^x to show that $D(e^x) = e^x$.

3.[R] Prove Theorem 12.4.2 by carrying out the necessary differentiations.

4.[R]

(a) Show that, for $|t| < 1$, $1/(1+t^2) = 1 - t^2 + t^4 - t^6 + \cdots$.

(b) Use Theorem 12.4.3 to show that, for $|x| < 1$, $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.

(c) Give the formula for the k^{th} term of the series in (b).

(d) How many terms of the series in (b) are needed to approximate $\arctan(1/2)$ to three decimal places?

(e) Use the formula in (b) to estimate $\arctan(1/2)$ to three decimal places.

NOTE: Exercise 21 shows that the series in (b) converges to $\arctan(x)$ also when $x = -1$ and $x = 1$.

5.[R]

(a) Using Theorem 12.4.3, show that for $|x| < 1$,

$$\int_0^x \frac{dt}{1+t^3} = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots.$$

(b) Use (a) to express $\int_0^{0.7} dt/(1+t^3)$ as a series of numbers.

(c) How many terms of the series in (a) are needed to estimate $\int_0^{0.7} dt/(1+t^3)$ to three decimal places?

(d) Use (b) to evaluate $\int_0^{0.7} dt/(1+t^3)$ to three decimal places.

(e) Describe how you would evaluate $\int_0^{0.7} dt/(1+t^3)$ using the fundamental theorem of calculus. (*Do not carry out the details.*)

(f) Use a computer algebra system to find the exact value of $\int_0^{0.7} dt/(1+t^3)$.

6.[R]

(a) Find the first four terms of the Maclaurin series for $e^x/(1-x)$ by division of series. HINT: Keep the first five terms of e^x.

(b) Find the first four terms of the Maclaurin series for $e^x/(1-x)$ by using the formula in terms of derivatives.

7.[R]

(a) Find the first three nonzero terms of the Maclaurin series for $\tan(x)$ by dividing the series for $\sin(x)$ by the series for $\cos(x)$.

(b) Find the first two nonzero terms of the Maclaurin series for $\tan(x)$ by using the formula for the k^{th} term, $b_k = f^{(k)}(0)/k!$.

8.[R]

(a) Find the first four terms of the Maclaurin series for $(1-\cos(x))/(1-x^2)$ by division of series.

(b) Find the first four terms of the Maclaurin series for $(1-\cos(x))/(1-x^2)$ by multiplication of series.

In Exercises 9 and 10, obtain the first three nonzero terms in the power series in x for the indicated functions by algebraic operations with known series. Also, identify the radius of convergence.

9.[R] $e^x \sin(x)$

10.[R] $\frac{x}{\cos(x)}$

In Exercises 11 to 16 use power series to determine the limits.

11.[R] $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

12.[R] $\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)}$

13.[R] $\lim_{x\to 0} \frac{\sin^2(x^3)}{(1-\cos(x^2))^3}$

14.[R] $\lim_{x\to 0} \left(\frac{1}{\sin(x)} - \frac{1}{\ln(1+x)}\right)$

15.[R] $\lim_{x\to 0} \frac{(e^x-1)^2}{\sin(x^2)}$

16.[R] $\lim_{x\to 0} \frac{\sin(x)(1-\cos(x))}{e^{x^3}-1}$

17.[R] Estimate $\int_0^{1/2} \sqrt{x}e^{-x}\,dx$ to four decimal places.

18.[R] Let $f(x) = \sum_{k=0}^{\infty} k^2 x^k$.

(a) What is the domain of f?

(b) Find $f^{(100)}(0)$.

19.[R] Let $f(x) = \arctan(x)$. Making use of the Maclaurin series for $\arctan(x)$, find

(a) $f^{(100)}(0)$

(b) $f^{(101)}(0)$.

20.[M]

(a) Give a numerical series equal to $\int_0^1 \sqrt{x}\sin(x)\,dx$.

(b) How many terms of the series in (a) are needed to approximate this integral to four decimal places?

(c) Use (a) to evaluate the integral to four decimal places.

21.[M] The Taylor series for $\arctan(x)$ is $\sum_0^\infty \frac{(-1)^k}{2k+1}x^{2k+1}$. While the interval of convergence of this power series is easily found to be $[-1,1]$, Theorem 12.4.3 tells us only that this series converges to $\arctan(x)$ on the open interval $(-1,1)$.

(a) Show that, when $x = 1$, the given series is the Maclaurin series for $\arctan(1)$. HINT: Look at the Lagrange Form for the Remainder.

(b) Repeat (a), using $x = -1$.

(c) Because $\arctan(1) = \pi/4$, the Maclaurin series for $\arctan(1)$ provides one way to obtain approximations to π. Approximate π using the first 5 non-zero terms in the Maclaurin series for $\arctan(1)$.

(d) Estimate the error in the approximation to π found in (c).

(e) How many terms in the Maclaurin series are needed to obtain an approximate value of π accurate to 2 decimal places? 4 decimal places? 12 decimal places?

22.[M]

(a) From the Maclaurin series for $\cos(x)$, obtain the Maclaurin series for $\cos(2x)$.

(b) Exploiting the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, obtain the Maclaurin series for $\sin^2(x)/x^2$.

(c) Estimate $\int_0^1 (\sin(x)/x)^2 \, dx$ using the first three nonzero terms of the series in (b).

(d) Find a bound on the error in the estimate in (c).

23.[M] Let $\sum_{k=0}^{\infty} b_k x^k$ and $\sum_{k=0}^{\infty} c_k x^k$ converge for $|x| < 1$. If, for all k, they converge to the same limit, must $b_k = c_k$?

24.[M] This exercise outlines a way to compute logarithms of numbers larger than 1.

(a) Show that every number $y > 1$ can be written in the form $(1 + x)/(1 - x)$ for some x in $(0, 1)$.

(b) When $y = 3$, find x.

(c) Show that if $y = (1+x)/(1-x)$, then $\ln(y) = 2(x + x^3/3 + \cdots + x^{2n+1}/(2n + 1) + \dots)$.

(d) Use (b) and (c) to estimate $\ln(3)$ to two decimal places. HINT: To control the error, compare a tail end of the series to an appropriate geometric series.

(e) Is the error in (d) less than the first omitted term?

25.[M] Sam has an idea: "I have a more direct way of estimating $\ln(y)$ for $y > 1$. I just use the identity $\ln(y) = -\ln(1/y)$. Because $1/y$ is in $(0, 1)$ I can write it as $1-x$, and x is still in (0,1). In short, $\ln(y) = -\ln(1/y) = -\ln(1-x) = x+x^2/2+x^3/3+\dots$. It's even an easier formula. And it's better because it doesn't have that coefficient 2 in front."

(a) Is Sam's formula correct?

(b) Use his method to estimate $\ln(3)$ to two decimal places.

(c) Which is better, Sam's method or the one in Exercise 24?

26.[M] Use the method of Exercise 24 to estimate $\ln(5)$ to two decimal places. Include a description of your procedure.

27.[C] Here are five ways to compute $\ln(2)$. Which seems to be the most efficient? least efficient? Explain.

(a) The series for $\ln(1+x)$ when $x = 1$.

(b) The series for $\ln(1+x)$ when $x = \frac{-1}{2}$. NOTE: This gives $\ln\left(\frac{1}{2}\right) = -\ln(2)$.

(c) The series for $\ln((1+x)/(1-x))$ when $x = \frac{1}{3}$.

(d) Simpson's method applied to the integral $\int_1^2 dx/x$.

(e) The root of $e^x = 2$. HINT: Use Newton's method.

28.[C] In the discussion of endpoints for the Maclaurin series for $\ln(1+x)$, we showed that the series converges for $x = 1$, but we did not show that its sum is $\ln(2)$. To show that it does equal $\ln(2)$, integrate both sides of the following equation over $[0, 1]$:

$$\frac{1 + (-x)^{n+1}}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n.$$

HINT: Separate the left-hand side into two separate integrals. Then, take the limit as $n \to \infty$.

29.[C] What theorem justifies the assertion that if the series has a nonzero radius of convergence then

$$\lim_{x\to 0} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) = a_0?$$

30.[C]

(a) For which x does $\sum_{k=0}^{\infty} k^2 x^k$ converge?

(b) Starting with the Maclaurin series for $x^2/(1-x)$, sum the series in (a).

(c) Does your formula seem to give the correct answer when $x = \frac{1}{3}$?

31.[C] This exercise uses power series to give a new perspective on l'Hôpital's rule. Assume that f and g can be represented by power series in some open interval containing 0:

$$f(x) = \sum_{k=0}^{\infty} b_k x^k \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Assume that $f(0) = 0$, $g(0) = 0$, and $g'(0) \neq 0$. Under these assumptions explain why

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}.$$

32.[C] If R. P. Feynman, *Lectures on Physics*, Addison-Wesley, Reading, MA, 1963, appears this remark:

> Thus the average velocity is
> $$\langle E\rangle = \frac{\hbar\omega(0+x+2x^2+3x^3+\cdots)}{1+x+x^2+\cdots}.$$
> Now the two sums which appear here we shall leave for the reader to play with and have some fun with. When we are all finished summing and substituting for x in the sum, we should get — if we make no mistakes in the sum —
> $$\langle E\rangle = \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}.$$
> This, then, was the first quantum-mechanical formula ever known, or ever discussed, and it was the beautiful culmination of decades of puzzlement.

Have the aforementioned fun, given that $x = e^{-\hbar\omega/kT}$.

33.[C] Justify this statement, found in a biological monograph:

> Expanding the equation
> $$a\cdot\ln(x+p) + b\cdot\ln(y+q) = M,$$
> we obtain
> $$a\left(\ln(p) + \frac{x}{p} - \frac{x^2}{2p^2} + \frac{x^3}{3p^3} - \cdots\right) + b\left(\ln(q) + \frac{y}{q} - \frac{y^2}{2q^2} + \frac{y^3}{3q^3} - \cdots\right) = M.$$

Exercises 34 to 37 outline a proof that the Maclaurin series associated with $(1+x)^r$ converges to $(1+x)^r$ for $|x|<1$. This justifies the assertion that $(1+x)^r = \sum_{k=0}^{\infty}\binom{n}{k}x^k$ for $|x|<1$. The notation $\binom{n}{k}$ stands for $\frac{n!}{k!(n-k)!}$.

34.[C] Show that
$$k\binom{r}{k} + (k+1)\binom{r}{k+1} = \binom{r}{k}.$$
(This is needed in Exercise 35.) HINT: First, rewrite the equation as $(k+1)\binom{r}{k+1} = (r-k)\binom{r}{k}$.

35.[C] Let $f(x) = \sum_{k=0}^{\infty}\binom{r}{k}x^k$.

(a) Find the interval of convergence for $f(x)$.

(b) Show that $(1+x)f'(x) = rf(x)$. HINT: First, write out the first four terms to see the pattern.

36.[C] Using the result from Exercise 35, show that the derivative of $f(x)/(1+x)^r$ is 0.

37.[C] Show that $f(x)/(1+x)^r = 1$, which implies that $\sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^r$. What is the interval of convergence

12.5 Complex Numbers

Let us think of the number line of real numbers as coinciding with the x-axis of the xy coordinate system. This number line, with its addition, subtraction, multiplication, and division, is a small part of a number system that occupies the plane, and which obeys the usual rules of arithmetic. This section describes that system, known as the **complex numbers**. One of the important properties of the complex numbers is that any nonconstant polynomial has a root; in particular, the equation $x^2 = -1$ has two solutions.

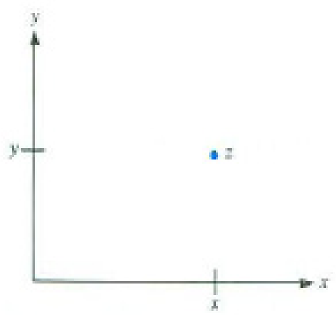


Figure 12.5.1:

The Complex Numbers

By a complex number z we mean an expression of the form $x + iy$ or $x + yi$, where x and y are real numbers and i is a symbol with the property that $i^2 = -1$. This expression will be identified with the point (x, y) in the xy plane, as in Figure 12.5.1. Every point in the xy plane may therefore be thought of as a complex number.

To add or multiply two complex numbers, follow the usual rules of arithmetic of real numbers, with one new proviso:

Whenever you see i^2, replace it by -1.

For instance, to add the complex numbers $3 + 2i$ and $-4 + 5i$, just collect like terms:

$$(3 + 2i) + (-4 + 5i) = (3 - 4) + (2i + 5i) = -1 + 7i.$$

(See Figure 12.5.2(a).) Addition does not make use of the fact that $i^2 = -1$. However, multiplication does, as Example 1 shows.

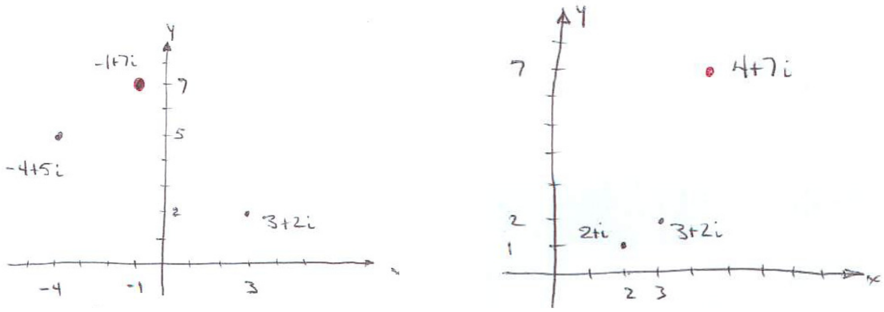
EXAMPLE 1 Compute the product $(2 + i)(3 + 2i)$.
SOLUTION We can multiply the complex numbers just as we would multiply binomials. (Recall the mnemonic FOIL for "first, outer, inner, last.") We have

$$(2+i)(3+2i) = 2\cdot3+2\cdot2i+i\cdot3+i\cdot2i = 6+4i+3i+2i^2 = 6+4i+3i-2 = 4+7i.$$

Figure 12.5.2(b) shows the complex numbers $2 + i$, $3 + 2i$, and their product $4 + 7i$. ◇

Real numbers are on the x-axis, imaginary on the y-axis.

Note that $(-i)(-i) = i^2 = -1$. Both i and $-i$ are square roots of -1. The symbol $\sqrt{-1}$ traditionally denotes i rather than $-i$.



(a) (b)

Figure 12.5.2:

A complex number that lies on the y-axis is called **imaginary**. Every complex number z is the sum of a real number and an imaginary number, $z = x + iy$. The number x is called the **real part of** z, and y is called the **imaginary part**. One often writes "$\operatorname{Re} z = x$" and "$\operatorname{Im} z = y$."

We have seen how to add and multiply complex numbers. Subtraction is straightforward. For instance,

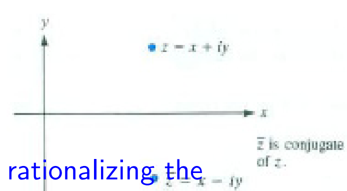
$$(3 + 2i) - (4 - i) = (3 - 4) + (2i - (-i)) = -1 + 3i.$$

Division of complex numbers requires rationalizing the denominator. This involves the **conjugate** of a complex number. The conjugate of the complex number $z = x + yi$ is the complex number $x - yi$, which is denoted $\bar{z}$. Note that

conjugate of z

$$\begin{aligned} z\bar{z} &= (x + yi)(x - yi) = x^2 + y^2 \\ z + \bar{z} &= (x + yi) + (x - yi) = 2x \\ \text{and} \qquad z - \bar{z} &= (x + yi) - (x - yi) = 2yi. \end{aligned}$$

Thus, $z\bar{z}$ and $z + \bar{z}$ are real, and $z - \bar{z}$ is imaginary. Figure 12.5.3 shows the relation between z and $\bar{z}$, which is that $\bar{z}$ is the mirror image of z reflected across the x-axis. To "rationalize the denominator" means to find an equivalent fraction with a real-valued denominator. If the fraction is $\frac{w}{z}$, the denominator can be rationalized by multiplying by $\frac{\bar{z}}{\bar{z}}$.



rationalizing the denominator

Figure 12.5.3:

EXAMPLE 2 Compute the quotient $\frac{1+5i}{3+2i}$.
SOLUTION To rationalize the denominator, we multiply by $\frac{3-2i}{3-2i}$:

$$\frac{1+5i}{3+2i} = \frac{1+5i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{3 - 2i + 15i + 10}{9 - 6i + 6i + 4i^2} = \frac{13 + 13i}{13} = 1 + i.$$

◇

Now All Polynomials Have Roots

Every polynomial has a root in the complex numbers.

The complex numbers provide the equation $x^2 + 1 = 0$ with two solutions, i and $-i$. This illustrates an important property of complex numbers: If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is any polynomial of degree $n \geq 1$, with real or complex coefficients, then there is a complex number z such that $f(z) = 0$. This fact, known as the **Fundamental Theorem of Algebra**, is illustrated in Example 3. Its proof requires advanced mathematics.

EXAMPLE 3 Solve the quadratic equation $z^2 - 4z + 5 = 0$.
SOLUTION By the quadratic formula, the solutions are

$$\begin{aligned} z &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} \\ &= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i. \end{aligned}$$

The two solutions are $2 + i$ and $2 - i$.

These solutions can be checked by substitution in the original equation. For instance,

$$\begin{aligned} (2+i)^2 - 4(2+i) + 5 &= (4 + 4i + i^2) - 8 - 4i + 5 \\ &= 4 + 4i - 1 - 8 - 4i + 5 = 0 + 0i = 0. \end{aligned}$$

Yes, it checks. The solution $2 - i$ can be checked similarly. ◇

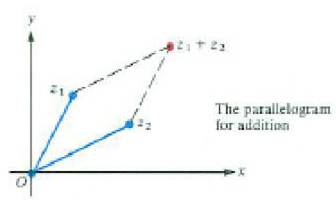


Figure 12.5.4:

The sum of the complex numbers z_1 and z_2 is the fourth vertex (opposite O) in parallelogram determined by the origin O and the points z_1 and z_2, as shown in Figure 12.5.4. The geometry of the product of z_1 and z_2 is more involved.

The Geometry of the Product

The geometric relation between z_1, z_2 and their product $z_1 z_2$ is easily described in terms of the magnitude and argument of a complex number. Each complex number z other than the origin is at a (positive) distance r from the origin and has a polar angle θ relative to the positive x-axis. The distance r is called the **magnitude of** z, and θ is called the **argument of** z. A complex number has an infinity of arguments differing from each other by an integer multiple of 2π. The complex number 0, which lies at the origin, has magnitude 0 and any polar angle as argument. In short, we may think of magnitude and argument as polar coordinates r and θ of z, with the restriction that r is nonnegative.

The symbols $|z|$ and $\arg(z)$

The magnitude of z is denoted $|z|$. The symbol $\arg(z)$ denotes any of the arguments of z, it being understood that if θ is an argument of z, then so is $\theta + 2\pi$ for any integer n.

EXAMPLE 4

(a) Draw all complex numbers with magnitude 3.

(b) Draw the complex number z of magnitude 3 and argument $\pi/6$.

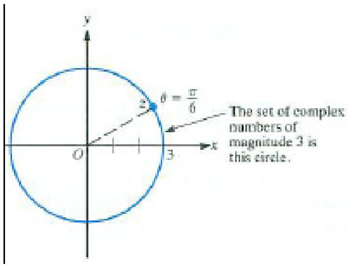


Figure 12.5.5: NOTE: Draw the for (b) in red.

SOLUTION

(a) The complex numbers of magnitude 3 form a circle of radius 3 with center at 0. (See Figure 12.5.5.)

(b) The complex number of magnitude 3 and argument $\pi/6$ is shown (in red) in Figure 12.5.5.

◇

Note that $|x+iy| = \sqrt{x^2+y^2}$, by the Pythagorean theorem. Each complex number $z = x+iy$ other than 0 can be written as the product of a positive real number and a complex number of magnitude 1. To show this, let $z = x+iy$ have magnitude r and argument θ. Recalling the relation between polar and rectangular coordinates, we conclude that

$$\begin{aligned} z &= r\cos(\theta) + r\sin(\theta) \\ &= r(\cos(\theta) + \sin(\theta)). \end{aligned}$$

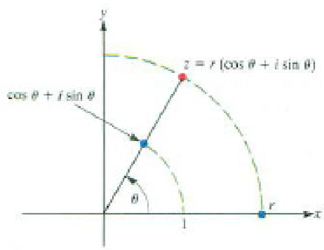


Figure 12.5.6: ARTIST: Draw the point for (b) in red.

The number r is a positive real number. The magnitude of the number $\cos(\theta) + i\sin(\theta)$ is $\sqrt{\cos(\theta)^2 + \sin(\theta)^2} = 1$. Figure 12.5.6 shows the number r and $\cos(\theta) + i\sin(\theta)$, whose product is z. (The expression $\cos(\theta) + i\sin(\theta)$ appears so frequently when working with complex numbers that the shorthand notation $\text{cis}(\theta)$ is used, that is, $\text{cis}(\theta) = \cos(\theta) + i\sin(\theta)$. While this is convenient, you have to be careful not to confuse "cis" with "cos.")

The next theorem describes *how to multiply two complex numbers if they are given in polar form,* that is, in terms of their magnitudes and arguments.

Theorem. *Assume that z_1 has magnitude r_1 and argument θ_1 and that z_2 has magnitude r_2 and argument θ_2. Then the product z_1z_2 has magnitude r_1r_2 and argument $\theta_1 + \theta_2$.*

Proof

The last step uses the identities for $\cos(u+v)$ and $\sin(u+v)$.

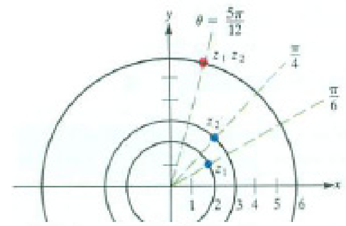
$$\begin{aligned} z_1z_2 &= r_1(\cos(\theta_1) + i\sin(\theta_1))r_2(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1r_2(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)) \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

Thus, the magnitude of z_1z_2 is r_1r_2 and the argument of z_1z_2 is $\theta_1 + \theta_2$. This proves the theorem. •

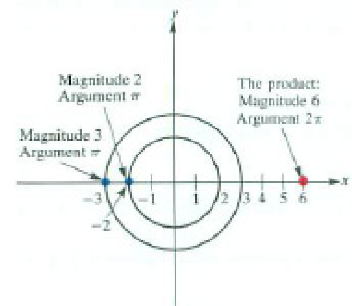
In practical terms, this theorem says:

"To multiply two complex numbers, add their arguments and multiply their magnitudes."

EXAMPLE 5 Find z_1z_2 for z_1 and z_2 in Figure 12.5.7.
SOLUTION z_1 has magnitude 2 and argument $\pi/6$; z_2 has magnitude 3 and argument $\pi/4$. Thus, z_1z_2 has magnitude $2 \cdot 3 = 6$ and argument $\pi/6 + \pi/4 = 5\pi/12$. (See Figure 12.5.7 ◇



EXAMPLE 6 Using the geometric description of multiplication, find the product of the real numbers -2 and -3.
SOLUTION The number -2 has magnitude 2 and argument π. The number -3 has magnitude 3 and argument π. Therefore $(-2) \cdot (-3)$ has magnitude $2 \cdot 3 = 6$ and argument $\pi + \pi = 2\pi$. The complex number with magnitude 6 and argument 2π is just our old friend, the real number 6. Thus $(-2) \cdot (-3) = 6$, in agreement with the statement "the product of two negative numbers is positive." (See Figure 12.5.8.) ◇



See Exercise 29.

Figure 12.5.8:

Division of Complex Numbers

Division of complex numbers given in polar form is similar, except that the magnitudes are divided and the arguments are subtracted:

$$\frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i\sin(\theta_2))} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right).$$

EXAMPLE 7 Let $z_1 = 6(\cos(\pi/2) + i\sin(\pi/2)$ and $z_2 = 3(\cos(\pi/6) + i\sin(\pi/6)$. Find *(a)* z_1z_2 and *(b)* z_1/z_2.
SOLUTION See Figure 12.5.9

(a)

$$\begin{aligned} z_1z_2 &= 6 \cdot 3\left(\cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right)\right) = 18\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) \\ &= 18\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -9 + 9\sqrt{3}i. \end{aligned}$$

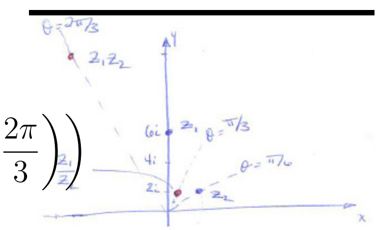


Figure 12.5.9:

(b)

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{6}{3}\left(\cos\left(\frac{\pi}{2} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\right) = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) \\ &= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i \end{aligned}$$

◇

EXAMPLE 8 Compute the product $(2+i)(3+2i)$ and check the answer in terms of magnitudes and arguments.
SOLUTION

$$(1+i)(3+2i) = 3+2i+3i+2i^2 = 3+2i+3i-2 = 1+5i.$$

To check this calculation, first verify that $|1+5i| = |1+i||3+2i|$. We have

$$\begin{aligned} |1+5i| &= \sqrt{1^2+5^2} = \sqrt{26}, \\ |1+i| &= \sqrt{1^2+1^2} = \sqrt{2}, \\ |3+2i| &= \sqrt{3^2+2^2} = \sqrt{13}. \end{aligned}$$

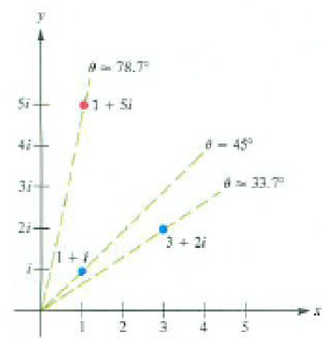


Figure 12.5.10: $\arg(x+iy) = \arctan(y/x)$ for $x+iy$ in the first or fourth quadrants.

Since $\sqrt{26} = \sqrt{2}\sqrt{13}$, the magnitude of $1+5i$ is the product of the magnitudes of $1+i$ and $3+2i$.

Next, consider the arguments. First, $\arg(1+5i) = \arctan(5) \approx 1.3734$. Similarly, $\arg(1+i) = \arctan(1) \approx 0.7854$ and $\arg(3+2i) = \arctan(2/3) \approx 0.5880$. Since $0.7854 + 0.5880 = 1.3734$, the argument of $1+5i$ is the sum of the arguments of $1+i$ and $3+2i$. (See also Figure 12.5.10.) ◇

Powers of z

How to compute z^n

When the polar coordinates of z are known, it is easy to compute the powers z^2, z^3, z^4, ... Let z have magnitude r and argument θ. Then $z^2 = z \cdot z$ has magnitude $r \cdot r = r^2$ and argument $\theta + \theta = 2\theta$. So, to square a complex number, just square it magnitude and double its argument (angle).

More generally, to compute z^n for any positive integer n, find $|z|^n$ and multiply the argument of z by n. In short, we have **DeMoivre's Law**:

$$\boxed{(r(\cos(\theta) + i\sin(\theta)))^n \,. = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)}$$

Example 9 illustrates the geometric view of computing powers.

EXAMPLE 9 Let z have magnitude 1 and argument $2\pi/5$. Compute and sketch z, z^2, z^3, z^4, z^5, and z^6.
SOLUTION Since $|z| = 1$, it follows that $|z^2| = |z|^2 = 1^2 = 1$. Similarly, for all positive integers n, $|z^n| = 1$; that is, z^n is a point on the unit circle with center O. All that remains is to examine the arguments of z^2, z^3, etc..

The argument of z^2 is twice the argument of z: $2(2\pi/5) = 4\pi/5$. Similarly, $\arg(z^3) = 6\pi/5$, $\arg(z^4) = 8\pi/5$, $\arg(z^5) = 10\pi/5 = 2\pi$, and $\arg(z^6) = 12\pi/5$. Observe that $z^5 = 1$, since it has magnitude 1 and argument 2π. Similarly, $z^6 = z$, since both z and z^6 have magnitude 1 and their arguments differ by an integer multiple of 2π. (Or, algebraically, $z^6 = z^{5+1} = z^5 \cdot z = 1 \cdot z = z$.) Figure 12.5.11 shows that the powers of z form the vertices of a regular pentagon. ◇

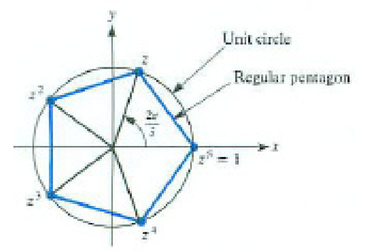


Figure 12.5.11:

The powers of i.

The equation $x^5 = 1$ has only one real root, namely, $x = 1$. However, it has five complex roots. For instance, the number z shown in Figure 12.5.11 is a solution of $x^5 = 1$ since $z^5 = 1$. Another root is z^2, since $(z^2)^5 = z^{10} = (z^5)^2 = 1^2 = 1$. Similarly, z^3 and z^4 are roots of $x^5 = 1$. There are five roots: 1, z, z^2, z^3, and z^4.

The powers of i will be needed in the next section. They are $i^2 = -1$, $i^3 = i^2 \cdot i = (-1)i = -i$, $i^4 = i^3 \cdot i = (-i)i = -i^2 = 1$, $i^5 = i^4 \cdot i = i$, and so on. They repeat in blocks of four: for any integer n, $i^{n+4} = i^n$.

It is often useful to express a complex number $z = x + iy$ in polar form. Recall that $|z| = \sqrt{x^2 + y^2}$. To find θ, it is best to sketch z in order to see in which quadrant it lies.

For instance, to put $z = -2 - 2i$ in polar form, first sketch z, as in Figure 12.5.12. We have $|z| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8}$ and $\arg(z) = 5\pi/4$. Thus

$$z = \sqrt{8}\left(\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)\right).$$

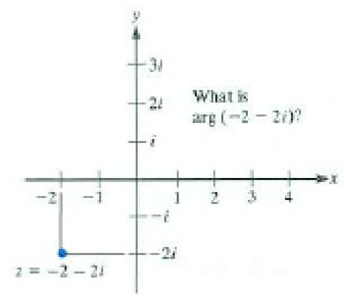


Figure 12.5.12:

Roots of z

Each complex number z, other than 0, has exactly n n^{th} roots for each positive integer n. These can be found by expressing z in polar coordinates. If $z = r(\cos(\theta) + i \sin(\theta))$, that is, has magnitude r and argument θ, then one n^{th} root of z is

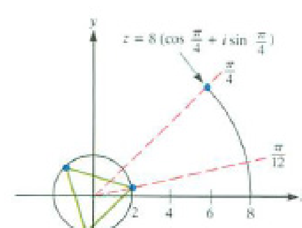
$$r^{1/n}\left(\cos\left(\frac{\theta}{n}\right) + i \sin\left(\frac{\theta}{n}\right)\right).$$

To check that this is an n^{th} root of z, just raise it to the n^{th} power.

To find the other n^{th} roots of z, change the argument z from θ to $\theta + 2k\pi$, where $k = 1, 2, \ldots, n-1$. Then

$$r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right)$$

is also an n^{th} root of z. (Why?)



The n roots of the equation $z^n = a$ are the vertices of a regular polygon with n sides.

Figure 12.5.13:

For instance, let $z = 8(\cos(\pi/4) + i \sin(\pi/4))$. Then the three cube roots of z all have magnitude $8^{1/3} = 2$. Their arguments are

$$\frac{\pi/4}{3} = \frac{\pi}{12}, \qquad \frac{\pi/4 + 2\pi}{3} = \frac{\pi}{12} + \frac{2\pi}{3}, \qquad \frac{\pi/4 + 4\pi}{3} = \frac{\pi}{12} + \frac{4\pi}{3}.$$

These three roots are shown in Figure 12.5.13, along with z.

Summary

The real numbers, with which we all grew up, are just a small part of the complex numbers, which fill up the xy plane. We add complex numbers by a "parallelogram law." To multiply them "we multiply their magnitudes and add their angles." Using the complex numbers we can see that "negative real time negative real is positive," since $180^\circ + 180^\circ = 360^\circ$, which describes the positive x-axis. We also saw how to raise a complex number to a power and how to take its roots. We can now view points in the xy plane as "numbers." However, mathematicians have shown that we cannot treat points in space as "numbers" that satisfy the usual rules of addition and multiplication.

EXERCISES for Section 12.5 *Key:* R–routine, M–moderate, C–challenging

In Execises 1 to 4 compute the given quantities:

1.[R]

(a) $(2+3i)+(5-2i)$

(b) $(2+3i)(2-3i)$

(c) $\frac{1}{2-i}$

(d) $\frac{3+2i}{4-i}$

2.[R]

(a) $(2+3i)^2$

(b) $\frac{4}{3-i}$

(c) $(1+i)(3-i)$

(d) $\frac{1+5i}{2-3i}$

3.[R]

(a) $(1+3i)^2$

(b) $(1+i)(1-i)$

(c) i^{-3}

(d) $\frac{4+\sqrt{2}i}{2+i}$

4.[R]

(a) $(1+i)^3$

(b) $\frac{i}{1-i}$

(c) $(3+i)^{-1}$

(d) $(5+2i)(5-2i)$

In Exercises 5 to 8 express the number in polar form $r(\cos(\theta) + i\sin(\theta))$ with θ is $[0, 2\pi]$.

5.[R] $\sqrt{3} + i$

6.[R] $\sqrt{3} - i$

7.[R] $\sqrt{2} + \sqrt{2}i$

8.[R] $-4 + 4i$

In Exercises 9 to 12 express the number in polar and rectangular form.

9.[R] $(-1 + i)^{10}$

10.[R] $(\sqrt{3} + i)^4$

11.[R] $(2 + 2i)^8$

12.[R] $1 - \sqrt{3}i)^7$

13.[R] Rationalize the denominator in each fraction. That is, express the fraction as an equivalent fraction whose denominator is an integer.

(a) $\frac{1}{1+\sqrt{2}}$

(b) $\frac{1}{2-i}$

(c) $\frac{2-\sqrt{3}}{\sqrt{3}+2}$

(d) $\frac{3+2i}{i-3}$

14.[R] For each equation, (i) find all solutions to the equation, (ii) plot all solutions in the complex plane, and (iii) check that the solution satisfies the equation.

(a) $x^2 + x + 1 = 0$

(b) $x^2 - 3x + 5 = 0$

(c) $2x^2 + x + 1 = 0$

(d) $3x^2 + 4x + 5 = 0$

15.[R] Let z_1 have magnitude 2 and argument $\pi/6$, and let z_2 have magnitude 3 and argument $\pi/3$.

(a) Plot z_1 and z_2.

(b) Find $z_1 z_2$ using the polar form.

(c) Write z_1 and z_2 in the rectangular form $x + yi$.

(d) With the aid of (c) compute $z_1 z_2$.

16.[R] Let z_1 have magnitude 2 and argument $\pi/4$, and let z_2 have magnitude 3 and argument $3\pi/4$.

(a) Plot z_1 and z_2.

(b) Find $z_1 z_2$ using the polar form.

(c) Write z_1 and z_2 in the form $x + yi$.

(d) With the aid of (c) compute $z_1 z_2$.

17.[R] The complex number z has argument $\pi/3$ and magnitude 1. Find and plot (a) z^2, (b) z^3, and (c) z^4.

18.[R] Find (a) i^3, (b) i^4, (c) i^5, and (d) i^{73}.

19.[R] If z has magnitude 2 and argument $\pi/6$, what are the magnitude and argument of (a) z^2, z^3, z^4, and z^n. (b) Sketch z, z^2, z^3, and z^4.

20.[R] Let z have magnitude 0.9 and argument $\pi/4$.

(a) Find and plot z^2, z^3, z^4, z^5, and z^6.

(b) What happens to z^n as $n \to \infty$?

21.[R] Find and plot all solutions of the equation $z^5 = 32\,(\cos(\pi/4) + i\sin(\pi/4))$.

22.[R] Find and plot all solutions of the equation $z^4 = 8 + 8\sqrt{3}i$. HINT: First draw $8 + 8\sqrt{3}i$.

23.[R] Let z have magnitude r and argument θ. Let w have magnitude $1/r$ and argument $-\theta$. Show that $zw = 1$. NOTE: w is called the **reciprocal of** z, and denoted z^{-1} or $1/z$.

24.[R] Find z^{-1} if $z = 4 + 4i$. NOTE: See Exercise 23.

25.[R]

(a) By substitution, verify that $2+3i$ is a solution of the equation $x^2-4x+13=0$.

(b) Use the quadratic formula to find all solutions of the equation $x^2-4x+13=0$.

26.[R]

(a) Use the quadratic formula to find all solutions of the equation $x^2+x+1=0$.

(b) Plot the solutions in (a).

(c) Check that the solutions in (a) satisfy $x^2+x+1=0$.

27.[R] Write in polar form

(a) $5+5i$,

(b) $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$,

(c) $-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$,

(d) $3+4i$.

28.[R] Write in rectangular form as simply as possible:

(a) $3\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)$,

(b) $2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$,

(c) $10\left(\cos\left(\pi\right)+i\sin\left(\pi\right)\right)$,

(d) $\frac{1}{5}\left(\cos\left(22^\circ\right)+i\sin\left(22^\circ\right)\right)$ Hint: Express the answer to at least three decimal places.

29.[R] Let z_1 have magnitude r_1 and argument θ_1, and let z_2 have magnitude r_2 and argument θ_2.

(a) Explain why the magnitude of z_1/z_2 is r_1/r_2.

(b) Explain why the argument of z_1/z_2 is $\theta_1-\theta_2$.

30.[R] Compute

$$\frac{\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)}$$

by two ways: (a) by the result of Exercise 29, (b) by rationalizing the denominator.

31.[R] Compute

(a) $(2+3i)(1+i)$

(b) $\frac{2+3i}{1+i}$

(c) $(7-3i)(\overline{7-3i})$

(d) $3(\cos(42°) + i\sin(42°)) \cdot 5(\cos(168°) + i\sin(168°))$

(e) $\frac{\sqrt{8}(\cos(147°)+i\sin(147°)}{\sqrt{2}(\cos(57°)+i\sin(57°))}$

(f) $1/(3-i)$

(g) $((\cos(52°) + i\sin(52°))^{-1}$

(h) $\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)^{12}$

32.[R] Compute

(a) $(4+3i)(4-3i)$

(b) $\frac{3+5i}{-2+i}$

(c) $\frac{1}{2+i}$

(d) $\left(\cos(\left(\frac{\pi}{12}\right) + i\sin(\left(\frac{\pi}{12}\right)\right)^{20}$

(e) $(r(\cos(\theta) + i\sin(\theta))^{-1}$

(f) $\mathrm{Re}\left((r(\cos(\theta) + i\sin(\theta)))^{10}\right)$

(g) $\frac{3\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)}{5-12i}$

33.[R] Find and plot all solutions of $z^3 = i$.

34.[R] Sketch all complex numbers z such that (a) $z^6 = 1$, (b) $z^6 = 64$, (c) $z^6 = -1$.

35.[R]

(a) Why is the symbol $\sqrt{-4}$ ambiguous?

(b) Draw all solutions of $z^2 = -4$.

36.[R] If z_k has argument θ_k and magnitude r_k, $k = 1, 2$, write each of the following in the form $r(\cos(\theta) + i\sin(\theta))$.

(a) z_1^2

(b) $1/z_1$

(c) $\bar{(}z_1)$

(d) $z_1 z_2$

(e) z_1/z_2

(f) $1/\overline{z_1}$

37.[R] Draw the six sixth roots of

(a) 1

(b) 64

(c) i

(d) -1

(e) $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$

38.[M] Using the fact that

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

find formulas for $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

39.[M]

(a) If $|z_1| = 1$ and $|z_2| = 1$, how large can $|z_1 + z_2|$ be? HINT: Draw some pictures.

(b) If $|z_1| = 1$ and $|z_2| = 1$, what can be said about $|z_1 z_2|$?

40.[M] Show that (a) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, (b) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

41.[M] If $\arg(z)$ is θ, what is an argument of (a) $\overline{z}$, (b) $1/z$.

42.[M] Let $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.

(a) Compute z^2 algebraically.

(b) Compute z^2 by putting z into polar form.

(c) Sketch the numbers z, z^2, z^3, z^4, and z^5.

43.[M] Let a, b, and c be complex numbers such that $a \neq 0$ and $b^2 - 4ac \neq 0$. Show that $ax^2 + bx + c = 0$ has two distinct roots.

44.[M] Find and plot the roots of $x^2 + ix + 3 - i = 0$.

45.[M] Compute the roots of the following equation and plot them relative to the same axes:

(a) $x^2 - 3x + 2 = 0$

(b) $x^2 - 3x + 2.25 = 0$

(c) $x^2 - 3x + 2.5 = 0$

(d) $x^2 - 3x + 1.5 = 0$

46.[M] The complex number $z = t + i$ (t a real number) lies on the line $y = 1$.

(a) Plot z^2 for, at least, $t = 0$, 1, and -1.

(b) Find the equation of the curve on which z^2 lies.

47.[M] For $x > 0$ the complex number $z = x + i/x$ lies on the curve $y = 1/x$. On what curve does z^2 lie?

(a) Plot z^2 for, at least, $x = 1$, 2, and 3.

(b) Determine the curve on which z^2 lies.

48.[M] The complex number $z = t + i$ (t a real number) lies on the line $y = 1$.

(a) Plot z^2 for, at least, $x = 0$, 1, and -1.

(b) Determine the curve on which z^2 lies.

49.[M] The complex number $z = 1 + ti$ (t a real number) lies on the line $x = 1$.

(a) Plot the points $1/z$ for $t = 0$, 1, -1, and 2.

(b) Determine the curve on which $1/z$ lies.

50.[M]

(a) Draw the curve on which $z = t + 2ti$ lies.

(b) Draw the curve on which z^2 lies.

51.[M] If z lies on the line $x + y = 1$, where does $1/z$ lie? HINT: Plot $1/z$ for a sample of points z on the original line.

52.[C] For which complex numbers z is $\overline{z} = 1/z$?

53.[C] Let $z = \frac{1}{2} + \frac{i}{2}$.

(a) Sketch the numbers z^n for $n = 1$, 2, 3, 4, and 5.

(b) What happens to z^n as $n \to \infty$?

54.[C] Let $z = 1 + i$.

(a) Sketch the numbers $z^n/n!$ for $n = 1$, 2, 3, 4, and 5.

(b) What happens to $z^n/n!$ as $n \to \infty$?

55.[C]

(a) Graph $r = \cos(\theta)$ in polar coordinates.

(b) Pick five points on the curve in (a). Viewing each as a complex number z, plot z^2.

(c) As z runs through the curve in (a), what curve does z^2 sweep out? HINT: Give its polar equation.

56.[C] The partial-fraction representation of a rational function is much simpler when we have complex numbers. No second-degree polynomial ax^2+bx+c is needed. This exercise indicates why this is the case.
Let z_1 and z_2 be the roots of $ax^2 + bx + c = 0$, $a \neq 0$.

(a) Using the quadratic formula (or by other means), show that $z_1 + z_2 = -b/a$ and $z_1 z_2 = c/a$.

(b) From (a) deduce that

$$ax^2 + bx + c = a(x - z_1)(x - z_2).$$

(c) With the aid of (b) show that

$$\frac{1}{ax^2 + bx + c} = \frac{1}{a(z_1 - z_2)} \left(\frac{1}{x - z_1} - \frac{1}{x - z_2} \right).$$

Part (c) shows that the theory of partial fractions, described in Section 8.4, becomes much simpler when complex numbers are allowed as the coefficients of the polynomials. Only partial fractions of the form $k/(ax + b)^n$ are needed.

57.[C] Let $f(x) = a_0 + a_1 x + q a_2 x^2 + a_3 x^3 + a_4 x^4$, where each coefficient is real.

(a) Show that if c is a root of $f(x) = 0$, then so is $\overline{c}$.

(b) Show that if c is a root of f and is not real, then $(x - c)(x - \overline{c})$ divides $f(x)$.

(c) Using the fundamental theorem of algebra, show that any fourth-degree polynomial with real coefficients can be expressed as the product of polynomials of degree at most 2 with real coefficients.

Exercise 58 is related to Exercise 90 on page 781. (See also Exercises 5 and 6 at the end of this chapter.)
58.[C] Let a point **0** be a distance $a \neq 1$ from the center of a unit circle.

(a) Show that the average value of the (natural) logarithm of the distance from **0** to points on the circumference is

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\ln(1+a^2-2a\cos(\theta))\ d\theta.$$

(b) Spend at least three minutes, but at most 5 minutes, trying to evaluate the integral in (a).

12.6 The Relation Between the Exponential and the Trigonometric Functions

With the aid of complex numbers, in 1743, Leonard Euler discovered that the trigonometric functions can be expressed in terms of the exponential function e^z, where z is complex. This section retraces his discovery. In particular, it will be shown that

Finding $\sin(x)$ and $\cos(x)$ in e^x.

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \text{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Complex Series

In order to relate the exponential function to the trigonometric functions, we will use infinite series such as $\sum_{k=0}^{\infty} z_k$, where the z_k's are complex numbers. Such a series is said to converge to S if its k^{th} partial sum S_k approaches S in the sense that $|S - S_k| \to 0$ as $k \to \infty$. It is shown in Exercise 35 that if $\sum_{k=0}^{\infty} |z_k|$ (a series with real-valued terms) converges, so does $\sum_{k=0}^{\infty} z_k$, and the series $\sum_{k=0}^{\infty} z_k$ is said to *converge absolutely*. If a series converges absolutely, we may rearrange the terms in any order without changing the sum.

Here, $|\cdot|$ refers to the magnitude.

Let $z_k = x_k + iy_k$, where x_k and y_k are real. If $\sum_{k=0}^{\infty} z_k$ converges, so do $\sum_{k=0}^{\infty} x_k$ and $\sum_{k=0}^{\infty} y_k$. If $\sum_{k=0}^{\infty} z_k = S = a + bi$, then $\sum_{k=0}^{\infty} x_k = a$ and $\sum_{k=0}^{\infty} y_k = b$. $\sum_{k=0}^{\infty} x_k$ is called the *real part* of $\sum_{k=0}^{\infty} z_k$ and $\sum_{k=0}^{\infty} y_k$ is the *imaginary part* of $\sum_{k=0}^{\infty} z_k$.

$\mathrm{Re}\left(\sum_{k=0}^{\infty} z_k\right) = \sum_{k=0}^{\infty} x_k$
$\mathrm{Im}\left(\sum_{k=0}^{\infty} z_k\right) = \sum_{k=0}^{\infty} y_k$

EXAMPLE 1 Determine for which complex numbers z, $\sum_{k=0}^{\infty} z^k/k!$ converges.

SOLUTION We will examine absolute convergence, that is, the convergence of $\sum_{k=0}^{\infty} |z^k|/k!$. This series has real terms. In fact, it is the Maclaurin series for $e^{|z|}$, which converges for all real numbers. Since $\sum_{k=0}^{\infty} z^n/n!$ converges absolutely for all z, it converges for all z. ◇

$|z|$ is a real number

Defining e^z

The Maclaurin series for e^x when x is real suggests the following definition:

In some treatments of exponentials e^z is defined as a power series and e is defined as the value of the power series when $z = 1$.

> **DEFINITION** *(e^z for complex z.)* Let z be a complex number. Define e^z to be the sum of the convergent series $\sum_{k=0}^{\infty} z^k/k!$.

It can be shown by multiplying the series for e^{z_1} and e^{z_2} that $e^{z_1+z_2} = e^{z_1}e^{z_2}$ in accordance with the basic law of exponents. When the expression for z is complicated, we sometimes write e^z as $\exp(z)$. For example, in exp notation the law of exponents becomes $\exp(z_1 + z_2) = (\exp(z_1))(\exp(z_2))$.

Euler's Formula: The Link between $e^{i\theta}$, $\cos(\theta)$, and $\sin(\theta)$

Euler's Formula

Figure 12.6.1: This is the license plate of the mathematician Martin Davis, whose e-mail signature is "eipye, add one, get zero."

The following theorem of Euler provides the key link between the exponential function e^z and the trigonometric functions $\cos(\theta)$ and $\sin(\theta)$.

Theorem 12.6.1. *Euler's Formula Let θ be a real number. Then*

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta).}$$

Proof

Recall that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, ...

By definition of e^z for any complex number,

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots. \\
&= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots. \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots. \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right) \quad \text{(rearranging)} \\
&= \cos(\theta) + i\sin(\theta).
\end{aligned}$$

Figure 12.6.2 shows $e^{i\theta}$, which lies on the standard unit circle. •

$e^{i\pi} = -1$

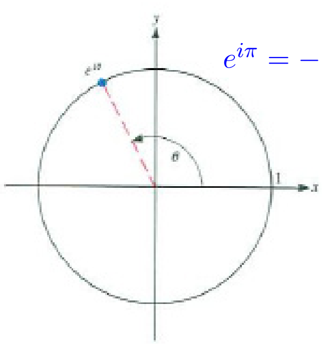


Figure 12.6.2:

There is an old saying: "God created the complex numbers; anything less is the work of man."

Theorem 12.6.1 asserts, for instance, that

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i \cdot 0 = -1.$$

The equation $e^{i\pi} = -1$ is remarkable in that it links e (the fundamental number in calculus), π (the fundamental number in trigonometry), i (the fundamental complex number), and the negative number -1. The history of that short equation would recall the struggles of hundreds of mathematicians to create the number system that we now take for granted. It is as important in mathematics as $F = ma$ or $E = mc^2$ in physics.

With the aid of Theorem 12.6.1, both $\cos(\theta)$ and $\sin(\theta)$ may be expressed in terms of the exponential function.

Theorem 12.6.2. *Let θ be a real number. Then*

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \textit{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Proof

We begin with Euler's formula (Theorem 12.6.1),

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{12.6.1}$$

Replacing θ by $-\theta$ in (12.6.1), we obtain

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta). \tag{12.6.2}$$

The sum of (12.6.1) and (12.6.2) yields

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta),$$

hence

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Subtraction of (12.6.2) from (12.6.1) yields

$$e^{i\theta} - e^{-i\theta} = 2i\sin(\theta),$$

hence

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

This establishes the two results in this theorem. •

sinh and cosh were defined in Section 4.1, see Exercises 49 to 52 on page 301.

The hyperbolic functions $\cosh(x)$ and $\sinh(x)$ were defined in terms of the exponential function by

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Theorem 12.6.2 shows the trigonometric functions could be similarly defined in terms of the exponential function — if complex numbers were available.

Just as Maxwell discovered the connection between light and electricity, Euler discovered the connection between the exponential and trigonometric functions.

Indeed, from the complex numbers and e^z we could even obtain the derivative formulas for $\sin(\theta)$ and $\cos(\theta)$. For instance,

$$\frac{d}{d\theta}\sin(\theta) = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)' = \frac{ie^{i\theta} + ie^{-i\theta}}{2i} = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos(\theta).$$

(That the familiar rules for differentiation extend to complex-valued functions is justified in a course in complex variables.)

Sketching e^z

Magnitude and argument of e^{x+iy}

If $z = x + iy$, the evaluation of e^z can be carried out as follows:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

The magnitude of e^{x+iy} is e^x and the argument of e^{x+iy} is y.

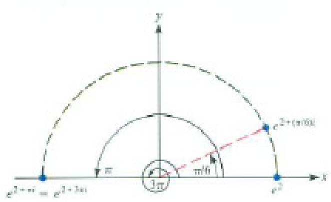


Figure 12.6.3:

EXAMPLE 2 Compute and sketch (a) $e^{2+(\pi/6)i}$, (b) $e^{2+\pi i}$, and (c) $e^{2+3\pi i}$.
SOLUTION (a) $e^{2+(\pi/6)i}$ has magnitude e^2 and argument $\pi/6$. (b) $e^{2+\pi i}$ has magnitude e^2 and argument π; it equals $-e^2$. (c) $e^{2+3\pi i}$ has magnitude e^2 and argument 3π, so is the same number as the number in (b). The results are sketched in Figure 12.6.3. ◇

The next example illustrates a typical computation in alternating currents. Electrical engineers frequently use j as the symbol for i (so they can use i to represent current).

EXAMPLE 3 Find the real part of $100e^{j(\pi/6)}e^{j\omega t}$. Here t refers to time, ω is a real constant related to frequency, and j is the mathematician's i.
SOLUTION

$$\begin{aligned}100e^{j(\pi/6)}e^{j\omega t} &= 100e^{j(\pi/6)+j\omega t}\\ &= 100e^{j(\pi/6+\omega t)}\\ &= 100\left(\cos\left(\frac{\pi}{6}+\omega t\right)+i\sin\left(\frac{\pi}{6}+\omega t\right)\right).\end{aligned}$$

Thus

$$\mathrm{Re}\left(100e^{j(\pi/6)}e^{j\omega t}\right) = 100\cos\left(\frac{\pi}{6}+\omega t\right).$$

◇

It is often convenient to think of $\cos(\theta)$ as $\mathrm{Re}\left(e^{i\theta}\right)$. The next example exploits this point of view.

EXAMPLE 4 Evaluate $\sum_{k=0}^{\infty}\frac{\cos(k\theta)}{k!}$.
SOLUTION Recall that $e^{ik\theta} = \cos(k\theta)+i\sin(k\theta)$. Hence $\cos(k\theta) = \mathrm{Re}\left(e^{ik\theta}\right)$,

and we have

Recall the definition of e^z.

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\cos(k\theta)}{k!} &= \sum_{k=0}^{\infty} \frac{\mathrm{Re}\left(e^{ik\theta}\right)}{k!} = \mathrm{Re}\left(\sum_{k=0}^{\infty} \frac{e^{ik\theta}}{k!}\right) \\
&= \mathrm{Re}\left(\sum_{k=0}^{\infty} \frac{\left(e^{i\theta}\right)^k}{k!}\right) = \mathrm{Re}\left(\sum_{k=0}^{\infty} \frac{\left(\exp(i\theta)\right)^k}{k!}\right) \\
&= \mathrm{Re}\left(e^{\exp(i\theta)}\right) = \mathrm{Re}\left(e^{\cos(\theta)+i\sin(\theta)}\right) \\
&= \mathrm{Re}\left(e^{\cos(\theta)}e^{i\sin(\theta)}\right) = e^{\cos(\theta)}\,\mathrm{Re}\left(e^{i\sin(\theta)}\right) \\
&= e^{\cos(\theta)}\,\mathrm{Re}\left(\cos(\sin(\theta)) + i\sin(\sin(\theta))\right) = e^{\cos(\theta)}\cos(\sin(\theta)).
\end{aligned}$$

Hence

$$\sum_{k=0}^{\infty} \frac{\cos(k\theta)}{k!} = e^{\cos(\theta)}\cos(\sin(\theta)).$$

◇

Summary

Using power series, we obtained the fundamental relation $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and showed that $\cos(\theta)$ and $\sin(\theta)$ can be expressed in terms of the exponential function. Since $\ln(x)$ is the inverse of e^x, it too is obtained from the exponential function. We may define even x^n, $x > 0$, in terms of the exponential function as $e^{n\ln(x)}$. Similarly, a^x, $a > 0$, can be defined as $e^{x\ln(a)}$. These observations suggest that the most fundamental function in calculus is e^x.

EXERCISES for Section 12.6 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 6 sketch the numbers given and state their real and imaginary parts.

1.[R] $e^{5\pi i/4}$

2.[R] $5e^{\pi i/4}$

3.[R] $2e^{\pi i/4} + 3e^{\pi i/6}$

4.[R] e^{2+3i}

5.[R] $e^{\pi i/6}e^{3\pi i/4}$

6.[R] $2e^{\pi i} \cdot 3e^{-\pi i/3}$

In Exercises 7 to 10 express the given numbers in the form $re^{i\theta}$ for a positive real number r and argument θ, where $-\pi < \theta \leq \pi$.

7.[R] $\frac{e^2}{\sqrt{2}} - \frac{e^2}{\sqrt{2}}i$

8.[R] $3\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

9.[R] $5\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) \cdot 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

10.[R] $7\left(\cos\left(\frac{7\pi}{3}\right) + i\sin\left(\frac{7\pi}{3}\right)\right)$

In Exercises 11 to 14 plot $\exp(z)$ for the given values of z:

11.[R] $z = 2$

12.[R] $\pi i/2$

13.[R] $2 - \pi i/3$

14.[R] $-1 + 17\pi i/6$

In Exercises 15 to 18 plot the given complex numbers:

15.[R] $\exp(\pi i/4 + 3\pi i)$

16.[R] $\exp(1 + 9\pi i/4$

17.[R] $\exp(2 - \pi i/3)$

18.[R] $\exp(-1 + 17\pi i/6)$

19.[R] Let $z = e^{a+bi}$. Find (a) $|z|$, (b) $\bar{z}$, (c) z^{-1}, (d) $\mathrm{Re}(z)$, (e) $\mathrm{Im}(z)$, and (f) $\arg(z)$. Note: In (f), assume a and b are positive.

20.[R] How far is $\exp(x + iy)$ from the origin?

21.[R] How far is $\exp(x + iy)$ from the x-axis? From the y-axis?

22.[R] For which values of a and b is $\lim_{n\to\infty}\left(e^{a+ib}\right)^n = 0$?

23.[R] Find all complex numbers z such that $e^z = 1$.

24.[R] Find all complex numbers z such that $e^z = -1$.

25.[R]

(a) Find $\left|e^{3+4i}\right|$.

(b) Plot the complex number e^{3+4i}.

26.[R]

(a) Plot all complex numbers of the form e^{x+4i}, x real.

(b) Plot all complex numbers of the form e^{3+yi}, y real.

27.[M] If z lies on the line $y = 1$, where does $\exp(z)$ lie?

28.[M] If z lies on the line $x = 1$, where does $\exp(z)$ lie?

29.[M] In Claude Garrod's *Twentieth Century Physics*, Faculty Publishing, Davis, Calif., p. 107, there is the remark: "Using the fact that

$$\left(e^{-i\omega_0 t}\right)^* \left(e^{-i\omega_0 t}\right) = 1,$$

we can easily evaluate the probability density for these standard waves." Justify this equation. NOTE: In this text, z^* denotes the conjugate of z and ω_0 is real.

30.[M] Use the fact that $1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos((n-1)\theta)$ is the real part of $1 + e^{\theta i} + e^{2\theta i} + \cdots + e^{(n-1)\theta i}$ to find a short formula for that trigonometric sum.

31.[M] Find all z such that $e^z = 3 + 4i$.

32.[M] Assuming that $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for complex numbers z_1 and z_2, obtain the trigonometric identities for $\cos(A + B)$ and $\sin(A + B)$.

33.[M] Evaluate

$$\sum_{k=0}^{\infty} \frac{\cos(n\theta)}{2^n}.$$

NOTE: First, show that the series converges (absolutely).

34.[M] Evaluate
$$\sum_{k=0}^{\infty} \frac{\sin(n\theta)}{n!}.$$

NOTE: First, show that the series converges (absolutely).

35.[M] This Exercise shows that if $\sum_{k=0}^{\infty} |z_k|$ converges, so does $\sum_{k=0}^{\infty} z_k$.

(a) Let $z_k = x_k + iy_k$. Show that $\sum_{k=0}^{\infty} |x_k|$ and $\sum_{k=0}^{\infty} |x_k|$ both converge. HINT: $|a| \leq \sqrt{a^2 + b^2}$

(b) Show that $\sum_{k=0}^{\infty} x_k$ and $\sum_{k=0}^{\infty} y_k$ both converge.

(c) Show that $\sum_{k=0}^{\infty}(x_k + iy_k)$ converges.

36.[C] For which z is

(a) $e^z = e^{-z}$,

(b) $e^{iz} = e^{-ix}$¡

(c) $\sin(z) = 0$.

Exercises 37 and 38 treat the complex logarithms of a complex number. They show that $z = \ln(w)$ is not single-valued.

37.[C] Let w be a nonzero complex number. Show that there are an infinite number of complex numbers z such that $e^z = w$. HINT: Use Euler's formula.

38.[C] (See Exercise 37.) When $e^z = w$, we write $z = \ln(w)$ although $\ln(w)$ is not a uniquely defined number. If b is a nonzero complex number and q is a complex number, define b^q to be $e^{q\ln(b)}$. Since $\ln(b)$ is not unique, b^q is usually not unique. List all possible values of (a) $(-1)^i$, (b) $10^{1/2}$, (c) 10^3,

39.[M] Let $f(z)$ be a polynomial with real coefficients.

(a) Show that if $f(a) = 0$, then $f(\overline{a}) = 0$. (This shows that roots of f occur in conjugate pairs.)

(b) Show that $\overline{e^z} = e^{\overline{z}}$.

(c) Show that $\overline{\sin(z)} = \sin(\overline{z})$.

40.[M] When z is real, $|\sin(z)| \leq 1$ and $|\cos(z)| \leq 1$. Do these inequalities hold for all complex z?

41.[M] Does the equation $\cos^2(z) + \sin^2(z) = 1$ hold for complex z?

42.[M] Let

$$z = \frac{1+i}{\sqrt{2}}.$$

(a) Plot z, $z^2/2!$, $z^3/3!$, and $z^4/4!$.

(b) Plot $1 + z + z^2/2! + z^3/3! + z^4/4!$, which is an estimate for $\exp\left((1+i)/\sqrt{2}\right)$.

(c) Plot $\exp\left((1+i)/\sqrt{2}\right)$ on the xy plane.

43.[M] An integral table lists $\int xe^{ax}\,dx = e^{ax}\left(ax - 1\right)/a^2$. At first glance, finding the integral of $xe^{ax}\cos(bx)$ may appear to be a much harder problem. However, by noticing that $\cos(bx) = \operatorname{Re}\left(e^{ibx}\right)$, we can reduce it to a simpler problem. Following this approach, find $\int xe^{ax}\cos(bx)\,dx$. HINT: The formula for $\int xe^{ax}\,dx$ holds when a is complex.

44.[M] In Section 4.1 we define $\cosh(x) = \left(e^x + e^{-x}\right)/2$ and $\sinh(x) = \left(e^x - e^{-x}\right)/2$. We can use the same definitions when x is complex. In view of Theorem 12.6.2, let us define sine and cosine for complex z by $\sin(z) = \left(e^{iz} - e^{-iz}\right)/(2i)$ and $\cos(z) = \left(e^{iz} + e^{-iz}\right)/2$. Establish the following links between the hyperbolic and trigonometric functions:

(a) $\cosh(z) = \cos(iz)$

(b) $\sinh(z) = -i\sin(iz)$

45.[M] Show that

(a) $\sin(z) = i\sinh(iz)$.

(b) $\cos(z) = \cosh(iz)$.

(c) $\cosh(z)^2 - \sinh(z)^2 = 1$

46.[C] Let z be a complex number and θ a real number. What is the relationship between z and $e^{i\theta}z$? Experiment, conjecture, and explain.

47.[C]

48.[C]

49.[M] Sam is at it again: "I don't need power series to define e^z. I just write z as $x+iy$ and define e^{x+iy} to be $e^x(\cos(y)+i\sin(y))$. That's all there is to it. If I call this function $E(z)$, then it's easy to check that $E(z_1+z_2)=E(z_1)E(z_2)$. Moreover, if z is real, then $y=0$ and $E(z)=e^x$, agreeing with our familiar $\exp(x)$."

(a) Is Sam right?

(b) Does his $E(z)$ obey the basic law of exponents, as he claims?

(c) Jane asks him, "But where did you get the idea for that definition? It seems to float in out of thin air." What is Sam's answer?

12.7 Fourier Series

In Section 5.4 we used sums of terms of the form ax^n, where n is a non-negative integer and a is a number, to represent a function. This required a function to have derivatives of all orders. Now, instead, we will use sums of terms of the form $a\cos(kx)$ and $b\sin(kx)$, where a, b, and k are numbers. This method applies to a much broader class of functions, even, for instance, the absolute value function, $f(x) = |x|$, which is not differentiable at 0, and some functions that are not even continuous. The technique, called **Fourier Series**, is used in such varied fields as heat conduction, electric circuits, the theory of sound and mechanical vibrations.

To listen to several tuning forks, go to `http://www.onlinetuningfork.com/`.

At first glance, the use of sine and cosine, which are periodic functions, may seem a surprising choice. However, if you think in terms of sound, it is quite plausible. Every tuning fork produces a pure pitch at a specific frequency. With a collection of such devices, each at a different pitch, struck simultaneously, you can approximate the sound made by a band or an orchestra. Each tuning fork corresponds to $\sin(kt)$ or $\cos(kt)$, where t is time. The one set at concert A vibrates at the rate of 440 cycles per second, that is, 440 Hertz (440 Hz). In this case the acoustic wave is expressed as $\sin(400\,(2\pi t))$, for, as t increases by $1/400$ second, the argument $400\,(2\pi t)$ increases by 2π, enabling the function to complete one cycle.

Periodic Functions

The function $\cos(x)$ (and $\sin(x)$) has period 2π, that is, $\cos(x + 2\pi) = \cos(x)$. Changing the input by 2π does not change the output. It follows that $\cos(x - 2\pi) = \cos(x)$, $\cos(x + 4\pi) = \cos(x)$, and, more generally, for any integer n, $\cos(x)$ has $n(2\pi)$ as a period. When we say "$\cos(x)$ has period 2π" we are emphasizing the smallest period. Its other periods are all integer multiples of that period.

EXAMPLE 1 Find the period of (a) $\cos(3\pi x)$, (b) $\cos(k\pi x/L)$, where k is a positive integer and L is a positive number.

SOLUTION In each case we ask, "How much must x change in order for the argument (the input) to change by 2π?"

(a) For $3\pi x$ to change by 2π, we solve the equation $3\pi x = 2\pi$, obtaining $x = 2/3$. Thus $\cos(3\pi x)$ has period $2/3$.

(b) For $\cos(k\pi x/L)$ the reasoning used in (a) leads us to conclude the period is $2L/k$.

Note that in (b) the larger L is, the longer the period, but the larger k is, the shorter the period. For each k, $2L$ is among its periods. ⋄

Fourier Series for Functions with Period 2π

We first treat the familiar case of functions that have period 2π. Then we consider the general case, where the period is $2L$, for any positive number L.

Let $f(x)$ have period 2π. Its values are determined by its values on the interval $[-\pi, \pi)$. We choose this interval rather than $[0, 2\pi)$ to simplify some computations that we will encounter momentarily.

Let $f(x)$ be a function of period 2π. The Fourier Series associated with this function is

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx)\right) \tag{12.7.1}$$

The formulas for a_k and b_k are known as "**Euler's formulas**." Euler published them in 1777, but Fourier was unaware of them.

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx \qquad k = 0, 1, 2, \ldots \tag{12.7.2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx \qquad k = 1, 2, \ldots. \tag{12.7.3}$$

(This assumes the integrals in (12.7.2) and (12.7.3) exist.)

After we compute two Fourier series, we will show why the coefficients are given by the integrals in (12.7.2) and (12.7.3).

Constant term is $a_0/2$

The numbers a_k and b_k are called the **Fourier coefficients** for $f(x)$. The formula for a_0 reduces to $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$. This means that the constant term $a_0/2$ is the average value of the function $f(x)$ over one period. Note that the formula for a_k in (12.7.2) also holds for $k = 0$ because the constant term in (12.7.1) is $a_0/2$. (The 2 was included so (12.7.2) would hold when $k = 0$.)

EXAMPLE 2 Find the Fourier series associated with the function defined by

$$f(x) = \begin{cases} -1 & -\pi < x \le 0 \\ 1 & 0 < x \le \pi. \end{cases}$$

Because $f(x)$ is (almost) an odd function, we expect only sines to appear in its Fourier series.

To make $f(x)$ have period 2π, just repeat the graph on every interval of the form $(-\pi + 2n\pi, \pi + 2n\pi]$. The graph of $f(x)$ is shown in Figure 12.7.1(a) and the extension of $f(x)$ is shown in Figure 12.7.1(b).

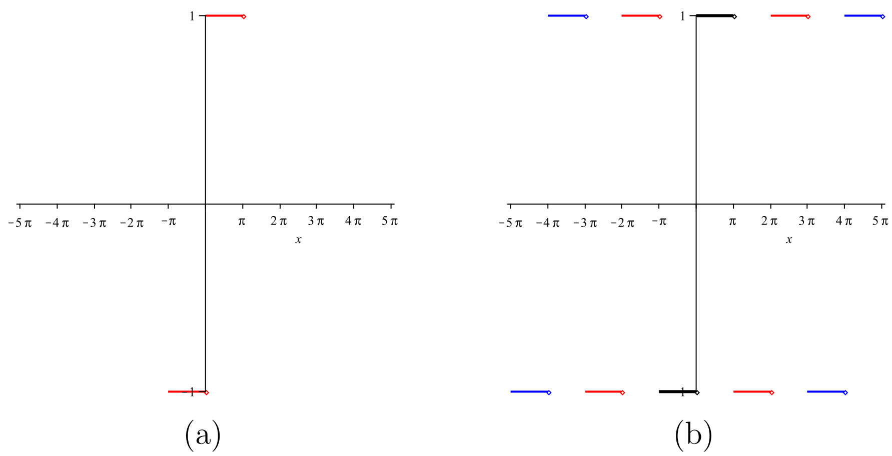


Figure 12.7.1:

SOLUTION

$$\begin{aligned}
a_0 &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx\\
&= \tfrac{1}{\pi}\int_{-\pi}^{0} f(x)\ dx + \tfrac{1}{\pi}\int_{0}^{\pi} f(x)\ dx\\
&= \tfrac{1}{\pi}\int_{-\pi}^{0} -1\ dx + \tfrac{1}{\pi}\int_{0}^{\pi} 1\ dx\\
&= \tfrac{1}{\pi}(-\pi) + \tfrac{1}{\pi}(\pi) = 0.
\end{aligned}$$

Similarly, for $k \geq 1$,

$$\begin{aligned}
a_k &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx\\
&= \tfrac{1}{\pi}\int_{-\pi}^{0} f(x)\cos(kx)\ dx + \tfrac{1}{\pi}\int_{0}^{\pi} f(x)\cos(kx)\ dx\\
&= \tfrac{1}{\pi}\int_{-\pi}^{0} -\cos(kx)\ dx + \tfrac{1}{\pi}\int_{0}^{\pi} \cos(kx)\ dx\\
&= \tfrac{1}{\pi}\ \tfrac{-\sin(kx)}{k}\Big|_{-\pi}^{0} + \tfrac{1}{\pi}\ \tfrac{\sin(kx)}{k}\Big|_{0}^{\pi} = 0 + 0 = 0
\end{aligned}$$

and

$$\begin{aligned}
b_k &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx\\
&= \tfrac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(kx)\ dx + \tfrac{1}{\pi}\int_{0}^{\pi} f(x)\sin(kx)\ dx\\
&= \tfrac{1}{\pi}\int_{-\pi}^{0} -\sin(kx)\ dx + \tfrac{1}{\pi}\int_{0}^{\pi} \sin(kx)\ dx\\
&= \tfrac{1}{\pi}\ \tfrac{\cos(kx)}{k}\Big|_{-\pi}^{0} + \tfrac{1}{\pi}\ \tfrac{-\cos(kx)}{k}\Big|_{0}^{\pi} = \tfrac{1}{\pi}\left(\tfrac{1-\cos(-k\pi)}{k}\right) + \tfrac{1}{\pi}\left(\tfrac{-\cos(k\pi)+1}{k}\right)
\end{aligned}$$

Because $\cos(-k\pi) = \cos(k\pi)$, we have

$$b_k = \tfrac{1}{k\pi}\left((1-\cos(k\pi)) + (1-\cos(k\pi))\right) = \tfrac{2(1-\cos(k\pi))}{k\pi}.$$

When k is even, $1-\cos(k\pi) = 1-1 = 0$. And, when k is odd, $1-\cos(k\pi) = 1-(-1) = 2$. Thus

$$b_k = \begin{cases} 0 & \text{when } k \text{ is even} \\ \frac{4}{k\pi} & \text{when } k \text{ is odd.} \end{cases}$$

The Fourier Series (12.7.1) in this case has only terms involving $\sin(kx)$

with k odd. It is

$$\frac{4}{\pi}\sin(x) + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x) + \dots.$$

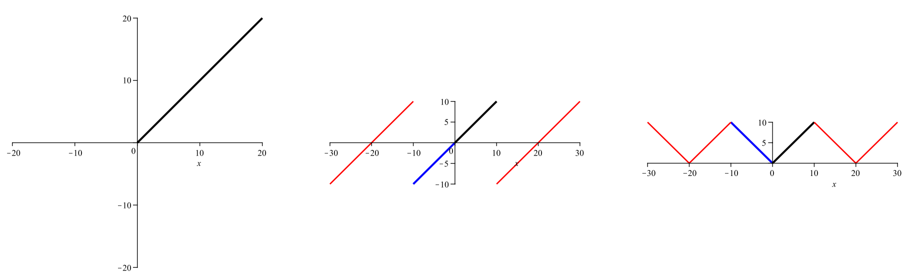
In particular, when $x = \pi/2$, $f(x) = 1$ and we have

$$\begin{aligned} 1 &= \tfrac{4}{\pi}\sin\left(\tfrac{(\pi}{2}\right) + \tfrac{4}{3\pi}\sin\left(\tfrac{3\pi}{2}\right) + \tfrac{4}{5\pi}\sin\left(\tfrac{5\pi}{2}\right) + \dots \\ 1 &= \tfrac{4}{\pi} - \tfrac{4}{3\pi} + \tfrac{4}{5\pi} - \dots \\ \text{Thus} \qquad \tfrac{\pi}{4} &= 1 - \tfrac{1}{3} + \tfrac{1}{5} - \dots. \end{aligned}$$

This result was obtained previously in Exercise 21 in Section 12.4 with the aid of the Maclaurin series for $\arctan(x)$. ◇

Fourier Series for Functions with Period $2L$

Sometimes we will want to develop the Fourier series for a function over an interval not of length 2π. For instance, we may want to obtain the Fourier series for $f(x) = x$ on the interval $[0, 10)$. Because the function is not periodic, the first step is to replace $f(x)$ with a function $g(x)$ that is periodic and coincides with $f(x)$ on $[0, 10)$. The graph of $y = f(x)$ on $[0, 10)$ is shown in Figure 12.7.2(a); two possible extensions of $f(x)$ are shown in Figure 12.7.2(b) and (c).



(a) (b) (c)

Figure 12.7.2:

The Fourier series for a function of period $2L$ has the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{k\pi x}{L}\right) + b_k\sin\left(\frac{k\pi x}{L}\right)\right) \qquad (12.7.4)$$

with coefficients given by

Note that the formula for a_k includes the case for a_0.

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx \qquad k = 0, 1, 2, \ldots \qquad (12.7.5)$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\,dx \qquad k = 1, 2, \ldots. \qquad (12.7.6)$$

EXAMPLE 3 Find the Fourier series of the triangular wave with period 20 shown in Figure 12.7.2(c).

SOLUTION Let $T(x)$ denote the triangular wave. In this case, $2L = 20$, so $L = 10$. To compute the Fourier series of $T(x)$ we need to know the definition of $T(x)$ on $[-L, L)$.

$T(x) = |x|$ for x in $[-10, 10)$

$$T(x) = \begin{cases} x & \text{for } 0 \le x \le 10 \\ -x & \text{for } -10 \le x < 0 \end{cases}.$$

Because $T(x)$ is an even function, $b_k = 0$ for $k = 1, 2, \ldots$. Then

If $T(x) = T(-x)$, then $\int_{-1}^{1} T(x)\,dx = 2\int_0^1 T(x)\,dx$.

$$a_0 = \tfrac{1}{10}\int_{-10}^{10} T(x)\,dx = \tfrac{2}{10}\int_0^{10} x\,dx = \tfrac{1}{10}\,x^2\Big|_0^{10} = 10.$$

The coefficients of the cosine terms are

$a_k = \tfrac{1}{10}\int_{-10}^{10} T(x)\cos(k\pi x/10)\,dx = \tfrac{2}{10}\int_0^{10} x\cos(k\pi x/10)\,dx$	because $x\cos(k\pi x/10)$ is even
$= \tfrac{2}{10}\left(\tfrac{x}{k\pi}\sin(k\pi x/10)\Big\vert_0^{10} - \tfrac{1}{k\pi}\int_0^{10}\sin(k\pi x/10)\,dx\right)$	int by parts: $u = x$, $dv = \cos(k\pi x/10)\,dx$
$= 2\left(0 + \tfrac{1}{(k\pi)^2}\cos(k\pi x/10)\Big\vert_0^1\right)$	$\sin(k\pi/10) = 0$ for all integers k
$= \frac{2(\cos(k\pi/10)-1)}{(k\pi/10)^2} = \frac{2((-1)^k-1)}{(k\pi/10)^2}$	

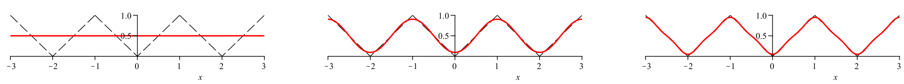
When k is an even integer, $a_k = 20((-1)^k - 1)/(k\pi)^2 = 0$. And, when k is an odd integer, $a_k = 20((-1)^k - 1)/(k\pi)^2 = -40/(k\pi)^2$.

The Fourier series for the triangular wave is

$$T(x) = 5 - \frac{40}{\pi^2}\left(\cos(\pi x/10) + \frac{1}{9}\cos(3\pi x/10) + \frac{1}{25}\cos(5\pi x/10) + \ldots\right).$$

◇

Figure 12.7.3 shows the partial Fourier sums for the triangular wave with 1, 2, and 5 terms. In an advanced calculus course it is proved that the partial sums converge to the wave for every real number.



(a) (b) (c)

Figure 12.7.3:

The Origins of the Formulas for a_k and b_k

We will derive the formulas for the Fourier coefficients in the special case when the period is 2π. Exercises 42 and 43 outline the similar argument for the general case when the period is $2L$.

These integrals were evaluated in Exercises 17 to 19.

The keys are the following three integrals:

$$\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} L & \text{if } m = k,\ k = 1, 2, \ldots \\ 0 & \text{if } m \neq k,\ k = 1, 2, \ldots \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} L & \text{if } m = k,\ k = 1, 2, \ldots \\ 0 & \text{if } m \neq k,\ k = 1, 2, \ldots \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx = 0 \quad \text{for any } m = 1, 2, \ldots \text{ and any } k = 1, 2, \ldots.$$

The third one is immediate, for the integrand, being the product of an odd function and an even function, is an odd function. The other two depend on trigonometric identities, and were developed in Exercises 17 to 19 inSection 8.5.

The formula for a_m, $m = 1, 2, \ldots$, is found by multiplying $f(x)$ by

$\cos(m\pi x)$ and integrating term-by-term over one period of length 2π:

We are assuming it's legal to switch the order, integrate term-by-term, then sum: "$\int_{-\pi}^{\pi}\sum_{k=1}^{\infty} = \sum_{k=1}^{\infty}\int_{-\pi}^{\pi}$".

$$\begin{aligned}
&\int\limits_{-\pi}^{\pi} f(x)\cos(mx)\ dx \\
&\quad = \int\limits_{-\pi}^{\pi}\left(\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(kx) + b_k\sin(kx)\right)\cos(mx)\right)\ dx \\
&\quad = \frac{a_0}{2}\int\limits_{-\pi}^{\pi}\cos(mx)\ dx \\
&\qquad + \sum_{k=1}^{\infty}\left(a_k\int\limits_{-\pi}^{\pi}\cos(kx)\cos(mx)\ dx + b_k\int\limits_{-\pi}^{\pi}\sin(kx)\cos(mx)\ dx\right).
\end{aligned}$$

Each integral in this last expression is zero — except the coefficient of a_m. This gives the equation

$$\begin{aligned}
\int\limits_{-\pi}^{\pi} f(x)\cos(mx) &= a_m\int\limits_{-\pi}^{\pi}\left(\cos(kx)\right)^2\ dx \\
&= a_m\pi.
\end{aligned}$$

Solving for a_m, we find that

$$a_m = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos(mx)\ dx.$$

The derivations of the formulas for a_0 and for b_k, $k = 1, 2, \ldots$ are similar. (See Exercises 42 and 43.)

Remarks on the Underlying Theory

Just as a Taylor series associated with a function may not represent the function, the Fourier series associated with a function may not represent it, even if the function is continuous. However, there are several theorems that assure us that for many functions met in applications the series does converge to the function. First, a couple of definitions.

Recall that the right-hand limit of $f(x)$ at a is defined as the limit of $f(x)$ as x approaches a through values larger than a, and is denoted $\lim_{x\to a^+} f(x)$. Similarly, the left-hand limit, denoted $\lim_{x\to a^-} f(x)$, is defined as the limit of $f(x)$ as x approaches a through values smaller than a. If both these limits exist at a and are different, we say that the function has a "jump discontinuity at a."

Theorem. *Let $f(x)$ have period $2L$. Assume that in the interval $[-L, L)$ (a) $f(x)$ is differentiable exept at a finite number of points, where there are jump discontinuities, and (b) at L the right-hand limit of $f(x)$ exists and at $-L$ the left-hand limit of $f(x)$ exists. Then, if the function is continuous at a, its associated Fourier series converges to $f(a)$. If $f(x)$ has a jump discontinuity at a, then the series converges to the average of the left- and right-hand limits at a. At the endpoints, L and $-L$, the Fourier series converges to the average of* $\lim_{x\to -L^+} f(x)$ *and* $\lim_{x\to L^-} f(x)$.

Note that there is no mention of the existence of any derivative, first-order, second-order, or any higher order derivatives.

The name Joseph Fourier (1768—1830) is attached to trigonometric series because he explored and applied them in his classic Analytic Theory of Heat, published in 1822. He came upon the formulas for the coefficients by an indirect route, starting with the Maclaurin series for $\sin(x)$ and $\cos(x)$. For the details, see Morris Kline's *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972 (especially pages 671–675, but see further references in its index). In the nineteenth and twentieth centuries mathematicians developed a variety of conditions that implied the series converges to the function. The most recent is due to Lenart Carleson (1928–) in 1966, which settled a famous conjecture.

Summary

While Taylor Series are useful for dealing with a function that is very smooth (having derivatives of all orders), Fourier series can represent a function that is not even continuous. While the coefficients in Taylor series are expressed in terms of derivatives, those in Fourier series are expressed in terms of integrals. Even non-periodic functions can be treated by Fourier series. For instance, to deal with x^2 on, say, $[0, 100)$ just extend its domain to the whole x-axis by defining a function of period 100 that agrees with x^2 on $[0, 100)$. If the even extension is used, the Fourier series has only cosine terms but, if the odd extension is used, the Fourier series has only sine terms.

pdflte**EXERCISES for Section 12.7** *Key:* R–routine, M–moderate, C–challenging

The following table of integrals will be helpful in evaluating some of the integrals in these exercises.

$$\begin{array}{rcl}
\int \sin(ax)\ dx &=& -\frac{1}{a}\cos(ax)+C \\
\int \cos(ax)\ dx &=& \frac{1}{a}\sin(ax)+C \\
\int x\sin(ax)\ dx &=& \frac{1}{a^2}\sin(ax)-\frac{x}{a}\cos(ax)+C \\
\int x\cos(ax)\ dx &=& \frac{1}{a^2}\cos(ax)+\frac{x}{a}\sin(ax)+C \\
\int x^2\sin(ax)\ dx &=& \frac{2}{a^3}\cos(ax)+\frac{2x}{a^2}\sin(ax)-\frac{x^2}{a}\cos(ax)+C \\
\int x^2\cos(ax)\ dx &=& \frac{-2}{a^3}\sin(ax)+\frac{2x}{a^2}\cos(ax)+\frac{x^2}{a}\sin(ax)+C \\
\int \sin(x)\sin(ax)\ dx &=& \frac{1}{2(a-1)}\sin((a-1)x)-\frac{1}{2(a+1)}\sin((a+1)x)+C \\
\int \sin(x)\cos(ax)\ dx &=& \frac{1}{2(a-1)}\cos((a-1)x)-\frac{1}{2(a+1)}\cos((a+1)x)+C \\
\int \cos(x)\sin(ax)\ dx &=& \frac{-1}{2(a-1)}\cos((a-1)x)-\frac{1}{2(a+1)}\cos((a+1)x)+C \\
\int \cos(x)\cos(ax)\ dx &=& \frac{1}{2(a-1)}\sin((a-1)x)+\frac{1}{2(a+1)}\sin((a+1)x)+C
\end{array}$$

In Exercises 1 to 6 give the period of the function

1.[R] $\tan(x)$

2.[R] $2/\cos^2(x)$

3.[R] $\sin(3x)$

4.[R] $\sin(2\pi x)$

5.[R] $\sin(\pi x/3)$

6.[R] $\sin(x/3)$

7.[R] Let $f(x)=x^2$ for x in $[-\pi,\pi)$ and have period 2π.

(a) Find $f(\pi)$, $f(2\pi)$, $f(-\pi)$, and $f(-2\pi)$.

(b) Graph $f(x)$ for x in $[-4\pi,4\pi]$.

(c) Show that the Fourier series of $f(x)$ is

$$2\sin(x)-\sin(2x)+\frac{2}{3}\sin(3x)-\frac{1}{2}\sin(4x)$$

(d) Why are there no sine terms in the Fourier series?

(e) What is the average value of $f(x)$ over any interval of length 2π?

8.[R] Let $f(x)=-x^2$ for x in $[-\pi,0)$ and x^2 for x in $[0,\pi)$ and have period 2π.

(a) Find $f(\pi)$, $f(2\pi)$, $f(-\pi)$, and $f(-2\pi)$.

(b) Graph $f(x)$ for x in $[-4\pi,4\pi]$.

(c) Show that the Fourier series of $f(x)$ is

$$2\frac{\pi^2-4}{\pi}\sin(x)-\pi\sin(2x)+2\frac{9\pi^2-4}{27\pi}\sin(3x)-\frac{\pi}{2}\sin(4x)+2\frac{25\pi^2-4}{125\pi}\sin(5x)-\frac{\pi}{3}\sin(6x)$$

(d) Why are there no cosine terms in the Fourier series?

(e) What is the average value of $f(x)$ over any interval of length 2π?

9.[R] Let $f(x) = x$ for x in $[-\pi, \pi)$ and have period 2π. NOTE: This function is known as a sawtooth function.

(a) Find $f(\pi)$, $f(2\pi)$, $f(-\pi)$, and $f(-2\pi)$.

(b) Graph $f(x)$ for x in $[-4\pi, 4\pi]$.

(c) Show that the Fourier series of $f(x)$ is

$$2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x).$$

(d) Why are there no sine terms in the Fourier series?

(e) What is the average value of $f(x)$ over any interval of length 2π?

(f) What does the series converge to at the jump discontinuities?

10.[R] Let $f(x) = x$ for x in $[-1, 1)$ and have period 2. NOTE: This function is known as a sawtooth function.

(a) Find $f(1)$, $f(2)$, $f(-1)$, and $f(-2)$.

(b) Graph $f(x)$ for x in $[-4, 4]$.

(c) Show that the Fourier series of $f(x)$ is

$$\frac{2}{\pi}\left(\sin(\pi x) - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x) - \frac{1}{4}\sin(4\pi x) + \cdots\right).$$

(d) Why are there no sine terms in the Fourier series?

(e) What is the average value of $f(x)$ over any interval of length 2π?

(f) What does the series converge to at the jump discontinuities?

(g) How does this Fourier series compare with the one in Exercise 9?

11.[R] Find the Fourier series of $f(x) = \sin(x)$ (for all x).

12.[R] Find the Fourier series of $f(x) = \cos(2x)$ (for all x).

In Exercises 13 to 22, compute the Fourier series of the indicated function. Sketch at least two periods of the function corresponding to the Fourier series. NOTE: In each case assume the function is periodic.

13.[R] $f(x) = x^2$, $-1 \leq x < 1$ (period 2)

14.[R] $f(x) = x^2$, $-2 \leq x < 2$ (period 4)

15.[R] $f(x) = \begin{cases} 0 & \text{for } -1 \leq x < 0 \\ 1 & \text{for } 0 \leq x < 1 \end{cases}$

16.[R] $f(x) = \begin{cases} 1 & \text{for } -1 \leq x < 0 \\ 0 & \text{for } 0 \leq x < 1 \end{cases}$

17.[R] $f(x) = \begin{cases} 0 & \text{for } -1 \leq x < 0 \\ x & \text{for } 0 \leq x < 1 \end{cases}$

18.[R] $f(x) = \begin{cases} 1 & \text{for } -1 \leq x < 0 \\ x & \text{for } 0 \leq x < 1 \end{cases}$

19.[R] $f(x) = \begin{cases} 0 & \text{for } -\pi \leq x < 0 \\ \sin(x) & \text{for } 0 \leq x < \pi \end{cases}$

20.[R] $f(x) = \begin{cases} 1 & \text{for } -\pi \leq x < 0 \\ \cos(x) & \text{for } 0 \leq x < \pi \end{cases}$

21.[R] $f(x) = \begin{cases} 0 & \text{for } -2\pi \leq x < 0 \\ \sin(x) & \text{for } 0 \leq x < 2\pi \end{cases}$

22.[R] $f(x) = \begin{cases} 1 & \text{for } -2\pi \leq x < 0 \\ \cos(x) & \text{for } 0 \leq x < 2\pi \end{cases}$

In Exercises 23 to 28, (a) extend the given function to be an odd periodic function with period $2L$, (b) compute the Fourier series of the function found in (a), (c) graph at least two periods of the first three non-zero terms of the Fourier series found in (b).

23.[R] $f(x) = 1$, $0 \leq x \leq 1$ $(L = 1)$

24.[R] $f(x) = x$, $0 \leq x \leq 1$ $(L = 1)$

25.[R] $f)x) = x^2$, $0 \leq x \leq 1$ $(L = 1)$

26.[R] $f)x) = |x - 1|$, $0 \leq x \leq 2$ $(L = 2)$

27.[R] $f)x) = \sin(x)$, $0 \leq x \leq \pi$ $(L = \pi)$

28.[R] $f)x) = \cos(x)$, $0 \leq x \leq \pi$ $(L = \pi)$

In Exercises 29 to 34, (a) extend the given function to be an even periodic function with period $2L$, (b) compute the Fourier series of the function found in (a), (c) graph at least two periods of the function corresponding to the Fourier series found in (b).

29.[R] $f(x)$ from Exercise 23

30.[R] $f(x)$ from Exercise 24

31.[R] $f(x)$ from Exercise 25

32.[R] $f(x)$ from Exercise 26

33.[R] $f(x)$ from Exercise 27

34.[R] $f(x)$ from Exercise 28

35.[M] Use the properties of even and odd functions to justifiy that:

(a) the product of two even functions is even.

(b) the product of two odd functions is even.

(c) the product of an even function and an odd function is odd.

36.[M] Determine which of the statements in Exercise 35 is true if the word "product" is replaced with "sum".

37.[M] Show that any function, $f(x)$, can be written as the sum of an even function (f_{even}) and an odd function (f_{odd}). HINT: Write $f(x) = f_{even}(x) + f_{odd}(x)$. Use the properties of f_{even} and f_{odd} to express $f(-x)$ in terms of $f_{even}(x)$ and $f_{odd}(x)$.

38.[M] Write each of the following functions as the sum of an even function and an odd function.

(a) $f(x) = x^2 + 2x$

(b) $f(x) = x^3 - 2x$

(c) $f(x) = x^3 + 3x^2 - 2x + 1$

(d) $f(x) = \sin(4x) - 3x^3$

(e) $f(x) = |x| \sin(x)$

(f) $f(x) = |x| \cos(x)$

(g) $f(x) = (\sin(x) + 1)^3$

(h) $f(x) = (\cos(x) + 1)^3$

39.[M] Show that

$$\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} L & \text{if } m = k,\ k = 1, 2, \ldots \\ 0 & \text{if } m \neq k,\ k = 1, 2, \ldots \end{cases}.$$

Hint: Use the trigonometric identity $\sin(u)\sin(v) = \frac{1}{2}\left(\cos(u-v) - \cos(u+v)\right)$.

40.[M] Show that

$$\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} L & \text{if } m = k,\ k = 1, 2, \ldots \\ 0 & \text{if } m \neq k,\ k = 1, 2, \ldots \end{cases}.$$

Hint: Use the trigonometric identity $\cos(u)\cos(v) = \frac{1}{2}\left(\cos(u-v) + \cos(u+v)\right)$.

41.[M] Show that

$$\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx = 0.$$

Hint: While you could use the trigonometric identity $\sin(u)\cos(v) = \frac{1}{2}\left(\sin(u-v) + \sin(u+v)\right)$ this exercise can be completed without finding any integrals.

42.[M] Derive the formula

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.$$

Hint: Integrate (12.7.4) term-by-term over the interval $[-L, L]$.

43.[M] Derive the formula

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\,dx$$

for $m = 1, 2, 3, \ldots$. Hint: Multiply (12.7.4) by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate term-by-term over the interval $[-L, L]$.

44.[M] In Section 11.6, Example 3, it is claimed that the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2}$$

converges to $\frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)$ for $0 \leq x \leq 2\pi$. Use Fourier series to verify this claim.

45.[C] Let $f(x)$ be a periodic function with period $2L$.

(a) Show that $\int_0^{2L} f(x)\ dx = \int_{-L}^{L} f(x)\ dx$.

(b) Show that $\int_a^{a+2L} f(x)\ dx = \int_{-L}^{L} f(x)\ dx$.

(c) Show that $\int_{-2L}^{0} f(x)\ dx = \int_{-L}^{L} f(x)\ dx$.

(d) Show that $\int_a^{a+2L} f(x)\ dx = \int_{-L}^{L} f(x)\ dx$ for any number a.

Exercise 46 Just as the complex numbers revealed a close tie between the exponential and trigonometric functions, they also reveal a relation between power series and Fourier series. Exercise 46 helps to make this connection.

46.[C] A Taylor series $\sum_{k=0}^{\infty} a_k z^k$ does not look like a Fourier series. However, when a_k is written as $b_k + ic_k$ and z is expressed as $r(\cos(\theta) + i\sin(\theta))$, one sees a close resemblence. Check that this is so. That is, write the series in the form $A+Bi$ where A and B are Fourier series.

12.S Chapter Summary

The Taylor polynomials first encountered in Section 5.4 suggested the powerful power series associated with a function that has derivatives of all orders at a, namely

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \qquad (12.S.1)$$

which certainly converges when x is a. It may even converge for other values of x, but not necessarily to $f(x)$. For the common functions e^x, $\sin(x)$, and $\cos(x)$ the corresponding power series does converge to the function for all values of x.

The error in using a front end up through the power $(x-a)^n$ to estimate $f(x)$ is given by Lagrange's formula,

$$f(x)-\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c \text{ between } x \text{ and } a. \qquad (12.S.2)$$

For some functions, such as $\tan(x)$, it is not easy to find the k^{th} derivative. So, we should be glad that e^x, $\sin(x)$, and $\cos(x)$ have such convenient higher derivatives.

Replace x by $-x^2$.

One can obtain a few terms of the Maclaurin series for $\tan(x)$ by dividing the series for $\sin(x)$ by the series for $\cos(x)$. The series for $1/(1+x^2)$ is easily found by massaging the sum of the geometric series $1/(1-x) = 1+x+x^2+\dots$. Integration of that series yields painlessly the Maclaurin series for $\arctan(x)$.

Each power series $\sum_{k=0}^{\infty} a_k(x-a)^k$ has a radius of convergence, R. For $|x-a| < R$, the series converges absolutely and for $|x-a| > R$ the series does not converge. If it converges for all x, then $R = \infty$. For $|x-a| < R$, one may safely differentiate and integrate a series, getting new series.

Estimating an integrand $f(x)$ by the front end of a power series, we can then estimate $\int_a^b f(x)\,dx$. Also, power series are of use in finding indeterminate limits, such as $\lim_{x\to 0} \frac{f(x)}{g(x)}$.

Maclaurin series, combined with complex numbers, exposed a fundamental relation between exponential and trigonometric functions:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Other important truths, not covered in this chapter, are revealed with the aid of complex numbers. For instance, if we allow complex coefficients, every polynomial can be written as the product of first-degree polynomials, thus simplifying the partial fractions of Section 8.4. Complex numbers can also help us find the radius of convergence. For instance, what is the radius of convergence of the Taylor series in powers of $x-3$ associated with $1/(1+x^2)$?

Function	Maclaurin Series	Interval of Convergence	How F
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	all x: $(-\infty,\infty)$	Taylor
$\sin(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	all x: $(-\infty,\infty)$	Taylor
$\cos(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	all x: $(-\infty,\infty)$	Taylor
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	$\lvert x\rvert < 1$	Geome
$\ln(1+x)$	$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{x^k}{k}$	$-1 < x \le 1$	Integra
$\arctan(x)$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$	$\lvert x\rvert \le 1$	Integra
$\arcsin(x)$	$x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$	$\lvert x\rvert \le 1$	Integra
$(1+x)^r$	$1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \cdots$	$\lvert x\rvert < 1$	Taylor
$\frac{1}{(1-x)^2}$	$\sum_{k=0}^{\infty} kx^{k-1}$	$\lvert x\rvert < 1$	Differe

Table 12.S.1:

Answer: it is the distance from the point $(3,0)$ to the nearest complex number at which $1/(1+x^2)$ "blows up," that is, when $1+x^2=0$. This occurs when x is i or $-i$, both of which, by the Pythagorean Theorem, are at a distance $\sqrt{1^2+3^2}=\sqrt{10}$ from $(3,0)$. So, $R=\sqrt{10}$.

The final section introduced Fourier series. In contrast to Taylor series, its coefficients are given by integrals, rather than by derivatives. Consequently, Fourier series apply to a larger class of functions. However, this method applies directly only to periodic functions. In the case of a non-periodic function, one restricts the domain to an interval $(-L,L)$ and extends the function to have period $2L$

EXERCISES for 12.S *Key:* R–routine, M–moderate, C–challenging

Exercise 1 provides additional detail for the historical discussion (see page 58) about Newton's calculation of the area under a hyperbola to more than 50 decimal places. (See also Exercises 29 and 30 in Section 6.5.)

1.[R] Let c be a positive constant.

(a) Show that the area under the curve $y=1/(1+x)$ above the interval $[0,c]$ is $-\displaystyle\sum_{k=1}^{\infty}\frac{(-c)^k}{k}$.

(b) Show that the area under the curve $y=1/(1+x)$ above the interval $[-c,0]$ is $\displaystyle\sum_{k=1}^{\infty}\frac{c^k}{k}$.

2.[M] Assume that a Maclaurin series $M(x)$ is associated with $f(x)$ for x in $(-a,a)$.

Show that $M(x^2)$ is the Maclaurin series associated with $g(x) = f(x^2)$ for x in $(-\sqrt{a}, \sqrt{a})$.

3.[M] The integral $\int_0^{2\pi} (1-\cos(x))/x dx$ occurs in the theory of antennas.

(a) Show that it is an improper integral.

(b) Show that there is a continuous function whose domain is $[0, 2\pi]$ that coincides with the integrand when x is not 0.

(c) The integrand does not have an elementary antiderivative. Why is the power series technique of approximation inconvenient here?

Exercises 4 to 6 use complex numbers to find the average value of the logarithm of a function. Exercise 4 is related to Exercise 90 on page 781.

4.[C] Let a point Q be a distance $a \neq 1$ from the center of a unit circle.

(a) Show that the average value of the (natural) logarithm of the distance from Q to points on the circumference is

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \ln(1 + a^2 - 2a\cos(\theta))\ d\theta.$$

(b) Spend at least three minutes, but at most 5 minutes, trying to evaluate the integral in (a).

5.[C] This algebraic exercise is needed in Exercise 6. Let $z_0, z_1, \ldots, z_{n-1}$ be the n n^{th} roots of 1. Then it is shown in an algebra course that

$$(z - z_0)(z - z_1)(z - z_2) \cdots (z - z_{n-1}) = z^n - 1.$$

Check that this equation holds when n is (a) 2, (b) 3, (c) 4.

6.[C] Let $z_0, z_1, \ldots, z_{n-1}$ be the n n^{th} roots of 1.

(a) Why is $\displaystyle\frac{1}{n}\sum_{i=0}^{n-1} \ln|a - z_i|$ an estimate of the average distance?

(b) Show that the average in (a) equals

$$\frac{1}{n} \ln|a^n - 1|. \tag{12.S.3}$$

(c) If $0 < a < 1$, show that the limit of (12.S.3) as $n \to \infty$ is 0.

(d) If $a > 1$, show that the limit of (12.S.3) as $n \to \infty$ is $\ln(a)$.

(e) Use the results in (c) and (d) to evaluate the integral in Exercise 4(a).

7.[C] Find $\lim_{x\to\infty} \frac{xe^x}{e^{x^2}}$.

8.[C] Does $\sum_{n=1}^{\infty}\left(1-\cos\left(\frac{1}{n}\right)\right)$ converge or diverge? Explain.

Calculus is Everywhere # 15
Sparse Traffic

Customers arriving at a checkout counter, cars traveling on a one-way road, raindrops falling on a street and cosmic rays entering the atmosphere all illustrate one mathematical idea — the theory of sparse traffic involving independent events. We will develop the mathematics, which is the basis of the study of waiting time – whether customers at the checkout counter or telephone calls at a switchboard.

First we sketch briefly a bit of probability theory.

Some Probability Theory

The probability that an event occurs is measured by a number p, which can be anywhere from 0 up to 1; $p = 1$ implies the event will certainly occur with negligible exceptions and $p = 0$ that it will not occur with negligible exceptions. The probability that a penny turns up heads is $p = 1/2$ and that a die turns up 2 is $p = 1/6$. (The phrase "certainly occurs with negligible exceptions" means, roughly, that the times the event does not occur are so rare that we may disregard them. Similarly, the phrase "certainly will not occur with negligible exceptions" means, roughly, that the times the event does not occur are so rare that we may disregard them.)

The probability that two events that are independent of each other both occur is the product of their probabilities. For instance, the probability of getting heads when tossing a penny and a 2 when tossing the die is $p = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{1}{12}$.

The probability that exactly one of several mutually exclusive events occurs is the sum of their probabilities. For instance, the probability of getting a 2 or a 3 with a die is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

With that thumbnail introduction, we will analyze sparse traffic on a one-way road. We will assume that the cars enter the traffic independently of each other and travel at the same speed. Finally, to simplify matters, we assume each car is a point.

The Model

To construct our model we introduce the functions $P_0, P_1, P_2, \ldots, P_n, \ldots$ where $P_n(x)$ shall be the probability that any interval of length x contains exactly n

cars (independently of the location of the interval). Thus $P_0(x)$ is the probability that an interval of length x is empty. We shall assume that

$$P_0(x) + P_1(x) + \cdots + P_n(x) + \cdots = 1 \qquad \text{for any } x.$$

We also shall assume that $P_0(0) = 1$ ("the probability is 1 that a given point contains no cars").

For our model we make the following two major assumptions:

(a) The probability that exactly one car is in any fixed short section of the road is approximately proportional to the length of the section. That is, there is some positive number k such that

$$\lim_{\Delta x \to 0} \frac{P_1(\Delta x)}{\Delta x} = k.$$

(b) The probability that there is more than one car in any fixed short section of the road is neglible, even when compared to the length of the section. That is,

$$\lim_{\Delta x \to 0} \frac{P_2(\Delta x) + P_3(\Delta x) + P_4(\Delta x) + \cdots}{\Delta x} = 0. \qquad \text{(C.15.1)}$$

We shall now put assumptions (a) and (b) into more useful forms. If we let

$$\epsilon = \frac{P_1(\Delta x)}{\Delta x} - k \qquad \text{(C.15.2)}$$

where ϵ depends on Δx, assumption (a) tells us that $\lim_{\Delta x} \epsilon = 0$. Thus, solving (C.15.2) for $P_1(\Delta x)$, we see that assumption (a) can be phrased as

$$P_1(\Delta x) = k\Delta x + \epsilon\Delta x \qquad \text{(C.15.3)}$$

where $\epsilon \to 0$ as $\Delta x \to 0$.

Since $P_0(\Delta x) + P_1(\Delta x) + \cdots + P_n(\Delta x) + \cdots = 1$, assumption (b) may be expressed as

$$\lim_{\Delta x \to 0} \frac{1 - P_0(\Delta x) - P_1(\Delta x)}{\Delta x} = 0. \qquad \text{(C.15.4)}$$

In light of assumption (a), equation (C.15.4) is equivalent to

$$\lim_{\Delta x \to 0} \frac{1 - P_0(\Delta x)}{\Delta x} = k. \qquad \text{(C.15.5)}$$

In the manner in which we obtained (C.15.3), we may deduce that

$$1 - P_0(\Delta x) = k\Delta x + \delta\Delta x,$$

where $\delta \to 0$ as $\Delta x \to 0$. Thus

$$P_0(\Delta x) = 1 - k\Delta x - \delta\Delta x, \quad \text{(C.15.6)}$$

where $\delta \to 0$ as $\Delta x \to 0$. *On the basis of (a) and (b), expressed in* (C.15.3) *and* (C.15.6)*, we shall obtain an explicit formula for each* P_n.

Let us determine P_0 first. Observe that a section of length $x+\Delta x$ is vacant if its left-hand part of length x is vacant and its right-hand part of length Δx is also vacant. Since the cars move independently of each other, the probability that the whole interval of length $x + \Delta x$ being empty is the product of the probabilities that the two smaller intervals of lengths x and Δx are both empty. (See Figure C.15.1.) Thus we have

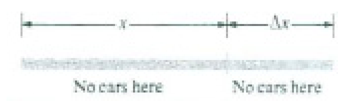


Figure C.15.1: No cars in a section of length $x + \Delta x$.

$$P_0(x + \Delta x) = P_0(x)P_0(\Delta x). \quad \text{(C.15.7)}$$

Recalling (C.15.6), we write (C.15.7) as

$$P_0(x + \Delta x) = P_0(x)(1 - k\Delta x - \delta\Delta x)$$

which a little algebra transforms to

$$\frac{P_0(x + \Delta x) - P_0(x)}{\Delta x} = -(k + \delta)P_0(x). \quad \text{(C.15.8)}$$

Taking limits on both sides of (C.15.8) as $\Delta x \to 0$, we obtain

$$P_0'(x) = -kP_0(x). \quad \text{(C.15.9)}$$

(Recall that $\delta \to 0$ as $\Delta x \to 0$.) From (C.15.9) it follows that there is a constant A such that $P_0(x) = Ae^{-kx}$. Since $1 = P_0(0) = Ae^{-k0} = A$, we conclude that $A = 1$, hence

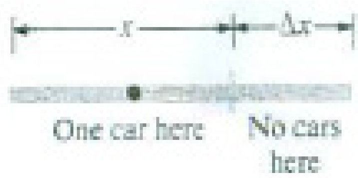
$$P_0(t) = e^{-kx}.$$

This explicit formula for P_0 is reasonable; e^{-kx} is a decreasing function of x, so that the larger an interval, the less likely that it is empty.

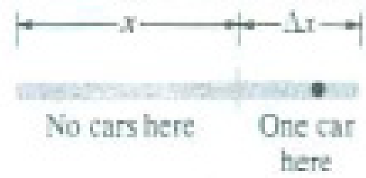
Now let us determine P_1. To do so, we examine $P_1(x + \Delta x)$ and relate it to $P_0(x)$, $P_0(\Delta x)$, $P_1(x)$, and $P_1(\Delta x)$, with the goal of finding an equation involving the derivative of P_1.

Again, imagine an interval of length $x + \Delta x$ cut into two intervals, the left-hand subinterval of length x and the right-hand subinterval of length Δx. Then there is precisely one car in the whole interval if *either* there is exactly one car in the left-hand interval and none in the right-hand subinterval *or* there is none in the left-hand subinterval and exactly one in the right-hand subinterval. (See Figure C.15.2.) Thus we have

$$P_1(x + \Delta x) = P_1(x)P_0(\Delta x) + P_0(x)P_1(\Delta x). \quad \text{(C.15.10)}$$



(a)



(b)

Figure C.15.2: The two ways to have exactly one care in an interval of length $x + \Delta x$.

In view of (C.15.3) and (C.15.6), we may write (C.15.10) as

$$P_1(x + \Delta x) = P_1(x)(1 - k\Delta x - \delta\Delta x) + P_0(x)(k\Delta x + \epsilon\Delta x)$$

which a little algebra changes to

$$\frac{P_1(x + \Delta x) - P_1(x)}{\Delta x} = -(k + \delta)P_1(x) + (k + \epsilon)P_0(x). \tag{C.15.11}$$

Letting $\Delta x \to 0$ in (C.15.11) and remembering that $\delta \to 0$ and $\epsilon \to 0$ as $\Delta x \to 0$, we obtain $P_1'(x) = -kP_1(x) + kP_0(x)$; recalling that $P_0(x) = e^{-kx}$, we deduce that

$$P_1'(x) = -kP_1(x) + ke^{-kx}. \tag{C.15.12}$$

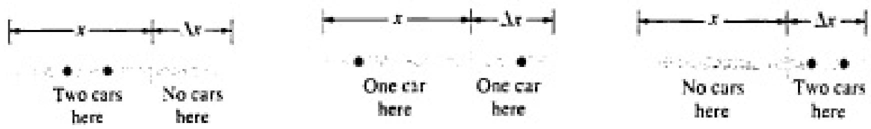
From (C.15.12) we shall obtain an explicit formula for $P_1(x)$. Since $P_0(x)$ involves e^{-kx} and so does (C.15.12). it is reasonable to guess that $P_1(x)$ involves e^{-kx}. Therefore let us express $P_1(x)$ as $g(x)e^{-kx}$ and determine the form of $g(x)$. (Since we have the identity $P_1(x) = \left(P_1(x)e^{kx}\right)e^{-kx}$, we know that $g(x)$ exists.)

According to (C.15.12) we have $\left(g(x)e^{-kx}\right)' = -kg(x)e^{-kx} + ke^{-kx}$; hence

$$g(x)(ke^{-kx}) + g'(x)e^{-kx} = -kg(x)e^{-kx} + ke^{-kx}$$

from which it follows that $g'(x) = k$. Hence $g(x) = kx + c_1$, where c_1 is some constant: $P_1(x) = (kx + c_1)e^{-kx}$. Since $P_1(0) = 0$, we have $P_1(0) = (k \cdot 0 + c_1)e^{-k\cdot 0} = c_1$ and hence $c_1 = 0$. Thus we have shown that

$$P_1(x) = kxe^{-kx} \tag{C.15.13}$$



(a) (b) (c)

Figure C.15.3: The three ways to have exactly two cars in an interval of length $x + \Delta x$.

and P_1 is completely determined.

To obtain P_2 we argue as we did in obtaining P_1. Instead of (C.15.10) we have

$$P_2(x + \Delta x) = P_2(x)P_0(\Delta x) + P_1(x)P_1(\Delta x) + P_0(x)P_2(\Delta x) \qquad \text{(C.15.14)}$$

an equation that records the three ways in which two cars in a section of length $x + \Delta x$ can be situated in a section of length x and a section of length Δx. (See Figure C.15.3.)

Similar reasoning shows that

See Exercise 8.

$$P_2(x) = \frac{k^2x^2}{2}. \qquad \text{(C.15.15)}$$

Then, applying the same reasoning inductively leads to

See Exercises 9 and 10.

$$P_n(x) = \frac{(kx)^n}{n!}e^{-kx}. \qquad \text{(C.15.16)}$$

We have obtained in (C.15.16) the formulas on which the rest of our analysis will be based. Note that these formujlas refer to a road section of any length, though the assumptions (a) and (b) refer only to short sections. What has enabled us to go from the "microscopic" to the "macroscopic" is the additional assumption that the traffic in any one section is independent of the traffic in any other section. The formulas (C.15.16) are known as the **Poisson formulas**.

The Meaning of k

The constant k was defined in terms of arbitrarily short intervals, at the "microscopic level". How would we compute k in terms of observable data, at the "macroscopic level"? It turns out that k records the traffic density: the average number of events during an interval of length x is kx.

The average number of events in a section of length x is defined as $\sum_{n=0}^{\infty} nP_n(x)$. This weights each possible number of events (n) with it's likelihood of occurring ($P_n(x)$). This average is

$$\begin{aligned}\sum_{n=0}^{\infty} nP_n(x) &= \sum_{n=1}^{\infty} nP_n(x) = \sum_{n=1}^{\infty} n\frac{(kx)^n e^{-kx}}{n!} \\ &= kxe^{-kx}\sum_{n=1}^{\infty}\frac{(kx)^{n-1}}{(n-1)!} \\ &= kxe^{-kx}\sum_{n=0}^{\infty}\frac{(kn)^n}{n!} = kxe^{-kx}e^{kx} = kx.\end{aligned}$$

Thus the expected number of cars in a section is proportional to the length of the section. This shows that the k appearing in assumption (a) is the measure of traffic density, the number of cars per unit length of road.

To estimate k, in the case of traffic for instance, divide the number of cars in a long section of the road by the length of that section.

EXAMPLE 4 (Traffic at a checkout counter.) Customers arrive at a checkout counter at the rate of 15 per hour. What is the probability that exactly five customers will arrive in any given 20-minute period?
SOLUTION We may assume that the probability of exactly one customer coming in a short interval of time is roughly proportional to the duration of that interval. Also, there is only a negligible probability that more than one customer may arrive in a brief interval of time. Therefore conditions (a) and (b) hold, if we replace "length of section" by "length of time". Without further ado, we conclude that the probability of exactly n customers arriving in a period of x minutes is given by (C.15.16). Moreover, the "customer density" is one per 4 minutes; hence $k = 1/4$, and thus the probability that exactly five customers arrive during a 20-minute period, $P_5(20)$, is

$$\left(\frac{1}{4}\cdot 20\right)^5 \frac{e^{-(1/4)\cdot 20}}{5!} = \frac{5^5 e^{-5}}{120} \approx 0.17547.$$

◇

Modeling of the type within this section is of use in predicting the length of waiting lines (or times) or the waiting time to cross. This is part of the theory of queues. See, for instance, Exercises 2 and 3. (See also Exercise 65 in the Summary Exercises in Chapter 4.)

EXERCISES

1.[R]

(a) Why would you expect that $P_0(a+b) = P_0(a) \cdot P_0(b)$ for any a and b?

(b) Verify that $P_0(x) = e^{-kx}$ satisfies the equation in (a).

2.[R] A cloud chamber registers an average of four cosmic rays per second.

(a) What is the probability that no cosmic rays are registered for 6 seconds?

(b) What is the probability that exactly two are registered in the next 4 seconds?

3.[R] Telephone calls during the busy hour arrive at a rate of three calls per minute. What is the probability that none arrives in a period of (a) 30 seconds, (b) 1 minute, (c) 3 minutes?

4.[R] In a large continually operating factory there are, on the average, two accidents per hour. Let $P_n(x)$ denote the probability that there are exactly n accidents in an interval of time of length x hours.

(a) Why is it reasonable to assume that there is a constant k such that $P_0(x)$, $P_1(x)$, ... satisfy 1 and 2 on page 1090?

(b) Assuming that these conditions are satisfied, show that $P_n(x) = (kx)^n e^{-kx}/n!$.

(c) Why must $k = 2$?

(d) Compute $P_0(1)$, $P_1(1)$, $P_2(1)$, $P_3(1)$, and $P_4(1)$.

5.[R] A typesetter makes an average of one mistake per page. Let $P_n(x)$ be the probability that a section of x pages (x need not be an integer) has exactly n errors.

(a) Why would you expect $P_n(x) = x^n e^{-x}/n!$?

(b) Approximately how many pages would be error-free in a 300-page book?

6.[R] In a light rainfall you notice that on one square foot of pavement there are an average of 3 raindrops. Let $P_n(x)$ be the probability that there are n raindrops on an area of x square feet.

(a) Check that assumptions 1 and 2 are likely to hold.

(b) Find the probability that an area of 3 square feet has exactly two raindrops.

(c) What is the most likely number of raindrops to find on an area of one square foot?

7.[R] Write x^2 in the form $g(x)e^{-kx}$.

8.[R] Show that $P_2(x) = \frac{k^2x^2}{2}e^{-kx}$.

9.[R] Show that $P_3(x) = \frac{(kx)^3}{3!}e^{-kx}$.

10.[M] Show that $P_n(x) = \frac{(kx)^n}{n!}e^{-kx}$.

11.[R]

(a) Why would you expect $P_3(a+b) = P_0(a)P_3(b) + P_1(a)P_2(b) + P_2(a)P_1(b) + P_3(a)P_0(b)$?

(b) Do functions defined in (C.15.16) satisfy the equation in (a)?

12.[R]

(a) Why would you expect $\lim_{n\to\infty} P_n(x) = 0$?

(b) Show that the functions defined in (C.15.16) have the limit in (a).

13.[R]

(a) Why would you expect $\lim_{x\to 0} P_1(x) = 1$ and, for all $n \geq 1$, $\lim_{x\to 0} P_n(x) = 0$?

(b) Show that the functions defined in (C.15.16) satisfy the limit in (a).

14.[R] We obtained $P_0(x) = e^{-kx}$ and $P_1(x) = kxe^{-kx}$. Verify that $\lim_{\Delta x\to 0} P_1(\Delta x)/\Delta x = k$, and $\lim_{\Delta x\to 0} P_0(\Delta x)/\Delta x = 1-k$. Hence show that $\lim_{\Delta x\to 0}(P_2(\Delta x) + P_3(\Delta x) + \cdots +)/\Delta x = 0$, and that assumptions 1 and 2 on page 1090 are indeed satisfied.

15.[R]

(a) Obtain assumption 1 from equation (C.15.3).

(b) Obtain equation (C.15.3) from assumption 2.

(c) Obtain assumption 2 from equation (C.15.6).

16.[M] What length of road is most likely to contain exactly one car? That is, what x maximizes $P_1(x)$?

17.[M] What length of road is most likely to contain three cars?

18.[M] For any $x \geq 0$, $\sum_{n=0}^{\infty} P_n(x)$ should equal 1 because it is certain that some number of cars is in a given section of length x (maybe 0 cars). Check that $\sum_{n=0}^{\infty} P_n(x) = 1$. NOTE: This provides a probabilistic argument that $e^u = \sum_{n=0}^{\infty} u^n/n!$ for $n \geq 0$.

19.[M] Planes arrive randomly at an airport at the rate of one per 2 minutes. What is the probability that more than three planes arrive in a 1-minute interval?

Chapter 13

Introduction to Differential Equations

13.1 Modeling and Differential Equations

We are now familiar with computing and interpreting derivatives of functions. The derivative of a function at a point gives the slope of the graph of the function at that point (assuming the derivative exists). Points where the derivative is zero (or does not exist) are the only possible locations for local extrema of a function.

In this section we will see another use of derivatives: differential equations. A **differential equation** is an equation that provides a relationship between the derivatives of a function, the function, and the independent variables (input to the function). Differential equations describe many physical situations, in fact, differential equations are often referred to as the "language of science and engineering".

Get some specific substances and decay rates.

Note that $k > 0$ and $U > 0$, so that $\frac{dU}{dt} < 0$.

EXAMPLE 1 The radioactive substance Uranium-238 decays into Thorium-234 with a half-life of 4.5×10^9 years. The rate of change of the concentration of Uranium-238 is always proportional to the current concentration. Let $U(t)$ denote the concentration of U-239 at time t. Then

$$\frac{dU}{dt} = -kU \tag{13.1.1}$$

with $k = \frac{\ln(2)}{4.5\times 10^9}$. ◇

Get specifics from Ledder.

EXAMPLE 2 One model for a population subject that grows proportional to its size but is also subject to a constant rate of reductions due to harvesting is given by

$$\frac{dP}{dt} = kP - h \tag{13.1.2}$$

where $P = P(t)$ is the size of the unknown population at time t, k is a positive growth rate, and h is a positive constant reflecting the harvesting rate. ◇

EXAMPLE 3 A model for the temperature of an object is

$$\frac{dT}{dt} = -k(T - S) \tag{13.1.3}$$

where $T = T(t)$ is the temperature of the object at time t, $k > 0$ is a constant reflecting the rate at which heat leaves the object and S is the temperature of the surrounding air. ◇

EXAMPLE 4 Newton's Second Law of Motion states that the total force on a moving object is equal to the product of the object's maas and acceleration: $F = ma$. For an object with height $y = y(t)$, the accelaration is $a = y''$.

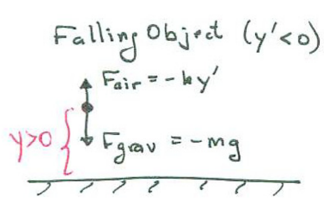


Figure 13.1.1:

If this object has mass m and is under the influence of both gravity and the resistance of the air $F = F_{grav} + F_{air}$. Let the object's height be measured from the ground. The force of gravity is constant and works to pull the object back to the ground, so $F_{grav} = -mg$. The force of air resistance is proportional to velocity and works to retard the current motion, thus $F_{air} = -ky'$. The differential equation that expresses Newton's Second Law of Motion is

$y = 0$ is on the ground;
$y > 0$ is above the ground
$g \approx 9.8$ ft/sec^2 ≈ 32 m/sec^2

$$my'' = -mg - ky'. \tag{13.1.4}$$

◇

The first three examples, (13.1.1), (13.1.2), and (13.1.3), are all first-order differential equations. The fourth example (13.1.4) is a second-order differential equation. In general, the **order** of a differential equation is the order of the highest derivative in the differential equation.

Differential equation (13.1.2) is a nonlinear differential equation because it involves N^2; the other three examples are all linear differential equations. More generally, a **linear differential equation** is a differential equation that is linear in the unknown function and its derivatives. A **nonlinear differential equation** involves nonlinear terms such as y^2, $e^{y'}$, or $\cos(y)$.

Our current interest in differential equations is to recognize a differential equation and to be able to make some basic classifications of the equation (order, linear / nonlinear). We also want to begin to develop the ability to write differential equations as a model of a real-world situation.

Absolute and Relative Rates of Change

When $y(t)$ is the size of an object at time t, the **absolute rate of change** of y is $\frac{dy}{dt}$. The **relateive rate of change** of y, $\frac{1}{y}\frac{dy}{dt}$, reflects the overall size of the object.

EXAMPLE 5 Find the differential equation for the size of a population that is growing at a constant absolute rate of change. Classify the differential equation. Find all solutions that satisfy this equation.

SOLUTION Let the size of the population at time t be denoted by $N = N(t)$. The assumption that the population grows at a constant absolute rate of change is expressed by

$$\frac{dN}{dt} = k \tag{13.1.5}$$

where k is a positive constant.

The differential equation (13.1.5) is both first-order and linear.

Any function whose first derivative is the constant k is a solution to (13.1.5). In other words, any antiderivative of k is a solution to this differential equation. Thus, $N(t) = kt + C$ for any choice of the constant C. ◇

The size of any population with constant absolute rate of change is a linear function. The slope of the solution is the constant k. The value of C is the size of the population at time $t = 0$.

EXAMPLE 6 Find the differential equation for the size of a population that is growing at a constant relative rate of change. Classify the differential equation. Find all solutions that satisfy this equation.

SOLUTION When a population grows with a constant relative rate of change k, $\frac{1}{N}\frac{dN}{dt} = k$ so that

Notice that the absolute rate of change is not constant — the larger N is, the faster N changes.

$$\frac{dN}{dt} = kN. \qquad (13.1.6)$$

The differential equation in (13.1.6) is also linear and first-order.

An explicit formula for the solutions to (13.1.6) can be found by noticing that

$$\frac{1}{N}\frac{dN}{dt} = \frac{d}{dt}\left(\ln|N(t)|\right).$$

Thus, $\frac{d}{dt}\left(\ln|N(t)|\right) = k$ so that $\ln|N(t)|$ must be an antiderivative of k. This means $\ln|N(t)| = kt+C$. Taking the exponential of both sides of this equation yields

$$\begin{aligned} e^{\ln|N(t)|} &= e^{kt+C} \\ |N(t)| &= e^{C}e^{kt} \\ N(t) &= \pm e^{C}e^{kt} \\ N(t) &= Ae^{kt} \end{aligned}$$

where $A = \pm e^{C}$ can be any real number. ◇

Any function whose relative rate of change is constant is an exponential function. When the relative rate of change, k, is positive the population grows exponentially; when k is negative the population decays exponentially.

EXERCISE: Half-life, doubling time

Summary

EXERCISES for Section 13.1 *Key:* R–routine, M–moderate, C–challenging

1.[C]

2.[C]

13.2 Using Slope Fields to Analyze Differential Equations

- Use presentation and examples from ODE PowerTool

Summary

EXERCISES for Section 13.2 *Key:* R–routine, M–moderate, C–challenging

1.[C] A

2.[C] B?

13.3 Separable Differential Equations

Summary

EXERCISES for Section 13.3 *Key:* R–routine, M–moderate, C–challenging

Exercises in other sections that involve separable ODEs include: Exercise 22 in Section 5.6

1.[C]

2.[C]

13.4 Euler's Method

Summary

EXERCISES for Section 13.4 *Key:* R–routine, M–moderate, C–challenging

1.[C]

2.[C]

13.5 Numerical Solutions to Differential Equations

This section will be written later.

Summary

EXERCISES for Section 13.5 *Key:* R–routine, M–moderate, C–challenging

1.[C]

2.[C]

13.6 Picard's Method

Summary

EXERCISES for Section 13.6 *Key:* R–routine, M–moderate, C–challenging

1.[C]

2.[C]

13.S Chapter Summary

The text and exercises for the summary will be written after the organization of the chapters is firmly settled.

EXERCISES for 13.S *Key:* R–routine, M–moderate, C–challenging

1.[M] Assume that the outdoors temperature increases linearly, $h(t) = t + 1$, for simplicity. The temperature of the house starts at time $t = 0$ to be $c < 0$. Then it warms up by Newton's law. If that temperature is $T(t)$, then $T'(t) = k(t - T(t))$.

(a) Find $T(t)$.

(b) Is the graph of $T(t)$ asymptotic to the graph of the outdoor temperature?

2.[C] Consider the differential equation $(y')^2 = 1 - y^2$ with $-1 < y(0) < 0$ and $y'(0) > 0$.

(a) Explain why y is never decreasing.

(b) Explain why y is bounded.

(c) What is the largest value y can be? (Call this value L.)

(d) Is it possible that $\lim_{t\to\infty} y(t) < L$?

(e) Explain why y must cross the t-axis.

(f) What can be said about the angle where y crosses the t-axis?

(g) When is the curve concave up? concave down? HINT: Differentiate the ode.

(h) What might the graph of the solution look like?

(i) Give an example of a specific function that satisfies the equation. HINT: Think trigonometry.

3.[C] In CIE 20 (Chapter 15) we found that the equation of a tractrix, which is the path of the rear wheel in the preceding exercise. That analysis depends on showing that

$$\left(\frac{dy}{dx}\right)^2 = \frac{y^2}{a^2 - y^2}. \tag{13.S.1}$$

Obtain the equation by differentiating both sides of the equation

$$y(s) = ke^{-s/a}$$

with respect to x.