Calculus

Sherman Stein Douglas Meade

October 29, 2007

Contents

Overview of Calculus I

There are two main concepts in calculus: the derivative and the integral. To introduce these ideas, consider the follow two complementary problems:

Scenario A

Setting	Your odometer is broken but you can still record your speed every second.
Question	How would you estimate the total distance?

Scenario B

Setting	Your speedometer is broken but the total distance covered every second can be recorded.
Question	How would you estimate our velocity throughout the trip?

To estimate the distance traveled in Scenario A, we use the observation that the velocity does not change very much during the one second between two velocity measurements. The product of the recorded velocity (with units of, say, meters per second) and the time between recordings (one second) provides an estimate of the distance traveled during this one second time interval (with units of meters). An estimate of the total distance traveled.

In Scenario A the speedometer is functional and the odometer is broken. Scenario B presents a complentary situation: the speedometer is broken and the odemeter works. Velocity is the rate of change of position with respect to time. Reasoning as before, the distance traveled between any two data recordings is the distance traveled over that second. This distance (with units of, say, meters), divided by the amount of time between the recordings (here, one second), gives an estimate of the velocity at any point during this one-second time interval. Note that the units for the velocity are meters per second.

Both the derivative and the integral are based on limits (Chapter 2. In both situations, the estimates become better if the data recordings are taken at more frequent times.

Derivatives, the rate of change of one variable with respect to another variable, are discussed in Chapters 3 and 4. Integrals are discussed in Chapters 5 and 6. The connection between these derivatives and integrals is very deep. The Fundamental Theorem of Calculus (Sections 5.4 and 5.5) explores this fundamental connection between derivatives and integrals.

Chapter 1

Pre-Calculus Review of Functions

Calculus is the study of functions. To understand the concepts introduced in this text, it is important to have a solid understanding of functions.

We begin this chapter with a review of the terminology for functions that will be used for the remainder of this book. In Section 1.2 three fundamental types of functions — and their inverses — are reviewed: power functions, exponentials, and logarithms. The final section describes how functions can be combined to create new functions.

1.1 Functions

This section begins with a brief review of the notion of a function. It then discusses one-to-one functions, inverse functions, increasing functions, and decreasing functions.

Definition of a Function

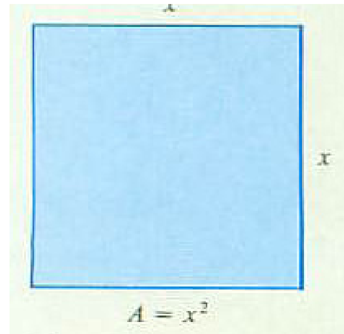


Figure 1.1.1:

The area of a square depends on the length of its side x and is given by the formula $A = x^2$. (See Figure 1.1.1.)

Similarly, the distance s (in feet) that a freely falling object drops in the first t seconds is described by the formula $s = 16t^2$. Each choice of t determines a specific value for s. For instance, when $t = 3$ seconds, $s = 16 \cdot 3^2 = 144$ feet.

Both of these formulas illustrate the mathematical notion of a function.

> **DEFINITION** *(Function.)* Let X and Y be sets. A **function** from X to Y is a rule (or method) for assigning one (and only one) element in Y to each element in X.

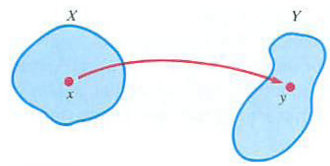


Figure 1.1.2:

The notion of a function is illustrated in Figure 1.1.2, where the element y in Y is assigned to the element x in X. Usually X and Y will be sets of numbers.

A function is often denoted by the symbol f. The element that the function assigns to the element x is denoted $f(x)$ (read "f of x"). In practice, though, almost everyone speaks interchangeably of the function f or the function $f(x)$.

If $f(x) = y$, x is called the **input** or **argument** and y is called the **output** or **value** of the function at x. Also, x is called the **independent variable** and y the **dependent variable**.

A function may be given by a formula, as in the function $A = x^2$. Because A depends on x, we say that "A is a function of x." Because A depends on only one number, x, it is called a function of a single variable, or **univariate function**. The first ten chapters concern functions of a single variable. The area A of a rectangle depends on its length l and width w of the rectangle; it is a function of *two* variables, $A = lw$. The last five chapters of this book extend calculus to **multivariate functions** and **vector-valued functions**.

Ways to write and talk about a function

The function that assigns to each argument x the value x^2 is usually described in a shorthand. For instance, we may write $x \mapsto x^2$ (and say "x goes to x^2" or "x is mapped to x^2"). Or we may say simply, "the formula x^2", "the function x^2", or, sometimes, just "x^2." Using this abbreviation, we might say, "How does x^2 behave when x is large?" Some people object to the shorthand

"x^2" because they fear that it might be misinterpreted as the number x^2, with no sense of a general assignment. In practice, the context will make it clear whether x^2 refers to a number or to a function.

EXAMPLE 1 Consider a circle of radius a, as shown in Figure 1.1.3. Let $f(x)$ be the length of chord AB of this circle at a distance x from the center of the circle. Find a formula for $f(x)$.

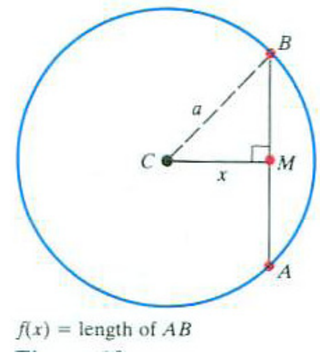


Figure 1.1.3:
Check the formula at $x = 0$ and $x = a$.

SOLUTION We are trying to find how the length $\overline{AB}$ varies as x varies. That is, we are looking for a *formula* for $\overline{AB}$, the length of AB, in terms of x.

Before searching for the formula, it is a good idea to calculate $f(x)$ for some easy inputs. These values can serve as a check on the formula we work out.

In this case $f(0)$ and $f(a)$ can be read at a glance at Figure 1.1.3: $f(0) = 2a$ and $f(a) = 0$. (Why?) Now let us find $f(x)$ for all x in $[0, a]$.

Let M be the midpoint of the chord AB and let C be the center of the circle. Observe that $\overline{CM} = x$ and $\overline{CB} = a$. By the Pythagorean theorem, $\overline{BM} = \sqrt{a^2 - x^2}$. Hence $\overline{AB} = 2\sqrt{a^2 - x^2}$. Thus

$$f(x) = 2\sqrt{a^2 - x^2}.$$

◇

Domain and Range

If f is a function from X to Y, we will often be concerned with the set of inputs (X) and the set of possible outputs (Y). For this reason we give names to the two sets.

> **DEFINITION** *(Domain and range)* Let X and Y be sets and let f be a function from X to Y. The set X is called the **domain** of the function. The set of all outputs of the function is called the **range** of the function.

The range is not necessarily all of Y.

When the function is given by a formula, the domain is usually understood to consist of all the numbers for which the formula is defined.

In Example 1 the domain is the closed interval $[0, a]$ and the range is the closed interval $[0, 2a]$.

For interval notation see Appendix A.

When using a calculator you must pay attention to the domain corresponding to a function key or command. If you enter a negative number as x and press the $\sqrt{x}$-key to calculate the square root of x your calculator will not be happy. It might display an E for "error" or start flashing, the calculator's standard signal for distress. Your error was entering a number not in the domain of the square root function.

Try it. What does your calculator do? Some advanced calculators go into "complex number" mode to handle square roots of negative numbers.

You can also get into trouble if you enter 0 and press the $1/x$-key. The domain of $1/x$, the reciprocal function, consists of all numbers — except 0.

Try it. *No* calculator, however advanced, can permit division by zero.

Graph of a Function

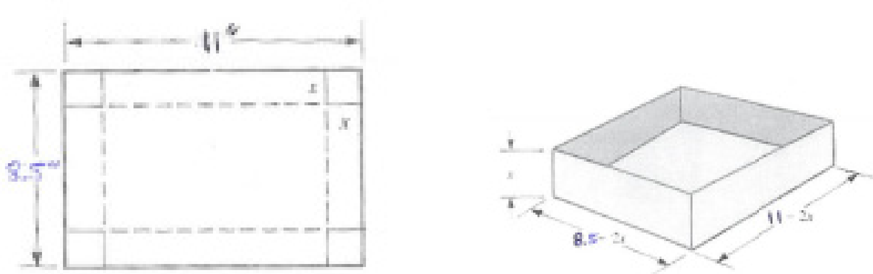
In case both the inputs and outputs of a function are numbers, we can draw a picture of the function, called its **graph**.

> **DEFINITION** *(Graph of a function)* Let f be a function whose inputs and output are numbers. The **graph** of f consists of those points (x, y) in the xy-plane such that $y = f(x)$.

The next example illustrates the usefulness of a graph. We will encounter this function again in Chapter 4.

EXAMPLE 2 A tray is to be made from a rectangular piece of paper. by cutting congruent squares from each corner and folding up the flaps. The dimensions of the rectangle are $8\frac{1}{2}'' \times 11''$. Find how the volume of the tray depends on the size of the cutout squares.

SOLUTION Let the side of the cutout square be x inches, as shown in Figure 1.1.4(a). The resulting tray is shown in Figure 1.1.4(b).



(a) (b)

Figure 1.1.4: (a) A rectangular sheet with a square cutout from each corner. (b) The tray formed when the sides are folded.

The volume $V(x)$ of the tray is the height, x, times the area of the base $(11 - 2x)(8.5 - 2x)$,

$$V(x) = x(11 - 2x)(8.5 - 2x). \tag{1}$$

The domain of V contains all values of x that lead to an actual tray. This means that x cannot be negative, and x cannot be more than half of the shortest side.

Note the peculiar trays that are obtained when $x = 0$ or $x = 4.25$.

Thus, the largest corners that can be cut out have sides of length 4.25". So, for this tray problem, the domain of interest is only the interval $[0, 4.25]$.

Of course we are free to graph (1) viewed simply as a polynomial whose domain is $(-\infty, \infty)$.

A short table of inputs and corresponding outputs will help us to sketch the graph. Figure 1.1.5 displays the graph of $V(x)$.

x (in)	-1	0	1	2	3	4	4.25	5	6
$V(x)$ (in^3)	-136.5	0	5.85	63	37.5	6	0	-7.5	21

Table 1.1.1:

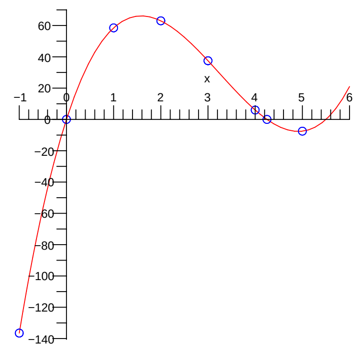


Figure 1.1.5:

When $11 - 2x = 0$, that is, when $x = \frac{11}{2} = 5.5$, $V(x) = 0$. When x is greater than $\frac{11}{2}$ all three factors in the formula for $V(x)$ are positive, and $V(x)$ becomes very large for large values of x.

For negative x, two factors in (1) are positive and one is negative. Thus $V(x)$ is negative and has large absolute value for negative inputs of large absolute value.

Which factor is negative?

Only the part of the graph above the interval $[0, 4.25]$ is meaningful in the tray problem. All other values of x have nothing to do with trays. ◇

The fact that $V(x) > 0$ for $x > 5.5$ is irrelevant. Why?

If you want to test whether some curve drawn in the xy-plane is the graph of a function, check that each vertical line meets the curve no more than once. If the vertical line $x = a$ meets the curve twice, say at (a, b) and (a, c), there would be the two outputs b and c for the single input a.

What does it mean if the vertical line $x = a$ never intersects the curve?

Vertical Line Test The input a is in the domain of f if and only if the vertical line $x = a$ intersects the graph of $y = f(x)$ exactly once. Otherwise, a is not in the domain of f.

In Example 2 the function is described by a single formula, $V(x) = x(11 - 2x)(8.5 - 2x)$. But a function may be described by different formulas for different intervals or individual points in its domain, as in the next example.

EXAMPLE 3 A hollow sphere of radius a has mass M, distributed uniformly throughout its surface. Describe the gravitational force it exerts on a particle of mass m at a distance r from the center of the sphere.

SOLUTION Let $f(r)$ be the force at a distance r from the center of the sphere. In an introductory physics course it is shown that the sphere exerts no force at all on objects in the interior of the sphere. Thus for $0 \le r < a$, $f(r) = 0$.

The sphere attracts an external particle as though all the mass of the sphere were at its center. Thus, for $r > a$, $f(r) = G\frac{Mm}{r^2}$, where G is the gravitational

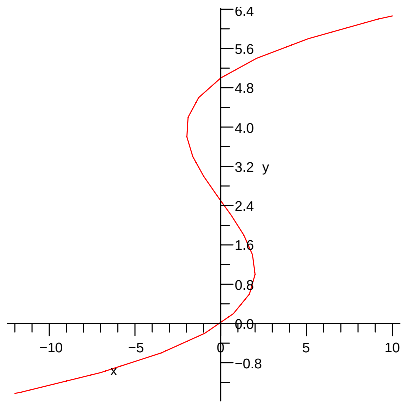


Figure 1.1.6: The input-output table corresponding to this graph would have three entries for each input $-2 < x < 2$, two entries for $x = -2$ and $x = 2$ and exactly one entry for each input $x < -2$ or $x > 2$.

constant, which depends on the units used for measuring length, time, mass, and force.

> In the SI system (Systeme International), the standard for length is the meter, for time is the second, and for mass is the kilogram. Force is measured in Newtons. One Newton, N, is the force required to impart an acceleration of one meter per second per second to a mass of one kilogram. $G \approx 6.67x10^{-11} Nm^2/kg^2$. (See `http://en.wikipedia.org/wiki/Gravitational_constant`.)

It can be shown by calculus that for a particle on the surface, that is, for $r = a$ the force is $G\frac{Mm}{2a^2}$. The graph of f is shown in Figure 1.1.7. A solid dot indicates that point is present; a hollow dot, that that point is absent. ⋄

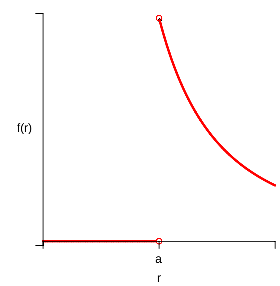


Figure 1.1.7:

Inverse Functions

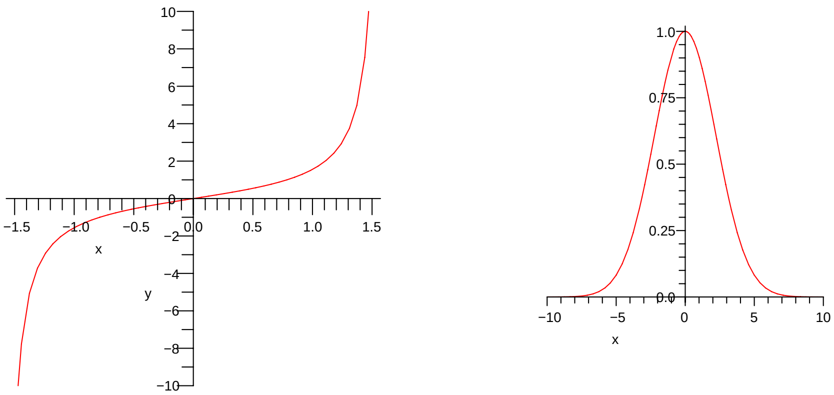
If you know a particular output of the function $f(x) = x^3$ you can figure out what the input must have been. For instance, if $x^3 = 8$, then $x = 2$ – you can go backwards from output to input. However, you cannot do this with the function $f(x) = x^2$. If you are told that $x^2 = 25$, you do not know what x is. It can be 5 or -5. However, if you are told that $x^2 = 25$ and that x is positive, then you know that x is 5.

This brings us to the notion of a one-to-one function.

See also Appendix B.

> **DEFINITION** *(One-to-one function)* A function f that does not assign the same output to two different inputs is **one-to-one**. That is, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

<u>Horizontal Line Test</u> The graph of a one-to-one function never meets a horizontal line more than once. (See Figure 1.1.8.)



(a) (b)

Figure 1.1.8: The function in (a) is one-to-one because it passes the horizontal line test. The function in (b) does not pass the horizontal line test, so it is not one-to-one.

The function $f(x) = x^3$ is one-to-one on the entire real line. A few entries in the tables for $f(x)$ and its inverse function are shown in Table 1.1.2(a) and (b), respectively.

input	1	2	$\frac{1}{2}$	3	-2
output	1	8	$\frac{1}{8}$	27	-8

(a)

input	1	8	$\frac{1}{8}$	27	-8
output	1	2	$\frac{1}{2}$	3	-2

(b)

Table 1.1.2: (a) Table of input and output value for $f(x) = x^3$. (b) Table of input and output values for the inverse of $f(x) = x^3$.

In this case an explicit formula for the inverse function can be found algebraically: if $y = x^3$ then $y^{1/3} = (x^3)^{1/3} = x$. Then $x = y^{1/3}$. Since it is customary to use the x-axis for the input and the y-axis for the output, it is convenient to rewrite $x = y^{1/3}$ as $y = x^{1/3}$.

Both say the same thing: "The output is the cube root of the input.".)

By the way, an inverse of a one-to-one function may not be given by a nice formula. For instance, $f(x) = 2x + \cos x$ is one-to-one, as will be easily shown by a technique presented in Chapter 4. However, the inverse function is not described by a convenient formula. Happily, we do not need to deal with an explicit formula for this particular inverse function.

The Graph of an Inverse Function

Suppose you know the graph of a one-to-one function. Then there is an easy way to draw the graph of the inverse function.

If (a, b) is a point on the graph of the function f, that is, $b = f(a)$, then (b, a) is a point on the graph of $\mathsf{inv}f$, shown in Figure 1.1.9.

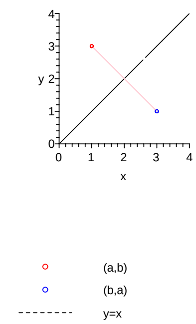


Figure 1.1.9: Geometrically, the point (b, a) is obtained by reflecting (a, b) around the line $y = x$.

Notation: The use of $\mathsf{inv}f$ to denote the inverse function of f is based on the fact that many calculators have a button marked inv to indicate the inverse of a function. The mathematical notation for the inverse function of f is f^{-1}. *Note that the -1 is not an exponent, and in general the inverse and reciprocal functions are different: f^{-1} is not equal to $\frac{1}{f}$.*

EXAMPLE 4 Draw the graphs of (a) the inverse of the cubing function given by $f(x) = x^3$, and (b) the squaring function $g(x) = x^2$ restricted to $x \geq 0$.

SOLUTION See Figure 1.1.10. ◇

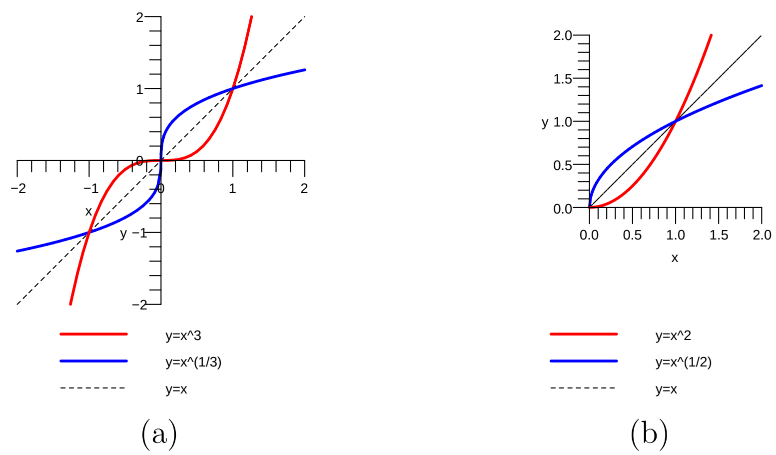


Figure 1.1.10: (a) Plots of $f(x) = x^3$ and $f^{-1}(x) = x^{1/3}$. (b) Plots of $g(x) = x^2$ $(x \geq 0)$ and $g^{-1}(x) = \sqrt{x}$.

EXAMPLE 5 Let $m \neq 0$ and b be constants and $f(x) = mx + b$. Show that f is one-to-one and describe its inverse function.

SOLUTION If $f(x_1) = f(x_2)$ we have

$$\begin{aligned} mx_1 + b &= mx_2 + b & \\ mx_1 &= mx_2 & \text{subtract } b \text{ from both sides} \\ x_1 &= x_2 & \text{divide both sides by } m \neq 0 \end{aligned}$$

Because $f(x_1) = f(x_2)$ only when $x_1 = x_2$, f is one-to-one.

This problem can also be analyzed graphically. The graph of $y = f(x)$ is the line with slope m and y-intercept b. (See Figure 1.1.11.)

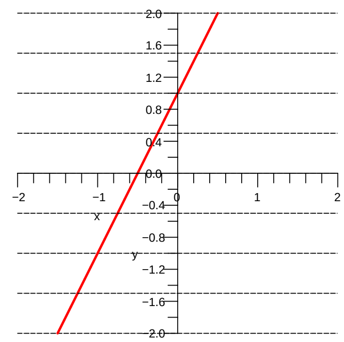


Figure 1.1.11:

To find the inverse function, solve the equation $y = f(x)$ to express x in terms of y:

$$\begin{aligned} y &= mx + b & & \\ y - b &= mx & & \text{subtract } b \text{ from both sides} \\ \tfrac{y-b}{m} &= x & & \text{divide by } m \neq 0 \\ x &= \tfrac{y}{m} - \tfrac{b}{m} & & \text{move } x \text{ to left-hand side} \\ y &= \tfrac{x}{m} - \tfrac{b}{m} & & \text{interchange } x \text{ and } y. \end{aligned}$$

Reversing the roles of x and y in the final step is done only to present the inverse function in a form where the input is called x and the output is called y. Thus the inverse function has the formula

$$f^{-1}(x) = \frac{x}{m} - \frac{b}{m}.$$

The graph of the inverse function is also a line; the slope is $1/m$ and the y-intercept is $-b/m$. ◇

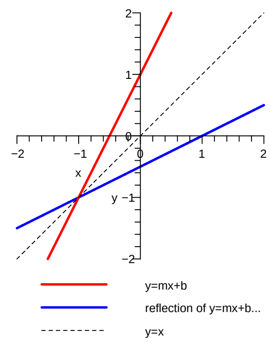


Figure 1.1.12:

OBSERVATION *(Reflecting a line of slope m)* If you reflect a line of slope $m \neq 0$ across the line $y = x$, you obtain a line of slope $1/m$, the reciprocal of m. This is shown in Figure 1.1.12.

This observation will play an important role in Section 3.5.

Decreasing and Increasing Functions

There is another way to check whether a function is one-to-one on an interval. It uses the following concepts.

A function is **increasing** on an interval if whenever x_1 and x_2 are in the interval and x_2 is greater than x_1, then $f(x_2)$ is greater than $f(x_1)$. As you move along the graph of f from left to right, you go up. This is shown in Figure 1.1.13(a). In the case of a **decreasing** function, you go down as you move from left to right: if $x_2 > x_1$ then $f(x_2) < f(x_1)$. (See Figure 1.1.13(b).)

For instance, consider $f(x) = \sin x$, whose graph is shown in Figure 1.1.14. On the interval $[-\pi/2, \pi/2]$ the values of $\sin x$ increase. On the interval $[\pi/2, 3\pi/2]$ the values of $\sin x$ decrease. The function x^3 increases on its entire domain $(-\infty, \infty)$.

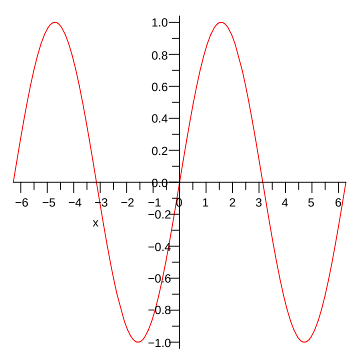
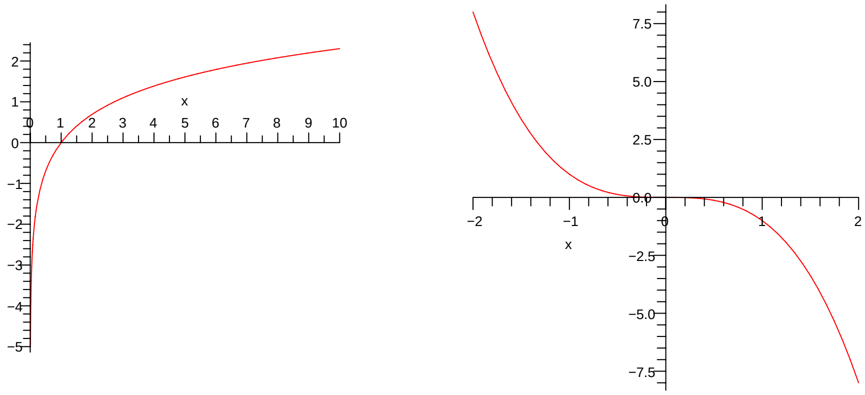


Figure 1.1.14:

EXAMPLE 6 For $k \neq 0$ and $x > 0$, x^k is a monotonic function. The inverse of x^k is $x^{1/k}$. If $k = 0$, we have a constant function, $x^0 = 1$. The

Power functions, x^k, will be discussed in greater detail in Section 2.2.



(a) (b)

Figure 1.1.13: Graph of (a) an increasing function and (b) a decreasing function.

constant function does not pass the Horizontal Line Test; therefore it has no inverse. ◇

A **monotonic** function is a function that is either only increasing or only decreasing.

Monotone is from the Greek, *mono*=single, *tonos*=tone, which also gives us the word 'monotonous').

Because strict inequalities are used in the definitions of increasing and decreasing, we sometimes say these functions are *strictly increasing* or *strictly decreasing* on an interval. A function f is said to be **non-decreasing** on an interval if $f(x_2) \geq f(x_1)$ for all $x_2 > x_1$ in that interval. The graph of a non-decreasing function is generally increasing except at points where it can be constant. Likewise, f is **non-increasing** on an interval if $f(x_2) \leq f(x_1)$ for all $x_2 > x_1$ in the interval.

Summary

This section introduced concepts that will be used throughout the coming chapters: function, domain, range, graph, one-to-one functions are invertible, increasing functions, decreasing functions, and monotonic functions. The graph of the inverse function is the reflection across the line $y = x$ of the graph of the original function.

A function can be described in several ways: by a formula, such as $V(x) = x(11 - 2x)(8.5 - 2x)$, by a table of values, or by words, such as "the volume of a tray depends on the size of the cut-out square."

EXERCISES for 1.1

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

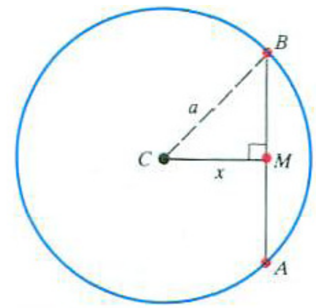


Figure 1.1.15: Exercises 1–4.

Exercises 1–4 refer to Figure 1.1.15.

1. Express the circle of triangle ABC as a function of $x = \overline{CM}$

2. Express the perimeter of triangle ABC as a function of x.

3. Express the area of triangle ABC as a function of θ.

4. Express the perimeter of triangle ABC as a function of θ.

In Example 2 a tray was formed from an 8" by 11" rectangle by removing squares from the corners. Find and graph the corresponding volume function for trays formed from sheets with the dimensions given in Exercises 5–8.

5. 4" by 13"

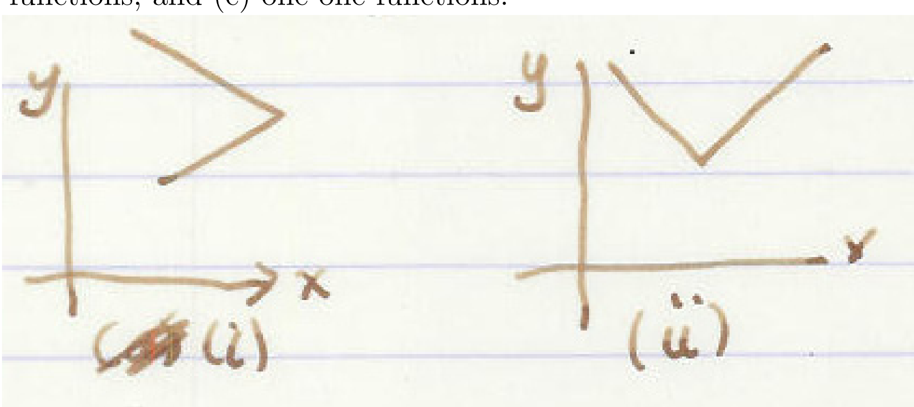
6. 5" by 7"

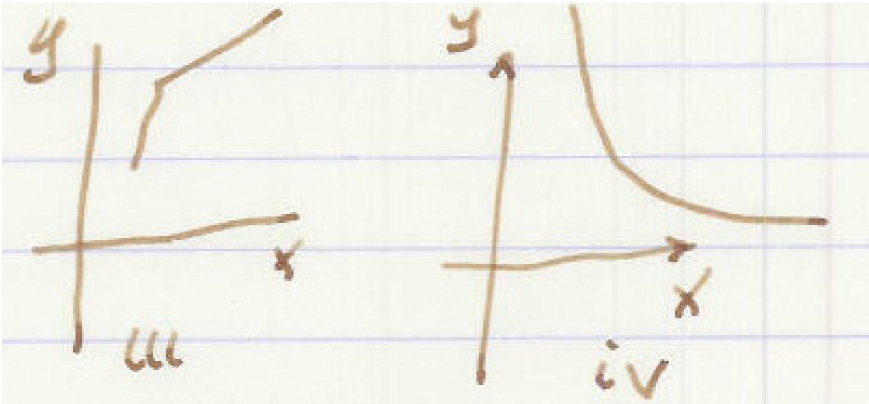
7. 6" by 6"

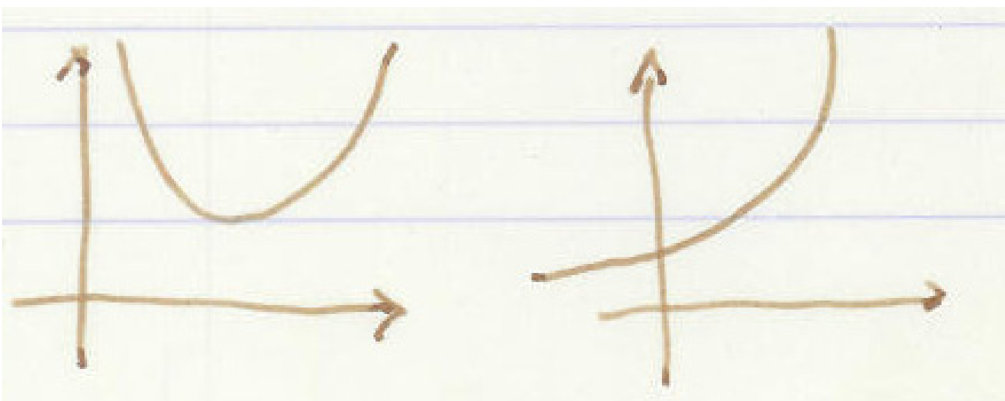
8. 5" by 5"

In Exercises 9 and 10 decide which curves are graphs of (a) functions, (b) increasing functions, and (c) one-one functions.

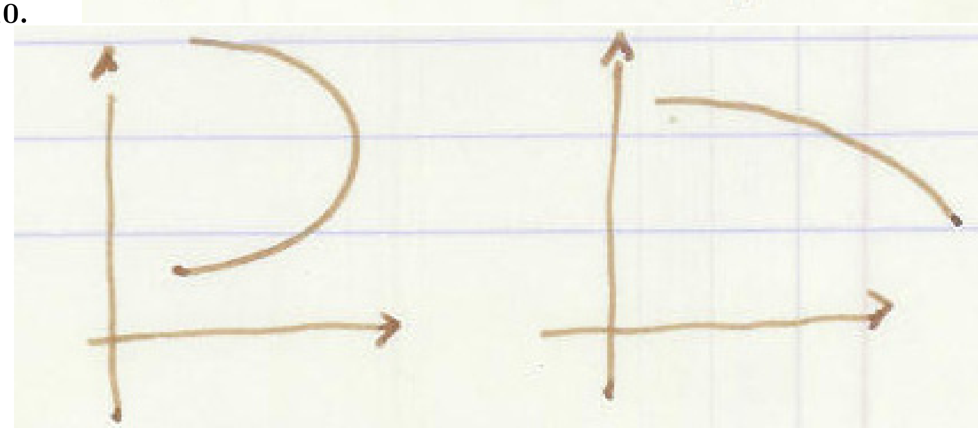
9.







10.



11. Let $f(x) = x^3$.

(a) Fill in this table

x	0	1/4	1/2	$-1/4$	$-1/2$	1	2
x^3							

(b) Graph f.

(c) Use the table in (a) to find seven points on the graph of f^{-1}.

(d) Graph f^{-1} (use the same axes as in (b)).

12. Let $f(x) = \cos x$, $0 \leq x \leq \pi$ (angles in radians).

(a) Fill in this table

x	0	$\pi/6$	$\pi/4$	$2\pi/3$	$\pi/2$	$3\pi/4$	π
$\cos x$							

(b) Graph f.

(c) Use the table in (a) to find seven points on the graph of $\mathsf{inv}\cos$.

(d) Graph $\mathsf{inv}\cos$ (use the same axes as in (b)).

In Exercises 13–18 the functions are one-one. Find the formula for each inverse function, expressed in the form $y = g(x)$, so that the independent variable is labeled x. NOTE: If you have trouble with the use of logarithms in Exercise 17 or Exercise 18, read Appendix D.

13. $y = 3x - 2$

14. $y = x/2 + 7$

15. $y = x^5$

16. $y = 3\sqrt{x}$

17. $y = 3^x$

18. $y = 5(2^x)$

In Exercises 19–23 the slope of line L is given. Let L' be the reflection of L across the line $y = x$. What is the slope of the reflected line, L'? In each case sketch a possible L and its reflection, L'.

19. L has slope 2.

20. L has slope 1.

21. L has slope 0.

22. L has slope -1/3.

23. L has slope -2.

24. A camper at A will walk to the river, put some water in a pail at P, and take it to the campsite at B. Where should P be located to minimize the length of the walk, $AP + PB$? (See Figure 1.1.16.) HINT: Reflect B across the line L. NOTE: Calculus-based methods for finding minima and maxima will be developed

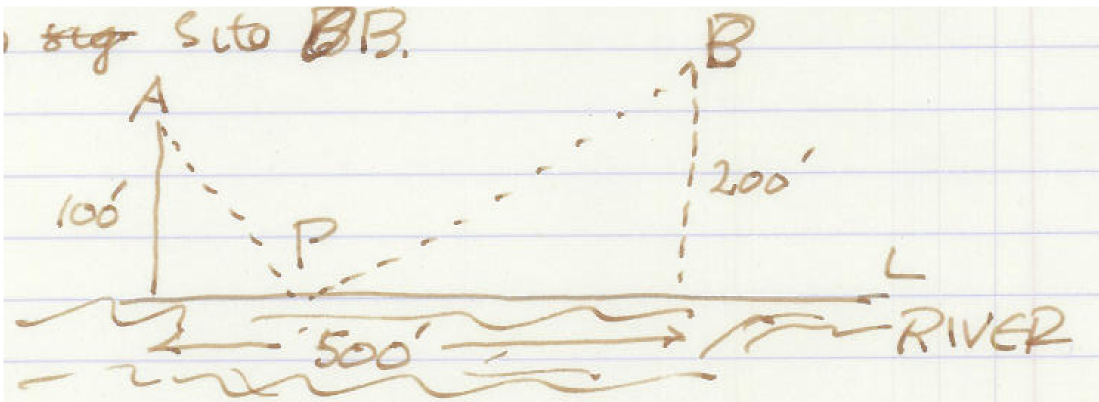
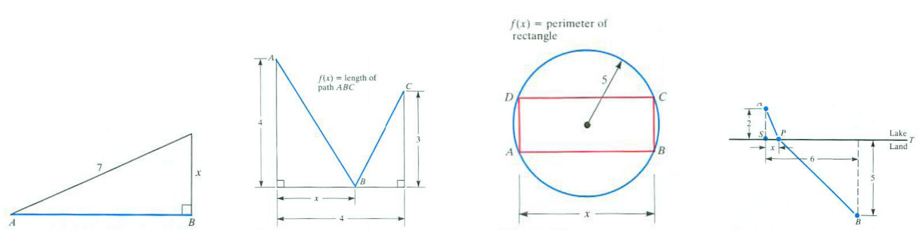


Figure 1.1.16: Sketch of situation in Exercise 24.

in Chapter 4.

In Exercises 25–34 state the formula for the function f and give the domain of the function.



(a) (b) (c) (d)

Figure 1.1.17:

25. $f(x)$ is the perimeter of a circle of radius x.

26. $f(x)$ is the area of a circle of radius x.

27. $f(x)$ is the perimeter of a square of side x.

28. $f(x)$ is the volume of a cube of side x.

29. $f(x)$ is the total surface area of a cube of side x.

30. $f(x)$ is the length of the hypotenuse of the right triangle whose legs have lengths 3 and x.

31. $f(x)$ is the length of the side AB in the triangle in Figure 1.1.17(a).

32. For $0 \leq x \leq 4$, $f(x)$ is the length of the path from A to B to C in Figure 1.1.17(b).

33. For $0 \leq x \leq 10$, $f(x)$ is the perimeter of the rectangle $ABCD$, one side of which has length x, inscribed in the circle of radius 5 shown in Figure 1.1.17(c).

34. A person at point A in a lake is going to swim to the shore ST and then walk to point B. She swims at 1.5 miles per hour and walks at 4 miles per hour. If she reaches the shore at point P, x miles from S, let $f(x)$ denote the time for her combined swim and walk. Obtain an algebraic formula for $f(x)$. (See Figure 1.1.17(d).)

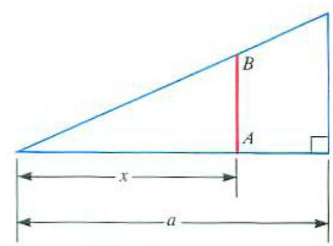


Figure 1.1.18: Exercise 35.

35. Let $f(x)$ be the length of the segment AB in Figure 1.1.18.

(a) What are $f(0)$ and $f(a)$?

(b) What is $f(a/2)$?

(c) Find the formula for $f(x)$ and explain your solution.

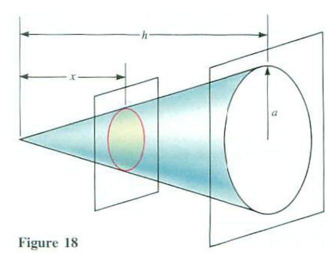


Figure 1.1.19: Exercise 36.

36. Let $f(x)$ be the area of the right circular cone cross section in Figure 1.1.19.

(a) What are $f(0)$ and $f(h)$?

(b) Find a formula for $f(x)$ and explain your solution.

In Exercises 37–41 give (a) three functions that satisfy the equation throughout their domain and (b) one function that does not.

37. $f(x+y) = f(x) + f(y)$.

38. $f(x+y) = f(x)f(y)$

39. $f(xy) = f(x) + f(y)$

40. $f(x+y) = f(x)f(y)$

41. $f(x) = f(y)$

42. The cost of life insurance depends on whether the person is a smoker or a non-smoker. The following chart lists the annual cost for a male for a million-dollar life insurance policy.

age (yrs)	20	30	40	50	60	70	80
cost for smoker ($)	1150	1164	1944	4344	9864	26500	104600
cost for non-smoker ($)	396	396	600	1490	3684	10900	41600

NOTE: A "smoker" is a person who has used tobacco during the previous three years.

(a) Plot the data and sketch the graphs on the same axes for both groups of males.

(b) A smoker at age 20 pays as much as a non-smoker of about what age?

(c) A smoker pays about how many times as much as a non-smoker of the same age?

43. The cost of a ride in a New York city taxi is described by this formula: NOTE: The cost also depends on other factors. For every two minutes stopped in traffic, 40 cents is added. During the evening rush, 4–8 pm, there is a surcharge of one dollar. Between 8 pm and 6 am there is a surcharge of 50 cents. So the cost, which depends on distance travelled, time stopped, and time of day, is actually a function of three variables.) At the start the meter reads $2.50. For every fifth of a mile, 40 cents is added. Graph the cost as a function of distance travelled.

44. If f is an increasing function, what, if anything, can be said about f^{-1}?

45. On a typical summer day in the Sacromento Valley the temperature is at a minimum of 60° at 7 AM and a maximum of 95° at 4 PM.

(a) Sketch a graph that shows how the temperature may vary during the twenty-four hours from midnight to midnight.

(b) A closed shed with little insulation is in the middle of a treeless field. Sketch a graph that shows how the temperature inside the shed may vary during the same period.

(c) Sketch a graph that shows how the temperature in a well-insulated house may vary. Assume that in the evening all the windows and skylights are opened when the outdoor temperature equals the indoor temperature, and closed in the morning when the two temperatures are again equal.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Air Temp (°)	56	60	68	76	83	88	91	89	85	77	69	60
Water Temp (°)	51	52	57	62	69	77	81	83	80	73	65	55

Table 1.1.3: Source: `http://www.myrtle-beach-resort.com/weather.htm`

46. The monthly average air and water temperatures in Myrtle Beach, SC, are shown in Table 1.1.3. NOTE: Assume, for convenience, that the temperatures in the table are the temperatures on the first day of each month.

(a) Sketch a graph that shows how the water temperature may vary during one calendar year, that is, from January 1 through December 31.

(b) Sketch a graph that shows how the difference between the air and water temperatures may vary during one calendar year. During what month is the water temperature difference greatest? least?

(c) During February, the water temperature increases 5° in 28 days so the average daily change is $5/28 \approx 0.1786°/\text{day}$. For each month, estimate the average daily change in the water temperature from one day to the next. NOTE: For example, During which month is this daily change greatest? least?

(d) Repeat (b) and (c) for the air temperature data.

47. This problem grew out of a question raised by the daughter of one of the authors, when cutting cloth for a dress. She wanted to cut out two congruent semicircles from a long strip of fabric 44 inches wide, as shown in Figure 1.1.20. The radius of the semicircles determines d, the length of fabric used.

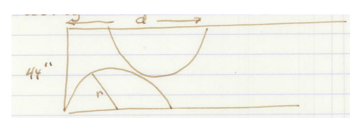


Figure 1.1.20: Exercise 47.

(a) Draw a picture to show $f(22) = 44$.

(b) For $0 \leq r \leq 22$, determine d as a function of r, $d = f(r)$.

(c) For $22 \leq r \leq 44$, determine d as a function of r, $d = f(r)$.

(d) Obtain an equation expressing r as a function of d.

(e) She had 104 inches of fabric, and guesed that the largest semicircle she could cut set has a radius of about 30 inches. Use (c) to see how good her guess is.

48. Let $g(d)$ be the radius of the largest pair of semicircles with diameters on the edge of the fabric, if the fabric is d inches long and 44 inches wide. The domain of

g is $(0, \infty)$. Find g and graph it. NOTE: This is related to Exercise 47.

49. A solar cooker for campers can be made out of a 60°-section of a sphere whose radius is 2 feet. A typical cross section is sketched in Figure 1.1.21. The cooker's axis AO is aimed toward the sum. A ray of light BD is reflected off the surface at B. Angle OBD equals angle OBC.

(a) Express the length OC as a function of θ.

(b) How long a hot dog can be placed on the radius AO and be heated by the cooker?

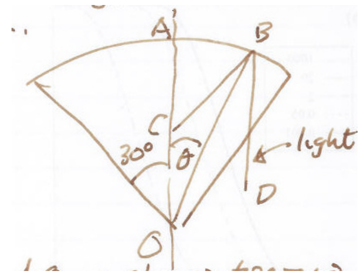


Figure 1.1.21: Exercise 49.

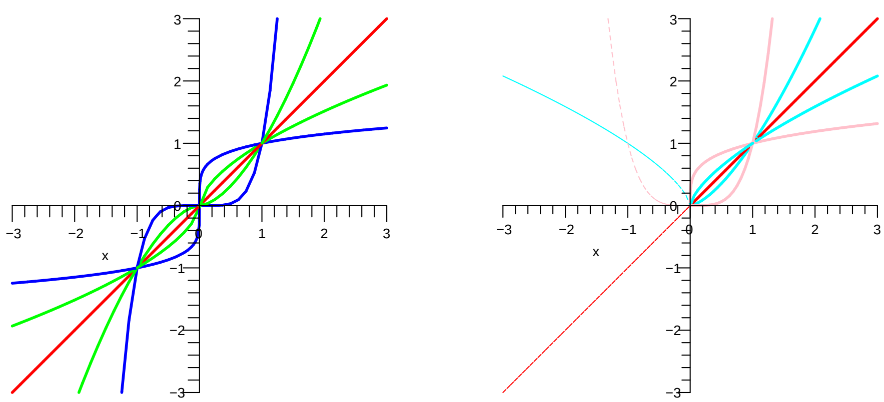
1.2 The basic functions of calculus

This section describes the basic functions in calculus. In the next section you will see how to use these functions as building blocks for more complicated functions.

The Power Functions

The first group of functions consists of the **power functions** x^k where the exponent k is a fixed non-zero number and the base x is the input. If k is an odd integer, then x^k has an inverse, $x^{1/k}$, another power function. If k is an even integer and we restrict the domain of x^k to the positive numbers, then it is one-to-one, and has an inverse, again $x^{1/k}$, with, again, a domain consisting of all positive numbers.

In Section 2.1 it was shown that the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ for all x. Notice, however, $g(x) = x^4$ does not pass the horizontal line test unless the domain is restricted to, say, positive inputs ($x \geq 0$). Thus, the inverse of $g(x) = x^4$ is $g^{-1}(x) = x^{1/4}$ only for $x \geq 0$.



(a) (b)

Figure 1.2.1: Graphs of power functions. (a) x^k for $k = 1$ (red), 5 (blue), 1/5 (blue), 5/3 (green), and 3/5 (green). (b) x^k for $k = 1$ (red), 4 (pink), 1/4 (pink), 3/2 (aqua), and 2/3 (aqua). Note that the pairs of blue and green graphs are inverses in (a), as are the pairs of (solid) pink and aqua graphs in (b). In (b) the graphs of x^4 and $x^{2/3}$ pass the horizontal line test only for $x \geq 0$, and the graphs of $x^{1/4}$ and $x^{3/2}$ are defined only for $x \geq 0$.

OBSERVATION *(Inverses of Power Functions)*

1. The inverse of a power function is another power function.

2. When $k = 0$, we obtain the function x^0, which is constant (with all outputs equal to 1), the very opposite of being one-to-one. Constant functions are discussed in more detail in Section 1.3.

3. When the exponent k is an even integer or a rational number (in lowest terms) whose numerator is even (2/3, 4/7, etc.) the graph of $y = x^k$ will not pass the horizontal line test unless the domain is restricted to $x \geq 0$.

When k is an even integer or a rational number (in lowest terms) whose numerator is even, $(-x)^k = x^k$. That's why these power functions are not one-to-one throughout $(-\infty, \infty)$.

A review of exponential and logarithmic functions is in Appendix D.

The Exponential and Logarithm Functions

Next we have the **exponential functions** b^x where the base b is fixed and the exponent x is the input. The inverses of exponential functions are not exponential functions. The inverses are called **logarithms** and are the next class of functions that we will consider.

Consider a function of the form b^x, where b is positive and fixed. In order to be concrete, let's take the case $b = 2$, that is, $f(x) = 2^x$.

As x increases, so does 2^x. So the function 2^x has an inverse function. In other words, if $y = 2^x$, then if we know the output y we can determine the input x, the exponent, uniquely. For instance, if $2^x = 8$ then $x = 3$. This is expressed as $3 = \log_2 8$ and it read as "the logarithm of 8, base 2, is 3." If $y = b^x$, then we write $x = \log_b y$.

Since we usually denote the independent variable (the input or argument) by x, and the dependent variable (the output, or value) by y, we will rewrite this as $y = \log_b x$.

The table of easy values of $\log_2 x$ in Table 1.2.1 will help us graph $y = \log_2 x$. Putting a smooth curve through the seven points in Table 1.2.1 yields the graph in Figure 1.2.2.

x	1	2	4	8	1/2	1/4	1/8
$\log_2 x$	0	1	2	3	-1	-2	-3

Table 1.2.1: Table of easy values of $y = \log_2 x$.

As x increases, $\log_2 x$ grows very slowly. For instance $\log_2 1024 = 10$, as every computer scientist knows. For x between 0 and 1, $\log_2 x$ is negative. As x moves from 1 towards 0, $|\log_2 x|$ grows very large. For instance, $\log_2 \frac{1}{1024} = -10$.

Because $y = 2^x$ and $y = \log_2 x$ are inverse functions, we could have sketched the graph of $y = \log_2 x$ by first sketching the graph of $y = 2^x$ and reflecting it around the line $y = x$.

For any positive base b, $\log_b x$ is defined similarly.

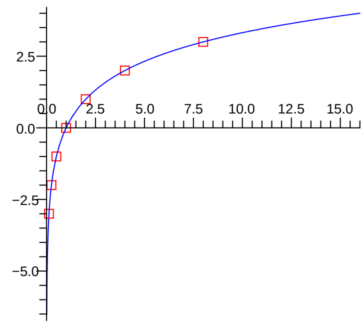


Figure 1.2.2: Plot of $y = \log_2 x$ based on data in Table 1.2.1.

For x and b both positive numbers, the logarithm of x to the base b, denoted $\log_b x$, is the power to which we must raise b to obtain x. By the very definition of the logarithm

$$b^{\log_b x} = x.$$

Whenever you see "$\log_b x$" you should think, "Ah, ha! The fancy name for an exponent."

The Trigonometric Functions and Their Inverses

So far we have the power functions, x^k, the exponential functions, b^x, and the logarithm functions, $\log_b x$. The last major group of important functions consists of the trigonometric functions, $\sin x$, $\cos x$, $\tan x$, and their inverses (after we shrink their domains to make the functions one-to-one).

$\sin x$ and its inverse

In calculus we generally measure angles in radians. See also Appendix E.

The graph of the **sine function** $\sin x$ has period 2π and is shown in Figure 1.2.2. The range is $[-1, 1]$. On the domain $[-\pi/2, \pi/2]$, $\sin x$ is increasing and its values for these inputs already sweep out the full range, $[-1, 1]$.

When we restrict the domain of the function $\sin x$ to $[-\pi/2, \pi/2]$ it is a one-to-one function with range $[-1, 1]$. This means the sine function has an inverse with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$. The inverse sine function is denoted by $\arcsin x$, $\sin^{-1} x$, or $\mathsf{inv}\sin x$.

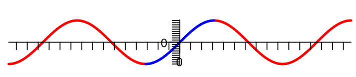


Figure 1.2.3:

Let's stop for a moment to summarize our findings: For x in $[-1, 1]$, $\arcsin x$ is the angle in $[-\pi/2, \pi/2]$ whose sine is x. In equations:

$$y = \arcsin x \iff \sin y = x.$$

For instance, $\arcsin 1 = \pi/2$ because the angle in $[-\pi/2, \pi/2]$ whose sine is 1 is $\pi/2$. Similarly, $\sin^{-1}(1/2) = \pi/6$, $\mathsf{inv}\sin 0 = 0$, $\arcsin(-1/2) = -\pi/6$, $\sin^{-1}(-1) = -\pi/2$.

We could graph $y = \arcsin x$ with the aid of these five values. However, it's easier just to reflect the graph of $y = \sin x$ around the line $y = x$. (See Figure 1.2.4.)

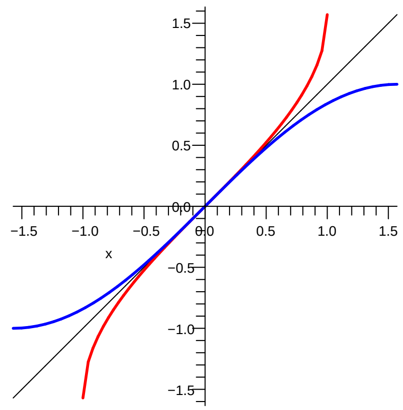


Figure 1.2.4: The graph of $y = \sin^{-1} x$ is the graph of $y = \sin x$ on $[-\pi/2, \pi/2]$ reflected around the line $y = x$.

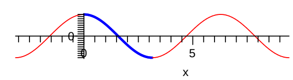


Figure 1.2.5:

$\cos x$ and its inverse

The graph of the **cosine function** $\cos x$ is shown in Figure 1.2.5.

It is clearly not one-to-one, even if we restrict the domain to the domain used for $\sin x$, namely $[-\pi/2, \pi/2]$. In this case note that $\cos x$ is a decreasing function on $[0, \pi]$. So the cosine function is one-to-one on $[0, \pi]$. Moreover, the values of $\cos x$ for x in $[0, \pi]$ sweep out all possible values of the cosine function, namely $[-1, 1]$.

Because $\cos x$ is a one-to-one function on the domain $[0, \pi]$, cosine has an inverse function, called $\arccos x$, $\mathsf{inv} \cos x$, or simply $\cos^{-1} x$. Each of these is short for "the angle (in radians) in $[0, \pi]$ whose cosine is x". For instance, $\cos^{-1} 0 = \pi/2$, $\cos^{-1} 1 = 0$, and $\cos^{-1}(-1) = \pi$. Moreover, because the range of the cosine function is the closed interval $[-1, 1]$, the domain of arccos is $[-1, 1]$. Figure 1.2.6 shows that the graph of $\cos^{-1} x$ is obtained by reflecting the graph of $\cos x$, with domain $[0, \pi]$, around the line $y = x$.

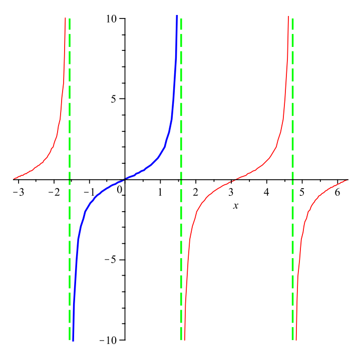


Figure 1.2.7:

$\tan x$ and its inverse

The range of the function $\tan x = \frac{\sin x}{\cos x}$ is $(-\infty, \infty)$, as Figure 1.2.7 shows.

When the inputs are restricted to $(-\pi/2, \pi/2)$, $\tan x$ is one-to-one, and therefore has an inverse function, denoted $\arctan x$, $\tan^{-1} x$, or $\mathsf{inv} \tan x$. The domain of the inverse tangent function is $(-\infty, \infty)$ and its range is $(-\pi/2, \pi/2)$.

For instance, $\tan^{-1}(0) = 0$, $\tan^{-1} 1 = \pi/4$, and as x increases, $\tan^{-1} x$ approaches $\pi/2$. Also, $\tan^{-1}(-1) = -\pi/4$, and when x is negative and becomes

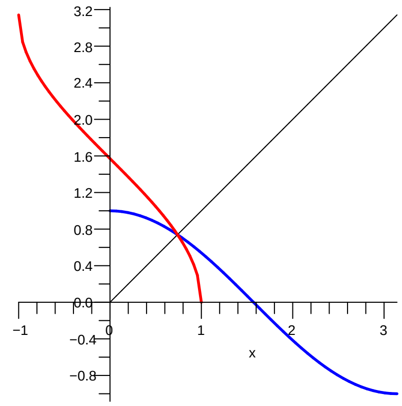


Figure 1.2.6: The graph of $y = \cos^{-1} x$ is defined for x in $[-1, 1]$. It is obtained by reflecting the graph of $y = \cos x$, with domain restricted to $[0, \pi]$, around the line $y = x$.

ever more negative (that is, $|x|$ becomes bigger and bigger) $\tan^{-1} x$ approaches $-\pi/2$. Figure 1.2.8 is the graph of $\tan^{-1} x$. It is the reflection of the blue part of the graph in Figure 1.2.7 across the line $y = x$. (See Figure 1.2.8.)

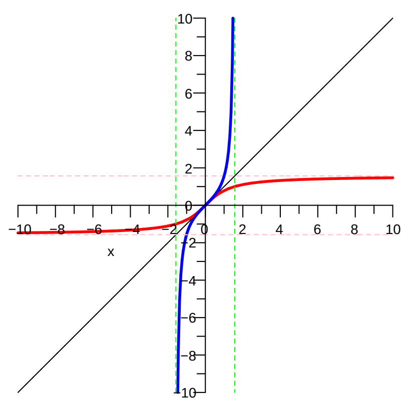


Figure 1.2.8:

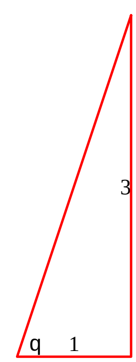


Figure 1.2.9: The traditional symbol for angles is the Greek letter θ (pronounced "theta").

EXAMPLE 1 Evaluate (a) $\sin(\sin^{-1} 0.3)$, (b) $\sin(\tan^{-1} 3)$, and (c) $\tan(\cos^{-1}(0$

SOLUTION

(a) The expression $\sin^{-1}(0.3)$ is short for the angle in the interval $[-\pi/2, \pi/2]$ whose sine is 0.3. So, the sine of $\sin^{-1}(0.3)$ is 0.3.

(b) To find $\sin(\tan^{-1} 3)$, first draw the angle θ whose tangent is 3 (and lies in the interval $[-\pi/2, \pi/2]$. Figure 1.2.9 shows a simple way to draw this angle. To find the sine of θ, recall that sine equals "opposite/hypotenuse."

By the Pythagorean Theorem, the hypotenuse is $\sqrt{3^2+1^2}=\sqrt{10}$. Thus, $\sin(\tan^{-1} 3) = 3/\sqrt{10}$.

(c) To evaluate $\tan(\cos^{-1} 0.4)$, first draw an angle whose cosine is $0.4 = \frac{2}{5}$, as in Figure 1.2.10, which is based on the fact that "$\cos\theta =$ adjacent/hypotenus To find the tangent of this angle, we need the length of the other leg in Figure 1.2.10. By the Pythagorean Theorem this length is $\sqrt{5^2-2^2} = \sqrt{21}$.

From the relation $\tan\theta =$ opposite/adjacent, we conclude that $\tan(\cos^{-1} 0.4) = \sqrt{21}/2 \approx 2.291$.

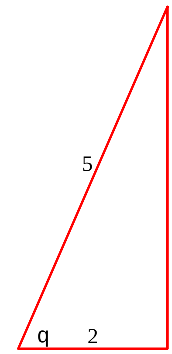


Figure 1.2.10:

$\diamond$

WARNING *(about notation for inverse functions)* We use "arcsin(x)" to denote "the angle whose sine is x." There is another common notation, "$\sin^{-1} x$." Unfortunately, this may be read as $(\sin x)^{-1}$, the reciprocal of $\sin x$. After all, $\sin^2 x$ means $(\sin x)^2$. Though the notation "$\sin^{-1} x$" is shorter than "$\arcsin x$," we prefer the latter to avoid the risk of misinterpretation. Similar comments apply to $\tan^{-1} x$ and $\arctan x$ and to $\cos^{-1} x$ and $\arccos x$. We will use both notations throughout this book.

$\csc x$, $\sec x$, and $\cot x$ and their inverses

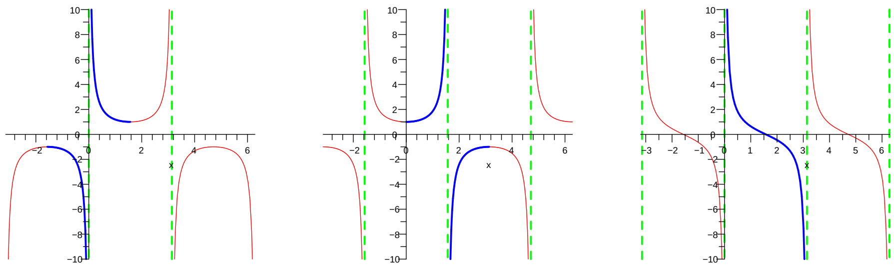
The **cosecant**, **secant**, and **cotangent** functions are defined in terms of the sine and cosine functions:

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \cot x = \frac{\sin x}{\cos x}.$$

While we could write $\csc x = (\sin x)^{-1}$, we do not because of the possible confusion with $\sin^{-1} x = \arcsin x$. Each of these functions is defined only when the denomonator is not zero. Figure 1.2.11 shows the graphs of these three functions.

Note that $|\sec x| \geq 1$ and $|\csc x| \geq 1$. In each case the range consists of two separate intervals: $[1,\infty)$ and $(-\infty,-1]$.

These three functions have inverses, when restricted to appropriate intervals. Table 1.2.2 contains a summary of the three inverse functions, $\csc^{-1} x$, $\sec^{-1} x$, and $\tan^{-1} x$. Figure 1.2.12 shows the graphs of csc, sec, and cot and their inverses.

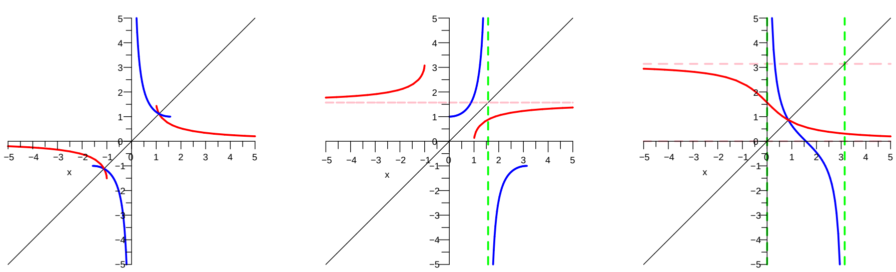


(a) (b) (c)

Figure 1.2.11: The graphs of (a) the cosecant, (b) the secant, and (c) the cotangent functions.

function	domain (input)	range (output)
$\csc^{-1} x$	$(-\infty, -1]$ and $[1, \infty)$	all of $[-\pi/2, \pi/2]$ except 0
$\sec^{-1} x$	$(-\infty, -1]$ and $[1, \infty)$	all of $[0, \pi]$ except 0
$\cot^{-1} x$	$(-\infty, \infty)$	the open interval $(0, \pi)$

Table 1.2.2: Summary of the inverse cosecant, inverse secant, and inverse cotangent functions.



(a) (b) (c)

Figure 1.2.12: Graphs of (a) $y = \csc x$ and $y = \csc^{-1} x$, (b) $y = \sec x$ and $y = \sec^{-1} x$, and (c) $y = \cot x$ and $y = \cot^{-1} x$. Notice how the inverse function is the reflection of the original function across the line $y = x$.

Summary

This section reviewed the basic functions in calculus, x^k, b^x, $\sin x$, $\cos x$, and $\tan x$, and their inverses $\log_b x$, $\arcsin x$, $\arccos x$, and $\arctan x$. (The inverse of x^k, $k \neq 0$, is just another power function $x^{1/k}$).

The functions that may be hardest to have a feel for are the logarithms. Now, $\log_2 x$ is typical of $\log_b x$, $b > 1$. These are its key features:

- its graph crosses the x-axis at $(1, 0)$ because $\log_2 1 = 0$ $(2^0 = 1)$,
- it is defined only for positive inputs, that is, the domain of $\log_2$ is $(0, \infty)$, because only positive numbers can be expressed in the form 2^x,
- it is always an increasing function,
- it grows very slowly as the argument increases: $\log_2 8 = 3$, $\log_2 16 = 4$, $\log_2 32 = 5$, $\log_2 64 = 6$, and $\log_2 1024 = 10$,
- for values of x in $(0, 1)$, $\log_2 x$ is negative ($x = 2^y < 1$ only when $y < 0$),
- as x gets near 0 (and is positive), $|\log_2 x|$ becomes very large.

The case when the base b is less than 1 is treated in Exercise 55.

EXERCISES for 1.2

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

1. Graph the power function $x^{3/2}$, $x \geq 0$, and its inverse.

2. Graph the power function x^5 and its inverse.

3. Explain your calculator's response when you try to calculate $\log_{10}(-3)$?

4. Explain your calculator's response when you try to calculate $\arcsin(2)$?

5.

(a) Graph 2^x and $(1/2)^x$ on the same axes.

(b) How could you obtain the second graph from the first?

6.

(a) Graph 3^x and $(1/3)^x$ are the same axes.

(b) How could you obtain the second graph from the first?

7. For any base b, $b^0 = 1$. What is the corresponding property of logarithms? Explain.

8. For any base b, $b^{x+y} = b^x b^y$. What is the corresponding property of logarithms?
NOTE: If you have trouble with Exercise 8, study Appendix D.

9. Explain why $\log_b(1/x) = -\log_b x$. ("The log of the reciprocal of x is the negative of the log of x.")

10. Explain why $\log_b(c^x) = x \log_b c$. ("The log of a number raised to a power x is x times the log of the number.")

11.

(a) Evaluate $\log_2 x$ and $\log_4 x$ at x=1, 2, 4, 8, 16, and 1/16.

(b) Graph $\log_2 x$ and $\log_4 x$ on the same axes (clearly label each point found in (a)).

(c) Compute $\frac{\log_4 x}{\log_2 x}$ for the six values of x in (a).

(d) Explain the phenomenon observed in (c).

(e) How would you obtain the graph of $\log_4 x$ from that for $\log_2 x$?

12.

(a) Evaluate $\log_2 x$ and $\log_8 x$ at x=1, 2, 4, 8, 16, and 1/8.

(b) Graph $\log_2 x$ and $\log_8 x$ on the same axes (clearly label each point found in (a)).

(c) Compute $\frac{\log_8 x}{\log_2 x}$ for the six values of x in (a).

(d) Explain the phenomena observed in (c).

(e) How would you obtain the graph of $\log_8 x$ from that for $\log_2 x$?

13. Evaluate

(a) $\log_{10} 1000$

(b) $\log_{100} 10$

(c) $\log_{10} 0.01$

(d) $\log_{10} \sqrt{10}$

(e) $\log_{10} 10$

14. Evaluate

(a) $\log_3(3^{17})$

(b) $\log_3(1/9)$

(c) $\log_3 1$

(d) $\log_3 \sqrt{3}$

(e) $\log_3 81$

15. Evaluate $5^{\log_5(17)}$.

16. Evaluate $3^{-\log_3(21)}$.

17. For large values of x, the three functions, $2^x, x^2$, and $\log_2 x$

18. For very large values of x what happens to the quotent $x^2/2^x$? Illustrate by using specific values for x.

19. What happens to $(\log_2 x)/x$ for large values of x? Illustrate by citing specific x.

20. What happens to $\frac{\log_4(x)}{\log_2(x)}$ as x increases.

21. Draw graphs of $\cos x$, x in $[0, \pi]$, and $\arccos x$ on the same axes.

22. Draw graphs of $\tan x$, x in $(-\pi/2, \pi/2)$, and $\arctan x$ on the same axes.

23. Evaluate

(a) $\sin^{-1}(1/2)$

(b) $\arcsin 1$

(c) $\mathsf{inv}\sin(-\sqrt{3}/2)$

(d) $\arcsin(\sqrt{2}/2)$

24. Evaluate

(a) $\cos^{-1} 0$

(b) $\mathsf{inv}\cos(-1)$

(c) $\arccos(1/2)$

(d) $\arccos(-1/\sqrt{2})$

25. Evaluate

(a) $\arctan 1$

(b) $\mathsf{inv}\tan(-1)$

(c) $\arctan\sqrt{3}$

(d) $\arctan 1000$ (approximately)

26. Evaluate

(a) $\operatorname{arcsec} 2$

(b) $\mathsf{inv}\sec(-2)$

(c) $\operatorname{arcsec}\sqrt{2}$

(d) $\sec^{-1} 1000$ (approximately)

27. Evaluate

(a) $\arcsin(0.3)$

(b) $\arccos(0.3)$

(c) $\arctan(0.3)$

(d) $\frac{\arcsin(0.3)}{\arccos(0.3)}$

NOTE: Observe that (c) and (d) are different.

28. Evaluate $\sin(\tan^{-1}(2))$.
29. Evaluate $\sin(\cos^{-1}(0.4))$.
30. Evaluate $\tan(\tan^{-1}(3))$.
31. Evaluate $\tan(\sin^{-1}(0.7))$.

32. Evaluate $\tan(\sec^{-1}(3))$.
33. Evaluate $\sec(\tan^{-1}(0.3))$.
34. Evaluate $\sin(\sec^{-1}(5))$.
35. Evaluate $\sec(\cos^{-1}(0.2))$.

36. Evaluate $\arctan(\tan \frac{\pi}{3})$.
37. Evaluate $\arcsin(\sin \frac{-3\pi}{4})$.
38. Evaluate $\arccos(\cos \frac{5\pi}{2})$.
39. Evaluate $\operatorname{arcsec}(\sec \frac{-\pi}{3})$.

In Exercises 40–43 express $\log_{10} f(x)$ as simple as possible.

40. $f(x) = \frac{(\cos x)^7\sqrt{(x^2+5)^3}}{4+(\tan x)^2}$
41. $f(x) = \sqrt{(1+x^2)^5(3+x)^4\sqrt(1+2x)}$
42. $f(x) = (x\sqrt{2+\cos x})^{x^2}$
43. $f(x) = \sqrt{\frac{x(1+x)}{\sqrt{1+2x}^3}}$

44. Imagine that your calculator fell on the floor and its multiplication and division keys stopped working. However, all the other keys, including the trigonometric, arithmetic, logarithmic, and exponential keys still functioned. Show how you would use your calculator to calculate the product and quotient of two positive numbers, a and b.

45. (Slide Rule) This exercise shows this may how to build a slide rule by exploring the indenty $\log_b xy = \log_b x + \log_b y$. We will use $\log_2$ for convenience.

Step 1. Mark on the bottom edge of a stick (or page) the numbers 2^0, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, and $2^4 = 16$, placing 2^n n cm from the left end. In other words, place each number x, at a distance $\log_2 x$ cm from the left edge. Figure 1.2.13 shows only numbers with convenient integer logarithms, with base 2.

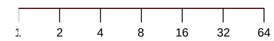


Figure 1.2.13:

Step 2. Do the same thing as the top edge of another stick or sheet of paper.

Step 3. You now have a slide rule. To compute 4×8, say, with your slide rule, slide the bottom stick along the top stick until its left edge is next to the 4 of the top stick. The product 4×8 appears above the 8 on the lower stick. Why?

46. (Richter Scale) In 1989, San Francisco and vicinity was struck by an earthquake that measured 7.1 on the **Richter scale**. The strongest earthquake in recent years had a Richter measure of 8.9 (Colombia-Equador in 1906 and Japan in 1933). A "major earthquake" typically has a measuer of at least 7.5.

In his *Introduction to the Theory of Seismology*, Cambridge, 1965, pp. 271–272, K. E. Bullen explains the Richter scale as follows:

"Gutenburg and Richter sought to connect the magnitude M with the energy E of an earthquake by the formula

$$aM = \log_{10}\left(\frac{E}{E_0}\right)$$

and after several revisions arrived in 1956 at the result $a = 1.5$, $E_0 = 2.5 \times 10^{11}$ ergs."
NOTE: Energy E is measured in ergs. M is the number assigned to the earthquake on the Richter scale. E_0 is the energy of the smallest instrumentally recorded earthquake.

(a) Deduce that $\log_{10} E \approx 11.4 + 1.5M$.

(b) What is the ratio between the energy of the earthquake that struck Japan in 1933 ($M = 8.9$) and the San Francisco earthquake of 1989 ($M = 7.1$)?

(c) What is the ratio between the energy of the San Francisco earthquake of 1906 ($M = 8.3$) and that of the San Francisco earthquake of 1989 ($M = 7.1$)?

(d) Find a formula for E in terms of M.

(e) If one earthquake has a Richter measure 1 larger than that of another earthquake, what is the ratio of their energies?

(f) What is the Richter measure of a 10-megaton H-bomb, that is, of an H-bomb whose energy is equivalent to that of 10 millon tons of TNT?

NOTE: One ton of TNT releases an energy of 4.2×10^6 ergs.

47. Translate the sentence, “She has a five-figure annual income” into logarithms. How small can the income be? How large?

48. As of 2006 the largest known prime was $2^{30402457} - 1$.

(a) When written in decimal notation, how many digits will it have?

(b) How many pages of this book would be needed to print it? (One page can hold about 6,400 digits.)

NOTE: There is a prize of $250,000 for the discovery of the first billion-digit prime. Do a Google search for “largest prime”.

49.

(a) In many calculators the log key refers to base-ten logarithms. You can use it to find logarithms to any base $b > 0$. To see why, start with the equation $b^{\log_b x} = x$ and then take $\log_{10}$ of both sides. This gives the formula

$$\log_b x = \frac{\log_{10} x}{\log_{10} b}.$$

(b) Use (a) to find $\log_3 7$. (Why should the result be between 1 and 2?)

(Semi-log graphs) In most graphs the scale on the y-axis is the same as the scale on the x-axis, or a constant multiple of it. However, to graph a rapidly increasing function, such as 10^x, it is convenient to “distort” the y-axis. Instead of plotting the point (x, y) at a height of, say, y inches, you plot it at a height of $\log_{10} y$ inches. So the datum $(x, 1)$ could be drawn with height O, the datum $(x, 10)$, would have height 1, and the datum $(x, 100)$ would have height 2 inches. Instead of graphs $y = f(x)$, you graph $Y = \log_{10} f(x)$. In particular, if $f(x) = 10^x$, $y = \log_{10} 10^x = x$: the graph would be a straight line. To avoid having to calculate a bunch of logarithms, it is convenient to use semi-log graph paper, shown in Figure 1.2.14.

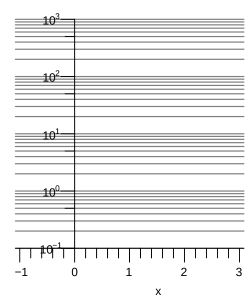


Figure 1.2.14:

50. Using semi-log paper, graph $y = 23^x$.

51. Using semi-log paper, graph $y = \frac{2}{3^x}$.

52. By the method described in Section 9.2, Newton computed the logarithms of 0.8, 0.9, 1.1, and 1.2 to 57 decimal places by hand.

(a) Show how to compute $\log 2$, using $\log 1.2$, $\log 0.8$ and $\log 0.9$.

(b) Show how to compute $\log 3$, using $\log 2$, $\log 1.2$ and $\log 0.8$.

(c) Show how to compute $\log 4$, using $\log 2$.

(d) Show how to compute $\log 5$, using $\log 2$ and $\log 0.8$.

(e) How would you then compute $\log 6$, $\log 8$, $\log 9$, and $\log 10$.

(f) How would you then estimate $\log 11$.

NOTE: You don’t need to know the base, why?

53. The graph of $y = \log_2 x$ consists of the part to the right of $(1, 0)$ and the part to the left of $(1, 0)$. Are the two parts congruent?

54. Say that you have drawn the graph of $y = \log_2 x$. Jane says that to get the graph of $y = \log_2(4x)$, you just raise that graph 2 units parallel to the y-axis. Sam says, "No, just shrink the x-coordinate of each part on the graph by a factor of 4." Who is right? Or are both wrong?

55. Answer the following questions about $y = \log_b(x)$ where $0 < b < 1$.

(a) Sketch the graphs of $y = b^x$ and $y = \log_b(x)$ on the same set of axes.

(b) What is the domain of $\log_b$?

(c) What is the x-intercept, that is, solve $\log_b(x) = 0$?

(d) For what values of x is $\log_b(x)$ positive? negative?

(e) Is the graph of $y = \log_b(x)$ an increasing or decreasing function?

(f) What can you say about the values of $\log_b(x)$ when x is close to zero (and in the domain)?

(g) What can you say about the values of $\log_b(x)$ when x is a large positive number?

(h) What can you say about the values of $\log_b(x)$ when x is a large negative number?

56. Let a, b, c, d be constants such that $ad - bc \neq 0$.

(a) Show that $y = (ax + b)/(cx + d)$ is one-to-one.

(b) For which a, b, c, d does the function in (a) equal its inverse function?

57. Prove that $\log_3 2$ is irrational. HINT: Assume that it is rational, that is, equal to m/n for some integers m and n, and obtain a contradiction.

1.3 Building more functions from basic functions

In this section we complete the list of functions needed for calculus. Our starting point is the basic functions introduced in Section 1.1. We will use just two general ideas – arithmetic and composition – to build more complicated functions from x^k, b^x, $\sin x$, $\cos x$, $\tan x$, and their inverses. For instance we will see how to obtain

$$f(x) = \frac{\sin(2x) + 3 + 4x + 5x^2}{\log_2 x + 3^{-5x} + \sqrt{1 + x^3}}. \tag{1}$$

Before we go into the details of how we construct new functions from old ones, we must introduce one more type of basic function. These functions are so simple, however, that they did not deserve to appear with the types in the preceding section. They are the constant functions. (See Figure 2.3.1.)

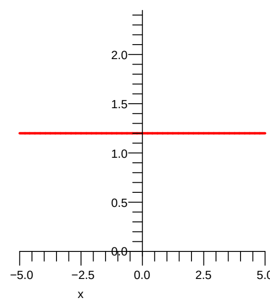


Figure 1.3.1:

The Constant Functions

DEFINITION *()* A function $f(x)$ is **constant** if there is a number C such that $f(x) = C$ for all x.

Using the Four Arithmetic Operations: $+$, $-$, x, $\div$

Given two functions f and g, we can produce other functions from them by using the four operations of arithmetic:

$f + g$ for an input value x, the function assigns $f(x) + g(x)$ as the output

$f - g$ for an input value x, the function assigns $f(x) - g(x)$ as the output

fg for an input value x, the function assigns $f(x)g(x)$ as the output

f/g for an input value x with $g(x) \neq 0$, the function assigns $f(x)/g(x)$ as the output

The domains of $f+g$, $f-g$, and fg consist of the numbers that belong to both the domain of f and the domain of g. The domain of f/g is a little different because division by zero is not defined. The function f/g is defined for all numbers x that belong to the domain of f and the domain of g with the extra condition that $g(x) \neq 0$.

With the aid of these constructions we can build any polynomial or rational function from the simple function $f(x) = x$, called the **identity function**, and the constant functions.

A **polynomial** is a function of the form $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where the coefficients a_0, a_1, a_2, $\ldots$, a_n are numbers. If a_n is not zero, the **degree** of the polynomial is n. A **rational function** is the quotient of two polynomials. The domain of a polynomial is all real numbers. The domain of a rational polynomial is all real numbers except those where the denominator is zero.

EXAMPLE 1 Use addition, subtraction, and multiplication to form the polynomial $F(x) = x^3 + 3x - 7$.

SOLUTION We first build each of the three terms: x^3, $3x$, and 7. The last of these is just a constant function. Multiplying the function x and the constant function 3 gives $3x$. The first term is obtained by first multiplying x and x to obtain x^2. Then multiplying x^2 and x yields x^3. Adding x^3 and $3x$ gives $x^3 + 3x$. Lastly, subtract the constant function 7 to obtain $x^3 + 3x - 7$.

Notice that each of the three functions involved in forming F is defined for all real numbers. As a result, the domain of F is also all real numbers, $(-\infty, \infty)$. ◇

Example 1 shows how to build any polynomial using $+$, $-$, and $\times$. Constructing rational functions also requires one use of the division operator.

But how would we build a function like $\sqrt{1+3x}$? This leads us to the most important technique for combining two or more functions to build more complicated functions.

Composite Functions

Given two functions f and g we can use the output of g as the input for f. That is, we can find $f(g(x))$. For instance, if $g(x) = 1+3x$ and f is the square root function, $f(x) = \sqrt{x}$, then $f(g(x)) = f(1+3x) = \sqrt{1+3x}$. This brings us to the definition of a composite function.

> **DEFINITION** *(Composition of functions)* Let X, Y, and Z be sets. Let g be a function from X to Y and let f be a function from Y to Z. Then the function that assigns to each element x in X the element $f(g(x))$ in Z is called the **composition** of f with g. It is denoted $f \circ g$, which is read as "f circle g" or as "f composed with g".

Some or all of the sets X, Y, and Z could be the same set.

Thinking of f and g as input-output machines we may consider $f \circ g$ as the machine built by hooking up the machine for f to process the outputs of the machine for g (see Figure 1.3.2).

Most functions we encounter are composititions. For instance, $\sin(2x)$ is the composition of $g(x) = 2x$ and $f(x) = \sin x$. Of course, we can hook up three or

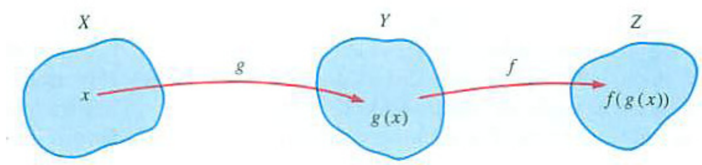


Figure 1.3.2: The output of the g machine, $g(x)$, becomes the input for the f machine. The result is the composition of f with g, $(f \circ g)(x) = f(g(x))$.

more functions to make even fancier functions. Consider $\sin^3(2x) = (\sin(2x))^3$. This function is built up as follows:

$$x \longrightarrow 2x \longrightarrow \sin(2x) \longrightarrow (\sin(2x))^3. \tag{2}$$

It is the composition of three functions: the first doubles the input, the second takes the sine of its input, and the third cubes its input.

The order does matter. If, instead, you cube first, then take the sine, and then double the input you obtain:

$$x \longrightarrow x^3 \longrightarrow \sin(x^3) \longrightarrow 2\sin(x^3). \tag{3}$$

When you enter a function on your calculator or on a computer, you have to be careful of the order in which the functions are applied as you evaluate a composite function. The specific way that you would evaluate $\sin(\log_{10} 240)$ on your calculator depends on your calculator. On a traditional scientific calculator you enter 240 followed by the log10 key, and finally the sin key. On many of the newer graphing calculators you would press the sin key followed by the log10 key, then 240, followed by two right parentheses,)), and, finally, the Enter key. Note that these two approaches are different. If you press the sin key before log10, you will get $\log_{10}(\sin 240)$. For most computer software it is necessary to use parentheses to indicate arguments to functions. In this case you might enter sin(log10(240)).

Before pressing the sin key, be sure to check that your calculator is in radians mode.
If your calculator is in degree mode, you will find that $\sin(240°) < 0$ and so $\log_{10}(\sin(240°))$ is not defined.

To describe the build-up of a composite function it is convenient to use various letters, not just x, to denote the variables. This is illustrated in Examples 2–4.

EXAMPLE 2 Show how the function $\sqrt{4 - x^2}$ is built up by the composition of functions. Find its domain.

SOLUTION The function $\sqrt{4 - x^2}$ is obtained by applying the square-root function to the function $4 - x^2$. To be specific, let

$$g(x) = 4 - x^2 \qquad \text{and} \qquad f(u) = \sqrt{u}\ (u \geq 0). \tag{4}$$

Then

$$f(g(x)) = f(4 - x^2) = \sqrt{4 - x^2}. \tag{5}$$

The square-root function is defined for all $u \geq 0$ and the polynomial $g(x)$ is defined for all numbers. So $f(g(x))$ is defined only when $g(x) \geq 0$:

$$\begin{aligned} g(x) &\geq 0 \\ 4 - x^2 &\geq 0 \\ 4 &\geq x^2 \\ 2 &\geq |x|. \end{aligned}$$

Thus, the domain of $\sqrt{4 - x^2}$ is the closed interval $[-2, 2]$. ◇

EXAMPLE 3 Express $1/\sqrt{1 + x^2}$ as a composition of three functions. Find the domain of this function.

SOLUTION Call the input x. First, we compute $1 + x^2$. Second, we take the square root of that output, getting $\sqrt{1 + x^2}$. Third, we take the reciprocal of that result, getting $1/\sqrt{1 + x^2}$. In summary, we form

$$u = 1 + x^2, \qquad \text{then} \quad v = \sqrt{u} \qquad \text{then} \quad y = \frac{1}{v}. \tag{6}$$

Given x, we first evaluate the polynomial $1 + x^2$, then apply the square-root function, then the reciprocal function.

The domain of a polynomial consists of all real numbers, the domain of the square-root function is $v \geq 0$, and the domain of the reciprocal function is all numbers except zero. Because $u = 1 + x^2 \geq 1$, v is defined for all x. Moreover, $v = \sqrt{1 + x^2} \geq 1$, so that $y = \frac{1}{v} = 1/\sqrt{1 + x^2}$ is defined for all real numbers x. ◇

The function in Example 3 can also be written as the composition of two functions: $x \longrightarrow 1 + x^2 \longrightarrow (1 + x^2)^{-1/2}$.

EXAMPLE 4 Let f be the cubing function and g the cube-root function. Compute $(f \circ g)(x)$, $(f \circ f)(x)$ and $(g \circ f)(x)$.

SOLUTION In terms of formulas, $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$.

$$(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x}) = \left(\sqrt[3]{x}\right)^3 = x. \tag{7}$$

$$(f \circ f)(x) = f(f(x)) = f(x^3) = \left(x^3\right)^3 = x^9. \tag{8}$$

$$(g \circ f)(x) = g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x. \tag{9}$$

Observe that the domains of f and g are $(-\infty, \infty)$. Therefore, each of $f \circ g$, $f \circ f$, and $g \circ f$ is defined for all real numbers.

Notice that both $f \circ g$ and $g \circ f$ are the identity function. Whenever g is the inverse of f, $f \circ g$ and $g \circ f$ are the identity function. Each function undoes the action of the other. ◇

EXAMPLE 5 Give two different ways of obtaining the function $1/f(x)$ from the function f.

SOLUTION The first approach is to view $1/f(x)$ as the quotient of the constant function 1 and the function $f(x)$.

This function can also be viewed as a composition. The quotient $1/f(x)$ can be obtained in two steps: First, evaluate $f(x)$. Second, take the reciprocal of the result. So, if $g(x) = 1/x$, then

$$\frac{1}{f(x)} = g(f(x)). \tag{10}$$

Regardless of the way in which $1/f(x)$ is constructed, the domain is all real numbers for which $f(x) \neq 0$. ◇

The observation that $1/f(x)$ can be expressed as a composition will be used in Section 3.4.

EXAMPLE 6 Show that every power function x^k, $x > 0$, can be constructed as a composition.

SOLUTION The first step is to write $x = 2^{\log_2 x}$. Then, $x^k = \left(2^{\log_2 x}\right)^k$ or, using the properties of exponentials, $x^k = 2^{k \log_2 x}$. So x^k is the composition of three functions: First, find $\log_2 x$, then multiply by the constant function k, and then raise 2 to the resulting power. ◇

That a power function can be expressed in terms of an exponential function will be used in Chapter 4.

OBSERVATION *(Consequences of Example 6)*

1. While the inputs in Example 6 are restricted to positive numbers, this construction of the power functions provides a meaning to functions like $x^{\sqrt{2}}$ and $x^{-\pi}$.
2. As a result of Example 6 we could remove the power functions from our list of basic functions in Section 1.1. We choose not to do so because power functions with integer exponents are so common and in many instances we want to define a power function for all numbers (not just positive numbers).
3. It might seem surprising that the power functions can be expressed in terms of exponentials (and logarithms). An even more astonishing result is that trigonometric functions, such as $\sin x$, can also be expressed in terms of exponentials, as shown in Section 12.8.

Why? Because the domain of $\log_2$ contains only positive numbers.

WARNING *(Traveler's Advisory)* Be careful when composing functions when one of them is a trigonometric function. For instance, what is meant by $\sin x^3$? Is it $\sin(x^3)$ or $(\sin x)^3$? Do we first cube x, then take the sine, or the other way around? There is a general agreement that $\sin x^3$ stands for $\sin(x^3)$; you cube first, then take the sine.

Spoken aloud, $\sin x^3$ is usually "the sine of x cubed," which is ambiguous. We can either insert a brief pause – "sine of (pause) x cubed" – to emphasize that x is cubed rather than $\sin x$, or rephrase it as "sine of the quantity x cubed."

On the other hand $(\sin x)^3$, which is by convention usually written as $\sin^3 x$, is spoken aloud as "the cube of $\sin x$" or "sine cubed of x."

Similar warnings apply to other trigonometric functions and logarithmic functions.

Summary

This section showed how to build more complicated functions from power, exponential, and trigonometric functions and their inverses and the constant functions. One method is to simply add, multiply, subtract, or divide outputs. The other method is the "composition of functions" in which one function operates on the output of a second function. Composite functions are extremely important, especially when we calculate derivatives and integrals beginning in Chapter 3.

EXERCISES for 1.3

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

The function $y = \sqrt{1+x^2}$ is the composition of $s = 1 + x^2$ and $y = \sqrt{s}$. In Exercises 1–12 use a similar format to build the given functions as the composition of two or more functions.

1. $\sin 2x$

2. $\sin^3 x$

3. $\sin 3x$

4. $\sin(x^3)$

5. $\sin^2(x^3)$

6. 2^{x^2}

7. $(x^2+3)^{10}$

8. $\log_{10}(1+x^2)$

9. $1/(x^2+1)$

10. $\cos^3(2x+3)$

11. $(\frac{2}{3x+5})^3$

12. $\arcsin(\sqrt{x})$

13. These tables show some of the values of functions f and g:

x	1	2	3	4	5
$f(x)$	6	8	9	7	10

x	6	7	8	9	10
$g(x)$	4	3	2	5	1

(a) Find $f(g(7))$.

(b) Find $g(f(3))$.

14. Figure 1.3.3 shows the graphs of functions f and g.

(a) Estimate $f(g(0.6))$.

(b) Estimate $f(g(0.3))$.

(c) Estimate $f(f(0.5))$.

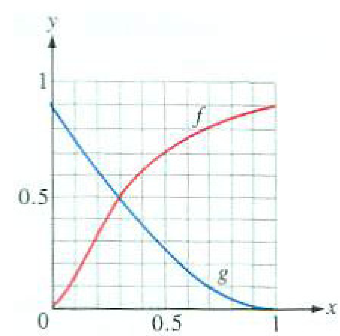


Figure 1.3.3:

In Exercises 15 and 16 write y as a function of x.

15. $u = \sin x$, $y = x^2$

16. $u = x^2$, $y = 2^x$

17. Let $f(x) = 2x^2-1$ and $g(x) = 4x^3-3x$. Show that $(f\circ g)(x) = (g\circ f)(x)$. [Rare indeed are pairs of polynomials that commute with each other under composition, as you may convince yourself by trying to find more examples.] NOTE: Of course, any two powers, such as x^3 and x^4, commute. (The composition of x^3 and x^4 in any order is x^{12}, as may be checked.)

18. Let $f(x) = 1/(1-x)$. What is the domain of f? of $f \circ f$? of $f \circ f \circ f$? Show that $(f \circ f \circ f)(x) = x$ for all x in the domain of $f \circ f \circ f$.

19. Let $g(x) = x^2$. Find all first degree polynomials $f(x) = ax + b$, $a \neq 0$, such that $f \circ g = g \circ f$, that is, $f(g(x)) = g(f(x))$.

20. Let $f(x) = x^3$. Is there a function $g(x)$ such that $(f \circ g)(x) = x$ for all numbers x? If so, how many such functions are there?

21. Let $f(x) = x^4$. Give two functions $g(x)$ such that $(f \circ g)(x) = x$ for all numbers x?

22. Figure 1.3.4 shows the graph of a function f whose domain is $[0, 1]$. Let $g(x) = f(2x)$.

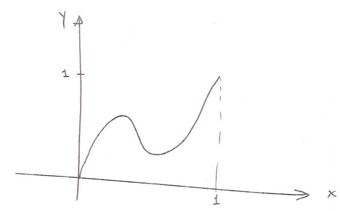


Figure 1.3.4:

(a) What is the domain of g?

(b) Graph $y = g(x)$

23. Show that there is a function $u(x)$ such that $\cos x = \sin u(x)$. NOTE: This shows that we didn't need to include $\cos x$ among our basic functions.

24. Find a function $u(x)$ such that $3^x = 2^{u(x)}$.

25. If f and g are one-to-one, must $f \circ g$ be one-to-one?

26. If $f \circ g$ is one-to-one, must f be one-to-one? Must g be one-to-one?

27. If f has an inverse, $\mathsf{inv}f$, compute $(f \circ \mathsf{inv}f)(x)$ and $((\mathsf{inv}f) \circ f)x$.

28. Let $g(x) = x^2$. Find all second-degree polynomials $f(x) = ax^2 + bx + c$, $a \neq 0$, such that $f \circ g = g \circ f$, that is, $f(g(x)) = g(f(x))$.

29. Let $f(x) = 2x + 3$. How many functions are there of the form $g(x) = ax + b$, a and b constants, such that $f \circ g = g \circ f$?

30. Let $f(x) = 2x + 3$. How many functions are there of the form $g(x) = ax^2 + bx + c$, a, b, and c constants, such that $f \circ g = g \circ f$?

31. Find all functions of the form $f(x) = 1/(ax+b)$, $a \neq 0$, such that $(f\circ f\circ f)(x) = x$ for all x in the domain of $f \circ f \circ f$. NOTE: See also Exercise 18. Ans: $1/(k - k^2x)$,

$k \neq 0$.

32. (Induction) This exercise rests as the identifies $\sin(x+y) = \sin x \cos y + \cos x \sin y$, $\cos(x+y) = \cos x \cos y - \sin x \sin y$, and $\cos^2 x + \sin^2 x = 1$.

(a) Show that $\sin 2x = 2 \sin x \cos x$ and $\cos 2x = 2\cos^2 x - 1$.

(b) Show that $\sin 3x = 3 \sin x - 4 \sin^3 x$ and $\cos 3x = 4 \cos 3x - 3 \cos x$.

(c) Show that $\sin 4x = \cos x(4 \sin x - 8 \sin^3 x)$ and $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$.

(d) Use induction to show that for each positive integer n, $\cos nx$ is a polynomial in $\cos x$. That is, there is a polynomial P_n such that $\cos nx = P_n(\cos x)$. NOTE: You will have to consider the form of $\sin nx$, n odd or even, in the induction.

(e) Explain why $P_n \circ P_m = P_m \circ P_n$. NOTE: This does not require the explicit formulas for P_n and P_m.

1.4 Graphing Functions Using A Graphing Calculator or Computer Algebra System

New technology has changed how and where mathematics is used. Graphing calculators provide an easy way to graph of a function. Computer algebra systems (CAS) such as Maple, Mathematica, and Derive can perform symbolic operations on mathematical expressions: for example, the factoring a polynomial

$$x^5 - 2*x^4 - 2*x^3 + 4*x^2 + x - 2 = (x-1)^2(x+1)^2(x-2),$$

expressing the quotient of two polynomials as the sum of simpler quotients

$$\frac{36}{x^5 - 2*x^4 - 2*x^3 + 4*x^2 + x - 2} = \frac{-3}{(x+1)^2} - \frac{9}{(x-1)^2} - \frac{4}{x+1} + \frac{4}{x-2},$$

solving nonlinear equations, such as

$$\arctan(x^2+1) = \pi/3 \qquad \text{and} \qquad \sin\left(\frac{\pi}{x}\right) - \frac{\pi}{x}\cos\left(\frac{\pi}{x}\right) = 0.$$

Some of these symbolic features are now available on some calculators, PDAs, telephones, and other handheld devices.

In this section we illustrate some of the problems that you may meet when using technology to graph a function. Calculus, especially the tools developed in Chapter **??**, will help you resolve those problems. We also provide general guidelines on the use of a graphing calculator or computer algebra system (CAS) to graph a function. The graphing utility needs to know the function and the **viewing window**. We will show by three examples some of the obstacles you may run into and how to avoid them.

The two basic pieces of information needed by the graphing routine are the function and the **viewing window**. The viewing window is the portion of the xy-plane to be displayed. We will say the viewing window is $[a,b]\times[c,d]$ when the window extends horizontally from $x=a$ to $x=b$ and vertically from $y=c$ to $y=d$. The graph of a function $y=f(x)$ is created by evaluating $f(x)$ for a sample of numbers x between a and b. The point $(x, f(x))$ is added to the plot. It is customary to connect these points to form the graph of $y=f(x)$. The examples in the remainder of this section demonstrate some of the things that can happen, and the steps you need to be prepared to take to avoid them.

EXAMPLE 1 Find a viewing window that shows the general shape of the graph of $y = x^4 + 6x^3 + 3x62 - 12x + 4$. Use graphs to estimate the location of the rightmost x intercept.

SOLUTION Figure 1.4.1 is typical of the first plot of a function. Choose

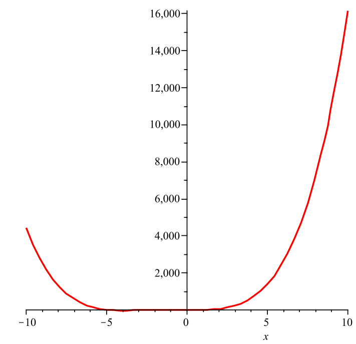


Figure 1.4.1:

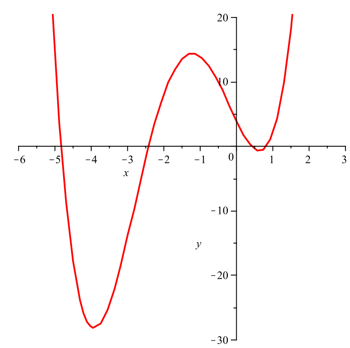


Figure 1.4.2:

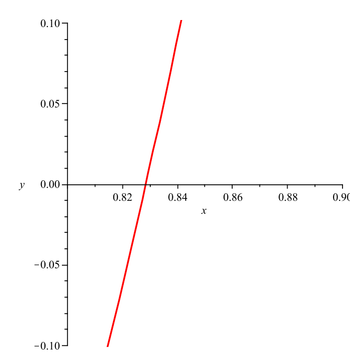


Figure 1.4.3:

a fairly wide x interval, here $[-10, 10]$, and let the graphing software choose an appropriate vertical range. While this view is useless for estimating any specific x intercept, it is clear that any x intercepts will be between $x = -6$ and $x = 3$. Figure 1.4.2 is the graph of this function on the viewing window $[-6, 3] \times [-30, 30]$. Now four x intercepts are visible. The rightmost one occurs around $x = 0.8$. Figure 1.4.3 is the result of zooming in on this part of the graph. From this view we estimate that the rightmost x intercept occurs when $x \approx 0.83$.

In fact, using a CAS, the four x intercepts for this function are found to occur at $x = 0.828427$, $x = 0.414213$, $x = -2.414213$, and $x = -4.828427$. ◇

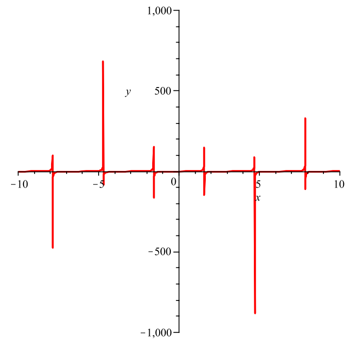


Figure 1.4.4:

Generating a collection of points and connecting the dots can sometimes lead to ridiculous results, as in Example 2.

EXAMPLE 2 Find a viewing window that clearly shows the general shape and periodicity of the graph of $y = \tan(x)$.

SOLUTION A computer-generated plot of $y = tan(x)$ for x between -10 and 10 with no vertical height of the viewing window is shown in Figure 1.4.4. This graph is not periodic; it looks more like an echocardiogram than the graph of one of the six standard trigonometric functions.

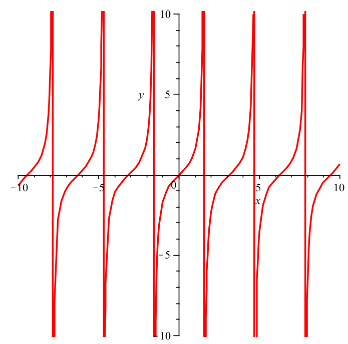


Figure 1.4.5:

Notice that the default vertical height is very long: $[-1000, 1000]$. Reducing this by a factor of 100, that is, to $[-10, 10]$, yields Figure 1.4.5. This graph is periodic and exhibits the expected behavior.

The "vertical" lines in Figure 1.4.5 are not really vertical. These are generated by the software by joining the last point to the left of the asymptote and the first point to the right of the asymptote. These segments are not really a part of the graph. Figure 1.4.3 shows the graph of $y = \tan(x)$ with these extraneous segments removed. ◇

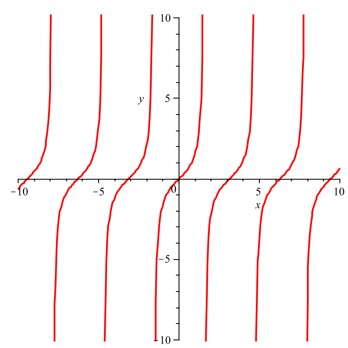


Figure 1.4.6:

Example 2 illustrates why we must never become complacent about using technology. We have to check the results and verify that they are consistent with the rest of our knowledge.

The last example shows that sometimes it is not possible to show all of the important features of a function in a single graph.

EXAMPLE 3 Use one or more graphs to show all major features on the graph $y = \sqrt[3]{x^2 - 8}e^{-x}$.

SOLUTION The graph of this function on the x interval $[-10, 10]$ with the vertical window chosen by the software is shown in Figure 1.4.7. In this window, the exponential function dominates the graph.

At $x = 0$ the value of the function is $(0 - 8)^{1/3}e^0 = -2$. To get enough detail to see both the positive and negative values of the function, zoom in by reducing the x interval to $[-5, 5]$. The result is Figure 1.4.8. Reducing the x interval to $[-4, 4]$ and specifying the y interval as $[-15, 15]$ gives Figure 1.4.9.

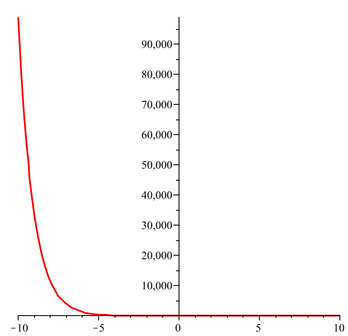


Figure 1.4.7:

We could continue to adjust the viewing window until we found suitable views. A more systematic approach is to look at the graphs of $y = \sqrt[3]{x^2 - 8}$ and $y = e^{-x}$ separately, but on the same pair of axes. (See Figure 1.4.10(a).) The exponential growth of e^{-x} for negative values of x stretches (vertically) the graph of $y = \sqrt[3]{x^2 - 8}$ to the left of the y axis while the exponential decay for $x > 0$ (vertically) compresses the graph of $y = \sqrt[3]{x^2 - 8}$ to the right of the y axis.

It is prudent to produce two separate plots to represent the sketch of this function. To the left of the y axis, with a viewing window of $[-4, 0] \times [-15, 100]$, the graph of the function is shown in Figure 1.4.10(b). To the right of the y axis, with a much shorter viewing window of $[0, 4] \times [-2.2, 0.2]$, the graph is as shown in Figure 1.4.10(c).

◇

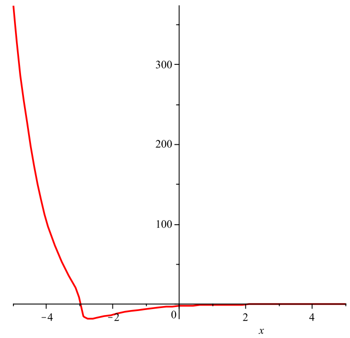


Figure 1.4.8:

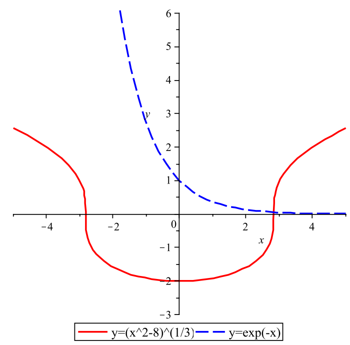
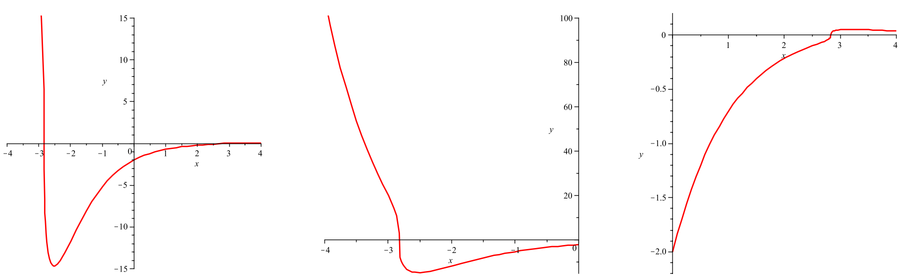


Figure 1.4.9:

Summary

The purpose of this brief section is two-fold. First, to give you some explicit pointers for using a graphing calculator or CAS to generate the graph of a function.

The second purpose is to make you aware that these tools do exist, and are likely to become more powerful and more widespread. These tools can change the way mathematics is used in the real world. The ability to factor a polynomial or to solve an equations will be less important than the ability to apply basic principles of mathematics and science to set up and analyze the equations.



(a) (b) (c)

Figure 1.4.10:

EXERCISES for 1.4

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

Use a graphing utility to sketch a graph of the functions in Exercise 1 to 10. Be sure to indicate the viewing window used to generate your graph.

1. $(x^2 + x - 6)\ln(x + 2)$

2. $(x^2 - x + 6)\ln(x + 2)$

3. $(x^2 - 4)\ln(x + 1)$

4. $(x^2 + 4)\ln(x + 1)$

5. $\frac{x^3 \arctan(x/5)}{x^2-4}$

6. $\frac{(x62-4)\arctan(x/5)}{x^3}$

7. $\frac{x^3-3x}{x^2-4}$

8. $\frac{x^3-2x}{x^2-4}$

9. $\frac{x^3-3x}{x^2-4}\arctan\left(\frac{x}{4}\right)$

10. $\frac{x^3-2x}{x^2-4}\arctan\left(\frac{x}{4}\right)$

Modern graphing utilities provide for the graphing of piecewise defined functions. Read the directions for your graphing software to learn how to do this. Then use your graphing utility to sketch a graph of the piecewise-defined functions in Exercises 11 and 12.

11. $y = \begin{cases} x^2 - x & x < 1 \\ \sqrt{x-1} & x \geq 1 \end{cases}$

12. $y = \begin{cases} \frac{\sin(x)}{x} & x < 1 \\ \sin x & 0 \leq x \geq \pi \\ x - 2 & x > \pi \end{cases}$

Some graphing utilities have trouble plotting functions with fractional exponents. Some general rules to follow when graphing $y = x^{p/q}$ where p/q is a positive fraction in lowest terms are:

- If p is even and q is odd, then graph $y = |x|^{p/q}$.
- If p and q are both odd, then graph $y = \frac{|x|}{x}|x|^{p/q}$.

Use these general facts to sketch the graph of each function in Exercises 13 to 16.

13. $y = x^{1/3}$

14. $y = x^{2/3}$

15. $y = x^{4/7}$

16. $y = x^{3/7}$

1.S Chapter Summary

The text and exercises for the summary will be written after the organization of the chapters is firmly settled.

EXERCISES for 1.S *Key:* R–routine, M–moderate, C–challenging

Chapter 2

Introduction to Calculus

There are two main concepts in calculus: the derivative and the integral. Underlying both is the concept of a limit. This chapter introduces limits, with an emphasis on developing both your intuitive understanding of limits and techniques for finding limits.

The four limits introduced in Section 2.1 provide a foundation for computing many other limits, particularly the limits needed in Chapter 3.

2.1 Four special limits

This section develops the notion of a limit of a function, using four examples that play a key role in Chapter 3.

A Limit Involving x^n

Let a and n be fixed numbers.

What happens to the quotient $\dfrac{x^n - a^n}{x - a}$ as x is chosen nearer and nearer to a? (1)

To keep the reasoning down-to-earth, let's look at a typical concrete case:

$$\text{What happens to } \frac{x^3 - 2^3}{x - 2} \text{ as } x \text{ gets closer and closer to 2?} \tag{2}$$

As x approaches 2, the numerator approaches $2^3 - 2^3 = 0$. Because 0 divided by almost anything is 0 we suspect that the quotient may approach 0. But the denominator approaches $2 - 2 = 0$. This is unfortunate because division by zero is not defined.

That $x^3 - 2^3$ approaches 0 as x approaches 2 may make the quotient small. The denominator approaches 0 as x approaches 2 may make the quotient very large. The balance between these two opposing forces contains the answer to what happens to (2) as x approaches 2.

We have already seen that it is pointless to replace x in (2) by 2 as this leads to $(2^3 - 2^3)/(2 - 2) = 0/0$, a meaningless expression.

Instead, let's do some experiments and see how the quotient behaves for specific values of x near 2; some less than 2, some more than 2. Table 2.1.1 shows the results as x increases from 1.9 to 2.1. You can add to the list with values of x even closer to 2.

Math is not a spectator sport. Check some of the calculuations reported in Table 2.1.1.

x	x^3	$x^3 - 2^3$	$x - 2$	$\frac{x^3-2^3}{x-2}$
1.90	6.859	-1.141	-0.1	11.41
1.99	7.8806	-0.1194	-0.01	11.94
1.999				
2.00	8.0000	0.0000	0.00	undefined
2.001				
2.01	8.1206	0.1206	0.01	12.06
2.10	9.261	1.261	0.1	12.61

Table 2.1.1: Table showing the steps in the evaluation of $\frac{x^3-2^3}{x-2}$ for four choices of x near 2. *Fill in the rows for 1.999 and 2.001.*

As x increases the quotient increases. The cases with $x = 1.99$ and 2.01, being closest to 2, should provide the best estimates of the quotient. This suggests that the quotient (2) approaches a number near 12 as x approaches 2, whether from below or from above.

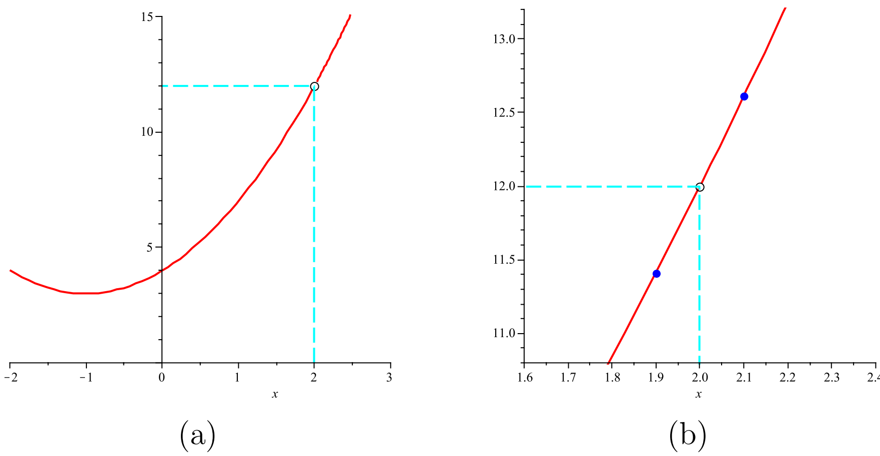


Figure 2.1.1: The graph of a $y = \frac{x^3-2^3}{x-2}$ suggests that the quotient approaches 12 as x approaches 2. In (b), zooming for x near 2 shows how the data in Table 2.1.1 also suggests the quotient approaches 12 as x approaches 2.

While the numerical and graphical evidence is very suggestive, this question can be answered once-and-for-all with a little bit of algebra. You can check that $x^3 - 2^3 = (x-2)(x^2 + 2x + 2^2)$. We have

$$\frac{x^3-2^3}{x-2} = \frac{(x-2)(x^2+2x+2^2)}{x-2} \qquad \text{for all } x \text{ other than 2.} \tag{3}$$

Check this factorization by multiplying the two terms in the numerator on the right-hand side.

For $x = 2$, both sides of (3) become the meaningless expression $0/0$.

Cancelling the $(x-2)$ in (3) shows that

$$\frac{x^3-2^3}{x-2} = x^2 + 2x + 2^2, \qquad x \neq 2.$$

Now it is easy to see what happens to $x^2 + 2x + 2^2$ as x gets nearer and nearer to 2: $x^2 + 2x + x^2$ approaches $4 + 4 + 4 = 12$. This agrees with the calculations (see Table 2.1.1).

We say "the limit of $(x^3 - 2^3)/(x-2)$ as x approaches 2 is 12" and use the shorthand

$$\lim_{x\to 2} \frac{x^3-2^3}{x-2} = \lim_{x\to 2}(x^2+2x+2^2) \tag{4}$$
$$= 3 \cdot 2^2 = 12. \tag{5}$$

Similar algebra yields the following limit, which will be used in the next chapter.

An outline of the algebra for the general case can be found in Exercises 16 and 17.

Theorem 2.1.1 *For any positive integer n and fixed number a,*

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1} \tag{6}$$

A Limit Involving b^x

What happens to $\frac{2^x-1}{x}$ and to $\frac{4^x-1}{x}$ as x approaches 0?

Consider $(2^x - 1)/x$ first: As x approaches 0, $2^x - 1$ approaches $2^0 - 1 = 1 - 1 = 0$. Since the numerator and denominator in $(2^x - 1)/x$ both approach 0 as x approaches 0, we face the same challenge as with $(x^3 - 2^3)/(x - 2)$. There is a battle between two opposing forces.

This time algebra will not help us. Instead, we will depend on our calculators. Table 2.1.2 records some results (rounded off), for four choices of x. You are invited to test values of x even closer to 0.

Fill in the missing entries in the rows for $x = \pm 0.0001$.

x	2^x	$2^x - 1$	$\frac{2^x-1}{x}$
−0.01	0.993093	−0.006907	0.691
−0.001	0.999307	−0.000693	0.693
−0.0001			
0.0001			
0.001	1.000693	0.000693	0.693
0.01	1.006956	0.006956	0.696

Table 2.1.2: Numerical evaluation of $(2^x - 1)/x$ for four different choices of x. The numbers in the last column are rounded to three decimal places. See also Figure 2.1.2.

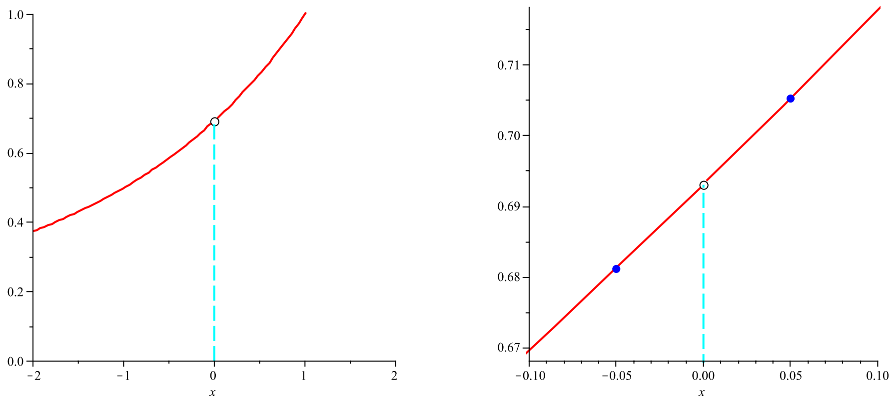
> **WARNING** *(Do not believe your eyes!)* The graphs in Figure 2.1.1(b) and Figure 2.1.2(b) are not the graphs of straight lines. They look straight only because the viewing windows are so small. Compare the labels on the axes in the two views in each of Figure 2.1.1 and Figure 2.1.2. The observation that the graphs of some functions do look straight as you zoom in on a point will be important in Section 3.1.

It seems that, as x approaches 0, $(2^x - 1)/x$ approaches a number whose decimal value begins 0.693. We write

$$\lim_{x \to 0} \frac{2^x - 1}{x} \approx 0.693 \qquad \text{rounded to three decimal places} \tag{7}$$

It is then a simple matter to find

$$\lim_{x \to 0} \frac{4^x - 1}{x}.$$



(a) (b)

Figure 2.1.2: (a) Graph of $y = (2^x - 1)/x$ for x near 0. (b) View for x nearer to 0, with the data points from Table 2.1.2. Note that there is no point for $x = 0$ since the quotient is not defined when x is 0.

Recalling the algebraic identity for the difference of two squares, $a^2 - b^2 = (a-b)(a+b)$, we have $4^x - 1 = (2^x)^2 - 1^2 = (2^x - 1)(2^x + 1)$. Hence

$$\frac{4^x - 1}{x} = \frac{(2^x - 1)(2^x + 1)}{x} = (2^x + 1)\frac{2^x - 1}{x}.$$

As $x \to 0$, $2^x + 1$ approaches $2^0 + 1 = 1 + 1 = 2$ and $(2^x - 1)/x$ approaches (approximately) 0.693. Thus,

$$\lim_{x\to 0} \frac{4^x - 1}{x} \approx 2 \cdot 0.693 \approx 1.386 \qquad \text{rounded to three decimal places.}$$

We now know $\lim_{x\to 0} \frac{b^x - 1}{x}$ for $b = 2$ and $b = 4$. They suggest that the larger b is, the larger the limit is. Since $\lim_{x\to 0} \frac{2^x - 1}{x}$ is less than 1 and $\lim_{x\to 0} \frac{4^x - 1}{x}$ is more than 1, it seems reasonable that there should be a value of b such that $\lim_{x\to 0} \frac{b^x - 1}{x} = 1$. This special number is called e, **Euler's number**. We know that e is between 2 and 4 and that $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$. It turns out that e is an irrational number with an endless decimal representation that begins $2.718281828\ldots$. In Chapter 3 we will see that e is as important in calculus as π is in geometry and trigonometry.

Euler named this constant e, but no one knows why he chose this name.

In Section 1.2 it was remarked that the logarithm base b, $\log_b$ can be defined for any base $b > 0$. The logarithm with base $b = e$ deserves special attention. The $\log_e(x)$ is called the **natural logarithm**, and is typically written as $\ln(x)$. Thus, in particular,

$$y = \ln(x) \quad \text{is equivalent to} \quad x = e^y.$$

Note that, as with any logarithm function, the domain of ln is the set of positive numbers $(0, \infty)$ and the range is the set of all real numbers $(-\infty, \infty)$.

We will see later that $\lim_{x \to 0} \frac{2^x - 1}{x}$ is the logarithm of 2, base e. More generally, $\lim_{x \to 0} \frac{b^x - 1}{x} = \log_e(b) = \ln(b)$ for any positive number b.

Three Important Bases for Logarithms

While logarithms can be defined for any positive base, three numbers have been used most often: 2, 10, and e. Logarithms to the base 2 are used in information theory, for they record the number of "yes – no" questions needed to pinpoint a particular piece of information. Base 10 was used for centuries to assist in computations. Since the decimal system is based on powers of 10, certain convenient numbers had obvious logarithms; for instance, $\log(1000) = \log(10^3) = 3$. Tables of logarithms to several decimal places facilitated the calculations of products, quotients, and roots. To multiply two numbers, you looked up their logarithms, and then searched the table for the number whose logarithm was the sum of the two logarithms. The calculator made the tables obsolete, just as it sent the slide rule into museums. However, a Google search for "slide rule" returns a list of more than 15 million websites full of history, instruction, and sentiment. The number e is the most convenient base for logarithms in calculus. Euler, as early as 1728, used e for the base of logarithms.

A Limit Involving $\sin(x)$

What happens to $\frac{\sin x}{x}$ as x gets nearer and nearer to 0?

Here x represents an angle, measured in radians. In Chapter 3 we will see that in calculus radians are much more convenient than degrees.

Appendix E includes a review of radians.

Consider first $x > 0$. Figure 2.1.3 identifies both x and $\sin x$ on a circle of radius 1, the **unit circle**.

To get an idea of the value of this limit, let's try $x = 0.1$. Setting our calculator in the "radian mode", we find

You may graph $y = \frac{\sin(x)}{x}$ on your calculator.

$$\frac{\sin 0.1}{0.1} \approx \frac{0.099833}{0.1} = 0.99833. \tag{8}$$

Likewise, with $x = 0.01$,

$$\frac{\sin 0.01}{0.01} \approx \frac{0.0099998}{0.01} = 0.99998. \tag{9}$$

These results lead us to suspect that maybe this limit is 1.

Geometry and a bit of trigonometry can be used to show that $\lim_{x \to 0} \frac{\sin x}{x}$ is indeed 1. First, using Figure 2.1.3, we show that $\frac{\sin x}{x}$ is less than 1 for all

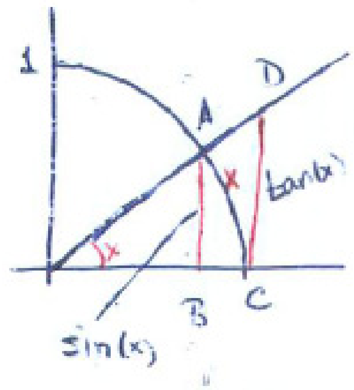


Figure 2.1.3: On the circle with radius 1, (a) x is the arclength subtended by an angle of x radians and $\sin x = \overline{AB}$.

positive x. Recall that $\sin x = \overline{AB}$. Now, $\overline{AB}$ is shorter than $\overline{AC}$, since a leg of a right triangle is shorter than the hypotenuse. Then $\overline{AC}$ is shorter than the circular arc joining A to C, since the shortest distance between two points is a straight line. Thus,

$$\sin x < \overline{AC} < x.$$

So $\sin x < x$. Since x is positive, dividing by x preserves the inequality. We have

$$\frac{\sin x}{x} < 1. \tag{10}$$

Next, we show that $\frac{\sin x}{x}$ is greater than something which gets near 1 as x approaches 0. Figure 2.1.3 again helps with this step.

The area of triangle OCD is greater than the area of the sector OCA. Thus

$$\underbrace{\frac{1}{2} \cdot 1 \cdot \tan(x)}_{\text{area of } \Delta OCD} > \underbrace{\frac{x \cdot 1^2}{2}}_{\text{area of sector } OCA}.$$

Multiplying this inequality by 2 simplifies it to

$$\tan x > x.$$

The area of a sector subtended by an angle x of a circle with radius r is $\pi r^2 \cdot \frac{x}{2\pi} = \frac{xr^2}{2}$.

In other words,

$$\frac{\sin x}{\cos x} > x.$$

Now, multiplying by $\cos x$ and dividing by x gives

$$\frac{\sin x}{x} > \cos x. \tag{11}$$

This step requires that both $x > 0$ and $\cos x > 0$. We have already assumed $x > 0$. To ensure $\cos x > 0$ requires that $x < \pi/2$. This additional assumption is not a problem because we are interested in the behavior of $\frac{\sin x}{x}$ for x near 0.

Putting (10) and (11) together we have

$$\cos x < \frac{\sin x}{x} < 1. \tag{12}$$

Since $\cos x$ approaches 1 as x approaches 0 *(through positive values)*, $\frac{\sin x}{x}$ is squeezed between 1 and something that gets closer and closer to 1, $\frac{\sin x}{x}$ must itself approach 1.

We still must look at $\frac{\sin x}{x}$ for $x < 0$ as x gets nearer and nearer to 0. Define u to be $-x$. Then u is positive, and

$$\frac{\sin x}{x} = \frac{\sin(-u)}{-u} = \frac{-\sin u}{-u} = \frac{\sin u}{u}.$$

As x is negative and approaches zero, u is positive and approaches 0. But, the previous analysis (with x replaced by u) shows that $\frac{\sin u}{u}$ approaches 1 as $u > 0$ approaches 0.

Recall that sine is an odd function. That is, $\sin(-u) = -\sin(u)$.

In short,

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \text{where the angle is measured in radians.}$$

The Meaning of $\lim_{x \to \to 0} \frac{\sin(x)}{x} = 1$

When x is near 0, $\sin(x)$ and x are both small. That their quotient is near 1 tells us much more, namely, that x is a "very good approximation of $\sin(x)$."

That means that the difference $\sin(x) - x$ is small, even in comparison to $\sin(x)$. In other words, the "relative error"

$$\frac{\sin(x) - x}{\sin x} \tag{13}$$

approaches 0 as x approaches 0.

To show that this is the case, we compute

$$\lim_{x \to 0} \frac{\sin(x) - x}{\sin(x)}.$$

We have

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin(x) - x}{\sin(x)} &= \lim_{x \to 0} \left(\frac{\sin(x)}{\sin(x)} - \frac{x}{\sin(x)} \right) \\
&= \lim_{x \to 0} \left(1 - \frac{x}{\sin(x)} \right) \\
&= \lim_{x \to 0} \left(1 - \frac{1}{\left(\frac{x}{\sin(x)} \right)} \right) \\
&= 1 - \frac{1}{1} = 0.
\end{aligned}$$

As you may check by graphing, the relative error in (13) stays less than 1 percent for x less than 0.24 radians, just under 14 degrees.

It is often quite useful to replace $\sin x$ by the much simpler quantity x. For instance, the force tending to return a swinging pendulum is proportional to $\sin \theta$, where θ is the angle that the pendulum makes with the vertical. As one physics book says, "If the angle is small, $\sin \theta$ is nearly equal to θ"; it then replaces $\sin(\theta)$ by θ.

A Limit Involving $\cos(x)$

Knowing that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$, we can show that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0. \tag{14}$$

All we will say about this limit here is that the numerator, $1 - \cos(x)$ is easily identified as the length of BC in Figure 2.1.3. Exercises 29 and 30 outline how to establish this limit.

Summary

This section examined four important limits:

$$\begin{aligned}
\lim_{x\to a} \frac{x^n - a^n}{x - a} &= na^{n-1} && (n \text{ a positive integer}) \\
\lim_{x\to 0} \frac{e^x - 1}{x} &= 1 && (e \approx 2.71828) \\
\lim_{x\to 0} \frac{\sin x}{x} &= 1 && (\text{angle in radians}) \\
\lim_{x\to 0} \frac{1 - \cos x}{x} &= 0 && (\text{angle in radians}).
\end{aligned}$$

Each of these limits will be needed in Chapter 3, which introduces the derivative of a function.

The next section examines the general notion of a limit. This is the basis for all of calculus.

EXERCISES for 2.1

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In each of Exercises 1–10 describe the two opposing forces involved in the limit. If you can figure out the limit on the basis of results in this section, give it. Otherwise, use a calculator to estimate the limit.

1. $\lim_{x \to 2} \frac{x^4 - 16}{x - 2}$

2. $\lim_{x \to 0} \frac{\sin(x)}{x \cos(x)}$

3. $\lim_{x \to 0} (1 - x)^{1/x}$

4. $\lim_{x \to 0} (\cos(x))^{1/x}$

5. $\lim_{x \to 0} x^x$

6. $\lim_{x \to 0} \frac{\arcsin(x)}{x}$

7. $\lim_{x \to 0} \frac{\tan(x)}{x}$ Hint: Write $\tan(x) = \sin(x)/\cos(x)$.

8. $\lim_{x \to 0} \frac{\tan(2x)}{x}$

9. $\lim_{x \to 0} \frac{8^x - 1}{2^x - 1}$ Hint: The numerator is the difference of two cubes; factor it!

10. $\lim_{x \to \infty} \frac{x^{1/3} - 2}{x - 8}$

Exercises 11–17 concern $\lim_{x \to a} \frac{x^n - a^n}{x - a}$.

11. Using the factorization $(x - a)(x + a) = x^2 - a^2$ find $\lim_{x \to a} \frac{x^2 - a^2}{x - a}$.

12. Using Exercise 11,

(a) find $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

(b) find $\lim_{x \to \sqrt{3}} \frac{x^2 - 3}{x - \sqrt{3}}$

13.

(a) By multiplying it out, show that $(x-a)(x^2+ax+a^2) = x^3 - a^3$.

(b) Use (a) to show that $\lim_{x\to a} \frac{x^3-a^3}{x-a} = 3a^2$.

(c) By multiplying it out, show that $(x-a)(x^3+ax^2+a^2x+a^3) = x^4 - a^4$.

(d) Use (c) to show that $\lim_{x\to a} \frac{x^4-a^4}{x-a} = 4a^3$.

14.

(a) What is the domain of $(x^3-8)/(x-2)$?

(b) Graph $(x^3-8)/(x-2)$.

NOTE: Use a hollow dot to indicate an absent point in the graph.

15.

(a) What is the domain of $(x^2-9)/(x-3)$?

(b) Graph $(x^2-9)/(x-3)$.

16. Let n be a positive integer. Find the polynomial $P(x)$, of degree $n-1$, such that $x^n - a^n = (x-a)P(x)$. NOTE: Check that your formula holds for $n=2$ (Exercise 11), $n=3$ (Exercise 13(a)), and $n=4$ (Exercise 13(c)).

17. Use Exercise 16 to show that $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$.

Exercises 18–21 concern $\lim_{x\to 0} \frac{a^x-1}{x}$.

18. What is a definition of the number e?

19. Use a calculator to estimate $(2.718^x-1)/x$ for $x = 0.1$, 0.01, and 0.001.

20. Use a calculator to compute $(2.7^x-1)/x$ and $(2.8^x-1)/x$ for $x = 0.001$. NOTE: This suggests that e is between 2.7 and 2.8.

21. Graph $y = (e^x-1)/x$ for $x \neq 0$.

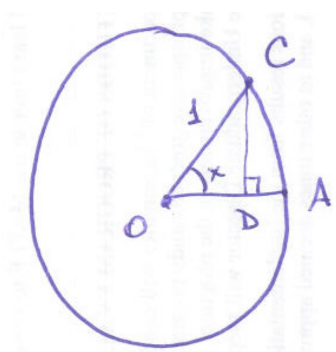
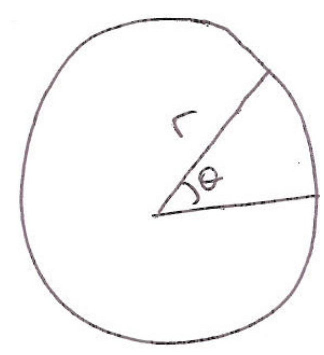


Figure 2.1.4:



Exercises 22–31 concern $\lim_{x\to 0} \frac{\sin(x)}{x}$ and $\lim_{x\to 0} \frac{1-\cos(x)}{x}$.

22. Find $\lim_{x\to 0} \frac{\tan x}{x}$. HINT: $\tan x = \sin(x)/\cos(x)$.

23. Using the fact that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, find the limits of the following as x

approaches 0.

(a) $\dfrac{\sin(3x)}{3x}$

(b) $\dfrac{\sin(3x)}{x}$

(c) $\dfrac{\sin(3x)}{\sin(x)}$

(d) $\dfrac{\sin^2(x)}{x}$

24. Why is the arc length from A to C in Figure 2.1.4 equal to x?

25. Why is the length of CD in Figure 2.1.4 equal to $\tan x$?

26. Why is the area of triangle OCD in Figure 2.1.4 equal to $(\tan x)/2$?

27. An angle of θ radians in a circle of radius r subtends a sector, as shown in Figure 2.1.5. What is the area of this sector? NOTE: If you need a review of trigonometry, see Appendix E.

28.

(a) Graph $(\sin(x))/x$ for x in $[-\pi, 0)$

(b) Graph $(\sin(x))/x$ for x in $(0, \pi]$.

(c) How are the graphs in (a) and (b) related?

(d) Graph $(\sin(x))/x$ for $x \neq 0$.

29. When $x = 0$, $(1 - \cos(x))/x$ is not defined. Estimate $\lim\limits_{x\to 0} \dfrac{1-\cos(x)}{x}$ by evaluating $(1 - \cos(x))/x$ at $x = 0.1$ (radians).

30. To find $\lim\limits_{x\to 0} \dfrac{1-\cos(x)}{x}$ first check this algebra and trigonometry:

$$\frac{1-\cos(x)}{x} = \frac{1-\cos(x)}{x}\frac{1+\cos(x)}{1+\cos(x)} = \frac{1-\cos^2 x}{x(1+\cos(x))} = \frac{\sin^2 x}{x(1+\cos(x))} = \frac{\sin(x)}{x}\frac{\sin(x)}{1+\cos(x)}.$$

Then show that

$$\lim_{x\to 0} \frac{\sin(x)}{x}\frac{\sin(x)}{1+\cos(x)} = 0.$$

31. NOTE: See Exercise 30. Show that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}.$$

Thus suggests that, for small values of x, $1-\cos(x)$ is close to $\frac{x^2}{2}$, so that $\cos(x)$ is approximately $1-\frac{x^2}{2}$.

(a) Use a calculator to compare $\cos(x)$ with $1-\frac{x^2}{2}$ NOTE: 0.2 radians is about 11°. for $x = 0.2$ and 0.1 radians.

(b) Use a graphing calculutor to compare the graphs of $\cos(x)$ and $1-\frac{x^2}{2}$ for x in $[-\pi, \pi]$.

(c) What is the largest interval on which the values of $\cos(x)$ and $1-\frac{x^2}{2}$ differ by no more than 0.1? That is, for what values of x is it true that $\left|\cos(x)-(1-\frac{x^2}{2})\right| < 0.1$?

32. In the design of a water sprinkler $\lim_{\theta\to 0} \frac{\sin 4\theta}{\sin\theta}$ appears. Find that limit.

33.

(a) What two influences operate on $(1+x)^{1/x}$ as x gets arbitrarily large?

(b) Using a calculator, examine $(1+x)^{1/x}$ for large positive values of x.

(c) Using a calculator, examine $(1+x)^{1/x}$ for x near -1 but $x > -1$.

(d) Investigate $(1+x)^{1/x}$ for x near 0. (See also Exercise 39.)

(e) On the basis of (a), (b), and (c), graph $y = (1+x)^{1/x}$ for $x > -1$ ($x \neq 0$).

34.

(a) We examined $(2^x-1)/x$ only for x near 0. When x is large and positive 2^x-1 is large. So both the numerator and denominator of $(2^x-1)/x$ are large. NOTE: The numerator influences the quotient to become large. The large denominator pushes the quotient toward 0. Use a calculator to see how the two forces balance for large values of x.

(b) Sketch the graph of $f(x) = (2^x-1)/x$ for $x > 0$. (Pay special attention to the behavior of the graph for large values of x.)

35.

(a) We examined $(2^x-1)/x$ only for x near 0. When x is large but negative, what happens to $(2^x-1)/x$?

(b) Sketch the graph of $f(x) = (2^x-1)/x$ for $x < 0$. (Pay special attention to the behavior of the graph for large negative values of x.)

36.

(a) Using a calculator explore what happens to $\sqrt{x^2+x} - x$ for large positive values of x.

(b) Show that for $x > 0$, $\sqrt{x^2+x} < x + 1/2$.

(c) Using algebra, find what number $\sqrt{x^2+x} - x$ approaches as x increases. HINT: Multiply $\sqrt{x^2+x} - x$ by $\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$.

37. Using a calculator, examine the behavior of the quotient $(\theta - \sin(\theta))/\theta^3$ for θ near 0.

38. Using a calculator, examine the behavior of the quotient $\left(\cos(\theta) - 1 + \dfrac{\theta^2}{2}\right)/\theta^4$ for θ near 0.

39. Suppose b is chosen so that $\frac{b^x-1}{x}$ is near 1 when x is near 0. We will write NOTE: Here "$\sim$" means "is close to."

$$\frac{b^x - 1}{x} \sim 1.$$

Multiplying by x gives $b^x - 1 \sim x$. Then, adding 1 to both sides yields $b^x \sim 1+x$. Raising both sides of this last relation to the power $1/x$ gives $b \sim (1+x)^{1/x}$. This suggests that b equals $\lim_{x\to 0}(1+x)^{1/x}$. (In other words, $b = \lim_{x\to 0}(1+x)^{1/x} = e$).

The conclusion turns out to be correct, but, while some of the steps in the reasoning are legitimate, there is a big leap at one step. What is that step?

Exercise 40–43 concern $f(x) = (1+x)^{1/x}$, x in $(-1, 0)$ and $(0, \infty)$.

40.

(a) Why is $(1+x)^{1/x}$ not defined when $x = 0$?

(b) For x near 0, $x > 0$, $1 + x$ is near 1. So we might expect $(1+x)^{1/x}$ to be near 1 then. However, the exponent $1/x$ is very large. So perhaps $(1+x)^{1/x}$ is also large. To see what happens, fill in this table.

x	1	0.5	0.1	0.01	0.001
$1+x$	2				
$1/x$	1				
$(1+x)^{1/x}$	2				

(c) For x near 0 but negative, investigate $(1+x)^{1/x}$ with the use of this table

x	-0.5	-0.1	-0.01	-0.001
$1+x$	0.5			
$1/x$	-2			
$(1+x)^{1/x}$	4			

41. Graph $y = (1+x)^{1/x}$ for x in $(-1,0)$ and $(0,10)$.

Exercises 40 and 41 show that $\lim_{x\to 0}(1+x)^{1/x}$ is about 2.718. Thus suggests that the number e may equal $\lim_{x\to 0}(1+x)^{1/x}$. In Section 3.2 we show that this is the case. However, the next two exercises give persuasive arguments for this fact. Unfortunately, each argument has a big hole or "unjustified leap," which you are asked to find.

42. Assume that all we know about the number e is that $\lim_{x\to 0} \frac{e^x-1}{x} = 1$. We will write this as

$$\frac{e^x - 1}{x} \sim 1,$$

and read this as "$(e^x - 1)/x$ is close to 1 when x is near 0." Multiplying both sides by x gives

$$e^x - 1 \sim x.$$

Adding 1 to both sides of this gives

$$e^x \sim 1 + x.$$

Finally, raising both sides to the power $1/x$ gives

$$(e^x)^{1/x} \sim (1+x)^{1/x}.$$

Hence

$$e \sim (1+x)^{1/x}.$$

This suggests that

$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

The conclusion is correct. Most of the steps are justified. Which step is the "big leap"?

43. Assume that $b = \lim\limits_{x\to 0}(1+x)^{1/x}$. We will "show" that

$$\lim_{x\to 0} \frac{b^x - 1}{x} = 1.$$

First of all, the x near (but not equal to) 0

$$b \sim (1+x)^{1/x}.$$

Then

$$b^x \sim 1 + x.$$

Hence

$$b^x - 1 \sim x.$$

Dividing by x gives

$$\frac{b^x - 1}{x} \sim 1.$$

Hence

$$\lim_{x\to 0} \frac{b^x - 1}{x} = 1.$$

where is the "suspect step" this time?

44. An intuitive argument suggested that $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$, which turned out to be correct. Try your intuition on another limit associated with the unit circle shown in Figure 2.1.6.

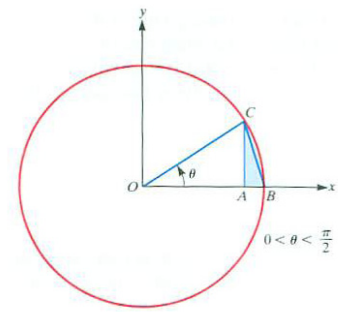


Figure 2.1.6:

(a) What do you think happens to the quotient

$$\frac{\text{Area of triangle } ABC}{\text{Area of shaded region}} \qquad \text{as } \theta \to 0?$$

More precisely, what does your intuition suggest is the limit of that quotient as $\theta \to 0$?

(b) Estimate the limit in (a) using $\theta = 0.01$.

Note: This problem is a test of your intuition. The limit is determined in Exercise 53 in Section 5.5. This question arose during some research in geometry. The authors guessed wrong, as has everyone we asked.

2.2 The Limit of a Function: The General Case

In Section 2.1 we examined four important limits:

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1},\quad \lim_{x\to 0}\frac{e^x-1}{x}=1,\quad \lim_{x\to 0}\frac{\sin x}{x}=1,\quad \lim_{x\to 0}\frac{1-\cos x}{x}=0.$$

Limits are fundamental to all of calculus. In this section, we pause to discuss the concept of a limit in greater detail. The first step towards understanding limits is the notion of a one-sided limit.

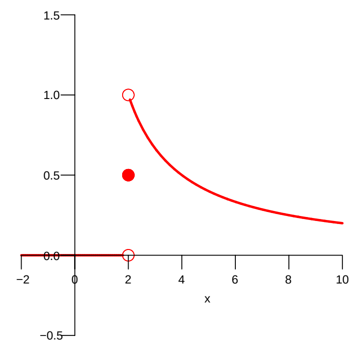


Figure 2.2.1:

One-Sided Limits

The domain of the function shown in Figure 2.2.1 is $(-\infty,\infty)$. In particular, the function is defined when $x=2$ and $f(2)=1/2$. This fact is conveyed by the solid dot at $(2,1/2)$ in the figure. The hollow dots at $(2,0)$ and $(2,1)$ indicate that these points are not on the graph of this function (but that some nearby points are on the graph).

Consider the part of the graph for inputs $x>2$, that is, for inputs to the right of 2. As x approaches 2 from the right, $f(x)$ approaches 1. This conclusion can be expressed as

$$\lim_{x\to 2^+} f(x)=1$$

and is read "the limit of f of x, as x approaches 2, from the right, is 1." Similarly, looking at the graph of f in Figure 2.2.1 for x to the left of 2, that is, for $x<2$, the values of $f(x)$ approach a different number, namely, 0. This is expressed with the shorthand

$$\lim_{x\to 2^-} f(x)=0.$$

It might sound strange to say the values of $f(x)$ "approach" 0 since the function values are exactly 0 for all inputs $x<2$. But, it is convenient, and customary, to use the word "approach" even for constant functions.

This illustrates the concept of the "right-hand" and "left-hand" limits, the two **one-sided limits**.

> **DEFINITION** *(Right-hand limit of $f(x)$ at a)* Let f be a function and a some fixed number. Assume that the domain of f contains an open interval (a,c). If, as x approaches a from the right, $f(x)$ approaches a specific number L, then L is called the **right-hand limit** of $f(x)$ as x approaches a. This is written
>
> $$\lim_{x\to a^+} f(x)=L$$
>
> or
>
> $$f(x)\to L \quad\text{as}\quad x\to a^+.$$

The assertion that

$$\lim_{x \to a^+} f(x) = L$$

is read "the limit of f of x as x approaches a from the right is L" or "as x approaches a from the right, $f(x)$ approaches L."

> **DEFINITION** *(Left-hand limit of $f(x)$ at a)* Let f be a function and a some fixed number. Assume that the domain of f contains an open interval (b, a). If, as x approaches a from the left, $f(x)$ approaches a specific number L, then L is called the **left-hand limit** of $f(x)$ as x approaches a. This is written
>
> $$\lim_{x \to a^-} f(x) = L$$
>
> or
>
> $$f(x) \to L \quad \text{as} \quad x \to a^-.$$

Notice that the definitions of the one-sided limits do not require that the number a be in the domain of the function f. If f is defined at a, we do not consider $f(a)$ when examining limits as x approaches a.

The Two-Sided Limit

The two-sided limit of $f(x)$ as x approaches a exists and $\lim_{x \to a} f(x) = L$ if both one-sided limits of $f(x)$ at $x = a$ exist and are equal:

$$\lim_{x \to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to a^+} f(x) = L.$$

For the function graphed in Figure 2.2.1 we found that $\lim_{x \to 2^+} f(x) = 1$ and $\lim_{x \to 2^-} f(x) = 0$. Because they are different, the two-sided limit of $f(x)$ at 2, $\lim_{x \to 2} f(x)$, does not exist.

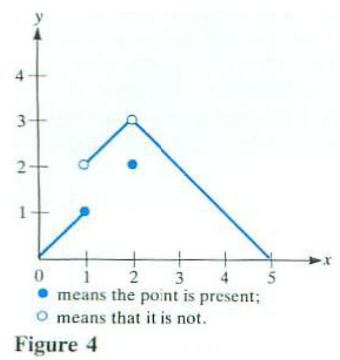


Figure 2.2.2:

EXAMPLE 1 Figure 2.2.2 shows the graph of a function f whose domain is the closed interval $[0, 5]$.

(a) Does $\lim_{x \to 1} f(x)$ exist?

(b) Does $\lim_{x \to 2} f(x)$ exist?

(c) Does $\lim_{x \to 3} f(x)$ exist?

SOLUTION

(a) Inspection of the graph shows that

$$\lim_{x\to 1^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 1^+} f(x) = 2.$$

Although the two one-sided limits exist, they are not equal. Thus, $\lim_{x\to 1} f(x)$ does not exist. In short, "f does not have a limit as x approaches 1."

(b) Inspection of the graph shows that

$$\lim_{x\to 2^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x\to 2^+} f(x) = 3.$$

Thus $\lim_{x\to 2} f(x)$ exists and is 3. That $f(2) = 2$ plays no role in our examination of the limit of $f(x)$ as $x \to 2$ (either one-sided or two-sided).

As indicated by the solid dot at $(2, 2)$.

(c) Inspection, once again, shows that

$$\lim_{x\to 3^-} f(x) = 2 \qquad \text{and} \qquad \lim_{x\to 3^+} f(x) = 2.$$

Thus $\lim_{x\to 3} f(x)$ exists and is 2. Incidentally, the fact that $f(3) = 2$ is irrelevant in determining $\lim_{x\to 3} f(x)$.

◇

We now define the (two-sided) limit without referring to one-sided limits.

DEFINITION *(Limit of $f(x)$ at a.)* Let f be a function and a some fixed number. Assume that the domain of f contains open intervals (b, a) and (a, c), as shown in Figure 2.2.3. If there is a number L such that as x approaches a, from both the right and the left, $f(x)$ approaches L, then L is called the **limit** of $f(x)$ as x approaches a. This is expressed as either

$$\lim_{x\to a} f(x) = L \qquad \text{or} \qquad f(x) \to L \quad \text{as} \quad x \to a.$$

Notice how the assumptions imply that the domain of f contains numbers arbitrarily close to a on both sides of a.

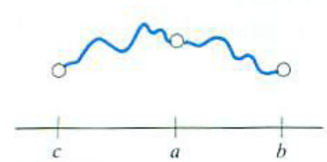


Figure 2

Figure 2.2.3: The function f is defined on open intervals on both sides of a.

EXAMPLE 2 Let f be the function defined by by $f(x) = \dfrac{x^n - a^n}{x - a}$ where n is a positive integer. This function is defined for all x except a. How does it behave for x near a?

SOLUTION In Section 2.1 and its Exercises we found that as x gets closer and closer to a, $f(x)$ gets closer and closer to na^{n-1}. This is summarized with the shorthand

$$\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1},$$

read as "the limit of $\frac{x^n-a^n}{x-a}$ as x approaches a is na^{n-1}." ◇

EXAMPLE 3 Investigate the one-sided and two-sided limits for the square root function at 0.

SOLUTION The function $\sqrt{x}$ is defined only for x in $[0,\infty)$. We can say that the right-hand limit at 0 exists since $\sqrt{x}$ approaches 0 as $x \to 0$ through positive values of x; that is, $\lim_{x\to 0^+}\sqrt{x} = 0$. Because $\sqrt{x}$ is not defined for any negative values of x, the left-hand limit of $\sqrt{x}$ at 0 does not exist. Consequently, the two-sided limit of $\sqrt{x}$ at 0, $\lim_{x\to 0}\sqrt{x}$, does not exist. ◇

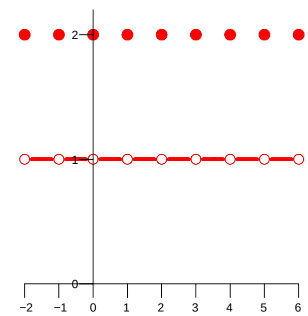


Figure 2.2.4:

EXAMPLE 4 Consider the function f defined so that $f(x) = 2$ if x is an integer and $f(x) = 1$ otherwise. For which a does $\lim_{x\to a} f(x)$ exist?
SOLUTION The graph of f, shown in Figure 2.2.4, will help us decide. If a is not an integer, then for all x sufficiently near a, $f(x) = 1$. So $\lim_{x\to a} f(x) = 1$. Thus the limit exists for all a that are not integers.

Now consider the case when a is an integer. In deciding whether $\lim_{x\to a} f(x)$ exists we never consider the value of f at a, namely $f(a)$. (In fact, $f(a)$ may not even be defined.) For all x sufficiently near an integer a, $f(x) = 1$. Thus, once again, $\lim_{x\to a} f(x) = 1$. The limit exists.

Thus, $\lim_{x\to a} f(x)$ exists and equals 1 for every number a. ◇

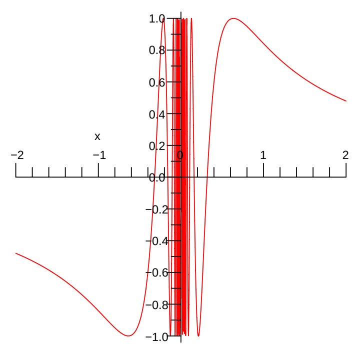


Figure 2.2.5: $y = g(x) = \sin(1/x)$.

EXAMPLE 5 Let $g(x) = \sin(1/x)$. For which a does $\lim_{x\to a} g(x)$ exist?

SOLUTION To begin, we graph the function. Notice that the domain of g is all x except 0. When x is very large, $1/x$ is very small, so $\sin(1/x)$ is small. As x approaches 0, $1/x$ becomes large. For instance, when $x = \frac{1}{2n\pi}$, for a non-zero integer n, $1/x = 2n\pi$ and therefore $\sin(1/x) = \sin(2n\pi) = 0$. Thus, the graph of $y = g(x)$ for x near 0 crosses the x-axis infinitely often. Similarly, $g(x)$ takes the values 1 and -1 infinitely often for x near 0. The graph is shown in Figure 2.2.5.

Does $\lim_{x\to 0} g(x)$ exist? Does $g(x)$ tend toward one specific number as $x \to 0$? No. The function oscillates, taking on all values from -1 to 1 (repeatedly) for x arbitrarily close to 0. Thus $\lim_{x\to 0}\sin(1/x)$ does not exist.

That g is not defined at 0 did not concern us.

At all other values of a, $\lim_{x\to a} g(x)$ does exist and equals $g(a) = \sin(1/a)$. ◇

Infinite Limits at a

A function may assume arbitrarily large values as x approaches a fixed number. One important example is the tangent function. As x approaches $\pi/2$ from

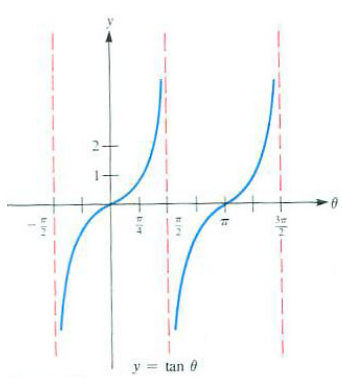


Figure 2.2.6:

the left, $\tan x$ takes on arbitrarily large positive values. (See Figure 2.2.6.) We write

$$\lim_{x \to \frac{\pi}{2}^-} \tan x = +\infty.$$

However, as $x \to \frac{\pi}{2}$ from inputs larger than $\pi/2$, $\tan x$ takes on negative values of arbitrarily large absolute value. We write

$$\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty.$$

DEFINITION *(Infinite limit of $f(x)$ at a)* Let f be a function and a some fixed number. Assume that the domain of f contains an open interval (a, c). If, as x approaches a from the right, $f(x)$ becomes <u>and remains</u> arbitrarily large and positive, then the limit of $f(x)$ as x approaches a is said to be positive infinity. This is written

$$\lim_{x \to a^+} f(x) = +\infty$$

or sometimes just

$$\lim_{x \to a^+} f(x) = \infty.$$

If, as x approaches a from the left, $f(x)$ becomes and remains arbitrarily large and positive, then we write

$$\lim_{x \to a^-} f(x) = +\infty.$$

Similarly, if $f(x)$ assumes values that are negative and these values remain arbitrarily large in absolute value, we write either

$$\lim_{x \to a^+} f(x) = -\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = -\infty,$$

depending upon whether x approaches a from the left or from the right.

Limits as $x \to \infty$

Sometimes it is useful to know how $f(x)$ behaves when x is a very large positive number (or a negative number of large absolute value).

EXAMPLE 6 Determine how $f(x) = 1/x$ behaves for

(a) large positive inputs

(b) negative inputs of large absolute value

(c) small positive inputs

(d) negative inputs of small absolute value

SOLUTION

(a) To get started, make a table of values as shown in the margin. As x becomes arbitrarily large, $1/x$ approaches 0: $\lim_{x\to\infty} \frac{1}{x} = 0$. This conclusion would be read as "as x approaches ∞, $f(x)$ approaches 0."

x	$1/x$
10	0.1
100	0.01
1000	0.001

(b) This is similar to *(a)*, except that the reciprocal of a negative number with large absolute value is a negative number with a small absolute value. Thus, $\lim_{x\to-\infty} \frac{1}{x} = 0$.

(c) For inputs that are positive and approaching 0, the reciprocals are positive and large: $\lim_{x\to 0^+} \frac{1}{x} = +\infty$.

(d) Lastly, the reciprocal of inputs that are negative and approaching 0 from the left are negative and arbitrarily large in absolute value: $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

◇

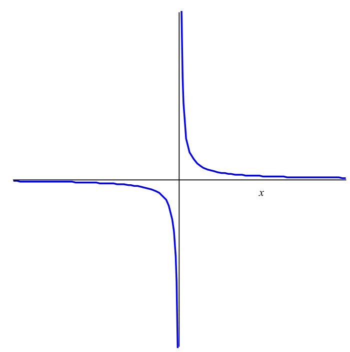


Figure 2.2.7:

More generally, for any fixed positive exponent p,

$$\lim_{x\to\infty} \frac{1}{x^p} = 0.$$

Limits of the form $\lim_{x\to\infty} P(x)$ and $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$, where P and Q, are polynomials are easy to treat.

Keep in mind that ∞ is not a number. It is just a symbol that tells us that something — either the inputs or the values of a function — are become arbitrarily large.

EXAMPLE 7 Find $\lim_{x\to\infty} 2x^3 - 5x^2 + 6x + 5$.

SOLUTION When x is large, x^3 is much larger than either x^2 or x. With this in mind, we rewrite the limit as

$$\begin{aligned} \lim_{x\to\infty} \frac{2x^3 - 5x^2 + 6x + 5}{x^3} \cdot x^3 &= \lim_{x\to\infty} \left(2 - \frac{5}{x} + \frac{6}{x^2} + \frac{5}{x^3}\right) \cdot x^3 \\ &= \lim_{x\to\infty} 2x^3 \\ &= \infty. \end{aligned}$$

◇

EXAMPLE 8 Find $\lim_{x\to\infty} \frac{2x^3-5x^2+6x+5}{7x^4+3x+2}$.

SOLUTION We use the same technique as in Example 7.

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^3-5x^2+6x+5}{7x^4+3x+2} &= \lim_{x\to\infty} \frac{\frac{2x^3-5x^2+6x+5}{x^3}\cdot x^3}{\frac{7x^4+3x+2}{x^4}\cdot x^4} \\
&= \lim_{x\to\infty}\left(\frac{2-\frac{5}{x}+\frac{6}{x^2}+\frac{5}{x^3}}{7+\frac{3}{x^3}+\frac{2}{x^4}}\right)\lim_{x\to\infty}\frac{x^3}{x^4} \\
&= \tfrac{2}{7}\cdot 0 \\
&= 0.
\end{aligned}$$

◇

As these two examples suggest, the limit of a quotient of two polynomials, $\frac{P(x)}{Q(x)}$, is completely determined by the limit of the quotient of the highest degree term in $P(x)$ and in $Q(x)$.

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

where a_n and b_m are not 0. Then

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)} = \lim_{x\to\infty}\frac{a_n x^n}{b_m x^m}.$$

In particular, if $m = n$, the limit is a_n/b_m. If $m > n$, the limit is 0. If $n > m$, the limit is infinite, either ∞ or $-\infty$, depending on the signs of a_n and b_n.

Summary

This section develops the concept of a limit and introduce notations for the various types of limits. One-sided limits are the foundation for the two-sided limit as well as for infinite limits and limits at infinity.

It is important to keep in mind that when deciding whether $\lim_{x\to a} f(x)$ exists, you never consider $f(a)$. Perhaps a isn't even in the domain of the function. Even if a is in the domain, the value $f(a)$ plays no role in deciding whether $\lim_{x\to a} f(x)$ exists.

EXERCISES for 2.2

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In Exercises 1 to 8 the limits exist. Find them.

1. $\lim_{x\to 3} \frac{x^2-9}{x-3}$

2. $\lim_{x\to 4} \frac{x^2-9}{x-3}$

3. $\lim_{x\to 0} \frac{\sin(x)}{x}$

4. $\lim_{x\to \frac{\pi}{2}} \frac{\sin(x)}{x}$

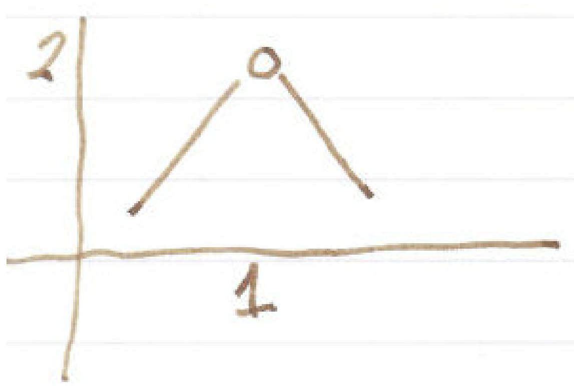
5. $\lim_{x\to 0} \frac{e^x-1}{2x}$

6. $\lim_{x\to 2} \frac{e^x-1}{2x}$

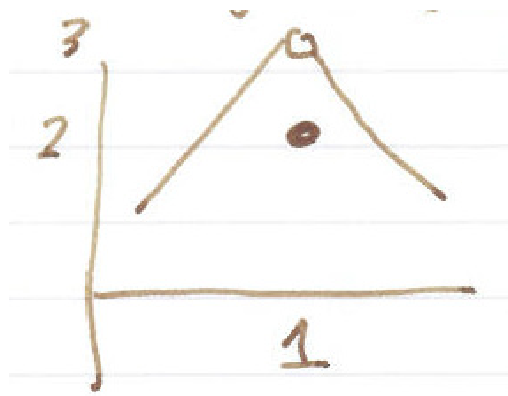
7. $\lim_{x\to 0} \frac{1-\cos(x)}{3x}$

8. $\lim_{x\to \pi} \frac{1-\cos(x)}{3x}$

In Exercises 9 to 12 the graph of a function $y = f(x)$ is given. Decide whether $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1^+} f(x)$, and $\lim_{x\to 1^+} f(x)$ exist. If they do exist, give their values.

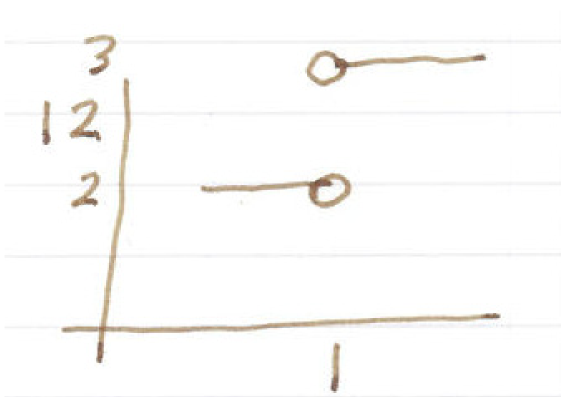


9.



10.

11.



12.

13.

(a) Sketch the graph of $y = \log_2(x)$.

(b) What are $\lim\limits_{x\to\infty} \log_2(x)$, $\lim\limits_{x\to 4} \log_2(x)$, and $\lim\limits_{x\to 0^+} \log_2(x)$?

14.

(a) Sketch the graph of $y = 2^x$.

(b) What are $\lim_{x\to\infty} 2^x$, $\lim_{x\to 4} 2^x$, and $\lim_{x\to-\infty} 2^x$?

15. Find $\lim_{x\to a} \frac{x^3-8}{x-2}$ for $a = 1$, 2, and 3.

16. Find $\lim_{x\to a} \frac{x^4-16}{x-2}$ for $a = 1$, 2, and 3.

17. Find $\lim_{x\to a} \frac{e^x-1}{x-2}$ for $a = -1$, 0, and 1.

18. Find $\lim_{x\to a} \frac{\sin(x)}{x}$ for $a = \frac{\pi}{6}$, $\frac{\pi}{4}$, and 0.

In Exercises 19 to 24, find the given limit (if it exists).

19. $\lim_{x\to\infty} 2^{-x}\sin(x)$

20. $\lim_{x\to\infty} 3^{-x}\cos(2x)$

21. $\lim_{x\to\infty} \frac{3x^5+2x^2-1}{6x^5+x^4+2}$

22. $\lim_{x\to\infty} \frac{13x^5+2x^2+1}{2x^6+x+5}$

23. $\lim_{x\to\infty} \frac{10x^6+x^5+x+1}{x^6}$

24. $\lim_{x\to\infty} \frac{25x^5+x^2+1}{x^3+x+2}$

In Exercises 25 to 27, information is given about functions f and g. In each case decide whether the limit asked for can be determined on the basis of that information. If it can, give its value. If it cannot, show by specific choices of f and g that it cannot.

25. Given that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 1$, discuss

(a) $\lim_{x\to\infty} (f(x)+g(x))$

(b) $\lim_{x\to\infty} (f(x)/g(x))$

(c) $\lim_{x\to\infty} (f(x)g(x))$

(d) $\lim_{x\to\infty} (g(x)/f(x))$

(e) $\lim_{x\to\infty} (g(x)/|f(x)|)$

26. Given that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, discuss

(a) $\lim_{x\to\infty} (f(x) + g(x))$

(b) $\lim_{x\to\infty} (f(x) - g(x))$

(c) $\lim_{x\to\infty} (f(x)g(x))$

(d) $\lim_{x\to\infty} (g(x)/f(x))$

27. Given that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} g(x) = \infty$, discuss

(a) $\lim_{x\to\infty} (f(x)/g(x))$

(b) $\lim_{x\to\infty} (f(x)g(x))$

(c) $\lim_{x\to\infty} (f(x) - 1)g(x)$

28. Graph $f(x) = \cos(1/x)$, following these steps.

(a) What is the domain of f?

(b) Fill in this table

(c) Does $\lim_{x\to 0} \cos(1/x)$ exist?

(d) Graph $f(x) = \cos(1/x)$.

29. Graph $f(x) = x \sin(1/x)$, following these steps.

(a) What is the domain of f?

(b) Graph the lines $y = x$ and $y = -x$.

(c) For which x does $f(x) = x$? When does $f(x) = -x$? (Notice that the graph of $y = f(x)$ goes back and forth between these lines.)

(d) Does $\lim_{x\to 0} f(x)$ exist? If so, what is it?

(e) Does $\lim_{x\to\infty} f(x)$ exist? If so, what is it?

(f) Graph $y = f(x)$.

30. Let $f(x) = \frac{|x|}{x}$, which is defined except at $x = 0$.

(a) What is $f(3)$?

(b) What is $f(-2)$?

(c) Graph $y = f(x)$.

(d) Does $\lim_{x\to 0} f(x)$ exist? If so, what is it?

(e) Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it?

(f) Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it?

In Exercises 31 to 34, find $\lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$ for the following functions.

31. $f(x) = 5x$

32. $f(x) = x^2$

33. $f(x) = e^x$

34. $f(x) = \sin(x)$ HINT: $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.

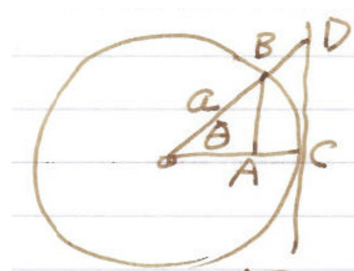


Figure 2.2.8: Exercise 35

35. Figure 2.2.8 shows a circle of radius a. Find

(a) $\lim_{\theta\to 0^+} \frac{AB}{arcCB}$

(b) $\lim_{\theta\to 0^+} \frac{AB}{CD}$

36. Let $f(x)$ be the diameter of the largest circle that fits in a $1 \times x$ rectangle.

(a) Find a formula for $f(x)$.

(b) Graph $y = f(x)$.

(c) Does $\lim_{x\to 1} f(x)$ exist?

37. Answer each question.

(a) "I am thinking of two numbers that are very near 0. What, if anything, can you say about their product?"

(b) "I am thinking of two numbers that are very near 0. What, if anything, can you say about their quotient?"

(c) "I am thinking of two numbers that are very near 0. What, if anything, can you say about their difference?"

(d) "I am thinking of two numbers that are very near 0. What, if anything, can you say about their sum?"

38. Answer each question.

(a) "I am thinking of two very large positive numbers. What, if anything, can you say about their product?"

(b) "I am thinking of two very large positive numbers. What, if anything, can you say about their quotient?"

(c) "I am thinking of two very large positive numbers. What, if anything, can you say about their difference?"

(d) "I am thinking of two very large positive numbers. What, if anything, can you say about their sum?"

39. Find $\lim_{x\to 0} \dfrac{e^{2x}-1}{x}$.

40. Sam and Jane are discussing

$$\frac{3x^2+2x}{x+5} - 3x.$$

Sam: For large x, $2x$ is small in comparison to $3x^2$, and 5 is small in comparison to x. So the quotient $\frac{3x^2+2x}{x+5}$ behaves like $\frac{3x^2}{x} = 3x$. Hence, the limit in question is 0.

Jane: "Nonsense. After all,

$$\frac{3x^2+2x}{x+5} = \frac{3x+2}{1+(5/x)}$$

which clearly behaves like $3x+2$ for large x. Thus the limit in question is 2, not 0.

Settle the argument.

41. Sam, Jane, and Wilber are arguing about limits in a case where $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = \infty$.

Sam: $\lim_{x\to\infty} f(x)g(x) = 0$, since $f(x)$ is going toward 0.

Jane: Rubbish! Since $g(x)$ gets large, it will turn out that $\lim_{x\to\infty} f(x)g(x) = \infty$.

Wilber: You're both wrong. The two influences will balance out and you will see that $\lim_{x\to\infty} f(x)g(x)$ is near 1.

Settle the argument.

42. Sam and Jane are arguing about limits in a case where $f(x) \geq 1$ for $x > 0$, $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0} g(x) = \infty$. What can be said about $\lim_{x\to 0^+} f(x)^{g(x)}$?

Sam: That's easy. Multiply a bunch of numbers near 1 and you get a number near 1. So the limit will be 1.

Jane: Rubbish! Since $f(x)$ may be bigger than 1 and you are multiplying it lots of times, you will get a really large number. There's no doubt in my mind: $\lim_{x\to 0} f(x)^{g(x)} = \infty$.

Settle the argument.

43. For a positive number n let $f(n)$ be the sum of the reciprocals of all the integers from n to $2n$:

$$f(n) = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}.$$

(a) Compute $f(n)$ for at least $n = 1, 2, \ldots, 10$.

(b) Show why $f(n)$ decreases as n increases.

(c) Show that $f(n) > 1/2$ for all n.

(d) The limit of $f(n)$ as n increases exists. In fact, it is a number met earlier in this chapter in a completely different context. What do you think that number is? (Try some values of n much larger than 10.)

44. An urn contains n marbles. One is green and the remaining $n-1$ are red. When picking one marble at random without looking, the probability is $1/n$ of getting the green marble. If you do this experiment n times, each time putting the chosen marble back, the probability of <u>not</u> ever getting the green marble is $((n-1)/n)^n$.

(a) Let $p(n) = \left(\frac{n-1}{n}\right)^n$. Compute $p(2)$, $p(3)$, and $p(4)$ to at least three decimal digits (to the right of the decimal point).

(b) Show that as $n \to \infty$, $p(n)$ approaches the reciprocal of $\lim_{x\to 0} (1+x)^{1/x}$.

2.3 Continuous Functions

This section introduces the notion of a continuous function. While almost all functions met in practice are continuous, we must always remain alert to situations where a function might not be continuous. We begin with an informal, intuitive description and then give a more useful working definition.

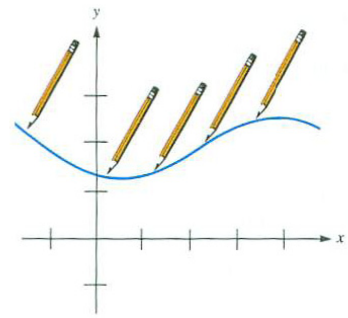


Figure 2.3.1:
The graph comes in one piece, like a length of wire. There are no gaps or jumps, although there may be sharp corners.

An Informal Introduction to Continuous Functions

When we draw the graph of a function defined on some interval, we usually do not have to lift the pencil off the paper. Figure 2.3.1 shows this typical situation.

A function is said to be **continuous** if, when considered on any interval in its domain, its graph has no jumps — it can he traced without lifting the pencil off the paper. (The domain of the function may consist of several intervals.) According to this definition any polynomial is continuous. So is each of the basic trigonometric functions, including $y = \tan(x)$, whose graph is shown in Figure 2.2.6 of Section 2.2. You may he tempted to say "But $\tan(x)$ blows up at $x = \pi/2$ and I have to lift my pencil off the paper to draw the graph." However, $x = \pi/2$ is not in the domain of the tangent function. *On every interval in its domain,* $\tan x$ *behaves quite decently;* on such an interval we can sketch its graph without lifting the pencil from the paper. That is why $\tan(x)$ is continuous. The function $1/x$ is also continuous, since it "explodes" only at a number not in its domain, namely at $x = 0$. The function whose graph is shown in Figure 2.3.2 is not continuous. It is defined throughout the interval $[-2, 3]$, but to draw its graph you must lift the pencil from the paper near $x = 1$. However, when you consider the function *only for* x *in* $[1, 3]$, *then it is continuous.* By the way, a formula for the function given graphically in Figure 2.3.2 is:

$$f(x) = \begin{cases} x+1 & \text{for } x \text{ in } [-2,1) \\ x & \text{for } x \text{ in } [1,2) \\ -x+4 & \text{for } x \text{ in } [2,3]. \end{cases}$$

It is pieced together from three different continuous functions.

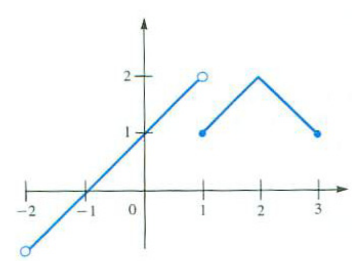


Figure 2.3.2:

The Definition of Continuity

Our informal "moving pencil" notion of a continuous function requires drawing a graph of the function. Our working definition does not require such a graph. Moreover, it easily generalizes to functions of more than one variable.

To get the feeling of this second definition, imagine that you had the information shown in the table in the margin about some function f. What would you expect the output $f(1)$ to be?

x	$f(x)$
0.9	2.93
0.99	2.9954
0.999	2.9999997

It would be quite a shock to be told that $f(1)$ is, say, 625. A reasonable function should present no such surprise. The expectation is that $f(1)$ will be 3. More generally, we expect the output of a function at the input a to be closely connected with the outputs of the function at inputs that are near a. The functions of interest in calculus usually behave that way. In short, "What you expect is what you get." With this in mind, we define the notion of *continuity at a number a*. We first assume that the domain of f contains an open interval around a.

DEFINITION *(Continuity at a number a)* Assume that $f(x)$ is defined in some open interval that contains the number a. Then the function f is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$. This means that

1. $f(a)$ is defined (that is, a is in the domain of f).
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x)$ equals $f(a)$.

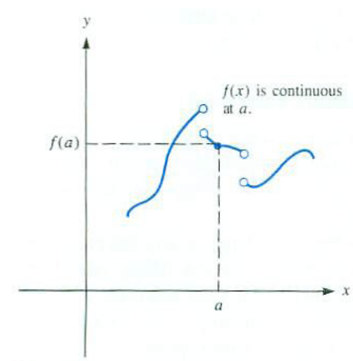


Figure 2.3.3:

As Figure 2.3.3 shows, whether a function is continuous at a depends on its behavior both at a and at inputs near a. Being continuous at a is a local matter, involving perhaps very tiny intervals about a.

To check whether a function f is continuous at a number a, we ask three questions:

Question 1: Is a in the domain of f?

Question 2: Does $\lim_{x\to a} f(x)$ exist?

Question 3: Does $f(a)$ equal $\lim_{x\to a} f(x)$?

If the answer is "yes" to each of these questions, we say that f is continuous at a.

If a is in the domain of f and the answer to Question 2 or to Question 3 is "no," then f is said to be **discontinuous at** a. If a is not in the domain of f, we do not define either continuity or discontinuity there.

We are now ready to define a continuous function.

DEFINITION *(Continuous function)* Let f be a function whose domain is the x-axis or is made up of open intervals. Then f is a **continuous function** if it is continuous at each number a in its domain.

EXAMPLE 1 Use the definition of continuity to decide whether $f(x) = 1/x$ is continuous.

SOLUTION Let a be in the domain of f. In other words, a is not 0. Since

$$\lim_{x \to a} \frac{1}{x} = \frac{1}{a},$$

the answer to Question 2 is "yes." Since

$$f(a) = \frac{1}{a},$$

the answer to Question 3 is also "yes." Thus $f(x) = 1/x$ is continuous at every number in its domain. Hence f is a continuous function.

This ensures the answer to Question 1 is "yes".

Note that the conclusion agrees with the "moving pencil" picture of continuity. ◇

Not every important function is continuous. For instance, let $f(x)$ be the greatest integer that is less than or equal to x. We have $f(1.8) = 1$, $f(1.9) = 1$, $f(2) = 2$, and $f(2.3) = 2$. This function is often used in number theory and computer science, where it is denoted $[x]$ or $\lfloor x \rfloor$ and called the **floor** of x. The next example examines where the floor function fails to be continuous.

People use the floor function every time they answer the question, "How old are you?"

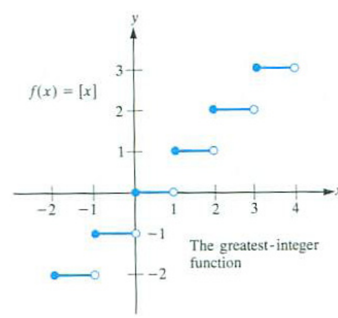


Figure 2.3.4:

EXAMPLE 2 Let f be the floor function, $f(x) = \lfloor x \rfloor$. Graph f and find where it is continuous. Is f a continuous function?

SOLUTION We begin with the following table to show the behavior of $f(x)$ for x near 1 or 2.

x	0	0.5	0.8	1	1.1	1.99	2	2.01
$\lfloor x \rfloor$	0	0	0	1	1	1	2	2

For $0 \le x < 1$, $\lfloor x \rfloor = 0$. But at the input $x = 1$ the output jumps to 1 since $\lfloor 1 \rfloor = 1$. For $1 \le x < 2$, $\lfloor x \rfloor$ remains at 1. Then at 2 it jumps to 2. More generally, $\lfloor x \rfloor$ has a jump at every integer, as shown in Figure 2.3.4.

Let us show that f is not continuous at $a = 2$ by seeing which of the three conditions in the definition are not satisfied. First of all, Question 1 is answered "yes" since 2 lies in the domain of the function; indeed, $f(2) = 2$.

What is the answer to Question 2? Does $\lim_{x \to 2} f(x)$ exist? We see that

$$\lim_{x \to 2^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = 2.$$

Since the left-hand and right-hand limits are not equal, $\lim_{x \to 2} f(x)$ does not exist. Question 2 is answered "no."

Already we know that the function is not continuous at $a = 2$. Since the limit does not exist there is no point in answering Question 3. Because there is one point in the domain where $\lfloor x \rfloor$ is not continuous, this is a discontinuous function. More specifically, the floor function is discontinuous at $x = a$, whenever a is an integer.

Is f continuous at a if a is not an integer? Let us take the case $a = 1.5$, for instance.

Question 1 is answered "yes," because $f(1.5)$ is defined. (In fact, $f(1.5) = 1$.)

Question 2 is answered "yes," since $\lim_{x\to 1.5} f(x) = 1$.

Question 3 is answered "yes," since $\lim_{x\to 1.5} f(x) = f(1.5)$. (Both values are 1.)

The floor function is continuous at $a = 1.5$. Similarly, f is continuous at every number a that is not an integer.

Note that $\lfloor x \rfloor$ is continuous on any interval that does not include an integer. For instance, if we consider the function only on the interval $(1.1, 1.9)$, it is continuous there. ◇

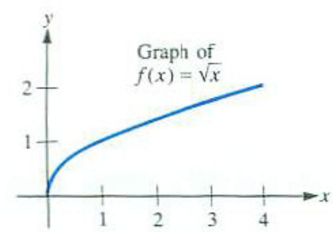


Figure 2.3.5:

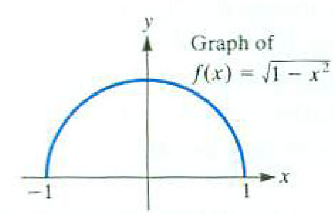


Figure 2.3.6:

Continuity at an Endpoint

The function $f(x) = \sqrt{x}$ is graphed in Figure 2.3.5 and $g(x) = \sqrt{1 - x^2}$ is graphed in Figure 2.3.6. We would like to consider both of these functions continuous. However, there is a slight technical problem. The number 0 is in the domain of f, but there is no open interval around 0 that lies completely in the domain, as our definition of continuity requires. Since $f(x) = \sqrt{x}$ is not defined for x to the left of 0, we are not interested in numbers x to the left of 0. Similarly, $g(x) = \sqrt{1 - x^2}$ is defined only when $1 - x^2 \geq 0$, that is, for $-1 \leq x \leq 1$. To cover this type of situation we utilize one-sided limits to define **one-sided continuity**.

DEFINITION *(Continuity from the right at a number a.)* Assume that $f(x)$ is defined in some closed interval $[a, c]$. Then the function f **is continuous from the right at** a if

1. $f(a)$ is defined
2. $\lim_{x\to a^+} f(x)$ exists
3. $\lim_{x\to a^+} f(x)$ equals $f(a)$

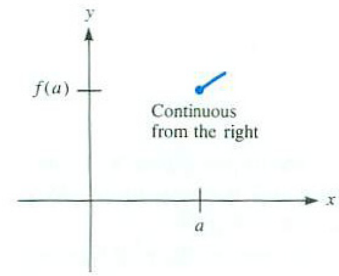


Figure 2.3.7:

Figure 2.3.7 illustrates this definition.

This definition also takes care of the continuity of $g(x) = \sqrt{1-x^2}$ at -1 in Figure 2.3.6. The next definition attends to the problem at the right-hand endpoint ($a = 1$).

> **DEFINITION** *(Continuity from the left at a number a.)* Assume that $f(x)$ is defined in some closed interval $[b, a]$. Then the function f **is continuous from the left at** a if
>
> 1. $f(a)$ is defined
> 2. $\lim_{x\to a^-} f(x)$ exists
> 3. $\lim_{x\to a^-} f(x)$ equals $f(a)$

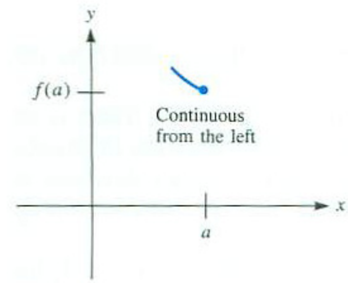


Figure 2.3.8:

Figure 2.3.8 illustrates this definition.

With these two extra definitions to cover some special numbers in the domain, we can extend the definition of continuous function to include those functions whose domains may contain endpoints. We say, for instance, that $\sqrt{1-x^2}$ is continuous because it is continuous at any number in $(-1, 1)$, is continuous from the right at -1, and continuous from the left at 1.

These special considerations are minor matters that will little concern us in the future. The key point is that $\sqrt{1-x^2}$ and $\sqrt{x}$ are both continuous functions. So are practically all the functions studied in calculus.

The following example reviews the notion of continuity.

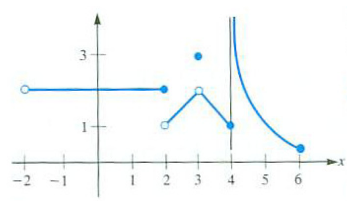


Figure 2.3.9:

EXAMPLE 3 Figure 2.3.9 is the graph of a certain (piecewise-defined) function $f(x)$ whose domain is the interval $(-2, 6]$. Discuss the continuity of $f(x)$ at (a) 6, (b) 4, (c) 3, (d) 2, (e) 1, and (f) -2.

SOLUTION

(a) Since $\lim\limits_{x\to 6-} f(x)$ exists and equals $f(6)$, f is continuous from the left at 6.

(b) Since $\lim\limits_{x\to 4} f(x)$ does not exist, f is not continuous at 4.

(c) Inspection of the graph shows that $\lim\limits_{x\to 3} f(x) = 2$. However, Question 3 is answered "no" because $f(3) = 3$, which is *not* equal to $\lim\limits_{x\to 3} f(x)$. Thus f is not continuous at 3.

(d) Though $\lim\limits_{x\to 2-} f(x)$ and $\lim\limits_{x\to 2+} f(x)$ both exist, they are not equal. (The left-hand limit is 2; the right-hand limit is 1.) Thus $\lim\limits_{x\to 2} f(x)$ does not exist, the answer to Question 2 is "no," and f is discontinuous at $x = 2$.

(e) At 1, "yes" is the answer to each of the three questions: $\lim_{x\to 1} f(x)$ exists (it equals 2) and equals $f(1)$. f is continuous at $x = 1$.

(f) Since -2 is not even in the domain of this function, we do not speak of continuity or discontinuity of f at $x = -2$.

$\diamond$

As Example 3 shows, a function can fail to be continuous at a given number a of its domain for either of two reasons:

1. $\lim_{x\to a} f(x)$ might not exist
2. when, $\lim_{x\to a} f(x)$ does exist, $f(a)$ might not be equal to that limit.

Continuity and Limits

Some limits are so easy that you can find them without any work; for instance, $\lim_{x\to 2} 5^x = 5^2 = 25$. Others offer a challenge; for instance, $\lim_{x\to 2} \frac{x^3-2^3}{x-2}$.

If you want to find $\lim_{x\to a} f(x)$, and you know f is a continuous function with a in its domain, then you just calculate $f(a)$. In such a case there is no challenge and the limit is called *determinate*.

The interesting case for finding $\lim_{x\to a} f(x)$ occurs when f is not defined at a. That is when you must consider the influences operating on $f(x)$ when x is near a. You may have to do some algebra or perform numerical computations. Such limits are called *indeterminate*.

Here are the most common types of **indeterminate limits**:

Type $0/0$**:** $\lim_{x\to a} \frac{g(x)}{h(x)}$, where $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} h(x) = 0$

Type ∞/∞**:** $\lim_{x\to a} \frac{g(x)}{h(x)}$, where $\lim_{x\to a} g(x) = \infty$ and $\lim_{x\to a} h(x) = \infty$

Type $0 \cdot \infty$**:** $\lim_{x\to a} g(x)h(x)$, where $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} h(x) = \infty$

Type 1^∞**:** $\lim_{x\to a} g(x)^{h(x)}$, where $\lim_{x\to a} g(x) = 1$ and $\lim_{x\to a} h(x) = \infty$

Type 0^0**:** $\lim_{x\to a} g(x)^{h(x)}$, where $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} h(x) = 0$

The four limits encountered in Section 2.1, $\displaystyle\lim_{x\to a} \frac{x^n - a^n}{x - a}$, $\displaystyle\lim_{x\to 0} \frac{b^x - 1}{x}$, $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$, and $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x}$ are all indeterminate of type $0/0$..

We list the properties of limits which are helpful in computing limits.

Theorem 2.3.1 (Properties of Limits) *Let g and h be two functions and assume that $\lim_{x\to a} g(x) = A$ and $\lim_{x\to a} h(x) = B$. Then*

Each of these properties remains valid when the two-sided limit is replaced with a one-sided limit.

Sum $\lim_{x\to a}(g(x)+h(x)) = \lim_{x\to a} g(x) + \lim_{x\to a} h(x) = A+B$
the limit of the sum is the sum of the limits

Difference $\lim_{x\to a}(g(x)-h(x)) = \lim_{x\to a} g(x) - \lim_{x\to a} h(x) = A-B$
the limit of the difference is the difference of the limits

Constant Multiple $\lim_{x\to a}(kg(x)) = k\left(\lim_{x\to a} g(x)\right) = kA$, *for any constant* k
the limit of the difference is the difference of the limits

Product $\lim_{x\to a}(g(x)h(x)) = \left(\lim_{x\to a} g(x)\right)\left(\lim_{x\to a} h(x)\right) = AB$
the limit of the product is the product of the limits

Quotient $\lim_{x\to a}\left(\frac{g(x)}{h(x)}\right) = \frac{(\lim_{x\to a} g(x))}{(\lim_{x\to a} h(x))} = \frac{A}{B}$, *provided* $B \neq 0$
the limit of the quotient is the quotient of the limits, provided the denominator is not 0

Power $\lim_{x\to a}\left(g(x)^{h(x)}\right) = \left(\lim_{x\to a} g(x)\right)^{(\lim_{x\to a} h(x))} = A^B$, *provided* $A > 0$
the limit of a varying base to a varying power

EXAMPLE 4 Find $\lim_{x\to 0} \frac{(x^4-16)\sin(5x)}{x^2-2x}$.

SOLUTION Notice that the denominator can be factored to obtain

$$\frac{(x^4-16)\sin(5x)}{x^2-2x} = \frac{x^4-2^4}{x-2}\cdot\frac{\sin(5x)}{x}.$$

Only the second factor is indeterminate at 0.

This allows the limit to be rewritten as

$$\lim_{x\to 0}\frac{x^4-2^4}{x-2}\cdot\lim_{x\to 0}\frac{\sin(5x)}{x}$$

where we have also used $16 = 2^4$. Now, $\lim_{x\to 0}\frac{x^4-2^4}{x-2} = 4\cdot 2^{4-1} = 32$. Putting this together with one of the intermediate results of the previous example, we conclude that

$$\lim_{x\to 0}\frac{(x^4-16)\sin(5x)}{x^2-2x} = \lim_{x\to 0}\frac{x^4-2^4}{x-2}\cdot\lim_{x\to 0}\frac{\sin(5x)}{x} = 32\cdot 5 = 160.$$

◇

Summary

This section opened with an informal view of continuous functions, expressed in terms of a moving pencil. It then gave the definition, phrased in terms of limits, that we will use throughout the text. Determinate and indeterminate limits were discussed.

The discussion concludes in the next section, with three important properties of continuous functions.

EXERCISES for 2.3

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In Exercises 1–11, tell whether each of these limits is *determinant* or *indeterminant*. Do not evaluate the limit.

1. $\lim_{x\to 0} 2^x - 1$

2. $\lim_{x\to \infty} 2^x - 1$

3. $\lim_{x\to 1} \dfrac{3^x-1}{2^x-1}$

4. $\lim_{x\to 2} \dfrac{3^x-1}{2^x-1}$

5. $\lim_{x\to \infty} \dfrac{x}{2^x}$

6. $\lim_{x\to 0} \dfrac{x}{2^x}$

7. $\lim_{x\to 0^+} \dfrac{x^2}{e^x-1}$

8. $\lim_{x\to \frac{\pi}{2}^-} (\sin(x))^{\tan(x)}$

9. $\lim_{x\to 0^+} x\log_2 x$

10. $\lim_{x\to 0^+} (2+x)^{3/x}$

11. $\lim_{x\to \infty} (2+x)^{3/x}$

In Exercises 12–15, evaluate the limit.

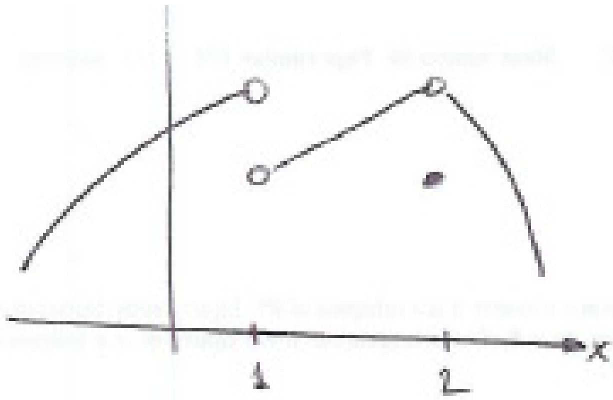
12. $\lim_{x\to \frac{\pi}{2}} \sin x \dfrac{e^x-1}{x}$

13. $\lim_{x\to 0} \dfrac{(\cos x)\,(e^x-1)}{x}$

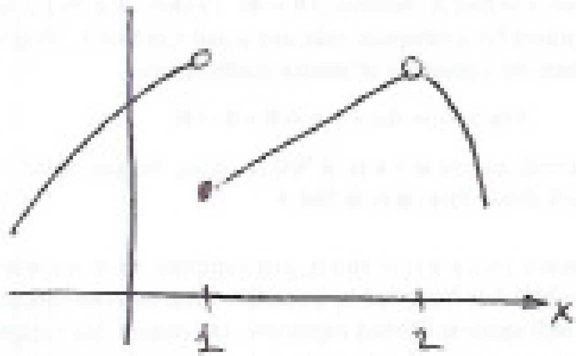
14. $\lim_{x\to 0} \dfrac{\sin(2x)}{x(\cos(3x))^2}$

15. $\lim_{x\to 1} \dfrac{(x-1)\cos x}{x^3-1}$

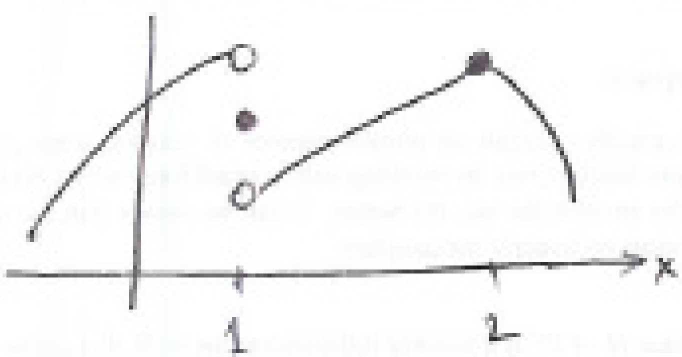
In Exercises 16 to 19 the graph of a function $y = f(x)$ is given. Determine all values of the number c for which $\lim_{x\to c} f(x)$ does not exist.



16.

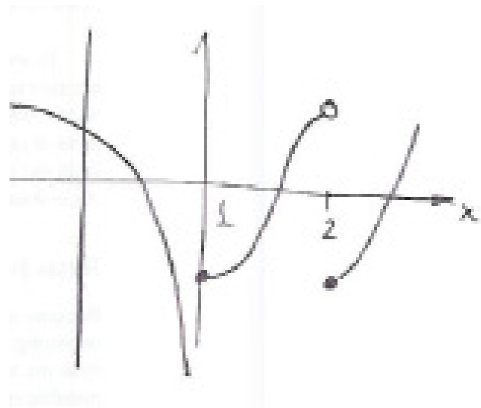


17.



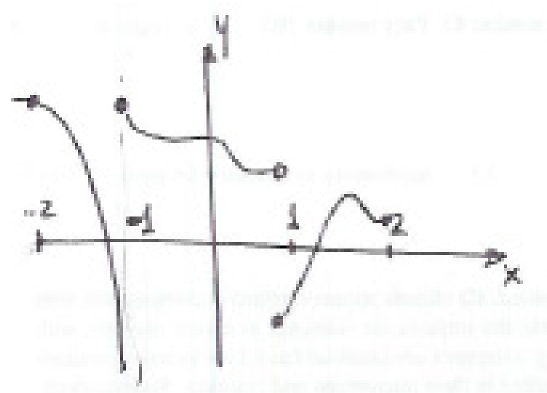
18.

19.



In Exercises 20 and 21 the graph of a function $y = f(x)$ and several intervals are given. For each interval, decide if the function is continuous on that interval.

20.



(a) $[-2, -1]$

(b) $(-2, -1)$

(c) $(-1, 1)$

(d) $[-1, 1)$

(e) $(-1, 1]$

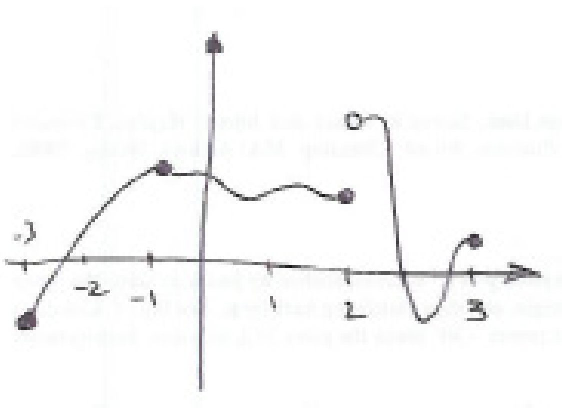
(f) $[-1, 1]$

(g) $(1, 2)$

(h) $[1, 2)$

(i) $(1, 2]$

(j) $[1, 2]$



21.

(a) $[-3, 2]$

(b) $(-1, 3)$

(c) $(-1, 2)$

(d) $[-1, 2)$

(e) $(-1, 2]$

(f) $[-1, 2]$

(g) $(2, 3)$

(h) $[2, 3)$

(i) $(2, 3]$

(j) $[2, 3]$

22. Let $f(x) = x + |x|$.

(a) Graph f.

(b) Is f continuous at -1?

(c) Is f continuous at 0?

23. Let $f(x) = 2^{1/x}$ for $x \neq 0$.

(a) Find $\lim_{x\to\infty} f(x)$.

(b) Find $\lim_{x\to-\infty} f(x)$.

(c) Does $\lim_{x\to 0^+} f(x)$ exist?

(d) Does $\lim_{x\to 0^-} f(x)$ exist?

(e) Graph f, incorporating the information from parts *(a)* to *(d)*.

(f) Is it possible to define $f(0)$ in such a way that f is continuous throughout the x-axis?

24. Let $f(x) = x\sin(1/x)$ for $x \neq 0$.

(a) Find $\lim_{x\to\infty} f(x)$.

(b) Find $\lim_{x\to-\infty} f(x)$.

(c) Find $\lim_{x\to 0} f(x)$.

(d) Is it possible to define $f(0)$ in such a way that f is continuous throughout the x-axis?

In Exercises 25–27 find and solve the equations that the parameters must satisfy for the function to be continuous.

25. $f(x) = \begin{cases} \frac{\sin(x)}{2x} & x \neq 0 \\ p & x = 0 \end{cases}$

26. $f(x) = \begin{cases} k & x \leq 0 \\ \arcsin(x) & 0 < x \leq \frac{\pi}{2} \\ p & x = 0 \end{cases}$

27. $f(x) = \begin{cases} \ln(x) & x > 1 \\ k - m\sqrt{x} & 0 < x \leq 1 \\ pe^{-x} & x \leq 0 \end{cases}$

28.

(a) Let f and g be two functions defined for all numbers. If $f(x) = g(x)$ when x is not 3, must $f(3) = g(3)$?

(b) Let f and g be two continuous functions defined for all numbers. If $f(x) = g(x)$ when x is not 3, must $f(3) = g(3)$?

Explain your answers.

29. *The reason* 0^0 *is not defined.* It might be hoped that if the positive number b and the number x are both close to 0, then b^x might be close to some fixed number. If that were so, it would suggest a definition for 0^0. Experiment with various choices of b and x near 0 and on the basis of your data write a paragraph on the theme, "Why 0^0 is not defined."

2.4 Three Important Properties of Continuous Functions

Continuous functions have three properties important in calculus: the "extreme-value" property, the "intermediate-value" property, and the "permanence" property. All three are quite plausible, and a glance at the graph of a typical continuous function may persuade us that they are true of all continuous functions. No proofs will he offered: they depend on the precise definitions of limits given in Sections 3.8 and 3.9 and are part of an advanced calculus course.

We will say that a function has a maximum at a point $(c, f(c))$ when $f(c) > f(x)$ for x near c. Likewise, a function has a minimum at a point $(c, f(c))$ when $f(c) < f(x)$ for x near c. Together, each maximum and minimum is referred to as an **extreme values** or **extremum** of the function.

The plural of extremum is extrema.

Extreme-Value Property

The first property is that a function continuous throughout the closed interval $[a, b]$ takes on a largest value somewhere in the interval.

Theorem 2.4.1 (Maximum-Value Property) *Let f be continuous throughout the closed interval $[a, b]$. Then there is at least one number in $[a, b]$ at which f takes on a maximum value. That is, for some number c in $[a, b]$, $f(c) \geq f(x)$ for all x in $[a, b]$.*

To persuade yourself that this is plausible, imagine sketching the graph of a continuous function. As your pencil moves along the graph from a to b it passes through a highest point. (See Figure 2.4.1.)

The maximum-value property guarantees that a maximum value exists, but it does *not tell how* to find it. The problem of finding it is addressed in Chapter 4.

There is also a **minimum-value property** that states that every continuous function on a closed interval takes on a smallest value somewhere in this interval. See Figure 2.4.1 for an ilustration of this property.

Theorem 2.4.2 (Extreme-Value Property) *Let f be continuous throughout the closed interval $[a, b]$. Then there is at least one number in $[a, b]$ at which f takes on a minimum value and there is at least one number in $[a, b]$ at which f takes on a maximum value. That is, for some numbers c and d in $[a, b]$, $f(d) \leq f(x) \leq f(c)$ for all x in $[a, b]$.*

EXAMPLE 1 Find all numbers in $[0, 3\pi]$ at which the cosine function, $f(x) = \cos x$, takes on a maximum value. Also, find all numbers in $[0, 3\pi]$ at which f takes on a minimum value.

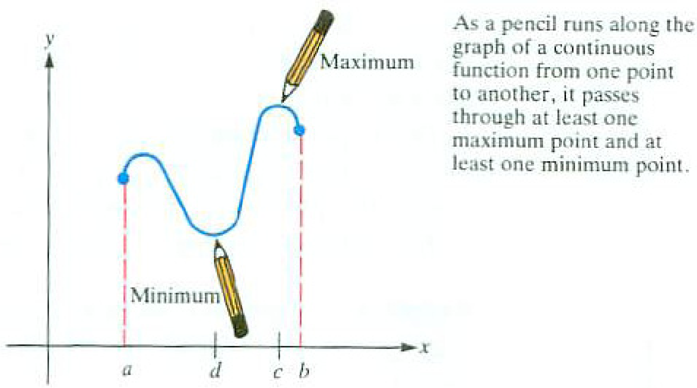


Figure 2.4.1: As a pencil runs along the graph of a continuous function from a to b it passes through at least one maximum point and at least one minimum point.

SOLUTION Figure 2.4.2 is a graph of $f(x) = \cos x$ for x in $[0, 3\pi]$. Inspection of the graph shows that the maximum value of $\cos x$ for $0 \leq x \leq 3\pi$ is 1, and it is attained twice: when $x = 0$ and when $x = 2\pi$. The minimum value is -1, which is also attained twice: when $x = \pi$ and when $x = 3\pi$. ◇

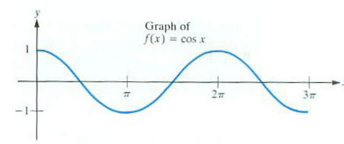


Figure 2.4.2:

The Extreme-Value Property has two assumptions: "f is continuous" and "the domain is a closed interval." If either of these conditions is removed, the conclusion need not hold.

Figure 2.4.3 shows the graph of a function that is *not* continuous, is defined on a closed interval, but has no maximum value. On the other hand $f(x) = \frac{1}{1-x^2}$ is continuous on $(-1, 1)$. It has no maximum value, as a glance at Figure 2.4.4 shows. This does not violate the Extreme-Value Property, since the domain $(-1, 1)$ is not a closed interval.

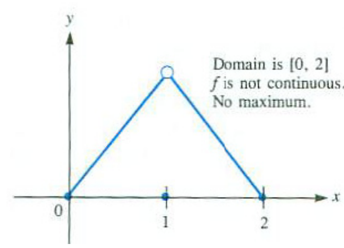


Figure 2.4.3:

Intermediate-Value Property

Imagine graphing a continuous function f defined on the closed interval $[a, b]$. As your pencil moves from the point $(a, f(a))$ to the point $(b, f(b))$ the y-coordinate of the pencil point goes through all values between $f(a)$ and $f(b)$. (Similarly, if you hike all day, starting at an altitude of 5.000 feet and ending at 11,000 feet, you must have been, say, at 7,000 feet at least once during the day. In mathematical terms, not in terms of a pencil (or a hike), "a function that is continuous throughout an interval takes on all values between any two of its values".

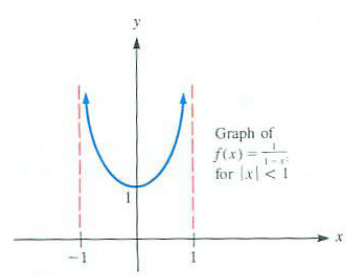


Figure 2.4.4:

Theorem 2.4.3 (Intermediate-Value Property) *Let f be continuous throughout the closed interval $[a, b]$. Let m be any number between $f(a)$ and $f(b)$. Then*

That is, $f(a) \leq m \leq f(b)$ if $f(a) \leq f(b)$, or $f(b) \leq m \leq f(a)$ if $f(b) \leq f(a)$.

there is at least one number c in $[a, b]$ such that $f(c) = m$.

Pictorially, the Intermediate-Value Property asserts that, if m is between $f(a)$ and $f(b)$, a horizontal line of height m must meet the graph of f at least once, as shown in Figure 2.4.5.

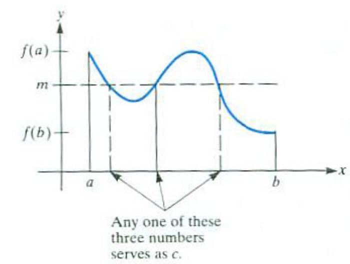


Figure 2.4.5:

Even though the property guarantees the existence of a certain number c, it does *not tell how* to find it. To find it we must solve an equation, namely, $f(x) = m$.

EXAMPLE 2 Use the Intermediate-Value Property to show that the equation $2x^3 + x^2 - x + 1 = 5$ has a solution in the interval $[1, 2]$.

SOLUTION Let $P(x) = 2x^3 + x^2 - x + 1$. Then

$$\begin{array}{rrcl} & P(1) & = & 2 \cdot 1^3 + 1^2 - 1 + 1 = 3 \\ \text{and} & P(2) & = & 2 \cdot 2^3 + 2^2 - 2 + 1 = 19. \end{array}$$

Since P is continuous (on $[1, 2]$) and $m = 5$ is between $P(1) = 3$ and $P(2) = 19$, the Intermediate-Value Property says there is at least one number c between 1 and 2 such that $P(c) = 5$.

To get a more accurate estimate for a number c such that $P(c) = 5$, find a shorter interval for which the Intermediate-Value Property can he applied. For instance, $P(1.2) = 4.696$ and $P(1.3) = 5.784$. By the Intermediate-Value Property, there is a number c in $[1.2.1.3]$ such that $P(c) = 5$. ◇

EXAMPLE 3 Show that the equation $-x^5 - 3x^2 + 2x + 11 = 0$ has at least one real root. That is, the function $-x^5 - 3x^2 + 2x + 11$ has an x-intercept.

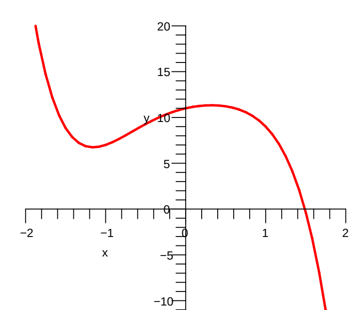
SOLUTION Let $f(x) = -x^5 - 3x^2 + 2x + 11$. We wish to show that there is a number c such that $f(c) = 0$. In order to use the Intermediate-Value Property, we need an interval $[a, b]$ for which 0 is between $f(a)$ and $f(b)$. Then we could apply that property, using $m = 0$.

One of $f(a)$ and $f(b)$ is positive and the other is negative.

We show that there are numbers a and b with $f(a) > 0$ and $f(b) < 0$. Because $\lim_{x\to\infty} f(x) = -\infty$, for x large and positive the polynomial $f(x)$ is negative. Thus, there is a number b such that $f(b) < 0$. Similarly, $\lim_{x\to-\infty} f(x) = \infty$, means that when x is negative and of large absolute value, $f(x)$ is positive. Thus there are numbers a and b, $a < b$, such that $f(a) > 0$ and $f(b) < 0$.

For definiteness, $a = 0$ and $b = 2$ are examples because $f(0) = 11 > 0$ and $f(2) = -29 < 0$.

The number 0 is between $f(a)$ and $f(b)$. Since f is continuous on the interval $[a, b]$, there is a number c in $[a, b]$ such that $f(c) = 0$. This number c is a solution to the equation $-x^5 - 3x^2 + 2x + 11 = 0$. ◇



Note that the argument in Example 3 applies to any polynomial of odd degree. Any polynomial of odd degree has a real root. The argument does not hold for polynomials of even degree; the equation $x^2 + 1 = 0$, for instance, has no real solutions.

EXAMPLE 4 Use the Intermediate-Value Property to show that there is a negative number such that $\ln(x + 4) = x^2 + 3$.

SOLUTION We wish to show that there is a negative number c where the function $\ln(x + 4)$ has the same value as the function $x^2 - 3$. The equation $\ln(x+4) = x^2 - 3$ is equivalent to $\ln(x+4) - x^2 + 3 = 0$. The problem reduces to showing that the function $f(x) = \ln(x+4) - x^2 + 3$ has the value 0 for some input c (with $c < 0$).

We will proceed, as we did in the previous example. We want to find numbers a and b (both negative) such that $f(a)$ and $f(b)$ have opposite signs.

Before beginning the search for a and b, note that $\ln(x + 4)$ is defined only for $x + 4 > 0$, that is, for $x > -4$. To complete the search for a and b, make a table of values of $f(x)$ for some sample arguments in $(-4, 0)$.

x	-3	-2	-1	0
$f(x)$	-6	-0.307	3.099	4.386

We see that $f(-2)$ is negative and $f(-1)$ is positive. Since $m = 0$ lies between $f(-2)$ and $f(-1)$, and f is continuous on $[-2, -1]$, the Intermediate-Value Property asserts that there is a number c, in $(-2, -1)$ such that $f(c) = 0$. It follows that $\ln(c + 4) = c^2 - 3$. ◇

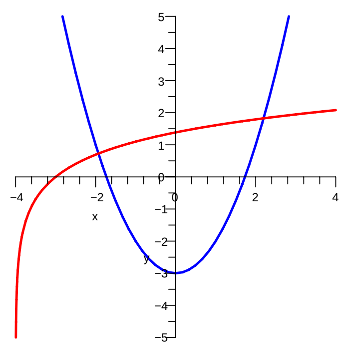


Figure 2.4.7: There are two points of intersection between these functions. The second intersection occurs for a positive value of x. (See Exercise 20.)

In Example 4 the Intermediate-Value Property does not tell what c is. The graphs of $\ln(x+1)$ and $x^2 - 3$ in Figure 2.4.7 suggest that there are two points of intersection, but only one with a negative input. The graph, and the table of values, suggest that the intersection point occurs when the input is close to -2. Calculations on a calculator or computer show that $c \approx -1.931$.

Permanence Property

The extrema property as well as the intermediate-value property involve the behavior of a continuous function throughout an interval. The next property concerns the "local" behavior of a continuous function.

Consider a continuous function f on an open interval that contains the number a. Assume that $f(a) = p$ is positive. Then it seems plausible that f remains positive in some open interval that contains a. We can say something stronger:

Theorem 2.4.4 (The Permanence Property) *Assume that the domain of a function f contains an open interval that includes the number a. Assume that f is continuous at a and that $f(a) = p$ is positive. Let q be any number less than p. Then there is an open interval including a such that $f(x) \geq q$ for all x in that interval.*

To persuade yourself that the permanence principle is plausible, imagine what the graph of $y = f(x)$ looks like near $(a, f(a))$, as in Figure 2.4.8.

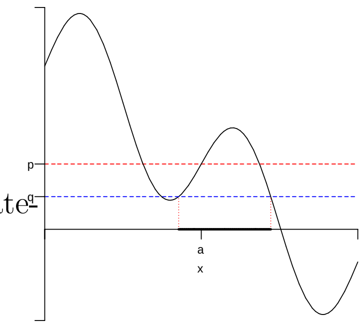


Figure 2.4.8:

Summary

This section statest (without proofs) the Extreme-Value Property, the Intermediate-Value Property, and the Permanence Property. In Chapter 4 limits will be used to develop the idea of a "derivative", one of the two fundamental tools in calculus.

EXERCISES for 2.4

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

1. For each of the given intervals, find the maximum value of $\cos x$ over that interval and find the value of x at which it occurs.

(a) $[0, \pi/2]$

(b) $[0, 2\pi]$

2. Does the function $\frac{x^3+x^4}{1+5x^2+x^6}$ have *(a)* a maximum value for x in $[1, 4]$? *(b)* a minimum value for x in $[1, 4]$? If so, where?

3. Does the function $2^x - x^3 + x^5$ have *(a)* a maximum value for x in $[-3, 10]$? *(b)* a minimum value for x in $[-3, 10]$? If so, where?

4. Does the function x^3 have a maximum value for x in *(a)* $[2, 4]$? *(b)* $[-3, 5]$? *(c)* $(1, 6)$? If so, where does the maximum occur and what is the maximum value?

5. Does the function x^4 have a minimum value for x in *(a)* $[-5, 6]$? *(b)* $(-2, 4)$? *(c)* $(3, 7)$? *(d)* $(-4, 4)$? If so, where does the minimum occur and what is the maximum value?

6. Does the function $2 - x^2$ have *(a)* a maximum value for x in $(-1, 1)$? *(b)* a minimum value for x in $(-1, 1)$? If so, where?

7. Does the function $2 + x^2$ have *(a)* a maximum value for x in $(-1, 1)$? *(b)* a minimum value for x in $(-1, 1)$? If so, where?

8. Show that the equation $x^5 + 3x^4 + x - 2 = 0$ has at least one solution in the interval $[0, 1]$.

9. Show that the equation $x^5 - 2x^3 + x^2 - 3x = -1$ has at least one solution in the interval $[1, 2]$.

In Exercises 10–14 verify the Intermediate-Value Property for the specified function f, the interval $[a, b]$, and the indicated value m. Find all c's in each case.

10. $f(x) = 3x + 5$, $[a, b] = [1, 2]$, $m = 10$.
11. $f(x) = x^2 - 2x$, $[a, b] = [-1, 4]$, $m = 5$.
12. $f(x) = \sin x$, $[a, b] = [\frac{\pi}{2}, \frac{11\pi}{2}]$, $m = -1$.
13. $f(x) = \cos x$, $[a, b] = [0, 5\pi]$, $m = \frac{\sqrt{3}}{2}$.
14. $f(x) = x^3 - x$, $[a, b] = [-2, 2]$, $m = 0$.

15. Show that the equation $2^x = 3x$ has a solution in the interval $[0, 1]$.

16. Does the equation $x + \sin x = 1$ have a solution?

17. Does the equation $x^3 = 2^x$ have a solution?

18. Use the Intermediate-Value Property to show that the equation $3x^3 + 11x^2 - 5x = 2$ has a solution.

19. Let $f(x) = 1/x$, $a = -1$, $b = 1$, $m = 0$. Note that $f(a) \le 0 \le f(b)$. Is there at least one c in $[a, b]$ such that $f(c) = 0$? If so, find c; if not, does this imply the Intermediate-Value Property sometimes does not hold?

20. Use the Intermediate-Value Property to show that there is a positive number such that $\ln(x + 4) = x^2 + 3$.

21. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of odd degree n and with positive leading coefficient a_n. Show that there is at least one real number r such that $P(r) = 0$.

22. (This continues Exercise 21.) The **factor theorem** from algebra asserts that the number r is a root of a polynomial $P(x)$ if and only if $x - r$ is a factor of $P(x)$. For instance, 2 is a root of the polynomial $x^2 - 3x + 2$ and $x - 2$ is a factor of the polynomial $x^2 - 3x + 2 = (x - 2)(x - 1)$. NOTE: See also Exercise 57.

(a) Use the factor theorem and Exercise 21 to show that every polynomial of odd degree has a factor of degree 1.

(b) Show that none of the polynomials x^2+1, $x^4 = 1$, or $x^{100}+1$ has a first-degree factor.

(c) Verify that $x^4 + 1 = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$. (It can be shown using complex numbers that every polynomial is the product of polynomials of degrees at most 2.)

Convex Sets and Curves

A set in the plane bounded by a curve is **convex** if for any two points P and Q in the set the line segment joining them also lies in the set. (See Figure 2.4.9.) The boundary of a convex set we will call a **convex curve**. (These ideas can generalize to a solid and its boundary surface.)

Disks, triangles, and parallelograms are convex sets. The quadrilateral shown in Figure 2.4.10 is not convex. Convex sets will be referred to in the following exercises and occassionally in the exercises in later chapters.

Exercises 23–34 concern convex sets and show how the Intermediate-Value Property can give geometric information. In these exercises you will need to define various functions geometrically. *You may assume these functions are continuous.*

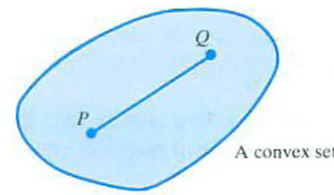


Figure 2.4.9: There are no dents in the boundary of a convex set.

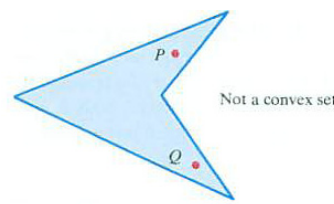


Figure 2.4.10: Not a convex set.

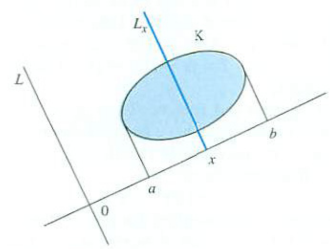


Figure 2.4.11:

23. Let L be a line in the plane and let K be a convex set. Show that there is a line parallel to L that cuts K into two pieces with equal areas.

Follow these steps.

(a) Introduce an x axis perpendicular to L with its origin on L. Each line parallel to L and meeting K crosses the x-axis at a number x. Label the line L_x. Let a be the smallest and b the largest of these numbers. (See Figure 2.4.11.) Let the area of K be A.

(b) Let $A(x)$ be the area of K situated to the left of the line corresponding to x. What is $A(a)$? $A(b)$?

(c) Use the Intermediate-Value Property to show that there is an x in $[a, b]$ such that $A(x) = \frac{A}{2}$.

(d) Why does *(c)* show that there is a line parallel to L that cuts K into two pieces of equal area?

24. Solve the preceding exercise by applying the Intermediate-Value Property to the function $f(x) = A(x) - B(x)$, where $B(x)$ is the area to the right of L_x.

25. Let P be a point in the plane and let K be a convex set. Is there a line through P that cuts K into two pieces of equal area?

26. Let K_1 and K_2 be two convex sets in the plane. Is there a line that simultaneously cuts K_1 into two pieces of equal area and cuts K_2 into two pieces of equal area? NOTE: This is known as the "two pancakes" question.

27. Let K be a convex set in the plane. Show that there is a line that simultaneously cuts K into two pieces of equal area and cuts the boundary of K into two pieces of equal length.

28. Let K be a convex set in the plane. Show that there are two perpendicular lines that cut K into four pieces of equal area. (It is not known whether it is always possible to find two perpendicular lines that divide K into four pieces whose areas are $\frac{1}{8}$, $\frac{1}{8}$, $\frac{3}{8}$, and $\frac{3}{8}$ of the area of K, with the parts of equal area sharing an edge, as in Figure 2.4.12.)

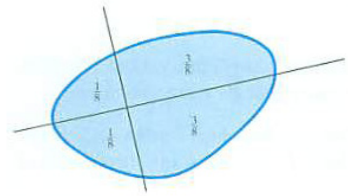


Figure 2.4.12:

29. Let K be a convex set in the plane whose boundary contains no line segments. A polygon is said to **circumscribe** K if each edge of the polygon is tangent to the boundary of K.

(a) Is there necessarily a circumscribing equilateral triangle? If so, how many?

(b) Is there necessarily a circumscribing rectangle? If so, how many?

(c) Is there necessarily a circumscribing square??

30. Let f be a continuous function whose domain is the x-axis and has the property that

$$f(x+y) = f(x) + f(y) \qquad \text{for all numbers } x \text{ and } y.$$

For any constant c, $f(x) = cx$ satisfies this equation since $c(x+y) = cx + cy$. This exercise shows that f must be of the form $f(x) = cx$ for some constant c.

(a) Let $f(1) = c$. Show that $f(2) = 2c$.

(b) Show that $f(0) = 0$.

(c) Show that $f(-1) = -c$.

(d) Show that that for any positive integer n, $f(n) = cn$.

(e) Show that that for any negative integer n, $f(n) = cn$.

(f) Show that $f(\frac{1}{2}) = \frac{c}{2}$.

(g) Show that that for any non-zero integer n, $f(\frac{1}{n}) = \frac{c}{n}$.

(h) Show that that for any intger m and any positive integer n, $f(\frac{m}{n}) = \frac{m}{n}c$.

(i) Show that for any irrational number x, $f(x) = cx$. This is where the continuity of f enters. Parts *(h)* and *(i)* together complete the solution.

Note: Verify that the function $f(x) = cx$ does satisfy the equation.

Exercises 31 and 32 illustrate the Permanence Principal.

31. Let $f(x) = 5x$. Then $f(1) = 5$. Find an interval (a, b) containing 1 such that $f(x) \geq 4.9$ for all x in (a, b).

32. Let $f(x) = x^2$. Then $f(2) = 4$. Find an interval (a, b) containing 2 such that $f(x) \geq 3.8$ for all x in (a, b).

33.

(a) Let f be a continuous function defined for all real numbers. Is there necessarily a number x such that $f(x) = x$?

(b) Let f be a continuous function with domain $[0, 1]$ such that $f(0) = 1$ and $f(1) = 0$. Is there necessarily a number x such that $f(x) = x$?

34. Let f be a continuous function defined on $(-\infty, \infty)$ such that $f(0) = 1$ and $f(2x) = f(x)$ for all numbers x.

(a) Give an example of such a function f.

(b) Find all functions satisfying these conditions.

Explain your answers.

35. Assume that $f(3)$ is 2, f is defined at least on an open interval containing 3, and that f is continuous at 3. Using the definition of continuity in terms of limits, explain why there must be an open interval around 3 in which $f(x)$ is always greater than or equal to 1. Hint: Assume that there is no such interval and show that f wouldn't be continuous at a.

2.S Chapter Summary

The text and additional exercises for the summary will be written after the organization of the chapters is firmly settled.

Sam: Why bother me with limits? In high school I learned the formulas for derivatives. The teacher told me they are rates of change. I can differentiate anything, using just 4 or 5 formulas. I love formulas.

Jane: Aren't you curious about why the formula for the derivative of a product is what it is?

Sam: No. It's been true for over three centuries. Just tell me what it is. If someone says the speed of light is 186,000 miles per second am I supposed to find a meter stick and clock and check it out?

Jane: But what if you forget the formula during a test?

Sam: That's not much of a reason.

Jane: But my physics class uses derivatives and limits to define basic concepts.

Sam: Oh?

Jane: Density of mass at a point or density of electric charge are defined as limits. And it uses derivatives all over the place. You will be lost if you don't know their definitions. Just look at the applications at the end of chapters ... for instance.

Sam: O.K., O.K. enough. I'll look.

EXERCISES for 2.S *Key:* R–routine, M–moderate, C–challenging In Exercises 1–17, tell whether each of these limits is *determinant* or *indeterminant*. Do not evaluate the limit.

1. $\lim_{x\to-\infty} 2^x - 1$

2. $\lim_{x\to 1} \frac{2^x-1}{3^x-1}$

3. $\lim_{x\to 0} 3^{-x}(x+2)$

4. $\lim_{x\to-\infty} 3^{-x}(x+2)$

5. $\lim_{x\to\frac{\pi}{4}} \frac{\sin(8x)}{\sin(4x)}$

6. $\displaystyle\lim_{x\to\frac{\pi}{4}} \frac{\sin(9x)}{\sin(4x)}$

7. $\displaystyle\lim_{x\to-\infty} \frac{1}{x3^x}$

8. $\displaystyle\lim_{x\to\frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)\tan(x)$

9. $\displaystyle\lim_{x\to\pi} \left(x-\frac{\pi}{2}\right)^3\cos(x)$

10. $\displaystyle\lim_{x\to\frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)^3\cos(x)$

11. $\displaystyle\lim_{x\to\frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)^3\sec(x)$

12. $\displaystyle\lim_{x\to0^+} \frac{x^2}{e^x-1}$

13. $\displaystyle\lim_{x\to0} (1+x)^{3/x}$

14. $\displaystyle\lim_{x\to\frac{\pi}{2}^-} (\tan(x))^{\sin(x)}$

15. $\displaystyle\lim_{x\to0^+} x\ln x$

16. $\displaystyle\lim_{x\to0^+} (2+x)^{3/x}$

17. $\displaystyle\lim_{x\to-1} (2+x)^{3/x}$

18. Define $f(x)=\begin{cases}\frac{x^3-3x^2-4x+k}{x-3} & x\neq 3\\ p & x=3\end{cases}$

(a) For what values of k and p is f continuous? (Justify your answer.)

(b) For these values of k and p, is f an even or odd function? (Justify your answer.)

19. Define $f(x)=\begin{cases}\frac{h(x)}{x-3} & x\neq 3\\ p & x=3\end{cases}$

(a) What conditions must be satisfied to make f continuous?

(b) What additional condition on h will make f an even function?

20. Assuming that $\lim_{x\to 0^+} x^x = 1$ and that $\lim_{x\to\infty} \ln(x) = \infty$, deduce each of the following limits:

(a) $\lim_{x\to 0} x\ln(x)$

(b) $\lim_{x\to\infty} \frac{\ln(x)}{x}$

(c) $\lim_{x\to\infty} x^{1/x}$

(d) $\lim_{x\to\infty} \frac{\ln(x)}{x^k}$, k a positive constant

(e) $\lim_{x\to\infty} \frac{x}{e^x}$

(f) $\lim_{x\to\infty} \frac{x^n}{e^x}$, n a positive integer

(g) $\lim_{x\to\infty} \frac{\ln(x)^n}{x}$, n a positive integer

21. (Contributed by G. D. Chakerian) This exercise obtains $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$ without using areas. Figure 2.4.1 shows a circle C of radius 1 center at the origin and a circle $C(r)$ of radius $r > 1$ that passes through the center of C. Let $S(r)$ be the part of $C(r)$ that lies within C. Its ends are P and Q. Let θ be the angle subtended by the top half of $S(r)$ at the center of $C(r)$. Note that as $r \to \infty$, $\theta \to 0$. Define $A(\theta)$ to be the length of the arc $S(r)$ as a function of θ.

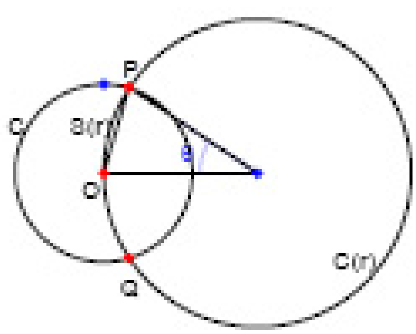


Figure 2.4.1:

(a) Looking at Figure 2.4.1, determine $\lim_{\theta\to 0} A(\theta)$. Hint: What happens to P as $r \to \infty$?

(b) Show that $A(\theta)$ is $2\frac{\theta/2}{\sin(\theta/2)}$.

(c) Combining (a) and (b), show that $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

Chapter 3

The Derivative

In this chapter we meet one of the two main concepts of calculus, the **derivative of a function**. The derivative tells how rapidly or slowly a function changes. For instance, if the function describes the position of a moving particle, the derivative tells us the velocity.

The definition of a derivative rests on the notion of a limit. The particular limits examined in Chapter 2 are the basis for finding the derivatives of all functions of interest. Fortunately, it is not necessary to evaluate a limit every time we find a derivative.

A few techniques allow us to find the derivative of almost any function that we will encounter. The goal of this chapter is twofold: to develop those techniques and an understanding of the meaning of a derivative.

3.1 Velocity and Slope: Two Problems with One Theme

This section discusses two problems which at first glance may seem unrelated. The first one concerns the slope of a tangent line to a curve. The second involves velocity. A little arithmetic will show that they are both just different versions of one mathematical idea: the *derivative.*

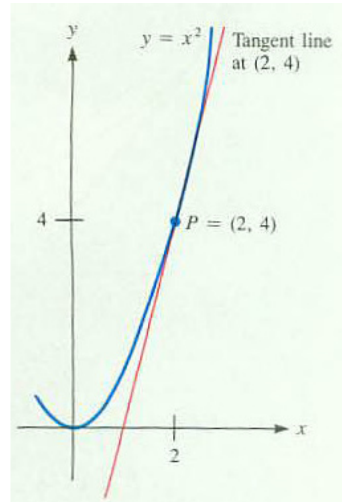


Figure 3.1.1:

Slope

Our first problem is important because it is related to finding the straight line that most closely resembles a given graph near a point on the graph.

EXAMPLE 1 What is the slope of the tangent line to the graph of $y = x^2$ at the point $P = (2, 4)$, as shown in Figure 3.1.1

For the present, the tangent line to a curve at a point P on the curve shall mean the line through P that has the "same direction" as the curve at P. (Look again at Figure 3.1.1.) This will he made precise in the next section.

SOLUTION If we know two points (x_1, y_1) and (x_2, y_2) on a straight line, we can compute the slope of that line. The slope is "change in y divided by change in x;" that is,

$$\text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}.$$

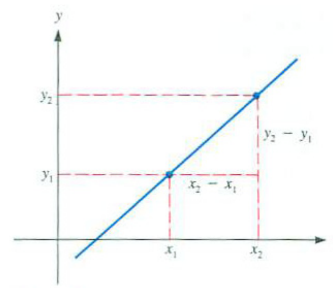


Figure 3.1.2:

Appendix B has a further discussion of the slope of a line.

See Figure 3.1.2. However, we know only one point on the tangent line at $(2, 4)$, namely, just the paint $(2, 4)$ itself. To get around this difficulty we will choose a point Q on the parabola $y = x^2$ near P and compute the slope of the line through P and Q. Such a line is called a **secant**. As Figure 3.1.3 suggests, such a secant line resembles the tangent line at $(2, 4)$. For instance, choose $Q = (2.1, 2.1^2)$ and compute the slope of the line through P and Q as shown in Figure 3.1.3(b).

$$\begin{aligned}\text{Slope of secant} &= \frac{\text{Change in } y}{\text{Change in } x}\\ &= \frac{2.1^2 - 2^2}{2.1 - 2}\\ &= \frac{4.41 - 4}{0.1}\\ &= \frac{0.41}{0.1}\\ &= 4.1.\end{aligned}$$

Thus an estimate of the slope of the tangent line is 4.1. Note that in making this estimate there was no need to draw the curve, or the secant.

We can also choose the point Q on the parabola to be to the left of $P = (2, 4)$. For instance, choose $Q = (1.9, 1.9^2)$. (See Figure 3.1.4.) Then

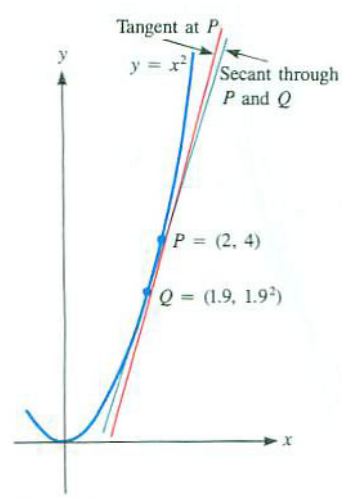
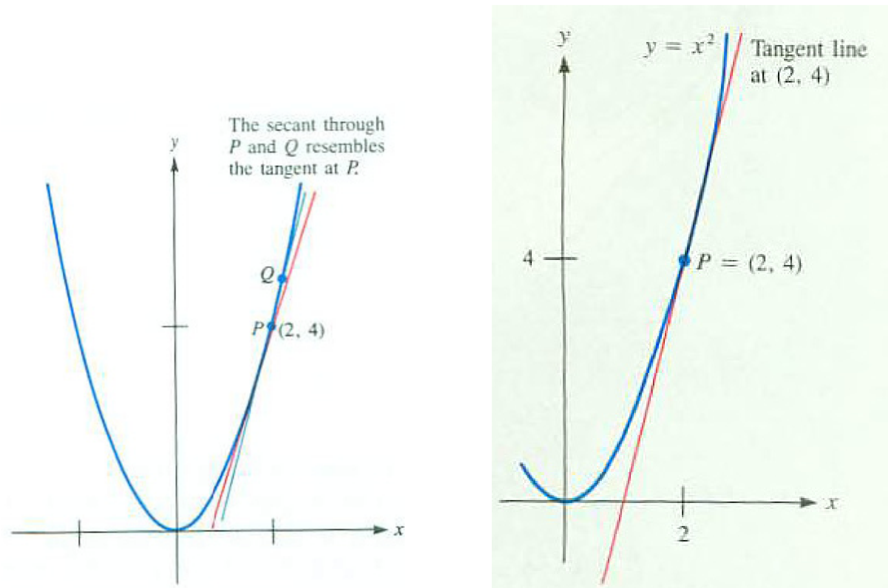


Figure 3.1.4:



(a) (b)

Figure 3.1.3:

$$\begin{aligned} \text{slope of secant} &= \tfrac{\text{Change in } y}{\text{Change in } x} \\ &= \tfrac{1.9^2-2^2}{1.9-2} \\ &= \tfrac{3.61-1}{-0.1} \\ &= \tfrac{-0.39}{-0.1} \\ &= 3.9. \end{aligned}$$

To obtain a better estimate, we could repeat the process using, for instance, the line through $P = (2, 4)$ and $Q = (2.01, 2.01^2)$. Rather than do this, it is simpler to consider a typical point Q. That is, consider the line through $P = (2, 4)$ and $Q = (x, x^2)$ when x is close to 2 but not equal to 2. (See Figures 3.1.3(a) and (b).) This line has slope

$$\frac{x^2 - 2^2}{x - 2}.$$

To find out what happens to this quotient as Q moves closer to P apply the techniques of limits (see Section 2.1). We have

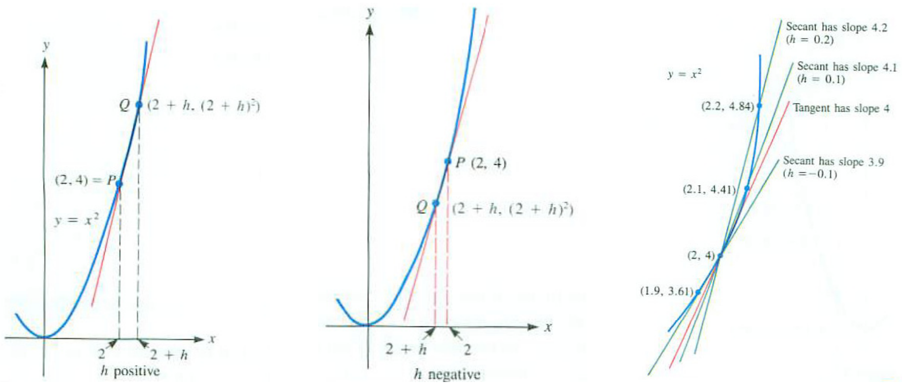
Q moves closer to P when x moves closer to 2.

Recall $a^2 - b^2 = (a + b)(a - b)$.

$$\lim_{x\to 2} \frac{x^2 - 2^2}{x - 2} = \lim_{x\to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x\to 2}(x + 2) = 4.$$

Thus, the tangent line to $y = x^2$ at $(2, 4)$ has slope 4.

Figure 3.1.5(c) shows how secant lines approximate the tangent line. It suggests a blowup of a small part of the curve $y = x^2$. ◇



(a) (b) (c)

Figure 3.1.5:

Velocity

If an airplane or automobile is moving at a constant velocity, we know that "distance traveled equals velocity times time." Thus

$$\text{velocity} = \frac{\text{distance traveled}}{\text{elapsed time}}.$$

If the vehicle's velocity is *not* constant, we still may speak of its "average velocity," which is defined as

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{elapsed time}}.$$

For instance, if you drive from San Francisco to Los Angeles, a distance of 400 miles, in 8 hours, the average velocity is 400 miles/8 hours = 50 miles per hour.

Suppose that up to time t_l you have traveled a distance D_1, while up to time t_2 you have traveled a distance D_2, where $t_2 > t_1$. Then during the time interval $[t_1, t_2]$ the distance traveled is $D_2 - D_1$. Thus the average velocity during the time interval $[t_1, t_2]$, which has duration $t_2 - t_1$, is

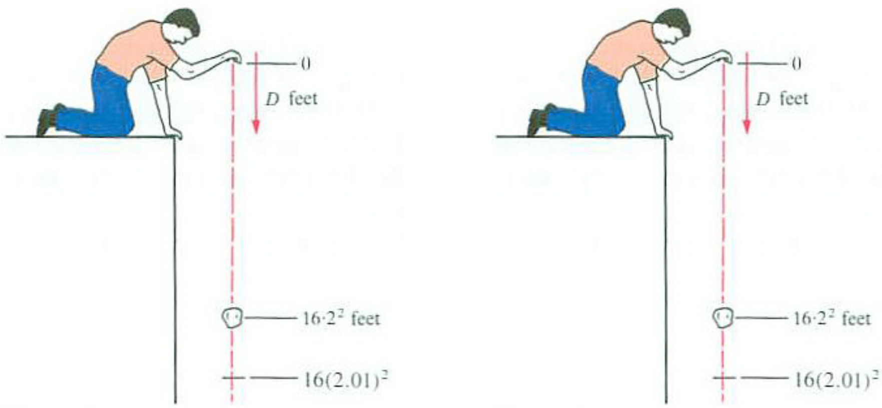
This observation shows the underlying similarity of the velocity and slope problems.

$$\text{average velocity} = \frac{D_2 - D_1}{t_2 - t_1}.$$

The computation of average velocity is the same as that for the slope of a line.

The next problem shows how to find the velocity at any instant for an object whose velocity is not constant.

EXAMPLE 2 A rock initially at rest falls $16t^2$ feet in t seconds. What is its velocity after 2 seconds?



(a) (b)

Figure 3.1.6: Note: (b) needs to have 2.01 replaced by t.

SOLUTION If the rock moves a distance of D feet during t seconds, we know what is meant by its average velocity during that time, namely, the quotient D/t feet per second. We will use this idea to deal with the abstract idea of "velocity at a given time," the so-called **instantaneous velocity**. In the next section the notion of velocity at a given instant will be made precise.

For practice, make an estimate by finding the average velocity of the rock during a short time interval, say from 2 to 2.01 seconds. At the start of this interval the rock has fallen $16(2^2) = 64$ feet. By the end it has fallen $16(2.01^2) = 16(4.0401) = 64.6416$ feet. So, during this interval of 0.01 seconds the rock fell 0.6416 feet. Its average velocity during this time interval is

$$\text{average velocity} = \frac{64.16 - 64}{2.00 - 2} = \frac{0.16}{0.01} = 16 \text{ feet per second},$$

an estimate of the velocity at time $t = 2$ seconds. (See Figure 3.1.6(a).)

Rather than make another estimate with the aid of a still shorter interval of time, let us consider the typical time interval from 2 to t seconds, $t > 2$. During this short time of $t-2$ seconds the rock travels $16(t^2) - 16(2^2) = 16(t^2 - 2^2)$ feet, as shown in Figure 3.1.6(b). The average velocity of the rock during this period is

$$\text{average velocity} = \frac{16t^2 - 16(2^2)}{t-2} = \frac{16(t^2 - 2^2)}{t-2} \text{ feet per second.}$$

Although we will keep $t > 2$, estimates could just as well be made with $t < 2$.

When t is close to 2, what happens to the average velocity? It approaches

$$\lim_{t\to 2} \frac{16(t^2 - 2^2)}{t-2} = 16 \lim_{t\to 2} \frac{t^2 - 2^2}{t-2} = 16 \lim_{t\to 2}(t+2) = 16 \cdot 4 = 64 \text{ feet per second.}$$

We say that the velocity at time $t = 2$ is 64 feet per second. ◇

Even though Examples 1 and 2 seem unrelated, their solutions turn out to be practically identical: The slope in Example 1 is approximated by the quotient

$$\frac{x^2 - 2^2}{x - 2}$$

and the velocity in Example 2 is approximated by the quotient

$$\frac{16t^2 - 16(2^2)}{t - 2} = 16 \cdot \frac{t^2 - 2^2}{t - 2}.$$

The only difference between the solutions is that the second quotient has an extra factor of 16 and x is replaced with t. This may not be too surprising, since the functions involved, x^2 and $16t^2$ differ by a factor of 16. (That the independent variable is named t in one case and x in the other does not affect the computations.)

A variable by any name is a variable.

The Derivative of a Function

In both the slope and velocity problems we were lead to studying similar limits. For the function x^2 it was

$$\frac{x^2 - 2^2}{x - 2} \quad \text{as } x \text{ approaches 2.}$$

For the function $16t^2$ it was

$$\frac{16t^2 - 16(2^2)}{t - 2} \quad \text{as } t \text{ approaches 2.}$$

In both cases we formed "change in outputs divided by change in inputs" and then found the limit as the change in inputs became smaller and smaller. This can be done for other functions, and brings us to one of the two key ideas in calculus, the *derivative of a function.*

DEFINITION *(Derivative of a function at a number a)* Let f be a function that is defined at least in some open interval that contains the number a. If

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists, it is called the **derivative of f at a**, and is denoted $f'(a)$. In this case the function f is said to be **differentiable at a**. If the limit does not exist, then f is **nondifferentiable at a**.

Read $f'(a)$ as "f prime at a" or "the derivative of f at a."

EXAMPLE 3 Find the derivative of $f(x) = 16x^2$ at 2.
SOLUTION In this case, $f(x) = 16x^2$ for any input x. By definition, the derivative of this function at 2 is

$$\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{16x^2-16(2^2)}{x-2} = 16 \lim_{x\to 2} \frac{x^2-2^2}{x-2} = 16 \lim_{x\to 2}(x+2) = 64.$$

We say that "the derivative of the function $f(x)$ at 2 is 4" and write $f'(2) = 4$.
◇

Differentiability and Continuity

A function that has a derivative at a is said to be **differentiable at** a. If it is differentiable at each point in its domain the function is said to be **differentiable**.

A small piece of the graph of a differentiable function at a looks almost like part of a straight line. You can check this by zooming in on the graph of a function of your choice. Differential calculus can be described as the study of functions where graphs locally look almost like a line.

It is no surprise that a differentiable function is always continuous. To show that a function is continuous at an argument a in its domain we must show that $\lim_{x\to a} f(x) - f(a) = 0$. To relate this limit to $f'(a)$ we rewrite the limit as

$$\begin{aligned} \lim_{x\to a} f(x) - f(a) &= \lim_{x\to a} \left(\tfrac{f(x)-f(a)}{x-a}(x-a)\right) \\ &= \lim_{x\to a} \left(\tfrac{f(x)-f(a)}{x-a}\right) \lim_{x\to a}(x-a) \\ &= f'(a)(0) \qquad \text{by the definition of th} \\ &= 0. \end{aligned}$$

So, f is continuous at a. Note the use of the definition of the derivative in the last step. This line of reasoning is valid provided the derivative exists.

A function can be continuous yet not differentiable. For instance, $f(x) = |x|$ is continuous but not differentiable at 0, as Figure 3.1.7 suggests.

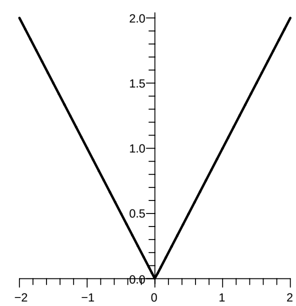


Figure 3.1.7:

differentiate *v.* to find the derivative of

Summary

From a mathematical point of view, the problems of finding the slope of the tangent line and the velocity of the rock are the same. In each case estimates lead to the same type of quotient, $\frac{f(x)-f(a)}{x-a}$. The behavior of this *difference quotient* is studied as x approaches a. In each case the answer is a limit, called the derivative of the function at a given number.

EXERCISES for 3.1

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

Exercises 1–4 review the concept of slope of a line. (See also Appendix B.)

1. What angle does a line make with the x-axis if its slope is

(a) 1?

(b) 2?

(c) 1/2?

NOTE: Use a calculator for (b) and (c).

2. What angle does a line make with the x-axis if its slope is

(a) -1?

(b) -2?

(c) $-1/2$?

HINT: Remember that the angle is in the second quadrant. NOTE: Use a calculator for (b) and (c).

3. Draw x- and y-axes and a line that is neither horizontal nor vertical. Using a ruler, estimate the slope of the line you drew.

4. Draw the line through $(1, 2)$ that has

(a) slope $\frac{3}{2}$.

(b) slope $\frac{-3}{2}$.

Exercises 5–12 concern slope. In each case use the technique of Example 1 to find the slope of the tangent line to the curve at the point.

5. $y = x^2$ at the point $(3, 3^2) = (3, 9)$

6. $y = x^2$ at the point $(\frac{1}{2}, \left(\frac{1}{2}\right)^2) = (\frac{1}{2}, \frac{1}{4})$

7. $y = x^2$ at the point $(-2, (-2)^2) = (-2, 4)$

8. $y = x^2$ at the point $(1, 1^2) = (1, 1)$

9. $y = x^3$ at the point $(2, 2^3) = (2, 8)$

10. $y = x^3$ at the point $(1, 1^3) = (1, 1)$

11.

(a) $y = x^2$ at the point $(0, 0)$

(b) Sketch the graph of $y = x^2$ and the tangent line at $(0, 0)$.

12.

(a) $y = x^3$ at the point $(0, 0)$

(b) Sketch the graph of $y = x^2$ and the tangent line at $(0, 0)$. NOTE: Be particularly careful when sketching the graph near $(0, 0)$. In this case the tangent line crosses the curve.

In Exercises 13–16 use the method of Example 2 to find the velocity of the rock after

13. 3 seconds

14. $\frac{1}{2}$ second

15. 1 second

16. $\frac{1}{4}$ second

17. A certain object travels t^3 feet in the first t seconds.

(a) How far does it travel during the time interval from 2 to 2.1 seconds?

(b) What is the average velocity during that time interval?

(c) Let h be any positive number. Find the average velocity of the object from time 2 to $2 + h$ seconds. HINT: To find $(2 + h)^3$, just multiply out the three terms.

(d) Find the velocity of the object at 2 seconds by letting h approach 0 in the result found in (c).

18. A certain object travels t^3 feet in the first t seconds.

(a) Find the average velocity during the time interval from 3 to 3.01 seconds?

(b) Find its average velocity of the object during the time interval from 3 to t seconds, $t > 3$.

(c) By letting t approach 3 in the result found in (b), find the velocity of the object at 3 seconds.

In the slope problem the nearby point Q was always pictured as being to the right of P. The point Q could just as well have been chosen to the left of P. Exercises 19 and 20 illustrate this case.

19. Consider the parabola $y = x^2$.

(a) Find the slope of the line through $P = (2, 4)$ and $Q = (1.99, 1.99^2)$.

(b) Find the slope of the line through $P = (2, 4)$ and $Q = (2.1, 2.1^2)$.

(c) Find the slope of the line through $P = (2, 4)$ and $Q = (2 + h, (2 + h)^2)$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (b) approaches 4.

20. Consider the curve $y = x^3$.

(a) Find the slope of the line through $P = (2, 8)$ and $Q = (1.9, 1.9^3)$.

(b) Find the slope of the line through $P = (2, 8)$ and $Q = (2.01, 2.01^3)$.

(c) Find the slope of the line through $P = (2, 8)$ and $Q = (2 + h, (2 + h)^3)$, where $h \neq 0$.

(d) Show that as h approaches 0, the slope in (b) approaches 12.

21.

(a) Find the slope of the tangent line to $y = x^2$ at $(4, 16)$.

(b) Use it to draw the tangent line to the curve at $(4, 16)$.

22.

(a) Find the slope of the tangent line to $y = x^2$ at $(-1, 1)$.

(b) Use it to draw the tangent line to the curve at $(-1, 1)$.

23.

(a) Use the method of this section to find the slope of the curve $y = x^3$ at $(1, 1)$.

(b) What does the graph of $y = x^3$ look like near $(1, 1)$?

24.

(a) Use the method of this section to find the slope of the curve $y = x^3$ at $(-1, -1)$.

(b) What does the graph of $y = x^3$ look like near $(-1, -1)$?

25. With the aid of a calculator, estimate the slope of $y = 2^x$ at $x = 1$, using the intervals

(a) $[1, 1.1]$

(b) $[1, 1.01]$

(c) $[0.9, 1]$

(d) $[0.99, 1]$

26. With the aid of a calculator, estimate the slope of $y = \frac{x+1}{x+2}$ at $x = 2$, using the intervals

(a) $[2, 2.1]$

(b) $[2, 2.01]$

(c) $[2, 2.001]$

(d) $[1.999, 2]$

The ideas common to both slope and velocity also appear in other applications. Exercises 27 to 30 present the same ideas in biology, economics, and physics.

27. A certain bacterial culture has a mass of t^2 grams after t minutes of growth.

(a) How much does it grow during the time interval $[2, 2.01]$?

(b) What is the average rate of growth during the time interval $[2, 2.01]$?

(c) What is the instantaneous rate of growth when $t = 2$?

28. A thriving business has a profit of t^2 million dollars in its first t years. Thus from time $t = 3$ to time $t = 3.5$ (the first half of its fourth year) it has a profit of $(3.5)^2 - 3^2$ million dollars, giving an annual rate of

$$\frac{(3.5)^2 - 3^2}{0.5} = 6.5 \text{ million dollars per year.}$$

(a) What is its annual rate of profit during the time interval $[3, 3.1]$?

(b) What is its annual rate of profit during the time interval $[3, 3.01]$?

(c) What is its instantaneous rate of profit after 3 years?

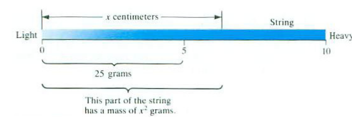


Figure 3.1.8:

Exercises 29 and 30 concern density.

29. The mass of the left-hand x centimeters of a nonhomogeneous string 10 centimeters long is x^2 grams, as shown in Figure 3.1.8. For instance, the string in the interval $[0, 5]$ has a mass of $5^2 = 25$ grams and the string in the interval $[5, 6]$ has mass $6^2 - 5^2 = 11$. The **density** of the string is the total mass divided by the length of the string. NOTE: density $= \frac{\text{total mass}}{\text{length}}$ grams.

(a) What is the mass of the string in the interval $[3, 3.01]$?

(b) Using the interval $[3, 3.01]$, estimate the density at 3.

(c) Using the interval $[2.99, 3]$, estimate the density at 3.

(d) By considering intervals of the form $[3, 3 + h]$, h positive, find the density at the point 3 centimeters from the left end.

(e) By considering intervals of the form $[3 + h, 3]$, h negative, find the density at the point 3 centimeters from the left end.

30. The left x centimeters of a string have a mass of x^2 grams.

(a) What is the mass of the string in the interval $[2, 2.01]$?

(b) Using the interval $[2, 2.01]$, estimate the density at 2.

(c) Using the interval $[1.99, 2]$, estimate the density at 2.

(d) By considering intervals of the form $[2, 2 + h]$, h positive, find the density at the point 2 centimeters from the left end.

(e) By considering intervals of the form $[2 + h, 2]$, h negative, find the density at the point 2 centimeters from the left end.

31.

(a) Graph the curve $y = 2x^2 + x$.

(b) By eye, draw the tangent line to the curve at the point $(1, 3)$. Using a ruler, estimate the slope of the tangent line.

(c) Sketch the line that passes through the point $(1, 3)$ and the point $(x, 2x^2 + x)$.

(d) Find the slope of the line in (c).

(e) Letting x get closer and closer to 1, find the slope of the tangent line at $(1, 3)$.

(f) How close was your estimate in (b)?

32. An object travels $2t^2 + t$ feet in t seconds.

(a) Find its average velocity during the interval of time $[1, x]$, where x is not 0.

(b) Letting x get closer and closer to 1, find the velocity at time 1.

(c) How close was your estimate in (b)?

33. Find the slope of the tangent line to the curve $y = x^2$ of Example 1 at the typical point $P = (x, x^2)$. To do this, consider the slope of the line through P and the nearby point $Q = (x + h, (x + h)^2)$ and let h approach 0.

34. Find the velocity of the falling rock of Example 2 at any time t. To do this, consider the average velocity during the time interval $[t, t + h]$ and then let h approach 0.

35. Does the tangent line to the curve $y = x^2$ at the point $(1, 1)$ pass through the point $(6, 12)$?

36.

(a) Graph the curve $y = 2^x$ as well as you can for $-2 \leq x \leq 3$.

(b) Using a straight edge, draw as well as you can a tangent to the curve at $(2, 4)$. Estimate the slope of this tangent by using a ruler to draw a "rise-and-run" triangle.

(c) Using a secant through $(2, 4)$ and $(x, 2^x)$, for x near 2, estimate the slope of the tangent to the curve at $(2, 4)$. Hint: Choose particular values of x and use your calculator to create a table of your results.

37.

(a) Using your calculator estimate the slope of the tangent line to the graph of $f(x) = \sin(x)$ at $(0, 0)$.

(b) At what (famous) angle do you think the curve crosses the x-axis at $(0, 0)$.

38.

(a) Sketch the curve $y = x^3 - x^2$.

(b) Using the method of the nearby point, find the slope of the tangent line to the curve at the point $(x, x^3 - x^2)$.

(c) Find all points on the curve where the tangent line is horizontal.

(d) Find all points on the curve where the tangent line has slope 1.

39. Repeat Exercise 38 for the curve $y = x^3 - x$.

40. An astronaut is traveling from left to right along the curve $y = x^2$. When she shuts off the engine, she will fly off along the line tangent to the curve at the point where she is at the moment the engines turn off. At what point should she shut off the engine in order to reach the point

(a) $(4, 9)$?

(b) $(4, -9)$?

41. See Exercise 40. Where can an astronaut who is traveling from left to right along $y = x^3 - x$ shut off the engine and pass through the point $(2, 2)$?

3.2 The Derivatives of the Basic Functions

In this section we use the definition of the derivative to find the derivatives of the important functions x^a (a rational), e^x, $\sin x$, and $\cos x$. We also introduce some of the standard notations for the derivative.

DEFINITION *(Derivative at a number)* Assume that the function f is defined at least in an open interval containing a. If

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \tag{1}$$

exists, it is called the **derivative of f at a**.

There are several notations for the quotient that appears in (1) and also for the derivative itself. Sometimes it is convenient to use $a+h$ instead of x and let h approach 0. Then, (1) reads

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}. \tag{2}$$

Expression (2) says the same thing as expression (1). "See how the quotient, change in output divided by change in input, behaves as the change in input gets smaller and smaller."

Sometimes it is useful to call the change in output "Δf" and the change in input "Δx." That is, $\Delta f = f(x) - f(a)$ and $\Delta x = x - a$. Then

The symbol Δ is Greek for "D"; it is pronounced "delta".

$$f'(a) = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}. \tag{3}$$

In mathematics, "Δ" generally indicates difference or change.

There is nothing sacred about the letters a, x, and h. One could say

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \tag{4}$$

or

$$f'(x) = \lim_{u\to x} \frac{f(u)-f(x)}{u-x}. \tag{5}$$

The symbol "$f'(a)$" is read aloud as "f prime at a" or "the derivative of f at a." The symbol $f'(x)$ is read similarly. However, the notation $f'(x)$ tells us that f', like f, is a function. For each input x the derivative, $f'(x)$, is the output. The derivative of the function f can also be written as $D(f)$.

The derivative of a specific function, such as x^2, is denoted $(x^2)'$ or $D(x^2)$. Then, $D(x^2) = 2x$, read aloud as "the derivative of x^2 is $2x$." This is shorthand for “the derivative of the function that assigns x^2 to x is the function that assigns $2x$ to x.” Since the value of derivative depends on x, it is a function.

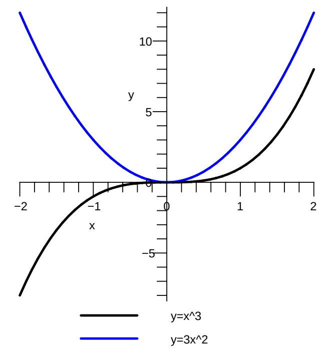


Figure 3.2.1:

EXAMPLE 1 Find the derivative of x^3 at a.
SOLUTION

$$\begin{aligned}(x^3)' &= \lim_{x\to a} \frac{x^3 - a^3}{x - a}\\ &= 3a^2\end{aligned}$$

Observe that this is one of the four limits in Section 2.1 (page 51)

We can write $(x^3)' = 3x^2$ or $D(x^3) = 3x^2$. ◇

In the same manner, Theorem 2.1.1 says that for any positive integer n, the derivative of x^n is nx^{n-1}. The exponent n becomes the coefficient and the exponent of x shrinks from n to $n-1$:

Theorem 3.2.1 (The derivative of x^n)

$$(x^n)' = nx^{n-1} \qquad \textit{where } n \textit{ is a positive integer.}$$

The next example treats an exponential function with a fixed base.
EXAMPLE 2 Find the derivative of 2^x.

SOLUTION

$$\begin{aligned}D(2^x) &= \lim_{h\to 0} \frac{2^{(x+h)} - 2^x}{h}\\ &= \lim_{h\to 0} \frac{2^x 2^h - 2^x}{h}\\ &= \lim_{h\to 0} 2^x \frac{2^h - 1}{h}\\ &= 2^x \lim_{h\to 0} \frac{2^h - 1}{h}.\end{aligned}$$

We found that $\lim_{h\to 0} \frac{2^h-1}{h} \approx 0.693$. Thus,

See Section 2.1, page 51.

$$D(2^x) \approx (0.693)2^x.$$

◇

No one wants to remember the (approximate) constant 0.693 when we use base 2. Watch what happens if the base is e, chosen because $\lim_{h\to 0} \frac{e^h-1}{h} = 1$.

We have

$$\begin{aligned} D(e^x) &= \lim_{h\to 0} \frac{e^{(x+h)} - e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h\to 0} e^x \frac{e^h - 1}{h} \\ &= e^x \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^x \cdot 1 = e^x. \end{aligned}$$

We have the important, and simple, formula

Theorem 3.2.2 (The derivative of e^x)

$$D(e^x) = e^x.$$

The function e^x has the remarkable property that the function is equal to its derivative.

Next, we turn to trigonometric functions.

EXAMPLE 3 Find the derivative of $\sin x$.

SOLUTION

$$\begin{aligned} D(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h\to 0} \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}. \end{aligned}$$

In Section 2.1 we found values for these two limits: $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$. Thus,

$$D(\sin x) = (\sin x)(0) + (\cos x)(1) = \cos x.$$

◇ We have the important formula

Theorem 3.2.3 (The derivative of $\sin(x)$)

$$D(\sin(x)) = \cos x.$$

If we graph $y = \sin(x)$ (see Figure 3.2.2), and consider its shape, the formula $D(\sin(x)) = \cos(x)$, is not a surprise. For instance, for x in $(-\pi/2, \pi/2)$ the slope is positive. So is $\cos(x)$. For x in $(\pi/2, 3\pi/2)$ the slope of the sine curve is negative. So is $\cos(x)$. Since $\sin(x)$ has period 2π, we would expect its derivatve also to have period 2π. Indeed, $\cos(x)$ does have period 2π.

In a similar manner, using the definition of the derivative and the identity $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, one can show that

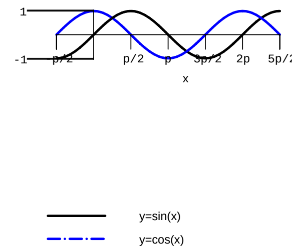


Figure 3.2.2:

Theorem 3.2.4 (The derivative of $\cos(x)$)

$$D(\cos(x)) = -\sin(x).$$

Derivatives of Other Power Functions

We showed that if n is a positive integer, $D(x^n) = nx^{n-1}$. Now let us find the derivative of power functions x^n where n is not a positive integer.

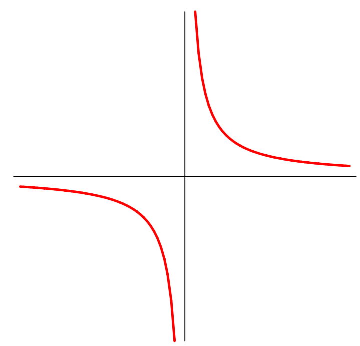


Figure 3.2.3:

EXAMPLE 4 Find the derivative of $x^{-1} = \frac{1}{x}$.
SOLUTION Before we calculate the necessary limit, let's pause to see how the slope of $y = 1/x$ behaves. A glance at Figure 3.2.3 shows that the slope is always negative. Also, for x near 0, the slope is large, but when $|x|$ is large, the slope is near 0.

Now, let's find the derivative of $1/x$:

$$\begin{aligned} D(1/x) &= \lim_{t \to x} \frac{1/t - 1/x}{t - x} \\ &= \lim_{t \to x} \frac{1}{t - x} \left(\frac{x - t}{xt} \right) \\ &= \lim_{t \to x} \frac{-1}{xt} \\ &= -\frac{1}{x^2}. \end{aligned}$$

As a check, note that $-1/x^2$ is always negative, is large when x is near 0, and is near 0 when $|x|$ is large. $\diamond$

It is worth memorizing that

Theorem 3.2.5 (Derivative of x^{-1})

$$D\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Or, written in exponential notation,

$$D(x^{-1}) = -x^{-2}.$$

The second form fits into the pattern established for positive integers n, $D(x^n) = nx^{n-1}$.

EXAMPLE 5 Find the derivative of $x^{2/3}$.
SOLUTION Once again we use the definition of the derivative:

$$D(x^{2/3}) = \lim_{t\to x} \frac{t^{2/3} - x^{2/3}}{t - x}.$$

A bit of algebra will help us find that limit. We write the four terms $t^{2/3}$, $x^{2/3}$, t, and x as powers of $t^{1/3}$ and $x^{1/3}$. Thus

$$D(x^{2/3}) = \lim_{t\to x} \frac{\left(t^{1/3}\right)^2 - \left(x^{1/3}\right)^2}{\left(t^{1/3}\right)^3 - \left(x^{1/3}\right)^3}.$$

Recalling that $a^2 - b^2 = (a-b)(a+b)$ and $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, we find

If you don't recall these formulas, multiply out $(a-b)(a+b)$ and $(a-b)(a^2+ab+b^2)$.

$$\begin{aligned}
D(x^{2/3}) &= \lim_{t\to x} \frac{\left(\left(t^{1/3}\right) - \left(x^{1/3}\right)\right)\left(\left(t^{1/3}\right) + \left(x^{1/3}\right)\right)}{\left(\left(t^{1/3}\right) - \left(x^{1/3}\right)\right)\left(\left(t^{1/3}\right)^2 + \left(t^{1/3}\right)\left(x^{1/3}\right) + \left(x^{1/3}\right)^2\right)} \\
&= \lim_{t\to x} \frac{\left(t^{1/3}\right) + \left(x^{1/3}\right)}{\left(t^{1/3}\right)^2 + \left(t^{1/3}\right)\left(x^{1/3}\right) + \left(x^{1/3}\right)^2} \\
&= \frac{\left(x^{1/3}\right) + \left(x^{1/3}\right)}{\left(x^{1/3}\right)^2 + \left(x^{1/3}\right)\left(x^{1/3}\right) + \left(x^{1/3}\right)^2} \\
&= \frac{2x^{1/3}}{3x^{2/3}} \\
&= \frac{2}{3}x^{-1/3}.
\end{aligned}$$

In short,

$$D(x^{2/3}) = \frac{2}{3}x^{-1/3}.$$

Note that this formula follows the pattern we found for $D(x^n)$ for $n = 1, 2, 3, \ldots$ and -1. The exponent of x becomes the coefficient and the exponent is lowered by 1. ◇

The method used in Example 5 applies to any rational exponent. We have the general formula

$$\boxed{\text{For any rational number } a,\ D(x^a) = ax^{a-1}.} \qquad (6)$$

This formula holds for values of x where both x^a and x^{a-1} are defined. For instance, $x^{1/2} = \sqrt{x}$ is defined for $x \geq 0$, but its derivative $\frac{1}{2}x^{-1/2}$ is defined only for $x > 0$.

The derivative of the square root function occurs so often, we emphasize its formula

$$\boxed{D(x^{1/2}) = \frac{1}{2}x^{-1/2}}$$

or, in terms of the usual square root sign,

$$\boxed{D(\sqrt{x}) = \frac{1}{2\sqrt{x}}.}$$

Another Notation for the Derivative

We have used the notations f' and $D(f)$ for the derivative of a function f. There is another notation that is sometimes convenient to use.

If $y = f(x)$, the derivative is denoted by the symbols

$$\frac{dy}{dx} \text{ or } \frac{df}{dx}.$$

The symbol $\frac{dy}{dx}$ is read as "the derivative of y with respect to x" or "dee y, dee x."

In this notation the derivative of x^3, for instance, is written

$$\frac{d\,(x^3)}{dx}.$$

If the function is expressed in terms of another letter, such as t, we would write

$$\frac{d\,(t^3)}{dt}.$$

In Section 5.4 a meaning will be given to dx and dy.

Keep in mind that in the notations df/dx and dy/dx, the symbols df, dy, and dx have no meaning by themselves. The symbol dy/dx should be thought of as a signle entity, just like the numeral 8, which we do not think of as formed of two 0's.

In the study of motion, Newton's **dot notation** is often used. If x is a function of time t, then $\dot{x}$ denotes the derivative dx/dt.

Summary

In this section we see why limits are important in calculus. We need them to define the derivative of a function. The definition can be stated in several ways, but each one says, informally, "look at how a small change in input changes the output." Here is the formal definition, in various costumes:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \qquad f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}.$$

The following derivatives should be memorized. However, if you forget a formula, you should be able to return to the definition and evaluate the necessary limit.

Function	Derivative	Comment
$f(x)$	$f'(x)$	
x^a	ax^{a-1}	a rational
e^x	e^x	
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	

EXERCISES for 3.2

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

1. Show that $D(\cos(x)) = -\sin(x)$. HINT: $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

Using the definition of the derivative, compute the appropriate limit to find the derivatives of the functions in Exercises 2–9.

2. x^4

3. 4^x

4. $x^{4/3}$

5. $5x^2$

6. $4\sin x$

7. $2e^x + \sin x$

8. $x^2 + x^3$

9. $1/(2x+1)$

In Exercises 10–15 use the definition of the derivative to find each derivative. NOTE: Use (6) to check the accuracy of your derivatives.

10. $D(1/x^2)$

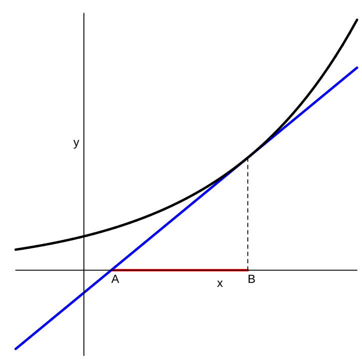
11. $D(x^{4/3})$

12. $D(\sin(2x))$

13. $D(\cos(x/2))$

14. $D(e^{-x})$

15. $D(e^{3x})$



The projection of the tangent onto the x-axis is called the **subtangent** of the graph. See also `en.wikipedia.org/wiki/Subtangent`.

16. Consider the following construction. Select a point $(a, f(a))$ on the graph of $y = f(x)$ and draw the tangent line to the graph at this point. The projection of the tangent line onto the x-axis is the line segment AB in Figure 3.2.

(a) When $f(x) = e^x$, show that the subtangent has constant length, that is, its length does not depend on a.

(b) Find the general formula for the subtangent of the graph of $y = f(x)$.

17. Using the formulas obtained for the derivatives of e^x, x^a, $\sin x$, and $\cos x$, evaluate the derivatives of the following functiosn at the given arguments.

(a) e^x at -1

(b) e^x at 0

(c) $\sin x$ at $\pi/3$

(d) $\sin x$ at $2\pi/3$

(e) $x^{1/3}$ at -8

(f) $\sqrt{x}$ at 25

(g) $\sqrt[3]{x}$ at 27

(h) $\cos x$ at $\pi/4$

(i) $\cos x$ at $-\pi$

18.

(a) Graph $y = x^3$ on the interval $[-2, 2]$.

(b) Find $D(x^3)$ at $x = 0$.

(c) What does (b) tell about the graph in (a)? HINT: Think about the tangent line to $y = x^3$ at $x = 0$.

(d) In view of (c), redraw the graph in (a) in the vicinity of $(0, 0)$.

19. Let $\mathrm{Sin}(x)$ denote the sine of an angle of x degrees and let $\mathrm{Cos}(x)$ denote the cosine of an angle of x degrees.

(a) Graph $y = \mathrm{Sin}(x)$ on the interval $[-180, 360]$.

(b) Find $\lim_{x\to 0} \frac{\mathrm{Sin}(x)}{x}$.

(c) Find $\lim_{x\to 0} \frac{1 - \mathrm{Cos}(x)}{x}$.

(d) Using the definition of the derivative, differentiate $\mathrm{Sin}(x)$.

NOTE: Now you see why in calculus angles are measured in radians.

3.3 Shortcuts for Computing Derivatives

This section develops methods for finding derivatives of functions, or what is called **differentiating** functions. With these methods it will he a routine matter to find, for instance, the derivative of

The verb is "differentiate."

$$\frac{(3 + 4x + 5x^2)e^x}{\sin(x)}$$

without going back to the definition of the derivative and (at great effort) finding the limit of a difference quotient.

Before developing the methods in this and the next two sections, it will be useful to find the derivative of any constant function.

The Derivative of a Constant Function

Theorem 3.3.1 *Derivative of Constant Function The derivative of a constant function $f(x) = C$ is 0.*

In other symbols, $\frac{d(C)}{dx} = 0$ and $D(C) = 0$.

Constant Rule

$$\boxed{(C)' = 0}$$

Proof

Let C be a fixed number and let f be the constant function, $f(x) = C$ for all inputs x. By the definition of derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Since the function f has the same output C for all inputs,

Δx is another name for h

$$f(x + \Delta x) = C \text{ and } f(x) = C.$$

Thus

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{C - C}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{0}{\Delta x} \\ &= \lim_{\Delta x \to 0} 0 \qquad \text{since } \Delta x \neq 0 \\ &= 0. \end{aligned}$$

This shows the derivative of any constant function is 0 for all x. •

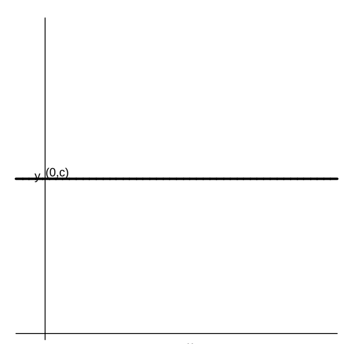


Figure 3.3.1:

From two points of view, Theorem 3.3.1 is no surprise: Since the graph of $f(x) = C$ is a horizontal line, it coincides with each of its tangent lines, as can he seen in Figure 3.3.1. Also, if we think of x as time and $f(x)$ as the position of a particle at time x, Theorem 3.3.1 implies that a stationary particle has zero velocity.

Functions can be built up from simpler functions by addition, subtraction, multiplication. and division, as the function in the opening paragraph illustrates. In order to develop the differentiation formulas that will make our life simple, we need the following delinitions.

DEFINITION *(Sum, difference, product, and quotient of functions)* Let f and g be two functions. The functions $f + g$, $f - g$, fg and f/g are defined as follows:

$$\begin{array}{ll}
(f+g)(x) = f(x) + g(x) & \text{for } x \text{ in the domains of both } f \text{ and } g \\
(f-g)(x) = f(x) - g(x) & \text{for } x \text{ in the domains of both } f \text{ and } g \\
(fg)(x) = f(x)g(x) & \text{for } x \text{ in the domains of both } f \text{ and } g \\
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} & \text{for } x \text{ in the domains of both } f \text{ and } g \textit{ and } g(x) \neq 0
\end{array}$$

Derivatives of $f + g$ and $f - g$

The next theorem asserts that if the functions, f and g have derivatives at a certain number, then so does their sum $f + g$ and

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

In other words, "the derivative of the sum is the sum of the derivatives." A similar formula holds for the derivative of $f - g$.

Equivalently, $(f+g)' = f' + g'$ or $D(f+g) = D(f) + D(g)$.

Theorem 3.3.2 (Derivative of Sum and Difference) *If f and g are differentiable functions, then so are $f + g$ and $f - g$. The **Sum Rule** and **Difference Rule** for computing their derivatives are*

$$\begin{array}{rcll}
(f+g)' & = & f' + g' & \textit{Sum Rule} \\
(f-g)' & = & f' - g' & \textit{Difference Rule}
\end{array}$$

Proof

To prove this theorem we must go back to the definition of the derivative. To begin, we give the function $f+g$ the name u, that is, $u(x) = f(x)+g(x)$. We have to examine

$$\lim_{\Delta x\to 0} \frac{u(x+\Delta x)-u(x)}{\Delta x} \tag{1}$$

or, equivalently,

$$\lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x}. \tag{2}$$

In order to evaluate (2), we will express Δu in terms of Δf and Δg. Here are the details:

$$\begin{array}{rcll}
\Delta u & = & u(x+\Delta x)-u(x) & \\
& = & (f(x+\Delta x)+g(x+\Delta x)) - (f(x)+g(x)) & \text{definition of } u \\
& = & (f(x+\Delta x)-f(x)) - (g(x+\Delta x)-g(x)) & \text{algebra} \\
& = & \Delta f + \Delta g & \text{definition of } \Delta f \text{ and } \Delta g
\end{array}$$

All told, $\Delta u = \Delta f + \Delta g$.

The change in u is the change in f plus the change in g.

The hard work is over. We can now evaluate (2):

$$\begin{array}{rcl}
\lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x} & = & \lim\limits_{\Delta x\to 0} \dfrac{\Delta f + \Delta g}{\Delta x} \\
& = & \lim\limits_{\Delta x\to 0} \dfrac{\Delta f}{\Delta x} + \lim\limits_{\Delta x\to 0} \dfrac{\Delta g}{\Delta x} \\
& = & f'(x) + g'(x).
\end{array}$$

Thus, $u = f+g$ is differentiable and

$$u'(x) = f'(x) + g'(x).$$

A similar argument applies to $f-g$. •

Theorem 3.3.2 extends to any finite number of differentiable functions. For example.

$$\begin{array}{rcl}
(f+g+h)' & = & f'+g'+h' \\
(f-g+h)' & = & f'-g'+h'
\end{array}$$

EXAMPLE 1 Using Theorem 3.3.2, differentiate $x^2 + x^3 + \cos(x) + 3$.

SOLUTION

$$\begin{array}{rcl}
D\left(x^2 + x^3 + \cos(x) + 3\right) & = & D(x^2) + D(x^3) + D(\cos(x)) + D(3) \\
& = & 2x^{2-1} + 3x^{3-1} + (-\sin(x)) + 0 \\
& = & 2x + 3x^2 - \sin(x).
\end{array}$$

◇

EXAMPLE 2 Differentiate $x^4 - \sqrt{x} - e^x$.

SOLUTION

$$\begin{array}{rcl}
\frac{d}{dx}\left(x^4 - \sqrt{x} - e^x\right) & = & \frac{d}{dx}\left(x^4\right) - \frac{d}{dx}\left(\sqrt{x}\right) - \frac{d}{dx}\left(e^x\right) \\
& = & 4x^3 - \frac{1}{2\sqrt{x}} - e^x
\end{array}$$

◇

The Derivative of fg

The following theorem, concerning the derivative of the product of two functions, may be surprising, for it turns out that the derivative of the product is *not* the product of the derivatives. The formula is more complicated than that for the derivative of the sum. It asserts that *"the derivative of the product is the derivative of the first function times the second plus the first function times the derivative of the second."*

Theorem 3.3.3 *If f and g are differentiable functions, then so is their product fg. Its derivative is given by the formula*

$$(fg)' = f'g + fg' \quad \textit{Product Rule}$$

Proof

The proof is similar to that for Theorem 3.3.2. This time we give the product fg the name u. Then we express Δu in terms of Δf and Δg. Finally, we determine $u'(x)$ by examining $\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$.

We have

$$u(x) = f(x)g(x) \quad \text{and} \quad u(x + \Delta x) = f(x + \Delta x)g(x + \Delta x).$$

Rather than subtract $u(x)$ from $u(x + \delta x)$ directly, first write

$$f(x + \Delta x) = f(x) + \Delta f \quad \text{and} \quad g(x + \Delta x) = g(x) + \Delta g.$$

Then

$$\begin{aligned} u(x + \Delta x) &= (f(x + \Delta x))\,(g(x + \Delta x)) \\ &= (f(x) + \Delta f)\,(g(x) + \Delta g) \\ &= f(x)g(x) + (\Delta f)g(x) + f(x)\Delta g + (\Delta f)(\Delta g). \end{aligned}$$

Hence

$$\begin{aligned} \Delta u &= u(x + \Delta x) - u(x) \\ &= f(x)g(x) + (\Delta f)g(x) + f(x)(\Delta g) + (\Delta f)(\Delta g) - f(x)g(x) \\ &= (\Delta f)g(x) + f(x)(\Delta g) + (\Delta f)(\Delta g) \end{aligned}$$

and

$$\begin{aligned} \frac{\Delta u}{\Delta x} &= \frac{(\Delta f)g(x) + f(x)(\Delta g) + (\Delta f)(\Delta g)}{\Delta x} \\ &= \frac{\Delta f}{\Delta x} g(x) + f(x)\frac{\Delta g}{\Delta x} + \Delta f \frac{\Delta g}{\Delta x} \end{aligned}$$

As $\Delta x \to 0$, $\Delta g/\Delta x \to g'(x)$ and $\Delta f/\Delta x \to f'(x)$. Furthermore, because f is differentiable, hence continuous, $\Delta f \to 0$ as $x \to 0$. It follows that

See the last subsection in Section 3.1.

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = f'(x)g(x) + f(x)g'(x) + 0 \cdot g'(x).$$

Therefore, u is differentiable and

$$u' = f'g + fg'.$$

The formula for $(fg)'$ was discovered by Leibniz in 1676. His first guess was incorrect.

•

Remark Figure 3.3.2 provides a picture to illustrate Theorem 3.3.3 and its proof. With f, Δf, g, and Δg taken to be positive, the inner rectangle has area $u = fg$ and the whole rectangle has area $u + \Delta u = (f + \Delta f)(g + \Delta g)$. The shaded region whose area is Δu is made up of rectangles of areas $f \cdot (\Delta g)$, $(\Delta f) \cdot g$, and $(\Delta f) \cdot (\Delta g)$. The little corner rectangle, of area $(\Delta f) \cdot (\Delta g)$, is negligible in comparison with the other two rectangles. Thus, $\Delta u \approx (\Delta f)g + f(\Delta g)$, which suggests the formula for the derivative of a product.

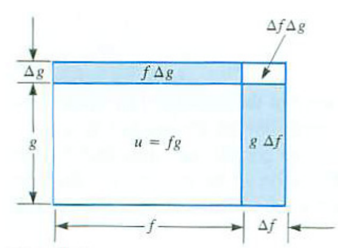


Figure 3.3.2:

EXAMPLE 3 Find $D\left(x^2+x^3+\cos(x)+3\right)\left(x^4-\sqrt{x}-e^x\right)$.

SOLUTION By the product rule,

$$\begin{aligned}
&D\left(x^2+x^3+\cos(x)+3\right)\left(x^4-\sqrt{x}-e^x\right)\\
&= \left(D\left(x^2+x^3+\cos(x)+3\right)\right)\left(x^4-\sqrt{x}-e^x\right)\\
&\quad+\left(x^2+x^3+\cos(x)+3\right)\left(D\left(x^4-\sqrt{x}-e^x\right)\right)\\
&= \left(2x+3x^2-\sin(x)\right)\left(x^4-\sqrt{x}-e^x\right)\\
&\quad+\left(x^2+x^3+\cos(x)+3\right)\left(4x^3-\frac{1}{2\sqrt{x}}-e^x\right)
\end{aligned}$$

◇

Note that the function to be differentiated is the product of the functions differentiated in Examples 1 and 2.

Derivative of Constant Times f

A special case of the formula for the product rule occurs so frequently that it is singled out in Theorem 3.3.4.

The derivative of Cf

Constant Multiple Rule

Theorem 3.3.4 *If C is a constant function and f is a differentiable function, the Cf is differentiable and its derivative is given by the formula*

$$\boxed{(Cf)' = C(f').}$$

The derivative of a constant times a function is the constant times the derivative of the function.

Proof

Because we are dealing with a product of two differentiable functions, C and f, we may use the product rule (Theorem 3.3.3). We have

$$\begin{aligned}
(Cf)' &= (C')f + C(f') && \text{derivative of a product}\\
&= 0\cdot f + cCf') && \text{derivative of constant is 0}\\
&= C(f').
\end{aligned}$$

•

Theorem 3.3.4 asserts that that "it is legal to move a constant factor outside the derivative symbol."

In other notations for the derivative, $\frac{d(Cf)}{dx} = C\frac{df}{dx}$ and $D(Cf) = CD(f)$.

EXAMPLE 4 Find $D(6x^3)$.

SOLUTION

$$\begin{aligned}
D(6x^3) &= 6D(x^3) && \text{6 is a constant}\\
&= 6\cdot 3x^2 && D(x^n)=nx^{n-1}\\
&= 18x^2.
\end{aligned}$$

With a little practice, one would simply write $D(6x^3) = 18x^2$. ◇

EXAMPLE 5 Find $D\left(\sqrt{x}/11\right)$.
SOLUTION

$$\begin{aligned} D\left(\frac{\sqrt{x}}{11}\right) &= D\left(\tfrac{1}{11}\sqrt{x}\right) \\ &= \tfrac{1}{11}D(\sqrt{x}) \\ &= \tfrac{1}{11}\tfrac{1}{2\sqrt{x}} \\ &= \tfrac{1}{22}x^{-1/2} \end{aligned}$$

◇

Example 5 generalizes to the fact that for a nonzero C,

$$\boxed{\left(\frac{f}{C}\right)' = \frac{f'}{C}, \qquad C \text{ is a constant function, } C \text{ not } 0.}$$

$\frac{d}{dx}\left(\frac{f}{C}\right) = \frac{1}{C}\frac{df}{dx}$

The formula for the derivative of the product extends to the product of several differentiable functions. For instance,

$$\boxed{(fgh)' = (f')gh + f(g')h + fg(h')}$$

In each summand only one derivative appears. The next example illustrates the use of this formula.

See Exercise 26.

EXAMPLE 6 Differentiate $\sqrt{x}e^x\sin(x)$.
SOLUTION

$$\begin{aligned} &\left(\sqrt{x}e^x\sin(x)\right)' \\ &= (\sqrt{x})'e^x\sin(x) + \sqrt{x}(e^x)'\sin(x) + \sqrt{x}e^x(\sin(x))' \\ &= \left(\frac{1}{2\sqrt{x}}\right)e^x\sin(x) + \sqrt{x}e^x\sin(x) + \sqrt{x}e^x\cos(x) \end{aligned}$$

◇

Any polynomial can be differentiated by the methods already developed.

EXAMPLE 7 Differentiate $6t^8 - t^3 + 5t^2 + \pi^3$.
SOLUTION Notice that the independent variable in this polynomial is t, and the polynomial is to be differentiated with respect to t.

Differentiate a polynomial "term-by-term".

$$\begin{aligned} \tfrac{d}{dt}\left(6t^8 - t^3 + 5t^2 + \pi^3\right) &= \tfrac{d}{dt}\left(6t^8\right) - \tfrac{d}{dt}\left(t^3\right) + \tfrac{d}{dt}\left(5t^2\right) + \tfrac{d}{dt}\left(\pi^3\right) \\ &= 48t^7 - 3t^2 + 10t + 0 \\ &= 48t^7 - 3t^2 + 10t \end{aligned}$$

π is a constant; so is π^3

◇

Derivative of $1/g$

Often one needs the derivative of the reciprocal of a function g, that is, $(1/g)'$.

Theorem 3.3.5 *If g is a differentiable function, then*

> ***Theorem 3.3.6*** *Reciprocal Rule*
>
> $$\left(\tfrac{1}{g}\right)' = -\tfrac{g'}{g^2}, \quad \textit{where } g(x) \neq 0$$

Proof
Again we must go back to the definintion of the derivative.

Assume $g(x) \neq 0$ and let $u(x) = 1/g(x)$. Then $u(x+\Delta x) = 1/g(x+\Delta x) = 1/(g(x) + \Delta g)$. Thus

$$\begin{aligned}
\Delta u &= u(x+\Delta x) - u(x) \\
&= \frac{1}{g(x)+\Delta g} - \frac{1}{g(x)} \\
&= \frac{g(x) - (g(x)+\Delta g)}{g(x)(g(x)+\Delta g)} \quad \text{common denominator} \\
&= \frac{-\Delta g}{g(x)(g(x)+\Delta g)}.
\end{aligned}$$

Then

$$\begin{aligned}
u'(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} &= \lim_{\Delta x \to 0} \frac{-\Delta g/\left(g(x)(g(x)+\Delta g)\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta g/\Delta x}{g(x)(g(x)+\Delta g)} && \text{algebra: } \tfrac{(a/b)}{c} = \tfrac{(a/c)}{b} \\
&= \frac{\lim_{\Delta x \to 0}\left(\frac{-\Delta g}{\Delta x}\right)}{\lim_{\Delta x \to 0}(g(x)(g(x)+\Delta g))} && \text{quotient rule for limits} \\
&= \frac{-g'(x)}{g(x)^2} && \text{because } g(x) \text{ is continuous}
\end{aligned}$$

•

EXAMPLE 8 Find $D\left(\frac{1}{\cos(x)}\right)$.
SOLUTION In this case, $g(x) = \cos(x)$ and $g'(x) = -\sin(x)$. Therefore,

$$\begin{aligned}
D\left(\frac{1}{\cos(x)}\right) &= \frac{-(-\sin(x))}{(\cos(x))^2} \\
&= \frac{\sin(x)}{\cos^2(x)} \qquad \text{for all } x \text{ with } \cos(x) \neq 0
\end{aligned}$$

◇

Example 8 gives a formula for the derivative of $\sec(x) = \frac{1}{\cos(x)}$. Its derivative is

$$\begin{aligned}\frac{\sin(x)}{\cos^2(x)} &= \frac{\sin(x)}{\cos(x)}\frac{1}{\cos(x)}\\ &= \tan(x)\sec(x)\end{aligned}$$

Therefore,

Memorize this formula!

Theorem 3.3.7 *Derivative of* $\sec(x)$

$$D(\sec(x)) = \sec(x)\tan(x)$$

The Derivative of f/g

EXAMPLE 9 Derive a formula for the derivative of the quotient f/g. What conditions are required for this formula to apply?

SOLUTION The quotient f/g can be written as a product $f \cdot \frac{1}{g}$. Assuming f and g are differentiable functions, we may use the product and reciprocal rules to find

$$\begin{aligned}\left(\tfrac{f(x)}{g(x)}\right)' &= \left(f(x)\tfrac{1}{g(x)}\right)' && \text{rewrite quotient as product}\\ &= f'(x)\left(\tfrac{1}{g(x)}\right) + f(x)\left(\tfrac{1}{g(x)}\right)' && \text{product rule}\\ &= f'(x)\left(\tfrac{1}{g(x)}\right) + f(x)\left(\tfrac{-g'(x)}{g(x)^2}\right) && \text{reciprocal rule, assuming } g(x) \neq 0\\ &= \tfrac{f'(x)}{g(x)} - \tfrac{f(x)g'(x)}{g(x)^2} && \text{algebra}\\ &= \tfrac{f'(x)g(x)-f(x)g'(x)}{g(x)^2} && \text{algebra: common denominator.}\end{aligned}$$

◇

Example 9 is the proof of the **quotient rule**. The quotient rule should be committed to memory. A simple case of the quotient rule has already been used to find the derivative of $\sec(x) = \frac{1}{\cos(x)}$ (Example 8). The full quotient rule will be used to find the derivative of $\tan(x) = \frac{\sin(x)}{\cos(x)}$ (Example 10). Because it used used so often, the quotient rule should be memorized.

Theorem 3.3.8 *Quotient Rule Let f and g be differentiable functions at x, and assume $g(x) \neq 0$. Then the quotient f/g is differentiable at x, and*

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \quad \textit{where } g(x) \neq 0.$$

A memory device for $(f/g)'$

Remark Because the numerator in the quotient rule is a difference, it is important to get the terms in the numerator in the correct order. Here is an easy way to remember the quotient rule.

Step 1. Write down the parts where g^2 and g appear:

$$\frac{g\qquad\quad}{g^2}.$$

This ensures that you get the denominator correct, and have a good start on the numerator.

Step 2. To complete the numerator, remember that it has a minus sign:

$$\frac{gf' - fg'}{g^2}.$$

EXAMPLE 10 Find the derivative of the tangent function.
SOLUTION

$$\begin{aligned}
(\tan(x))' &= \left(\frac{\sin(x)}{\cos(x)}\right)' \\
&= \frac{\cos(x)(\sin(x))' - \sin(x)(\cos(x))'}{(\cos(x))^2} && \text{quotient rule} \\
&= \frac{(\cos(x))\cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} && \sin^2(x) + \cos^2(x) = 1 \\
&= \sec^2(x) && \sec(x) = 1/\cos(x)
\end{aligned}$$

This result is valid whenever $\cos(x) \neq 0$. ◇

Memorize!

Theorem 3.3.9 *Derivative of* $\tan(x)$

$$D(\tan(x)) = \sec^2(x) \quad \textit{for all } x \textit{ in the domain of } \tan(x).$$

It is now a simple matter to find $(x^n)'$ when n is a negative integer. Let $n = -p$ where p is positive. Then

$$\begin{aligned} D(x^n) = D(x^{-p}) &= D\left(\frac{1}{x^p}\right) \\ &= -\frac{D(x^p)}{(x^p)^2} \\ &= -\frac{px^{p-1}}{x^{2p}} \\ &= -px^{-p-1} \\ &= nx^{n-1}. \end{aligned}$$

EXAMPLE 11 Use the Quotient Rule to differentiate x^{-3}.
SOLUTION

$$\begin{aligned} D(x^{-3}) &= D\left(\frac{1}{x^3}\right) \\ &= \frac{(x^3)(1)' - (1)(x^3)'}{(x^3)^2} \\ &= \frac{-3x^2}{x^6} \\ &= \frac{-3}{x^4} \\ &= -3x^{-4} \end{aligned}$$

◇

In the same way, the Power Rule can be extended to x^a for any non-zero rational number a.

Theorem 3.3.10 *Power Rule Let a be a non-zero rational number,*

$$D(x^a) = ax^{a-1} \quad \textit{for any non-zero rational number } a$$

EXAMPLE 12 Compute $(x^2/(x^3+1))'$, showing each step..

SOLUTION

$$\begin{aligned}
\left(\frac{x^2}{x^3+1}\right)' &= \frac{(x^3+1)\cdots}{(x^3+1)^2} && \text{write denominator and start numerator} \\
&= \frac{(x^3+1)(x^2)' - (x^2)(x^3+1)'}{(x^3+1)^2} && \text{complete numerator, remembering the minus sign} \\
&= \frac{(x^3+1)(2x) - (x^2)(3x^2)}{(x^3+1)^2} && \text{compute derivatives} \\
&= \frac{2x^4+2x-3x^4}{(x^3+1)^2} && \text{algebra} \\
&= \frac{2x-x^4}{(x^3+1)^2} && \text{algebra: collecting}
\end{aligned}$$

◇

As Example 12 illustrates, the techniques for differentiating polynomials and quotients can be combined to differentiate any **rational function**, that is, any quotient of polynomials.

Summary

Let f and g be two differentiable functions and let C be a constant. We obtained formulas for differentiating $f+g$, $f-g$, fg, Cf, $1/f$, and f/g.

Rule	Formula	Comment
Constant Rule	$C' = 0$	C a constant
Sum Rule	$(f+g)' = f' + g'$	
Difference Rule	$(f-g)' = f' - g'$	
Product Rule	$(fg)' = (f')g + f(g')$	
Constant Multiple Rule	$(Cf)' = C(f')$	
Reciprocal Rule	$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$	$g \neq 0$
Quotient Rule	$\left(\frac{f}{g}\right)' = \frac{g(f')-f(g')}{g^2}$	$g \neq 0$

Table 3.3.1:

With the aid of the formulas in Table 3.3.1, we can differentiate $\sec(x)$, $\csc(x)$, $\tan(x)$, and $\cot(x)$ using $(\sin(x))' = \cos(x)$ and $(\cos(x))' = -\sin(x)$. These results are among the others we obtained.

Function	Derivative	Comment
x^a	ax^{a-1}	a is non-zero rational number
$\tan(x)$	$\sec^2(x)$	
$\sec(x)$	$\sec(x)\tan(x)$	

Table 3.3.2:

EXERCISES for 3.3

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In Exercises 1–15 differentiate the given function. *Use only the formulas presented in this and earlier sections.*

1. $5x^3$

2. $5x^3 - 7x + 2^3$

3. $3\sqrt{x} - \sqrt[3]{x}$

4. $1/\sqrt{x}$

5. $(5+x)(x^2 - x + 7)$

6. $\sin(x)\cos(x)$

7. $3\tan(x)$

8. $3(\tan(x))^2$ HINT: Write $(\tan(x))^2 = \tan(x)\tan(x)$

9. $\dfrac{x^3-1}{2x+1}$

10. $\dfrac{\sin(x)}{e^x}$

11. $\dfrac{3x^2 + x + \sqrt{2}}{\cos(x)}$

12. $\dfrac{2}{x^3} + \dfrac{3}{x^4}$

13. $x^2 \sin(x) e^x$

14. $\sqrt{x}\cos(x)$

15. $\sqrt{x}\cos(x)$

16. Find the derivative of the following functions:

(a) $\dfrac{(1+\sqrt{x})(x^3+\sin(x))}{x^2+5x+3e^x}$

(b) $\dfrac{(3+4x+5x^2)e^x}{\sin(x)}$

17. Use the quotient rule to verify the following derivatives.

(a) $D(\tan(x)) = (\sec(x))^2$

(b) $D(\cot(x)) = -(\csc(x))^2$

(c) $D(\sec(x)) = \sec(x)\tan(x)$

(d) $D(\csc(x)) = -\csc(x)\cot(x)$

NOTE: There is a pattern here. The negative goes with each "co" function (cos, cot, csc).

18. Find $\left(e^{2x}\right)'$ by writing e^{2x} as $e^x e^x$.

19. Find $\left(e^{3x}\right)'$ by writing e^{3x} as $e^x e^x e^x$.

20. Find $(e^{-x})'$ by writing e^{-x} as $\frac{1}{e^x}$.

21. Find $\left(e^{-2x}\right)'$ by writing $e^{-2x} = e^{-x}e^{-x}$. (See Exercise 18

22. Find $\left(e^{-2x}\right)'$ by writing $e^{-2x} = \frac{1}{e^{2x}}$. (See Exercise 20

23. In Section 3.1 we showed that $D(1/x) = -1/x^2$. Obtain this same formula by using the Quotient Rule.

24. If you had lots of time, how would you differentiate $(1+2x)^{100}$ using the formulas developed so far? NOTE: In Section 3.5 we will obtain a shortcut for differentiating $(1+2x)^{100}$.

25. At what point on the graph of $y = xe^{-x}$ is the tangent horizontal?

26. Using the formula for the derivative of a product, obtain the formula for $(fgh)'$. HINT: First write fgh as $(f)(gh)$. Then use the Product Rule twice.

27. Obtain the formula for $(f-g)'$ by first writing $f-g$ as $f+(-1)g$.

28. Using the definition of the derivative, show that $(f-g)' = f' - g'$.

29. Using the version of the definition of the derivative that makes use of both x and $x+h$, obtain the formula for differentiating the sum of two functions.

30. Using the version of the definition of the derivative in the form $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$, obtain the formula for differentiating the product of two functions.

31. In Section 3.2 we show that $D(x^n) = nx^{n-1}$, where n is a positive integer. Now that we have the formula for the derivative of a product of two functions we can obtain this result much more easily.

(a) Show, using the definition of the derivative, that the formula $D(x^n) = nx^{n-1}$ holds when $n = 1$.

(b) Using (a) and the formula for the derivative of a product, show that it holds when $n = 2$. HINT: $x^2 = x \cdot x$.

(c) Using (b) and the formula for the derivative of a product, show that it holds when $n = 3$.

(d) Show that if it holds for some positive integer n, it also holds for the integer $n + 1$.

(e) Combine (c) and (d) to show that the formula holds for $n = 4$.

(f) Why must it hold for $n = 5$?

(g) Why must it hold for all positive integers?

32. Using induction, as in Exercise 31, show that for each positive integer n, $D\left(x^{-n}\right) = -nx^{-n-1}$.

33. We obtained the formula for $(f/g)'$ by writing f/g as the product of f and $1/g$. Obtain $(f/g)'$ directly from the definition of the derivative. HINT: First review how we obtained the formula for the derivative of a product.

3.4 The Chain Rule

We come now to the most frequently used formula for computing derivatives. For example, it will help us to find the derivative of $(1+x^2)^{100}$ without having to multiply out one hundred copies of $(1+x^2)$ or of e^{3x} without writing $e^{3x} = e^x e^x e^x$. You might be tempted to guess that the derivative of $(1+x^2)^{100}$ would be $100(1+x^2)^{99}$. *This cannot be right!* After all, when you expand $(1+x^2)^{100}$ you get a polynomial of degree 200, so its derivative is a polynomial of degree 199. But when you expand $(1+x^2)^{99}$ you get a polynomial of degree 198. Something is wrong.

At this point we know the derivative of $\sin(x)$, but what is the derivative of $\sin(x^2)$? It is not the cosine of x^2. In this section we obtain a way to differentiate these functions easily — and correctly.

The key is to recognize that both of these functions is the composition of two simpler functions which are easy to differentiate.

$$\begin{aligned}(1+x^2)^{100} &= f(g(x)) = (f\circ g)(x) \quad \text{for } f(x)=x^{100} \text{ and } g(x)=1+x^2\\ \sin(x^2) &= f(g(x)) = (f\circ g)(x) \quad \text{for } f(x)=\sin(x) \text{ and } g(x)=x^2\end{aligned}$$

How to Differentiate a Composite Function

To see how to differentiate a composite function, once again we must go back to the definition of the derivative. We have $u = g(x)$ and $y = f(u)$. Thus $y = f(g(x))$. First of all, $g'(x)$ is approximated by $\Delta u/\Delta x$, where Δu is the change in u corresponding to the change Δx in x. If Δu is not 0, this change in u induces a change Δy in $f(u)$. The quotient $\Delta y/\Delta u$ approximates $f'(u)$. Figure 3.4.1 describes Δx, Δu, and Δy. Now,

It could happen that $\Delta u = 0$, as it would, for instance, if g were a constant function.

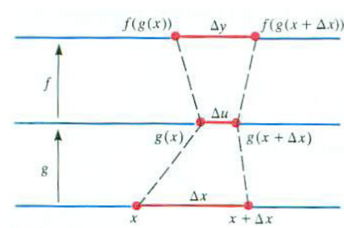


Figure 3.4.1:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}. \tag{1}$$

Then,

$$\begin{aligned}(f\circ g)'(x) &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}.\end{aligned}$$

Since g is continuous, $\Delta u \to 0$ as $\Delta x \to 0$. So we have

$$\begin{aligned}(f\circ g)'(x) &= \lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\\ &= f'(u)g'(x).\end{aligned}$$

Which gives us

Theorem 3.4.1 *Chain Rule Let g be differentiable at x and f be differentiable at $g(x)$, then*

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

This formula tells us how to differentiate a composite function, $f \circ g$:

The Chain Rule is the technique most frequently used in finding derivateives.

Step 1. Compute the derivative of the outer function f, evaluated at the inner function. This is $f'(g(x))$.

The first term in the formula.

Step 2. Compute the derivative of the inner function, $g'(x)$.

The second term in the formula.

Step 3. The derivative of the composite function is the product of the expressions found in Steps 1. and 2.

Examples

EXAMPLE 1 Find $D\left((1+x^2)^{100}\right)$.
SOLUTION Here $g(x) = 1 + x^2$ (the inside function) and $f(u) = u^{100}$ (the outside function). The first step is to compute $f'(u) = 100u^{99}$, which gives us $f'(g(x)) = 100(1 + x^2)^{99}$. The second step is to find $g'(x) = 2x$. Then,

$$(f \circ g)'(x) = f'(\underbrace{u}_{u=g(x)})g'(x) = \underbrace{100u^{99}}_{f'(g(x))} \cdot \underbrace{2x}_{g'(x)} = 100(1+x^2)^{99} \cdot 2x = 200x(1+x^2)^{99}.$$

The answer is not just $100(1 + x^2)^{99}$ — the derivative of the outer function. There is an extra factor of $2x$ that comes from the derivative of the inner function. ◇

As expected, the derivative is a polynomial. What is its degree?

The same example, done with Leibniz notation, looks like this:

$$y = (1 + x^2)^{100} = u^{100}, \qquad u = 1 + x^2.$$

Then the Chain Rule reads simply

$$\begin{array}{rcll} \frac{dy}{dx} & = & \frac{dy}{du}\frac{du}{dx} & \text{Chain Rule} \\ & = & 100u^{99} \cdot 2x & \\ & = & 100(1+x^2)^{99}(2x) & \text{using } u = 1 + x^2 \\ & = & 200x(1+x^2)^{99}. & \end{array}$$

WARNING *(Notation)* We avoided using Leibniz notation earlier, in particular, during the derivation of the Chain Rule, because it tempts the reader to cancel the du's in (1). However, the expressions dy, du, and dx are meaningless — in themselves. In Leibniz's time in the late seventeenth century their meaning was fuzzy, standing for a quantity that was zero and also vanishingly small at the same time. Bishop Berkeley poked fun at this, asking "may we not call them the ghosts of departed quantities?"

George Berkeley, 1734, *The Analyst: A Discourse Addressed to an Infidel Mathematician.* See also `http://muse.jhu.edu/journals/configurations/v004/4.1paxson.html`.

With practice, you wil be able to do the whole calculuation without introducing extra symbols, such as u, which do not apear in the final answer. You will be writing just

$$D\left((1+x^2)^{100}\right) = 100(1+x^2)^{99}\cdot 2x = 200x(1+x^2)^{99}.$$

But this skill, like playing a musical instrument, takes practice, which the exercises at the end of this section (and chapter) provide.

When we write $\frac{dy}{du}$ and $\frac{du}{dx}$, the u serves two ways. In $\frac{dy}{du}$ it denotes an independent variable while in $\frac{du}{dx}$ u is a dependent variable. This double role usually causes no problem in computing derivatives.

EXAMPLE 2 If $y = \sin(x^2)$, find $\frac{dy}{dx}$.
SOLUTION Here, starting from the outside, let $y = \sin(u)$ and $u = x^2$. Then, using the Chain Rule,

$$\begin{aligned}(\sin(x^2))' &= \frac{dy}{dx} \\ &= \frac{dy}{du}\frac{du}{dx} && \text{by the Chain Rule} \\ &= \cos(u)\cdot 2x \\ &= \cos(x^2)\cdot 2x \\ &= 2x\cos(x^2).\end{aligned}$$

◇

The Chain Rule holds for compositions of more than two functions. We illustrate this in the next example.

EXAMPLE 3 Differentiate $y = \sqrt{\sin(x^2)}$.
SOLUTION In this case the function is the composition of three functions:

$$u = x^2 \qquad v = \sin(u) \qquad y = \sqrt{v} \text{ (provided } v \geq 0).$$

Then

Do this example yourself *without introducing any auxiliary symbols (u, v, and y).*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dv}\frac{dv}{dx} && \text{Chain Rule}\\
&= \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx} && \text{Chain Rule, again}\\
&= \frac{1}{2\sqrt{v}}\cdot\cos(u)\cdot 2x\\
&= \frac{1}{2\sqrt{\sin(x^2)}}\cdot\cos(x^2)\cdot 2x\\
&= \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}
\end{aligned}$$

◇

EXAMPLE 4 Let $y = 2^x$. Find y'.
SOLUTION In its original form, 2^x is not a composite function. However, we can write $2 = e^{\ln(2)}$ and then 2^x equals $(e^{\ln(2)})^x = e^{\ln(2)x}$. Now we see that 2^x can be expressed as the composite function:

$b = e^{\ln(b)}$ for any $b > 0$

$$y = e^u, \text{ where } u = (\ln(2))x.$$

Then

$$y' = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\cdot\ln(2) = e^{\ln(2)x}\ln(2) = 2^x\ln(2).$$

In Example 2 (Section 3.2), using a calculator, we found $D(2^x) \approx (0.693)2^x$. We have just learned that the exact formula for this derivative is $D(2^x) = 2^x\ln(2)$. This means we have just learned that 0.693 is an approximation of $\ln(2)$. ◇

Verify that $\ln(2) \approx 0.693$.

EXAMPLE 5 Find $D\left(x^3\tan(x^2)\right)$.
SOLUTION The function $x^3\tan(x^2)$ is the product of two functions. We first apply the Product Rule to obtain:

Product Rule:
$(fg)' = f'\cdot g + f\cdot g'$

$$\begin{aligned}
D\left(x^3\tan(x^2)\right) &= (x^3)'\tan(x^2) + x^3\left(\tan(x^2)\right)'\\
&= 3x^2\tan(x^2) + x^3\left(\tan(x^2)\right)'.
\end{aligned}$$

Since "the derivative of the tangent is the square of the secant", the Chain Rule tells us that

$(\tan(x))' = \sec^2(x)$

$$\left(\tan(x^2)\right)' = \sec^2(x^2)(x^2)' = 2x\sec^2(x^2).$$

Thus,

$$\begin{aligned}
D\left(x^3\tan(x^2)\right) &= 3x^2\tan(x^2) + x^3\left(\tan(x^2)\right)'\\
&= 3x^2\tan(x^2) + x^3\left(2x\sec^2(x^2)\right)\\
&= 3x^2\tan(x^2) + 2x^4\sec^2(x^2).
\end{aligned}$$

◇

In the computation of $D(\tan(x^2))$ we did not introduce any new symbols. That is how your computations should look, once you get the rythym of the Chain Rule.

Famous Composite Functions

Certain types of composite functions occur so frequent that it is worthwhile memorizing thesir derivatives. Here is a list:

Function	Derivative	Example
$(g(x))^n$	$ng(x)^{n-1}g'(x)$	$((1+x^2)^{100})' = 100(1+x^2)^{99}(2x)$
$\frac{1}{g(x)}$	$\frac{-g'(x)}{(g(x))^2}$	$D\left(\frac{1}{\cos(x)}\right) = \frac{-(-\sin(x))}{(\cos(x))^2}$
$\sqrt{g(x)}$	$\frac{g'(x)}{2\sqrt{g(x)}}$	$\left(\sqrt{\tan(x)}\right)' = \frac{(\sec(x))^2}{2\sqrt{\tan(x)}}$
$e^{g(x)}$	$e^{g(x)}g'(x)$	$\left(e^{x^2}\right)' = e^{x^2}(2x)$

Table 3.4.1:

Summary

This section presented the single most important tool for computing derivatives: the Chain Rule. It began with a description of composite functions and showed the composition of continuous functions is continuous and the composition of differentiable functions is differentiable. To be specific, the derivative of $f \circ g$ at x is

$$\underbrace{f'(g(x))}_{\text{derivative of outer function evaluated at the innner function}} \quad \text{times} \quad \underbrace{g'(x)}_{\text{derivative of inner function}}$$

Introducing the symbol u, we describe the Chain Rule for $y = f(u)$ and $u = g(x)$ with the brief notation

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dy}.$$

When the function is built up from more than two functions, such as $y = f(u)$, $u = g(v)$, and $v = h(x)$. Then we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dy},$$

a chain of more derivatives.

With practice, applying the Chain Rule can become second nature.

EXERCISES for 3.4

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In Exercises 1–4, repeat the specified example from this section *without* introducing an extra variable (such as u).

1. Example 3.

2. Example 4.

3. Example 5.

4. Example 6.

In Exercises 5–26 differentiate the given function.

5. $(5x^2+3)^{10}$

6. $(\sin(3x))^3$

7. $\frac{1}{5t^2+t+2}$

8. $\frac{1}{e^{5s}+s}$

9. $\sqrt{4+u^2}$

10. $\sqrt{\cos(2\theta)}$

11. e^{5x^3}

12. $\sin^2(3x)$

13. $e^{\tan(3t)}$

14. $\sqrt{\tan(2u)}$

15. $\sqrt[3]{\tan(s^2)}$

16. $v^3\tan(2v)$

17. $e^{2r}\sin(3r)$

18. $\frac{\cos(2x)}{x^2}$

19. $e^{\sin(x)}$

20. $\frac{(3t+2)^4}{\sin(2t)}$

21. $e^{-5s}\tan(3s)$

22. 3^x

23. $(\sin(2u))^5(\cos(3u))^6$

24. $\left(x+3^{3x}\right)^2\left(\sin(\sqrt{x})\right)^3$

25. $\frac{t^3}{\left(t+\sin^2(3t)\right)}$

26. $\frac{(3x+2)^4}{(x^3+x+1)^2}$

Exercise 27 and 28 illustrate how differentiation can be used to obtain one trigonometry identity from another.

27.

(a) Differentiate both sides of the identity $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$. What trigonometric identity do you get?

(b) Differentiate the identity found in (a) to obtain another trigonometric identity. What identity is obtained?

28. Let k be a constant. Differentiate both sides of the identity $\sin(x+k) = \sin(x)\cos(k) + \cos(x)\sin(k)$ to obtain the corresponding identity for $\cos(x+k)$.

29. Differentiate $(e^x)^3$

(a) directly, by the Chain Rule

(b) after writing the function as $e^x \cdot e^x \cdot e^x$

(c) after writing the function as e^{3x}

(d) Which of these approaches to you prefer? Why?

30. Find $D(x^x)$, $x > 0$. Hint: Rewrite the base as $e^{\ln(x)}$.

31. In Theorem 3.3.5 (Section 3.3) we obtained the derivative of $1/g(x)$ by using the definition of the derivative. Obtain the formula for the Reciprocal Rule by using the Chain Rule.

32. Find the domain of $\sqrt{\sin(x^2)}$.

33. In our proof of the Chain Rule we had to assume that Δu is not 0 when Δx is sufficiently small. Show that if the derivative of g is not 0 at the argument x, then the proof is valid.

34. Here is an example of a differentiable g not covered by the proof of the Chain Rule given in the text. Define $g(x)$ to be $x^2 \sin\left(\frac{1}{x}\right)$ for x different from 0 and $g(0)$ to be 0.

(a) Sketch the part of the graph of g near the origin.

(b) Show that there are arbitrarily small values of Δx such that $\Delta u = g(\Delta x) - g(0) = 0$.

(c) Show that g is differentiable at 0.

35. Here is a proof of the Chain Rule that manages to avoid division by $\Delta u = 0$. Let $f(u)$ be differentiable at $g(a)$, where g is differentiable at a. Let $\Delta f = f(g(a) + \Delta u) - f(g(a))$. Then $\frac{\Delta f}{\Delta u} - f'(g(a))$ is a function of Δu, which we call $p(\Delta u)$. This function is defined for $\Delta u \neq 0$. By the definition of f', $p(\Delta u)$ tends to 0 as Δu approaches 0. Define $p(0)$ to be 0. Note that p is continuous at 0.

(a) Show that $\Delta f = f'(g(a))\Delta u + p(\Delta u)\Delta u$ when Δu is different than 0, and also when $\Delta u = 0$.

(b) Define $q(\Delta x) = \frac{\Delta u}{\Delta x} - g'(a)$. Observe that $q(\Delta x)$ approaches 0 as Δx approaches 0. Show that $\Delta u = g'(a)\Delta x + q(\Delta x)\Delta x$ when Δx is not 0.

(c) Combine (a) and (b) to show that

$$\Delta f = f'(g(a))\left(g'(a)\Delta x + q(\Delta x)\Delta x\right) + p(\Delta u)\Delta u.$$

(d) Using (c), show that

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(g(a))g'(a).$$

(e) Why did we have to define $p(0)$ but not $g(0)$?

3.5 Derivative of an Inverse Function

In this section we obtain the derivatives of the inverse functions of e^x and of the six trigonometric functions. This will complete the inventory of basic derivatives. The first issue to resolve is that the inverse of a differentiable function is itself differentiable. The Chain Rule will then be used to find the specific differentiation formulas for the most important inverse functions: the logarithm and the inverse trigonometric functions.

Differentiability of Inverse Functions

As mentioned in Section 1.1 the graph of an inverse function is an exact copy of the graph of the original function. One graph is obtained from the other by reflection across the line $y = x$. If the original function, f, is differentiable at a point (a, b), $b = f(a)$, then the graph of $y = f(x)$ has a tangent line at (a, b). In particular, the reflection of the tangent line to the original function is the tangent line to the inverse function at (b, a). Thus, we expect that the inverse function is differentiable at (b, a).

The Chain Rule will now be used to find the derivatives of $\log_e(x)$ and of the six inverse trigonometric functions. Throughout this discussion we assume that the inverse function is differentiable.

$b = f(a)$ means $a = f^{-1}(b)$

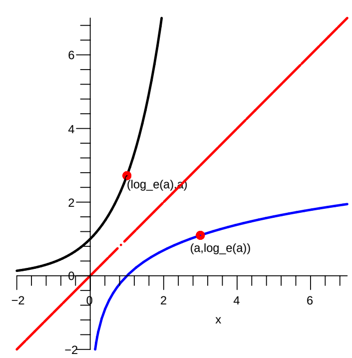


Figure 3.5.1:

The Derivative of $\log_e(x)$

Let $y = \log_e(x)$. Figure 3.5.1 shows the graphs of $y = e^x$ and inverse function $y = \log_e(x)$. We want to find $y' = \frac{dy}{dx}$. By the definition of logarithm as the inverse of the exponential function

$$x = e^y. \tag{1}$$

We differentiate both sides of (1) with respect to x:

$$\begin{array}{rcll} \frac{d(x)}{dx} &=& \frac{d(e^y)}{dx} & e^y \text{ is a function of } x \text{, since } y \text{ is} \\ 1 &=& \frac{d(e^x)}{dx} & \frac{dx}{dx} = 1 \\ 1 &=& e^y \frac{dy}{dx} & \text{Chain Rule.} \end{array}$$

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

This shows that

The derivative of the inverse function is the reciprocal of the derivative of the original function.

Memorize!

Theorem 3.5.1

$$(\log_e(x))' = \frac{1}{x}, \qquad x > 0.$$

See Exercises 27 and 28.

It may come as a surprise that such a "complicated" function has such a simple derivative. It may also be a surprise that $\log_e(x)$ is one of the most important functions in calculus, mainly because it has the derivative $1/x$.

For k not equal to -1, an antiderivative of x^k is just a constant times another power function.

EXAMPLE 1 Find $(\log_b)'$ for any $b > 0$.
SOLUTION The function $\log_b x$ is just a constant times $\log_e(x)$:

$$(\log_b(x)) = (\log_b(e)) \log_e(x).$$

Therefore

$$(\log_b(x))' = (\log_b(e)) \frac{1}{x}. \tag{2}$$

If b is not e, then $\log_b(e)$ is not 1. If e is chosen as the base for the logarithm, then the coefficient in front of the $\frac{1}{x}$ becomes $\log_e(e) = 1$, simplifying our lives. That is why we prefer e as the base for logarithms in calculus ◇

We call $\log_e(x)$ the **natural logarithm**, denoted $\ln(x)$. The exponential function e^x also has its own name, exp. This notation comes in handy when the exponent is messy. For instance, $e^{\frac{\sin^3(2x)}{x}}$ may be written $\exp\left(\frac{\sin^3(2x)}{x}\right)$.

Many calculators and computer languages use exp to name the exponential function with base e.

WARNING *(Logarithm Notation)* $\ln(x)$ is often written simply as $\log(x)$, with the base understood to be e. Any references to the base-10 logarithm will be written as $\log_{10}$.

Another View of e

For each choice of the base b ($b > 0$), we obtain a certain value for $\lim_{x\to 0} \frac{b^x - 1}{x}$. We defined e to be the base for which that limit is as simple as possible, namely 1.

Now that we know that the derivative of $\ln x = \log_e x$ is $1/x$, we can obtain a new view of e.

We know that the derivative of $\ln(x)$ at 1 is $1/1 = 1$. By the definition of the derivative, that means

$$\lim_{h\to 0} \frac{\ln(1+h) - ln(1)}{h} = 1.$$

Since $\ln(1) = 0$, we have

$$\lim_{h \to 0} \frac{\ln(1+h)}{h} = 1.$$

By a property of logarithms, we may rewrite the limit as

$c \ln p = \ln(p^c)$

$$\lim_{h \to 0} \ln\left((1+h)^{1/h}\right) = 1.$$

Writing e^x as $\exp x$ for convenience, we conclude that

$$\exp\left(\lim_{h \to 0} \ln\left((1+h)^{1/h}\right)\right) = \exp(1) = e.$$

Since exp is a continuous function,we may switch the order of exp and lim, getting

$$\lim_{h \to 0} \left(\exp\left(\ln\left((1+h)^{1/h}\right)\right)\right) = e.$$

But, $\exp(\ln(p)) = p$ for any positive number, by the very definition of a logarithm. That tells us that

$$\lim_{h \to 0}(1+h)^{1/h} = e.$$

This is a much more direct view of e than the one we had in Section 2.1. As a check, let $h = 1/1000 = 0.001$, then $(1 + 1/1000)^{1000} \approx 2.717$, and values of h that are closer to 0 give even better estimates for e.

The Derivative of $\arcsin(x)$

For x in $[-\pi/2, \pi/2]$ $\sin(x)$ is one-to-one and therefore has an inverse function, $\arcsin(x)$. This function gives you the angle, in radians, if you know the sine of the angle. For instance, $\arcsin(1) = \pi/2$, $\arcsin(\sqrt{2}/2) = \pi/4$, $\arcsin(-1/2) = -\pi/6$, and $\arcsin(-1) = -\pi/2$. The domain of $\arcsin(x)$ is $[-1, 1]$; its range is $[-\pi/2, \pi/2]$. For convenience we include the graphs of $y = \sin(x)$ and $y = \arcsin(x)$ in Figure 3.5.2, but will not need them as we find $(\arcsin(x))'$.

For additional review of arcsin, see Section 1.2.

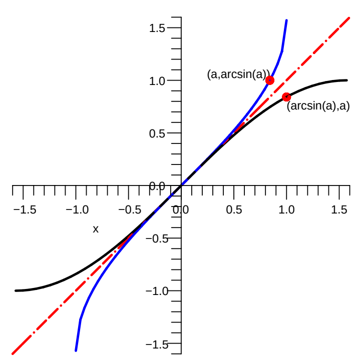


Figure 3.5.2:

To find $(\arcsin(x))'$, we proceed exactly we did when finding $(\log_e(x))'$. Let $y = \arcsin(x)$, then

$$x = \sin(y). \tag{3}$$

Differentiating with respect to x gives

$$1 = \frac{d(x)}{dx} = \frac{d(\sin(y))}{dx} = \cos(y)\frac{dy}{dx}.$$

Thus

$$\frac{dy}{dx} = \frac{1}{\cos(y)}. \tag{4}$$

All that is left is to use the relationship $\sin(y) = x$ to express $\cos(y)$ in terms of x.

Figure 3.5.3 displays the diagram that defines the sine of an angle. The line segment AB represents $\cos(y)$ and the line segment BC represents $\sin(y)$. Observe that the cosine is positive for angles y in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, the first and fourth quadrants and sine is positive for angles y in the first and third quadrants. When $x = \sin(y)$, $AB^2 + BC^2 = 1$ gives $\cos(y) = \sqrt{1-x^2}$. Consequentely, by (4), we find

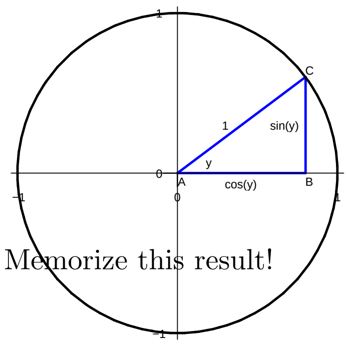


Figure 3.5.3:

Memorize this result!

Theorem 3.5.2 *Derivative of* $\arcsin(x)$

$$\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}}, \qquad |x| < 1.$$

Note at $x = 1$ or -1, the derivative is not defined. However, for x near 1 or -1 the derivative is very large (in absolute value), telling us that the $\arcsin x$ curve is very steep near its two ends. That is a reflection of the fact that the $\sin(x)$ curve is horizontal at $x = -\pi/2$ and $x = \pi/2$.

A transcendental function can have an algebraic derivative. An algebraic function always has an algebraic derivative.

$\frac{d}{dx}\sin(x)\big|_{x=\pm\pi/2} = \cos(\pm\pi/2) = 0$

EXAMPLE 2 Differentiate $\frac{1}{2}\left(x\sqrt{x^2-a^2} + a^2\arcsin\left(\frac{x}{a}\right)\right)$ where a is a constant.

SOLUTION

$$\begin{aligned}
&D\left(\frac{1}{2}\left(x\sqrt{x^2-a^2} + a^2\arcsin\left(\frac{x}{a}\right)\right)\right) && \\
&= \tfrac{1}{2}D\left(\left(x\sqrt{x^2-a^2} + a^2\arcsin\left(\tfrac{x}{a}\right)\right)\right) && \\
&= \tfrac{1}{2}\left(D\left(x\sqrt{x^2-a^2}\right) + a^2 D\left(\arcsin\left(\tfrac{x}{a}\right)\right)\right) && \\
&= \tfrac{1}{2}\left(\left((1)\sqrt{x^2-a^2}\right) + \left(x\left(\frac{-\left(\frac{1}{2}\right)(2x)}{\sqrt{x^2-a^2}}\right)\right) + a^2\left(\frac{\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right)\right) && D(\arcsin(x)) = \tfrac{1}{\sqrt{1-x^2}},\ \text{Product and Chain Rules} \\
&= \tfrac{1}{2}\left(\sqrt{x^2-a^2} - \frac{-x^2}{\sqrt{x^2-a^2}} + \frac{a^2}{\sqrt{a^2-x^2}}\right) && \text{algebra} \\
&= \tfrac{1}{2}\left(\frac{a^2-x^2-x^2+a^2}{\sqrt{x^2-a^2}}\right) && \text{common denominator} \\
&= \sqrt{x^2-a^2} &&
\end{aligned}$$

◇

The Derivative of $\arctan(x)$

For x in $(-\pi/2, \pi/2)$ $\tan(x)$ is one-to-one and has an inverse function, $\arctan(x)$. This inverse function tells us the angle, in radians, if we know the sine of the angle. For instance, $\arctan(1) = \pi/4$, $\arctan(0) = 0$, and $\arctan(-1) = -\pi/4$.

When x is a large positive number, $\arctan(x)$ is near, and smaller than, $\pi/2$. When x is a large negative number, $\arctan(x)$ is near, and larger than, $-\pi/2$. Figure 3.5.4 shows the graph of $y = \arctan(x)$ and $y = \tan(x)$. We will not need this graph in our derivation of the derivative of $\arctan(x)$, but it serves as a check on the formula, which should give values near 0 when $|x|$ is large.

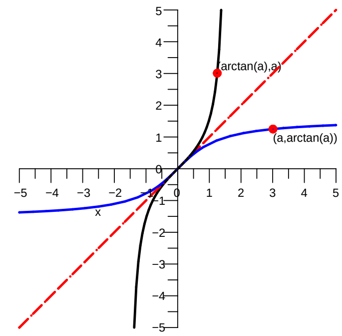


Figure 3.5.4:
See Exercise 28.

To find $(\arctan(x))'$, we again call on the Chain Rule. Starting with

$$y = \arctan(x),$$

we proceed as before:

$$\begin{aligned}
x &= \tan(y). \\
\frac{d(x)}{dx} &= \frac{d(\tan(y))}{dx} && \text{differentiate with respect to } x \\
1 &= \left(\sec^2(y)\right) y' && \text{Chain Rule} \\
y' &= \frac{1}{\sec^2(y)} && \text{algebra} \\
y' &= \frac{1}{1+\tan^2(y)} && \text{trigonometric identity} \\
y' &= \frac{1}{1+x^2} && x = \tan(y).
\end{aligned}$$

So we have

Memorize!

Theorem 3.5.3 *Derivative of* $\arctan(x)$

$$D(\arctan(x)) = \frac{1}{1+x^2} \qquad \textit{for all inputs } x$$

EXAMPLE 3 Find $D\left(\tan^{-1}(3x)\right)$.
SOLUTION By the Chain Rule

$$\begin{aligned}
D\left(\tan^{-1}(3x)\right) &= \frac{1}{1+(3x)^2}\frac{d(3x)}{dx} \\
&= \frac{3}{1+9x^2}.
\end{aligned}$$

◇

EXAMPLE 4 Find $D\left(x\tan^{-1}(x) - \frac{1}{2}\ln\left(1+x^2\right)\right)$.
SOLUTION

$$\begin{aligned}
D\left(x\tan^{-1}(x) - \tfrac{1}{2}\ln\left(1+x^2\right)\right) &= D\left(x\tan^{-1}(x)\right) - \tfrac{1}{2}D\left(\ln\left(1+x^2\right)\right) \\
&= \left(\tan^{-1}(x) + \frac{x}{1+x^2}\right) - \frac{1}{2}\frac{2x}{1+x^2} \\
&= \tan^{-1}(x).
\end{aligned}$$

◇

Do not memorize this formula!

More on $\ln(x)$

We showed that for $x > 0$, $\ln(x)$ is an antiderivative of $1/x$. But what if we needed an antiderivative of $1/x$ for negative x? The next example answers this question.

Recall that $\ln(x)$ is not defined for $x < 0$.

EXAMPLE 5 Show that for negative x, $\ln(-x)$ is an antiderivative of $1/x$.
SOLUTION Let $y = \ln(-x)$. By the Chain Rule,

$$\frac{dy}{dx} = \left(\frac{1}{-x}\right)\frac{d(-x)}{dx} = \frac{1}{-x}(-1) = \frac{1}{x}.$$

So $\ln(-x)$ is an antiderivative of $1/x$ when x is negative. ◇

In view of Example 5, we can say

$$\boxed{D(\ln|x|) = \frac{1}{x} \qquad \text{for } x \neq 0.}$$

This formula covers both cases: positive x and negative x.

We know the derivative of x^a for any rational number a. To extend this result to x^k for any number k, the key is that any positive number x can be written as $x = e^{\ln(x)}$.

EXAMPLE 6 Find $D(x^k)$ for $x > 0$ and any constant $k \neq 0$, rational or irrational.
SOLUTION For $x > 0$ we can write $x = e^{\ln(x)}$. Then

$$x^k = \left(e^{\ln(x)}\right)^k = \left(e^{\ln(x)}\right)^k = e^{k\ln(x)}.$$

Now, $y = e^{k\ln(x)}$ is a composite function, $y = e^u$ where $u = k\ln(x)$. Thus,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\frac{k}{x} = x^k\frac{k}{x} = kx^{k-1}.$$

◇

The preceding example shows that for positive x and any fixed exponent k, $\left(x^k\right)' = kx^{k-1}$. It probably does not come as a surprise. In fact you may wonder why we worked so hard to get the derivative of x^a when a is an integer or rational number when this example covers all exponents. We had two reasons for treating the special cases. First, they include cases when x is negative. Second, they were simpler and helped us introduce the derivative.

The Derivatives of the Six Inverse Trigonometric Functions

Each of the six trigonometric functions has an associated inverse function. We have found the derivatives of the ones used most often.

There are six inverse trigonometric functions. The most important arcsin and arctan. The other four are treated in Exercises 18–21. Table 3.5.1 summarizes all six derivatives. There is no reason to memorize all six of these formulas. If we need, say, an antiderivative of $\frac{-1}{1+x^2}$, we do not have to use $\operatorname{arccot}(x)$. Instead, $-\arctan(x)$ would do. So, for finding antiderivatives, we don't need arccot — or any of the inverse co-functions. You should memorize the formulas for the derivatives of arcsin, arctan, and arcsec.

Note that the negative signs go with the "co-" functions.

$$\begin{array}{rclrcll}
D(\sin^{-1}(x)) & = & \dfrac{1}{\sqrt{1-x^2}} & D(\cos^{-1}(x)) & = & -\dfrac{1}{\sqrt{1-x^2}} & (-1 < x < 1) \\
D(\tan^{-1}(x)) & = & \dfrac{1}{1+x^2} & D(\cot^{-1}(x)) & = & -\dfrac{1}{1+x^2} & (-\infty < x < \infty) \\
D(\sec^{-1}(x)) & = & \dfrac{1}{x\sqrt{x^2-1}} & D(\csc^{-1}(x)) & = & -\dfrac{1}{x\sqrt{x^2-1}} & (x > 1 \text{ or } x < -1)
\end{array}$$

Table 3.5.1: Derivatives of the six inverse trigonometric functions.

This chapter provides you with the tools to differentiate any elementary function. The result will always be an elementary function. Happily, for users of calculus, it is not necessary to go back to the definition of a derivative as a limit to compute a derivative. Even so, we should remember that the definition of derivative is based on limits.

The foundation of calculus is the limit concept.

Summary

A geometric argument is given to suggest that the inverse of every differentiable function is differentiable.

The derivatives of $\ln(x)$, $\sin^{-1}(x)$, and $\tan^{-1}(x)$ and of the other four inverse trigonometric functions were found using the Chain Rule.

EXERCISES for 3.5

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In Exercises 1–6 evaluate the function and its derivative at the given argument.

1. $\arcsin(x)$; $1/2$

2. $\arcsin(x)$; $-1/2$

3. $\arctan(x)$; -1

4. $\arctan(x)$; $\sqrt{3}$

5. $\ln(x)$; e

6. $\ln(x)$; 1

In Exercises 7–16 differentiate the function.

7. $\sin^{-1}(3x)\sin(3x)$

8. $\tan^{-1}(5x)\tan(5x)$

9. $e^{2x}\ln(3x)$

10. $e^{\left(\ln(3x)x^{\sqrt{2}}\right)}$

11. $\log_{10}(x)$ HINT: Express $\log_{10}$ in terms of the natural logarithm.

12. $\log_x(10)$ HINT: Express $\log_x$ in terms of the natural logarithm.

13. $x^2\sin^{-1}(x^2)$

14. $\left(\sin^{-1}(3x)\right)^2$ HINT: Recall that $\sin^{-1}(3x) = \arcsin(3x)$.

15. $\frac{\tan^{-1}(2x)}{1+x^2}$

16. $\frac{x^3}{\tan^{-1}(6x)}$ NOTE: $\tan^{-1}(6x)$ is *not* the reciprocal of $\tan(6x)$.

17. Let $b > 0$. This problem provides some additional experience with the development of the formula $\log_b(x) = \log_b(e)\log_e(x)$. Recall that $\log_b(a) = \frac{\log_e(a)}{\log_e(b)}$.

(a) Show that $\log_b(e) = 1/\log_e(b)$.

(b) Conclude that $\log_b(x) = \log_b(e)\log_e(x)$.

NOTE: This result is used in Example 1.

18. Prove that $(\operatorname{arcsec}(x))' = \frac{1}{|x|\sqrt{x^2-1}}$ for all $x < -1$ or $x > 1$.

19. Prove that $(\arccos(x))' = \frac{-1}{\sqrt{1-x^2}}$ for all $-1 < x < 1$.

20. Prove that $(\operatorname{arccot}(x))' = \frac{-1}{1+x^2}$ for all x.

21. Prove that $(\operatorname{arccsc}(x))' = \frac{-1}{|x|\sqrt{x^2-1}}$ for all $x < -1$ or $x > 1$.

22. Verify that $D\left(2(\sqrt{x}-1)e^{\sqrt{x}}\right) = e^{\sqrt{x}}$.

In Exercises 23–26 use the Chain Rule to obtain the given derivative.

23. $(\cos^{-1}(x))' = \frac{-1}{\sqrt{1-x^2}}$

24. $(\sec^{-1}(x))' = \frac{1}{x\sqrt{x^2-1}}$

25. $(\cot^{-1}(x))' = \frac{-1}{1+x^2}$

26. $(\csc^{-1}(x))' = \frac{-1}{x\sqrt{x^2-1}}$

27. We have used the equation $\sec^2(x) = 1 + \tan^2(x)$.

(a) Derive this equation from the equation $\cos^2(x) + \sin^2(x) = 1$.

(b) Derive the equation $\cos^2(x) + \sin^2(x) = 1$ from the Pythagorean Theorem.

28.

(a) Evaluate $\lim_{x\to\infty} \dfrac{1}{1+x^2}$ and $\lim_{x\to-\infty} \dfrac{1}{1+x^2}$.

(b) What do these results tell you about the graph of the arctangent function?

29.

Sam: I say that $D(\log_b(x)) = \frac{1}{x\ln(b)}$. It's simple. Let $y = \log_b(x)$. That tells me $x = b^y$. I differentiate both sides of that, getting $1 = b^y(\ln(b))y'$. So $y' = \frac{1}{b^y \ln(b)} = \frac{1}{x\ln(b)}$.

Jane: Well, not so fast. I start with the equation $\log_b(x) = (\log_b(e))\ln(x)$. So $D(\log_b(x)) = \frac{\log_b(e)}{x}$.

Sam: Something is wrong. Where did you get that equation you started with?

Jane: Just take $\log_b$ of both sides of $x = e^{\ln(x)}$.

Sam: I hope this won't be on the next midterm exam!

Settle this argument.

30. Find $D(\ln^3(x))$

(a) by the Chain Rule and

(b) by first writing $\ln^3(x)$ as $\ln(x) \cdot \ln(x) \cdot \ln(x)$.

Which method do you prefer? Why?

In Exercises 31–42 use differentiation to check that the first function is an antiderivative of the second function. (The symbols a, b, c, and d are constants.)

31. $-a\cos\left(\frac{x}{a}\right)$; $\sin\left(\frac{x}{a}\right)$

32. $-\frac{1}{2}\ln\left(\frac{1+\cos(x)}{1-\cos(x)}\right)$; $\frac{1}{\sin(x)} = \csc(x)$

33. $\frac{1}{a^2}(ax-1)e^{ax}$; xe^{ax}

34. $\frac{1}{a^3}(a^2x^2-2ax+2)e^{ax}$; x^2e^{ax}

35. $\frac{1}{a^2+b^2}e^{au}(\sin(bu)-\cos(bu))$; $e^{au}\sin(bu)$

36. $\frac{1}{a^2+b^2}e^{au}(\sin(bu)+\cos(bu))$; $e^{au}\cos(u)$

37. $\frac{1}{b^2}\left(a+bx-a\ln(a+bx)\right)$; $\frac{x}{ax+b}$, $\quad a+bx>0$

38. $\frac{1}{b^3}\left(a+bx-2a\ln(a+bx)-\frac{a^2}{a+bx}\right)$; $\frac{x^2}{(a+bx)^2}$, $\quad a+bx>0$

39. $\frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right)$; $\frac{1}{a^2+b^2x^2}$

40. $\frac{x}{2a^2(a^2+x^2)}+\frac{1}{2a^2}\tan^{-1}\left(\frac{x}{a}\right)$; $\frac{1}{(a^2+x^2)^2}$

41. $\frac{1}{2a^2}\tan^{-1}\left(\frac{x^2}{a^2}\right)$; $\frac{x}{a^4+x^4}$

42. $\frac{2\sqrt{x}}{b^2-\frac{2a}{b}^2}\tan^{-1}\left(\frac{b\sqrt{x}}{a}\right)$; $\frac{\sqrt{x}}{a^2+b^2x}$

We did not need the Chain Rule to find the derivatives of inverse functions. Instead, we could have taken a geometric approach, using the "slope of the tangent line" interpretation of the derivative. When we reflect the graph of f around the line $y=x$ to obtain the graph of f^{-1}, the reflection of the tangent line to the graph of f with slope m is the tangent line to the graph of f^{-1} with slope $1/m$. (See Section 1.1.) Exercises 43–47 use this approach to develop formulas obtained in this section.

43. Let $f(x) = \ln(x)$. The slope of the curve $y = \ln(x)$ at $(a, \ln(a))$, $a > 0$, is the reciprocal of the slope of the curve $y = e^x$ at $(\ln(a), a)$. Use this fact to show that the slope of the curve $y = \ln(x)$ when $x = a$ is $1/a$.

In Exercises 44–47 use the technique illustrated in Exercise 43 to differentiate the given function.

44. $f(x) = \tan^{-1}(x)$.

45. $f(x) = \sin^{-1}(x)$.

46. $f(x) = \sec^{-1}(x)$.

47. $f(x) = \cos^{-1}(x)$.

3.6 Antiderivatives and Slope Fields

So far in this chapter we have started with a function and found its derivative. In this section we will go in the opposite direction: Given a function f, we will be interested in a function F whose derivative is f. Why? Because this procedure of going from the derivative back to the function plays a central role in **integral calculus**.

Antiderivatives

If $F' = f$, F is called an **antiderivative** of f. The more practice we have in calculating derivatives, the easier we can find antiderivatives.

EXAMPLE 1 Find an antiderivative of x^6.
SOLUTION When we differentiate x^a we get ax^{a-1}. The exponent in the derivative, $a-1$, is one less than the original exponent, a. So we expect an *anti*derivative of x^6 to involve x^7.

Now, $(x^7)' = 7x^6$. This means x^7 is an antiderivative of $7x^6$, not of x^6. We must get rid of that coefficient of 7 in front of x^6. To accomplish this, divide x^7 by 7. We then have

$$\begin{aligned} \left(\frac{x^7}{7}\right)' &= \frac{7x^6}{7} && \text{because } \left(\frac{f}{C}\right)' = \frac{f'}{C} \\ &= x^6. && \text{canceling common factor 7 from numerator and denominator} \end{aligned}$$

We can state that $\frac{1}{7}x^7$ is an antiderivative of x^6.

However, $\frac{1}{7}x^7$ is not the only antiderivative of x^6. For instance,

$$\left(\frac{1}{7}x^7 + 2011\right)' = \frac{1}{7}7x^6 + 0 = x^6.$$

We can add any constant to $\frac{1}{7}x^7$ and the result is always an antiderivative of x^6. ◇

A constant added to any antiderivative of a function f gives another antiderivative of f.

The same reasoning as in this example suggests that $\frac{1}{a+1}x^{a+1}$ is an antiderivative of x^a. This formula is meaningless when $a+1=0$. We have to expect a different formula for antiderivatives of $x^{-1} = \frac{1}{x}$. In Section 3.5 we learned that $(\ln(x))' = 1/x$.

Theorem 3.6.1 *Power Rule for Antiderivatives For any number $a \neq -1$, antiderivatives of x^a are*

$$\frac{1}{a+1}x^{a+1} + C \quad \textit{for any constant } C.$$

For $a = -1$, antiderivatives of $x^{-1} = \frac{1}{x}$ are

$$\ln(x) + C \quad \textit{for any constant } C.$$

Every time you compute a derivative, you are also finding an antiderivative. For instance, since $D(\sin(x)) = \cos(x)$, $\sin(x)$ is an antiderivative of $\cos(x)$. So is $\sin(x)+C$ for any constant C. There are tables of antiderivatives that go on for hundreds of pages. Here is a miniature table with entries corresponding to the derivatives that we have found so far.

Search Google for "antiderivative table".

Function (f)	Antiderivative (F)	Comment
x^a	$\frac{1}{a+1}x^{a+1}$	for $a \neq -1$
$x^{-1} = \frac{1}{x}$	$\ln(x)$	
e^x	e^x	
$\cos(x)$	$\sin(x)$	
$\sin(x)$	$-\cos(x)$	
$\sec^2(x)$	$\tan(x)$	see Example 8
$\sec(x)\tan(x)$	$\sec(x)$	see Example 10
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$	see Theorem 3.5.2 (Section 3.5)
$\frac{1}{1+x^2}$	$\arctan(x)$	see Theorem 3.5.3 (Section 3.5)

Table 3.6.1: Miniature table of antiderivatives ($F' = f$).

An **elementary function** is a function that can be expressed in terms of polynomials, powers, trigonometric functions, exponentials, logarithms, and compositions. The derivative of an elementary function is elementary. We might expect that every elementary function would have an antiderivative that is also an elementary function.

Joseph Liouville (1809–1882)

In 1833 Joseph Liouville proved beyond a shadow of a doubt that there are elementary functions that do not have elementary antiderivatives. Elementary functions that do not have elementary antiderivatives include

e^{x^2} is important in statisticians' **bell curve**

$$e^{x^2} \qquad \frac{\sin(x)}{x} \qquad x\tan(x)\sqrt{x}\sqrt[3]{1+x} \qquad \sqrt[4]{1+x^2}$$

There are two types of elementary functions: the **algebraic** and the **transcendental**. Algebraic functions consist of polynomials, quotients of polynomials (the rational functions), and all functions that can be built up by the four operations of algebra and taking roots. For instance, $\frac{\sqrt{x+\sqrt[3]{x}+x^2}}{(1+2x)^5}$ is algebraic; while functions such as $\sin(x)$ and 2^x are not algebraic. Such functions are **transcendental**.

The four operations of algebra are $+$, $-$, $\times$ and $\div$.

See Exercise 1.

It is difficult to tell whether a given elementary function has an elementary antiderivative. For instance, $x\sin(x)$ does, namely $-x\cos(x)+\sin(x)$, as you may readily check; but $x\tan(x)$ does not. The function e^{x^2} does not, as mentioned earlier. However, $e^{\sqrt{x}}$, which looks more frightening, does have an elementary antiderivative.

See Exercise 22.

The table of basic antiderivatives will continue to expand as more derivatives are obtained in the rest of Chapter 3. The importance of antiderivatives will be revealed in Chapter 5. Specific techniques for finding antiderivatives are developed in Chapter 7.

It is often difficult to decide when an elementary function has an elementary antiderivative. There are algorithms implemented in software on a computers, PDAs, and calculators that can answer this question. The most well-known is the **Risch algorithm**, developed in 1968, based on differential equations and abstract algebra. A Google search for "risch antiderivative elementary symbolic" produces links related to the Risch algorithm.

How computers find antiderivatives

Reference: `http://en.wikipedia.org/wiki/Risch_algorithm`

Picturing Antiderivatives

Now that you are convinced that it is not possible to find an explicit formula for the antiderivative of many (most!) elementary functions, why do we continue to believe that these functions do have antiderivatives? This section puts the answer directly in front of your eyes. In Chapter 5 we will show how to construct antiderivatives.

The **slope field** for a function $f(x)$ shows a short line segment with slope $f(x)$ at selected points (x, y). By drawing a slope field you can get a feel for the graph of the antiderivatives of $f(x)$. The fact that an antiderivative can be graphed is very strong evidence that the antiderivative does exist. In Chapter 5 we will show that each continuous function has an antiderivative — which may not be elementary.

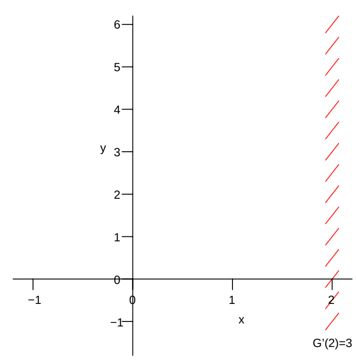
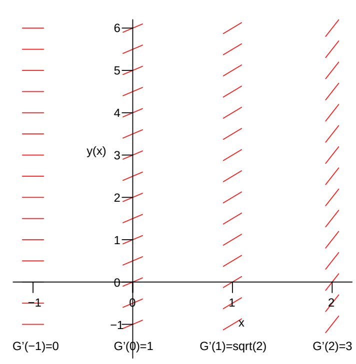


Figure 3.6.1:



EXAMPLE 2 Imagine that you are looking for an antiderivative $F(x)$ of $\sqrt{1+x^3}$. You want $F'(x)$ to be $\sqrt{1+x^3}$. Or, to put it geometrically, you want the slope of the curve $y=F(x)$ to be $\sqrt{1+x^3}$. For instance, when $x=2$, you want the slope to be $\sqrt{1+x^3}=3$. We do not know what $F(2)$ is, but at least we can draw a short piece of the tangent line at all points for which $x=2$; they all have slope 3. (See Figure 3.6.1.)

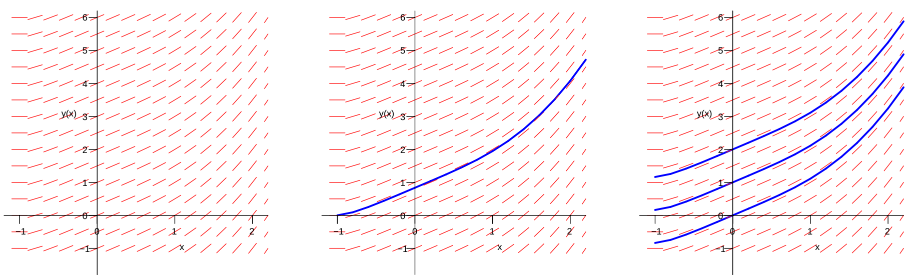
When $x=1$, $\sqrt{1+x^3}=\sqrt{2}\approx 1.4$. So we draw short lines with slope $\sqrt{2}$ on the vertical line $x=1$. When $x=0$, $\sqrt{1+x^3}=1$; the tangent lines for

$x = 0$ all have slope 1. When $x = -1$, the slopes are $\sqrt{1+x^3} = 0$ so the tangent lines are all horizontal. (See Figure 3.6.2.)

lines with slope 1 make an angle of $\frac{\pi}{4}$ or 45° with the horizontal

The plot of a slope field is most commonly done with the aid of a graphing device. Some graphing calculators have this facility and there are a number of software products for creating a computer-generated plot of a slope field. These automatic plotters have the precision to plot line segments with accurate slopes and patience to plot many line segments. A typical plot of a slope field is shown in Figure 3.6.3(a).

For a sample of available resources, search Google for "calculus slope field plot".



(a) (b) (c)

Figure 3.6.3: (a) Slope field for $f(x) = \sqrt{1+x^3}$. (b) Includes the antiderivative with $F(-1) = 0$. (c) Adds two more antiderivatives.

You can almost see the curves that follow the slope field for $f(x) = \sqrt{1+x^3}$. Start at a point, say $(-1, 0)$. At this point the slope is $F'(-1) = f(-1) = 0$ so the curve starts moving horizontally to the right. As soon as the curve leaves this initial point the slope, as given by $F'(x) = f(x)$, becomes slightly positive. This pushes the curve upward. The slope continues to increase as x increases. The curve in Figure 3.6.3(b) is the graph of an antiderivative of $f(x) = \sqrt{1+x^3}$.

The curve is given by $y = F(x)$ where $F'(x) = f(x) = \sqrt{1+x^3}$.

If you start from a different initial point, you will obtain a different antiderivative. Three antiderivatives are shown in Figure 3.6.3(c). Many other antiderivatives for $f(x) = \sqrt{1+x^3}$ are visible in the slope field. None of these functions are elementary. ⋄

It appears that different antiderivatives of a function differ by a constant: the graph of one antiderivative is simply the graph of another antiderivative raised or lowered by their constant difference. This observation is confirmed in Chapter 5

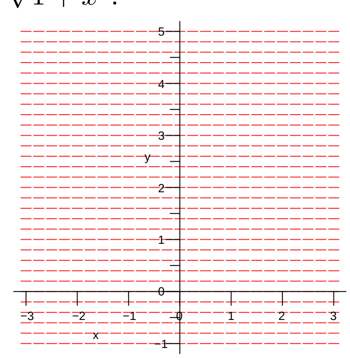
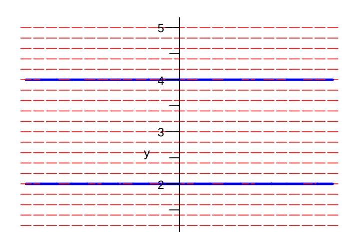


Figure 3.6.4:

EXAMPLE 3 Draw the slope field for $\frac{dy}{dx} = 0$.

SOLUTION Since the slope is 0 everywhere, each of the short tangent lines will be represented by a horizontal line segment, as in Figure 3.6.4.

In Figure 3.6.5 two possible antiderivatives of 0 are shown, namely the constant functions $f(x) = 2$ and $g(x) = 4$.



It appears as though the only functions whose derivatives are 0 everywhere are the constant functions. While this will not be completely established until Section 5.4(?), we will make use of this fact as early as Section 3.3. ⋄

Summary

The *antiderivative* was introduced as the inverse operation of differentiation. If $F' = f$, then F is an antiderivative of f. If F is an antiderivative of f, then so is $F + C$ for any constant C.

$(F(x) + C)' = F'(x) = f$.

We introduced the notion of an *elementary function*. These are the functions built up from polynomials, logarithms, exponentials, and the trigonometric functions by the four operations $+$, $-$, $\times$, $\div$, and the most important operation, composition. While the derivative of an elementary function is elementary, its antiderivative does not need to be elementary. Each elemetary function is either algebraic or transcendental. Algebraic functions, such as x^2 and $(x + 3)/(x^3 - 2x + 4)$, are built from the operations of algebra, starting with the function x; transcendental functions are the non-algebraic functions.

We showed how a *slope field* can help us analyze an antiderivative even though we may not know whether it is elementary. Slope fields appear again in Section 13.2 when we study equations that involve unknown functions and their derivatives.

EXERCISES for 3.6

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

1. Verify that $-x\cos(x) + \sin(x)$ is an antiderivative of $x\sin(x)$.

In Exercises 2–11 give two antiderivatives for each given function.

2. x^3

3. x^4

4. x^{-2}

5. $\frac{1}{x^3}$

6. $\sqrt[3]{x}$

7. $\frac{2}{x}$

8. $\sec(x)\tan(x)$

9. $\sin(x)$

10. e^{-x}

11. $\sin(2x)$

In Exercises 12–20

(a) draw the slope fields for the given derivative.

(b) then draw two curves that they suggest

12. $f'(x) = 2$

13. $f'(x) = x$

14. $f'(x) = \frac{-x}{2}$

15. $f'(x) = \frac{1}{x}$, $x > 0$

16. $f'(x) = \cos(x)$

17. $f'(x) = \sqrt{x}$

18. $f'(x) = e^{-x}$, $x > 0$

19. $f'(x) = 1/x^2$, $x \neq 0$

20. $f'(x) = 1/(x-1)$, $x \neq 1$

In Exercises 21–30 use differentiation to check that the first function is an antiderivative of the second function.

21. $2x\sin(x) - (x^2 - 2)\cos(x)$; $x^2\sin(x)$

22. $(4x^3 - 24x)\sin(x) - (x^4 - 12x^2 + 24)\cos(x)$; $x^4 \sin(x)$

23. $\frac{1}{x^3}$; $\frac{-1}{2x^2}$

24. $\frac{1}{x^{3/2}}$; $\frac{-2}{\sqrt{x}}$

25. $(x-1)e^x$; xe^x

26. $(x^2 - 2x + 2)e^x$; $x^2 e^x$

27. $\frac{1}{2}e^u(\sin(u) - \cos(u))$; $e^u \sin(u)$

28. $\frac{1}{2}e^u(\sin(u) + \cos(u))$; $e^u \cos(u)$

29. $\frac{x}{2} - \frac{\sin(x)\cos(x)}{2}$; $\sin^2(x)$

30. $2x\cos(x) - (x^2 - 2)\sin(x)$; $x^2 \cos(x)$

31.

(a) Draw the slope field for $\frac{dy}{dx} = e^{-x^2}$.

(b) Draw the curve of the antiderivative of $\frac{dy}{dx} = e^{-x^2}$ that passes through the point $(0, 1)$.

32.

(a) Draw the slope field for $\frac{dy}{dx} = f(x)$ where $f(x) = \frac{\sin(x)}{x}$, $x \neq 0$, and $f(x) = 1$ for $x = 0$.

(b) What is the slope for any point on the y-axis?

(c) Draw the curve of the antiderivative of $\frac{dy}{dx} = f(x)$ that passes through the point $(0, 1)$.

33. A table of antiderivatives lists two antiderivatives of $\frac{1}{x^2(a+bx)}$, where a and b are constants, namely

$$\frac{-1}{a^2}\left(\frac{a+bx}{x} - b\ln\left(\frac{a+bx}{x}\right)\right) \quad \text{and} \quad -\frac{1}{ax} + \frac{b}{a^2}\ln\left(\frac{a+bx}{x}\right).$$

Assume $\frac{a+bx}{x} > 0$.

(a) By differentiating both expressions, show that both are correct.

(b) Show that the two expressions differ by a constant, by finding their difference.

34. If $F(x)$ is an antiderivative of $f(x)$, what function is an antiderivative of $g(x) = f(2x)$?

3.7 Motion and the Second Derivative

In an official drag race Melanie Troxel reached a speed of 324 miles per hour in a mere 4.539 seconds. By comparison, a 1968 Fiat 850 Idromatic could reach a speed of 60 miles per hour in 25 seconds and a 1997 Porsche 911 Turbo S in a mere 3.6 seconds.

That's 475 feet per second!

Since Troxel increased her speed from 0 feet per second to 475 feet per second in 4.539 seconds her speed was increasing at the rate of $\frac{475}{4.539} \approx 105$ feet per second per second, assuming she kept the motor at maximum power throughout the time interval. That acceleration is more than three times the accelaration due to gravity at sea level (32 feet per second per second). Ms. Troxel must have felt quite a force as her seat pressed against her back.

Source: `http://web.missouri.edu/~apcb20/times.html`.

Put "automobile accera-tion" into Google to check out accelaration data for other cars.

This brings us to the formal definition of **acceleration** and an introduction to the higher derivatives.

In Sections 3.1 and 3.2 we saw that the velocity of an object moving on a line is represented by a derivative. In this section we examine the acceleration mathematically.

Acceleration

The sign of the velocity indicates direction. **Speed**, the absolute value of velocity, does not provide any indication of direction.

Velocity is the rate at which position changes. The rate at which velocity changes is called **acceleration**, denoted a. Thus if $y = f(t)$ denotes position on a line at time t, then the derivative $\frac{dy}{dt}$ equals the velocity, and the derivative of the derivative equals the acceleration. That is,

$$v = \frac{dy}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

The second derivative

The derivative of the derivative of a function $y = f(x)$ is called the **second derivative**. It is denoted many different ways, including:

$$\frac{d^2y}{dx^2}, \quad D^2y, \quad y'', \quad f'', \quad D^2f, \quad f^{(2)}, \quad \text{or} \quad \frac{d^2f}{dx^2}.$$

If $y = f(t)$, where t denotes the time, the first and second derivatives dy/dt, and d^2y/dt^2 are sometimes denoted $\dot{y}$ and $\ddot{y}$, respectively.

For instance, if $y = x^3$,

$$\frac{dy}{dx} = 3x^2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x$$

Other ways of denoting the second derivative of this function are

$$D^2(x^3) = 6x, \; \frac{d^2(x^3)}{dx^2} = 6x, \quad \text{and} \quad (x^3)'' = 6x.$$

The table in the margin lists dy/dx, the first derivative, and d^2y/dx^2, the second derivative, for a few functions. Most functions f met in applications of calculus can be differentiated repeatedly in the sense that Df exists, the derivative of Df, namely, D^2f, exists, the derivative of D^2f exists, and so on.

y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
x^3	$3x^2$	$6x$
$\frac{1}{x}$	$\frac{-1}{x^2}$	$\frac{2}{x^3}$
$\sin(5x)$	$5\cos(5x)$	$-25\sin(5x)$

The derivative of the second derivative is called the **third derivative** and is denoted many ways, such as

$$\frac{d^3y}{dx^3}, \quad D^3y, \quad y''', \quad f''', \quad f^{(3)}, \quad \text{or} \quad \frac{d^3f}{dx^3}.$$

The fourth derivative is defined similarly, as the derivative of the third derivative. In the same way we can define the n^{th} derivative for any positive integer n and denote this by such symbols as

$$\frac{d^ny}{dx^n}, \quad D^ny, \quad f^{(n)}, \quad \text{or} \quad \frac{d^nf}{dx^n}.$$

It is read as "the n^{th} derivative with respect to x." For instance, if $f(x) = 2x^3 + x^2 - x + 5$, we have

$$\begin{aligned}
f^{(1)}(x) &= 6x^2 + 2x - 1 \\
f^{(2)}(x) &= 12x + 2 \\
f^{(3)}(x) &= 12 \\
f^{(4)}(x) &= 0 \\
f^{(n)}(x) &= 0 \qquad \text{for } n \geq 5.
\end{aligned}$$

The "higher-order" derivatives will be needed in Chapter 5.

EXAMPLE 1 Find $D^n(e^{-2x})$ for each positive integer n.
SOLUTION

$$\begin{aligned}
D^1\left(e^{-2x}\right) &= D\left(e^{-2x}\right) = -2e^{-2x} \\
D^2\left(e^{-2x}\right) &= D\left(-2e^{-2x}\right) = (-2)^2e^{-2x} \\
D^3\left(e^{-2x}\right) &= D\left((-2)^2e^{-2x}\right) = (-2)^3e^{-2x}
\end{aligned}$$

At each differentiation another (-2) becomes part of the coefficient. Thus

$$D^n\left(e^{-2x}\right) = (-2)^ne^{-2x}.$$

The power $(-1)^n$ records a "plus" if n is even and a "minus" is n is odd.

This can also be written

$$D^n\left(e^{-2x}\right) = (-1)^n2^ne^{-2x}.$$

◇

Finding Velocity and Acceleration from Position

EXAMPLE 2 A falling rock drops $16t^2$ feet in the first t seconds. Find its velocity and acceleration.

SOLUTION Place the y-axis in the usual position, with 0 at the beginning of the fall and the part with positive values above 0, as in Figure 3.7.1. At time t the object has the y coordinate

$$y = -16t^2.$$

The velocity is $v = (-16t^2)' = -32t$ feet per second, and the acceleration is $a = (-32t)' = -32$ feet per second per second. The velocity changes at a constant rate. That is, the acceleration is constant. ◇

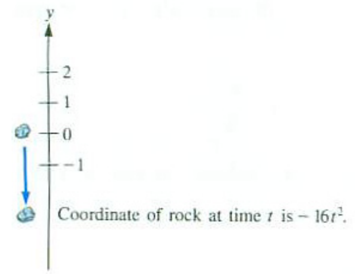


Figure 3.7.1:

Finding Position from Velocity and Acceleration

To calculate the position of a moving object at any time it is enough to know the object's acceleration at all times, the initial position, and the initial velocity. This will be demonstrated in the next two examples, where acceleration is constant. In the first example, the acceleration is 0.

EXAMPLE 3 In the simplest motion, no forces act on a moving particle, hence its acceleration is 0. Assume that a particle is moving on the x-axis and no forces act on it. Let its location at time t seconds be $x = f(t)$ feet. See Figure 3.7.2. If at time $t = 0$, $x = 3$ feet and the velocity is 5 feet per second, determine $f(t)$.

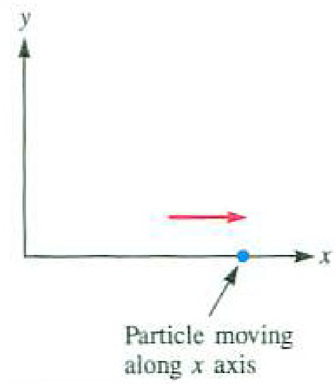


Figure 3.7.2:

SOLUTION The assumption that no force operates on the particle tells us that $d^2x/dt^2 = 0$. Call the velocity v. Then

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = 0$$

Now, v is itself a function of time. Since its derivative is 0, v must be constant:

Any constant function is an antiderivative of 0.

$$v(t) = C$$

for some constant C. Since $v(0) = 5$, the constant C must be 5.

To find the position x as a function of time, note that

$$\frac{dx}{dt} = 5$$

This equation implies that $x = f(t)$ must be in the form

$$x = 5t + K$$

for some constant K. Now, when $t = 0$, $x = 3$. Thus $K = 3$. In short, at time t seconds, the particle is at $x = 5t + 3$ feet. ◇

Since $5t$ is an antiderivative of 5, any other antiderivative must be of the form $5t + K$.

The next example concerns the case in which the acceleration is constant, but not zero.

EXAMPLE 4 A ball is thrown straight up, with an initial speed of 64 feet per second, from a cliff 96 feet above a beach. Where is the ball t seconds later? When does it reach its maximum height? How high above the beach does the ball rise? When does the ball hit the beach? Assume that there is no air resistance and that the acceleration due to gravity is constant.

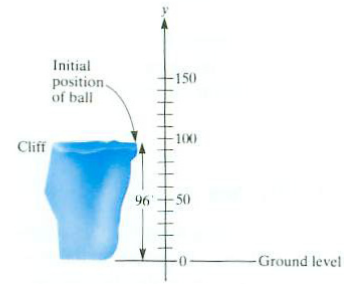


Figure 3.7.3:

SOLUTION Introduce a vertical coordinate axis to describe the position of the ball. It is more natural to call it the y-axis, and so the velocity is dy/dt and acceleration is d^2y/dt^2. Place the origin at ground level and let the positive part of the y-axis be above the ground, as in Figure 3.7.3. At time $t = 0$, the velocity $dy/dt = 64$, since the ball is thrown up at a speed of 64 feet per second. As time increases, dy/dt decreases from 64 to 0 (when the ball reaches the top of it path and begins its descent) and continues to decrease through larger and larger negative values as the ball falls to the ground. Since v is decreasing, the acceleration dv/dt is negative. The (constant) value of dv/dt, gravitational acceleration, is approximately -32 feet per second per second.

If it had been thrown down dy/dt would be -64.

From the equation

$$a = \frac{dv}{dt} = -32,$$

it follows that

$$v = -32t + C,$$

Velocity is an antiderivative of acceleration.

where C is some constant. To find C, recall that $v = 64$ when $t = 0$. Thus

$$64 = -32 \cdot 0 + C,$$

The $-32t + C$ is an antiderivative of the constant function, -32.

and $C = 64$. Hence $v = -32t + 64$ for any time t until the ball hits the beach. Next, $v = dy/dt$, so

$$\frac{dy}{dt} = v = -32t + 64.$$

Position is an antiderivative of velocity.

Since the position function y is an antiderivative of the velocity, $-32t+64$, we have

$$y(t) = -16t^2 + 64t + K,$$

where K is a constant. To find K, make use of the fact that $y = 96$ when $t = 0$. Thus

$$96 = -16 \cdot 0^2 + 64 \cdot 0 + K,$$

and $K = 96$.

We have obtained a complete description of the position of the ball at any time t while it is in the air:

$$y = -16t^2 + 64t + 96.$$

This, together with $v = -32t + 64$, provides answers to many questions about the ball's flight.

When does it reach its maximum height? When it is neither rising nor falling. In other words, the velocity is neither positive nor negative, but must be 0. When $v = 0$; that is, when $-32t + 64 = 0$, or when $t = 2$ seconds.

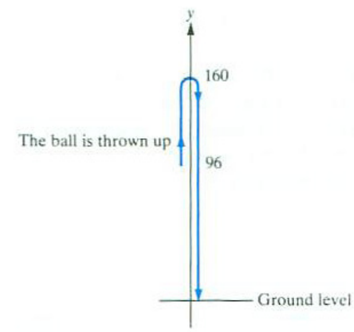


Figure 3.7.4:

How high above the ground does the ball rise? Simply compute y when $t = 2$. This gives $-16 \cdot 2^2 + 64 \cdot 2 + 96 = 160$ feet. (See Figure 3.7.4.)

When does the ball hit the beach? When $y = 0$. Find t such that

$$y = -16t^2 + 64t + 96 = 0$$

Division by -16 yields the simpler equation $t^2 - 4t - 6 = 0$, which has the solutions

$$t = \frac{4 \pm \sqrt{16 + 24}}{2} = 2 \pm \sqrt{10}.$$

Since $2-\sqrt{10}$ is negative and the ball cannot hit the beach before it is thrown, the only physically meaningful solution is $2+\sqrt{10}$. The ball lands $2+\sqrt{10}$ seconds after it is thrown; it is in the air for about 5.2 seconds.

The graphs of position, velocity, and acceleration as functions of time provide another perspective on the motion of the ball, as shown in Figure 3.7.4.

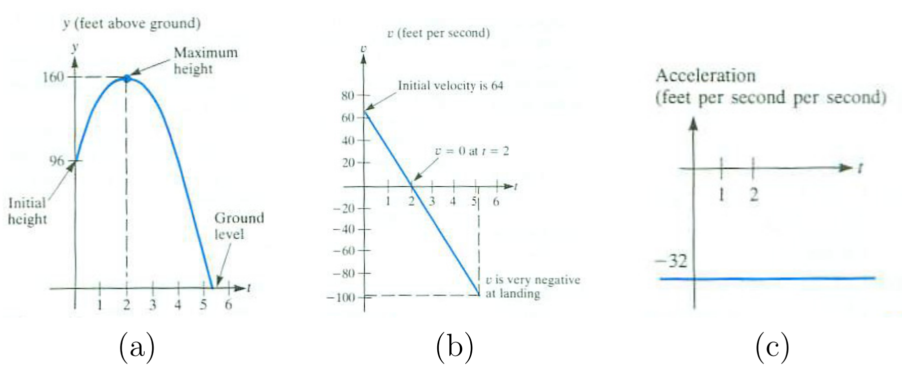


Figure 3.7.5: (a) Position, (b) velocity, and (c) acceleration for the object in Example 4.

◇

Reasoning like that in Examples 3 and 4 establishes the following description of motion in all cases where the acceleration is constant.

a, v_0, and y_0 must be given in consistent units.

OBSERVATION *(Motion Under Constant Acceleration)* Assume that a particle moving on the y axis has a constant acceleration a at any time. Assume that at time $t = 0$ it has an initial velocity v_0 and has the initial y-coordinate y_0. Then at any time $t \geq 0$ its y-coordinate is

$$y = \frac{a}{2}t^2 + v_0 t + y_0.$$

In Example 3, $a = 0$, $v_0 = 5$, and $y_0 = 3$; in Example 4, $a = -32$ $v_0 = 64$, and $y_0 = 96$.

Summary

We defined the higher derivatives of a function. They are obtained by repeatedly differentiating. The second derivative is the derivative of the derivative, the third derivative being the derivative of the second derivative, and so on. The first and second derivatives, $D(f)$ and $D^2(f)$, are used in many applications. We also analyzed motion under constant acceleration.

EXERCISES for 3.7

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In Exercises 1–16 find the first and second derivatives of the given functions.

1. $y = 2x + 3$

2. $y = 5x - 7$

3. $y = x^5$

4. $y = x^6$

5. $y = 2x^3 + x + 2$

6. $y = 4x^3 - x^2 + x$

7. $y = \frac{x}{x+1}$

8. $y = \frac{x^2}{x-1}$

9. $y = x\cos(x^2)$

10. $y = \frac{x}{\tan(3x)}$

11. $y = (x-2)^4$

12. $y = (x+1)^3$

13. $y = \sin(3x)$

14. $y = \tan(x^2)$

15. $y = x^2\arctan(3x)$

16. $y = -\frac{\arcsin(2x)}{x^2}$

17. Find $D^3\left(5x^2 - 2x + 7\right)$.

18. Find $D^4(\sin(2x))$.

19. Find $D^n(e^x)$.

20. Find $D(\sin(x))$, $D^2(\sin(x))$, $D^3(\sin(x))$, and $D^4(\sin(x))$.

21. Find $D(\cos(x))$, $D^2(\cos(x))$, $D^3(\cos(x))$, and $D^4(\cos(x))$.

22. Find $D(\ln(x))$, $D^2(\ln(x))$, $D^3(\ln(x))$, and $D^4(\ln(x))$.

23. Find $D^4(x^4)$ and $D^5(x^4)$.

24. Find $D^{200}(\sin(x))$

25. Find $D^{200}(e^x)$

26. Find $D^3(5^x)$

27. Find all functions f such that $D^2 f = 0$ for all x.

28. Find all functions f such that $D^3(f) = 0$ for all x.

29. Give the most general formula you can think of for functions f such that $D^2(f) = 4f$.

30. Give the most general formula you can think of for functions f such that $D^2(f) = -4f$.

31. Use calculus, specifically derivatives, to restate the following reports about the Leaning Tower of Pisa.

(a) "Until 2001, the tower's angle from the vertical was increasing more rapidly."

(b) "Since 2001, the tower's angle from the vertical has not changed."

HINT: Let $\theta = f(t)$ be the angle of deviation from the vertical at time t. NOTE: Incidently, the tower, begun in 1174 and completed in 1350, is 179 feet tall and leans about 14 feet from the vertical. Each day it leaned on the average, another $\frac{1}{5000}$ inch until the tower was propped up in 2001.

Exercises 32–34 concern Example 4.

32.

(a) How long after the ball in Example 4 is thrown does it pass by the top of the hill?

(b) What are its speed and velocity at this instant?

33. Suppose the ball in Example 4 had simply been dropped from the cliff. Find the position y as a function of time. How long would it take the ball to reach the beach?

34. In view of the result of Exercise 33, provide a physical interpretation of the three terms on the right-hand side of the formula $y = -16t^2 + 64t + 96$.

35. At time $t = 0$ a particle is at $y = 3$ feet and has a velocity of -3 feet per second; it has a constant acceleration of 6 feet per second per second. Find its position at any time t.

36. At time $t = 0$ a particle is at $y = 10$ feet and has a velocity of 8 feet per second; it has a constant acceleration of -8 feet per second per second.

(a) Find its position at any time t.

(b) What is its maximum y coordinate.

37. At time $t = 0$ a particle is at $y = 0$ feet and has a velocity of 0 feet per second. Find its position at any time t if its acceleration is always -32 feet per second per second.

38. At time $t = 0$ a particle is at $y = -4$ feet and has a velocity of 6 feet per second; it has a constant acceleration of -32 feet per second per second.

(a) Find its position at any time t.

(b) What is its largest y coordinate.

39. A jetliner begins its descent 120 miles from the airport. Its velocity when the descent begins is 500 miles per hour and its landing velocity is 180 miles per hour. Assuming a constant deceleration, how long does the descent take?

40. Let $y = f(t)$ describe the motion on the y-axis of an object whose acceleration has the constant value a. Show that

$$y = \frac{a}{2}t^2 + v_0 t + y_0$$

where v_0 is the velocity when $t = 0$ and y_0 is the position when $t = 0$.

41. Which has the highest acceleration? Melanie Troxel's dragster, a 1997 Porsche 911 Turbo S, or an airplane being launched from an aircraft carrier? The plane reaches a velocity of 180 miles per hour in 2.5 seconds, within a distance of 300 feet. Hint: Assume each acceleration is a constant.

42. Why do engineers call the third derivative of position with respect to time the **jerk**?

43. A model rocket is launched upward from the surface of the earth from rest with acceleration given by $a = 6t - t^2$ feet per second per second.

(a) At what time is the rocket's acceleration zero? When is the rocket accelerating? decelerating?

(b) What is the rocket's velocity, as a function of time?

(c) At what time does the rocket reach its maximum height?

(d) What is the rocket's position, as a function of time?

(e) What is the maximum height attained by this rocket?

(f) At what time does the rocket return to earth? What are the velocity and acceleration at impact?

44. A car accelerates with constant acceleration from 0 (rest) to 60 miles per hour in 15 seconds. How far does it travel in this period? NOTE: Be sure to do your computations either all in seconds, or all in hours; for instance, 60 miles per hour is 88 feet per second.

45. Show that a ball thrown straight up from the ground takes as long to rise as to fall back to its initial position. How does the velocity with which it strikes the ground compare with its initial velocity? How do the initial and landing speeds compare?

3.8 Precise Definition of Limits at Infinity: $\lim_{x\to\infty} f(x) = L$

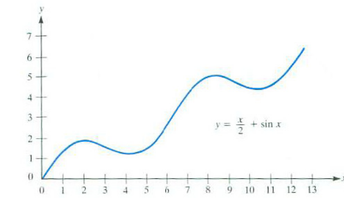


Figure 3.8.1:

How would you vote?

One day I drew on the board the graph shown in Figure 3.8.1. It is the graph of $x/2 + \sin(x)$. Then I asked my class whether they thought that

$$\lim_{x\to\infty} f(x) = \infty.$$

A third of the class voted "No" because "it keeps going up and down." A third voted "Yes" because "the function tends to get very large as x increases." A third didn't vote. Such a variety of views on such a fundamental concept suggests that we need a more precise definition of a limit than the ones developed in Sections 1.1 and 1.2.

In the definitions of the limits considered in Sections 2.1 and 2.2 appear such phrases as "x approaches a," "$f(x)$ approaches a specific number," "as a gets larger," and "$f(x)$ becomes and remains arbitrarily large." Such phrases, although appealing to the intuition and conveying the sense of a limit, are not precise. The definitions seem to suggest moving objects and call to mind the motion of a pencil point as it traces out the graph of a function.

This informal approach was adequate during the early development of calculus, from Leibniz and Newton in the seventeenth century through the Bernoullis, Euler, and Gauss in the eighteenth and early nineteenth centuries. But by the mid-nineteenth century, mathematicians, facing more complicated functions and more difficult theorems, no longer could depend solely on intuition. They realized that glancing at a graph was no longer adequate to understand the behavior of functions — especially if theorems covering a broad class of functions were needed.

It was Weierstrass who developed, over a period of 16 years, a way to define limits without any hint of motion or pencils tracing out graphs. His approach, on which he lectured after joining the faculty at the University of Berlin in 1859, has since been followed by pure and applied mathematicians throughout the world. Even an undergraduate advanced calculus course depends on Weierstrass's approach.

1841–1856

In this section we examine how Weierstrass would define the concepts for "limits at infinity:"

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = L.$$

In the next section we consider finite limits at finite points:

$$\lim_{x\to a} f(x) = L.$$

The Precise Definition of $\lim_{x\to\infty} f(x) = \infty$

Recall the definition of $\lim_{x\to\infty} f(x) = \infty$ given in Section 2.1.

Informal definition of $\lim_{x\to\infty} f(x) = \infty$

1. $f(x)$ is defined for all x beyond some number
2. As x gets large through positive values, $f(x)$ becomes and remains arbitrarily large and positive.

To take us part way to the precise definition, let us reword the informal definition, paraphrasing it in the following definition, which is still informal.

Reworded informal definition of $\lim_{x\to\infty} f(x) = \infty$

1. Assume that $f(x)$ is defined for all x greater than the number c.
2. If x is sufficiently large and positive, then $f(x)$ is necessarily large and positive.

The precise definition paralles the reworded definition.

DEFINITION *(Precise definition of* $\lim_{x\to\infty} f(x) = \infty$*)*

1. Assume the $f(x)$ is defined for all x greater than some number c.
2. For each number E there is a number D such that for all $x > D$ it is true that $f(x) > E$.

The "challenge and reply" approach to limits

Think of the number E as a challenge and D as the reply. The *larger* E is, the *larger* D must usually be. Only if a number D (which depends on E) can he found for *every* number E can we make the claim that $\lim_{x\to\infty} f(x) = \infty$. In other words, D could be expressed as a function of E. To picture the idea behind the precise definition, consider the graph in Figure 3.8.2 of a function f for which $\lim_{x\to\infty} f(x) = \infty$. For each possible choice of a horizontal line, say, at height E, if you are far enough to the right on the graph of f, you stay above that horizontal line. That is, there is a number D such that if $x > D$, then $f(x) > E$, as illustrated in Figure 3.8.3.

The number D in Figure 3.8.4 is not a suitable reply. It is too small since there are (some) values of $x > D$ such that $f(x) \le E$.

Examples 1 and 2 illustrate how the precise definition is used.

EXAMPLE 1 Using the precise definition, show that $\lim_{x\to\infty} 2x = \infty$.
SOLUTION Let E be any (positive) number. We must show that there is a number D such that whenever $x > D$ it follows that $2x > E$. The number D will depend on E. Our goal is find a formula for D for any value of E.

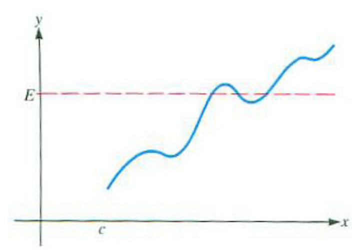


Figure 3.8.2:

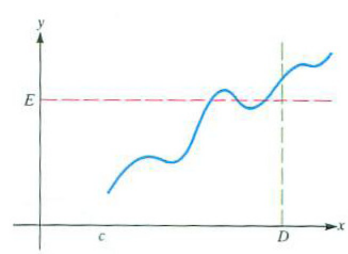
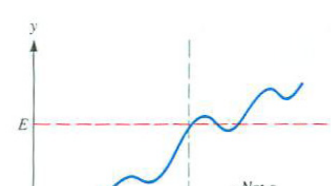


Figure 3.8.3:



Now, the inequality $2x > E$ is equivalent to

$$x > \frac{E}{2}.$$

In other words, if $x > E/2$, then $2x > E$ (and vice versa). So choosing $D = E/2$ will suffice. To verify this: when $x > D$ $(= E/2)$, $2x > 2D = 2\frac{E}{2} = E$. This allows us to conclude immediately that

D depends on E

$$\lim_{x\to\infty} 2x = \infty.$$

◇

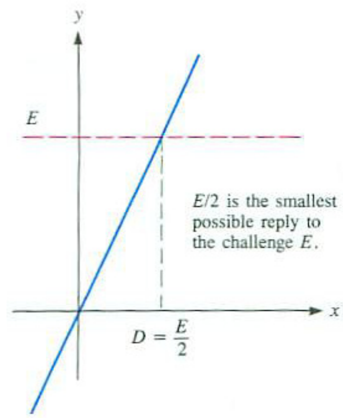


Figure 3.8.5:

In Example 1 a formula was provided for a suitable D in terms of E, namely, $D = E/2$ (see Figure 3.8.5. For instance, when challenged with $E = 1000$, the response $D = 500$ suffices. In fact, any larger value of D also is suitable. If $x > 600$, it is still the case that $2x > 1000$ (since $2x > 1200$). If one value of D is a satisfactory response to a given challenge E, then any larger value of D also is a satisfactory response.

Now that we have a precise definition of $\lim_{x\to\infty} f(x) = \infty$ we can settle the question, "Is $\lim_{x\to\infty}(x/2 + \sin(x)) = \infty$?"

EXAMPLE 2 Using the precise definition, show that $\lim_{x\to\infty} \frac{x}{3} + \sin(x) = \infty$.

SOLUTION Let E be any number. We must exhibit a number D, depending on E, such that $x > D$ forces

$$\frac{x}{3} + \sin(x) > E. \qquad (1)$$

Now, $\sin(x) \geq -1$ for all x. So, if we can force

$$\frac{x}{3} + (-1) > E \qquad (2)$$

then it will follow that

$$\frac{x}{3} + \sin(x) > E.$$

The smallest value of x that satisfies inequality (1) can be found as follows:

$$\begin{array}{lll} \frac{x}{3} & > & E+1 \quad \text{add 1 to both sides} \\ x & > & 3(E+1) \quad \text{multiply by a positive constant.} \end{array}$$

Thus $D = 3(E + 1)$ will suffice. That is,

D depends on E

$$\text{If } x > 3(E+1), \text{ then } \frac{x}{3} + \sin(x) > E.$$

To verify this assertion we must check that $D = 3(E+1)$ is a satisfactory reply to E. Assume that $x > D = 3(E+1)$. Then

$$\begin{array}{lrcl} & \frac{x}{3} & > & E+1 \\ \text{and} & \sin(x) & \geq & -1. \end{array}$$

Adding these last two inequalities gives

If $a > b$ and $c \geq d$, then $a + c > b + d$.

$$\begin{array}{lrcl} & \frac{x}{3} + \sin(x) & > & (E+1) + (-1) \\ \text{or simply} & \frac{x}{3} + \sin(x) & > & E, \end{array}$$

which is inequality (1). That is,

$$\lim_{x\to\infty} \frac{x}{2} + \sin(x) = \infty.$$

As x increases, the function does *become* and *remain* large, despite the small dips downward. ◇

The Precise Definition of $\lim_{x\to\infty} f(x) = L$

Next, recall the definition of $\lim_{x\to\infty} f(x) = L$ given in Section 2.1.

L is a finite number.

Informal definition of $\lim_{x\to\infty} f(x) = L$

1. $f(x)$ is defined for all x beyond some number
2. As x gets large through positive values, $f(x)$ approaches L.

Again we reword this definition before offering the precise definition.

Reworded informal definition of $\lim_{x\to\infty} f(x) = L$

1. Assume that $f(x)$ is defined for all x greater than some number c.
2. If x is sufficiently large, then $f(x)$ is necessarily near L.

Once again, the precise definition parallels the reworded definition. In order to make precise the phrase "$f(x)$ is necessarily near L," we shall use the absolute value of $f(x) - L$ to measure the distance from $f(x)$ to L. The following definition says that "if x is large enough, then $|f(x) - L|$ is as small as we please".

DEFINITION *(Precise definition of* $\lim_{x\to\infty} f(x) = L$*)*

1. Assume the $f(x)$ is defined for all x beyond some number c.

2. For each positive number ϵ there is a number D such that for all $x > D$ it is true that

$$|f(x) - L| < \epsilon.$$

Draw two lines parallel to the x-axis, one of height $L + \epsilon$ and one of height $L - \epsilon$. They are the two edges of an endless band of width 2ϵ and centered at $y = L$. Assume that for each positive ϵ, a number D can be found such that the part of the graph to the right of $x = D$ lies within the band. Then we say that "as x approaches ∞, $f(x)$ approaches L" and write

$$\lim_{x\to\infty} f(x) = L.$$

The positive number ϵ is the challenge, and D is a reply. The smaller ϵ is, the narrower the band is, and the larger D usually must be chosen. The geometric meaning of the precise definition of $\lim_{x\to\infty} f(x) = L$ is shown in Figure 3.8.6.

The number ϵ is the challenge. The number D is a reply.

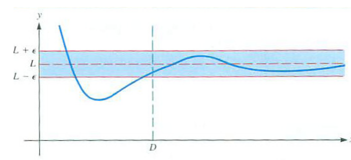


Figure 3.8.6:

"ϵ" (epsilon) is the Greek letter corresponding to the English letter "e"

EXAMPLE 3 Use the precise definition of "$\lim_{x\to\infty} f(x) = L$" to show that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right) = 1.$$

SOLUTION Here $f(x) = 1+1/x$, which is defined for all $x \neq 0$. In particular, observe that $f(x)$ is defined for all $x > 0$. The number L is 1. We must show that for each positive number ϵ, however small, there is a number D such that, for all $x > D$,

$$\left|\left(1 + \frac{1}{x}\right) - 1\right| < \epsilon. \tag{3}$$

Inequality (3) reduces to

$$\left|\frac{1}{x}\right| < \epsilon.$$

Since we shall consider only $x > 0$, this inequality is equivalent to

$$\frac{1}{x} < \epsilon. \tag{4}$$

Multiplying inequality (4) by the positive number x yields the equivalent inequality

$$1 < x\epsilon. \tag{5}$$

Division of inequality (5) by the positive number ϵ yields

$$\frac{1}{\epsilon} < x \quad \text{or} \quad x > \frac{1}{\epsilon}.$$

D depends on ϵ.

These steps are reversible. This shows that $D = 1/\epsilon$ is a suitable reply to the challenge ϵ. If $x > 1/\epsilon$, then

$$\left|\left(1 + \frac{1}{x}\right) - 1\right| < \epsilon.$$

That is, inequality (3) is satisfied.

According to the precise definition of "$\lim_{x\to\infty} f(x) = L$", we conclude that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right) = 1.$$

◇

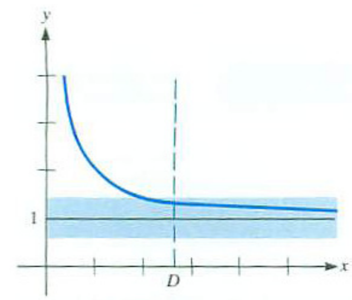


Figure 3.8.7:

The graph of $f(x) = 1+1/x$, shown in Figure 3.8.7, reinforces the argument. It seems plausible that no matter how narrow a band someone may place around the line $y = 1$, it will always be possible to find a number D such that the part of the graph to the right of $x = D$ stays within that band. In Figure 3.8.7 the typical band is shown shaded.

The precise definitions can also be used to show that some claim about an alleged limit is false. The next example illustrates how this is done.

EXAMPLE 4 Show that the claim that $\lim_{x\to\infty} \sin(x) = 0$ is false.

SOLUTION To show that the claim is *false*, we must exhibit a challenge $\epsilon > 0$ for which no response D can be found. That is, we must exhibit a positive number ϵ such that no D exists for which $|\sin(x) - 0| < \epsilon$ for all $x > D$.

Recall that $\sin(\pi/2) = 1$ and that $\sin(x) = 1$ whenever $x = \pi/2 + 2n\pi$ for any integer n. This means that there are arbitrarily large values of x for which $\sin(x) = 1$. This suggests how to exhibit an $\epsilon > 0$ for which no response D can be found. Simply pick ϵ to be some positive number less than or equal to 1. For instance, $\epsilon = 0.7$ will do.

For any number D there is always a number $x^* > D$ such that we have $\sin(x^*) = 1$. This means that $|\sin(x^*) - 0| = 1 > 0.7$. Hence no response can he found for $\epsilon = 0.7$. Thus the claim that $\lim_{x\to\infty} \sin(x) = 0$ is false. ◇

To conclude this section, we show how the precise definition of the limit can be used to obtain information about new limits.

EXAMPLE 5 Use the precise definition of "$\lim_{x\to\infty} f(x) = L$" to show that if f and g are defined everywhere and $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to\infty} g(x) = 3$, then $\lim_{x\to\infty}(f(x) + g(x)) = 5$.

SOLUTION The objective is to show that for each positive number ϵ, however small, there is a number D such that, for all $x > D$,

$$|(f(x) + g(x)) - 5| < \epsilon.$$

Observe that $|(f(x)+g(x))-5|$ can be written as $|(f(x)-2)+(g(x)-3))|$, and this is no larger than the sum $|f(x)-2|+|g(x)-3|$. If we can show that for all x sufficiently large that both $|f(x)-2|<\epsilon/2$ and $|g(x)-3|<\epsilon/2$, then there sum will be no larger than $\epsilon/2+\epsilon/2=\epsilon$.

Here is how this plan can be implemented.

The fact that $\lim_{x\to\infty} f(x)=2$ means for any given $\epsilon>0$ there exists a number D_1 with the property that $|f(x)-2|<\epsilon/2$ for all $x>D_1$. Likewise, the fact that $\lim_{c\to\infty} g(x)=3$ means for any given $\epsilon>0$ there exists a number D_2 with the property that $|g(x)-2|<\epsilon/2$ for all $x>D_2$.

$D=\max\{D_1,D_2\}$

Let D refer to the larger of D_1 and D_2. For any x greater than D we know that

$$|f(x)+g(x)-5|<|f(x)-2|+|g(x)-3|<\epsilon/2+\epsilon/2=\epsilon.$$

According to the precise definition of a limit at infinity, we conclude that

$$\lim_{x\to\infty}(f(x)+g(x))=2+3=5.$$

◇

EXERCISES for 3.8

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

1. Let $f(x) = 3x$.

(a) Find a number D such that, for $x > D$, it follows that $f(x) > 600$.

(b) Find another number D such that, for $x > D$, it follows that $f(x) > 600$.

(c) What is the smallest number D such that, for all $x > D$, it follows that $f(x) > 600$?

2. Let $f(x) = 4x$.

(a) Find a number D such that, for $x > D$, it follows that $f(x) > 1000$.

(b) Find another number D such that, for $x > D$, it follows that $f(x) > 1000$.

(c) What is the smallest number D such that, for all $x > D$, it follows that $f(x) > 1000$?

3. Let $f(x) = 5x$. Find a number D such that, for all $x > D$,

(a) $f(x) > 2000$,

(b) $f(x) > 10,000$.

4. Let $f(x) = 6x$. Find a number D such that, for all $x > D$,

(a) $f(x) > 1200$,

(b) $f(x) > 1800$.

In Exercises 5 to 12 use the precise definition of the assertion "$\lim_{x\to\infty} f(x) = \infty$" to establish each limit.

5. $\lim_{x\to\infty} 3x = \infty$

6. $\lim_{x\to\infty} 4x = \infty$

7. $\lim_{x\to\infty} (x + 5) = \infty$

8. $\lim_{x\to\infty}(x-600)=\infty$

9. $\lim_{x\to\infty}(2x+4)=\infty$

10. $\lim_{x\to\infty}(3x-1200)=\infty$

11. $\lim_{x\to\infty}(4x+100\cos(x))=\infty$

12. $\lim_{x\to\infty}(2x-300\cos(x))=\infty$

13. Let $f(x)=x^2$.

(a) Find a number D such that, for all $x>D$, $f(x)>100$.

(b) Let E be any nonnegative number. Find a number D such that, for all $x>D$, it follows that $f(x)>E$.

(c) Let E be any negative number. Find a number D such that, for all $x>D$, it follows that $f(x)>E$.

(d) Using the precise definition of "$\lim_{x\to\infty} f(x)=\infty$", show that $\lim_{x\to\infty} x^2=\infty$.

14. Using the precise definition of "$\lim_{x\to\infty} f(x)=\infty$", show that $\lim_{x\to\infty} x^3=\infty$. Hint: See Exercise 13.

Exercises 15 to 22 concern the precise definition of "$\lim_{x\to\infty} f(x)=L$".

15. Let $f(x)=3+1/x$ if $x\neq 0$.

(a) Find a number D such that, for all $x>D$, it follows that $|f(x)-3|<\frac{1}{10}$.

(b) Find another number D such that, for all $x>D$, it follows that $|f(x)-3|<\frac{1}{10}$.

(c) What is the smallest number D such that, for all $x>D$, it follows that $|f(x)-3|<\frac{1}{10}$?

(d) Using the precise definition of "$\lim_{x\to\infty} f(x)=L$", show that $\lim_{x\to\infty}(3+1/x)=3$.

16. Let $f(x)=2/x$ if $x\neq 0$.

(a) Find a number D such that, for all $x>D$, it follows that $|f(x)-0|<\frac{1}{100}$.

(b) Find another number D such that, for all $x>D$, it follows that $|f(x)-0|<\frac{1}{100}$.

(c) What is the smallest number D such that, for all $x>D$, it follows that $|f(x)-0|<\frac{1}{100}$?

(d) Using the precise definition of "$\lim_{x\to\infty} f(x)=L$", show that $\lim_{x\to\infty}(2/x)=0$.

In Exercises 17 to 22 use the precise definition of "$\lim_{x\to\infty} f(x) = L$" to establish each limit.

17. $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$ HINT: $|\sin(x)| \leq 1$ for all x.

18. $\lim_{x\to\infty} \frac{x+\cos(x)}{x} = 1$

19. $\lim_{x\to\infty} \frac{4}{x^2} = 0$

20. $\lim_{x\to\infty} \frac{2x+3}{x} = 2$

21. $\lim_{x\to\infty} \frac{1}{x-100} = 0$

22. $\lim_{x\to\infty} \frac{2x+10}{3x-5} = \frac{2}{3}$

23. Using the precise definition of "$\lim_{x\to\infty} f(x) = \infty$," show that the claim that $\lim_{x\to\infty} x/(x+1) = \infty$ is false.

24. Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that the claim that $\lim_{x\to\infty} \sin(x) = \frac{1}{2}$ is false.

25. Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that the claim that $\lim_{x\to\infty} 3x = 6$ is false.

26. Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that for every number L the assertion that $\lim_{x\to\infty} 2x = L$ is false.

In Exercises 27 to 30 develop precise definitions of the given limits. Phrase your definitions in terms of a challenge number E or ϵ and a reply D. Show the geometric meaning of your definition on a graph.

27. $\lim_{x\to\infty} f(x) = -\infty$

28. $\lim_{x\to-\infty} f(x) = \infty$

29. $\lim_{x\to-\infty} f(x) = -\infty$

30. $\lim_{x\to\infty} f(x) = L$

31. Let $f(x) = 5$ for all x.

(a) Using the precise definition of "$\lim_{x\to\infty} f(x) = L$," show that $\lim_{x\to\infty} f(x) = 5$.

(b) Using the precise definition of "$\lim_{x\to-\infty} f(x) = L$," show that $\lim_{x\to\infty} f(x) = 5$. (See Exercise 30

32. Is this argument correct? "I will prove that $\lim_{x\to\infty}(2x + \cos(x)) = \infty$. Let E be given. I want

$$\begin{array}{lrcl} & 2x + \cos(x) & > & E \\ \text{or} & 2x & > & E - \cos(x) \\ \text{so} & x & > & \frac{E-\cos(x)}{2}. \end{array}$$

Thus, if $D = \frac{E-\cos(x)}{2}$, then $2x + \cos(x) > E$."

33. Use the precise definition of "$\lim_{x\to\infty} f(x) = L$," to prove this version of the sum law for limits: if $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty} g(x) = B$, then $\lim_{x\to\infty}(f(x) + g(x)) = A + B$. HINT: See Example 5.

34. Use the precise definition of "$\lim_{x\to\infty} f(x) = L$," to prove this version of the product law for limits: if $\lim_{x\to\infty} f(x) = A$, then $\lim_{x\to\infty}(f(x)^2) = A^2$. HINT: See Exercise 33.

35. Use the precise definition of "$\lim_{x\to\infty} f(x) = L$," to prove this version of the product law for limits: if $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty} g(x) = B$, then $\lim_{x\to\infty}(f(x)g(x)) = AB$. HINT: See Example 34.

3.9 Precise Definition of Limits at a Finite Point: $\lim\limits_{x \to a} f(x) = L$

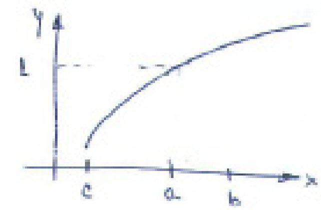


Figure 3.9.1:

To conclude the discussion of limits, we extend the ideas developed in Section 3.8 to limits with a finite limit point.

Informal definition of $\lim_{x \to a} f(x) = L$

Let f be a function and a some fixed number. (See Figure 3.9.1.)

1. Assume that the domain of f contains open intervals (c, a) and (a, b) for some number $c < a$ and some number $b > a$.
2. If, as x approaches a, either from the left or from the right, $f(x)$ approaches a specific number L, then L is called the **limit** of $f(x)$ as x approaches a. This is written

$$\lim_{x \to a} f(x) = L.$$

Keep in mind that a need not be in the domain of f. Even if it happens to be in the domain of f, the value of $f(a)$ plays no role in determining whether $\lim_{x \to a} f(x) = L$.

Reworded informal definition of $\lim_{x \to a} f(x) = L$

Let f be a function and a some fixed number.

1. Assume that the domain of f contains open intervals (c, a) and (a, b) for some number $c < a$ and some number $b > a$.
2. If x is is sufficiently close to a but not equal to a, then $f(x)$ is necessarily near L.

The precise definition parallels the reworded informal definition.

DEFINITION *(Precise definition of* $\lim_{x \to a} f(x) = L$*)* Let f be a function and a some fixed number.

The "ϵ, δ" definition of "$\lim_{x \to a} f(x) = L$"

1. Assume that the domain of f contains open intervals (c, a) and (a, b) for some number $c < a$ and some number $b > a$.
2. For each positive number ϵ there is a positive number δ such that for all x that satisfy the inequality

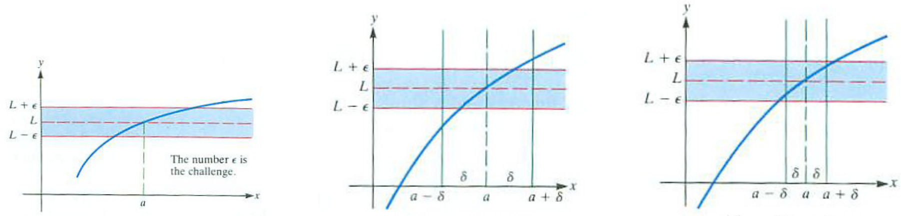
$$\begin{array}{lrcl} & 0 < |x - a| & < & \delta \\ \text{it is true that} & |f(x) - L| & < & \epsilon \end{array}$$

"δ" (delta) is the Greek letter corresponding to the English letter "d."

The meaning of $0 < |x - a| < \delta$

The inequality $0 < |x-a|$ that appears in the definition is just a fancy way of saying "x is not a." The inequality $|x - a| < \delta$ asserts that x is within a distance δ of a. The two inequalities may be combined as the single statement $0 < |x - a| < \delta$, which describes the open interval $(a - \delta, a + \delta)$ from which a is deleted. This deletion is made since the value $f(a)$ pIays no role in the definition of $\lim_{x\to a} f(x)$.

Once again ϵ is the challenge. The reply is δ. Usually, the smaller ϵ is, the *smaller* δ will have to be.



(a) (b) (c)

Figure 3.9.2: (a) The number ϵ is the challenge. (b) δ is not small enough. (c) δ is small enough.

The geometric significance of the precise definition of "$\lim_{x\to a} f(x) = L$" is shown in Figure 3.9. The narrow horizontal band of width 2ϵ is again the challenge (see Figure 3.9(a)). The desired response is a sufficiently narrow vertical band, of width 2δ, such that the part of the graph within that vertical band (except perhaps at $x = a$) also lies in the horizontal band of width 2ϵ. In Figure 3.9(b) the vertical band shown is not narrow enough to meet the challenge of the horizontal band shown. But the vertical band shown in Figure 3.9(c) is sufficiently narrow.

Assume that for each positive number ϵ it is possible to find a positive number δ such that the parts of the graph between $x = a - \delta$ and $x = a$ and between $x = a$ and $x = a + \delta$ lie within the given horizontal band. Then we say that "as x approaches a, $f(x)$ approaches L". The narrower the horizontal band around the line $y = L$, the smaller δ usually must be.

EXAMPLE 1 Use the precise definition of "$\lim\limits_{x\to a} f(x) = L$" to show that $\lim\limits_{x\to 0} x^2 = 0$.

SOLUTION In this case $f(x) = x^2$, $a = 0$ and $L = 0$. Let ϵ be a positive number. We wish to find a positive number δ such that for $0 < |x - 0| < \delta$ it follows that $|x^2 - 0| < \epsilon$. Since $|x|^2 = |x^2|$, we are asking: "For which x is $|x|^2 < \epsilon$?" This inequality is satisfied when

$$|x| < \sqrt{\epsilon}.$$

In other words, when $|x| < \sqrt{\epsilon}$, it follows that $|x^2 - 0| < \epsilon$. Thus, $\delta = \sqrt{\epsilon}$ suffices. ◇

For instance, when $\epsilon = 1$, $\delta = \sqrt{1} = 1$ is a suitable reply. When the challenge is $\epsilon = 0.01$, a reply of $\delta = 0.1$ suffices.

EXAMPLE 2 Use the precise definition of "$\lim_{x\to a} f(x) = L$" to show that $\lim_{x\to 2}(3x+5) = 11$.

SOLUTION Here $f(x) = 3x+5$, $a = 2$, and $L = 11$. Let ϵ be a positive number. We wish to find a number $\delta > 0$ such that for $0 < |x-2| < \delta$ we have $|(3x+5)-11| < \epsilon$.

So let us find out for which x it is true that $|(3x+5)-11| < \epsilon$. This inequality is equivalent to

$$\begin{array}{rrcl} & |3x-6| & < & \epsilon \\ \text{or} & 3|x-2| & < & \epsilon \\ \text{or} & |x-2| & < & \frac{\epsilon}{3}. \end{array}$$

Thus $\delta = \epsilon/3$ is a suitable response. If $0 < |x-2| < \epsilon/3$, then $|(3x+5)-11| < \epsilon$. ◇

The algebra of finding a response δ can be much more involved for other functions, such as $f(x) = ax^2 + bx + c$. The precise definition of limit can actually be easier to apply in more general situations where f and a are not given explicitly. To illustrate, we present a proof of the Permanence Principle.

Mathematicians call this a "proof by handwaving".

When the Permanence Principle was first introduced in Section 2.4, the only justification we provided was a picture and an appeal to your intuition that a continuous function cannot jump instantaneously from a positive value to zero or a negative value — the function has to remain positive on some open interval.

EXAMPLE 3 Prove the Permanence Principle: If $f(a) > 0$ and f is continuous on an open interval containing a, then there exists an open interval containing a on which $f(x) > 0$.

SOLUTION Let p be the positive value of the function at $x = a$: $p = f(a) > 0$. The assumption that f is continuous on an open interval containing a implies f is continuous at $x = a$. That is, $\lim_{x\to a} f(x) = f(a) = p$. Now, because this limit exists, given any positive ϵ, there exists a $\delta > 0$ with the property that

$$|f(x) - p| < \epsilon \quad \text{for all} \quad 0 < |x-a| < \delta.$$

In particular, choose $\epsilon = p/2$ and let δ be the corresponding response.

Let x be any number within δ of p, that is, $|x-p| < \delta$. The definition of "$\lim_{x\to a} f(x) = p$" with $p = f(a)$ and $\epsilon = p/2$ assures us that whenever

$|x-a|<p/2$ we know that

$$\begin{array}{lrcl} & |f(x)-p| & < & \epsilon \\ \text{that is} & |f(x)-p| & < & \frac{p}{2} \\ \text{or} & \frac{-p}{2}<f(x)-p & < & \frac{p}{2} \\ \text{so} & p+\frac{-p}{2}<f(x) & < & p+\frac{p}{2} \\ \text{thus} & \frac{p}{2}<f(x) & < & \frac{3p}{2}. \end{array}$$

Since $p=f(a)>0$, the left-hand inequality in this last inequality tells us that

$$\text{if} \quad |x-a|<\delta \quad \text{then} \quad f(x)>\frac{p}{2}>0.$$

Thus, we can now say that $f(x)$ is positive on the open interval centered at $x=a$ with width p, that is for x between $a-\frac{p}{2}$ and $a+\frac{p}{2}$). ◇

The choice of $\epsilon=p/2$ is not special. We can choose ϵ to be any number less than p. (See also Exercise 25.)

EXERCISES for 3.9

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In Exercises 1 to 4 use the precise definition of "$\lim_{x\to a} f(x) = L$" to justify each statement.

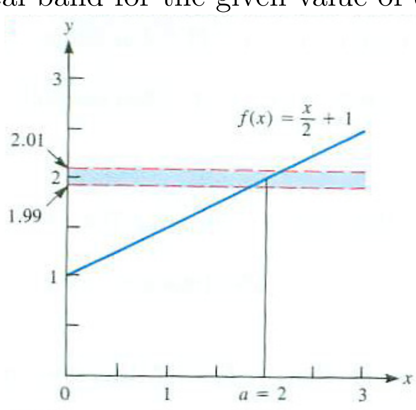
1. $\lim_{x\to 2} 3x = 6$

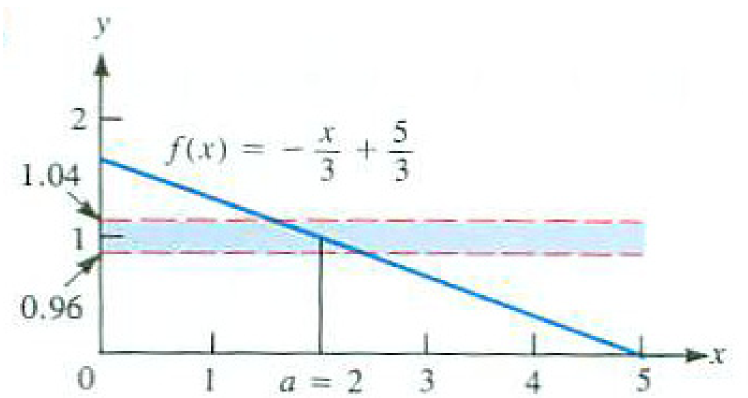
2. $\lim_{x\to 3}(4x-1) = 11$

3. $\lim_{x\to 1}(x+2) = 3$

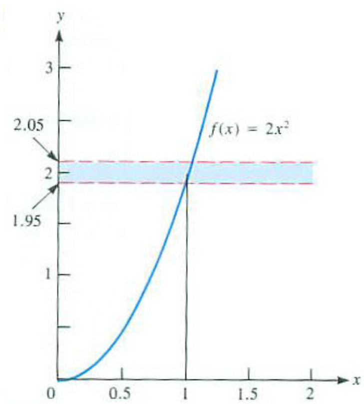
4. $\lim_{x\to 5}(2x-3) = 7$

In Exercises 5 and 8 find a number δ such that the point $(x, f(x))$ lies in the shaded band for all x in the interval $(a-\delta, a+\delta)$. HINT: Draw suitable vertical band for the given value of ϵ.

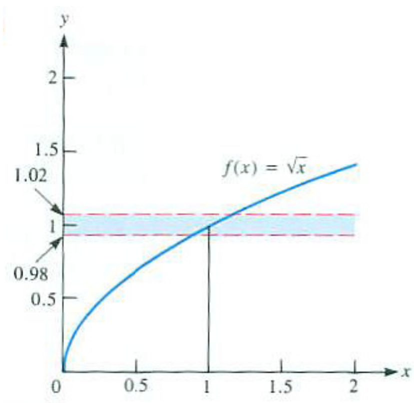
5. 

6. 

7.



8.



In Exercises 9 and 12 use the precise definition of "$\lim_{x\to a} f(x) = L$" to justify each statement.

9. $\lim\limits_{x\to 0} \dfrac{x^2}{4} = 0$

10. $\lim\limits_{x\to 0} 4x^2 = 0$

11. $\lim\limits_{x\to 1} (3x+5) = 8$

12. $\lim\limits_{x\to 1} \dfrac{5x+3}{4} = 2$

13. Give an example of a number $\delta > 0$ such that $|x^2 - 4| < 1$ if $0 < |x-2| < \delta$.

14. Give an example of a number $\delta > 0$ such that $|x^2+x-2| < 0.5$ if $0 < |x-1| < \delta$.

Develop precise definitions of the given limits in Exercises 15 to 20. Phrase your definitions in terms of a challenge, E or ϵ, and a response, δ.

15. $\lim\limits_{x\to a^+} f(x) = L$

16. $\lim\limits_{x\to a^-} f(x) = L$

17. $\lim_{x\to a} f(x) = \infty$

18. $\lim_{x\to a} f(x) = -\infty$

19. $\lim_{x\to a^+} f(x) = \infty$

20. $\lim_{x\to a^-} f(x) = \infty$

21. Let $f(x) = 9x^2$.

(a) Find $\delta > 0$ such that, for $0 < |x - 0| < \delta$, it follows that $|9x^2 - 0| < \frac{1}{100}$.

(b) Let ϵ be any positive number. Find a positive number δ such that, for $0 < |x - 0| < \delta$ we have $|9x^2 - 0| < \epsilon$.

(c) Show that $\lim_{x\to 0} 9x^2 = 0$.

22. Let $f(x) = x^3$.

(a) Find $\delta > 0$ such that, for $0 < |x - 0| < \delta$, it follows that $|x^3 - 0| < \frac{1}{1000}$.

(b) Show that $\lim_{x\to 0} x^3 = 0$.

23. Show that the assertion "$\lim_{x\to 2} 3x = 5$" is false. To do this, it is necessary to exhibit a positive number ϵ such that there is no response number $\delta > 0$
HINT: Draw a picture.

24. Show that the assertion"$\lim_{x\to 2} x^2 = 3$" is false.

25. Review the proof of the Permanence Principle given in Example 3.

(a) Revise the proof for the choice $\epsilon = p/4$.

(b) How does the value of δ obtained with $\epsilon = p/4$ compare with the value of δ obtained with $\epsilon = p/2$?

(c) Why does the proof not work when $\epsilon = p$?

(d) Show that the proof remains valid for any $0 < \epsilon < p$.

26. The Permanence Principle discussed in Example 3 and Exercise 25 pertains to limits at a finite point a. State, and prove, a version of the Permanence Principle that is valid at ∞.

27.

(a) Show that, if $0 < \delta < 1$ and $|x - 3| < \delta$, then $|x^2 - 9| < 7\delta$.

(b) Use (a) to deduce that $\lim_{x\to 3} x^2 = 9$.

28.

(a) Show that, if $0 < \delta < 1$ and $|x - 4| < \delta$, then

$$|\sqrt{x} - 2| < \frac{\delta}{\sqrt{3} + 2}.$$

HINT: Rationalize $\sqrt{x} - 2$.

(b) Use (a) to deduce that $\lim_{x\to 4} \sqrt{x} = 2$.

29.

(a) Show that, if $0 < \delta < 1$ and $|x - 3| < \delta$, then $|x^2 + 5x - 24| < 12\delta$.
HINT: Factor $x^2 + 5x - 24$.

(b) Use (a) to deduce that $\lim_{x\to 3}(x^2 + 5x) = 24$.

30.

(a) Show that, if $0 < \delta < 1$ and $|x - 2| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{\delta}{2}.$$

(b) Use (a) to deduce that $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

31. Assume that $f(x)$ is continuous at a and is defined at least on an open interval containing a. Assume that $f(x) = p > 0$. Using the precise definitino of a limit, show that there is an open interval, I, containing a such that $f(x) > p/2$ for all x in I.

3.S Chapter Summary

In this chapter we defined the derivative of a function, developed ways to compute derivatives, and applied derivatives to graphs and motion.

The derivative of a function f at a point $x = a$ is defined as the limit of the slopes of secant lines through the points $(a, f(a))$ and $(b, f(b))$ as the input b is taken closer and closer to the input a.

Algebraically, the derivative is the limit of a quotient, "the change in the output divided by the change in the input". The limit is usually written in one of the following forms:

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}, \quad \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}, \quad \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}.$$

The derivative is denoted in several ways, such as f', $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, and $D(f)$.

For the functions most frequently encountered in applications, this limit exists. Geometrically speaking, the derivative exists whenever the graph of the function on a very small interval looks almost like a straight line. The slope of this straight line is the value of the derivative at this point.

The derivative records how fast something changes. For instance, the velocity of a moving object is defined as the derivative of the object's position. Also, the derivative gives the slope of the tangent line to the graph of a function.

We then developed ways to compute the derivative of functions expressible in terms of the functions met in algebra and trigonometry, including exponentials with a fixed base and logarithms; the so-called "elementary functions". That development was based on three specific limits:

$$\begin{aligned}
\lim_{x\to a}\frac{x^n-a^n}{x-a} &= na^{n-1}, \qquad n \text{ a positive integer}\\
\lim_{x\to 0}\frac{e^x-1}{x} &= 1\\
\lim_{x\to 0}\frac{\sin(x)}{x} &= 1.
\end{aligned}$$

Using these limits, we obtained the derivatives of x^n, e^x, and $\sin(x)$. We showed, if we knew the derivatives of two functions, how to compute the derivatives of their sum, difference, product, and quotient. Naturally, this was based on the definition of the derivative as a limit.

The next step was the development of the most important computational tool: the Chain Rule. This enables us to differentiate a composite function, such as $\cos^3(x^2)$. It tells us that this derivative is $3\cos^2(x^2)(-\sin(x^2))(2x)$.

Differentiating inverse functions enabled us to show that the derivative of $\ln|x|$ is $\frac{1}{x}$ and the derivative of $\arcsin(x)$ is $\frac{1}{\sqrt{1-x^2}}$. The following list of derivatives of key functions should be memorized.

function	derivative
x^a (a constant)	ax^{a-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	e^x
a^x (a constant)	write $a^x = e^{x(\ln(a))}$
$\ln(x)$ ($x > 0$)	$\frac{1}{x}$
$\ln\lvert x\rvert$ ($x \neq 0$)	$\frac{1}{x}$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$\frac{1}{1+x^2}$
$\sqrt{(x)}$	$\frac{1}{2\sqrt{x}}$
$\frac{1}{x}$	$\frac{-1}{x^2}$

Table 3.9.1: Table of Common Functions. Notice that the minus sign appears in the "co" functions.

As you compute and apply derivatives you may begin to think of them as slope or velocity or rate of change, and forget the underlying definition as a limit. However, we will from time to time return to the definition in terms of limits as we develop more applications of the derivative.

We also introduced the antiderivative and, closely related to it, the slope field. While the derivative of an elementary function is again elementary, its antiderivative often is not. For instance, $\sqrt{1+x^3}$ does not have an elementary antiderivative. However, as we will see in Chapter 6, it does have an antiderivative. Chapter 8 will present a few ways to find antiderivatives.

The derivative of the derivative is the second derivative. I nthe case of motion, the second derivative dsecribes acceleration. It is denoted several ways, such as $D^2(f)$, $\frac{d^2f}{dx^2}$, f'', and $f^{(2)}$. While the first and second derivatives suffice for most applications, higher derivatives of all orders are used in Chapter 5, where we estimate the error when approximating a function by a polynomial.

The final two sections returned to the notion of a limit, providing a precise definition of limit.

EXERCISES for 3.S *Key:* R–routine, M–moderate, C–challenging

In Exercises 1–21 use the properties of derivatives to verify each formula. It may be necessary to simplify the derivative algebraically. The letters a, b, and c denote constants. NOTE: These problems provide good practice in differentiation and algebra. Note that each differentiation formula has a corresponding antiderivative formula. Note how important $\ln(x)$ is in supplying antiderivatives.

1. $\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)=\frac{1}{a^2+x^2}$

2. $D\left(\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)\right)=\frac{1}{a^2-x^2}$

3. $\left(\ln\left(x+\sqrt{a^2+x^2}\right)\right)'=\frac{1}{\sqrt{a^2+x^2}}$

4. $\frac{d}{dx}\left(\frac{1}{a}\ln\left(\frac{x+\sqrt{a^2-x^2}}{x}\right)\right)=\frac{1}{x\sqrt{a^2+x^2}}$

5. $D\left(\frac{-1}{b(a+bx)}\right)=\frac{1}{(a+bx)^2}$

6. $\left(\frac{1}{b^2}\left(a+bx-a\ln(a+bx)\right)\right)'=\frac{x}{a+bx}$

7. $\frac{d}{dx}\left(\frac{1}{b^2}\left(\frac{a}{2(a+bx)^2}-\frac{1}{a+bx}\right)\right)=\frac{x}{(a+bx)^3}$

8. $D\left(\frac{1}{ab'-a'b}\ln\left(\frac{a'+b'x}{a+bx}\right)\right)=\frac{1}{(a+bx)(a'+b'x)}$ $\quad (a,\ b,\ a',\ b'$ constants)

9. $\left(\frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2cx+b}{\sqrt{4ac-b^2}}\right)\right)'=\frac{1}{a+bx+cx^2}$ $\quad (4ac>b^2)$

10. $\frac{d}{dx}\left(\frac{-2}{\sqrt{b^2-4ac}}\ln\left(\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}}\right)\right)=\frac{1}{a+bx+cx^2}$ $\quad (4ac<b^2)$

11. $D\left(\frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right)\right)=\frac{1}{x\sqrt{x^2-a^2}}$

12. $\left(\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\left(\frac{x}{a}\right)\right)\right)'=\sqrt{a^2-x^2}$ $\quad (|x|<|a|)$

13. $\frac{d}{dx}\left(\frac{-x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)\right)=\frac{x^2}{\sqrt{a^2-x^2}}$ $\quad (|x|<|a|)$

14. $D\left(-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\left(\frac{x}{a}\right)\right)=\frac{\sqrt{a^2-x^2}}{x^2}$ $\quad (|x|<|a|)$

15. $\left(\arcsin(x)-\sqrt{1-x^2}\right)'=\sqrt{\frac{1+x}{1-x}}$ $\quad (|x|<1)$

16. $\frac{d}{dx}\left(\frac{x}{2}-\frac{1}{2}\cos(x)\sin(x)\right)=\sin^2(x)$

17. $D\left(x\arcsin x+\sqrt{1-x^2}\right)=\arcsin(x)$ $\quad (|x|<1)$

18. $\left(x\tan^{-1}(x)-\frac{1}{2}\ln(1+x^2)\right)'=\arctan(x)$

19. $\frac{d}{dx}\left(\frac{e^{ax}}{a^2}\left(a^2-1\right)\right)=xe^{ax}$

20. $D\left(x-\ln(1+e^x)\right)=\frac{1}{1+e^x}$

21. $\left(\frac{e^{ax}(a\sin(bx)-b\cos(bx))}{a^2+b^2}\right)'=e^{ax}\sin(bx)$

In Exercises 22–25 give two antiderivatives for each given function.

22. xe^{x^2}

23. $(x^2+x)e^{x^3+3x}$

24. $\cos^3(x)\sin(x)$

25. $\sin(2x)$

26. Verify that $2(\sqrt{x}-1)e^{\sqrt{x}}$ is an antiderivative of $e^{\sqrt{x}}$.

27. The antiderivative of $1/x$ that passes through $(1,0)$ is $\ln(x)$. One would expect that for t near 1, the antiderivative of $1/x^t$ that passes through $(1,0)$ would look much like $\ln(x)$ when x is near 1. To verify that this is true

(a) graph the slope fields for $1/x^t$ with $t = 1.1$

(b) graph the antiderivative of $1/x^t$ that passes through $(1,0)$ for $t = 1.1$

(c) repeat (a) and (b) for $t = 0.9$

(d) repeat (a) and (b) for $t = 1.01$

(e) repeat (a) and (b) for $t = 0.99$

The slope field for $1/x$ and the antiderivative of $1/x$ passing through $(1,0)$ are shown in Figure 3.9.1.

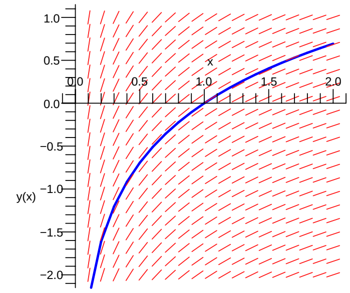


Figure 3.9.1:

This exercise suggests that we could have expected $\ln(x)$ to be the missing antiderivative of $1/x$.

28. (See Exercise 27.)

(a) Verify that the antiderivative of $1/x^t$ that passes through $(1,0)$ is $\frac{x^{1-t}-1}{1-t}$.

(b) Holding x fixed and letting t approach 1, show that

$$\lim_{t\to 1} \frac{x^{1-t}-1}{1-t} = \ln(x).$$

HINT: Recognize the limit as the derivative of a certain function at a certain input. Keep in mind that x is constant.

29. Let $y = x^{m/n}$, where $x > 0$ and m and $n \neq 0$ are integers. Show that $\frac{dy}{dx} = \frac{m}{n} x^{\frac{m}{n}-1}$ by starting with $y^n = x^m$ and differentiating both y^n and x^m with respect to x.

30. Define f as follows:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$$

(a) What does the graph of f look like?

(b) Does $\lim_{x\to 1} f(x)$ exist?

(c) Does $\lim_{x\to\sqrt{2}} f(x)$ exist?

(d) Does $\lim_{x\to 0} f(x)$ exist?

(e) For which numbers a does $\lim_{x\to a} f(x)$ exist?

A dotted curve may be used to indicate that points are missing.

31. Define f as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ x^3 & \text{if } x \text{ is irrational.} \end{cases}$$

(a) What does the graph of f look like?

See the advice for the previous problem.

(b) Does $\lim_{x\to 2} f(x)$ exist?

(c) Does $\lim_{x\to 1} f(x)$ exist?

(d) Does $\lim_{x\to 0} f(x)$ exist?

(e) For which numbers a does $\lim_{x\to a} f(x)$ exist?

32. Use precise definitions of limits to prove: if f and g are both continuous at a, then their product, fg, is also continuous at a.

Chapter 4

Derivatives and Curve Sketching

When you graph a function you typically plot a few points and connect them with (generally) straight line segments. Most electronic graphing devices use the same approach, and obtain better results by plotting more points and using shorter segments. The more points used, the smoother the graph will appear. In this chapter we will also learn how to choose the key points when sketching a graph.

Three properties of the derivative are developed in Section 4.1, and proved in Section 4.4. In Section 4.2 these properties enable us to exploit the first derivative when graphing a function. In Section 4.3 we see what the second derivative tells us about a graph.

4.1 Three Theorems about the Derivative

This section is based on two plausible observations about the graphs of differentiable functions, which we restate as theorems. These ideas will then be combined, in Section 4.2, to sketch a graph of functions.

See Section 2.4.

An effective approach to sketching the graph of a function is to identify the extreme values of the function. That is, the points where the function takes on its largest and smallest values.

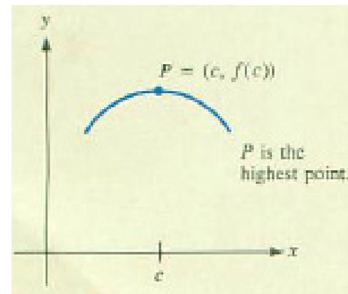


Figure 4.1.1:

OBSERVATION *(Tangent Lines at an Extreme Value)* Suppose that a function $f(x)$ attains its largest value when $x = c$, that is, $f(c)$ is the largest value of $f(x)$ over a given interval. Figure 4.1.1 illustrates this situation. The maximum occurs at a point $(c, f(c))$, which we call P. If $f(x)$ is differentiable, at least in some open interval containing the number c, then the tangent line at P will exist. What can we say about it?

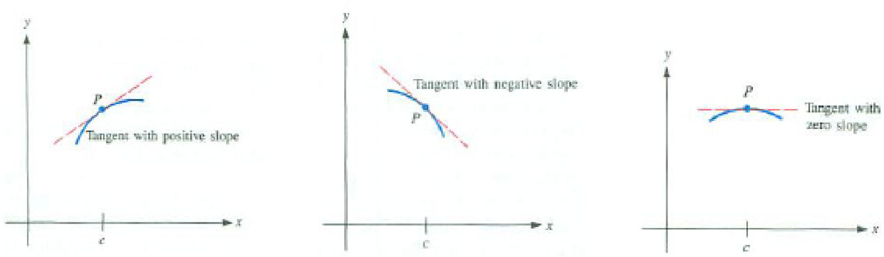


Figure 4.1.2:

If the tangent at P were *not* horizontal (that is, not parallel to the x-axis), then it would be tilted. So a small piece of the graph around P – which is almost straight — would appear as shown in Figure 4.1.2(a) or (b).

This suggests that the tangent line at the highest point must be horizontal.

In the first case P could not be the highest point on the curve because there would be higher points *to the right* of P. In the second case P could not be the highest point because there would be higher points *to the left* of P. Therefore the tangent at P must be horizontal, as shown in Figure 4.1.2(c). That is, $f'(c) = 0$.

This observation is the foundation for simple criteria for identifying local extrema.

Theorem of the Interior Extremum

Theorem 4.1.1 (Theorem of the Interior Extremum) *Let f be a function defined at least on the open interval (a, b). If f takes on an extreme value at a number c in this interval, then either*

1. $f'(c) = 0$ *or*
2. $f'(c)$ *does not exist.*

If an extreme value occurs within an open interval and the derivative exists there, the derivative must be 0 there. This idea will be used in Section 4.2 to find the maximum and minimum values of a function.

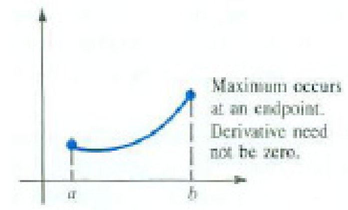


Figure 4.1.3:

WARNING *(Two Cautions about Theorem 4.1.1)*

1. If in Theorem 4.1.1 the open interval (a, b) is replaced by a closed interval $[a, b]$ the conclusion may not hold. A glance at Figure 4.1.3 shows why — the extreme value could occur at an endpoint ($x = a$ or $x = b$).
2. The converse of Theorem 4.1.1 is not true. Having the derivative equal to 0 at a point does *not* guarantee that there is an extremum at this point. Figure 4.1.4, which shows the graph of $y = x^3$, shows why. Since $f'(x) = 3x^2$, $f'(0) = 0$. While the tangent line is indeed horizontal at $(0, 0)$, it crosses the curve at this point. The graph has neither a maximum nor a minimum at the origin.

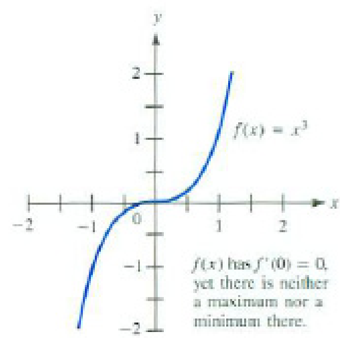


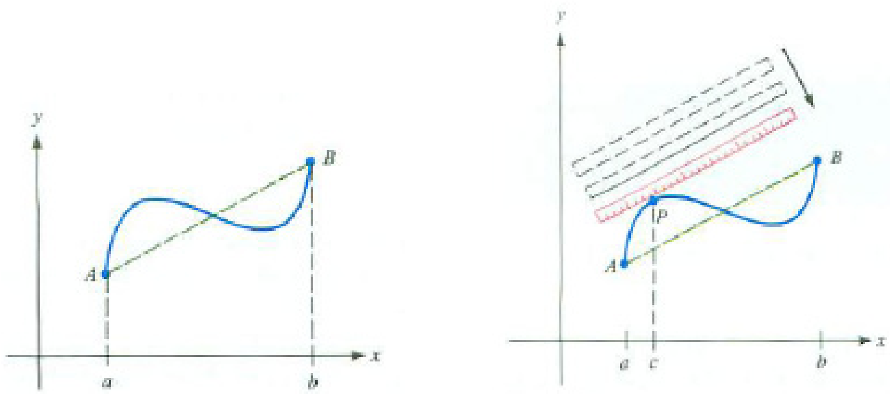
Figure 4.1.4:

Though the next observation is phrased in terms of slopes, we will see that it has implications for velocity and any changing quantity.

OBSERVATION *(Chord and Tangent Line with Same Slope)* Let $A = (a, f(a))$ and $B = (b, f(b))$ be two points on the graph of a differentiable function f defined at least on the interval $[a, b]$, as shown in Figure 4.1.5(a). Draw the line segment AB joining A and B. Imagine holding a ruler parallel to AB and lowering it until it just touches the graph of $y = f(x)$, as in Figure 4.1.5(b). The ruler then touches the curve at a point P and appears to lie along the tangent at P. At that point $f'(c)$ is equal to the slope of AB. (In Figure 4.1.5(b) there are two such numbers between a and b.)

A line segment that joins two points on the graph of a function f is called a **chord** of f.

It is customary to state two separate theorems based on the observation about chords and tangent lines. The first, Rolle's Theorem, is a special case of the second, the Mean-Value Theorem.

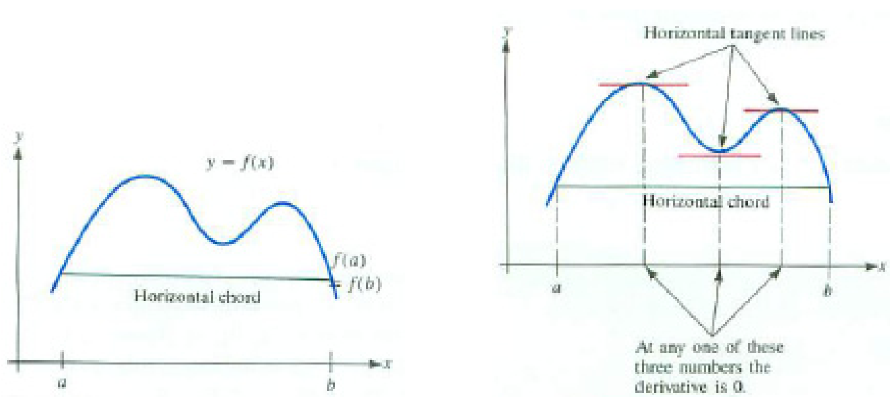


(a) (b)

Figure 4.1.5:

Rolle's Theorem

The next theorem is suggested by a special case of the second observation. When the points A and B in Figure 4.1.5(a) have the same y coordinate, the chord AB has slope 0. (See Figure 4.1.6.) In this special case, the observation tells us there must a horizontal tangent to the graph. Expressed in terms of derivatives, this gives us Rolle's Theorem[1]



(a) (b)

Figure 4.1.6:

[1]Michel Rolle (1652–1719) was a French mathematician. In addition to his discovery of Rolle's Theorem in 1691, he also invented the current standardized notation to denote the nth root of x: $\sqrt[n]{x}$. Source: `http://en.wikipedia.org/wiki/Michel_Rolle`.

Theorem 4.1.2 (Rolle's Theorem) *Let f be a continuous function on the closed interval $[a, b]$ and have a derivative at all x in the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.*

EXAMPLE 1 Verify Rolle's Theorem for the case with $f(t) = (t^2-1)\ln\left(\frac{t}{\pi}\right)$ on $[1, \pi]$.
SOLUTION The function $f(t)$ is differentiable whenever t/π is positive, that is, for all $t > 0$. In particular, $f(t)$ is differentiable on the closed interval $[1, \pi]$. Notice that $f(1) = 0$ and, because $\ln(1) = 0$, $f(\pi) = 0$. Therefore, by Rolle's Theorem, there must be a value of c between 1 and π where $f'(c) = 0$.

Recall: $\ln(x)$ is differentiable for all $x > 0$. See Section 2.4.

The derivative $f'(t) = 2t\ln\left(\frac{t}{\pi}\right) + \frac{t^2-1}{t}$ is a pretty complicated function. Even though it is not possible to find the exact value of c with $f(c) = 0$, Rolle's Theorem guarantees that there is at least one such value of c. Figure 4.1.7 confirms that there is only one solution to $f'(c) = 0$ on $[1, \pi]$. ◇

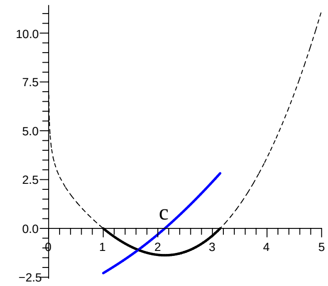


Figure 4.1.7: Graph of $y = f(t)$ (black) and $y = f'(t)$ (blue).

Remark Assume that $f(x)$ is a differentiable function such that $f'(x)$ is never 0 for x in an interval. Then the equation $f(x) = 0$ *can have at most one solution in that interval.* If it has two solutions, a and b, then $f(a) = 0$ and $f(b) = 0$. (See Figure 4.1.8.)

In an interval in which the derivative $f'(x)$ is never 0, the graph of $y = f(x)$ can have no more than one x-intercept.

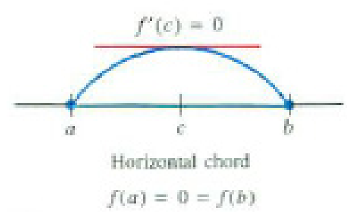


Figure 4.1.8:

Example 2 illustrates this observation.

EXAMPLE 2 Use Rolle's Theorem to determine how many real roots there are for the equation

$$x^3 - 6x^2 + 15x + 3 = 0. \quad (1)$$

SOLUTION Since $f(x) = x^3 - 6x^2 + 15x + 3$ is a polynomial of odd degree, there is at least one real number r such that $f(r) = 0$. Could there be another root s? If so, by Rolle's Theorem, there would be a number c (between r and s) at which $f'(c) = 0$.

Recall the argument in Section 2.4 based on the Intermediate Value Theorem. We will learn how to approximate the unique solution to (1) in Chapter 10.

To check, we compute the derivative of $f(x)$ and see if it is ever equal to 0. We have $f'(x) = 3x^2 - 12x + 15$. To find when $f'(x)$ is 0, we solve the equation $3x^2 - 12x + 15 = 0$ by the quadratic formula, obtaining

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(15}}{6} = \frac{12 \pm \sqrt{-36}}{6} = 2 \pm \sqrt{-1}$$

and the equation $f'(x) = 0$ has no real solutions. It follows that $x^3 - 6x^2 + 15x + 3$ has only one real root. ◇

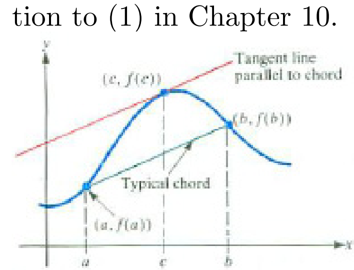


Figure 4.1.9:

Mean-Value Theorem

The "mean-value" theorem, is a generalization of Rolle's Theorem in that it applies to any chord, not just horizontal chords.

In geometric terms, the theorem asserts that if you draw a chord for the graph of a well-behaved function (as in Figure 4.1.9), then somewhere above or below that chord the graph has at least one tangent line parallel to the chord. (See Figure 4.1.5(a).) Let us translate this geometric statement into the language of functions. Call the ends of the chord $(a, f(a))$ and $(b, f(b))$. The slope of the chord is

$$\frac{f(b)-f(a)}{b-a}.$$

Since the tangent line and the chord are parallel, they have the same slopes. If the tangent line is at the point $(c, f(c))$, then

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Specifically, we have

Theorem 4.1.3 (Mean-Value Theorem) *Let f be a continuous function on the closed interval $[a, b]$ and have a derivative at every x in the open interval (a, b). Then there is at least one number c in the open interval (a, b) such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

EXAMPLE 3 Verify the Mean-Value Theorem for $f(t) = \sqrt{4-t^2}$ on the interval $[0, 2]$.

SOLUTION Because $4-t^2 \geq 0$ for t between -2 and 2 (including these two endpoints), f is continuous on $[0, 2]$ and is differentiable on $(0, 2)$. The slope of the chord through $(a, f(a)) = (0, 2)$ and $(b, f(b)) = (2, 0)$ is

$$\frac{f(b)-f(a)}{b-a} = \frac{0-2}{2-0} = -1.$$

According to the Mean-Value Theorem, there is at least one number c between 0 and 2 where $f'(c) = m = -1$.

Let us try to find c. Since $f'(t) = \dfrac{-2t}{2\sqrt{4-t^2}}$, we need to solve the equation

$$\begin{array}{rcll}
\dfrac{-c}{\sqrt{4-c^2}} & = & -1 & \\
-c & = & -\sqrt{4-c^2} & \text{[multiply both sides by } \sqrt{4-t^2}\text{]} \\
c^2 & = & 4-c^2 & \text{[square both sides]} \\
2c^2 & = & 4 & \\
c^2 & = & 2. &
\end{array}$$

There are two solutions: $c = \sqrt{2}$ and $c = -\sqrt{2}$. Only $c = \sqrt{2}$ is in $(0, 2)$. Hence, there is only one number, namely $c = \sqrt{2}$, whose existence is guaranteed by the Mean-Value Theorem. ◇

The iterpretation of the derivative as slope suggested the Mean-Value Theorem. What does the Mean-Value Theorem say when the function when the function describes the position of a moving object and the derivative its velocity? This question is considered in Example 4.

EXAMPLE 4 A car moving on the x-axis has the x-coordinate $x = f(t)$ at time t. At time a its position is $f(a)$. At some later time b its position is $f(b)$. What does the Mean-Value Theorem assert for this car?
SOLUTION In this case the quotient

$$\begin{array}{rccc} & \dfrac{f(b)-f(a)}{b-a} & \text{equals} & \dfrac{\text{Change in position}}{\text{Change in time}} \\ \text{that is,} & f'(c) & = & \dfrac{f(b)-f(a)}{b-a}, \end{array}$$

or "average velocity" for the interval of time $[a, b]$. The Mean-Value Theorem asserts that at some time c, $f'(c)$ is equal to the quotient $\dfrac{f(b)-f(a)}{b-a}$. This says that the velocity at time c is the same as the average velocity during the time interval $[a, b]$. To be specific, if a car travels 210 miles in 5 hours, then at some time its speedometer must read 42 miles per hour. ◇

Consequences of the Mean-Value Theorem

There are several ways of writing the Mean-Value Theorem. For example, the equation

$$\begin{array}{rrcll} & f'(c) & = & \dfrac{f(b)-f(a)}{b-a} & \\ \text{is equivalent to} & f(b)-f(a) & = & (b-a)f'(c) & \\ \text{and hence to} & f(b) & = & f(a)+(b-a)f'(c). & (2) \end{array}$$

In this last form, the Mean-Value Theorem asserts that $f(b)$ is equal to $f(a)$ plus a quantity that involves the derivative f' at some number c between a and b. The following important corollaries are based on this alternative view of the Mean-Value Theorem.

A different view of the Mean-Value Theorem

Corollary 4.1.1 *If the derivative of a function is 0 throughout an interval I, then the function is constant on the interval.*

Proof

Let a and b be any two numbers in the interval I and let the function be denoted by f. To prove this corollary, it suffices to prove that $f(a) = f(b)$, for that is the defining property of a constant function.

Using the conclusion of the Mean-Value Theorem as stated in (2), there is a number c between a and b such that

$$f(b) = f(a) + (b - a)f'(c).$$

But $f'(c) = 0$, since $f'(x) = 0$ for all x in I. Hence

$$f(b) = f(a) + (b - a)(0)$$

which proves that

$$f(b) = f(a).$$

•

When Corollary 4.1.1 is interpreted in terms of motion, it is quite plausible. It asserts that if an object has zero velocity for a period of time, then it does not move during that time.

EXAMPLE 5 Use calculus to show that $f(x) = (e^x + e^{-x})^2 - e^{2x} - e^{-2x}$ is a constant. Find the constant.

SOLUTION The function f is differentiable for all numbers x. Its derivative is

$$\begin{aligned} f'(x) &= 2(e^x + e^{-x})(e^x - e^{-x}) - 2e^{2x} + 2e^{-2x} \\ &= 2(e^{2x} - e^{-2x}) - 2e^{2x} + 2e^{-2x} \\ &= 0 \end{aligned}$$

Because $f'(x)$ is always zero, f must be a constant.

To find the constant, just evaluate $f(x)$ for a convenient value of x. Here we choose $x = 0$: $f(0) = (e^0 + e^0)^2 - e^0 - e^0 = 2^2 - 2 = 2$. Thus,

$$(e^x + e^{-x})^2 - e^{2x} - e^{-2x} = 2 \qquad \text{for all numbers } x.$$

This result can also be derived by squaring $e^x + e^{-x}$.

◇

Corollary 4.1.2 *If two functions have the same derivatives throughout an interval, then they differ by a constant. That is, if $F'(x) = G'(x)$ for all x in an interval, then there is a costant C such that $F(x) = G(x) + C$.*

Proof

Define a third function h by the equation $h(x) = F(x) - G(x)$. Then

$$h'(x) \quad = \quad F'(x) - G'(x) = 0. \quad \text{[since } F'(x) = G'(x)\text{]}$$

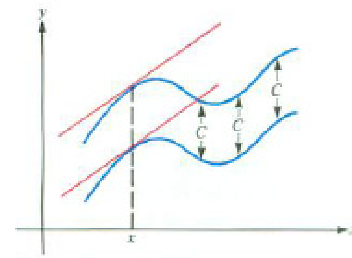


Figure 4.1.10:

Since the derivative if h is 0, Corollary 4.1.1 implies that h is constant, that is, $h(x) = C$ for some fixed number C. Thus

$$F(x) - G(x) = C \qquad \text{or} \qquad F(x) = G(x) + C,$$

and Corollary 4.1.2 is proved. •

Is Corollary 4.1.2 plausible when the derivative is interpreted as slope? In this case, the corollary asserts that if the graphs of two functions have the property that their tangent lines at points with the same x coordinate are parallel, then one graph can be obtained from the other by raising (or lowering) it by a constant amount C. If you sketch two such graphs (as in Figure 4.1.10, you will see that the corollary is reasonable.

In the language of Section 3.5, any antiderivative of $2x$ must be of the form $x^2 + C$.

EXAMPLE 6 What functions have a derivative equal to $2x$ everywhere?
SOLUTION One such solution is x^2; another is $x^2 + 25$. For any constant C, $D(x^2 + C) = 2x$. Are there any other possibilities? Corollary 4.1.2 tells us there are not, for if F is a function such that $F'(x) = 2x$, then $F'(x) = (x^2)'$ for all x. Thus the functions f and x^2 differ by a constant, say C, that is,

$$F(x) = x^2 + C.$$

The only antiderivatives of $2x$ are of the form $x^2 + C$. ◇

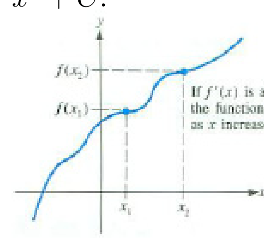


Figure 4.1.11:

Corollary 4.1.1 asserts that if $f'(x) = 0$ for all x, then f is a constant. What can be said about f if $f'(x)$ is *positive* for all x in an interval? In terms of the graph of f, this assumption implies that all the tangent lines slope upward. It is reasonable to expect that as we move from left to right on the graph in Figure 4.1.11, the y-coordinate increases.

Increasing and decreasing are defined in Section 1.1.

Corollary 4.1.3 *If f is continuous on the closed interval $[a, b]$ and has a positive derivative on the open interval (a, b), then f is increasing on the interval $[a, b]$.*

If f is continuous on the closed interval $[a, b]$ and has a negative derivative on the open interval (a, b), then f is decreasing on the interval $[a, b]$.

Proof
We prove the "increasing" case; the other case is handled in Exercise 43. Take two numbers x_1 and x_2 such that

$$a \leq x_1 < x_2 \leq b.$$

The goal is to show that $f(x_2) > f(x_1)$.

By the Mean-Value Theorem, there is some number c between x_1 and x_2 such that

$$f(x_2) = f(x_1) + (x_2 - x_1)f'(c).$$

Now, since $x_2 > x_1$, we know $x_2 - x_1$ is positive. Since $f'(c)$ is assumed to be positive, it follows that

$$(x_2 - x_1)f'(c) > 0.$$

The product of two positive numbers is positive.

Thus, $f(x_2) > f(x_1)$, and so $f(x)$ is an increasing function. •

EXAMPLE 7 Determine whether $2x + \sin(x)$ is an increasing function, a decreasing function, or neither.

SOLUTION The funcion $2x + \sin(x)$ is the sum of two simpler functions: $2x$ and $\sin(x)$. The "$2x$" part is an increasing function. The second term, "$\sin(x)$", increases for x between 0 and $\pi/2$ and decreases for x between $\pi/2$ and π. It is not clear what type of function you will get when you add $2x$ and $\sin(x)$. Let's see what Corollary 4.1.3 tells us.

The derivative of $2x + \sin(x)$ is $2 + \cos(x)$. Since $\cos(x) \geq -1$ for all x,

$$(2x + \sin(x))' = 2 + \cos(x) \geq 2 + (-1) = 1.$$

Since $(2x + \sin(x))'$ is positive for all numbers x, $2x + \sin(x)$ is an increasing function. Figure 4.1.12 shows the graph of $2x+\sin(x)$ together with the graphs of $2x$ and $\sin(x)$. ◇

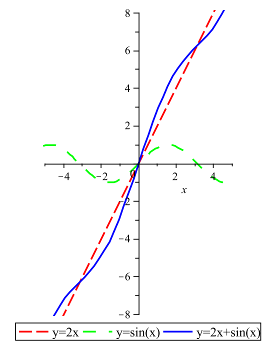


Figure 4.1.12:

Remark Increasing/Decreasing at a point

1. Corollary 4.1.3, and the definitions of increasing and decreasing, are stated in terms of intervals. When we talk about a function f being increasing (or decreasing) at a point c, here is what we really mean: there is an interval (a, b) with $a < c < b$ where f is increasing on (a, b). That is, "a function is increasing at c" is shorthand for "a function is increasing in an interval that contains c."
2. When $f'(c) > 0$ and f' is continuous, the Permanence Property (Theorem 2.4.4 in Section 2.4) tells us there is an interval (a, b) containing c where $f'(x) > 0$ for all numbers x in (a, b). Thus, f is increasing on (a, b), and hence increasing at c.

Summary

This section focused on three theorems, which we state informally.

The Theorem of the Interior Extremum tells us that the local extrema of a differentiable function occurs at a critical number, where the derivative is zero.

Rolle's Theorem says that if a differentiable function has equal values at two different inputs, then its derivative must equal zero at least at one number between the inputs. From this we deduced the Mean-Value Theorem, which states that for any chord on the graph of a differentiable function, there is a parallel tangent line. In symbols, it asserts that for given a and b, there is c between them such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. This also says that $f(b) = f(a) + f'(c)(b-a)$.

From the Mean-Value Theorem it follows that where a derivative is positive, a function is increasing, where it is negative it is decreasing, and where it stays at the value zero, it is constant. The last assertion implies that two antiderivatives of the same integrand differ at most by a constant.

EXERCISES for 4.1

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

Exercises 1 — 6 concern the interior extremum.

1. Consider the function $f(x) = x^2$ only for x in $[-1, 2]$.

(a) Graph the function $f(x)$ for x in $[-1, 2]$.

(b) What is the maximum value of $f(x)$ for x in the interval $[-1, 2]$?

(c) Does $f'(x)$ exist at the maximum?

(d) Does $f'(x)$ equal zero at the maximum?

(e) Does $f'(x)$ equal zero at the minimum?

2. Consider the function $f(x) = \sin(x)$ only for x in $[0, \pi]$.

(a) Graph the function $f(x)$ for x in $[0, \pi]$.

(b) What is the maximum value of $f(x)$ for x in the interval $[0, \pi]$?

(c) Does $f'(x)$ exist at the maximum?

(d) Does $f'(x)$ equal zero at the maximum?

(e) Does $f'(x)$ equal zero at the minimum?

3.

(a) Repeat Exercise 1 on the interval $[1, 2]$.

(b) Repeat Exercise 1 on the interval $(-1, 2)$.

(c) Repeat Exercise 1 on the interval $(1, 2)$.

(d) Repeat Exercise 2 on the interval $[0, 2\pi]$.

(e) Repeat Exercise 2 on the interval $(0, \pi)$.

(f) Repeat Exercise 2 on the interval $(0, 2\pi)$.

4.

(a) Graph $y = -x^2 + 3x + 2$ for x in $[0, 2]$.

(b) Looking at the graph, estimate the x coordinate where the maximum value of y occurs for x in $[0, 2]$.

(c) Find where $dy/dx = 0$.

(d) Using (c), determine exactly where the maximum occurs.

5.

(a) Graph $y = 2x^2 - 3x + 1$ for x in $[0, 1]$.

(b) Looking at the graph, estimate the x coordinate where the maximum value of y occurs for x in $[0, 1]$. At which value of x does it occur?

(c) Looking at the graph, estimate the x coordinate where the minimum value of y occurs for x in $[0, 12]$.

(d) Find where $dy/dx = 0$.

(e) Using (d), determine exactly where the minimum occurs.

6. For each of the following functions, (a) show that the derivative of the given function is 0 when $x = 0$ and (b) decide whether the function has an extremum at $x = 0$.

(a) $x^2 \sin(x)$

(b) $1 - \cos(x)$

(c) $e^x - x$

(d) $x^2 - x^3$

Exercises 7 – 15 concern Rolle's Theorem.

7.

(a) Graph $f(x) = x^{2/3}$ for x in $[-1, 1]$.

(b) Show that $f(-1) = f(1)$.

(c) Is there a number c in $(-1, 1)$ such that $f'(c) = 0$?

(d) Why does this not contradict Rolle's Theorem?

8.

(a) Graph $f(x) = 1/x^2$ for x in $[-1, 1]$.

(b) Show that $f(-1) = f(1)$.

(c) Is there a number c in $(-1, 1)$ such that $f'(c) = 0$?

(d) Why does this not contradict Rolle's Theorem?

In Exercises 9 — 14, verify that the given function satisfies Rolle's Theorem for the given interval. Find all numbers c that satisfy the conclusion of the theorem.

9. $f(x) = x^2 - 2x - 3$ and $[0, 2]$

10. $f(x) = x^3 - x$ and $[-1, 1]$

11. $f(x) = x^4 - 2x^2 + 1$ and $[-2, 2]$

12. $f(x) = \sin(x) + \cos(x)$ and $[0, 4\pi]$

13. $f(x) = e^x + e^{-x}$ and $[-2, 2]$

14. $f(x) = x^2 e^{-x^2}$ and $[-2, 2]$

15. Let $f(x) = \ln(x^2)$. Note that $f(-1) = f(1)$. Is there a number c in $(-1, 1)$ such that $f'(c) = 0$? If so, find at least one such number. If not, why is this not a contradiction of Rolle's Theorem?

Exercises 16 — 21 concern the Mean-Value Theorem.

In Exercises 16 — 19, find explicitly all values of c which satisfy the Mean-Value Theorem for the given functions and intervals.

16. $f(x) = x^2 - 3x$ and $[1, 4]$

17. $f(x) = 2x^2 + x + 1$ and $[-2, 3]$

18. $f(x) = 3x + 5$ and $[1, 3]$

19. $f(x) = 5x - 7$ and $[0, 4]$

20.

(a) Graph $y = \sin(x)$ for x in $[\pi/2, 7\pi/2]$.

(b) Draw the chord joining $(\pi/2, f(\pi/2))$ and $(7\pi/2, f(7\pi/2))$.

(c) Draw all tangents to the graph parallel to the chord drawn in (b).

(d) Using (c), determine how many numbers c there are in $(\pi, 7\pi/2)$ such that

$$f'(c) = \frac{f(7\pi/2) - f(\pi/2)}{7\pi/2 - \pi/2}.$$

(e) Use the graph to estimate the values of the c's.

21.

(a) Graph $y = \cos(x)$ for x in $[0, 9\pi/2]$.

(b) Draw the chord joining $(0, f(0))$ and $(9\pi/2, f(9\pi/2))$.

(c) Draw all tangents to the graph that are parallel to the chord drawn in (b).

(d) Using (c), determine how many numbers c there are in $(0, 9\pi/2)$ such that

$$f'(c) = \frac{f(9\pi/2) - f(0)}{9\pi/2 - 0}.$$

(e) Use the graph to estimate the values of the c's.

22. State Rolle's Theorem, using as few mathematical symbols as you can.

23. Use Rolle's Theorem to determine how many real roots there are for the equation $x^3 - 6x^2 + 15x + 3 = 0$.

24. Use Rolle's Theorem to determine how many real roots there are for the equation $3x^4 + 4x^3 - 12x^2 + 4 = 0$. Give intervals on which there is exactly one root.

25. At time t seconds a thrown ball has the height $f(t) = -16t^2 + 32t + 40$ feet.

(a) Show that after 2 seconds it returns to its initial height.

(b) What does Rolle's Theorem imply about the velocity of the ball?

(c) Verify Rolle's Theorem in this case by computing the numbers c which it asserts exist.

26. Find all points where $f(x) = 2x^3(x-1)$ can have an extreme value on the following intervals

(a) $(-1/2, 1)$

(b) $[-1/2, 1]$

(c) $[-1/2, 1/2]$

(d) $(-1/2, 1/2)$

27. Let $f(x) = |2x - 1|$.

(a) Explain why $f'(1/2)$ does not exist.

(b) Find $f'(x)$. Hint: Write the absolute value in two parts, one for $x < 1/2$ and the other for $x > 1/2$.

(c) Does the Mean-Value Theorem apply on the interval $[-1, 2]$?

28. Express the Mean-Value Theorem in symbols, where the function is denoted g and the interval is $[e, f]$.

29. Express the Mean-Value Theorem in words, using no symbols to denote the function or the interval.

30. The year is 2015. Because a gallon of gas costs six dollars and Highway 80 is full of tire-wrecking potholes, the California Highway Patrol no longer patrols the 77 miles between Sacramento and Berkeley. Instead it uses two cameras. One, in Sacramento, records the license number and time of a car on the freeway, and another does the same in Berkeley. A computer processes the data instantly. Assume that the two cameras show that a car that was in Sacramento at 10:45 reached

Berkeley at 11:40. Show that the Mean-Value Theorem justifies giving the driver a ticket for exceeding the 70 mile-per-hour speed limit. (Of course, intuition justifies the ticket, but mentioning the Mean-Value Theorem is likely to impress a judge who studied calculus.)

31. What is the shortest time for the trip from Berkeley to Sacramento for which the Mean-Value Theorem does not convict the driver of speeding? NOTE: See Exercise 30.

32. Verify the Mean-Value Theorem for $f(t) = x^2 e^{-x/3}$ on $[1, 10]$. NOTE: See Example 1.

33. Find all antiderivatives of each of the following functions. Check your answer by differentiation.

(a) $3x^2$

(b) $\sin(x)$

(c) $\frac{1}{1+x^2}$

(d) e^x

34. Find all antiderivatives of each of the following functions. Check your answer by differentiation.

(a) $\cos(x)$

(b) $\sec(x)\tan(x)$

(c) $1/x$ $(x > 0)$

(d) $\sqrt{x}$ $(x > 0)$

35. Find all functions whose second derivative is 0 for all x in $(-\infty, \infty)$.

36. If two functions have the same second derivative for all x in $(-\infty, \infty)$, what can be said about the relation between the two functions?

37.

(a) Differentiate $\sec^2(x)$ and $\tan^2(x)$.

(b) The derivatives in (a) are equal. Corollary 4.1.2 then asserts that there exists a constant C such that $\sec^2(x) = \tan^2(x) + C$. Find the constant.

38. Show that $f(x) = \sec^2(3x) - \tan^2(3x)$ is a constant for all values of x. Find the constant.

39. Show that $f(x) = \ln(x/5) - \ln(5x)$ is a constant for all values of x. Find the constant.

40. Use Rolle's Theorem to determine how many real roots there are for the polynomial $f(x) = 3x^4 + 4x^3 - 12x^2 + A$. Give intervals on which there is exactly one root. [ANS: A¡=5: 4 solutions, A=5: 4 solutions (double root at x=1), A¿5: 2 solutions]

41. Consider the equation $x^3 - ax^2 + 15x = 3 = 0$. The number of real roots to this equation depends on the value of a.

(a) Find all values of a when the equation has 3 real roots.

(b) Find all values of a when the equation has 1 real root.

(c) Are there any values of a with exactly two real roots?

[ANS: (a) $|A| > 45$, (b) $|A| \leq \sqrt{45}$, (c) No.]

42. If f is differentiable for all real numbers and $f'(x) = 0$ has three solutions, what can be said about the number of solutions of $f(x) = 0$? of $f(x) = 5$?

43. Prove the "decreasing" case of Corollary 4.1.3.

Exercises 44 – 47 involve the **hyperbolic functions**. The **hyperbolic sine** function is $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and the **hyperbolic cosine** function is $\cosh(x) = \frac{e^x + e^{-x}}{2}$.

44.

(a) Show that $\frac{d}{dx}\sinh(x) = \cosh(x)$.

(b) Show that $\frac{d}{dx}\cosh(x) = \sinh(x)$.

45. Define $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{2}{e^x + e^{-x}}$ and $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

(a) Show that $\frac{d}{dx}\tanh(x) = (\operatorname{sech}(x))^2$.

(b) Show that $\frac{d}{dx}\operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x)$.

Note: $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ and $\operatorname{sech}(x) = \frac{1}{\cosh(x)}$

46. Use calculus to show that $(\cosh(x))^2 - (\sinh(x))^2$ is a constant. Find the constant.

47. Use calculus to show that $(\operatorname{sech}(x))^2 + (\tanh(x))^2$ is a constant. Find the constant.

48. For which values of the constant k is the function $7x + k\sin(2x)$ always increasing?

49. Which of the corollaries to the Mean-Value Theorem implies that

(a) if two cars on a straight road have the same velocity at every instant, they remain a fixed distance apart?

(b) If all tangents to a curve are horizontal, the curve is a horizontal line.

Explain each answer.

50. If a function f is differentiable for all x and c is a number, is there necessarily a chord of the graph of f that is parallel to the tangent line at $(c, f(c))$? Explain.

4.2 The First-Derivative and Graphing

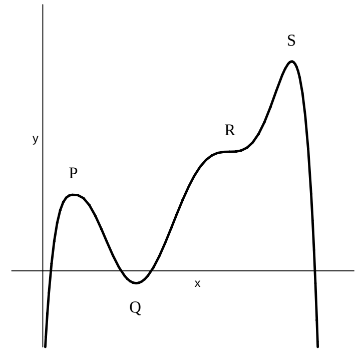


Figure 4.2.1:

Section 4.1 showed the connection between extrema and the places where the derivative is zero. In this section we use this connection to find high and low points on a graph.

The graph of a differentiable function f defined for all real numbers x is shown in Figure 4.2.1. The points P, Q, R, and S are of special interest. S is the highest point on the graph *for all x in the domain*. We call it a *global maximum* or *absolute maximum*. The point P is higher than all points *near it* on the graph; it is called a *local maximum* or *relative maximum*. Similarly, Q is called a *local minimum* or *relative minimum*. The point R is not a relative extremum.

If you were to walk left to right along the graph in Figure 4.2.1, you would call P the top of a hill, Q the bottom of a valley, and S the highest point on your walk (it is also a top of a hill). You might notice R, for you get a momentary break from climbing from Q to S. For just this one instant it would be like walking along a horizontal path.

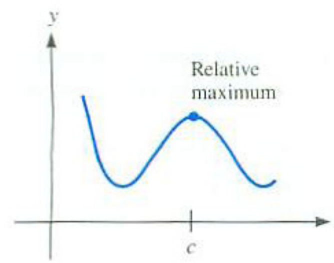


Figure 4.2.2:

These important aspects of a function and its graph are made precise in the following definitions. These definitions are phrased in terms of a general domain. In most cases the domain of the function will be an interval — open, closed, or half-open.

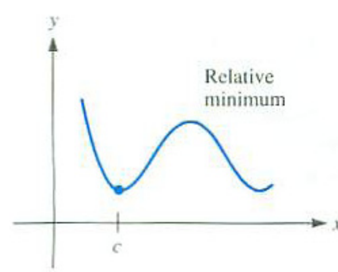


Figure 4.2.3:

> **DEFINITION** *(Relative Maximum (Local Maximum))* The function f has a **relative maximum** (or **local maximum**) at a number c if there is an open interval around c such that $f(c) \geq f(x)$ for all x in that interval that lie in the domain of f.

> **DEFINITION** *(Relative Minimum (Local Minimum))* The function f has a **relative minimum** (or **local minimum**) at a number c if there is an open interval around c such that $f(c) \leq f(x)$ for all x in that interval that lie in the domain of f.

DEFINITION *(Absolute Maximum (Global Maximum))* The function f has a **absolute maximum** (or **global maximum**) at a number c if $f(c) \geq f(x)$ for all x in the domain of f.

DEFINITION *(Absolute Minimum (Global Minimum))* The function f has a **absolute minimum** (or **global minimum**) at a number c if $f(c) \leq f(x)$ for all x in the domain of f.

Each global extremum is also a local extremum.

A local extremum is like the summit of a single mountain or an individual valley. A global maximum corresponds to Mt. Everest; a global minimum corresponds to the Mariana Trench in the Pacific Ocean. At 11km below sea level, the Mariana Trench is the lowest point on the Earth's crust.

In this section it is assumed that the functions is differentiable. If a function is not differentiable at an isolated point, this point will need to be considered separately.

DEFINITION *(Critical Number and Critical Point)* A number c at which $f'(c) = 0$ is called a **critical number** for the function f. The corresponding point $(c, f(c))$ on the graph of f is a **critical point** on that graph.

Remark Some texts define a critical number as a number where the derivative is 0 or else is not defined. Since we emphasize differentiable functions, a critical number is defined to be a number where the derivative is 0.

The Theorem of the Interior Extremum, in Section 4.1, says that every local maximum and minimum of a function f occurs where the tangent line to the curve either is horizontal or does not exist.

Some functions have extreme values, and others do not. The following theorem gives simple conditions under which both a global maximum and a global minimum are guaranteed to exist.

A continuous function on a closed interval has both a global maximum and a global minimum.

Theorem 4.2.1 (Extreme Value Theorem) *Let f be a continuous function on a closed interval $[a, b]$. Then f attains an absolute maximum value $M = f(c)$ and an absolute minimum value $m = f(d)$ at some numbers c and d in $[a, b]$.*

EXAMPLE 1 Find the absolute extrema on the interval $[0, 2]$ of the function whose graph is shown in Figure 4.2.4.
SOLUTION The function has an absolute maximum value of 2 but no absolute minimum value. The range is $(-1, 2]$. This function takes on values that are arbitrarily close to -1, but -1 is not in the range of this function. This occurs only because this function is not continuous at $x = 1$. ◇

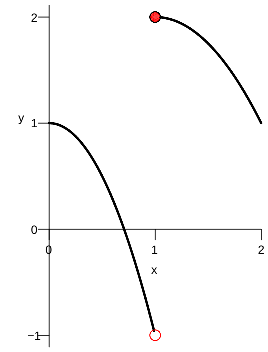


Figure 4.2.4:

The function in Figure 4.2.4 has an absolute minimum but not an absolute maximum. Even though the function values increase as x moves closer to 1, from the right, there is no absolute maximum value. Observe that this function is not defined at $x = 1$ so neither is it continuous at $x = 1$.

differentiable implies continuous, so not continuous implies not differentiable

Recall that Corollary 4.1.3 provides a convenient test to determine if a function is increasing or decreasing at a point: if $f'(c) > 0$ then f is increasing at $x = c$ and if $f'(c) < 0$ then f is decreasing at $x = c$.

EXAMPLE 2 Let $f(x) = x \ln(x)$ for all $x > 0$. Determine the intervals on which f is increasing, decreasing, or neither.
SOLUTION The function is increasing at numbers x where $f'(x) > 0$ and decreasing where $f'(x) < 0$. More effort is needed to determine the behavior at points where $f'(x) = 0$ (or does not exist). The Product Rule allows us to find

The natural domain of f is $x > 0$.

$$f'(x) = \ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1.$$

In order to find where $f'(x)$ is positive or is negative, we first find where it is zero. At such numbers the derivative may switch sign. Even though f is neither increasing nor decreasing when $f'(x) = 0$, the numbers where $f'(x) = 0$ will be the endpoints of the intervals where f is increasing and decreasing.

$$\begin{aligned} \ln(x) + 1 &= 0 \\ \ln(x) &= -1 \\ e^{\ln(x)} &= e^{-1} \\ x &= e^{-1}. \end{aligned}$$

So, $x = e^{-1}$, is the only place f is neither increasing nor decreasing. When x is larger than e^{-1}, $\ln(x)$ is larger than -1 so that $f'(x) = \ln(x) + 1$ is positive and f is increasing. Finally, f is decreasing when x is a number between 0 and e^{-1} because $\ln(x) < -1$ which makes $f'(x) = \ln(x) + 1$ negative.

$e^{-1} \approx 0.367879$

The graph of this function will be obtained in Exercise 36 (Section 4.3). ◇

Using Critical Numbers to Identify Local Extrema

The previous examples show there is a close connection between critical points and local extrema. Notice that, generally, just to the left of a local extreme the function is increasing, while just to the right it is decreasing. The opposite holds for a local minimum. The First-Derivative Test for a Local Extreme Value at $x = c$ gives a precise statement of this result.

First-Derivative Test for a Local Extreme Value at $x = c$

Theorem 4.2.2 *Let f be a function and let c be a number in its domain. Suppose f is continuous on an open interval that contains c and is differentiable on that interval, except possibly at c. Then:*

1. *If f' changes from positive to negative as x moves from left to right through the value c,*
   then f has a local maximum at c.
2. *If f' changes from negative to positive as x moves from left to right through the value c,*
   then f has a local minimum at c.
3. *If f' does not change sign at c,*
   then f does not have a local extremum at $x = c$.

EXAMPLE 3 Classify all critical numbers of $f(x) = 3x^5 - 20x^3 + 10$ as a local maximum, local minimum, or neither.
SOLUTION To identify the critical numbers of f, we find and factor the derivative:

$$f'(x) = 15x^4 - 60x^2 = 15x^2(x^2 - 4) = 15x^2(x - 2)(x + 2).$$

The critical numbers of f are $x = 0$, $x = 2$, and $x = -2$. To determine if any of these numbers provide local extrema it is necessary to know where f is increasing and where it is decreasing.

Find critical numbers

Because f' is continuous the three critical numbers are the only places the sign of f' can possibly change. All that remains is to determine if f is increasing or decreasing on the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$. This is easily answered from table of function values shown in the first two rows of Table 4.2.1. Observe that $f(-2) = 74 > 10 = f(0)$; this means f

x	$\to -\infty$	-2	0	2	$\to \infty$
$f(x)$	$-\infty$	74	10	-54	∞
$f'(x)$		0	0	0	

Table 4.2.1:

is decreasing on $(-2, 0)$. Likewise, f must be decreasing on $(0, 2)$ because $f(0) = 10 > -54 = f(2)$. For the two unbounded intervals, limits at $\pm\infty$ must be used but the overall idea is the same. Since $\lim_{x\to-\infty} f(x) = -\infty$, the function must be increasing on $(-\infty, -2)$. Likewise, in order to have $\lim_{x\to\infty} f(x) = +\infty$, f must be increasing on $(2, \infty)$.

To conclude, because the graph of f changes from increasing to decreasing at $x = -2$, there is a local maximum at $(-2, 74)$. At $x = 2$ the graph changes from decreasing to increasing, so a local minimum occurs at $(2, -54)$. Because the derivative does not change sign at $x = 0$, this critical number is not a local extreme. ◇

The graph of this function will be obtained in Exercise 37 (Section 4.3).

EXAMPLE 4 Find all local extrema of $f(x) = (x + 1)^{2/7}e^{-x}$.

SOLUTION The Product and Chain Rules for derivatives can be used to obtain

The natural domain of f is $(-\infty, \infty)$.

$$\begin{aligned} f'(x) &= \tfrac{2}{7}(x+1)^{-5/7}e^{-x} + (x+1)^{2/7}e^{-x}(-1) \\ &= \tfrac{2}{7}(x+1)^{-5/7}e^{-x} - (x+1)^{2/7}e^{-x} \\ &= (x+1)^{-5/7}e^{-x}\left(\tfrac{2}{7} - (x+1)\right) \\ &= (x+1)^{-5/7}e^{-x}\left(-x - \tfrac{5}{7}\right) \\ &= \frac{-x-\frac{5}{7}}{(x+1)^{5/7}e^{x}}. \end{aligned}$$

The only solution to $f'(x) = 0$ is $x = -5/7$, so $c = -5/7$ is the only critical number. In addition, because the denominator of $f'(x)$ is zero when $x = -1$, f is not differentiable for $x = -1$. Thus, using the information in Table 4.2.2,

x	$\to -\infty$	-1	$-5/7$	$\to \infty$
$f(x)$	∞	0	$(2/7)^{(2/7)}e^{5/7} \approx 1.42811$	0
$f'(x)$		*dne*	0	

Table 4.2.2:

f is decreasing on $(-\infty, -1)$, increasing on $(-1, -5/7)$, and decreasing on $(-5/7, \infty)$. By the First-Derivative Test, f has a local minimum at $(-1, 0)$ and a local maximum at $(-5/7, (2/7)^{(2/7)}e^{5/7}) \approx (-0.71429, 1.42811)$.

Notice that the First-Derivative Test applies at $x = -1$ even though f is not differentiable for $x = -1$. A graph of $y = f(x)$ is shown in Figure 4.2.5. (See also Exercise 25 in Section 4.3.) ◇

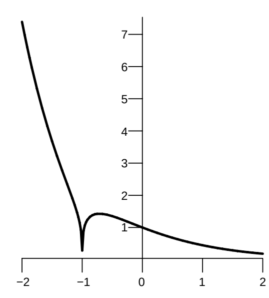


Figure 4.2.5:

Extreme Values on a Closed Interval

Many problems involve a continuous function only on a particular closed interval $[a, b]$. The Extreme Value Theorem guarantees the function attains both a maximum and a minimum at some point in the interval. The extreme values occur either at See Section 4.1

1. an endpoint ($x = a$ or $x = b$),
2. a critical number ($x = c$ where $f'(c) = 0$), or
3. where f is not differentiable ($x = c$ where $f'(c)$ is not defined).

EXAMPLE 5 Find the absolute maximum and minimum values of $f(x) = x^4 - 8x^2 + 1$ on the interval $[-1, 3]$.
SOLUTION The function is continuous on a closed and bounded interval. The absolute maximum and minimum values occur either at a critical point or at an endpoint of the interval. The endpoints are $x = -1$ and $x = 3$. To find the critical points we solve $f'(x) = 0$:

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2) = 0.$$

There are three critical numbers, $x = 0$, 2, and -2, but only $x = 0$ and $x = 2$ are in the interval. The intervals where the graph of $y = f(x)$ is increasing and decreasing can be determined from the information in Table 4.2.3.

x	-1	0	2	3
$f(x)$	-6	1	-15	10
$f'(x)$		0	0	0

Table 4.2.3:

Since we are looking only for global extrema on a closed interval, it is unnecessary to determine these intervals or to classify critical points as local extrema. Instead, we simply scan the list of function values at the endpoints and at the critical numbers – row 2 of Table 4.2.3 – for the largest and smallest values of $f(x)$. The largest value is 10, so the global maximum occurs at $x = 3$. The smallest value is -15, so the global minimum occurs at $x = 2$. ◇

In Example 5 it was not necessary to determine the intervals on which the function is increasing and decreasing, nor did we need to identify the local extreme values. This information can be obtained as in the earlier examples See Exercise 8.

Summary

This section shows how to use the first derivative to find extreme values of a function. Namely, identify when the derivative is zero, positive, and negative.

EXERCISES for 4.2

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

For each function in Exercises 1 – 14, sketch the graph of the function. Find all critical numbers, determine the intervals where the graph of $y = f(x)$ is increasing and where it is decreasing, and identify all local extreme values.

1. $f(x) = x^5$

2. $f(x) = x^6$

3. $f(x) = (x-1)^3$

4. $f(x) = (x-1)^4$

5. $f(x) = 3x^4 + x^3$

6. $f(x) = 2x^3 + 3x^2$

7. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 18$

8. $f(x) = x^4 - 8x^2 + 1$

9. $f(x) = xe^{-x/2}$

10. $f(x) = xe^{x/3}$

11. $f(x) = e^{-x^2}$

12. $f(x) = xe^{-x^2/2}$

13. $f(x) = x\sin(x) + \cos(x)$

14. $f(x) = x\cos(x) - \sin(x)$

In Exercises 15 to 22 sketch the general shape of the graph, using the given information. Assume the function and its derivative are defined for all x and are continuous. Explain your reasoning.

15. Critical point $(1, 2)$, $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$.

16. Critical point $(1, 2)$ and $f'(x) < 0$ for all x except $x = 1$.

17. x intercept -1, critical points $(1, 3)$ and $(2, 1)$, $\lim_{x\to\infty} f(x) = 4$, $\lim_{x\to-\infty} f(x) = -1$.

18. y intercept 3, critical point $(1, 2)$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 4$.

19. x intercept -1, critical points $(1, 5)$ and $(2, 4)$, $\lim_{x\to\infty} f(x) = 5$, $\lim_{x\to-\infty} f(x) = -\infty$.

20. x intercept 1, y intercept 2, critical points $(1, 0)$ and $(4, 4)$, $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to-\infty} f(x) = \infty$.

21. x intercepts 2 and 4, y intercept 2, critical points $(1, 3)$ and $(3, -1)$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 1$.

22. No x intercepts, y intercept 1, no critical points, $\lim_{x\to\infty} f(x) = 2$, $\lim_{x\to-\infty} f(x) = 0$.

In Exercises 23 – 38 graph the given functions, showing any intercepts, asymptotes, critical points, or local or global extrema.

23. $f(x) = x^3 - 3x^2 + 3x$

24. $f(x) = x^4 - 4x^3 + 4x^2$

25. $f(x) = x^4 - 4x + 3$

26. $f(x) = x^5 + 5x$

27. $f(x) = x^2 - 6x + 5$

28. $f(x) = 2x^2 + 3x + 5$

29. $f(x) = x^4 + 2x^3 - 3x^2$

30. $f(x) = 2x^3 + 3x^2 - 6x$

31. $f(x) = \dfrac{3x+1}{3x-1}$

32. $f(x) = \dfrac{x}{x+1}$

33. $f(x) = \dfrac{x}{x^2+1}$

34. $f(x) = \dfrac{x}{x^2-1}$

35. $f(x) = \dfrac{1}{2x^2-x}$

36. $f(x) = \dfrac{1}{x^2-3x+2}$

37. $f(x) = \dfrac{x^2+3}{x^2-4}$

38. $f(x) = \dfrac{\sqrt{x^2+1}}{x}$

Exercises 39 – 54 concern functions whose domains are restricted to closed intervals. In each, find the maximum and minimum value for the given function on the given interval.

39. $f(x) = x^2 - x^4$ on $[0, 1]$

40. $f(x) = 4x - x^2$ on $[0, 5]$

41. $f(x) = 2x^2 - 5x$ on $[-1, 1]$

42. $f(x) = x^3 - 2x^2 + 5x$ on $[-1, 3]$

43. $f(x) = \dfrac{x}{x^2+1}$ on $[0, 3]$

44. $f(x) = x^2 + x^4$ on $[0, 1]$

45. $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$ on $[0, 3]$

46. $f(x) = \sin(x) + \cos(x)$ on $[0, \pi]$

47. $f(x) = \sin(x) - \cos(x)$ on $[0, \pi]$

48. $f(x) = x + \sin(x)$ on $[-\pi/2, \pi/2]$

49. $f(x) = x + \sin(x)$ on $[-\pi, 2\pi]$

50. $f(x) = x/2 + \sin(x)$ on $[-\pi, 2\pi]$

51. $f(x) = 2\sin(x) - \sin(2x)$ on $[-\pi, \pi]$

52. $f(x) = \sin(x^2) + \cos(x^2)$ on $[0, \sqrt{2\pi}]$

53. $f(x) = \sin(x) - \cos(x)$ on $[-2\pi, 2\pi]$

54. $f(x) = \sin^2(x) - \cos^2(x)$ on $[-2\pi, 2\pi]$

In Exercises 55 to 61 graph the function.

55. $f(x) = \dfrac{\sin(x)}{1 + 2\cos(x)}$

56. $f(x) = \dfrac{\sqrt{x^2 - 1}}{x}$

57. $f(x) = \dfrac{1}{(x-1)^2(x-2)}$

58. $f(x) = \dfrac{3x^2 + 5}{x^2 - 1}$

59. $f(x) = 2x^{1/3} + x^{4/3}$

60. $f(x) = \dfrac{3x^2 + 5}{x^2 + 1}$

61. $f(x) = \sqrt{3}\sin(x) + \cos(x)$

62. Graph $f(x) = (x^2 - 9)^{1/3} e^{-x}$. Hint: This function is difficult to graph in one picture. Instead, create separate sketches for $x > 0$ and for $x < 0$. Watch out for the points where f is not differentiable.

63. Let f and g be polynomials without a common root.

(a) Show that if the degree of g is odd, the graph of f/g has a vertical asymptote.

(b) Show that if f and g have the same degree, the graph of f/g has a horizontal asymptote.

(c) Show that if the degree of f is less than the degree of g, the graph of f/g has a horizontal asymptote.

64.

(a) Graph $y = x$ and $y = \tan(x)$ relative to the same axes.

(b) Use (a) to find how many solutions there are to the equation $x = \tan(x)$.

(c) Graph $y = \frac{\sin(x)}{x}$ showing intercepts and asymptotes.

(d) Write a short commentary on the critical points of $\sin(x)/x$. Hint: Part (b) may come in handy.

65. A certain differentiable function has $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$. Moreover, $f(0) = 3$, $f(1) = 1$, and $f(2) = 2$.

(a) What is the minimum value of $f(x)$ for x in $[0, 2]$? Why?

(b) What is the maximum value of $f(x)$ for x in $[0, 2]$? Why?

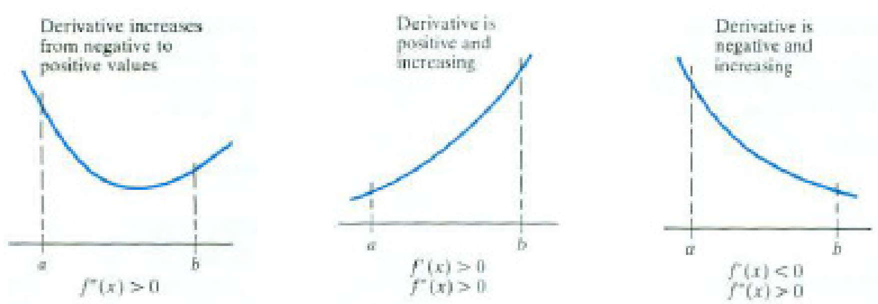
66. What is the minimum value of $y = (x^3 - x)/(x^2 - 4)$ for $x > 2$?

4.3 The Second Derivative and Graphing

The sign of the first derivative tells whether a function is increasing or decreasing. In this section we examine what the sign of the second derivative tells us about a function and its graph. This information will be used to help graph functions and also to provide an additional way to test whether a critical point is a maximum or minimum.

Concavity and Points of Inflection

The second derivative is the derivative of the first derivative. Thus, the sign of the second derivative determines if the first derivative is increasing or decreasing. For example, if $f''(x)$ is positive for all x in an interval (a, b), then f' is an increasing function throughout the interval (a, b). In other words, the slope of the graph of $y = f(x)$ increases as x increases from left to right on that part of the graph corresponding to (a, b). The slope may increase from



(a) (b) (c)

Figure 4.3.1:

The graph of a concave up function bends to the left.

negative values to zero to positive values, as in Figure 4.3.1(a). Or the slope may be positive throughout (a, b), as in Figure 4.3.1(b). Or the slope may be negative throughout (a, b), as in Figure 4.3.1(c).

In the same way, if $f''(x)$ is negative on the interval (a, b) then f' is decreasing on (a, b). The slope of the graph of $y = f(x)$ decreases as x increases from left to right on that part of the graph corresponding to (a, b).

> **DEFINITION** *(Concave Up and Concave Down)*
>
> A function f whose first derivative is increasing throughout the open interval (a, b) is called **concave up** in that interval.
>
> A function f whose first derivative is decreasing throughout the open interval (a, b) is called **concave down** in that interval.

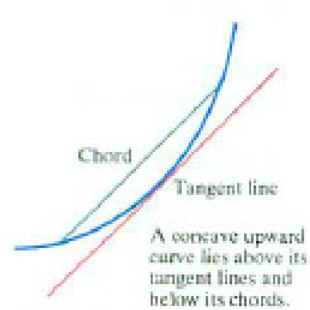


Figure 4.3.2:

When a curve is concave up, it lies above its tangent lines and below its chords. The graph of a concave up function is shaped like a cup. See Figure 4.3.2.

When a curve is concave down, it lies below its tangent lines and above its chords. The graph of a concave down function is shaped like a frown. See Figure 4.3.3.

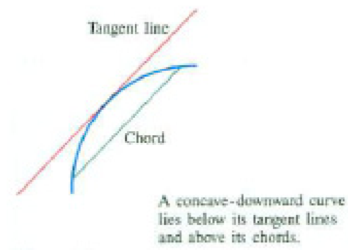


Figure 4.3.3:

EXAMPLE 1 Where is the graph of $f(x) = x^3$ concave up? concave down?
SOLUTION First, compute the second derivative: $f'(x) = 3x^2$ and $f''(x) = 6x$. Clearly, $6x$ is positive when x is positive and negative when x is negative. Thus, the graph is concave up for $x > 0$ and is concave down for $x < 0$. Note that the sense of concavity changes at $x = 0$, where $f''(x) = 0$. ⋄

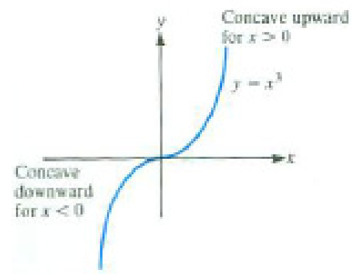


Figure 4.3.4:

In an interval where $f''(x)$ is positive, the function $f'(x)$ is increasing, and so the function f is concave up. However, if a function is concave up, $f''(x)$ need not be positive for all x in the interval. For instance, consider $y = x^4$. The derivative $4x^3$ is increasing on any interval, so the graph is concave up over any interval.

Any point where the graph of a function changes concavity is important.

> **DEFINITION** *(Inflection Number and Inflection Point)* Let f be a function and let a be a number. Assume there are numbers b and c such that $b < a < c$ and
>
> 1. f is continuous on the open interval (b, c)
> 2. f is concave up on (b, a) and concave down on (a, c)
>   *or*
> f is concave down on (b, a) and concave up on (a, c).
>
> Then, the point $(a, f(a))$ is called an **inflection point** or **point of inflection** of f. The number a is called an **inflection number** of a.

Notice that having $f''(a) = 0$ does not automatically make a an inflection number of f. To be an inflection number, the concavity has to change at a. (This is different from the use of critical number. Some authors define a number a with $f''(a) = 0$ as a **possible inflection number**.)

Observe that if the second derivative changes sign at the number a, then a is an inflection number. If the second derivative exists at an inflection number, it must be 0. But there can be an inflection point if $f''(a)$ is not defined. This is illustrated in the next example.

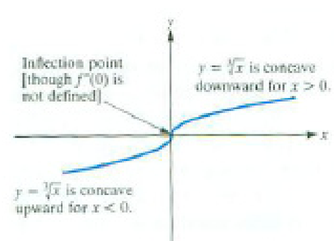


Figure 4.3.5:

EXAMPLE 2 Examine the concavity of the graph of $y = x^{1/3}$.
SOLUTION Here $y' = \frac{1}{3}x^{-2/3}$ and $y'' = \frac{1}{3}\left(\frac{-2}{9}\right)x^{-5/3}$. Althought $x = 0$ is in

the domain of this function, neither y' nor y'' is defined for $x = 0$. When x is negative, y'' is positive; when x is positive, y'' is negative. Thus, the concavity changes from concave up to concave down at $x = 0$. This means $x = 0$ is an inflection number and $(0, 0)$ is an inflection point. See Figure 4.3.5. ⋄

The simplest way to look for inflection points is to use both the first and second derivatives:

To find inflection points of $y = f(x)$:

1. Compute $f'(x)$ and $f''(x)$.
2. Look for numbers a such that f'' is not defined at a.
3. Look for numbers a such that $f''(a) = 0$
4. For each interval defined by the numbers a identified in Steps 2. and 3., determine if the first derivative is increasing or decreasing.

This process can be implemented using the same ideas used in Section 4.2 to identify critical points, as Example 3 shows.

EXAMPLE 3 Find the inflection point(s) of $f(x) = x^4 - 8x^3 + 18x^2$.
SOLUTION First, $f'(x) = 4x^3 - 24x^2 + 36x$ and

$$\begin{aligned} f''(x) &= 12x^2 - 48x + 36 \\ &= 12(x^2 - 4x + 3). \\ \text{Thus} \qquad &= 12(x-1)(x-3). \end{aligned}$$

f'' is defined for all real numbers so the only candidate for inflection numbers are the solutions to $f''(a) = 0$. Solving $f''(x) = 0$ yields:

$$0 = 12(x-1)(x-3).$$

Hence $x - 1 = 0$ or $x - 3 = 0$, and $x = 1$ or $x = 3$.

Table 4.3.1 is used to determine whether each of these number is, in fact, an inflection number of f. Because $-\infty < 16$, f' increases on $(-\infty, 1)$ and

x	$\to -\infty$	1	3	$\to \infty$
$f'(x)$	$-\infty$	16	0	∞
$f''(x)$		0	0	

Table 4.3.1:

the graph of $y = f(x)$ is concave up on this interval. Similarly, becuase $f'(x)$

decreases on $(1, 3)$, the graph is concave down on this interval. The graph changes from concave up to concave down at $x = 1$ so there is an inflection point at $(1, f(1)) = (1, 11)$.

The first derivative increases on the interval $(3, \infty)$ so the graph of $y = f(x)$ is concave up on $(3, \infty)$. This confirms that there is a change in concavity at $x = 3$. The second inflection point is $(3, f(3)) = (3, 27)$. These are the only two inflection points of $f(x) = x^4 - 6x^3 + 12x^2$. ⋄

Using Concavity in Graphing

The second derivative, together with the first derivative and the other tools of graphing, can help us sketch the graph of a function. Example 4 continues Example 3.

EXAMPLE 4 Graph $f(x) = x^4 - 8x^3 + 18x^2$.

SOLUTION The function $x^4 - 8x^3 + 18x^2$ is neither even nor odd. Because f is defined for all real numbers and $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty$, neither does it have any asymptotes. Since $f(0) = 0^4 - 8(0^3) + 18(0^2)$, its y intercept is 0. To find its x intercepts we look for solutions to the equation

Analysis based on $f(x)$

$$\begin{aligned} x^4 - 8x^3 + 18x^2 &= 0 \\ x^2(x^2 - 8x + 18) &= 0. \end{aligned}$$

Thus $x = 0$ or $x^2 - 8x + 18 = 0$. The quadratic equation can be solved by the quadratic formula. The discriminant is $(-8)^2 - 4(1)(18) = -8$ which is negative, so there are no real solutions of $x^2 - 8x + 18 = 0$. The only x intercept of $y = f(x)$ is $x = 0$.

The **discriminant** of $ax^2 + bx + c$ is $b^2 - 4ac$.

In Example 3 the first derivative was found:

Analysis based on $f'(x)$

$$\begin{aligned} f'(x) &= 4x^3 - 24x^2 + 36x \\ &= 4x(x^2 - 6x + 9) \\ &= 4x(x - 3)^2. \end{aligned}$$

Thus, $f'(x) = 0$ only when $x = 0$ and $x = 3$. The two critical points are $(0, 0)$ and $(3, f(3)) = (3, 27)$. The information in Table 4.3.2 allows us to conclude that the graph of $y = f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ with a local minimum at $(0, 0)$.

The analysis based on the second derivative was completed in Example 3. $f''(x) = 12(x - 1)(x - 3) = 0$ only when $x = 1$ or $x = 3$. The information in Table 4.3.1 was used to conclude that each of these is an inflection number and that the graph of $y = f(x)$ is concave up on $(-\infty, 1)$ and $(3, \infty)$ and concave down on $(1, 3)$.

Analysis based on $f''(x)$

x	$\to -\infty$	0	3	$\to \infty$
$f(x)$	∞	0	27	∞
$f'(x)$		0	0	

Table 4.3.2:

To begin to sketch the graph of $y = f(x)$ we plot the three points $(0, 0)$, $(1, 11)$, and $(3, 27)$. These three points divide the domain into four intervals. On $(-\infty, 0)$ the graph is decreasing and concave up; on $(0, 1)$ the graph is increasing and concave up; on $(1, 3)$ the graph is increasing and concave down; and on $(3, \infty)$ the graph is once again increasing and concave up. The final graph is shown in Figure 4.3.6. ⋄

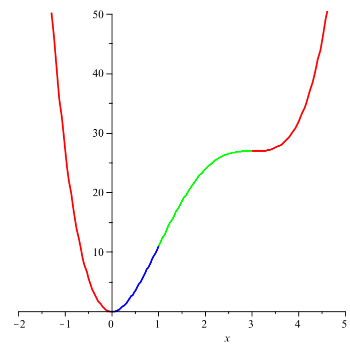


Figure 4.3.6:

(a) (b) (c) (d)

Figure 4.3.7: The general shape of a function that is (a) increasing and concave up, (b) increasing and concave down, (c) decreasing and concave up, and (d) decreasing and concave down

The procedure demonstrated in Example 4 has several advantages. Note that it was only necessary to evaluate $f(x)$ for a few "important" inputs x. These inputs cut the domain into intervals where neither the first derivative nor the second derivative changes sign. On each of these intervals the graph of the function will have one of the four shapes shown in Figure 4.3.7. A graph is usually made by piecing together these four shapes on adjoining intervals.

Local Extrema and the Second-Derivative Test

The second derivative is also useful in testing whether a critical number corresponds to a relative minimum or relative maximum. For this, we will use the relationships between concavity and tangent lines shown in Figures 4.3.2 and 4.3.3.

Let a be a critical number for the function f. Assume, for instance, that $f''(a)$ is negative. If f'' is continuous in some open interval that contains a, then (by the Permanence Property) $f''(x)$ remains negative for a suitably

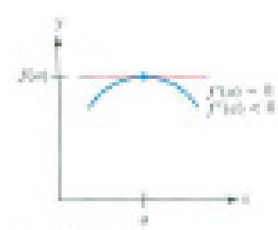


Figure 4.3.8:

The Permanence Property is Theorem 2.4.4 in Section 2.4.

small open interval that contains a. This means the graph of f is concave down near $(a, f(a))$, hence it lies below its tangent lines. In particular, it lies below the horizontal tangent line at the critical point $(a, f(a))$, as illustrated in Figure 4.3.8. Thus the function f has a *relative maximum* at the critical number a. Similarly, if $f'(a) = 0$ and $f''(a) > 0$, the critical point $(a, f(a))$ is a relative minimum because the graph of f is concave up and lies above the horizontal tangent line at $(a, f(a))$. These observations suggest the following test for a relative extreme.

Theorem 4.3.1 *Second-Derivative Test for Relative Extreme Values Let f be a function such that $f'(x)$ is defined at least on some open interval containing the number a. Assume that $f''(a)$ is defined.*

If $f'(a) = 0$ and $f''(a) < 0$, then f has a relative minimum at $(a, f(a))$.

If $f'(a) = 0$ and $f''(a) > 0$, then f has a relative maximum at $(a, f(a))$.

EXAMPLE 5 Use the Second-Derivative Test to classify all local extrema of the function $f(x) = x^4 - 8x^3 + 18x^2$.

Compare with Examples 3 and 4.

SOLUTION This is the same function analyzed in Examples 3 and 4. The two critical points are $(0, 0)$ and $(3, 27)$. The second derivative is $f''(x) = 12x^2 - 48x + 36$. At $x = 0$ we have

$$f''(0) = 12(0^2) - 48(0) + 36 = 36,$$

which is positive. Since $f'(0) = 0$ and $f''(0) > 0$, f has a local minimum at $(0, 0)$. At $x = 3$ we have

$$f''(3) = 12(3^2) - 48(3) + 36 = 0.$$

Since $f''(3) = 0$, the Second-Derivative Test tells us nothing about the critical number 3.

This is consistent with our previous findings. The point at $(3, 27)$ is an inflection point and not a local extreme point. ◇

Summary

Table 4.3.3 shows the meaning of the signs of $f(x)$, $f'(x)$, and $f''(x)$ in terms of the graph of $y = f(x)$.

	is positive (> 0).	is negative (< 0).	changes sign.	is zero ($= 0$).
Where the ordinate $f(x)$	the graph is above the x axis.	the graph is below the x axis.	the graph crosses the x axis.	there is an x intercept.
Where the slope $f'(x)$	the graph slopes upward.	the graph slopes downward.	the graph has a horizontal tangent and a relative maximum or minimum.	there is a critical point.
Where $f''(x)$	the graph is concave up (like a cup).	the graph is concave down (like a frown).	the graph has an inflection point.	there may be an inflection point.

Table 4.3.3:

Keep in mind that the graph has an inflection point at $(a, f(a))$ when the sign of $f''(x)$ has different signs on either side of $x = a$. This can occur when either $f''(a) = 0$ or when $f''(a)$ is not defined. Similarly, a graph can have a maximum or minimum at $(a, f(a))$ when either $f'(a) = 0$ or $f'(a)$ is not defined.

EXERCISES for 4.3

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In Exercises 1 to 16 describe the intervals where the function is concave up and concave down and give any inflection points.

1. $f(x) = x^3 - 3x^2 + 2$

2. $f(x) = x^3 - 6x^2 + 1$

3. $f(x) = x^2 + x + 1$

4. $f(x) = 2x^2 - 5x$

5. $f(x) = x^6$

6. $f(x) = x^5$

7. $f(x) = x^4 - 4x^3$

8. $f(x) = 3x^5 - 5x^4$

9. $f(x) = \frac{1}{1+x^2}$

10. $f(x) = \frac{1}{1+x^4}$

11. $f(x) = x^3 = 6x^2 - 15x$

12. $f(x) = \frac{x^2}{2} + \frac{1}{x}$

13. $f(x) = \tan(x)$

14. $f(x) = \sin(x) + \sqrt{3}\cos(x)$

15. $f(x) = \cos(x)$

16. $f(x) = \cos(x) + \sin(x)$

In Exercises 17 to 27 graph the polynomials, showing inflection points, critical points, and intercepts.

17. $f(x) = x^3 + 3x^2$

18. $f(x) = 2x^3 + 9x^2$

19. $f(x) = x^4 - 4x^3 + 6x^2$

20. $f(x) = x^4 + 4x^3 + 6x^2 - 2$

21. $f(x) = x^4 - 6x^3 + 12x^2$

22. $f(x) = 2x^6 - 10x^4 + 10$

23. $f(x) = 2x^6 + 3x^5 - 10x^4$

24. $f(x) = 3x^4 + 4x^3 - 12x^2 + 4$

25. $f(x) = 3x^5 - 20x^3 + 10$ NOTE: This function was first encountered in Example 3.

26. $f(x) = 3x^4 + 4x^3 - 12x^2 + 4$

27. $f(x) = 2x^6 - 15x^4 + 20x^3 - 20x + 10$

In each of Exercises 28 to 35 sketch the general appearance of the graph of the given function near $(1, 1)$ on the basis of the information given. Assume that f, f', and f'' are continuous.

28. $f(1) = 1$, $f'(1) = 0$, $f''(1) = 1$

29. $f(1) = 1$, $f'(1) = 0$, $f''(1) = -1$

30. $f(1) = 1$, $f'(1) = 0$, $f''(1) = 0$ NOTE: Sketch four possibilities.

31. $f(1) = 1$, $f'(1) = 0$, $f''(1) = 0$, $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$

32. $f(1) = 1$, $f'(1) = 0$, $f''(1) = 1$ and $f''(x) < 0$ for x near 1

33. $f(1) = 1$, $f'(1) = 1$, $f''(1) = -1$

34. $f(1) = 1$, $f'(1) = 1$, $f''(1) = 0$, $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$

35. $f(1) = 1$, $f'(1) = 1$, $f''(1) = 0$ and $f''(x) > 0$ for x near 1

36. Find all inflection points of $f(x) = x\ln(x)$. On what intervals is the graph of $y = f(x)$ concave up? concave down? [On what intervals is the graph increasing? decreasing?] NOTE: This function was first encountered in Example 2.

37. Find all inflection points of $f(x) = (x + 1)^{2/7}e^{-x}$. On what intervals is the graph of $y = f(x)$ concave up? concave down? [On what intervals is the graph increasing? decreasing?] NOTE: This function was first encountered in Example 4.

38. Find the critical points and infelction points of $f(x) = x^2e^{-x/3}$. NOTE: See Example 1.

In Exercises 39 to 40 sketch a graph of a hypothetical function that meets the given conditions. Assume f' and f'' are continuous. Explain your reasoning.

39. Critical point $(2, 4)$; inflection points $(3, 1)$ and $(1, 1)$; $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$

40. Critical points $(-1, 1)$ and $(3, 2)$; inflection point $(4, 1)$; $\lim_{x\to0^+} f(x) = -\infty$ and $\lim_{x\to0^-} f(x) = \infty$ $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = \infty$

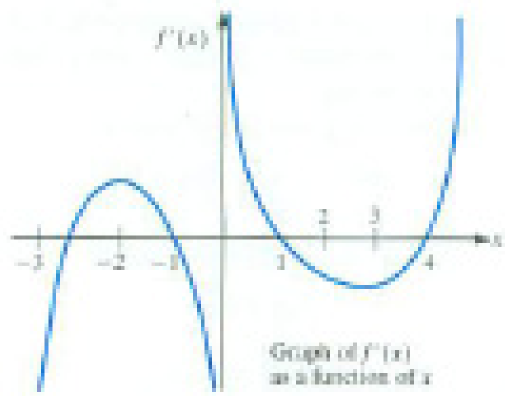


Figure 4.3.9:

41. *(Contributed by David Hayes)* Let f be a function that is continuous for all x and differentiable for all x other than 0. Figure 4.3.9 is the graph of its derivative $f'(x)$ as a function of x.

(a) Answer the following questions about f (*not* about f'). Where is f increasing? decreasing? concave up? concave down? What are the critical numbers? Where do any relative extrema occur? Explain.

(b) Assuming that $f(0) = 1$, graph a hypothetical function f that satisfies the conditions given.

(c) Graph $f''(x)$.

42. Graph $y = 2(x-1)^{5/3} + 5(x-1)^{2/3}$, paying particular attention to points where y' does not exist.

43. Graph $y = x + (x+1)^{1/3}$.

44. Find the critical points and inflection points in $[0, 2\pi]$ of $f(x) = \sin^2(x)\cos(x)$.

45. Can a polynomial of degree 6 have *(a)* no inflection points? *(b)* exactly one inflection point? Explain.

46. Can a polynomial of degree 5 have *(a)* no inflection points? *(b)* exactly one inflection point? Explain.

47. In the theory of **inhibited growth** it is assumed that the growing quantity y approaches some limiting size M. Specifically, one assumes that the rate of growth is proportional both to the amount present and to the amount left to grow:

$$\frac{dy}{dt} = ky(M - y).$$

Prove that the graph of y as a function of time has an inflection point when the amount y is exactly half the limiting amount M.

48. Let f be a function such that $f''(x) = (x - 1)(x - 2)$.

(a) For which x is f concave up?

(b) For which x is f concave down?

(c) List its inflection number(s).

(d) Find a specific function f whose second derivative is $(x - 1)(x - 2)$.

49. A certain function $y = f(x)$ has the property that

$$y' = \sin(y) + 2y + x.$$

Show that at a critical number the function has a local minimum.

50. Assume that the domain of $f(x)$ is the entire x axis, and $f'(x)$ and $f''(x)$ are continuous. Assume that $(1, 1)$ is the only critical point and that $\lim_{x\to\infty} f(x) = 0$.

(a) Must $f(x)$ be decreasing for $x > 1$?

(b) Must $f(x)$ have an inflection point?

4.4 Proofs of the Three Theorems

In Section 4.1 two observations about tangent lines led to the Theorem of the Interior Extremum (Theorem 4.1.1), Rolle's Theorem (Theorem 4.1.2), and the Mean-Value Theorem (Theorem 4.1.3). These results allow us to determine properties of a function that can be used to sketch a graph of the function. Now, using the definition of the derivative, we proof these theorems.

Proof of the Theorem of the Interior Extremum

Suppose the maximum of f on the open interval (a, b) occurs at the number c. This means that $f(c) \geq f(x)$ for each number x between a and b.

Proof of Theorem 4.1.1 $f'(c) = 0$ at the maximum or minimum on an open interval.

Assume that f is differentiable at c.

Our challenge is to use only this information and the definition of the derivative as a limit to show that $f'(c) = 0$. We will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, forcing $f'(c)$ to be zero.

Recall that

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

The assumption that f is differentiable on (a, b) means that $f'(c)$ exists. Consider the difference quotient

$$\frac{f(c + \Delta x) - f(c)}{\Delta x}. \tag{1}$$

when Δx is so small that $c+\Delta x$ is in the interval (a, b). Then $f(c+\Delta x) \leq f(c)$. Hence $f(c + \Delta x) - f(c) \leq 0$. Therefore, when Δx is positive, the difference quotient in (1) will be negative, or 0. Consequently, as $\Delta x \to 0$ through *positive* values,

$\frac{\text{negative}}{\text{positive}} = \text{negative}$

$$f'(c) = \lim_{\Delta x \to 0^+} \frac{f(c + \Delta x) - f(c)}{\Delta x} \leq 0. \tag{2}$$

If, on the other hand, Δx is negative, then the difference quotient in (3) will be positive, or 0. Hence, as $\Delta x \to 0$ through *negative* values,

$\frac{\text{negative}}{\text{negative}} = \text{positive}$

$$f'(c) = \lim_{\Delta x \to 0^-} \frac{f(c + \Delta x) - f(c)}{\Delta x} \geq 0. \tag{3}$$

The only way $f'(c) \leq 0$ and $f'(c) \geq 0$ can both hold is when $f'(c) = 0$. This proves that if f has a maximum on (a, b), then $f'(c) = 0$.

See Exercise 15.

The proof for the case when f has a minimum on (a, b) is essentially the same. •

The proofs of Rolle's Theorem and the Mean-Value Theorem are related. Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

Proof of Rolle's Theorem

Proof of Theorem 4.1.2
If $f(a) = f(b)$, then $f'(c) = 0$ for at least one number between a and b.

The goal here is to use the facts that f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$ to conclude that there must a number c in (a, b) with $f'(c) = 0$.

Since f is continuous on the closed interval $[a, b]$, it has a maximum value M and a minimum value m on that interval. There are two cases to consider: $m < M$ and $m = M$.

Case 1: If $m = M$, f is constant and $f'(x) = 0$ for all x in $[a, b]$. Then any number in (a, b) will serve as the desired number c.

Case 2: Suppose $m < M$. Then, $f(a) = f(b)$, the minimum and maximum cannot both occur at the ends of the interval. At least one of the extrema occurs at a number c strictly between a and b. By assumption, f is differentiable at c, so $f'(c)$ exists. Thus, by the Theorem of the Interior Extremum, $f'(c) = 0$. This completes the proof of Rolle's Theorem. •

The idea behind the proof of the Mean-Value Theorem is to define a function to which Rolle's Theorem can be applied.

Proof of the Mean-Value Theorem

Proof of Theorem 4.1.3
$f'(c) = \frac{f(b)-f(a)}{b-a}$ for at least one number between a and b.

Let $y = L(x)$ be the equation of the chord through the two points $(a, f(a))$ and $(b, f(b))$. The slope of this line is $L'(x) = \dfrac{f(b) - f(a)}{b - a}$. Define $h(x) = f(x) - L(x)$. Note that $h(a) = h(b) = 0$ because $f(a) = L(a)$ and $f(b) = L(b)$.

By assumption, f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Because L a linear function, it is differentiable. So, h is continuous on $[a, b]$ and differentiable on (a, b).

Rolle's Theorem applies to h on the interval $[a, b]$. Therefore, there is at least one number c in (a, b) where $h'(c) = 0$. Now, $h'(c) = f'(c) - L'(c)$ so that

$$f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a}.$$

•

Summary

Using only the definition of the derivative and the assumption that a continuous function defined on a closed interval assumes maximum and minimum values, we proved the Theorem of the Interior Extrema, Rolle's Theorem, and the Mean-Value Theorem. Note that we did not appeal to any pictures or to our geometric intuition.

EXERCISES for 4.4

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In each of Exercises 1 — 6 sketch a graph of a differentiable function that meets the given conditions. (Just draw the graph; there is no need to come up with a formula for the function.)

1. $f'(x) < 0$ for all x

2. $f'(3) = 0$ for for $x \neq 3$, $f'(x) < 0$

3. x intercepts at 1 and 5; y intercept at 2; $f'(x) < 0$ for $x < 4$; $f'(x) > 0$ for $x > 4$

4. x intercepts at 2 and 5; y intercept at 3; $f'(x) > 0$ for $x < 1$ and for $x > 3$; $f'(x) < 0$ for x in $(1, 3)$

5. $f'(x) = 0$ only when $x = 1$ or 4; $f(1) = 3$, $f(4) = 1$; $f'(x) < 0$ for $x < 1$; $f'(x) > 0$ for $x > 4$

6. $f'(x) = 0$ only when $x = 1$ or 4; $f(1) = 3$, $f(4) = 1$; $f'(x) > 0$ for $x < 1$ and for $x > 4$

In Exercises 7 to 10 explain why no differentiable function satisfies all the conditions.

7. $f(1) = 3$, $f(2) = 4$, $f'(x) < 0$ for all x

8. $f(2) = 5$, $f(3) = -1$, $f'(x) \geq 0$ for all x

9. x intercepts only at 1 and 2; $f(3) = -1$, $f(4) = 2$

10. $f(x) = 2$ only when $x = 0$, 1, and 3; $f'(x) = 0$ only when $x = \frac{1}{4}$, $\frac{3}{4}$, and 4.

11. In "Surely You're Joking, Mr. Feynmann!," Norton, New York, 1985, Nobel laureate Richard P. Feynmann writes:

> I often liked to play tricks on people when I was at MIT. One time, in mechanical drawing class, some joker picked up a French curve (a piece of plastic for drawing smooth curves — a curly funny-looking thing) and said, "I wonder if the curves on that thing have some special formula?"
>
> I thought for a moment and said, "Sure they do. The curves are very special curves. Lemme show ya," and I picked up my French curve and began to turn it slowly. "The French curve is made so that at the lowest point on each curve, no matter how you turn it, the tangent is horizontal."
>
> All the guys in the class were holding their French curve up at different angles, holding their pencil up to it at the lowest point and laying it down, and discovering that, sure enough, the tangent is horizontal.

How was Feynmann playing a trick on his classmates?

12.

(a) Show that the equation $5x - \cos(x) = 0$ has exactly one solution.

(b) Find a specific interval which contains the solution.

13. What can be said about the number of solutions of the equation $f(x) = 3$ for a differentiable function if

(a) $f'(x) > 0$ for all x?

(b) $f'(x) > 0$ for $x < 7$ and $f'(x) < 0$ for $x > 7$?

14. Consider the function $f(x) = x^3 + ax^2 + c$. Show that if $a < 0$ and $c > 0$, then f has exactly one negative root.

15. Prove the Theorem of the Interior Extremum when the minimum of f on (a, b) occurs at c.

16. With the book closed, obtain the Mean-Value Theorem from Rolle's Theorem.

17. Show that a polynomial $f(x)$ of degree n, $n \geq 1$, can have at most n distinct real roots, that is, solutions to the equation $f(x) = 0$.

(a) Use algebra to show that the statement holds for $n = 1$ and $n = 2$.

(b) Use calculus to show that the statement holds for $n = 3$.

(c) Use calculus to show that the statement holds for $n = 4$ and $n = 5$.

(d) Why does it hold for all positive integers n?

18. To keep your differentiation skills sharp, differentiate each of the following expressions:

(a) $\sqrt{1-x^2}\sin(3x)$

(b) $\dfrac{\sqrt[3]{x}}{x^2+1}$

(c) $\tan\left(\dfrac{1}{(2x+1)^2}\right)$

(d) $\ln\left(\dfrac{(x^2+1)^3\sqrt{1-x^2}}{\sec^2(x)}\right)$

19. Is there a differentiable function f whose domain is the x axis such that f is increasing and yet the derivative is *not* positive for all x?

20. Consider the function f given by the formula $f(x) = x^3 - 3x$.

(a) At which numbers x is $f'(x) = 0$?

(b) Use the theorem of the Interior Extremum to show that the maximum value of $x^3 - 3x$ for x in $[1, 5]$ occurs either at 1 or at 5.

21.

(a) Recall the definition of $L(x)$ in the proof of the Mean-Value Theorem, and show that

$$L(x) = f(a) + \frac{x-a}{b-a}\left(f(b) - f(a)\right).$$

(b) Using (a), show that

$$L'(x) = \frac{f(b)-f(a)}{b-a}.$$

22. Show that Rolle's Theorem is a special case of the Mean-Value Theorem.

23. In the proof of the Mean-Value Theorem, L is the line through the two points $(a, f(a))$ and $(b, f(b))$ on the graph of $y = f(x)$. Find the formula for $L(x)$.

24. Is this proposed proof of the Mean-Value Theorem correct?
Proof T

ilt the x and y axes until the x axis is parallel to the given chord. The chord is now "horizontal," and we may apply Rolle's Theorem. •

25. Find all functions $f(x)$ such that $f'(x) = 2$ for all x and $f(1) = 4$.

26. Find all differentiable functions such that $f(1) = 3$, $f'(1) = -1$, and $f''(1) = e^x$.

27. Assume that $f(x)$ is a continuous function defined for all real numbers and that $f(1) = 3$. Using the definition of continuity in terms of limits, explain why there is an interval $(1, b)$ such that $f(x) \geq 2$ for all x in the interval. HINT: Assume that there is no such interval and get a contradiction.

28.

Sam: The key to this section is that at an extremum the derivative is 0. I don't like the book's proof, with its use of limits.

Jane: So what's yours?

Sam: Simple. Either the derivative is 0, positive, or negative. Right?

Jane: That's a no-brainer.

Sam: If it's positive, a little piece of the graph would look like a line with a positive slope. Right?

Jane: Well, yes.

Sam: So just to the right of the extreme point, the graph would have larger values since the line rises from left to right.

Jane: O.K. so far.

Sam: That rules out a positive derivative. And in the same way I could rule out a negative derivative.

Jane: I have to agree.

Sam: So the derivative must be 0. I'll e-mail this proof to the authors.

Jane: Better hold off. I smell a fish.

What is the fish?

29. Prove: If f has a negative derivative on (a, b) then f is decreasing on the interval $[a, b]$.

Exercises 30–32 provide analytic justification for the statement in Section 4.3 that "[W]hen a curve is concave up, it lies above its tangent lines and below its chords."

30. Show that in an open interval in which f'' is positive, tangents to the graph of f lie below the curve. HINT: Why do you want to show that if a and x are in the interval, then $f(x) > f(a) + f'(a)(x - a)$? Treat the cases $a < x$ and $x > a$ separately. NOTE: See also Exercises 34 and 35 in Section 5.4.

31. Assume that $f''(x)$ is positive for x in an open interval. Let $a < b$ be two numbers in the interval. Show that the chord joining $(a, f(a))$ and $(b, f(b))$ lies above the graph of f. HINT:

A. Why does one want to prove that

B. How does it help to know that

C. Show that the function on the right-hand side of the inequality in (b) is increasing for $a < x < b$. Why does this show that the chords lie above the curve?

32.

Sam: I can do Exercise 31 more easily. I'll show that (b) is true. By the Mean-Value Theorem, I can write the left side as $f'(c)$ where c is in $[a, b]$ and the right side as $f'(d)$ where d is in $[a, x]$. Since $b > x$, I know $c > d$, hence $f'(c) > f'(d)$. Nothing to it.

Is Sam's reasoning correct?

33. We stated, in Section 4.3, that if $f(x)$ is defined in an open interval around the critical number a and $f''(a)$ is negative, then $f(x)$ has a relative maximum at a. Explain why this is so, following these steps.

(a) Why is $\lim_{\Delta x \to 0} \dfrac{f'(a + \Delta x) - f'(a)}{\Delta x}$ negative?

(b) Deduce that if Δx is small and positive, then $f'(a + \Delta x)$ is negative.

(c) Show that if Δx is small and negative, then $f'(a + \Delta x)$ is positive.

(d) Show that $f'(x)$ changes sign from positive to negative at a. By the First-Derivative Test for a Relative Maximum, $f(x)$ has a relative maximum at a.

4.S Chapter Summary

The text and exercises for the summary will be written after the organization of the chapters is firmly settled.

Chapter 5

More Applications of Derivatives

This chapter samples some of the many applications of the derivative in the "real world" and within mathematics. For instance, the derivative is of use in choosing the most economical route, finding limits, approximating functions, such as $\sin(x)$ and e^x, by polynomials, and in estimating the error in the approximation.

5.1 Applied Maximum and Minimum Problems

In Chapter 4, we saw how the derivative and second derivative are of use in finding maxima and minima of a given function – the locally high and low points on its graph. Now we will use these same techniques to find extrema in "applied" problems. Though the examples will be drawn mainly from geometry they illustrate the general procedure. The main challenge in these situations is figuring out the formula for the function that describes the quantity to be maximized (or minimized).

The General Procedure

The general procedure runs something along these lines.

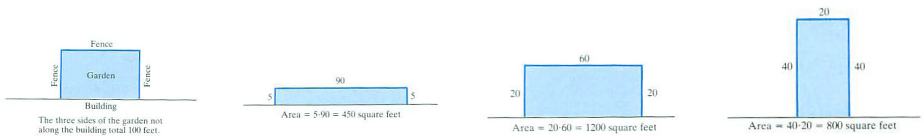
1. Get a feel for the problem (experiment with particular cases.)
2. Devise a formula for the function whose maximum or minimum you want to find.
3. Determine the domain of the function – that is, the inputs *that make sense in the application*.
4. Find the maximum or minimum of the function for inputs that are in the domain identified in Step 3.

The most important step is finding a formula for the function. To become skillful at doing this, practice.

A Large Garden

EXAMPLE 1 A couple have enough wire to construct 100 feet of fence. They wish to use it to form three sides of a rectangular garden, one side of which is along a building, as shown in Figure 5.1.1(a). What shape garden should they choose in order to enclose the largest possible area?

SOLUTION *Step* 1. First make a few experiments. Figures 5.1.1(b)–(d) show some possible ways of laying out the 100 feet of fence. In the first case the side parallel to the building is very long, in an attempt to make a large area. However, doing this forces the other sides of the garden to be small. The area is $90 \times 5 = 450$ square feet. In the second case, the garden has a larger area, $60 \times 20 = 1200$ square feet. In the third case, the side parallel to



(a) (b) (c) (d)

Figure 5.1.1:

the building is only 20 feet long, but the other sides are longer. The area is $20 \times 40 = 800$ square feet.

Clearly, we may think of the area of the garden as a function of the length of the side parallel to the building.

Step 2. Let $A(x)$ be the area of the garden when the length of the side parallel to the building is x feet, as in Figure 5.1.2. The other sides of the garden have length y. But y is completely determined by x since the fence is 100 feet long:

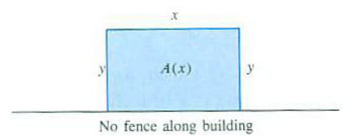


Figure 5.1.2:

$$x + 2y = 100.$$

Thus $y = (100 - x)/2$.

Figure 5.1.3:

Since the area of a rectangle is its length times its width,

$$\begin{aligned} A(x) &= xy \\ A(x) &= x\left(\frac{100-x}{2}\right) \end{aligned}$$

or

$$A(x) = 50x - \frac{x^2}{2}. \qquad (1)$$

(See Figure 5.1.1.) We now have the function.

Step 3. Which values of x in (5.1.1) correspond to possible gardens? Since there is only 100 feet of fence, $x \leq 100$. Furthermore, it makes no sense to have a negative amount of fence; hence $x \geq 0$. Therefore the domain on which we wish to consider the function (5.1.1) is the closed interval $[0, 100]$.

Step 4. To maximize $A(x) = 50x - x^2/2$ on $[0, 100]$ we examine $A(0)$, $A(100)$, and the value of $A(x)$ at any critical numbers. To find critcial numbers, differentiate $A(x)$:

$$\begin{aligned} A(x) &= 50x - \frac{x^2}{2} \\ A'(x) &= 50 - x. \end{aligned}$$

Setting $A'(x) = 0$ gives

$$\begin{aligned} 0 &= 50 - x \\ \text{or} \qquad x &= 50. \end{aligned}$$

There is one critical number, 50.

All that is left is to find the largest of $A(0)$, $A(100)$, and $A(50)$. We have

$$\begin{aligned} A(0) &= 50 \cdot 0 - \frac{0^2}{2} = 0, \\ A(100) &= 50 \cdot 100 - \frac{100^2}{2} = 0, \\ \text{and} \qquad A(50) &= 50 \cdot 50 - \frac{50^2}{2} = 1250. \end{aligned}$$

The maximum possible area is 1250 square feet, and the fence should be laid out as shown in Figure 5.1.4. ◇

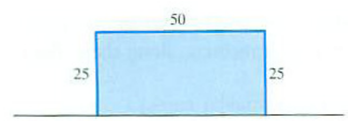
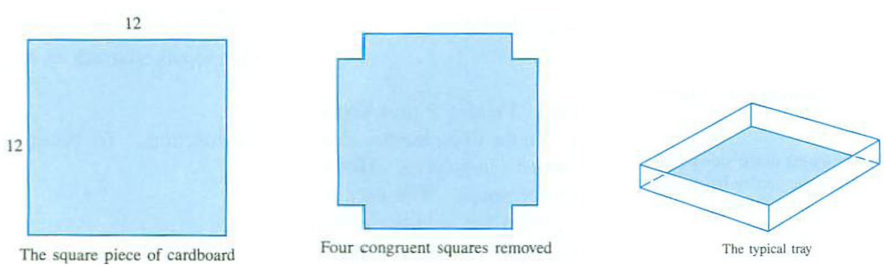


Figure 5.1.4:

A Large Tray

EXAMPLE 2 If we cut four congruent squares out of the corners of a square piece of cardboard 12 inches on each side, we can fold up the four remaining flaps to obtain a tray without a top. What size squares should be cut in order to maximize the volume of the tray? (See Figure 5.1.5.)

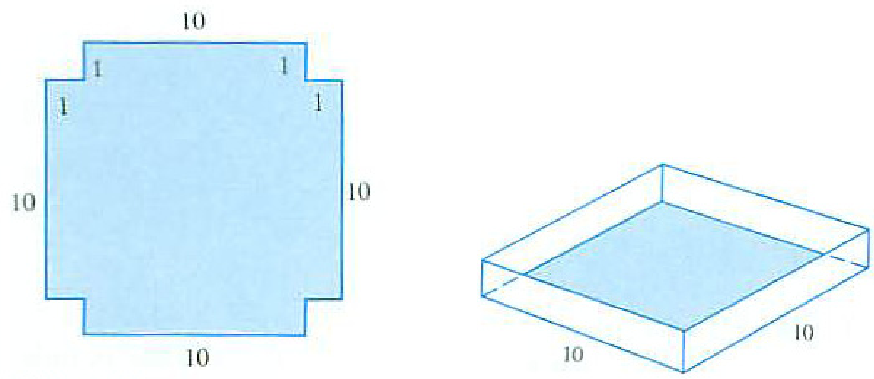


(a) (b) (c)

Figure 5.1.5:

SOLUTION *Step* 1. First we get a feel for the problem. Let us make a couple of experiments.

Say that we remove small squares that are 1 inch by 1 inch, as in Figure 5.1.6(a). When we fold up the flaps we obtain a tray whose base is a



(a) (b)

Figure 5.1.6:

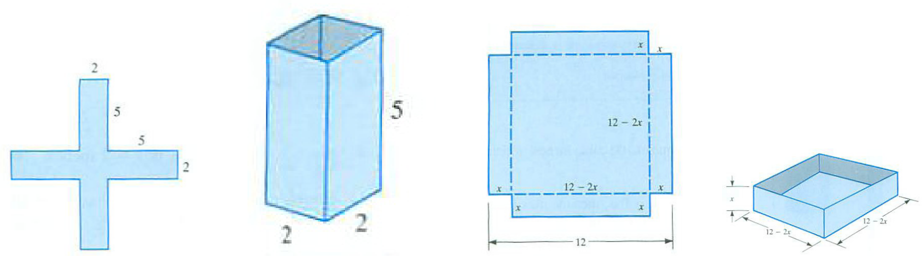
10-inch by 10-inch square and whose height is 1 inch, as in Figure 5.1.6(b). The volume of the tray is

$$\text{Area of base} \times \text{height} = \underbrace{10 \times 10}_{\text{base area}} \times \underbrace{1}_{\text{height}} = 100 \text{ cubic inches.}$$

For our second experiment, let's try cutting out a large square, say 5 inches by 5 inches, as in Figure 5.1.7(a). When we fold up the flaps, we get a very tall tray with a very small base, as in Figure 5.1.7(b). It volume is

$$\text{Area of base} \times \text{height} = 2 \times 2 \times 5 = 20 \text{ cubic inches.}$$

Clearly *volume depends on the size of the cut-out squares*. The function we will investigate is of the type $V = f(x)$, where V is the volume of the tray formed by removing four squares whose sides all have length x.



(a) (b) (c) (d)

Figure 5.1.7:

Step 2. To find the formula for $f(x)$ we make a *large*, clear diagram of the typical case, as in Figure 5.1.7(c) and Figure 5.1.7(d). Now

$$\begin{aligned}\text{Volume of tray} &= \text{length} \cdot \text{width} \cdot \text{height} \\ &= (12-2x)(12-2x)x \\ &= (12-2x)^2 x,\end{aligned}$$

hence

$$V(x) = 4x^3 - 48x^2 + 144x. \tag{2}$$

We have obtained a formula for volume as a function of the length of the sides of the cut-out squares.

Step 3. Next determine the domain of the function $V(x)$ that is *meaningful* in the problem.

The smallest that x can be is 0. In this case the tray has height 0 and is just a flat piece of cardboard. (Its volume is 0.) The size of the cut is not more than 6 inches, since the cardboard has sides of length 12 inches. The cut can be as near 6 inches as we please, and the nearer it is to 6 inches, the smaller is the base of the tray. For convenience of our calculations, allow cuts with $x = 6$, when the area of the base is 0 square inches and the height is 6 inches. (The volume is 0 cubic inches.) Therefore the domain of the volume function $V(x)$ is the closed interval $[0, 6]$.

Step 4. To maximize $V(x) = 4x^3 - 48x^2 + 144x$ on $[0, 6]$, evaluate $V(x)$ at critical numbers in $[0, 6]$ and at the endpoints of $[0, 6]$.

We have

$$\begin{aligned}V'(x) &= 12x^2 - 96 + 144 \\ &= 12(x^2 - 8x + 12) \\ &= 12(x-2)(x-6).\end{aligned}$$

A critical number satisfies the equation

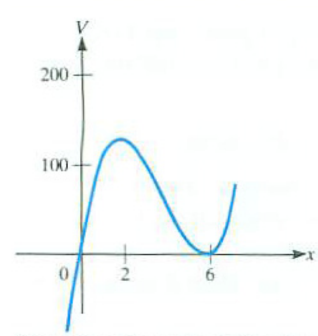
$$0 = 12(x-2)(x-6).$$

Hence $x - 2 = 0$ or $x - 6 = 0$. The critical numbers are 2 and 6.

The endpoints of the interval $[0, 6]$ are 0 and 6. Therefore the maximum value of $V(x)$ for x in $[0, 6]$ is the largest of $V(0)$, $V(2)$, and $V(6)$. Since $V(0) = 0$ and $V(6) = 0$, the largest value is

$$V(2) = 4(2^3) - 48(2^2) - 144 \cdot 2 = 128 \text{ cubic inches.}$$

The cut that produces the tray with the largest volume is $x = 2$ inches. ⋄



Only values of x in the portion above $[0, 6]$ correspond to physically realizable trays.

Figure 5.1.8:

As a matter of interest, let us graph the function V, showing its behavior for all x, not just for values of x significant in the problem. Note in Figure 5.1.8 that at $x = 2$ and $x = 6$ the tangent is horizontal.

Remark: In Example 2 you might say $x = 0$ and $x = 6$ don't really correspond to what you would call a tray. If so, you would restrict the domain of $V(x)$ to the open interval $(0, 6)$. You would then have to examine the behavior of $V(x)$ for x *near* 0 and for x *near* 6. By making the domain $[0, 6]$ from the start, you avoid the extra work of examining $V(x)$ for x *near* the ends of the interval.

The key step in these two examples and in any applied problem is Step 2, findng a formula for the quantity whose extremum you are seeking. In case the problem is geometrical, the following chart may be of aid.

Setting Up the Function

1. Draw and label the appropriate diagrams.
   (Make them large enough so that there is room for labels.)
2. Label the varous quantities by letters, such as x, y, A, V.
3. Identify the quantity to be maximized (or minimized).
4. Express the quantity to be maximized (or minimized) in terms of one or more of the other variables.
5. Finally, express that quantity in terms of only one variable.

An Economical Can

EXAMPLE 3 Of all the tin cans that enclose a volume of 100 cubic inches, which requires the least metal?

SOLUTION The can may be flat or tall. If the can is flat, the side uses little metal, but then the top and bottom bases are large. If the can is shaped like a mailing tube, then the two bases require little metal, but the curved side requires a great deal of metal. (See Figure 5.1.9, where r denotes the radius and h the height of the can.) What is the ideal compromise between these two extremes?

The surface area S of the can is given by

$$S = 2\pi r^2 + 2\pi rh \qquad (3)$$

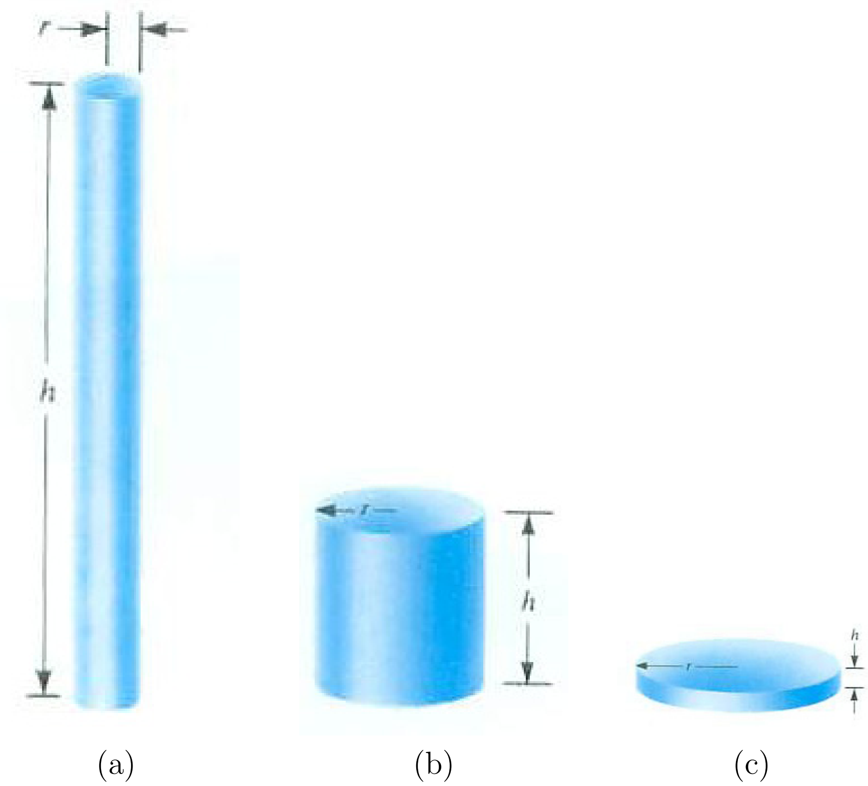


Figure 5.1.9:

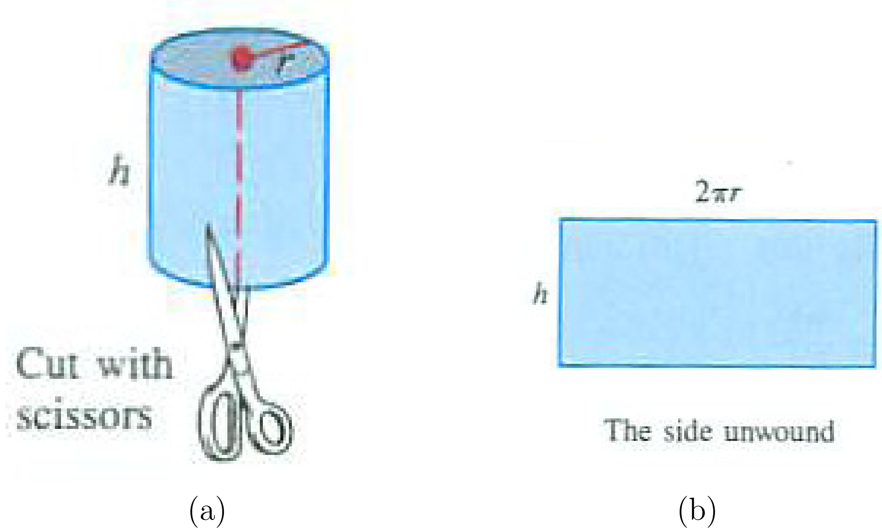


Figure 5.1.10:

which accounts for the two circular bases and the side. Figure 5.1.10 shows why the area of the side is $2\pi rh$. Since the amount of metal in the can is proportional to S, it suffices to minimize S.

Equation (3) gives S as a function of two variables, but we can express one of the variables in terms of the other. In the tin cans under consideration, the radius and height are related by the equation

$$\pi r^2 h = 100, \tag{4}$$

since their volume is 100 cubic inches. In order to express S as a function of one variable, use (4) to eliminate either r or h. Choosing to eliminate h, we solve (4) for h,

$$h = \frac{100}{\pi r^2}.$$

Substitution into (3) yields

$$S = 2\pi r^2 + 2\pi r \frac{100}{\pi r^2} \text{or} S = 2\pi r^2 + \frac{200}{r}. \tag{5}$$

Equation (5) expresses S as a function of just one variable, r.

The domain of this function for our purposes is $(0, \infty)$, since the tin can has a positive radius.

Compute dS/dr:

$$\frac{dS}{dr} = 4\pi r - \frac{200}{r^2} = \frac{4\pi r^3 - 200}{r^2}. \tag{6}$$

Set the derivative equal to 0 to find any critical numbers. We have

In this case the only critical numbers are where the derivative is 0.

$$\begin{array}{lrcl} & 0 & = & \dfrac{4\pi r^3 - 200}{r^2}, \\ \text{hence} & 0 & = & 4\pi r^3 - 200 \\ \text{or} & 4\pi r^3 & = & 200 \\ & r^3 & = & \dfrac{200}{4\pi} \\ & r & = & \sqrt[3]{\dfrac{50}{\pi}}. \end{array}$$

There is only one critical number. Does it provide a minimum? Let's check it two ways, first by the first-derivative test, then by the second-derivative test.

The first derivative is

$$\frac{dS}{dr} = \frac{4\pi r^3 - 200}{r^2}. \tag{7}$$

When $r = \sqrt[3]{50\pi}$, the numerator in (7) is 0. When $r < \sqrt[3]{50\pi}$ the numerator is negative and when $r > \sqrt[3]{50\pi}$ the numerator is positive. (The denominator is always positive.) Since $dS/dr < 0$ for $r < \sqrt[3]{50\pi}$, and $dS/dr > 0$ for $r > \sqrt[3]{50\pi}$, the function $S(r)$ decreases for $r < \sqrt[3]{50\pi}$ and increases for $r > \sqrt[3]{50\pi}$. That shows that a global minimum occurs at $\sqrt[3]{50\pi}$. (See Figure 5.1.11.)

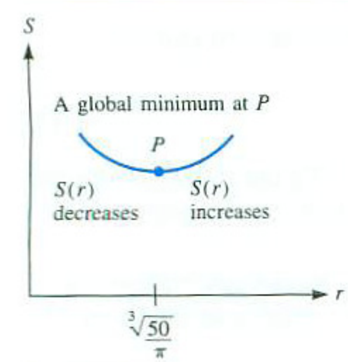


Figure 5.1.11:

Let us instead use the second-derivative test. Differentiation of (6) gives

$$\frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}. \tag{8}$$

Inspection of (8) shows that for all r in $(0, \infty)$, which is the domain that is meaningful for tin cans, d^2S/dr^2 is positive. (The function is concave up as shown in Figure 5.1.12.) Not only is P a relative minimum, it is a global minimum, since the graph lies above its tangents, in particular, the tanget at P.

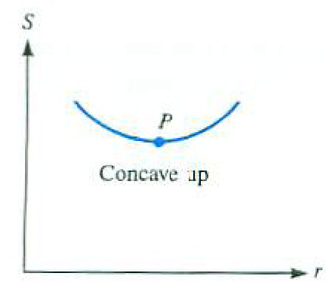


Figure 5.1.12:

The minimum of $S(r)$ is shown in Figure 5.1.13.

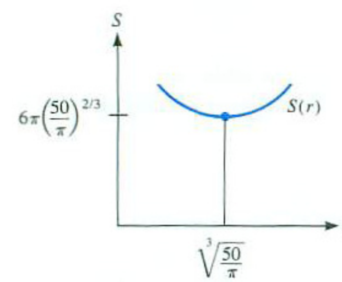


Figure 5.1.13:

To find the height of the most economical can, solve (8) for h:

$$\begin{aligned} h = \frac{100}{\pi r^2} &= \frac{100}{\pi(\sqrt[3]{50/\pi})^2} \\ &= \frac{100}{\pi(\sqrt[3]{50/\pi})^2}\frac{\sqrt[3]{50/\pi}}{\sqrt[3]{50/\pi}} \text{rationalize the denomitor} \\ &= \frac{100}{\pi(50/\pi)}\sqrt[3]{\frac{50}{\pi}} = 2\sqrt[3]{\frac{50}{\pi}}. \end{aligned}$$

The height of the can is equal to twice its radius, that is, its diameter. ⋄

Summary

We showed how to use calculus to solve applied problems: experiment, set up a function, find its domain, and its critical points. Then test the critical points and endpoints of the domain to determine the extrema.

EXERCISES for 5.1

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

Some of these exercises will be moved to the chapter summary.

1. A gardener wants to make a rectangular garden with 100 feet of fence. What is the largest area the fence can enclose?

2. Of all rectangles with area 100 square feet, find the one with the shortest perimeter.

3. Solve Example 1, expressing A in terms of y instead of x.

4. A gardener is going to put a rectangular garden inside one arch of the cosine curve, as shown in Figure 5.1.14. What is the garden with the largest area.

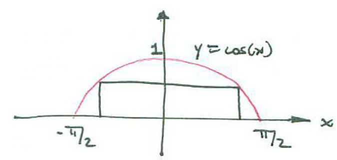


Figure 5.1.14:

Exercises 5 to 8 are related to Example 2. In each case find the length of the cut that maximizes the volume of the tray. The dimensions of the cardboard are given.

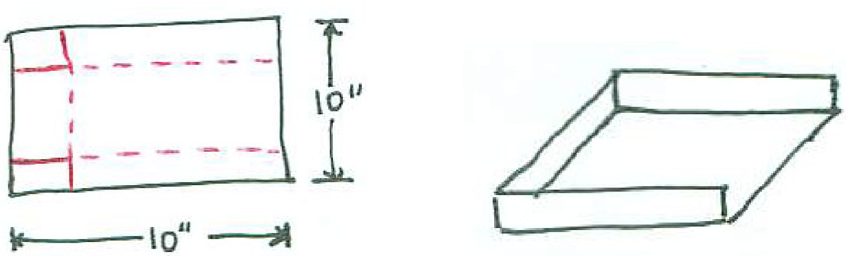
5. 5 inches by 5 inches

6. 5 inches by 7 inches

7. 4 inches by 8 inches,

8. 6 inches by 10 inches,

9. Starting with a square piece of paper 10″ on a side, Sam wants to make a paper



(a) (b)

Figure 5.1.15:

holder with three sides. The pattern he will use is shown in Figure 5.1.15 along with the tray. He will remove two squares and fold up three flaps.

(a) What size square maximizes the volume of the tray?

(b) What is that volume?

10. A chef wants to make a cake pan out of a circular piece of aluminum of radius 12 inches. To do this he plans to cut the circular base from the center of the piece and then cut the side from the remainder. What should the radius and height be to maximize the volume of the pan?

Figure 5.1.16:

11. Solve Example 3, expressing S in terms of h instead of r.

12. Of all cylindrical tin cans *without a top* that contains 100 cubic inches, which requires the least material?

13. Of all enclosed rectangular boxes with square bases that have a volume of 1000 cubic inches, which uses the least material?

14. Of all topless rectangular boxes with square bases that have a volume of 1000 cubic inches, which uses the least material?

15. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius a. The typical rectangle is shown in Figure 5.1.17. HINT: Express the area in terms of the angle θ shown.

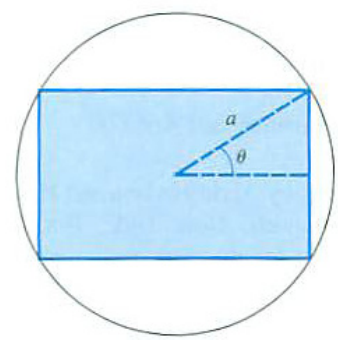


Figure 5.1.17:

16. Solve Exercise 15, expressing the area in terms of half the width of the rectangle, x. HINT: Square the area to avoid square roots.

17. Find the dimensions of the rectangle of largest perimeter that can be inscribed in a circle of radius a.

18. Show that of all rectangles of a given area, the square has the shortest perimeter. *Suggestion:* Call the fixed area A and keep in mind that it is a constant.

19. A rancher wants to construct a rectangular corral. He also wants to divide the corral by a fence parallel to one of the sides. He has 240 feet of fence. What are the dimensions of the corral of largest area he can enclose?

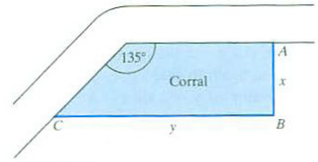


Figure 5.1.18:

20. A river has a 45° turn, as indicated in Figure 5.1.18. A rancher wants to construct a corral bounded on two sides by the river and on two sides by 1 mile of fence ABC, as shown. Find the dimensions of the corral of largest area.

21.

(a) How should one choose two nonnegative numbers whose sum is 1 in order to maximize the sum of their squares?

(b) To minimize the sum of their squares?

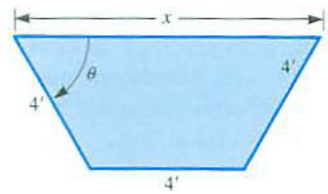


Figure 5.1.19:

22. How should one choose two nonnegative numbers whose sum is 1 in order to maximize the product of the square of one of them and the cube of the other?

23. An irrigation channel made of concrete is to have a cross section in the form of an isosceles trapezoid, three of whose sides are 4 feet long. See Figure 5.1.19. How should the trapezoid be shaped if it is to have the maximum possible area? HINT: Consider the area as a function of x and solve.

24.

(a) Solve Exercise 23 expressing the area as a function of θ instead of x.

(b) Do the answers in (a) and Exercise 23 agree? Explain.

In Exercises 25 to 28 use the fact that the combined length and girth (distance around) of a package to be sent through the mail by the United States Postal Service (USPS) cannot exceed 108 inches. NOTE: The combined length and girth of a packages sent as "parcel post" is 130 inches. The United Parcel Service (UPS) limit is 165 inches for combined length and girth with the length not exceeding 108 inches. Why do you think they have this restriction?

25. Find the dimensions of the right circular cylinder of largest volume that can be sent through the mail.

26. Find the dimensions of the right circular cylinder of largest surface area that can be sent through the USPS.

27. Find the dimensions of the rectangular box with square base of largest volume that can be sent through the USPS.

28. Find the dimensions of the rectangular box with square base of largest surface area that can be sent throught the USPS.

29.

(a) Repeat Exercise 25 with for a package sent by UPS.

(b) Generalize your solutions to Exercise 25 for a packages subject to a combined length and girth that does not exceed M inches.

30.

(a) Repeat Exercise 26 with for a package sent by UPS.

(b) Generalize your solutions to Exercise 26 for a packages subject to a combined length and girth that does not exceed M inches.

Exercises 31 to 37 concern "minimal cost" problems.

31. A cylindrical can is to be made to hold 100 cubic inches. The material for its top and bottom costs twice as much per square inch as the material for its side. Find the radius and height of the most economical can. *Warning:* This is not the same as Example 3.

(a) Would you expect the most economical can in this problem to be taller or shorter than the solution to Example 3? (Use common sense, not calculus.)

(b) For convenience, call the cost of 1 square inch of the material for the side k cents. Thus the cost of 1 square inch of the material for the top and bottom is $2k$ cents. (The precise value of k will not affect the answer.) Show that a can of radius r and height h costs

$$C = 4k\pi r^2 + 2k\pi rh \text{ cents.}$$

(c) Find r that minimizes the functions C in (b). Keep in mind during any differentiation that k is constant.

(d) Find the corresponding h.

32. Sam is at the edge of a circular lake of radius one mile and Jane is at the edge, directly opposite. Sam wants to visit Jane. He can walk 3 miles per hour. He has a canoe. What mix of paddling and walking should Sam use to minimize the time needed to reach Jane if

(a) he paddles at least three miles an hour?

(b) he paddles at 1.5 miles per hour?

(c) he paddles at 2 miles per hour?

33. Consider a right triangle ABC, with C being at the right angle. There are two routes from A to B. One is direct, along the hypotenuse. The other is along the two legs, from A to C and then to B. Now, the shortest path between two points is the straight one. That raises this question: What is the largest percentage saving possible by walking along the hypotenuse instead of along the two legs? For which shape right triangle does this savings occur?

34. A rectangular box with a square base is to hold 100 cubic inches. Material for the top of the box costs 2 cents per square inch; material for the sides costs 3 cents per square inch; material for the bottom costs 5 cents per square inch. Find the dimensions of the most economical box.

35. The cost of operating a certain truck (for gasoline, oil, and depreciation) is $(20 + s/2)$ cents per mile when it travels at a speed of s miles per hour. A truck driver earns $18 per hour. What is the most economical speed at which to operate the truct during a 600 mile trip?

(a) If you considered only the truck, would you want s to be small or large?

(b) If you, the employer, considered only the expense of the driver's wages, would you want s to be small or large?

(c) Express cost as a function of s and solve. (Be sure to put the costs all in terms of cents or all in terms of dollars.)

(d) Would the answer be different for a 1000 mile trip?

36. A government contractor who is removing earth from a large excavation can route trucks over either of two roads. There are $10,000$ cubic yards of earth to move. Each truck holds 10 cubic yards. On one road the cost per truckload is $1+2x^2$ cents, when x trucks use that raod; the function records the cost of congestion. On the other road the cost is $2+x^2$ cents per truckload when x trucks use that road. How many trucks should be dispatched to each of the two roads?

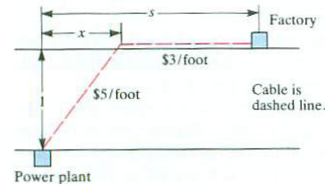


Figure 5.1.20:

37. On one side of a river 1 mile wide is an electric power station; on the other side, s miles upstream, is a factory. (See Figure 5.1.20.) It costs 3 dollars per foot to run cable over land and 5 dollars per foot under water. What is the most economical way to run cable from the station to the factory?

(a) Using no calculus, what do you think would be (approximately) the best route if s were very small? if s were very large?

(b) Solve with the aid of calculus, and draw the routes for $s = \frac{1}{2}$, $\frac{3}{4}$, 1, and 2.

(c) Solve for arbitrary s.

Warning: Minimizing the length of cable is not the same as minimizing its cost.

38. (From *Dynamics of Airplanes*, by John E. Younger and Baldwin M. Woods.) "Recalling that

$$I = A\cos^2\theta + C\sin^2\theta - 2E\cos\theta\sin\theta,$$

we wish to find θ when I is a maximum or a minimum." Show that at an extremum of I,

$$\tan 2\theta = \frac{2\ E}{C-A}.\ (\text{assume that} A \neq C)$$

39. (From a physics text) "By differentiating the equation for the horizontal range,

$$R = \frac{v_0^2\sin(2\theta)}{g},$$

show that the initial elevation angle θ for maximum range is 45°." In the formula for R, v_0 and g are constants. (R is the horizontal distance a baseball covers if you throw it at an angle θ with speed v_0. Air resistance is disregarded.)

(a) Using calculus, show that the maximum range occurs when $\theta = 45°$.

(b) Solve the same problem without calculus.

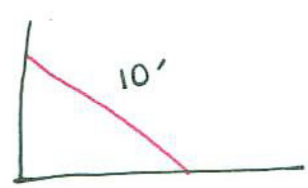
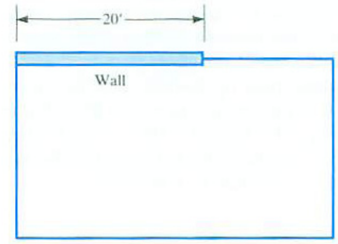


Figure 5.1.21:

40. A gardener has 10 feet of fence and wishes to make a triangular garden next to two buildings, as in Figure 5.1.21. How should he place the fence to enclose the maximum area?



41. Fencing is to be added to an existing wall of length 20 feet, as shown in Figure 5.1.22. How should the extra fence be added to maximum the area of the enclosed rectangle if the additional fence is

(a) 40 feet long?

(b) 80 feet long?

(c) 60 feet long?

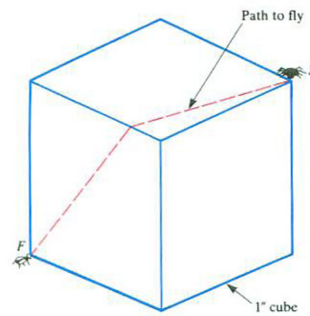


Figure 5.1.23:

42. Let A and B be constants. Find the maximum and mimimum values of $A\cos t + B\sin t$.

43. A spider at corner S of a cube of side 1 inch wishes to capture a fly at the opposite corner F. (See Figure 5.1.23.) The spider, who must walk on the surface of the solid cube, wishes to find the shortest path.

(a) Find a shortest path without the aid of calculus.

(b) Find a shortest path with calculus.

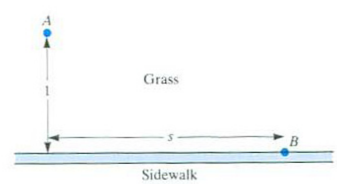


Figure 5.1.24:

44. A ladder of length b leans against a wall of height a, $a < b$. What is the maximal horizontal distance that the ladder can extend beyond the wall if its base rests on the horizontal ground?

45. A woman can walk 3 miles per hour on grass and 5 miles per hour on sidewalk. She wishes to walk from point A to point B, shown in Figure 5.1.24, in the least time. What route should she follow if s is

(a) $\frac{1}{2}$?

(b) $\frac{3}{4}$?

(c) 1?

46. The potential energy in a diatomic molecule is given by the formula

$$U(r) = u_0\left(\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right),$$

where U_0 and r_0 are constants and r is the distance between the atoms. For which value of r is $U(r)$ a minimum?

47. What are the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius a?

48. The stiffness of a rectangular beam is proportional to the product of the width and the cube of the height of its cross section. What shape beam should be cut from a log in the form of a right circular cylinder of radius r in order to maximize its stiffness.

49. A rectangular box-shaped house is to have a square floor. Three times as much heat per square foot enters through the roof as through the walls. What shape should the house be if it is to enclose a volume of 12,000 cubic feet and minimize heat entry. (Assume no heat enters through the floor.)

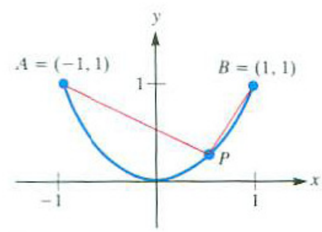


Figure 5.1.25:

50. (See Figure 5.1.25.) Find the coordinates of the points $P = (x, y)$, with $y \leq 1$, on the parabola $y = x^2$, that

(a) minimize $\overline{PA}^2 + \overline{PB}^2$,

(b) maximize $\overline{PA}^2 + \overline{PB}^2$.

51. The speed of traffic through the Lincoln Tunnel in New York City depends on the amount of traffic. Let S be the speed in miles per hour and let D be the amount of traffic measured in vehicles per mile. The relation between S and D was seen to be approximated closely, for $D \leq 100$, by the formula

$$S = 42 - \frac{D}{3}.$$

(a) Express in terms of S and D the total number of vehicles that enter the tunnel in an hour.

(b) What value of D will maximize the flow in (a)?

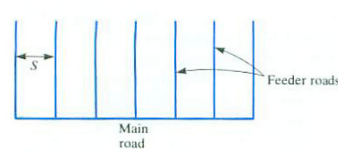


Figure 5.1.26:

52. When a tract of timber is to be logged, a main logging road is built from which small roads branch off as feeders. The question of how many feeders to build arises in practice. If too many are built, the cost of construction would be prohibitive. If too few are built, the time spent moving the logs to the roads would be prohibitive. The formula for total cost,

$$y = \frac{CS}{4} + \frac{R}{VS},$$

is used in a logger's manual to find how many feeder roads are to be built. R, C, and V are known constants: R is the cost of road at "unit spacing"; C is the cost of moving a log a unit distance; V is the value of timber per acre. S denotes the distance between the regularly spaced feeder roads. (See Figure 5.1.26.) Thus the cost y is a function of S, and the object is to find that value of S that minimizes y. The manual says, "To find the desired S set the two summands equal to each other and solve

$$\frac{CS}{4} = \frac{r}{VS}."$$

Show that the method if valid.

53. A delivery service is deciding how many warehouses to set up in a large city. The warehouses will serve similarly shaped regions of equal area A and, let us assume, an equal number of people.

(a) Why would transportation costs per item presumably be proportional to $\sqrt{A}$?

(b) Assuming that the warehouse cost per item is inversely proportional to A, show that C, the cost of transportation and storage per item, is of the form $t\sqrt{A} + w/A$, where t and w are appropraite constants.

(c) Show that C is a minimum when $A = (2w/t)^{2/3}$.

Exercises 54 and 55 are related.

54. A pipe of length b is carried down a long corridor of width $a < b$ and then around corner C. (See Figure 5.1.27.) During the turn y starts out at 0, reaches a maximum, and then returns to 0. (Try this with a short stick.) Find that maximum in terms of a and b. *Suggestion:* Express y in terms of a, b, and θ; θ is a variable, while a and b are constants.

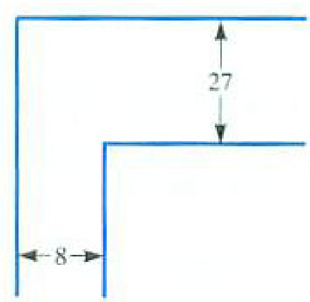
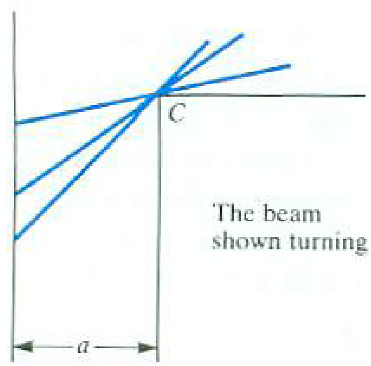


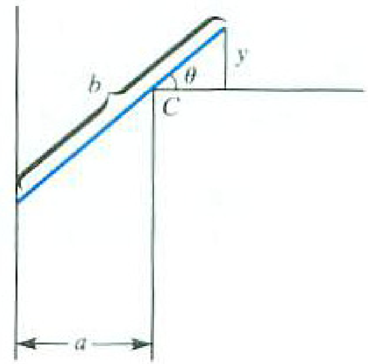
Figure 5.1.28:

55. Figure 5.1.28 shows two corridors meeting at right angle. One has width 8; the other, width 27. Find the length of the longest pipe that can be carried horizontally from one hall, around the corner and into the other hall. *Suggestion:* Do Exercise 54 first.

56. Two houses, A and B, are a distance p apart. They are distances q and r, respectively, from a straight road, and on the same side of the road. Find the length



(a)



(b)

Figure 5.1.27:

of the shortest path that goes from A to the road, and then on to the other house B.

(a) Use calculus.

(b) Use only elementary geometry. *Hint:* Introduce an imaginary house C such that the midpoint of B and C is on the road and the segment BC is perpendicular to the road; that is, "reflect" B across the road to become C.

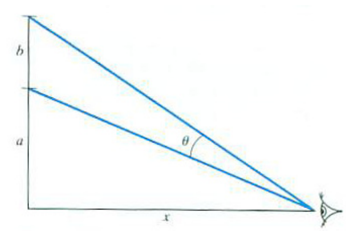


Figure 5.1.29:

57. The base of a painting on a wall is a feet above the eye of an observer, as shown in Figure 5.1.29. The vertical side of the painting is b feet long. How far from the wall should the ovserver stand to maximize the angle that the painting subtends? *Hint:* It is more convenient to maximize $\tan\theta$ than θ itself. HINT: Recall that $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.

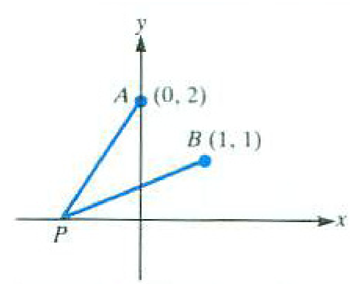
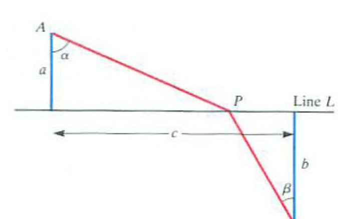


Figure 5.1.30:

58. Find the point P on the x axis such that the angle APB in Figure 5.1.30 is maximal. *Suggestion:* Note hint in Exercise 57.

59. *(Economics)* Let p denote the price of some commodity and y the number sold at that price. To be concrete, assume that $y = 250 - p$ for $0 \leq p \leq 250$. Assume that it costs the producer $100 + 10y$ dollares to manufacture y units. What price p should the producer choose in order to maximize total profit, that is, "revenue minus cost"?

60. *(Leibniz on light)* A ray of light travels from point A to point B in Figure 5.1.31



in minimal time. The point A is in one medium, such as air or a vacuum. The point B is in another medium, such as water or glass. In the first medium, light travels at velocity v_1 and in the second at velocity v_2. The media are separated by line L. Show that for the path APB of minimal time,

$$\frac{\sin\alpha}{v_1} = \frac{\sin(\beta)}{v_2}.$$

Leibniz solved this problem with calculus in a paper published in 1684. (The result is called **Snell's law of refraction**.)

Leibniz then wrote, "other very learned men have sought in many devious ways what someone versed in this calculus can accomplish in these lines as by magic." (See C. H. Edwards Jr., *The Historical Development of the Calculus*, p. 259, Springer-Verlag, New York, 1979.)

61. The following calculation occurs in an article by Manfred Kochen, "On Determining Optimum Size of New Cities": The net utility to the total client-centered system is

$$U = \frac{RLv}{A} n^{1/2} - nK - \frac{ALc}{v} n^{-1/2}.$$

All symbols except U and n are constant; n is a measure of decentralization. Regarding U as a differentiable function of n, we can determine when $dU/dn = 0$. This occurs when

$$\frac{RLv}{2A} n^{-1/2} - K + \frac{ALc}{2v} n^{-3/2} = 0.$$

This is a cubic equation for $n^{-1/2}$.

(a) Check that the differentiation is correct.

(b) Of what cubic polynomial is $n^{-1/2}$ a root?

62. Consider the curve $y = x^{-2}$ in the first quadrant. A tangent to this curve, together with axes, determine a triangle.

(a) What is the largest area of such a triangle?

(b) The smallest area?

63. Let f be a differentiable function that is never zero on its domain. Define the function g so that $g(x) = (f(x))^2$. Show that the functions f and g have the same critical numbers.

64. Let f be a differentiable function. Define the function g by $g(x) = \tan(f(x))$. Show that the functions f and g have the same critical numbers.

65. Let f and g be two differentiable functions. Define F to be the composition of f and g: $F(x) = f(g(x))$. Under what additional condition on g' do f and F have the same critical numbers?

5.2 Implicit Differentiation and Related Rates

Sometimes a function $y = f(x)$ is given indirectly by an equation that involves y and x. This section shows how to differentiate y without solving for y explicitly in terms of x.

We will apply this technique to determine how the rate at which one quantity changes influences the rate at which another quantity changes.

A Function Given Implicitly

Consider the equation

$$x^2 + y^2 = 25. \tag{1}$$

This equation describes a circle of radius 5 and center at the origin, as in Figure 5.2.1. This circle is not the graph of a function, since a vertical line can meet the circle in two points. However, the top half is the graph of a function and so is the bottom half. To find these functions explicitly, solve (1) for y:

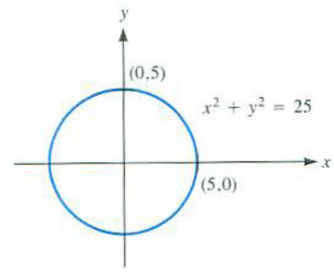


Figure 5.2.1:

$$\begin{aligned} y^2 &= 25 - x^2 \\ y &= \pm\sqrt{25 - x^2}. \end{aligned}$$

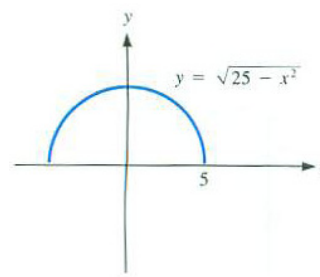


Figure 5.2.2:

So either $y = \sqrt{25 - x^2}$ or $y = -\sqrt{25 - x^2}$. The graph of $y = \sqrt{25 - x^2}$ is the top semicircle (see Figure 5.2.2); the graph of $y = -\sqrt{25 - x^2}$ is the bottom semicircle (see Figure 5.2.3). There are thus two continuous functions that satisfy (1).

The equation $x^2 + y^2 = 25$ is said to describe the function $y = f(x)$ **implicitly**. The equations

$$y = \sqrt{25 - x^2} \qquad \text{and} \qquad y = -\sqrt{25 - x^2}$$

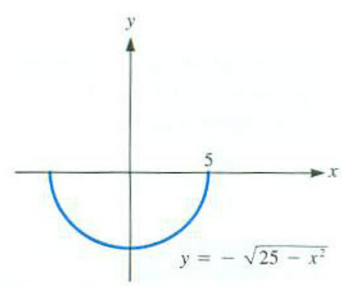


Figure 5.2.3:

describe the function $y = f(x)$ **explicitly**.

Differentiating an Implicit Function

It is possible to differentiate a function given implicitly without having to solve for the function and express it explicitly. An example will illustrate the method, which is to differentiate both sides of the equation that defines the function implicitly. This procedure is called **implicit differentiation**.

EXAMPLE 1 Let $y = f(x)$ be the continuous function that satisfies the equation

$$x^2 + y^2 = 25$$

such that $y = 4$ when $x = 3$. Find dy/dx when $x = 3$ and $y = 4$.
SOLUTION (We could, of course, solve for y, $y = \sqrt{25 - x^2}$, and differentiate directly. However, the algebra would be more involved since square roots would appear.) Differentiating both sides of the equation

$$x^2 + y^2 = 25$$

with respect to x yields

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(25), \\ 2x + \frac{d(y^2)}{dx} &= 0. \end{aligned}$$

To differentiate y^2 with respect to x, write $w = y^2$, where y is a function of x.

By the chain rule	$\frac{dw}{dx}$	$=$	$\frac{dw}{dy}\frac{dy}{dx}$,
which gives us	$\frac{d(y^2)}{dx}$	$=$	$2y\frac{dy}{dx}$.
Thus	$2x + 2y\frac{dy}{dx}$	$=$	0,
or	$x + y\frac{dy}{dx}$	$=$	$= 0$.
In particular, when $x = 3$ and $y = 4$,	$3 + 4\frac{dy}{dx}$	$=$	0,
and therefore,	$\frac{dy}{dx}$	$=$	$-\frac{3}{4}$.

◇

Observe that the algebra involves no square roots.

If you look back at Section 3.5, you will see that we used implicit differentiation to find derivatives of inverse functions. For instance, we differentiated both sides of $y = e^x$ with respect to y, obtaining $1 = e^x(dx/dy)$. Then $dx/dy = 1/e^x = 1/y$. In short, $D(\ln(y)) = 1/y$.

In the next example implicit differentiation is the only way to find the derivative, for in this case there is no formula expressible in terms of trigonometric and algebraic functions giving y explicitly in terms of x.

EXAMPLE 2 Assume that the equation

$$2xy + \pi \sin(y) = 2\pi$$

defines a function $y = f(x)$. Find dy/dx when $x = 1$ and $y = \pi/2$.

Verify that the equation is satisfied when $x = 1$ and $y = \pi/2$.

SOLUTION Implicit differentiation yields

$$\begin{aligned}
\tfrac{d}{dx}(2xy + \pi \sin y) &= \tfrac{d(2\pi)}{dx}, \\
\left(2x\tfrac{dy}{dx} + 2\tfrac{dx}{dx}y\right) + \pi(\cos y)\tfrac{dy}{dx} &= 0,
\end{aligned}$$

by the formula for the derivative of a product and the chain rule. Hence

$$2x\tfrac{dy}{dx} + 2y + \pi(\cos y)\tfrac{dy}{dx} = 0.$$

Solving for the derivative, dy/dx, we get

$$\tfrac{dy}{dx} = \tfrac{-2y}{2x+\pi\cos y}.$$

In particular, when $x = 1$ and $y = \pi/2$,

$$\begin{aligned}
\tfrac{dy}{dx} &= -\tfrac{2\cdot\frac{\pi}{2}}{2\cdot 1+\pi\cos\frac{\pi}{2}} \\
&= -\tfrac{\pi}{2+\pi\cdot 0} = -\tfrac{\pi}{2}.
\end{aligned}$$

◇

Implicit Differentiation and Extrema

Example 3 of Section 5.1 answered the question, "Of all the tin cans that enclose a volume of 100 cubic inches, which requires the least metal?" The radius of the most economical can is $\sqrt[3]{50/\pi}$. From this and the fact that its volume is 100 cubic inches, its height was found to be $2\sqrt[3]{50/\pi}$, exactly twice the radius. In the next example implicit differentiation is used to answer the same question. Not only will the algebra be simpler but it will provide the shape – the proportion between height and radius – easily.

EXAMPLE 3 Of all the tin cans that enclose a volume of 100 cubic inches, which requires the least metal?

SOLUTION The height h and radius r of any can of volume 100 cubic inches are related by the equation

$$\pi r^2 h = 100. \tag{2}$$

The surface area S of the can is

$$S = 2\pi r^2 + 2\pi r h \tag{3}$$

Consider h, and hence S, as functions of r. Differentiation of (2) and (3) with respect to r yields

It is not necessary to find h and S explicitly in terms of r.

$$\pi r^2 \frac{dh}{dr} + 2\pi r h = \frac{d(100)}{dr} = 0 \tag{4}$$

and

$$\frac{dS}{dr} = 4\pi r + 2\pi r \frac{dh}{dr} + 2\pi h. \tag{5}$$

When S is a minimum, $dS/dr = 0$, so we have

$$0 = 4\pi r + 2\pi r \frac{dh}{dr} + 2\pi h. \tag{6}$$

Equations (4) and (6) yield, with a little algebra, a relation between h and r, as follows:

Factoring πr out of (4) and 2π out of (6) shows that

$$r\frac{dh}{dr} + 2h = 0 \qquad \text{and} \qquad 2r + r\frac{dh}{dr} + h = 0. \tag{7}$$

Elimination of dh/dr from (7) yields

$$2r + r\left(\frac{-2h}{r}\right) + h = 0,$$

which simplifies to

$$2r = h. \tag{8}$$

This approach obtains the shape before the specific dimensions are used.

Equation (8) asserts that the height of the most economical can is the same as its diameter. Moreover, this is the ideal shape, no matter what the prescribed volume happens to be.

The specific dimensions of the most economical can are found by eliminating h from equations (2) and (4). This shows that

$$\pi r^2(2r) = 100 \qquad \text{or} \qquad r^3 = \frac{50}{\pi}.$$

Hence

$$r = \sqrt[3]{\frac{50}{\pi}} \qquad \text{and} \qquad h = 2r = 2\sqrt[3]{\frac{50}{\pi}}$$

◇

The procedure illustrated in Example 3 is quite general. It may be of use when maximizing (or minimizing) a quantity that at first is expressed as a function of two variable which are linked by an equation. The equation that links them is called the **constraint**. In Example 3, the constraint is $\pi r^2 h = 100$.

General procedure for using implicit differentiation in an applied extremum problem.

Using Implicit Differentiation in an Extremum Problem

1. Name the various quantities in the problem by letters, such as x, y, h, r, A, V.
2. Identify which quantity is to be maximized (or minimized).
3. Express the quantity to be maximized (or minimized) in terms of other quantities, such as x and y.
4. Obtain an equation relating x and y.
   (This equation is called a constraint.)
5. Differentiate implicity both the constraint and the expression to be maximized (or minimized), interpreting all quantities to be functions of a single variable (which you choose).
6. Set the derivative of the expression to be maximized (or minimized) equal to 0 and combine with the derivative of the constraint to obtain an equation relating x and y at a maximum (or minimum).
7. Step 6 gives only a relation or proportion between x and y at an extremum. If the explicit values of x and y are desired, find them by using the fact that x and y also satisfy the constraint.

Warning: Sometimes an extremum occurs where a derivative, such as dy/dx, is not defined.

Exercise 22 illustrates this possibility.

Related Rates

Implicit differentiation also comes in handy when showing how the rate of change of one quantity affects the rate of change of another.

EXAMPLE 4 An angler has a fish at the end of his line, which is reeled in at 2 feet per second from a bridge 30 feet above the water. At what speed is the fish moving through the water when the amount of line out is 50 feet? 31 feet? Assume the fish is at the surface of the water. (See Figure 5.2.4.)

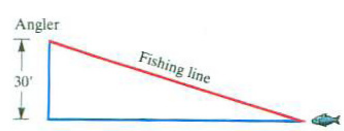


Figure 5.2.4:

SOLUTION Our first impression might be that since the line is reeled in at a constant speed, the fish at the end of the line moves through the water at a constant speed. As we will see, this is not the case.

Let s be the length of the line and x the horizontal distance of the fish from the bridge. (See Figure 5.2.5.)

Figure 5.2.5:

Since the line is reeled in at the rate of 2 feet per second, s is shrinking, and

$$\frac{ds}{dt} = -2.$$

The rate at which the fish moves through the water is given by the derivative, dx/dt. The problem is to find dx/dt when $s = 50$ and also when $s = 31$.

We need an equation that relates s and x at *any* time, not just when $x = 50$ or $x = 31$. If we consider only $x = 50$ or $x = 31$, there would be no motion, and no chance to use derivatives.

This equation is the heart of the example.

The quantities x and s are related by the Pythagorean Theorem:

$$x^2 + 30^2 = s^2.$$

Both x and s are functions of time t. Thus both sides of the equation may be differentiated with respect to t, yielding

$$\begin{array}{lrcl} & \frac{d(x^2)}{dt} + \frac{d(30^2)}{dt} & = & \frac{d(s^2)}{dt} \\ \text{or} & 2x\frac{dx}{dt} + 0 & = & 2s\frac{ds}{dt}. \\ \text{Hence} & x\frac{dx}{dt} & = & s\frac{ds}{dt}. \end{array}$$

This last equation provides the tool for answering the questions.

Since $ds/dt = -2$,

$$\begin{array}{lrcl} & x\frac{dx}{dt} & = & (s)(-2). \\ \text{Hence} & \frac{dx}{dt} & = & \frac{-2s}{x}. \\ \text{When } s = 50, & x^2 + 30^2 & = & 50^2, \end{array}$$

so $x = 40$. Thus when 50 feet of line is out, the speed is

$$\left|\frac{dx}{dt}\right| = \frac{2s}{x} = \frac{2 \cdot 50}{40} = 2.5 \text{ feet per second.}$$

$$\begin{array}{lrcl} \text{When } s = 31, & x^2 + 30^2 & = & 31^2. \\ \text{Hence} & x = \sqrt{31^2 - 30^2} & = & \sqrt{961 - 900} = \sqrt{61}. \end{array}$$

Thus when 31 feet of line is out, the fish is moving at the speed of

$$\frac{dx}{dt} = \frac{2s}{x} = \frac{2 \cdot 31}{\sqrt{61}} = \frac{62}{\sqrt{61}} \approx 7.9 \text{ feet per second.}$$

Let us look at the situation from the fish's point of view. When it is x feet from the point in the water directly below the bridge, its speed is $2s/x$ feet per second. Since s is larger than x, its speed is always greater than 2 feet per second. When x is very large, s/x is near 1 so the fish is moving through the water only a little faster than the line is reeled in. However, when the fish is almost at the point under the bridge, x is very small; then $2s/x$ is huge, and the fish finds itself moving at huge speeds, but according to Einstein, not faster than the speed of light. ⋄

Label all the lengths or quantities that can change with letters x, y, s, and so on, even if not all are needed in the solution. Only after you finish differentiating do you determine what the rates are at a specified value of the variable.

In Example 4 it would be a tactical mistake to indicate in Figure 5.2.5 that the hypotenuse of the triangle is 50 feet long, for if one leg is 30 feet and the hypotenuse is 50 feet, the triangle is determined; there is nothing left free to vary with time.

The General Procedure

The method used in Example 4 applies to many related rate problems. This is the general procedure, broken into steps:

Warning: Differentiate, then substitute the specific numbers for the variables. If you reversed the order, you would just be differentiating constants.

Procedure for Finding a Related Rate

1. Find an equation that relates to the varying quantities.
   (If the quantities are geometric, draw a picture and label the varying quantities with letters.)

2. Differentiate both sides of the equation with respect to time, obtaining an equation that relates the various rates of change.

3. Solve the equation obtained in Step 2 for the unknown rate.
   (Only at this step do you substitute constants for variable.)

Finding an Acceleration

The method described in Example 4 for determining unknown rates from known ones extends to finding an unknown acceleration. Just differentiate another time. Example 5 illustrates the procedure.

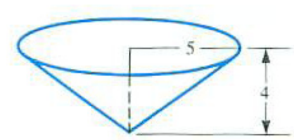


Figure 5.2.6:

EXAMPLE 5 Water flows into a conical tank at the constant rate of 3 cubic meters per second. The radius of the cone is 5 meters and its height is 4 meters. Let $h(t)$ represent the height of the water above the bottom of the cone at time t. Find dh/dt (the rate at which the water is rising in the tank) and d^2h/dt^2 (the rate at which that rate changes) when the tank is filled to a height of 2 meters. (See Figure 5.2.6 and Figure 5.2.7.)

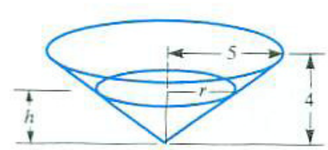


Figure 5.2.7:

SOLUTION Let $V(t)$ be the volume of water in the tank at time t. The fact that water flows into the tank at 3 cubic meters per second is expressed as

$$\frac{dV}{dt} = 3,$$

and, since this rate is constant,

$$\frac{d^2V}{dt^2} = 0.$$

To find dh/dt and d^2h/dt^2, first obtain an equation relating V and h.

When the tank is filled to the height h, the water forms a cone of height h and radius r. (See Figure 5.2.7.) By similar triangles,

$$\frac{r}{h} = \frac{5}{4} \qquad \text{or} \qquad r = \frac{5h}{4}.$$

Thus

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \left(\frac{5}{4}h\right)^2 h \\ &= \frac{25}{48}\pi h^3. \end{aligned}$$

The equation relating V and h is

$$V = \frac{25\pi}{48} h^3. \tag{9}$$

From here on, just differentiate as often as needed.

Differentiating both sides of (9) once (using the chain rule) yields

$$\frac{dV}{dt} = \frac{25\pi}{48} \frac{d(h^3)}{dh} \frac{dh}{dt}$$

or

$$\frac{dV}{dt} = \frac{25\pi}{16} h^2 \frac{dh}{dt}.$$

Since $dV/dt = 3$ all the time,

$$3 = \frac{25\pi h^2}{16} \frac{dh}{dt},$$

from which it follows that

$$\frac{dh}{dt} = \frac{48}{25\pi h^2} \text{ meters per second.} \tag{10}$$

As (10) shows, the larger h is, the slower the water rises. (Why is this to be expected?)

Even though the water enters the tank at a constant rate, it does not rise at a constant rate.

To find dh/dt when $h = 2$ meters, substitute 2 for h in (10), obtaining

$$\frac{dh}{dt} = \frac{48}{25\pi 2^2} = \frac{12}{25\pi} \approx 0.15279 \text{ meters per second.}$$

Now we turn to the acceleration, d^2h/dt^2. *We do not* differentiate the equation $dh/dt = 12/(25\pi)$ since this equation holds only when $h = 2$. We must go back to (10), which holds at any time.

Differentiating (10) with respect to t yields

$$\begin{aligned} \frac{d^2h}{dt^2} &= \frac{48}{25\pi}\frac{d}{dt}\left(\frac{1}{h^2}\right), \\ \frac{d^2h}{dt^2} &= \frac{48}{25\pi}\frac{-2}{h^3}\frac{dh}{dt}, \end{aligned}$$

or

$$\frac{d^2h}{dt^2} = \frac{-96}{25\pi h^3}\frac{dh}{dt}. \tag{11}$$

The last equation expresses the acceleration in terms of h and dh/dt. Substituting (10) into (11) gives

$$\frac{d^2h}{dt^2} = \frac{-96}{25\pi h^3}\frac{48}{25\pi h^2}$$

or

$$\frac{d^2h}{dt^2} = \frac{-(96)(48)}{(25\pi)^2 h^5} \text{ meters per second per second.} \tag{12}$$

Equation (12) tells us that, since d^2h/dt^2 is negative, the rate at which the water rises in the tank is decreasing.

The problem also asked for the value of d^2h/dt^2 when $h = 2$. To find that value, replace h by 2 in (12), obtaining

$$\frac{d^2h}{dt^2} = \frac{-(96)(48)}{(25\pi)^2 2^5}$$

or

$$\frac{d^2h}{dt^2} = \frac{-144}{625\pi^2} \approx -0.02334 \text{ meters per second per second.}$$

◇

Logarithmic Differentiation

If $\ln(f(x))$ is simpler than $f(x)$, there is a technique for finding $f'(x)$ that saves labor. Example 6 illustrates this method, which depends on implicit differentiation.

EXAMPLE 6 Let $y = \frac{\cos(3x)}{\left(\sqrt[3]{x^2+5}\right)^4}$. Find $\frac{dy}{dx}$.

SOLUTION The solution to this problem by **logarithmic differentiation** begins by simplifying $\ln(y)$ using the properties of logarithms:

$$\begin{aligned} \ln(y) &= \ln\left(\cos(3x)\right) - \ln\left(\left(\sqrt[3]{x^2+5}\right)^4\right) && [\ln(A/B) = \ln(A) - \ln(B)] \\ &= \ln\left(\cos(3x)\right) - \tfrac{4}{3}\ln\left(x^2+5\right) && [\ln(A^B) = B\ln(A)]. \end{aligned}$$

Next, since $\frac{d}{dx}(\ln(y)) = \frac{1}{y}\frac{dy}{dx}$ by the Chain Rule, we have

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}\left(\ln\left(\cos(3x)\right) - \tfrac{4}{3}\ln\left(x^2+5\right)\right) \\ &= \frac{-3\sin(3x)}{\cos(3x)} - \frac{4}{3}\frac{2x}{x^2+5}. \end{aligned}$$

Therefore

$$\frac{dy}{dx} = (y)\left(-3\tan(3x) - \frac{4}{3}\frac{2x}{x^2+5}\right).$$

Finally, replace y by its formula, getting

$$\frac{dy}{dx} = \frac{\cos(3x)}{\left(\sqrt[3]{x^2+5}\right)^4}\left(-3\tan(3x) - \frac{4}{3}\frac{2x}{x^2+5}\right).$$

To appreciate logarithmic differentiation, get the derivative directly, as requested in Exercise 51. ◇

If you want to differentiate $\ln(f(x))$ for some function f, first see if you can simplify the expression by using the properties of a logarithm.

Properties of Logarithms

$\ln(AB) = \ln(A) + \ln(B)$ $(A > 0,\ B > 0)$

$\ln(A/B) = \ln(A) - \ln(B)$ $(A > 0,\ B > 0)$

$\ln(A^B) = B\ln(A)$ $(A > 0)$

Summary

We described "implicit differentiation," in which you differentiate a function without having an explicit formula for it. The function appears in an equation linking it and another variable. To find its derivative, just differentiate both sides of the equation, carefully using the chain rule.

We applied these techniques in findng extrema and the relation between the rates of change of quantities linked by an equation. We also saw how the properties of logarithms can simplify finding the derivatives of some functions, particularly those involving products, quotients, and powers.

EXERCISES for 5.2

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In Exercises 1 to 4 find dy/dx at the indicated values of x and y in two ways: explicitly (solving for y first) and implicitly.

1. $xy = 4$ at $(1, 4)$

2. $x^2 - y^2 = 3$ at $(2, 1)$

3. $x^2y + xy^2 = 12$ at $(3, 1)$

4. $x^2 + y^2 = 100$ at $(6, -8)$

/eexgrp

In Exercises 5 to 8 find dy/dx at the given points by implicit differentiation.

5. $\frac{2xy}{\pi} + \sin y = 2$ at $(1, \pi/2)$

6. $2y^3 + 4xy + x^2 = 7$ at $(1, 1)$

7. $x^5 + y^3x + yx^2 + y^5 = 4$ at $(1, 1)$

8. $x + \tan(xy) = 2$ at $(1, \pi/4)$

9. Solve Example 3 by implicit differentiation, but differentiate (2) and (3) with respect to h instead of r.

10. What is the shape of the cylindrical can of largest volume that can be constructed with a given surface area? Do not find the radius and height of the largest can; find the ration between them. *Suggestion:* Call the surface area S and keep in mind that it is constant.

11. Using implicit differentiation, find $D(\arctan x)$. *Hint:* Start with $x = \tan(y)$.

12. Using implicit differentiation, find $D(\arcsin x)$. *Hint:* Start with $x = \sin(y)$.

In Exercises 13 to 16 find dy/dx at a general point (x, y) on the given curve.

13. $xy^3 + \tan(x+y) = 1$

14. $\sec(x+2y) + \cos(x-2y) + y = 2$

15. $-7x^2 + 48xy + 7y^2 = 25$

16. $\sin^3(xy) + \cos(x+y) + x = 1$

17. Assume that $y(x)$ is a differentiable function of x and that $x^3y + y^4 = 2$. Assume that $y(1) = 1$. Find $y''(1)$, following these steps.

(a) Show that $x^3y' + 3x^2y + 4y^3y' = 0$.

(b) Use (a) to find $y'(1)$.

(c) Differentiate the equation in (a) and thereby show that $x^3y'' + 6x^2y' + 6xy + 4y^3y'' + 12y^2(y')^2 = 0$.

(d) Use the equation in (c) to find $y''(1)$. [*Hint:* $y(1)$ and $y'(1)$ are known.]

18. Find $y''(1)$ if $y(1) = 2$ and $x^5 + xy + y^5 = 35$.

19. Find $y'(1)$ and $y''(1)$ if $y(1) = 0$ and $\sin y = x - x^3$.

20. Find $y''(2)$ if $y(2) = 1$ and $x^3 + x^2y - xy^3 = 10$.

21. Use implicit differentiation to find the highest and lowest points on the ellipse $x^2 + xy + y^2 = 12$. HINT: What do you know about dy/dx at the highest and lowest points on the graph of a function?

22.

(a) What difficulty arises when you use implicit differentiation to maximize x^2+y^2 subject to $x^2 + 4y^2 = 16$?

(b) Show that a maximum occurs when dy/dx is not defined. What is the maximum of $x^2 + y^2$ subject to $x^2 + 4y^2 = 16$?

(c) The problem can be viewed geometrically as "Maximize $x^2 + y^2$ for points on the ellipse $x^2 + 4y^2 = 16$." Sketch the ellipse and interpret (b) in terms of it.

23. How fast is the fish in Example 4 moving through the water when it is 1 foot horizontally from the bridge?

24. The angler in Example 4 decides to let the line out as the fish swims away. The fish swims away at a constant speed of 5 feet per second relative to the water. How fast is the angler paying out his line when the horizontal distance from the bridge to the fish is

(a) 1 foot?

(b) 100 feet?

25. A 10-foot ladder is leaning against a wall. A person pulls the base of the ladder away from the wall at the rate of 1 foot per second.

(a) Draw a neat picture of the situation and label the varying lengths by letters and the fixed lengths by numbers.

(b) Obtain an equation involving the variables in (a).

(c) Differentiate it with respect to time.

(d) How fast is the top going down the wall when the base of the ladder is 6 feet from the wall? 8 feet from the wall? 9 feet from the wall?

26. A kite is flying at a height of 300 feet in a horizontal wind.

(a) Draw a neat picture of the situation of label the varying lengths by letters and the fixed lengths by numbers.

(b) When 500 feet of string is out, the kite is pulling the string out at a rate of 20 feet per second. What is the kite's velocity? (Assume the string remains straight.)

27. A beachcomber walks 2 miles per hour along the shore as the beam from a rotating light 3 miles offshore follows him. (See Figure 5.2.8.)

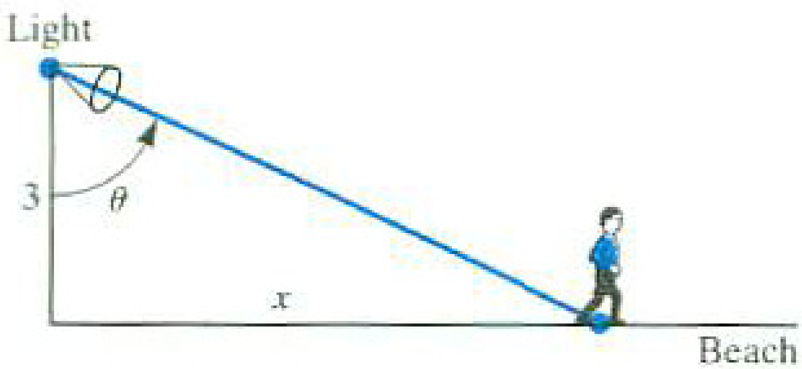


Figure 5.2.8:

(a) Intuitively, what do you think happens to the rate at which the light rotates as the beachcomber walks further and further along the shore away from the lighthouse?

(b) Let x describe the distance of the beachcomber from the point on the shore nearest the light and θ the angle of the light, obtain an equation relating θ and x.

(c) With the aid of (b), show that $d\theta/dt = 6/(9 + x^2)$ (radians per hour).

(d) Does the formula in (c) agree with your guess in (a)?

28. A man 6 feet tall walks at the rate of 5 feet per second away from a street lamp that is 20 feet high. At what rate is his shadow lengthening when he is

(a) 10 feet from the lamp?

(b) 100 feet from the lamp?

29. A large spherical balloon is being inflated at the rate of 100 cubic feet per minute. At what rate is the radius increasing when the radius is

(a) 10 feet?

(b) 20 feet?

(The volume of a sphere of radius r is $V = 4\pi r^3/3$.)

30. A shrinking spherical balloon loses air at the rate of 1 cubic inch per second. At what rate is its radius changing when the radius is

(a) 2 inches

(b) 1 inch?

31. Bulldozers are moving earth at the rate of $1,000$ cubic yards per hour onto a conically shaped hill whose height of the hill increasing when the hill is

(a) 20 yards high?

(b) 100 yards high?

(The volume of a cone of radius r and height h is $V = \pi r^2 h/3$.)

32. The lengths of the two legs of a right triangle depend on time. One leg, whose length is x, increaes at the rate of 5 feet per second, while the other, of length y, decreases at the rate of 6 feet per second. At what rate is the hypotenuse changing when $x = 3$ feet and $y = 4$ feet? Is the hypotenuse increasing or decreasing then?

33. Two sides of a triangle and their included angle are changing with respect to time. The angle increases at the rate of 1 radian per second, one side increases at the rate of 3 feet per second, and the other side decrease at the rate of 2 feet per second. Find the rate at which the area is changing when the angle is $\pi/4$, the first side is 4 feet long, and the second side is 5 long. Is the area decreasing or increasing then?

34. The length of a rectangle is increasing at the rate of 7 feet per second, and the width is decreasing at the rate of 3 feet per second. When the length is 12 feet and the width is 5 feet, find the rate of change of

(a) the area,

(b) the perimeter

(c) the length of the diagonal.

Exercises 35 to 38 concern acceleration.

35. What is the acceleration of the fish described in Example 4 when the length of line is

(a) 300 feet?

(b) 31 feet?

NOTE: The notation $\dot{x}$ for dx/dt, $\dot{\theta}$ for $d\theta/dt$, $\ddot{x}$ for d^2x/dt^2, and $\ddot{\theta}$ for $d^2\theta/dt^2$ was introduced by Newton and is still common in physics.

36. Find $\ddot{\theta}$ in Example 5 when the horizontal distance from the jet is

(a) 7 miles,

(b) 1 mile.

37. A particle moves on the parabola $y = x^2$ in such a way that $\dot{x} = 3$ throughout the journey. Find the formulas for

(a) $\dot{y}$ and

(b) $\ddot{y}$.

38. Call one acute angle of a right triangle θ. The adjacent leg has length x and the opposite leg has length y.

(a) Obtain an equation relating x, y and θ.

(b) Obtain an equation involving $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ (and other variables).

(c) Obtain an equation involving $\ddot{x}$, $\ddot{y}$, and $\ddot{\theta}$ (and other variables).

39. A two-piece extension ladder leaning against a wall is collapsing at the rate of 2 feet per second and the base of the ladder is moving away from the wall at the rate of 3 feet per second. How fast is the top of the ladder moving down the wall when it is 8 feet from the ground and the foot is 6 feet from the wall? (See Figure 5.2.9.)

40. At an altitude of x kilometers, the atmospheric pressure decreases at a rate of $128(0.88)^x$ millibars per kilometer. A rocket is rising at the rate of 5 kilometers per

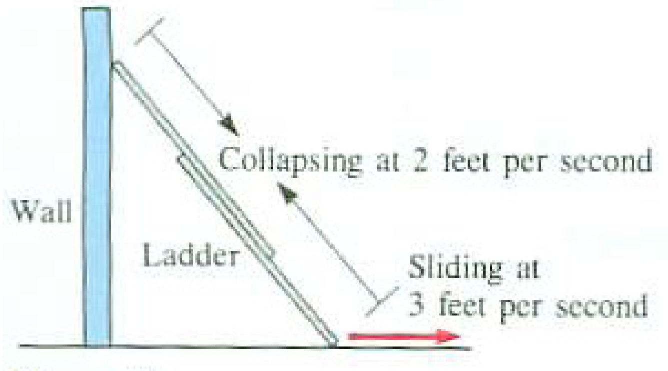


Figure 5.2.9:

second vertically. At what rate is the atmospheric pressure changing (in millibars per second) when the altitude of the rocket is

(a) 1 kilometer?

(b) 50 kilometers?

41. A woman is walking on a bridge that is 20 feet above a river as a boat passes directly under the center of the bridge (at a right angle to the bridge) at 10 feet per second. At that moment the woman is 50 feet from the center and approaching it at the rate of 5 feet per second.

(a) At what rate is the distance between the boat and woman changing at that moment?

(b) Is the rate at which they are approaching or separating increasing or is it decreasing?

42. A spherical raindrop evaporates at a rate proportional to its surface area. Show that the radius shrinks at a constant rate.

43. A couple is on a Ferris wheel when the sun is directly overhead. The diameter of the wheel is 50 feet, and its speed is 0.01 revolution per second.

(a) What is the speed of their shadows on the ground when they are at a two-o'clock position?

(b) A one-o'clock position?

(c) Show that the shadow is moving its fastest when they are at the top or bottom, and its slowest when they are at the three-o'clock or nine-o'clock position.

44. A woman on the ground is watching a jet through a telescope as it approaches at a speed of 10 miles per minute at an altitude of 7 miles. At what rate (in radians per minute) is the angle of the telescope changing when the horizontal distance of the jet from the woman is 24 miles? When the jet is directly above the woman?

45. Does the tangent line to the curve $x^3 + xy^2 + x^3y^5 = 3$ at the point $(1, 1)$ pass through the point $(-2, 3)$?

Exercises 46 and 47 obtain by implicit differentiation the formulas for differentiating $x^{1/n}$ and $x^{m/n}$ with the assumption that they are differentiable functions. Here m and n are integers.

46. Let n be a positive integer. Assume that $y = x^{1/n}$ is a differentiable function of x. From the equation $y^n = x$ deduce by implicit differentiation that $y' = (1/n)x^{1/n-1}$.

47. Let m be a nonzero integer and n a positive interger. Assume that $y = x^{m/n}$ is a differentiable function of x. From the equation $y^n = x^m$ deduce by implicit differentiation that $y' = (m/n)x^{m/n-1}$.

48. Water is flowing into a hemispherical bowl of radius 5 feet at the constant rate of 1 cubic foot per minute.

(a) At what rate is the top surface of the water rising when it height above the bottom of the bowl is 3 feet? 4 feet? 5 feet?

(b) If $h(t)$ is the depth in feet at time t, find $\ddot{h}$ when $h = 3, 4$, and 5.

49. A man in a hot-air balloon is ascending at the rate of 10 feet per second. How fast is the distance from the balloon to the horizon (that is, the distance the man

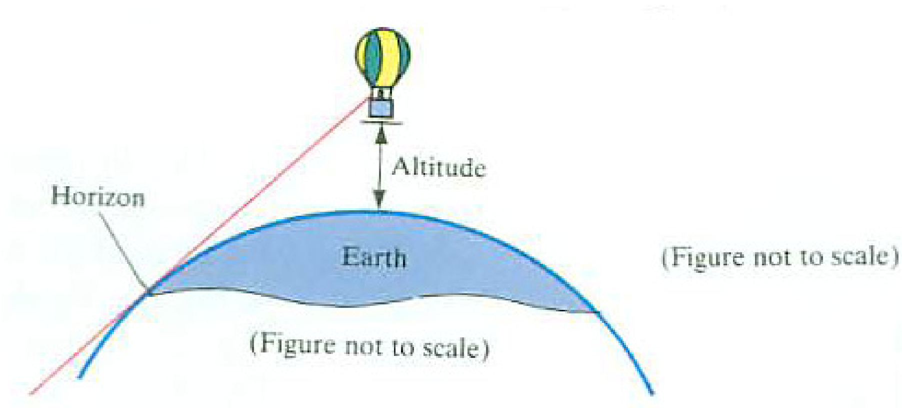


Figure 5.2.10:

can see) increasing when the balloon is $1,000$ feet high? Assume that the earth is a ball of radius $4,000$ miles. (See Figure 5.2.10.)

50. (Contributed by Keith Sollers, when an undergaduate at the University of California at Davis.) We quote from his note to us. "The numbers are ugly, but I think it's a good problem nevertheless. I didn't think it up myself. The Medical Center eye group gave me the problem and asked me to solve it. They were going to put a gas bubble in someone's eye."

The volume of a gas bubble changes from 0.4 cc to 1.6 cc in 74 hours. Assuming that the rate of change of the radius is constant, find,

(a) The rate at which the radius changes;

(b) The rate at which the volume of the bubble is increasing at any volume V;

(c) The rate at which the volume is increasing when the volume is 1 cc.

NOTE: Assume the bubble is spherical. The volume of a ball of radius r is $V = 4\pi r^3/3$.

In Exercises 51 and 52 the derivative of the given function will be found in two different ways.

(a) Find $f'(x)$ by direct calculation.

(b) Repeat (a), except this time simplify $f(x)$ using the properties of logarithms *before* taking the derivative.

$$\begin{aligned}
\ln(AB) &= \ln(A) + \ln(B) \quad (A > 0,\, B > 0) \\
\ln(A/B) &= \ln(A) - \ln(B) \quad (A > 0,\, B > 0) \\
\ln(A^B) &= B\ln(A) \quad (A > 0)
\end{aligned}$$

51. Differentiate the function in Example 6 directly, without taking logarithms first.

52. Suppose $f(x) = \ln\left(\dfrac{\left(\sqrt{1+x^2}\right)^3 \left(e^{3x}+1\right)}{1+\sin(2x)}\right)$.

In Exercises 53–56 first simplify the formula for the function with the aid of properties of logarithms. Then, find dy/dx.

53. $y = \ln\left(\left(\sqrt{1+\sin(2x)}\right)^3\right)$

54. $y = \ln\left(\frac{(x^3+2)^5}{(x^2+5)^2}\right)$

55. $y = \ln\left((\sin(2x))^3\sqrt{\tan^{-1}(3x)}\right)$

56. $y = \ln\left(\frac{(\ln(x^2))^5(\sin^{-1}(3x))^5}{(\tan(5x)^2}\right)$

In Exercises 57–62 differentiate the given function by logarithmic differentiation.

57. $y = x^3\sin^2(2x)$

58. $y = \sqrt{\sin(2x)}\sqrt[3]{1+x^3}$

59. $y = \frac{x^3\cos(2x)}{(1+x^2)^4}$

60. $y = \frac{\tan^3(5x)}{\sqrt[3]{e^{x^2}}\sin^{-1}(5x)}$

61. $y = \frac{(x^3+2x)(\tan^{-1}(3x)}{1+e^{2x}}$

62. $y = \frac{\left(\sqrt{\ln(2x)}\right)^3(\sin(3x))^5}{(x^3+x)^2}$

63. Find $D(x^k)$, $x > 0$, by logarithmic differentiation of $y = x^k$.

64. Let $y = x^x$.

(a) Find y' by logarithmic differentiation. That is, first take the logarithm of both sides.

(b) Find y' by first writing the base as $e^{\ln(x)}$. That is, write $y = x^x = \left(e^{\ln(x)}\right)^x = e^{x\ln(x)}$.

65. Find the first and second derivatives of $y = \sec(x^2)\frac{\sin(x^2)}{x}$.

66. Keith Sollers, when an undergraduate at the University of California at Davis, wrote the following in a note to one of the authors:

> The nubers are ugly, but I think it's a good problem nevertheless. I didn't think it up myself. The Medical Center eye group gave me the problem and asked me to solve it. They were going to put a gas bubble in someone's eye.
>
> The volume of a gas bubble changes from 0.4 cc to 1.6 cc in 74 hours. Assuming that the rate of change of the radius is constant, find
>
> (a) the rate at which the radius changes,
>
> (b) the rate at which the volume of the bubble is increasing at any volume V,
>
> (c) the rate at which the volume is increasing when the volume is 1 cc.

NOTE: Assume the gas bubble is spherical. The volume of a ball of radius r is $V = 4\pi r^3/3$.

5.3 Higher Derivatives and the Growth of A Function

The only higher derivative we've used so far is the second derivative. In the study of motion, if y denotes position then y'' is acceleration. In the study of graphs, the second derivative determines whether the graph is concave up ($y'' > 0$) or down ($y'' < 0$). Later, in Section 9.6, the second derivative will appear in a formula that measures the curviness of a curve.

Now we will see how the higher derivatives (including the second derivative) influence the growth of a function. In the next section this will be applied to estimate the error in approximating a function by a polynomial.

Introduction

Imagine that you are in a car motionless at the origin of the x-axis. Then you put your foot to the gas pedal and accelerate. The greater the acceleration, the faster the speed increases; the greater the speed, the further you travel in a given time. So the acceleration, which is the second derivative of the position function, influences the function itself. This illustrates how a higher derivative of a function influences the growth of a function. In this section we examine this influence in more detail.

The following lemma is the basis for our analysis.

If $a > b$, then $f(x) \geq g(x)$. See Exercise 32.

Lemma 5.3.1 *Let $f(x)$ and $g(x)$ be differentiable functions on an interval I. Let a be a number in I where $f(a) = g(a)$. Assume that $f'(x) \leq g'(x)$ for x in I. Then $f(x) \leq g(x)$ for all x in I to the right of a and $f(x) \geq g(x)$ for all x in I to the left of a.*

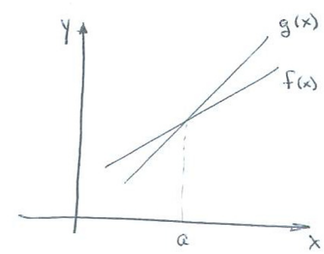


Figure 5.3.1:

Figure 5.3.1 makes this plausible, when the graphs of f and g are straight lines. To the right of $x = a$ the steeper line lies above the other line. To the left of $x = a$ the steeper line lies below the other line.

Proof of Lemma 5.3.1

Consider the case when $x > a$. Let $h(x) = f(x) - g(x)$. Then $h(a) = 0$ and $h'(x) = f'(x) - g'(x) \leq 0$. Thus, h is a non-increasing function. Since $h(a) = 0$, it follows that $h(x) \leq 0$ for $x \geq a$. That is, $f(x) - g(x) \leq 0$, or $f(x) \leq g(x)$ for $x > a$. •

Repeated application of Lemma 5.3.1 will enable us to establish connections between higher derivatives and the function itself.

Higher Derivatives and the Growth of a Function

In the following theorem we name the function $R(x)$ because that will be the notation in the next section when $R(x)$ is the "remainder" function. The notation $n!$ (read: "n factorial") for a positive integer n is shorthand for the product of all integers from 1 through n: $n! = n(n-1)\cdots 3\cdot 2\cdot 1$. The symbol $0!$ is usually defined to be 1.

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120.$

Theorem 5.3.1 (Growth Theorem) *Assume that at a the function R and its first n derivatives are zero,*

$$R(a) = R'(a) = R''(a) = R^{(3)}(a) = \cdots = R^{(n)}(a) = 0.$$

Assume also that $R(x)$ has continuous derivatives up through the derivative of order $n+1$ in some open interval I containing the number a. Also, assume that there is a number M such that $\left|R^{(n+1)}(x)\right| \le M$ for all x in I. Then

$\left|R^{(n+1)}(x)\right| \le M$ means $-M \le R^{(n+1)}(x) \le M$

$$|R(x)| \le M\frac{|x-a|^{n+1}}{(n+1)!} \tag{1}$$

for all x in the interval I.

Before giving the straightforward proof, we illustrate the result in the specific case with $n = 2$.

EXAMPLE 1 Assume that $R(5) = R'(5) = R''(5) = 0$ and $\left|R^{(3)}(x))\right| \le 4$ for x in the interval $(3, 7)$. Show that $|R(x)| \le 2|x-5|^3/3$ for x in $(3, 7)$.
SOLUTION This is a special case of the Growth Theorem when $a = 5$, $n = 2$, and $M = 4$. Thus,

$$|R(x)| \le \frac{4|x-5|^3}{3!} = \frac{4|x-5|^3}{6} = \frac{2}{3}|x-5|^3.$$

For instance, $|R(5.1)| \le \frac{2}{3}(0.1)^3 \approx 0.000667$. $\diamond$

Proof of the Growth Theorem

We illustrate the proof in the case $n = 2$. Assume $R(a) = R'(a) = R''(a) = 0$ and $\left|R^{(3)}(x)\right| \le M$. We have $R^3(x) \le M$ and will assume $x > a$.

See Exercises 34 and 35. The case with $x < a$ is messier because $(x-a)^n$ introduces negative signs when n is odd.

Starting with the inequality $R^{(3)}(x) \le M$, we apply the lemma repeatedly, gradually working up from $R^{(3)}(x)$ to $R(x)$. Our objective is to show that

$$-M\frac{(x-a)^3}{3!} \le R(x) \le M\frac{(x-a)^3}{3!}.$$

First rewrite $R^{(3)}(x) \le M$ as

$$\frac{d}{dx}R''(x) \le \frac{d}{dx}M(x-a). \tag{2}$$

Note that both $R''(x)$ and $M(x-a)$ equal 0 when $x = a$. The lemma, applied to (2), gives

$$R''(x) \le M(x-a).$$

We do not want the obvious antiderivative of M, namely Mx, because Mx does not equal 0 when $x = a$.

Rewrite this as

$$\frac{d}{dx}R'(x) \le \frac{d}{dx}\left(\frac{M(x-a)^2}{2}\right). \tag{3}$$

Since both $R'(x)$ and $\frac{M(x-a)^2}{2}$ are zero when $x = a$, the lemma can be applied — this time to (3). Thus,

$$R'(x) \le \frac{M(x-a)^2}{2}.$$

This result can be rewritten as

$$\frac{d}{dx}R(x) \le \frac{d}{dx}\frac{M(x-a)^3}{3\cdot 2}. \tag{4}$$

As before, both $R(x)$ and $\frac{M(x-a)^3}{3\cdot 2}$ are zero when $x = a$. A third application of the lemma, this time to (4), yields

$$R(x) \le \frac{M(x-a)^3}{3\cdot 2} = \frac{M(x-a)^3}{3!}.$$

A similar argument, starting with $-M \le R^{(3)}(x)$, shows that

$$\frac{-M(x-a)^3}{3!} \le R(x).$$

Thus,

$$|R(x)| \le M\frac{(x-a)^3}{3!}.$$

•

EXAMPLE 2 Show that $|e^x - 1 - x| \le \frac{e}{2}x^2$ for x in $(-1, 1)$.
SOLUTION Let $R(x) = e^x - 1 - x$. Then $R(0) = e^0 - 1 - 0 = 0$. And,

since $R'(x) = e^x - 1$, $R'(0) = e^0 - 1 = 0$ also. $R''(x) = e^x$. For x in $(-1, 1)$, $0 < e^x < e^1 = e$. By the Growth Theorem, with $a = 0$, $n = 1$, and $M = e$,

$$|e^x - 1 - x| \leq e\frac{|x-0|^2}{2!} = \frac{e}{2}x^2$$

for each x in $(-1, 1)$. ◇

For instance, $\left|e^{0.1} - 1 - 0.1\right| \leq \frac{e}{2}(0.1)^2 = \frac{e}{200}$. so the estimate 1.1 for $e^{0.1}$ is off by at most $\frac{e}{200} \approx 0.013591$.

EXAMPLE 3 Let $R(x) = \cos(x) - 1 + \frac{x^2}{2}$. Show that $|R(x)| \leq \frac{|x^3|}{6}$.
SOLUTION Since powers of $x = (x - 0)$ appear in $R(x)$, this suggests examining $R(x)$ at $a = 0$:

$$\begin{aligned} R(x) &= \cos(x) - 1 - \frac{x^2}{2}, &&\text{so} \quad R(0) = 1 - 1 + 0 = 0. \\ R'(x) &= -\sin(x) + x, &&\text{so} \quad R'(0) = 0 + 0 = 0. \\ R''(x) &= -\cos(x) + 1, &&\text{so} \quad R''(0) = -1 + 1 = 0. \end{aligned}$$

Thus $|R(x)| \leq \left|M\frac{(x-0)^3}{3!}\right| = \left|M\frac{x^3}{6}\right|$, where M is the maximum value of $|R^{(3)}(t)|$ in the interval $[0, x]$. Now, $R^{(3)}(t) = \sin(t)$. Since $|\sin(t)| \leq 1$ for all values of t, $M \leq 1$. Then

$$|R(x)| \leq \left|(1)\frac{x^3}{6}\right| = \frac{|x|^3}{6}.$$

◇

If $x < 0$, then consider the interval $[x, 0]$; again, $M \leq 1$.

Example 3 provides a good estimate for values of the cosine function for small angles. For instance, if $x = 0.1$ radians, we have

$$\left|\cos(0.1) - 1 + \frac{0.1^2}{2}\right| \leq \frac{0.1^3}{6} = 0.00016667 = 1.6667 \times 10^{-4}.$$

0.1 radians $= 0.1\frac{180°}{\pi}$ radians $\approx 5.72958°$

Thus, $1 - \frac{0.1^2}{2} = 1 - 0.005 = 0.995$ is an estimate of $\cos(0.1)$ with an error less than 0.00016667.

$\cos(0.1) \approx 0.9950041653$

Remark An even better bound on the growth of $R(x)$ in Example 3 is possible. In addition to $R(0) = R'(0) = R''(0) = 0$, notice that $R^{(3)}(0) = \sin(0) = 0$. This means that $|R(x)| \leq \left|M_4\frac{(x-0)^4}{4!}\right|$ where M_4 is the maximum value of $R^{(4)}(t) = \cos(t)$ in the interval $[0, x]$. As in Example 3, $M \leq 1$. Thus,

$$|R(x)| \leq \left|(1)\frac{x^4}{4!}\right| = \frac{x^4}{24}.$$

This means the difference between the exact value of $\cos(0.1)$ and the estimate $1 - \frac{0.1^2}{2} = 0.995$ is no more than $\frac{0.1^4}{24} = 4.16667 \times 10^{-6}$. This improves the estimate in Example 3 by a factor of forty.

In fact, $|\cos(0.1) - 0.995| \approx 4.16528 \times 10^{-6}$.

In any case, $1 - \frac{x^2}{2}$ is a good estimate of $\cos(x)$ for small values of x.

A Refinement of the Growth Theorem

The Growth Theorem is based on an estimate on the size of the derivative of the form $|R'(x)| \le M$, or $-M \le R'(x) \le M$. When more is known about the size of the derivative, say $m \le R'(x) \le M$, a stronger statement can be made about the size of the function, R, as the next theorem shows. The proof is similar to the one given previously for the Growth Theorem (Theorem 5.3.1).

Theorem 5.3.2 *Assume that $R(x)$ has continuous derivatives up through the derivatives of order $n+1$ in some open interval I containing the number a. Assume that, at a, $R(x)$ and its first n derivatives are all zero:*

Observe that $(x-a)^{n+1} > 0$ for $x < a$ and $n+1$ even and $(x-a)^{n+1} < 0$ for $x < a$ and $n+1$ odd.

$$R(a) = R'(a) = R''(a) = \cdots = R^{(n)}(a) = 0.$$

Also assume that for all x in that interval

$$m \le R^{(n+1)}(x) \le M.$$

Then, for $x \ge a$ in that interval,

$$m\frac{(x-a)^{n+1}}{(n+1)!} \le R(x) \le M\frac{(x-a)^{n+1}}{(n+1)!} \qquad \begin{array}{l}\text{for all } x \text{ in } I \text{ with } x < a, \text{ when } n+1 \text{ is even} \\ \text{for all } x \text{ in } I \text{ with } x > a\end{array} \tag{5}$$

$$m\frac{(x-a)^{n+1}}{(n+1)!} \ge R(x) \ge M\frac{(x-a)^{n+1}}{(n+1)!} \qquad \text{for all } x \text{ in } I \text{ with } x < a, \text{ when } n+1 \text{ is odd.}$$

EXAMPLE 4 Let $R(x) = e^x - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})$. Show that $\frac{1}{1152} \le R(\frac{1}{2}) \le \frac{1}{128}$. Use this estimate to obtain approximations, with error bounds, for $\sqrt{e} = e^{1/2}$ and e.

See Exercise 1.

SOLUTION

$$\begin{aligned} R(0) &= e^0 - 1 - 0. \\ R'(x) &= e^x - (1 + x + \frac{x^2}{2!}), &&\text{so} \quad R'(0) = 0. \\ R''(x) &= e^x - (1 + x), &&\text{so} \quad R''(0) = 0. \\ R^{(3)}(x) &= e^x - 1, &&\text{so} \quad R^{(3)}(0) = 0. \\ R^{(4)}(x) &= e^x, &&\text{and} \quad R^{(4)}(0) = 1 \ne 0. \end{aligned}$$

But, for x in $I = (-1, 1)$, $\frac{1}{3} \le e^{-1} \le e^x \le e^1 < 3$. Theorem 5.3.2, with $a = 0$, $n = 3$, $m = \frac{1}{3}$, $M = 3$, and $x = \frac{1}{2}$ gives

We assume $e < 3$.

$$\begin{array}{llcl} & \frac{1}{3}\frac{(1/2)^4}{4!} & \le R(1/2) \le & 3\frac{(1/2)^4}{4!} \\ \text{Then,} & \frac{1}{1152} & \le \sqrt{e} - \left(1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!}\right) \le & \frac{1}{128} \\ \text{or} & \frac{79}{48} + \frac{1}{1152} & \le \sqrt{e} \le & \frac{79}{48} + \frac{1}{128} \\ \text{so} & 1.64670 & \le \sqrt{e} \le & 1.65365. \end{array}$$

As you can check with your calculator, $\sqrt{e} \approx 1.64872$ to five decimal places. ⋄

A Convenient Form for $R(x)$

Assume that $R(a) = R'(a) = \cdots = R^{(n)}(a) = 0$ and that that M is the maximum value of $R^{(n+1)}(x)$ on a closed interval I with a in its interior and m is the minimum value of $R^{(n+1)}(x)$ on I. In view of the inequalities (5), the Intermediate-Value Theorem for continuous functions tells us that there must be a number c between a and x and in the interval I such that

See Theorem 2.4.3 in Section 2.4.

This holds when $x - a$ is positive or negative.

$$R(x) = R^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!}. \tag{6}$$

This is the form for the remainder we will use in the next section.

Approximation and Interpolation

We might expect that an effective way to approximate a function $f(x)$ throughout an interval $[a, b]$ would be to force the polynomial to match the function values at specific points. To be specific, divide the interval $[a, b]$ into n sections of equal length by $n+1$ points, of which the leftmost is a and the rightmost is b. There is a unique polynomial $P(x)$ of degree n, called an **interpolant** of $f(x)$, that coincides with the given function at these $n + 1$ inputs. We would expect that when n is large, $|f(x) - P(x)|$ would be small for all x in $[a, b]$.

This is *not* the case, even for such a pleasant function as $f(x) = 1/(1 + x^2)$ and the interval $[-5, 5]$. In numerical analysis it is proved that for large n the interpolating polynomial does not stay near $1/(1 + x^2)$. Some of its values become arbitrarily large as n increases. In fact, if $n = 5m+1$ and m is odd the polynomial $P(x)$ differs from $1/(1+x^2)$ at $x = 4.875$ by more than $1.8^m/451$, a quantity that grows exponentially as m increases. It is surprising phenomena such as this that show why intuition is no substitute for proof.

Summary

We showed that under certain conditions bounds on the size of the derivative of a function limit the growth of the function itself. When this observation is applied repeatedly we showed that if a function $R(x)$ and its first n derivatives are all zero at a, then

$$R(x) = R^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!} \qquad \text{(for some } c \text{ between } a \text{ and } x\text{)}.$$

The number c depends on n, not just on a, x, and the function $R(x)$.

EXERCISES for 5.3

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

1. If $f'(x) \geq 3$ for all $x \in (-\infty, \infty)$ and $f(0) = 0$, what can be said about $f(2)$? about $f(-2)$?

2. If $f'(x) \geq 2$ for all $x \in (-\infty, \infty)$ and $f(1) = 0$, what can be said about $f(3)$? about $f(-3)$?

3. State the Growth Theorem for $x \geq a$ in the case where R has at least five continuous derivatives and $R(a) = R'(a) = R''(a) = R^{(3)}(a) = R^{(4)}(a) = 0$.

4. State the Growth Theorem in words, using as little math notation as possible.

5. If $R(1) = R'(1) = R''(1) = 0$ and $R^{(3)}(x)$ is continuous on an interval that includes 1 and $R^{(3)}(x) \leq 2$, what can be said about $R(4)$?

6. What can be said about $f(2)$ if $f(1) = 0$, $f'(1) = 0$, and $2.5 \leq f''(x) \leq 2.6$ for all x?

7. What can be said about $f(4)$ if $f(1) = 0$, $f'(1) = 0$, and $2.9 \leq f''(x) \leq 3.1$ for all x?

8. A car starts from rest and travels for 4 hours. Its acceleration is always at least 5 miles per hour per hour, but never exceeds 12 miles per hour per hour. What can you say about the distance traveled after 4 hours?

9. A car starts from rest and travels for 6 hours. Its acceleration is always at least 4.1 miles per hour per hour, but never exceeds 15.5 miles per hour per hour. What can you say about the distance traveled after 6 hours?

10. If $R(3) = R'(3) = R''(3) = R^{(3)}(3) = R^{(4)}(3) = 0$ and $R^{(5)}(x) \leq 6$, what can be said about $R(3.5)$?

11. Let $R(x) = \sin(x) - \left(x - \frac{x^3}{6}\right)$. Show that

(a) $R(0) = R'(0) = R''(0) = R^{(3)}(0) = 0$.

(b) $R^{(4)}(x) = \sin(x)$.

(c) $|R(x)| \le \frac{x^4}{24}$.

(d) Use $x - \frac{x^3}{6}$ to approximate $\sin(x)$ for $x = 1/2$.

(e) Use (c) to estimate the difference between the exact value for $\sin\left(\frac{1}{2}\right)$ and the approximation obtained in (d).

(f) Explain why $|R(x)| \le \frac{|x|^5}{120}$. How can this be used to obtain a better estimate of the difference between the exact value for $\sin\left(\frac{1}{2}\right)$ and the approximation obtained in (d)?

(g) By how much does the estimate in (d) differ from $\sin\left(\frac{1}{2}\right)$?

NOTE: An angle of $\frac{1}{2}$ radian is about 29°.

12. Let $R(x) = \cos(x) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)$. Show that

(a) $R(0) = R'(0) = R''(0) = R^{(3)}(0) = R^{(4)}(0) = R^{(5)}(0) = 0$.

(b) $R^{(6)}(x) = -\cos(x)$.

(c) $|R(x)| \le \frac{x^6}{6!}$.

(d) Use $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ to estimate $\cos(x)$ for $x = 1$.

(e) By how much does the estimate in (d) differ from $\cos(1)$?

NOTE: An angle of 1 radian is about 57°.

13. Let $R(x) = (1+x)^5 - (1 + 5x + 10x^2)$. Show that

(a) $R(0) = R'(0) = R''(0) = 0$.

(b) $R^{(3)}(x) = 60(1+x)^2$.

(c) $|R(x)| \le 80x^3$ (on $[-1, 1]$)

(d) Use $1 + 5x + 10x^2$ to estimate $(1+x)^5$ for $x = 0.2$.

(e) By how much does the estimate in (d) differ from $(1.2)^5$?

14. If $f(3) = 0$ and $f'(x) \geq 2$ for all $x \in (-\infty, \infty)$, what can be said about $f(1)$? Explain.

15. If $f(0) = 3$ and $f'(x) \geq -1$ for all $x \in (-\infty, \infty)$, what can be said about $f(2)$ and about $f(-2)$? Explain.

16. What functions f has the property that $D^5(f(x)) = 0$ for all x?

17. Find constants a_0, a_1, a_2, and a_3 such that if $R(x) = \tan(x) - \left(a_0 + a_1x + a_2x^2 + a_3x^3\right)$ then $R(0) = R'(0) = R''(0) = R^{(3)}(0) = 0$.

18. Find constants a_0, a_1, a_2, and a_3 such that if $R(x) = \sqrt{1+x} - \left(a_0 + a_1x + a_2x^2 + a_3x^3\right)$ then $R(0) = R'(0) = R''(0) = R^{(3)}(0) = 0$.

19. Find constants a_0, a_1, a_2, and a_3 such that if

$$R(x) = \sin x - \left(a_0 + a_1\left(x - \frac{\pi}{6}\right) + a_2\left(x - \frac{\pi}{6}\right)^2 + a_3\left(x - \frac{\pi}{6}\right)^3\right)$$

then $R\left(\frac{\pi}{6}\right) = R'\left(\frac{\pi}{6}\right) = R''\left(\frac{\pi}{6}\right) = R^{(3)}\left(\frac{\pi}{6}\right) = 0$.

20. Because $e > 1$, it is known that $e^x \geq 1$ for every $x \geq 0$.

(a) Use Lemma 5.3.1 to deduce that $e^x > 1 + x$, for $x > 0$.

(b) Use (a) and Lemma 5.3.1 to deduce that, for $x > 0$, $e^x > 1 + x + \frac{x^2}{2!}$.

(c) Use (b) and Lemma 5.3.1 to deduce that, for $x > 0$, $e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

(d) In view of (a), (b), and (c), what is the general inequality that can be proved by this approach?

21. Let k be a fixed positive number. For x in $[0, k]$, $e^x \leq e^k$.

(a) Deduce that $e^x \leq 1 + e^k x$ for x in $[0, k]$.

(b) Deduce that $e^x \leq 1 + x + e^k\frac{x^2}{2!}$ for x in $[0, k]$.

(c) Deduce that $e^x \leq 1 + x + \frac{x^2}{2!} + e^k\frac{x^3}{3!}$ for x in $[0, k]$.

(d) In view of (a), (b), and (c), what is the general inequality that can be proved by this approach?

22. Combine the results of Exercises 20 and 21 to estimate $e = e^1$. NOTE: Assume $e \le 3$.

23. What properties of e^x did you use in Exercises 20 and 21?

24. Let $E(x)$ be a function such that $E(0) = 1$ and $E'(x) = E(x)$ for all x.

(a) Show that $E(x) \ge 1$ for all $x \ge 0$.

(b) Use (a) to show that $E(x)$ is an increasing function for all $x \ge 0$. HINT: Show that $E'(x) \ge 1$.

25. Consider the following proposal by Sam: "As usual, I can do things more simply than the text. For instance, say $R(a) = R'(a) = R''(a) = 0$ and $R^{(3)}(x) \le M$. I'll show how M affects the size of $R(x)$, for $x > a$.

By the Mean-Value Theorem, $R(x) = R'(c_1)(x - a)$ for some c_1 in $[a, x]$. Then I just use the MVT again, this time finding $R'(c_1) = R'(c_1) - R'(a) = R''(c_2)(c_1 - a)$ for some c_2 in $[a, c_1]$. One more application of this idea then gives $R''(c_2) = R''(c_2) - R''(a) = R^{(3)}(c_3)(c_3 - a)$.

Then I put these all together, getting

$$R(x) \le M(x - a)(c_2 - a)(c_3 - a).$$

Since c_1, c_2, and c_3 are in $[a, x]$, I can certainly say that

$$R(x) \le M(x - a)^3.$$

I didn't need that lemma about two functions."

Is Sam correct? Is this a valid substitute for the text's treatment? Explain.

Exercises 26–31 show that $\lim_{x\to\infty} \frac{x}{e^x}$, $\lim_{x\to\infty} \frac{\ln(y)}{y}$, $\lim_{x\to 0^+} x\ln(x)$, $\lim_{x\to\infty} \frac{x^k}{b^x}$ $(b > 1)$, and $\lim_{x\to 0^+} x^x$ are closely connected. (In fact, if you know one of them you can deduce the other three.)

In Exercise 20 it is shown that $e^x > 1 + x + \frac{x^2}{2}$ for all $x > 0$. Use this fact in Exercises 26–27.

26. Evaluate $\lim_{x\to\infty} \frac{x}{e^x}$.

27. Evaluate $\lim_{y\to\infty} \frac{\ln(y)}{y}$. HINT: Let $y = e^x$ and compare with Exercise 26.

Exercise 28 provides a proof of the fact that the exponential function grows faster than any power of x.

28. Write $\frac{x^n}{e^x} = \left(\frac{x}{e^{x/n}}\right)\left(\frac{x}{e^{x/n}}\right)\cdots\left(\frac{x}{e^{x/n}}\right)$. Let $y = x/n$ so that $\frac{x}{e^{x/n}} = \frac{ny}{e^y}$. Use Exercise 26 ($n$ times) to show that $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$.

29. Evaluate $\lim_{x\to 0^+} x\ln(x)$ as follows: Let $x = 1/t$, where $t \to \infty$. Then $x\ln(x) = \frac{1}{t}\ln\left(\frac{1}{t}\right) = \frac{-\ln(t)}{t}$. and refer to Exercise 27.

30. Evaluate $\lim_{x\to 0^+} x^x$ as follows: Let $y = x^x$. Then $\ln(y) = x\ln(x)$, a limit that was evaluated in Exercise 29. Explain why $\ln(y) \to 0$ implies $y \to 1$.

31. Evaluate $\lim_{x\to\infty} \frac{x^k}{b^x}$ for any $b > 1$ and k is a positive integer, HINT: Use the result obtained in Exercise 28.

32. Explain why $f(a) = g(a)$ and $f'(x) \le g'(x)$ on $[a, b]$ with $a > b$ implies $f(x) \ge g(x)$.

33. In Example 2 it is shown that $|e^x - 1 - x| \le \frac{e}{2}x^2$ for all x in $(-1, 1)$.

(a) Find a bound for $R(x) = e^x - 1 - x - \frac{x^2}{2}$ on $(-1, 1)$.

(b) Find a bound for $R(x) = e^x - 1 - x$ on $(-2, 1)$.

(c) Find a bound for $R(x) = e^x - 1 - x$ on $(-1, 2)$.

(d) Find a bound for $R(x) = e^x - 1 - x - \frac{x^2}{2}$ on $(-2, 1)$.

(e) Find a bound for $R(x) = e^x - 1 - x - \frac{x^2}{2}$ on $(-1, 2)$.

34. State and prove the refinement of the Growth Theorem when the assumption $\left|R^{(n+1)}(x)\right| \le M$ is replaced by $m \le R^{(n+1)}(x) \le M$.

35. Apply the lemma to the case when $R(a) = R''(a) = 0$, $\left|R^{(3)}\right| \le M$, but $R'(a) = 5$.

5.4 Taylor Polynomials and their Errors

We spend years learning how to add, subtract, multiply, and divide. These same operations are built into any calculator or computer. Both we and machines can evaluate a polynomial, such as

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

when x and the coefficients a_0, a_1, a_2, …, a_n are given numbers. Only multiplication and addition are needed. But how do we evaluate e^x or $\sin(x)$? We resort to our calculators or look in a table that lists values of e^x. If e^x were a polynomial in disguise, then it would be easy to evaluate it by finding the polynomial and evaluating it instead. But e^x cannot be a polynomial, even over a very short interval. Why? Since we cannot write e^x as a polynomial, we settle for the next best thing. Let's look for a polynomial that closely *approximates* e^x. However, no polynomial can be a good approximation of e^x for *all* x, since e^x grows too fast as $x \to \infty$. We search, instead, for a polynomial that is close to e^x for x in some short interval.

Three different reasons: 1. Because e^x equals its own derivative and no polynomial equals its own derivative (other than the polynomial that has constant value 0). 2. When you differentiate a non-constant polynomial, you get a polynomial with a lower degree. 3. Also, $e^x \to 0$ as $x \to -\infty$ and no non-constant polynomial has this property.

In this section we develop a method to construct polynomial approximations to functions. The accuracy of these approximations can be determined using the Growth Theorem from the previous section. Not surprisingly, higher derivatives play a pivotal role.

Fitting a Polynomial Locally, Near 0

Suppose we want to find a polynomial that closely approximates a function $y = f(x)$ for x near the input 0. For instance, what polynomial $p(x)$ of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ might produce a good fit?

First we insist that

$$p(0) = f(0) \tag{1}$$

so the approximation is exact when $x = 0$.

Second, we would like the slope of the graph of $p(x)$ to be the same as that of $f(x)$ when x is 0. Therefore, we require

$$p'(0) = f'(0). \tag{2}$$

There are many polynomials that satisfy these two conditions. To find the best choices for the four numbers a_0, a_1, a_2, and a_3 we need to have four equations. To have four equations in the four unknowns we will continue the pattern started by (1) and (2). So we also insist that

$$p''(0) = f''(0) \tag{3}$$

and

$$p^{(3)})(0) = f^{(3)}(0). \quad (4)$$

Equation (3) forces the polynomial $p(x)$ to have the same sense of concavity as the function $f(x)$ at $x = 0$. We expect the graphs of $f(x)$ and such a polynomial $p(x)$ to resemble each other for x close to a.

$p(x)$ and its derivatives	Their values at 0	Equation for a_k	Formula for a_k
$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$	$p(0) = a_0$	$a_0 = f(0)$	$a_0 = f(0)$
$p^{(1)}(x) = a_1 + 2a_2x + 3a_3x^2$	$p^{(1)}(0) = a_1$	$a_1 = f^{(1)}(0)$	$a_1 = f^{(1)}(0)$
$p^{(2)}(x) = 2a_2 + 3 \cdot 2a_3x$	$p^{(2)}(0) = 2a_2$	$2a_2 = f^{(2)}(0)$	$a_2 = \frac{1}{2}f^{(2)}(0)$
$p^{(3)}(x) = 3 \cdot 2a_3$	$p^{(3)}(0) = 3 \cdot 2a_3$	$3 \cdot 2a_3 = f^{(3)}(0)$	$a_3 = \frac{1}{3\cdot 2}f^{(3)}(0)$

Table 5.4.1:

To find the unknowns a_0, a_1, a_2, and a_3 we first compute $p(x)$, $p'(x)$, $p''(x)$, and $p^{(3)}(x)$ at 0. Table 5.4.1 displays the computations that express the unknowns, a_0, a_1, a_2, and a_3, in terms of $f(x)$ and its derivatives. For example, note how we compute $p''(x) = 2a_2 + 3 \cdot 2a_3x$ and evaluate it at 0 to obtain $p''(0) = 2a_2 + 3 \cdot 2a_3 \cdot 0 = 2a_2$. Then we obtain an equation for a_2 by equating $p''(0)$ and $f''(0)$; that is, $2a_2 = f''(0)$, so $a_2 = \frac{1}{2}f''(0)$.

Factorials appear in the denominator.

We can write a general formula for a_k if we let $f^{(0)}(x)$ denote $f(x)$ and recall that $0! = 1$ (by definition), $1! = 1$, $2! = 2 \cdot 1 = 2$, and $3! = 3 \cdot 2$. According to Table 5.4.1,

$$a_k = \frac{f^{(k)}(0)}{k!}, \qquad k = 0, 1, 2, 3.$$

Therefore

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)})(0)}{3!}x^3.$$

The coefficient of x^k is *completely determined* by the k^{th} derivative of f evaluated at 0. It equals the k^{th} derivative of f at 0 divided by $k!$. In Example 1 we illustrate this observation by finding a polynomial $p(x)$ of degree 3 that approximates e^x for x near 0.

EXAMPLE 1 Find the polynomial of degree 3 that agrees with the value e^x and its first three derivatives at $x = 0$.

SOLUTION The first step is to compute e^x and its first three derivatives, then evaluate them at $x = 0$. Dividing these values by a suitable factorial gives us the coefficients of the polynomial. Table 5.4.2 records the computations, which are especially simple because the derivative of e^x is e^x itself.

at x			at 0		
$f^{(0)}(x)$	$=$	e^x	$f^{(0)}(0)$	$=$	1
$f^{(1)}(x)$	$=$	e^x	$f^{(1)}(0)$	$=$	1
$f^{(2)}(x)$	$=$	e^x	$f^{(2)}(0)$	$=$	1
$f^{(3)}(x)$	$=$	e^x	$f^{(3)}(0)$	$=$	1

Table 5.4.2: Derivatives of $f(x) = e^x$

So the third-degree approximating polynomial is

$$p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

Figure 5.4.1 shows the graphs of both e^x and $p(x)$. Notice how the polynomial closely matches the exponential near $x = 0$ but is not as good an approximation for $|x| > 1$. ◇

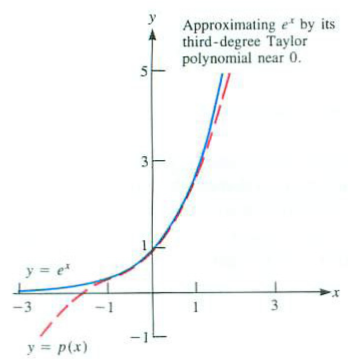


Figure 5.4.1:

The Taylor Polynomials at a

Example 1 illustrates the general procedure for finding polynomials that behave much like the given function near 0. These approximating polynomials are given a name in the following definition.

The n^{th}-order Taylor polynomial has degree at most n.

DEFINITION *(Taylor Polynomials at* 0*)* Let n be a non-negative integer and let f be a function with derivatives at 0 of all orders through n. Then the polynomial

$$f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \tag{5}$$

is called the **n^{th}-order Taylor polynomial of f centered at 0** and is denoted $P_n(x, 0)$.

Whether $P_n(x, 0)$ approximates $f(x)$ for x near 0 is not obvious. We will show that the Taylor polynomials centered at $x = 0$ for e^x and $\sin(x)$ do provide good approximations of the functions for x near 0. The polynomial found in Example 1 is $P_3(x, 0)$ for e^x. Figure 5.4.1 suggests that $P_3(x, 0)$ does a fairly good job of approximating e^x near 0. In general, the bigger n is, the better the approximation and the longer the interval where the approximation is good.

Taylor polynomials centered at 0, as in the definition above, are called the **Maclaurin polynomials**.

Taylor Polynomials Centered at a

We may be interested in estimating a function $f(x)$ near a number a, not just near 0. In that case, we express the approximating polynomial in terms of powers of $x-a$ instead of powers of $x = x - 0$ and make the derivatives of the approximating polynomial, evaluated at a, coincide with the derivatives of the function at a. Calculuations similar to those that gave us the polynomial (5) produce the polynomial called the "Taylor polynomial centered at a".

The n^{th}-order Taylor polynomial of f centered at a is denoted $P_n(x,a)$.

DEFINITION *(Taylor Polynomials of degree b, $P_n(x,a)$)* If the function f has derivatives through order n at a, then the **n^{th}-order Taylor polynomial of f centered at a** is defined as

$$f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and is denoted $P_n(x,a)$.

EXAMPLE 2 Compute the fourth-order Taylor polynomial at $a = 1$ for the function $f(x) = \frac{1}{x}$.
SOLUTION Table 5.4.3 shows the values of $1/x$ and its first 4 derivatives

at x			at 1		
$f^{(0)}(x)$	$=$	$1/x$	$f^{(0)}(0)$	$=$	1
$f^{(1)}(x)$	$=$	$-1/x^2$	$f^{(1)}(0)$	$=$	-1
$f^{(2)}(x)$	$=$	$2/x^3$	$f^{(2)}(0)$	$=$	2
$f^{(3)}(x)$	$=$	$-6/x^4$	$f^{(3)}(0)$	$=$	-6
$f^{(4)}(x)$	$=$	$24/x^5$	$f^{(4)}(0)$	$=$	24

Table 5.4.3: Derivatives of $f(x) = 1/x$

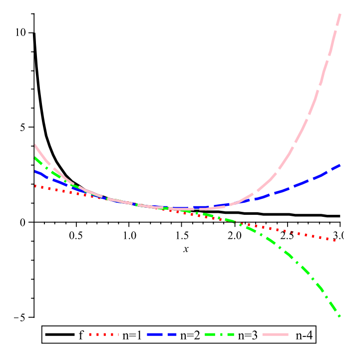


Figure 5.4.2:

at 1. Thus,

$$\begin{aligned}
P_4(x;1) &= f^{(0)}(1) + f^{(1)}(1)(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 \\
&= 1 + (-1)(x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4 \\
&= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4.
\end{aligned}$$

Figure 5.4.2 shows the graphs of $y = 1/x$ and its first four Taylor polynomials centered at $x = 1$. ◇

x	$1/x$	$P_4(x;1)$
1.0	1.000000	1.000000
1.1	0.909091	0.909100
1.5	0.666667	0.687500
2.0	0.500000	1.000000
0.5	2.000000	1.937500

Table 5.4.4 compares the value of the polynomial $P_4(x;1)$ found in Example 2 with the function $f(x) = 1/x$ for a few inputs near 1.

EXAMPLE 3 Find the Taylor polynomial $P_5(x,0)$ associated with the function $f(x) = \sin(x)$.
SOLUTION Again we make a table for computing the coefficients of the Taylor polynomial centered at 0. (See Table 5.4.5.)

at x			at 0		
$f^{(0)}(x)$	$=$	$\sin(x)$	$f^{(0)}(0)$	$=$	$\sin(0) = 0$
$f^{(1)}(x)$	$=$	$\cos(x)$	$f^{(1)}(0)$	$=$	$\cos(0) = 1$
$f^{(2)}(x)$	$=$	$-\sin(x)$	$f^{(2)}(0)$	$=$	$-\sin(0) = 0$
$f^{(3)}(x)$	$=$	$-\cos(x)$	$f^{(3)}(0)$	$=$	$-\cos(0) = -1$
$f^{(4)}(x)$	$=$	$\sin(x)$	$f^{(4)}(0)$	$=$	$\sin(0) = 0$
$f^{(5)}(x)$	$=$	$\cos(x)$	$f^{(5)}(0)$	$=$	$\cos(0) = 1$

Table 5.4.5: Derivatives of $f(x) = \sin(x)$

Thus

$$\begin{aligned}
P_5(x,0) &= f^{(0)}(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 \\
&= 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\
&= x - \frac{x^3}{6} + \frac{x^5}{120}.
\end{aligned}$$

Figure 5.4.3 illustrates the graphs of $P_5(x;1)$ and $\sin(x)$ near 0. ◇

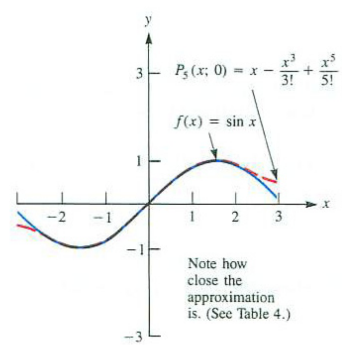


Figure 5.4.3:

Having found the fifth-order Taylor polynomial for $\sin(x)$ at $a = 0$, let us see how good an approximation it is of $\sin(x)$. Table 5.4.6 compares their values to six decimal place accuracy for inputs both near 0 and far from 0. As we see, the closer x is to 0, the better the Taylor approximation is. When x is large, $P_5(x,0)$ gets very large, but the value of $\sin(x)$ stays between -1 and 1.

x	$\sin(x)$	$P_5(x,0)$
0.0	0.000000	0.000000
0.1	0.099833	0.099833
0.5	0.479426	0.479427
1.0	0.841471	0.841667
2.0	0.909297	0.933333
π	0.000000	0.524044
2π	0.000000	46.546732

Table 5.4.6:

The Error in Using A Taylor Polynomial

There is no point using $P_n(x,a)$ to estimate a function $f(x)$ if we have no idea how large the difference between $f(x)$ and $P_n(x,a)$ may be. So let us take a closer look at the difference.

Define the **remainder** of this approximation to be the difference between the exact function, $f(x)$, and the Taylor polynomial, $P_n(x,a)$. Denote the remainder as $R_n(x,a)$. Then,

$$f(x) = P_n(x,a) + R_n(x,a).$$

We will be interested in the absolute value of the remainder. We call $|R_n(x,a)|$ the **error** between the function $f(x)$ and its n^{th}-order Taylor polynomial at a, $P_n(x,a)$.

We do not care whether $P_n(x,a)$ is larger or smaller than the exact value.

We will soon verify that the k^{th} derivative of $\frac{d^k}{dx^k}R_n(x,a)$ for k equal to 0, 1, 2, ..., n equals zero when evaluated at $x = a$. This means we can use the Growth Theorem (in Section 5.3) to estimate $|R_n(x,a)|$.

Theorem 5.4.1 (Lagrange's Form of the Remainder) *Assume that a function $f(x)$ has continuous derivatives of orders through $n+1$ in an interval that includes the numbers a and x. Let $P_n(x,a)$ be the n^{th}-order Taylor polynomial associated with $f(x)$ in powers of $x-a$. Then there is a number c_n between a and x such that*

$$R_n(x,a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}.$$

Proof of Theorem 5.4.1
By definition

$$R_n(x,a) = f(x) - P_n(x,a). \tag{6}$$

Thus, since $P_n(a,a) = f(a)$,

$$R_n(a,a) = f(a) - P_n(a,a) = f(a) - f(a) = 0.$$

Similarly, repeated differentiation of (6), leads to

$$R_n^{(k)}(x,a) = f^{(k)}(x) - P_n^{(k)}(x,a), \tag{7}$$

for each integer k, $1 \le k \le n$. Then, from the definition of $P_n(x,a)$,

$$R_n^{(k)}(a,a) = f^{(k)}(a) - P_n^{(k)}(a,a) = 0.$$

$P_n^{(k)}(a,a) = f^{(k)}(a)$, $k = 0$, 1, ..., n.

Thus, $R_n(x,a)$ is a function that meets all the criteria for the Growth Theorem.

See Theorem 5.3.1 in Section 5.3.

Since $P_n(x,a)$ is a polynomial of degree at most n, its $(n+1)^{\text{st}}$ derivative is 0. As a result, the $(n+1)^{\text{st}}$ derivative of $R_n(x,a)$ is the same as the $(n+1)^{\text{st}}$ derivative of $f(x)$. Recalling (6) from Section 5.3, we see there is a number c_n between a and x such that

$R_n^{(n+1)}(x,a) = f^{(n+1)}(x)$

$$R_n(x,a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}.$$

The subscript n in c_n is introduced since this constant depends not only on a and x but also on n.

This is called the **Lagrange form of the remainder**. •

EXAMPLE 4 Discuss the error in using $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ to estimate $\sin(x)$ for $x > 0$.

SOLUTION Example 3 showed that $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is the Taylor polynomial centered at 0, $P_5(x,0)$, associated with $\sin(x)$. Thus the error $|R_5(x,0)|$ is at most

$$\frac{\left|\frac{d^6}{dx^6}\sin(x)\right|}{6!}x^6 \leq \frac{x^6}{6!},$$

because every derivative of $\sin(x)$ is either $\pm\sin(x)$ or $\pm\cos(x)$. Then

$$\left|\sin(x) - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right| \leq \frac{|x|^6|}{6!} = \frac{x^6}{720}.$$

For instance, with $x = 1/2$,

$$\left|\sin\left(\frac{1}{2}\right) - \left(\left(\frac{1}{2}\right) - \frac{\left(\frac{1}{2}\right)^3}{6} + \frac{\left(\frac{1}{2}\right)^5}{120}\right)\right| \leq \frac{\left(\frac{1}{2}\right)^6}{720} = \frac{1}{(64)(720)} = \frac{1}{46,080} \approx 0.0000217 = 2.17\times10^{-5}$$

So the approximation

$$P_5(1/2,0) = \frac{1}{2} - \frac{1}{48} + \frac{1}{3840} = \frac{1841}{3840} \approx 0.4794271$$

differs from $\sin(1/2)$ (the sine of half a radian) by less than 2.17×10^{-5}; this means at least the first four decimal places are correct. The exact value of $\sin(1/2)$, to ten decimal places is 0.4794255386 and our estimate is correct to five decimal places. By comparison, a calculator gives $\sin(1/2) \approx 0.479426$ which is also correct to five decimal places. ◇

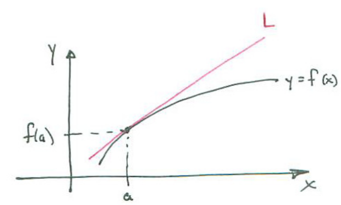


Figure 5.4.4: (Insert label for point $(a, f(a))$.)

The Linear Approximation $P_1(x,a)$

The graph of the Taylor polynomial $P_1(x,a) = f(a) + f'(a)(x-a)$ is a line that passes through the point $(a, f(a))$ and has the same slope as f does at a. That means that the graph of $P_1(x,a)$ is the *tangent line* to the graph of f at $(a, f(a))$. It is customary to call $P_1(x,a) = f(a) + f'(a)(x-a)$ the **linear approximation** to $f(x)$ for x near a. It is often denoted $L(x)$. Figure 5.4.4 shows the graphs of f and L near the point $(a, f(a))$.

Let x be a number close to a and define $\Delta x = x - a$ and $\Delta y = f(a+\Delta x) - f(a)$, quantities used in the definition of the derivative: $f'(a) = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$.

Often Δx is denoted by dx and $f'(a)dx$ is defined to be "dy", as shown in Figure 5.4.5. Note that dy is an approximation to Δy, and $f(a) + dy$ is an approximation to $f(a+\Delta x) = f(a) + \Delta y$.

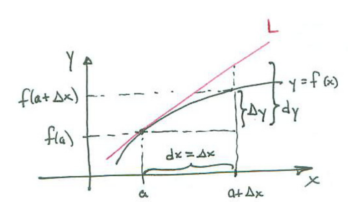


Figure 5.4.5:

In Section 8.2 we will use $dy = f'(x)dx$ and dx as bookkeeping tools to simplify the search for antiderivatives.

The expressions "dx" and "dy" are called **differentials**. In the seventeenth century, dx and dy referred to "infinitesimals", infinitely small numbers. Leibniz viewed the derivative as the quotient $\frac{dy}{dx}$, and that notation for the derivative persists more than three centuries later.

> **WARNING** *()*The derivative is not a quotient. The derivative is the limit of a quotient.

The next example uses the linear approximation to estimate $\sqrt{x}$ near $x = 1$.

EXAMPLE 5 Use $P_1(x;1)$ to estimate $\sqrt{x}$ for x near 1. Then discuss the error.

SOLUTION In this case $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, and $f'(1) = 1/2$. The linear approximation of $f(x)$ near $a = 1$ is

$$P_1(x;1) = f(1) + f'(1)(x-1) = 1 + \frac{1}{2}(x-1)$$

and the remainder is

$$R_1(x;1) = \sqrt{x} - \left(1 + \frac{1}{2}(x-1)\right).$$

Table 5.4.7 shows how rapidly $R_1(x;1)$ approaches 0 as $x \to 1$ and compares

x	$R_1(x;1)$					$(x-1)^2$	$R_1(x;1)/(x-1)^2$
2.0	$\sqrt{2}$	$-$	$\left(1+\frac{1}{2}(2-1)\right)$	$\approx$	-0.08578643	1	-0.08579
1.5	$\sqrt{1.5}$	$-$	$\left(1+\frac{1}{2}(1.5-1)\right)$	$\approx$	-0.02525512	0.25	-0.10102
1.1	$\sqrt{1.1}$	$-$	$\left(1+\frac{1}{2}(1.1-1)\right)$	$\approx$	-0.00119115	0.01	-0.11912
1.01	$\sqrt{1.01}$	$-$	$\left(1+\frac{1}{2}(1.01-1)\right)$	$\approx$	-0.00001243	0.0001	-0.12438

Table 5.4.7:

this difference with $(x-1)^2$.

The final column in Table 5.4.7 shows that $\frac{R_1(x;1)}{(x-1)^2}$ is nearly constant. Because $(x-1)^2 \to 0$ as $x \to 0$, this means $R_1(x;1)$ approaches 0 at the same rate as the square of $(x-1)$.

Since $R_1(x;1)$ is approximately $\frac{1}{2}f''(1)(x-1)^2$ when x is near 1, $\frac{R_1(x;1)}{(x-1)^2}$ should be near $\frac{1}{2}f''(1)$ when x is near 1. Just as a check, compute $\frac{1}{2}f''(1)$. We have $f''(x) = \frac{-1}{4}x^{-3/2}$. Thus $\frac{1}{2}f''(1) = \frac{1}{2}\left(\frac{-1}{4}\right) = \frac{-1}{8} = -0.125$. This is consistent with the final column of Table 5.4.7. ◇

Summary

Given a function f with n derivatives on an interval that contains the number a we defined the n^{th}-order Taylor polynomial at a, $P_n(x,a)$. The first n derivatives of the Taylor polynomial of degree n coincide with the first n derivatives

If we define the "zeroth derivative" of a function to be the function itself *and* start counting from 0, then we could say simply that the derivatives $P_n^{(k)}(x,a)$ coincide with $f^{(k)}(a)$ for $k = 0, 1, \ldots, n$.

of the given function f at a. Also, $P_n(x,a)$ has the same function value at a that f does.

$$P_n(x,a) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The remainder in using the Taylor polynomial of degree n to estimate a function involves the $(n+1)^{\text{st}}$ derivative of the function:

$$R_n(x,a) = f(x) - P_n(x,a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}$$

where c_n is a number between a and x. The error is the absolute value of the remainder, $|R_n(x,a)|$.

The linear approximation to a function at a is denoted

$$L(x) = P_1(x,a) = f(a) + f'(a)(x-a).$$

The differentials are $dx = x - a$ and $dy = f'(a)dx$. While $dx = \Delta x$, $dy \approx \Delta y = f(x + \Delta x) - f(x)$.

EXERCISES for 5.4

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

HINT: Use a graphing calculator or computer algebra computer algebra system to assist with the computations and with the graphing.

In Exercises 1 to 12 compute the Taylor polynomials $P_n(x;a)$. Graph $f(x)$ and $P_n(x;a)$ on the same axes on a domain centered at a. Keep in mind that the graph of $P_1(x;a)$ is the tangent line at the point $(a, f(x))$.

1. $f(x) = 1/(1+x)$, $P_1(x;0)$ and $P_2(x;0)$

2. $f(x) = 1/(1+x)$, $P_1(x;1)$ and $P_2(x;1)$

3. $f(x) = \ln(1+x)$, $P_1(x;0)$, $P_2(x;0)$ and $P_3(x;0)$

4. $f(x) = \ln(1+x)$, $P_1(x;1)$, $P_2(x;1)$ and $P_2(x;1)$

5. $f(x) = e^x$, $P_1(x;0)$, $P_2(x;0)$, $P_3(x;0)$, and $P_4(x;0)$

6. $f(x) = e^x$, $P_1(x;2)$, $P_2(x;2)$, $P_3(x;2)$, and $P_4(x;2)$

7. $f(x) = \arctan(x)$, $P_1(x;0)$, $P_2(x;0)$, and $P_3(x;0)$

8. $f(x) = \arctan(x))$, $P_1(x;-1)$, $P_2(x;-1)$, and $P_3(x;-1)$

9. $f(x) = \cos(x)$, $P_2(x;0)$ and $P_4(x;0)$

10. $f(x) = \sin(x)$, $P_7(x;0)$

11. $f(x) = \cos(x)$, $P_1(x;\pi/4)$

12. $f(x) = \sin(x)$, $P_3(x;\pi/4)$

13. Can there be a polynomial $p(x)$ such that $\sin(x) = p(x)$ for all x in the interval $[1, 1.0001]$? Explain.

14. Can there be a polynomial $p(x)$ such that $\ln(x) = p(x)$ for all x in the interval $[1, 1.0001]$? Explain.

15. State the Lagrange formula for the error in using a Taylor polynomial as an estimate of the value of a function. Use as little mathematical notation as you can.

16. Let $f(x) = \sqrt{x}$.

(a) What is the linear approximation, $P_1(x; 4)$, to $\sqrt{x}$ at $x = 4$?

(b) Fill in the following table.

x	$R_1(x; 4) = f(x) - P_1(x; 4)$	$(x-4)^2$	$\frac{R_1(x;4)}{(x-4)^2}$
5.0			
4.1			
4.01			
3.99			

(c) Compute $f''(4)/2$. Explain the relationship between this number and the entries in the fourth column of the table in (b).

17. Repeat Exercise 16 for the linear approximation to $\sqrt{x}$ at $x = 3$. Use $x = 4$, 3.1, 3.01, and 2.99.

18. Assume $f(x)$ has continuous first and second derivatives and that $4 \leq f''(x) \leq 5$ for all x.

(a) What can be said about the error in using $f(2) + f'(2)(x - 2)$ to approximate $f(x)$?

(b) How small should $x - 2$ be to be sure that the error — the absolute value of the remainder — is less than or equal to 0.005? NOTE: This ensures the approximate value is correct to 2 decimal places.

19. Let $f(x) = 2 + 3x + 4x^2$.

(a) Find $P_2(x; 0)$.

(b) Find $P_3(x; 0)$.

(c) Find $P_2(x; 5)$.

20. Find $P_4(x;0)$ and $P_5(x;0)$ for $f(x) = \sin(x)$.

21. Find $P_5(x;0)$ and $P_6(x;0)$ for $f(x) = \cos(x)$.

22. Let $f(x) = 2 + 3x - 4x^2$. Find $P_1(x;0)$, $P_2(x;0)$, and $P_3(x;0)$.

23.

(a) What can be said about the degree of the polynomial $P_n(x;0)$?

(b) When is the degree of $P_n(x;0)$ less than n?

(c) When is the degree of $P_n(x;a)$ less than n? $(a \neq 0)$

Exercises 24 to 29 are related.

24. Let $f(x) = (1+x)^3$.

(a) Find $P_3(x;0)$ and $R_3(x;0)$.

(b) Check that your answer to (a) is correct by multiplying out $(1+x)^3$.

25. Let $f(x) = (1+x)^4$.

(a) Find $P_4(x;0)$ and $R_4(x;0)$.

(b) Check that your answer to (a) is correct by multiplying out $(1+x)^4$.

26. Let $f(x) = (1+x)^5$. Using $P_5(x;0)$, show that

$$(1+x)^5 = 1 + 5x + \frac{5 \cdot 4}{1 \cdot 2}x^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}x^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}x^5.$$

For a positive integer n and a non-negative integer k, with $k \leq n$, the symbol $\binom{n}{k}$ denotes the **binomial coefficient**:

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdots k} = \frac{n!}{k!(n-k)!}.$$

Thus

$$(1+x)^5 = \binom{5}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5.$$

Using $P_n(x;0)$ one can show that, for any positive integer n,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

This is the basis for the **binomial theorem**,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

NOTE: Recall that $\binom{n}{0} = \frac{n!}{0!n!} = 1$ and $\binom{n}{n} = \frac{n!}{n!0!} = 1$.

In Exercises 27 and 28, use a calculator or computer to help evaluate the Taylor polynomials

27. Let $f(x) = e^x$.

(a) Find $P_{10}(x;0)$.

(b) Compute $f(x)$ and $P_{10}(x;0)$ at $x = 1$, $x = 2$, and $x = 4$.

28. Let $f(x) = \ln(x)$.

(a) Find $P_{10}(x;1)$.

(b) Compute $f(x)$ and $P_{10}(x;1)$ at $x = 1$, $x = 2$, and $x = 4$.

29. Obtain the binomial theorem from the special case $(1+x)^k = \sum_{k=0}^{n} \binom{n}{k} x^k$.

30. Using algebra, but no calculus, derive the binomial theorem for $(a+b)^3$ from the binomial theorem for $(1+x)^3$.

31.

(a) Which polynomials are even functions?

(b) If f is an even function, is $P_n(x;0)$ necessarily an even function? Explain.

32.

(a) Which polynomials are odd functions?

(b) If f is an odd function, is $P_n(x;0)$ necessarily an odd function? Explain.

33. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$.

(a) Find $f'(0)$.

(b) Find $f''(0)$.

(c) Find $P_2(x;0)$.

NOTE: HINT: : Recall the definition of the derivative.

34. Show that in an open interval in which f''' is positive, that $f(x) > f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$. HINT: Treat the cases $a < x$ and $x > a$ separately. NOTE: See also Exercise 30 in Section 4.4.

35. Show that in an open interval in which $f^{(n+1)}$ is positive (n a positive integer), that

$$f(x) > f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n.$$

HINT: Treat the cases $a < x$ and $x > a$ separately. NOTE: See also Exercise 34.

5.5 L'Hôpital's Rule for Finding Certain Limits

There are two types of limits in calculus: those that you can evaluate at a glance, and those that require some work to evaluate. For instance $\lim_{x\to\pi/2} \frac{\sin(x)}{x}$ is clearly $1/(\pi/2) = 2/\pi$. That's easy. But $\lim_{x\to 0}(\sin(x))/x$ is not obvious. Back in Section 2.1 we used a diagram of circles, sectors, and triangles, to show that this limit is 1.

In this section we describe a technique for evaluating some limits that are not visible at a glance, for instance

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

when both $f(x)$ and $f(x)$ approach 0 as x approaches a. The numerator is trying to drag $f(x)/g(x)$ toward 0, at the same time as the denominator is trying to make the quotient large. L'Hôpital's rule helps determine which term wins or whether there is a compromise.

L'Hôpital is pronounced lope-e-tal.

Indeterminate Limits

The following limits are called "indeterminate" because you can't determine them without knowing more about the functions of f and g.

$$\lim_{x\to a} \frac{f(x)}{g(x)}, \text{ where } \lim_{x\to a} f = 0 \text{ and } \lim_{x\to a} g = 0$$

$$\lim_{x\to a} \frac{f(x)}{g(x)}, \text{ where } \lim_{x\to a} f = \infty \text{ and } \lim_{x\to g} = \infty$$

L'Hôpital's Rule provides a way for dealing with these limits (and limits that can be transformed to those forms.) In short, l'Hôpital's rule applies only when you need it.

Theorem 5.5.1 (L'Hôpital's Rule (zero-over-zero case)) *Let a be a number and let f and g be differentiable over some open interval that contains a. Assume also that $g'(x)$ is not 0 for any x in that interval except perhaps at a. If*

$$\lim_{x\to a} f(x) = 0, \ \lim_{x\to a} g(x) = 0, \text{ and } \lim_{x\to a} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

In short, "to evaluate the limit of a quotient, that's indeterminant, evaluate the limit of the quotient of their derivatives." We will discuss the proof after some examples.

You evaluate the limit of the quotient of the derivatives, not the derivative of the quotient.

EXAMPLE 1 Find $\lim_{x\to 1}(x^5-1)/(x^3-1)$.
SOLUTION In this case,

$$a=1,\ f(x)=x^5-1,\ \text{and } g(x)=x^3-1.$$

All the assumptions of l'Hôpital's rule are satisfied. In particular,

First checking that the assumption of l'Hôpital's Rule holds

$$\lim_{x\to 1}(x^5-1)=0 \text{ and } \lim_{x\to 1}(x^3-1)=0.$$

According to l'Hôpital's rule,

$$\lim_{x\to 1}\frac{x^5-1}{x^3-1} \overset{\text{l'H}}{=} \lim_{x\to 1}\frac{(x^5-1)'}{(x^3-1)'}$$

if the latter limit exists. Now,

$$\begin{array}{rcll} \lim_{x\to 1}\frac{(x^5-1)'}{(x^3-1)'} & = & \lim_{x\to 1}\frac{5x^4}{3x^2} & \text{differentiation of numerator and differentiation of denominator} \\ & = & \lim_{x\to 1}\frac{5}{3}x^2 & \text{algebra} \\ & = & \frac{5}{3}. & \end{array}$$

Thus

$$\lim_{x\to 1}\frac{x^5-1}{x^3-1}=\frac{5}{3}.$$

◇

Sometimes it may be necessary to apply l'Hôpital's Rule more than once, as in the next example.

EXAMPLE 2 Find $\lim_{x\to 0}(\sin(x)-x)/x^3$.
SOLUTION As $x\to 0$, both numerator and denominator approach 0. By l'Hôpital's Rule,

$$\begin{aligned} \lim_{x\to 0}\frac{\sin(x)-x}{x^3} &\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{(\sin(x)-x)'}{(x^3)'} \\ &= \lim_{x\to 0}\frac{\cos(x)-1}{3x^2}. \end{aligned}$$

But as $x\to 0$, both $\cos(x)-1\to 0$ and $3x^2\to 0$. So use l'Hôpital's Rule again:

Repeated application of l'Hôpital's Rule

$$\begin{aligned} \lim_{x\to 0}\frac{\cos(x)-1}{3x^2} &\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{(\cos(x)-1)'}{(3x^2)'} \\ &= \lim_{x\to 0}\frac{-\sin(x)}{6x}. \end{aligned}$$

Both $\sin(x)$ and $6x$ approach 0 as $x \to 0$. Use l'Hôpital's Rule yet another time:

Or recall from Section 2.1 that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

$$\begin{aligned} \lim_{x\to 0} \frac{-\sin(x)}{6x} &\overset{\text{l'H}}{=} \lim_{x\to 0} \frac{(-\sin(x))'}{(6x)'} \\ &= \lim_{x\to 0} \frac{-\cos(x)}{6} \\ &= \frac{-1}{6}. \end{aligned}$$

So after three applications of l'Hôpital's Rule we find that

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = -\frac{1}{6}.$$

◇

Sometimes a limit may be simplified before l'Hôpital's Rule is applied. For instance, consider

$$\lim_{x\to 0} \frac{(\sin(x) - x)\cos^5(x)}{x^3}.$$

Since $\lim_{x\to 0} \cos^5(x) = 1$, we have

$$\lim_{x\to 0} \frac{(\sin(x) - x)\cos^5(x)}{x^3} = \left(\lim_{x\to 0} \frac{\sin(x) - x}{x^3}\right) \cdot 1,$$

which, by Example 2, is $-\frac{1}{6}$. This shortcut saves a lot of work, as may be checked by finding the limit using l'Hôpital's Rule without separating $\cos^5(x)$.

Replacing a by ∞ in Theorem 5.5.1, etc.

Theorem 5.5.1 concerns limits as $x \to a$. L'Hôpital's Rule also applies if $x \to \infty$, $x \to -\infty$, $x \to a^+$, or $x \to a^-$. In the first case, we would assume that $f(x)$ and $g(x)$ are differentiable in some interval (c, ∞) and $f'(x)$ is not zero there. In the case of $x \to a^+$, assume that $f(x)$ and $g(x)$ are differentiable in some open interval (a, b) and $g'(x)$ is not 0 there.

Infinity-over-Infinity Limits

"Infinity-over-infinity" ($\frac{\infty}{\infty}$) is indeterminate.

Theorem 5.5.1 concerns the limit of $f(x)/g(x)$ when both $f(x)$ and $g(x)$ approach 0. But a similar problem arises when both $f(x)$ and $g(x)$ get arbitrarily large as $x \to a$ or as $x \to \infty$. The behavior of the quotient $f(x)/g(x)$ will be influenced by how rapidly $f(x)$ and $g(x)$ become large.

In short, if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a}(f(x)/g(x))$ is an indeterminate form.

Another indeterminate form

The next theorem presents a form of l'Hôpital's Rule that covers the case in which $f(x) \to \infty$ and $g(x) \to \infty$.

Theorem 5.5.2 (L'Hôpital's Rule (infinity-over-infinity case) *Let f and g be defined and differentiable for all x larger than some fixed number. Then, if*

L'Hôpital's rule for the infinity-over-infinity case

$$\lim_{x\to\infty} f(x) = \infty, \ \lim_{x\to\infty} g(x) = \infty, \ \textit{and} \ \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L,$$

it follows that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

A similar result holds for $x \to a$, $x \to a^-$, $x \to a^+$, or $x \to -\infty$. Moreover, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ could both be $-\infty$, or one could be ∞ and the other $-\infty$.

EXAMPLE 3 Find $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$.

SOLUTION Since $\ln(x) \to \infty$ and $x^2 \to \infty$ as $x \to \infty$, we may use l'Hôpital's Rule in the "infinity-over-infinity" form.

We have

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln(x)}{x^2} &\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{(\ln(x))'}{(x^2)'} \\ &= \lim_{x\to\infty} \frac{1/x}{2x} \\ &= \lim_{x\to\infty} \frac{1}{2x^2} \\ &= 0. \end{aligned}$$

Hence $\lim_{x\to\infty}((\ln(x))/x^2) = 0$. This says that $\ln(x)$ grows much more slowly than x^2 does as x gets large. ◇

EXAMPLE 4 Find

$$\lim_{x\to\infty} \frac{x - \cos(x)}{x}. \tag{1}$$

SOLUTION Both numerator and denominator approach ∞ and $x \to \infty$. Trying l'Hôpital's Rule, we obtain

$$\begin{aligned} \lim_{x\to\infty} \frac{x-\cos(x)}{x} &\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{(x-\cos(x))'}{x'} \\ &= \lim_{x\to\infty} \frac{1+\sin(x)}{1}. \end{aligned}$$

L'Hôpital's Rule may fail to provide an answer

But $\lim_{x\to\infty}(1+\sin(x))$ does not exist, since $\sin(x)$ oscillates back and forth from -1 to 1 as $x \to \infty$

What can we conclude about the limit in (1)? Nothing at all. L'Hôpital's Rule says that if $\lim_{x\to\infty} f'(x)/g'(x)$ exists, then $\lim_{x\to\infty} f(x)/g(x)$ exists and has the same value. It say nothing about the case when $\lim_{x\to\infty} f'(x)/g'(x)$ does not exist.

It is not difficult to evaluate (1) directly, as follows:

Moral: Look carefully at a limit before you decide to use l'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to\infty} \tfrac{x-\cos(x)}{x} &= \lim_{x\to\infty}\left(1-\tfrac{\cos(x)}{x}\right) && \text{algebra}\\
&= 1-0 && \text{since } |\cos(x)|\le 1\\
&= 1.
\end{aligned}$$

◇

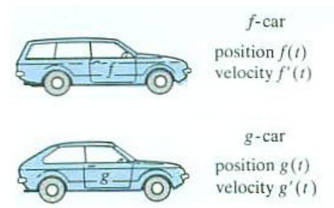


Figure 5.5.1:

Two cars can help make Theorem 5.5.2 plausible. Imagine that $f(t)$ and $g(t)$ describe the locations on the x axis of two cars at time t. Call the cars the f-car and the g-car. See Figure 5.5.1. Their velocities are therefore $f'(t)$ and $g'(t)$. These two cars are on endless journeys. But let us assume that as time $t\to\infty$ the f-car tends to travel at a speed closer and closer to L times the speed of the g-car. That is, assume that

$$\lim_{t\to\infty}\frac{f'(t)}{g'(t)} = L.$$

No matter how the two cars move in the short run, it seems reasonable that in the long run the f-car will tend to travel about L times as far as the g-car; that is,

$$\lim_{t\to\infty}\frac{f(t)}{g(t)} = L.$$

Transforming Limits So You Can Use l'Hôpital's Rule

Many limits can be transformed to limits to which l'Hôpital's Rule applies. For instance, the problem of finding

zero-times-infinity $(0\cdot\infty)$ case

$$\lim_{x\to 0^+} x\ln(x)$$

does not fit into l'Hôpital's Rule, since it does not involve the quotient of two functions. As $x\to 0^+$, one factor, x, approaches 0 and the other factor $\ln(x)$, approaches $-\infty$. So this is another type of indeterminate limit, a small number times a large number ("zero-times-infinity"). It is not obvious how this product, $x\ln(x)$, behaves as $x\to 0^+$. (Such a limit can turn out to be "zero, medium, large, or infinite"). A little algebra transforms the zero-times-infinity case into a problem to which l'Hôpital's Rule applies, as the next example illustrates.

EXAMPLE 5 Find $\lim_{x\to 0^+} x\ln(x)$.
SOLUTION Rewrite $x\ln(x)$ as a quotient, $\frac{\ln(x)}{(1/x)}$. Note that

$$\lim_{x\to 0^+} \ln(x) = -\infty \text{ and } \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

By l'Hôpital's Rule,

$$\begin{aligned} \lim_{x\to 0^+} \frac{\ln(x)}{1/x} &\overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to 0^+} (-x) \\ &= 0. \end{aligned}$$

Thus

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = 0,$$

from which it follows that $\lim_{x\to 0^+} x\ln(x) = 0$. ◇

The factor x, which approaches 0, dominates the factor $\ln(x)$ which "slowly grows towards $-\infty$."

The final example illustrates another type of limit that can be found by first relating it to limits to which l'Hôpital's Rule applies.

Try this on your calculator first.

EXAMPLE 6 $\lim_{x\to 0^+} x^x$.
SOLUTION Since this limit involves an exponential, not a quotient, it does not fit directly into l'Hôpital's Rule. But a little algebra changes the problem to one covered by l'Hôpital's Rule.

Let	y	$=$	x^x.
Then	$\ln(y)$	$=$	$\ln(x^x) = x\ln(x)$
By Example 5,	$\lim_{x\to 0^+} x\ln(x)$	$=$	0.

Therefore,$\lim_{x\to 0^+} \ln(y) = 0$ so y must approach 1 as $x \to 0^+$. Thus, using the definition of $\ln(y)$ and the continuity of e^x:

$$\text{Thus,} \quad \begin{aligned} \lim_{x\to 0^+} y &= \lim_{x\to 0^+} e^{\ln(y)} \\ &= e^{\lim_{x\to 0^+} \ln(y)} \\ &= e^0 \\ &= 1. \end{aligned}$$

Hence $x^x \to 1$ as $x \to 0^+$. ◇

Concerning the Proof

Argument for a special case of Theorem 5.5.1

A complete proof of Theorem 5.5.1 may be found in Exercises 70 to 72. The following argument is intended to make the theorem plausible. To do so, consider the *special case* where f, f', g, and g' are all continuous throughout an open interval containing a — in particular, all four functions are defined at a. Assume that $g'(x) \neq 0$ throughout the interval. Since we have $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, it follows by continuity that $f(a) = 0$ and $g(a) = 0$.

Differentiable functions are automatically continuous.

Assume that $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Then

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \tfrac{f(x)-f(a)}{g(x)-g(a)} && \text{since } f(a)=0 \text{ and } g(a)=0 \\
&= \lim_{x\to a} \tfrac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} && \text{algebra} \\
&= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} && \text{limit of quotient = quotient of limits} \\
&= \frac{f'(a)}{g'(a)} && \text{definitions of } f'(a) \text{ and } g'(a) \\
&= \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} && f' \text{ and } g' \text{ are continuous, by assumption} \\
&= \lim_{x\to a} \tfrac{f'(x)}{g'(x)} && \text{quotient of limits = limit of quotients} \\
&= L && \text{by assumption.}
\end{aligned}$$

Consequently,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Summary

We described l'Hôpital's Rule, which is a technique for dealing with limits of the indeterminate form "zero-over-zero" $\left(\frac{0}{0}\right)$ and "infinity-over-infinity" $\left(\frac{\infty}{\infty}\right)$. In both of these cases

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if the latter limit exists. Note that it concerns the quotient of two derivatives, *not* the derivative of the quotient.

Table 5.5.1 shows how some limits of other indeterminate forms can be converted into either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate Forms	Name	Conversion Method	New Form
$f(x)g(x)$; $f(x) \to 0$, $g(x) \to 0$	Zero-times-infinity ($0 \cdot \infty$)	Write as $\frac{f(x)}{1/g(x)}$ or $\frac{g(x)}{1/f(x)}$	$\frac{0}{0}$ or $\frac{\infty}{\infty}$
$f(x)^{g(x)}$; $f(x) \to 1$, $g(x) \to \infty$	One-to-the-infinity (1^∞)	Let $y = f(x)^{g(x)}$; take $\ln(y)$, find limit of $\ln(y)$, and then find limit of $y = e^{\ln(y)}$	$\ln(y)$ has form $\infty \cdot 0$
$f(x)^{g(x)}$; $f(x) \to 0$, $g(x) \to 0$	Zero-to-the-zero (0^0)	Same as for 1^∞	$\ln(y)$ has form $0 \cdot \infty$.

Table 5.5.1:

EXERCISES for 5.5

Exercises will be ordered by increasing difficulty. Some exercises will be added and others will be moved to a Chapter Summary or to another section.

In Exercises 1 to 16 check that l'Hôpital's Rule applies and use it to find the limits. *Identify all uses of l'Hôpital's Rule, including the type of indeterminant form.*

1. $\displaystyle\lim_{x\to 2}\frac{x^3-8}{x^2-4}$

2. $\displaystyle\lim_{x\to 1}\frac{x^7-1}{x^3-1}$

3. $\displaystyle\lim_{x\to 0}\frac{\sin(3x)}{\sin(2x)}$

4. $\displaystyle\lim_{x\to 0}\frac{\sin(x^2)}{(\sin(x))^2}$

5. $\displaystyle\lim_{x\to 0}\frac{\sin(5x)\cos(3x)}{x}$

6. $\displaystyle\lim_{x\to 0}\frac{\sin(5x)\cos(3x)}{x-\frac{\pi}{2}}$

7. $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{\sin(5x)\cos(3x)}{x}$

8. $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{\sin(5x)\cos(3x)}{x-\frac{\pi}{2}}$

9. $\displaystyle\lim_{x\to \infty}\frac{x^3}{e^x}$

10. $\displaystyle\lim_{x\to \infty}\frac{x^5}{3^x}$

11. $\displaystyle\lim_{x\to 0}\frac{1-\cos(x)}{x^2}$

12. $\displaystyle\lim_{x\to 0}\frac{\sin(x)-x}{(\sin(x))^3}$

13. $\displaystyle\lim_{x\to 0}\frac{\tan(3x)}{\ln(1+x)}$

14. $\displaystyle\lim_{x\to 1}\frac{\cos(\pi x/2)}{\ln(x)}$

15. $\displaystyle\lim_{x\to 2}\frac{(\ln(x))^2}{x}$

16. $\lim_{x\to 0} \frac{\arcsin(x)}{e^{2x}-1}$

In each of Exercises 17 to 22 transform the problem into one to which l'Hôpital's Rule applies; then find the limit. *Identify all uses of l'Hôpital's Rule, including the type of indeterminant form.*

17. $\lim_{x\to 0}(1-2x)^{1/x}$

18. $\lim_{x\to 0}(1+\sin(2x))^{\csc(x)}$

19. $\lim_{x\to 0^+}(\sin(x))^{(e^x-1)}$

20. $\lim_{x\to 0^+} x^2\ln(x)$

21. $\lim_{x\to 0^+}(\tan(x))^{\tan(2x)}$

22. $\lim_{x\to 0^+}(e^x-1)\ln(x)$

> **WARNING** *(R)*emember that l'Hôpital's Rule, carelessly applied, may give a wrong answer or no answer.

In Exercises 23 to 50 find the limits. Use l'Hôpital's Rule only if it applies. *Identify all uses of l'Hôpital's Rule, including the type of indeterminant form.*

23. $\lim_{x\to\infty} \frac{2^x}{3^x}$

24. $\lim_{x\to\infty} \frac{2^x+x}{3^x}$

25. $\lim_{x\to\infty} \frac{\log_2(x)}{\log_3(x)}$

26. $\lim_{x\to 1} \frac{\log_2(x)}{\log_3(x)}$

27. $\lim_{x\to\infty} \left(\frac{1}{x}-\frac{1}{\sin(x)}\right)$

28. $\lim_{x\to\infty} \left(\sqrt{x^2+3}-\sqrt{x^2+4x}\right)$

29. $\lim_{x\to\infty} \frac{x^2+3\cos(5x)}{x^2-2\sin(4x)}$

30. $\lim_{x\to\infty} \frac{e^x - 1/x}{e^x - 1/x}$

31. $\lim_{x\to 0} \frac{3x^3 + x^2 - x}{5x^3 + x^2 + x}$

32. $\lim_{x\to\infty} \frac{3x^3 + x^2 - x}{5x^3 + x^2 + x}$

33. $\lim_{x\to\infty} \frac{\sin(x)}{4 + \sin(x)}$

34. $\lim_{x\to\infty} x\sin(3x)$

35. $\lim_{x\to 1^+} (x-1)\ln(x-1)$

36. $\lim_{x\to\pi/2} \frac{\tan(x)}{x - (\pi/2)}$

37. $\lim_{x\to 0} (\cos(x))^{1/x}$

38. $\lim_{x\to 0^+} x^{1/x}$

39. $\lim_{x\to\infty} \frac{\sin(2x)}{\sin(3x)}$

40. $\lim_{x\to 1} \frac{x^2 - 1}{x^3 - 1}$

41. $\lim_{x\to 0} \frac{xe^x(1+x)^3}{e^x - 1}$

42. $\lim_{x\to 0} \frac{xe^x\cos^2(6x)}{e^{2x} - 1}$

43. $\lim_{x\to 0} (\csc(x) - \cot(x))$

44. $\lim_{x\to 0} \frac{\csc(x) - \cot(x)}{\sin(x)}$

45. $\lim_{x\to 0} \frac{5^x - 3^x}{\sin(x)}$

46. $\lim_{x\to 0} \frac{(\tan(x))^5 - (\tan(x))^3}{1 - \cos(x)}$

47. $\lim_{x\to 2} \frac{x^3 + 8}{x^2 + 5}$

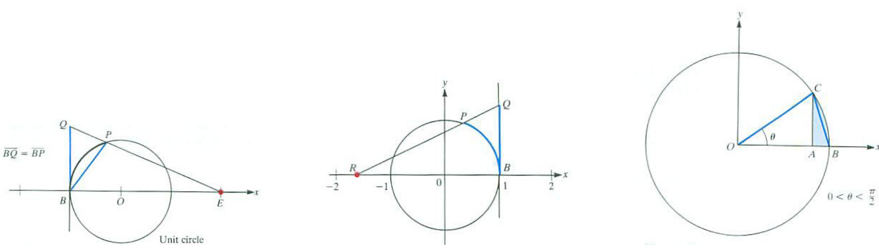
48. $\lim_{x \to \pi/4} \frac{\sin(5x)}{\sin(3x)}$

49. $\lim_{x \to 0} \left(\frac{1}{1 - \cos(x)} - \frac{2}{x^2} \right)$

50. $\lim_{x \to 0} \frac{\arcsin(x)}{\arctan(2x)}$

51. In Figure 5.5.2(a) the unit circle is centered at O, BQ is a vertical tangent line, and the length of BP is the same as the length of BQ. What happens to the point E as $Q \to B$?

52. In Figure 5.5.2(b) the unit circle is centered at the origin, BQ is a vertical tangent line, and the length of BQ is the same as the arc length $\overset{\frown}{BP}$. Prove that the x-coordinate of R approaches -2 as $P \to B$.



(a) (b) (c)

Figure 5.5.2:

53. Exercise 44 of Section 2.1 asked you to guess a certain limit. Now that limit will be computed.

> **WARNING** *(A)*s Albert Einstein observed, "Common sense is the deposit of prejudice laid down in the mind before the age of 18."

In Figure 5.5.2(c), which shows a circle, let

$f(\theta)$ = area of triangle $ABCg(\theta)$ = area of shaded region formed by deleting triangle OAC from sector OBC.

Clearly, $0 < f(\theta) < g(\theta)$.

(a) What would you guess is the value of $\lim_{\theta \to 0} f(\theta)/g(\theta)$?

(b) Find $\lim_{\theta \to 0} f(\theta)/g(\theta)$.

54. In Eugene Silberberg, *The Structure of Economics*, McGraw-Hill, New York, 1978, the following argument appears:

"Consider the production function

$$y = k\left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-1/\rho},$$

where k, α, x_1, and x_2 are positive constants and $\alpha < 1$. Taking the limit as $\rho \to 0^+$, we find that

$$\lim_{\rho\to 0^+} y = kx_1^{\alpha}x_2^{1-\alpha},$$

which is the Cobb-Douglas function, as expected."

Fill in the details.

55. Sam proposes the following proof for Theorem 5.5.1: "Since

$$\lim_{x\to a^+} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a+} g(x) = 0,$$

I will define $f(a) = 0$ and $g(a) = 0$. Next I consider $x > a$ but near a. I now have continuous functions f and g defined on the closed interval $[a, x]$ and differentiable on the open interval (a, x). So, using the Mean-Value Theorem, I conclude that there is a number c, $a < c < x$, such that

$$\frac{f(x) - f(a)}{x-a} = f'(c) \qquad \textit{and} \qquad \frac{g(x) - g(a)}{x-a} = g'(c).$$

Since $f(a) = 0$ and $g(a) = 0$, these equations tell me that

$$f(x) = (x-a)f'(c)\text{and} \qquad g(x) = (x-a)g'(c)$$

$$\begin{array}{lrcl} \text{Thus} & \frac{f(x)}{g(x)} & = & \frac{f'(c)}{g'(c)} \\ \text{Hence} & \lim_{x\to a^+}\frac{f(x)}{g(x)} & = & \lim_{x\to a^+}\frac{f'(c)}{g'(c)}. \end{array}$$

Alas, Sam made one error. What is it?

56. Find $\lim_{x\to 0}\left(\frac{1+2^x}{x}\right)^{1/x}$.

57. R. P. Feynman, in *Lectures in Physics*, wrote: "Here is the quantitative answer of what is right instead of kT. This expression

$$\frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

should, of course, approach kT as $\omega \to 0$. . . . . See if you can prove that it does — learn how to do the mathematics."

Do the mathematics.

58. Graph $y = x^x$ for $0 < x \leq 1$, showing its minimum point.

In Exercises 59 to 61 graph the specified function, being sure to show *(a)* where the function is increasing and decreasing, *(b)* where the function has any asymptotes, and *(c)* how the function behaves for x near 0.

59. $f(x) = (1 + x)^{1/x}$ for $x > -1$, $x \neq 0$

60. $y = x \ln(x)$

61. $y = x^2 \ln(x)$

62. In which cases below is it possible to determine $\lim_{x \to a} f(x)^{g(x)}$ without further information about the functions?

(a) $\lim_{x \to a} f(x) = 0$; $\lim_{x \to a} g(x) = 7$

(b) $\lim_{x \to a} f(x) = 2$; $\lim_{x \to a} g(x) = 0$

(c) $\lim_{x \to a} f(x) = 0$; $\lim_{x \to a} g(x) = 0$

(d) $\lim_{x \to a} f(x) = 0$; $\lim_{x \to a} g(x) = \infty$

(e) $\lim_{x \to a} f(x) = \infty$; $\lim_{x \to a} g(x) = 0$

(f) $\lim_{x \to a} f(x) = \infty$; $\lim_{x \to a} g(x) = -\infty$

63. In which cases below is it possible to determine $\lim_{x \to a} f(x)/g(x)$ without further information about the functions?

(a) $\lim_{x \to a} f(x) = 0$; $\lim_{x \to a} g(x) = \infty$

(b) $\lim_{x \to a} f(x) = 0$; $\lim_{x \to a} g(x) = 1$

(c) $\lim_{x \to a} f(x) = 0$; $\lim_{x \to a} g(x) = 0$

(d) $\lim_{x \to a} f(x) = \infty$; $\lim_{x \to a} g(x) = -\infty$

64. Jane says, "I can get $\lim_{x \to 0} \frac{e^x - 1}{x}$ easily. It's just the derivative of e^x evaluated at 0. I don't need l'Hôpital's Rule." Is Jane right, or has Sam's influence affected her ability to reason?

65.

$$\begin{array}{lrcl} \text{If} & \lim_{t\to\infty} f(t) = & \infty & = \lim_{t\to\infty} g(t) \\ \text{and} & \lim_{t\to\infty} \frac{f(t)}{g(t)} & = & 3, \end{array}$$

what can be said about

$$\lim_{t\to\infty} \frac{\ln(f(t))}{\ln(g(t))}?$$

NOTE: Do *not* assume f and g are differentiable.

66. Give an example of a pair of functions f and g such that we have $\lim_{x\to 0} f(x) = 1$, $\lim x \to 0 g(x) = \infty$, and $\lim_{x\to 0} f(x)^{g(x)} = 2$.

67. Sam is angry. "Now I know why calculus books are so long. They spend all of page 55 showing that $\lim_{x\to 0} \frac{\sin(x)}{x}$ is 1. They could have saved space (and me a lot of trouble) if they had just used l'Hôpital's approach."

Is Sam right, for once?

68. Obtain l'Hôpital's Rule for $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ from the case $\lim_{t\to 0^+} \frac{f(t)}{g(t)}$.
HINT: Let $t = 1/x$.

69. Find the limit of $(1^x + 2^x + 3^x)^{1/x}$ as

(a) $x \to 0$

(b) $x \to \infty$

(c) $x \to -\infty$.

The proof of Theorem 5.5.1, to be outlined in Exercise 72, depends on the following generalized mean-value theorem.

Generalized Mean-Value Theorem. Let f and g be two functions that are continuous on $[a, b]$ and differentiable on (a, b). Furthermore, assume that $g'(x)$ is never 0 for x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

70. During a given time interval one car travels twice as far as another car. Use the Generalized Mean-Value Theorem to show that there is at least one instant when the first car is traveling exactly twice as fast as the second car.

71. To prove the Generalized Mean-Value Theorem, introduce a function h defined by

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)). \qquad (2)$$

Show that $h(b) = 0$ and $h(a) = 0$. Then apply Rolle's Theorem to h on (a, b). NOTE: Rolle's Theorem is Theorem 4.1.2 in Section 4.1.

> **Remark** The function h in (2) is similar to the function h used in the proof of the Mean-Value Theorem (Theorem 4.1.3 in Section 4.1). Check that $h(x)$ is the vertical distance between the point $(g(x), f(x))$ and the line through $(g(a), f(a))$ and $(g(b), f(b))$.

72. Assume the hypotheses of Theorem 5.5.1. Define $f(a) = 0$ and $g(a) = 0$, so that f and g are continuous at a. Note that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)},$$

and apply the Generalized Mean-Value Theorem from Exercise 70. NOTE: This Exercise proves Theorem 5.5.1, l'Hôpital's Rule in the zero-over-zero case.

73.

$$\begin{array}{lrcl} \text{If} & \lim_{t\to\infty} f(t) = & \infty & = \lim_{t\to\infty} g(t) \\ \text{and} & \lim_{t\to\infty} \frac{\ln(f(t))}{\ln(g(t))} & = & 1, \\ \text{must} & \lim_{t\to\infty} \frac{f(t)}{g(t)} & = & 1? \end{array}$$

Explain.

74.

Sam: I bet I can find $\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$ by using the Taylor polynomial $P_2(x; 0)$ for e^x and paying attention to the error.

Is Sam right?

5.S Chapter Summary

The text and additional exercises for the summary will be written after the organization of the chapters is firmly settled.

EXERCISES for 5.S *Key:* R–routine, M–moderate, C–challenging

1. In Example 4 an approximate value for Euler's constant is obtained by applying Theorem 5.3.2 to $R(x) = e^x - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$ on the interval $(-1, 1)$ with $a = 0$ and $x = \frac{1}{2}$.

(a) Compare this estimate with the estimate obtained by applying Theorem 5.3.2 to $R(x)$ on the interval $(-1, 2)$ with $a = 0$ and $x = 1$.

(b) Why is it not permissible to apply Theorem 5.3.2 in this case on the interval $(0, 1)$?

Remember to start counting derivatives at zero: $f^{(0)}(x) = f(x)$.

2. Supply all the steps to show that the polynomial $a_0 + a_1(x - a) + a_2(x - a)^2$ whose first two derivatives coincide with those for $f(x)$ is given by $f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$.