Chapter 5

The Definite Integral

Up to this point we have been concerned with the derivative, which gives local information, such as the slope at a particular point on a curve or the velocity at a particular time. The present chapter introduces the second major concept of calculus, the definite integral. In contrast to the derivative, the definite integral gives overall global information, such as the area under a curve.

The derivative turns out to be one of the tools for evaluating definite integrals.

Section 5.1 motivates the definite integral through three of its applications. Section 5.2 defines the definite integral. Section 5.3 presents ways to estimate the definite integral. Sections 5.4 and 5.5 provide the connection between the derivative and the definite integral, which culminates in the Fundamental Theorem of Calculus.

Chapters 1 to 4 form the core of calculus. Later chapters are mostly variations or applications of the key ideas in these five chapters.

5.1 Three Problems That Are One Problem

This chapter introduces the integral by three problems. At first glance these problems may seem unrelated, but by the end of the section it will be clear that they represent one basic problem in various guises. They lead up to the concept of the definite intgra, which is defined in the next section.

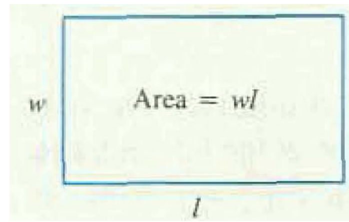


Figure 5.1.1:

Estimating an Area

It is easy to find the exact area of a rectangle. Just multiply its length by its width (see Figure 5.1.1). But, how do you find the area of the region in Figure 5.1.2? In this section we will show how to make very accurate *estimates* of the area. The technique we use will lead up to the definition of the definite integral of a function in the next section.

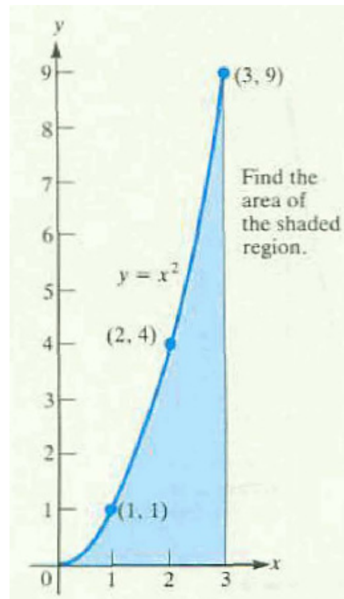


Figure 5.1.2:

PROBLEM 1 Estimate the area of the region bounded by the curve $y = x^2$, the x-axis, and the vertical line $x = 3$, as shown in Figure 5.1.2.

Since we know how to find the area of a rectangle, we will use rectangles to approximate the region in Figure 5.1.2. Figure 5.1.3(a) shows an approximation by six rectangles whose total area is less than the area under the parabola. Figure 5.1.3(b) shows a similar approximation whose area is less than the area under the parabola.

In each case we break the interval $[0, 3]$ into six shorter intervals, all of width $\frac{1}{2}$. In order to find the area of the overestimate and the area of the underestimate, we must find the height of each rectangle. That height is determined by the curve $y = x^2$. Let us examine only the overestimate, leaving the underestimate for the Exercises.

There are six rectangles in the overestimate shown in Figure 5.1.3(a). The smallest rectangle is shown in Figure 5.1.3(c). The height of the rectangle in Figure 5.1.3c) is equal to the value of x^2 when $x = \frac{1}{2}$. The height is therefore $\left(\frac{1}{2}\right)^2$ and the area is $\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)$, the product of its height and its width. The areas of the other five rectangles can be found similarly, in each case evaluating x^2 at the right end of the rectangle's base in order to find the height. There total area is

$$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{2}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{6}{2}\right)^2\left(\frac{1}{2}\right),$$

which is

$$\frac{1}{8}\left(1^2+2^2+3^2+4^2+5^2+6^2\right) = \frac{91}{8} = 11.375. \tag{5.1}$$

The area under the parabolas is therefore less that 11.375.

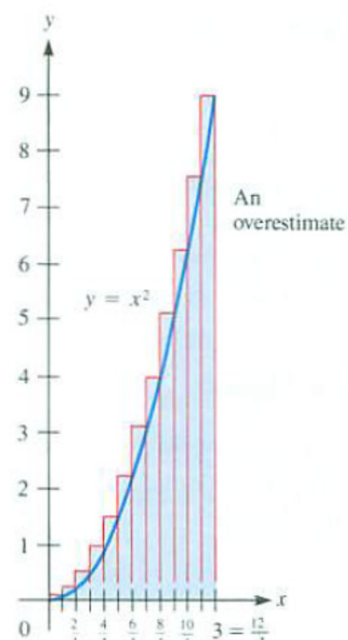
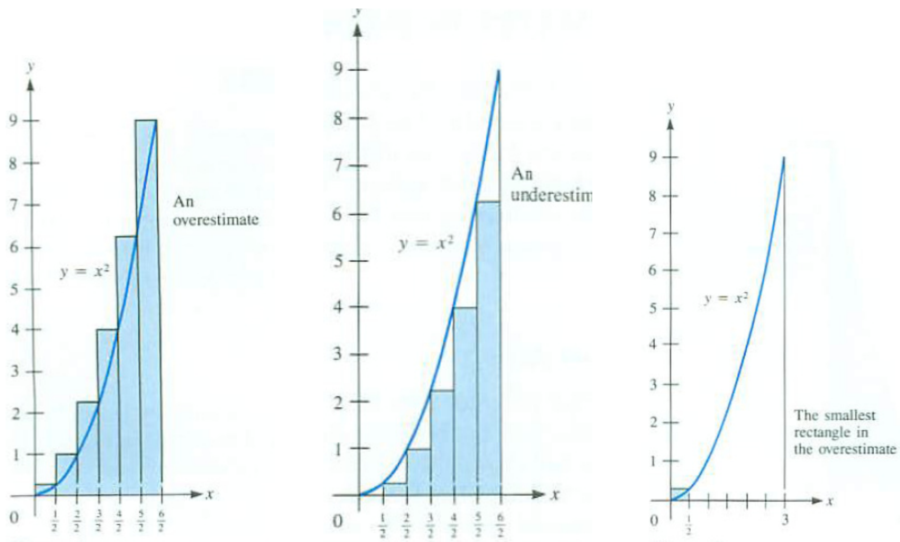


Figure 5.1.4:



(a) (b) (c)

Figure 5.1.3:

To get a closer estimate we should use more rectangles. Figure 5.1.4 shows an overestimate in which there are 12 rectangles. Each has width $\frac{3}{12} = \frac{1}{4}$. The total area of the overestimate is

$$\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{2}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) + \cdots + \left(\frac{12}{4}\right)^2\left(\frac{1}{4}\right),$$

which is

$$\frac{1}{64}\left(1^2 + 2^2 + 3^2 + \cdots + 12^2\right) = \frac{650}{64} = 10.15625. \tag{5.2}$$

Now we know the area under the parabolas is less that 10.15625.

To get closer estimates we would cut the interval $[0, 3]$ into more sections, maybe 100 or 10000 or more, and calculuate the total area of the corresponding rectangles. (This is an easy computation on a computer.)

Figure 5.1.5:

In general, we would divide $[0, 3]$ into n sections of equal length. The length of each section is then $\frac{3}{n}$. Their endpoints are shown in Figure 5.1.5.

Then, for each integer $i = 1,\ 2,\ \ldots, n$, the i^{th} section from the left has endpoints $(i-1)\left(\frac{3}{n}\right)$ and $i\left(\frac{3}{n}\right)$, as shown in Figure 5.1.6.

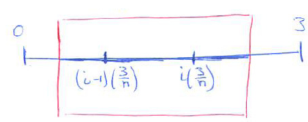


Figure 5.1.6: [Redraw to give effect of zooming in on ith interval]

To make an overestimate we observe that x^2 is increasing for $x > 0$ and evaluate x^2 at the right endpoint of each interval and multiply the result by the width of the interval, getting

$$\left(i\left(\frac{3}{n}\right)\right)^2\frac{3}{n} = 3^3\frac{i^2}{n^3}.$$

Then, sum these overestimates for all n intervals:

$$3^3\frac{1^2}{n^3} + 3^3\frac{2^2}{n^3} + 3^3\frac{3^2}{n^3} + \cdots + 3^3\frac{(n-1)^2}{n^3} + 3^3\frac{n^2}{n^3}$$

which simplifies to

$$3^3\left(\frac{1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2}{n^3}\right). \tag{5.3}$$

In the summation notation described in Appendix C(?), this equals

$$3^3\frac{\sum_{i=1}^{n} i^2}{n^3}.$$

We have already seen that these overestimates become more and more accurate as the number of intervals increases. We would like to know what happens to the overestimate as n gets larger and larger. More specifically, does

$$\lim_{n\to\infty}\frac{1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2}{n^3} \tag{5.4}$$

exist?

The numerator gets large, tending to make the fraction large. But, the denominator also gets large, which tends to make the fraction small. Once again we are encountering one of the "limit battles" that occurs in the foundation of calculus.

Archimedes, some 2000 years ago, found a short formula for the numerator in (5.3), enabling him to limit in (5.4

). See, for instance, S. Stein, "Archimedes: What did he do besides cry Eureka?".

For $n = 6$, with the calculation in (5.1), we have that (5.3) equals $\frac{91}{6^3} = \frac{91}{216} \approx 0.421$. For $n = 12$, with the calculation in (5.2), we have that (5.3) equals $\frac{650}{12^3} = \frac{650}{1728} \approx 0.376$.

If we knew the limiit in (5.3) we would know the area under the parabola and above the interval $[0, 3]$. If the limit is L, then the area is 3^3L. Since we do not know L, we don't know the area.

Estimating a Distance Traveled

Notice how the units simplify: $\frac{\text{mi}}{\text{hr}} \times \text{hr} = \text{mi}$.

If you drive a constant speed of v miles per hour for a period of t hours, you would travel vt miles:

$$\text{Distance} = \text{Speed} \times \text{Time} = vt \text{ miles}.$$

But, how would you compute the total distance traveled if your speed were not constant? (Imagine that your odometer, which records distance traveled, was broken. However, your speedometer and clock are still working fine, so you know your speed at any instant.) The next problem illustrates how you could make very accurate estimates of the total distance traveled.

PROBLEM 2 A snail is crawling about for three minutes. This remarkable snail knows that she is traveling at the rate of t^2 feet per minute at time t minutes. For instance, after half a minute, she is slowly moving at the rate of $\left(\frac{1}{2}\right)^2$ feet per minute. At the end of her journey she is moving along at 3^2 feet per minute. Estimate how far she travels during the three minutes.

Figure 5.1.7:

The speed during the three minute trip increases from 0 to 9 feet per minute. During shorter time intervals, such a wide fluctuation will not occur. As in Problem 1, cut the three minutes of the trip into six equal intervals each $\frac{1}{2}$ minute long, and use them to make an estimate of the total distance covered. Represent time by a line segment cut into six parts of equal length, as in Figure 5.1.7.

Speed increases as t increases.

Consider the distance she travels during one of the six half-minute intervals, say the interval $[\frac{3}{2}, \frac{4}{2}]$. At the beginning of this time interval her speed was $\left(\frac{3}{2}\right)^2$ feet per minute; at the end she was going $\left(\frac{4}{2}\right)^2$ feet per minute. The highest speed during this half hour was $\left(\frac{4}{2}\right)^2$ feet per minute. Therefore, she traveled at most $\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)$ feet during this period. Thus

$$\text{Distance traveled during time interval } \left[\frac{3}{2}, \frac{4}{2}\right] \leq \left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right).$$

Similar reasoning applies to the other five half-minute periods. Adding up these upper estimates for the distance traveled on each section of time, we get

$$\text{Total distance traveled} \leq \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{2}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{6}{2}\right)^2\left(\frac{1}{2}\right).$$

If we divide the time interval into n equal sections of duration $\frac{3}{n}$, the right endpoint of the i^{th} interval is $i\left(\frac{3}{n}\right)$. At that time the speed is $\left(i^{\text{th}}\right)^2$ feet per minute. The total overestimate is then

$$3^3\frac{1^2}{n^3}+3^3\frac{2^2}{n^3}+3^3\frac{3^2}{n^3}+\cdots+3^3\frac{(n-1)^2}{n^3}+3^3\frac{n^2}{n^3}$$

or

$$3^3\left(\frac{1^2+2^2+3^2+\cdots+(n-1)^2+n^2}{n^3}\right). \tag{5.5}$$

The calculuations in the area problem, (5.3), and the distance problem, (5.5), are the same. Thus, the area and distance have the same upper estimates. Their lower estimates are also the same. They are really the same problem.

Estimating a Volume

The volume of a rectangular box is easy to compute; it is the product of its length, width, and height. See Figure 5.1.8. But finding the area of a pyramid or ball requires more work. The next example illustrates how we can estimate the volume of a certain tent.

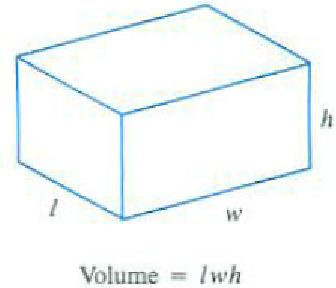
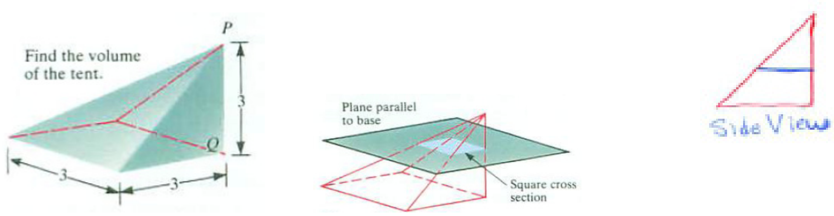


Figure 5.1.8:

PROBLEM 3 Find the volume inside a tent with a square floor of side 3 feet, whose vertical pole, 3 feet long, rises above *a corner of the floor*. The tent is shown in Figure 5.1.9(a).

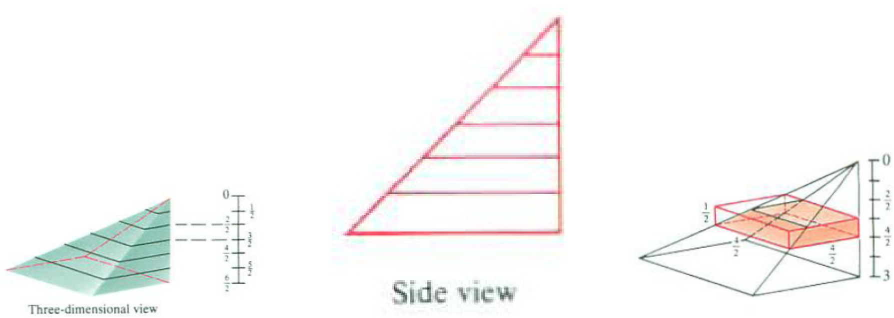
SHERMAN: Is there any problem putting the pole in the center of the tent and computing the volume of a quarter of the tent?



(a) (b) (c)

Figure 5.1.9:

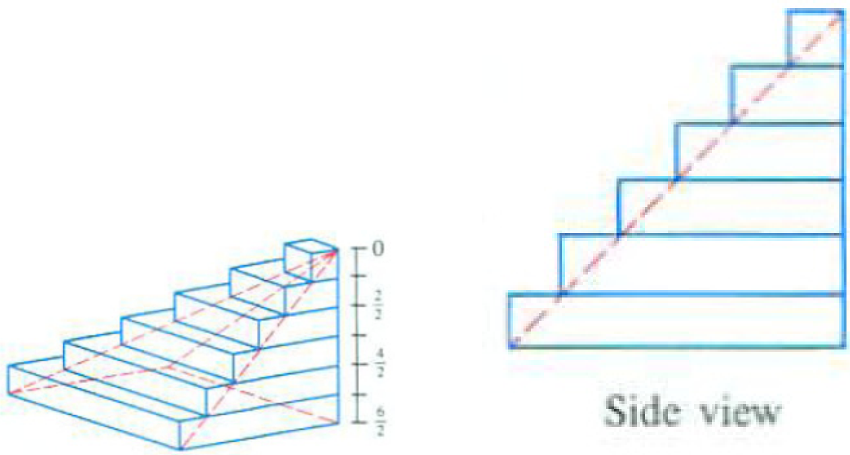
Observe that the cross-section of the tent made by any plane parallel to the base is a square, as shown in Figure 5.1.9(b). The width of the square equals its distance from the top of the pole. Using this fact we can approximate the with rectangular boxes with a square cross-section.



(a) (b) (c)

Figure 5.1.10:

We begin by cutting a vertical line, representing the pole, into six sections of equal length, each $\frac{1}{2}$ foot long. Then we draw the corresponding square cross section of the tent, as in Figure 5.1.10(a).



(a) (b)

Figure 5.1.11:

We use these square cross-sections to form rectangular boxes. Consider the part of the tent corresponding to the interval $[\frac{3}{2}, \frac{4}{2}]$ on the pole. The base of this section is a square with sides $\frac{4}{2}$ feet. The box with this square as a base and height $\frac{1}{2}$ foot encloses completely the part of the tent corresponding to $[\frac{3}{2}, \frac{4}{2}]$. (See Figure 5.1.10(c).) The volume of this box is $\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)$ cubic feet. Figure 5.1.11(a) shows six such boxes, whose total volume is greater than the volume of the tent.

Since the volume of a box is just the area of its base times its height, the total volume of the six boxes is

$$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{2}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{4}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{6}{2}\right)^2\left(\frac{1}{2}\right) \text{ cubic feet.} \tag{5.6}$$

This sum, which we have encountered twice before, equals 11.375. It is an *overestimate* of the volume of the tent. Better (over)estimates can be obtained by cutting the pole into shorter pieces.

We know that the number describing the volume of the tent is the same as the number describing the area under the parabola and the length of the snail's journey. The arithmetic of the estimates isthe same in all three cases.

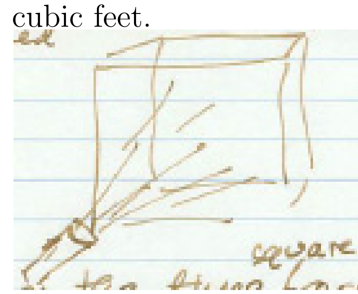


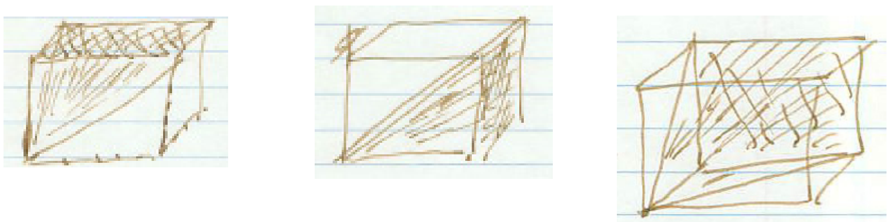
Figure 5.1.12:

A Neat Bit of Geometry

If we knew the limit L, we would then find the answers to all three problems. But we haven't found L. Luckily, there is a trick fro finding the volume of the tent without knowing L.

This trick is similar to the way the area of a triangle is found by assembling two copies of the triangle to form a parallelogram.

The key is that three identical copies of the tent fill up a cube of side 3 feet. To see why, imagine a flashlight at one corner of the cube, aimed into the cube, as in Figure 5.1.12.



(a) (b) (c)

Figure 5.1.13:

The flashlight illuminates the three square faces not meeting the corner at the flashlight. The rays from the flashlight to each of the faces fill out a copy of the tent, as shown in Figure 5.1.13.

Since three copies of the tent fill a cube of volume $3^3 = 27$ cubic feet, the tent has volume 9 cubic feet. From this, we see that the area under the parabola above $[0, 3]$ is 9 and the snail travels 9 feet. Incidentally, the limit L must be $\frac{1}{3}$, that is

$$\lim_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2}{n^3} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} i^2}{n^3}. = \frac{1}{3} \tag{5.7}$$

Summary

Using upper estimates, we showed that problems concerning area, distance traveled, and volume were the same problem in various disguises. We were really studying a single problem concerning a particular function, x^2, over a particular interval $[0, 3]$. We solve the problem by cutting a cube into three congruent pieces. By the end of this chapter you will learn general techniques that will make use of such a special device unnecessary.

EXERCISES for 5.1 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 – 4 develop the underestimates for each of the problems considered in this section.

1.[R] In Problem 1 we found overestimates for the area under the parabola x^2 over the interval $[0, 3]$. Here, we will obtain underestimates for this area as follows.

(a) Break $[0, 3]$ into six sections of equal length and draw the six rectangles whose total area is smaller than the area under the curve.

(b) Because x^2 is increasing on $[0, 3]$, the left endpoint of each section determines the height of each rectangle. Show the height and width of each rectangle you drew in (a).

(c) Find the total area of the six rectangles.

2.[R] Repeat Exercise 1 with 12 subintervals.

3.[M] Extend Exercise 1(c) and 3(c) to n subintervals.

4.[M] To make an estimate of the area in Problem 1 you divide the interval $[0, 3]$ into n sections of equal length. Using the right-hand endpoint of each of the n sections you then obtain an overestimate. Using the left-hand endpoint, you obtain an underestimate.

(a) Show that these two estimates differ by $\frac{27}{n}$.

(b) How large should n be chosen in order to be sure the difference between the upper estimate and the area under the parabola is less than 0.01?

Exercises 5 – 22 concern estimates of areas under curves.

5.[R] In Problem 1 we broke the interval $[0, 3]$ into six sections. Instead, break $[0, 3]$ into four sections of equal length and estimate the area under $y = x^2$ and above $[0, 3]$ as follows.

(a) Draw the four rectangles whose total area is larger than the area under the curve. The value of x^2 at the right endpoint of each section determines the height of each rectangle.

(b) On the diagram in (a), show the height and width of each rectangle.

(c) Find the total area of the four rectangles.

6.[R] Like Exercise 5, but this time obtain an underestimate of the area by the value of x^2 at the left endpoint of each section to determine the height of the rectangles.

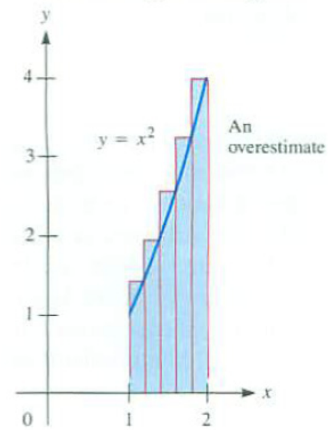


Figure 5.1.14:

7.[R] Cutting the interval $[0,3]$ into five sections, estimate the area in Problem 1 by finding the sum of the areas of the five rectangles whose heights are determined by (a) left endpoints and (b) right endpoints. (c) Using the information found in (a) and (b), complete the sentence:
The area in Problem 1 is certainly less than ______ but larger than ______.

8.[R] Estimate the area under $y = x^2$ and above $[1,2]$ using the five rectangles in Figure 5.1.14.

9.[R] Estimate the area under $y = x^2$ and above $[1,2]$ using the five rectangles in Figure 5.1.15.

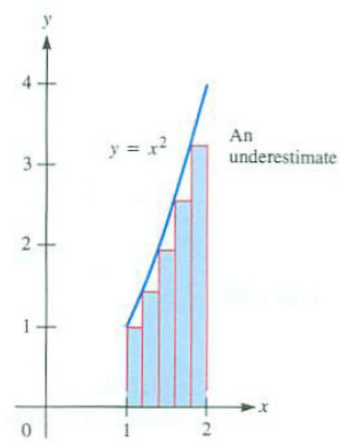


Figure 5.1.15:

10.[R] Estimate the area in Problem 1, using the division of $[0,3]$ into four sections with endpoints 0, 1, $\frac{5}{3}$, $\frac{11}{4}$, and 3 (see Figure 5.1.16).

(a) Estimate the area when the right-hand endpoints of each section are used to find the height of the rectangles.

(b) Repeat (a) using the left-hand endpoints of each section to find the height of the rectangles.

(c) Repeat (a) computing the heights of the rectangles at the points $\frac{1}{2}$, $\frac{3}{2}$, 2, and $\frac{14}{5}$ (one of these is in each of the four sections).

Figure 5.1.16:

11.[R] Like Exercise 10, but use midpoints to determine the heights of the rectangles.

12.[R] Figure 5.1.17(a) shows the curve $y = \frac{1}{x}$ above the interval $[1,2]$ and an approximation to the area under the curve by five rectangles of equal width.

(a) Make a large copy of Figure 5.1.14.

(b) On your diagram show the height and width of each rectangle.

(c) Find the total area of the five rectangles.

13.[R] Repeat Exercise 12 with the five rectangles in Figure 5.1.17(b).

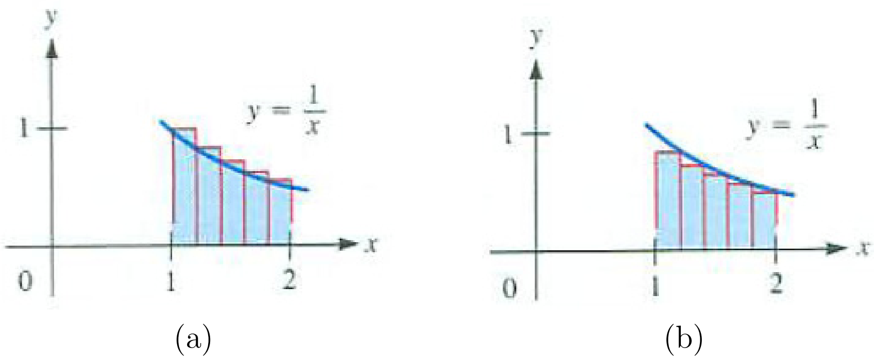


Figure 5.1.17:

In each of Exercises 14–19

(a) Draw the region.

(b) Draw six rectangles of equal width whose total area overestimates the area of the region.

(c) On your diagram indicate the height and width of each rectangle.

(d) Find the total area of the six rectangles. (Give this answer accurate to four decimal places.)

14.[R] Under $y = x^2$, above $[2, 3]$.

15.[R] Under $y = \frac{1}{x}$, above $[2, 3]$.

16.[R] Under $y = x^3$, above $[0, 1]$.

17.[R] Under $y = \sqrt{x}$, above $[1, 4]$.

18.[M] Under $y = \sin(x)$, above $[0, \pi]$. (Be careful that your rectangles give an overestimate of the area. See also Exercise 20.)

19.[M] Under $y = \ln(x)$, above $[1, e]$.

20.[M] Estimate the area under $y = x^2$ and above $[-1, 2]$ by dividing the interval into six sections.

(a) Draw the six rectangles that form an overestimate for the area under the curve. Note that you cannot do this using only left- or right-endpoints.

(b) Find the total area of all six rectangles.

(c) Repeat (a) and (b) to find an underestimate for this area.

21.[M] Estimate the area between the curve $y = x^3$, the x-axis, and the vertical line $x = 6$ using a division in to

(a) three sections of equal length with midpoints;

(b) six sections of equal length with midpoints;

(c) six sections of equal length with left endpoints;

(d) six sections of equal length with right endpoints.

22.[M] Estimate the area below the curve $y = \frac{1}{x^2}$ and above $[1, 7]$ following the directions in Exercise 21.

23.[R] An electron is being accelerated in such a way that its velocity is t^3 kilometers per second after t seconds, $t \geq 0$. Estimate how far it travels in the first 4 seconds, as follows:

(a) Draw the interval $[0, 4]$ as the time axis and cut it into eight sections of equal length.

(b) Using the sections in (a), make an estimate that is too large.

(c) Using the sections in (a), make an estimate that is too small.

24.[M] A business which now shows no profit is to increase its profit flow gradually in the next 3 years until it reaches a rate of 9 million dollars per year. At the end of the first half year the rate is to be $\frac{1}{4}$ million dollars per year; at the end of 2 years, 4 million dollars per year. In general, at the end of t years, where t is any number between 0 and 3, the rate of profit is to be t^2 million dollars per year. Estimate the total profit during the next 3 years if the plan is successful using

(a) six intervals and left endpoints;

(b) six intervals and right endpoints;

(c) six intervals and midpoints.

25.[C] A right circular cone has a height of 3 feet and a radius of 3 feet, as shown in Figure 5.1.18. Estimate its volume by the sum of the volumes of six cylindrical slabs, just as we estimated the volume of the tent with the aid of six rectangular slabs.

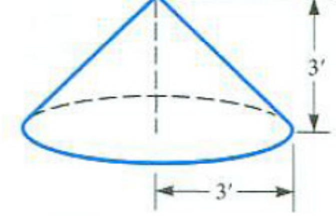


Figure 5.1.18:

(a) Make a large and neat diagram that shows the six cylinders used in making an overestimate.

(b) Compute the total volume of the six cylinders in (a).

(c) Make a separate diagram showing a corresponding underestimate.

(d) Compute the total volume of the six cylinders in (c). (Note: One of the cylinders nas radius 0.)

26.[M] Oil is leaking out of a tank at the rate of 2^{-t} gallons per minute after t minutes. Describe how you would estimate how much oil leaks out during the first 10 minutes. Illustrate your procedure by computing one estimate.

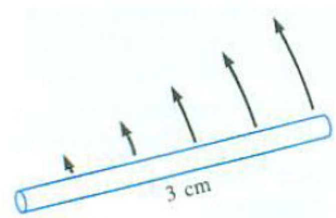


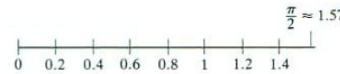
Figure 5.1.19:

27.[C] The kinetic energy of an object, for example, a bullet or car, of mass m grams and speed v centimeters per second is defined as $\frac{1}{2}mv^2$ ergs. Now, in a certain machine a uniform rod 3 centimeters long and weighing 32 grames rotates once per second around one of its ends as shown in Figure 5.1.19. Estimate teh kinetic energy of this rod by cutting it into six sections, each $\frac{1}{2}$ centimeter long, and taking as the "speed of a section" the speed of its midpoint.

28.[C] Let $y = f(x)$ be a function such that $f(x) \geq 0$, $f'(x) \geq 0$, and $f''(x) \geq 0$ for all x in $[1, 4]$. An estimate of the area under $y = f(x)$ is made by dividing the interval into sections and forming rectangles. The height of each rectangle is the value of $f(x)$ at the midpoint of the corresponding section.

(a) Show that the estimate is less than or equal to the area under the curve. Hint: Draw a tangent to the curve at each of the midpoints.

(b) How does the estimate compare to the area under the curve if $f''(x) \leq 0$ for all x in $[1, 4]$?



29.[M] Estimate the area of the region under the curve $y = \sin(x)$ and above the interval $[0, \frac{\pi}{2}]$, cutting the interval as shown in Figure 5.1.20 and using

(a) left endpoints

(b) right endpoints

(c) midpoints.

(Note that all but the last section are of the same size.)

30.[M] Make three copies of teh tent in Problem 3 by folding a pattern as shown in Figure 5.1.21. Check that they do fill up a cube.

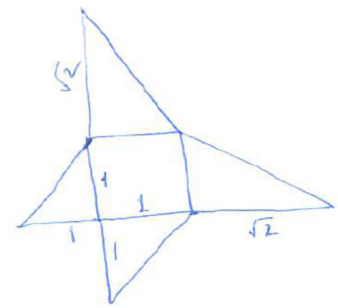


Figure 5.1.21:

31.[R] Evaluate

(a) $\sum_{i=1}^{4} i^2$

(b) $\sum_{i=1}^{4} 2^i$

(c) $\sum_{n=3}^{4}(n-3)$

32.[R] Evaluate

(a) $\sum_{i=1}^{4} i^3$

(b) $\sum_{i=2}^{5} 2^i$

(c) $\sum_{k=1}^{4}(k^3 - k^2)$

33.[C] The function f is increasing for x in the interval $[a, b]$ and is positive. To estimate the area under the graph of $y = f(x)$ and above $[a, b]$ you divide the interval $[a, b]$ into n sections of equal length. You then form an overestimate B (for "big") using right-hand endpoints of the sections and an underestimate S (for "small") using left-hand endpoints. Find the difference between the two estimates $B - S$.

34.[C] Archimedes showed that $\sum_{i=1}^{n} = \frac{n(n+1)(2n+1)}{6}$. You can prove this as follows:

This method of proof is known as **mathematical induction**.

(a) Check that the formula is true for $n = 1$.

(b) Show that if the formula is true for the integer n, it is also true for the next integer, $n + 1$.

(c) Why do (a) and (b) together show that Archimedes' formula holds for all positive integers n?

35.[C] Archimedes' derivation of surface area and volume of a sphere. Approximate the surface with triangles. Approximate the volume with pyramids. The total area of the triangles approaches the surface area of the sphere, S. The total volume of the pyramids is $\frac{r}{3}$ times the total surface area of the triangles. Thus, in the limit, the volume of the sphere is found to be $\frac{r}{3}S = \frac{r}{3}4\pi r^2 = \frac{4}{3}\pi r^3$.

SHERMAN: Defer this until later in this chapter. Do you have a good wording for this?

36.[R] Differentiate for practice:

(a) $(1+x^2)^{4/3}$

(b) $\frac{(1+x^3)\sin(3x)}{\sqrt[3]{5x}}$

(c) $\frac{3x}{8} + \frac{3x\sin(4x)}{32} + \frac{\cos^3(2x)\sin(2x)}{8}$

(d) $\frac{3}{8(2x+3)^2} - \frac{1}{4(2x+3)}$

(e) $\frac{\cos^3(2x)}{6} - \frac{\cos(2x)}{2}$

(f) $x^3\sqrt{x^2-1}\tan(5x)$

37.[R] Give an antiderivative of

(a) $(x+2)^3$

(b) $(x^2+1)^2$

(c) $x\sin(x^2)$

(d) $x^3 + \frac{1}{x^3}$

(e) $\frac{1}{\sqrt{x}}$

(f) $\frac{3}{x}$

(g) e^{3x}

(h) $\frac{1}{1+x^2}$

(i) $\frac{1}{x^2}$

(j) 2^x

5.2 The Definite Integral

We now introduce the other main concept in calculus, the "definite integral of a function over an interval".

The preceding section was not really about area under a curve, distance a snail traveled, and volume of a tent. The common theme of all three was a procedure we carried out with the function x^2 and the interval $[0, 3]$. Cut the interval into small pieces, evaluate the function somewhere in each section, formed a certain sum, and then we're interested in how those sums behaved as we chose the sections smaller and smaller.

This is the general procedure. We have a function f defined at least on an interval $[a, b]$. We divide thh interval into n sections by the numbers $x_0 = a$, x_1, x_2, ..., x_{n-1}, $x_n^3 = b$, as in Figure 5.2.1. The sections need not all be of

Figure 5.2.1:

the same length, though usually, for our convenience, they will be.

Then we pick a number in each interval, c_1 in $[x_0, x_1]$, c_2 in $[x_1, x_2]$, ..., c_i in $[x_{i-1}, x_i]$, ..., c_n in $x_{n-1}, x_n]$ (also as in Figure 5.2.1).

In Section 5.1 the c_i's were generally either right-hand or left-hand end-points or midpoint. However, they can be anywhere in each section.

Next we bring in the particular function f. We evaluate $f(x)$ at each sampling number c_i and form the sum

In Section 5.1 it was $f(x) = x^2$.

$$f(c_1)(x_1-x_0)+f(c_2)(x_2-x_1)+\cdots+f(c_i)(x_i-x_{i-1})+\cdots+f(c_{n-1}(x_{n-1}-x_{n-2})+f(c_n)(x_n-x_{n-1}). \quad (5.1)$$

Rather than continue to write out such a long expression, we choose to take advantage of the fact that each term in (5.1) follows the same general pattern: for each of the n sections, multiply the function value at the sampling point and the length of the section. This type of pattern is easily expressed in the shorthand of Σ-notation:

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \tag{5.2}$$

or, if the length of section i is written as $\Delta x_i = x_i - x_{i-1}$,

$$\sum_{i=1}^{n} f(c_i)\Delta x_i. \tag{5.3}$$

If all the sections have the same length, each $x_i - x_{i-1}$ equals $(b-a)/n$, since the length of $[a, b]$ is $b-a$. In that case every Δx_i is equal. We will write (5.2) and (5.3) as

$$\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) = \sum_{i=1}^{n} f(c_i)\Delta x \tag{5.4}$$

where $\Delta x = \frac{b-a}{n}$.

SHERMAN: You appear to be unsure whether to use $\Delta_i x$ or Δx_i. The latter is certainly more common, but the former might be more mathematically informed. I have chosen the latter for now, but could be convincd to change.

$\Delta x = \frac{b-a}{n}$

The final step is to investigate what happens to the sums of the form (5.3) as the lengths of all the sections approach 0. That is, we try to find

$$\lim_{\text{all } x_i - x_{i-1} \text{ approach } 0} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}). \tag{5.5}$$

The sum in 5.1)–(5.4) are all called **Riemann sums** in honor of the nineteenth century mathematician, Bernhard Riemann. (See `http://en.wikipedia.org/wiki/Bernhard_Riemann`.)

In advanced mathematics it is proved that if f is continuous on $[a, b]$ then the limit in (5.5) does approach a single number. This brings us to the definition of the definite integral.

The Definite Integral

DEFINITION *(Definite Integral)* Let f be a continuous function defined at least on the interval $[a, b]$. The limit of sums of the form $\sum_{i=1}^{n} f(c_i)\Delta x_i$, as *all* Δx_i approach 0, exists. The limiting value is called the **definite integral of f over the interval** $[a, b]$ and is denoted

$$\int_a^b f(x)\, dx.$$

NOTE: The "dx", strictly speaking, is not needed. It, too, was introduced by Gottfried Liebniz. (See `http://en.wikipedia.org/wiki/Gottfried_Leibniz`.) It survives because it serves as a matching bracket to delimit the function f and the "x" indicates the independent variable in the problem. Thus, it simplifies certain calculations.

EXAMPLE 1 Express the area under $y = x^2$ and above $[0, 3]$ as a definite integral.

SOLUTION Here the function is $f(x) = x^2$ and the interval is $[0, 3]$. As we saw in the previous section, the area equals the limit of Riemann sums

$$\lim_{\Delta x \to 0^+} \sum_{i=1}^{n} x_i^2 \Delta x = \int_0^3 x^2 \, dx. \tag{5.6}$$

◇

The symbol $\int_a^b x^2 \, dx$ is read as "the integral from a to b of x^2". Freeing ourselves from the variable x, we could say, "the integral from a to b of the squaring function". There is nothing special about the symbol x in "x^2". We could just as well have used the letter t — or any other letter. (We would typically pick a letter near the end of the alphabet, since letters near the beginning are customarily used to denote constants.) The notations

$$\int_a^b x^2 \, dx, \qquad \int_a^b t^2 \, dt, \qquad \int_a^b z^2 \, dz, \qquad \int_a^b u^2 \, du, \qquad \int_a^b \theta^2 \, d\theta$$

all denote the same number, that is, "the definite integral of the squaring function from a to b". Notice that $\int_a^b u^2 \, dx$ is a different definite intgral. While we could write just $\int_a^b x^2$, notice that the dx is a matching delimiter for the $\int_a^b$ and the x indicates the independent variable for the integral. Taken to the extreme, we could express (5.6) as

$$\int_a^b (\ )^2 \, d(\ ).$$

Usually, however, we find it more convenient to use some letter to name the independent varaible. Since the letter chosen to represent the variable has no significance of its own, ti is called a **dummy variable**. While x is a typical choice for the dummy variable, we will occasionally find it convenient to use other letters, such as t for "time". Later in this chapter there will be cases where the interval of integration is $[a, x]$ instead of $[a, b]$. Were we to write $\int_a^x x^2 \, dx$, you might think that there is some relation between the x in x^2 and the x in the interval of integration. To avoid possible confusion, we prefer to use a different dummy variable and write, for example, $\int_a^b t^2\ dt$ in such cases.

The symbol $\int$ comes from the letter "s" in sum. The dx traditionally suggests the length of a small section of the x axis and denotes the **variable of integration** (usually x, as in this case). The function $f(x)$ is called the **integrand**, while the numbers a and b are called the **limits of intgration**; a is the **lower limit of integration** and b is the **upper limit of integration**.

Limits of integration refer to the endpoints of the interval $[a, b]$ and are not limits in the usual mathematical sense. This terminology is traditional and difficult to avoid.

It is important to realize the area, distance traveled, and volume are merely applications of the definite integral. (It is a mistake to link the definite integral too closely with one of its applications, just as it narrows our understanding of the number 2 to link it always with the idea of two fingers.) The definite integral $\int_a^b f(x)\ dx$ is also call the **Riemann intgral**.

Slope and velocity are particular interpretations or applications of the derivative, which is a purely mathematical concept defined as a limit:

derivatives are limits

$$\text{derivative of } f \text{ at } x = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Similarly, area, total distance, and volume are just particular interpretations of the definite integral, which is also defined as a limit:

definite integrals are also limits

$$\text{definite integral of } f \text{ over } [a, b] = \lim_{\text{as all } \Delta x_i \to 0^+} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}).$$

The Definite Integral of a Constant Function

To bring the definition down to earth,, let us use it to evaluate the definite integral of a constant function.

EXAMPLE 2 Let f be the function whose value at any number x is 4; that is, f is the constant function given by the formula $f(x) = 4$. Use only the definition of the definite integral to compute

$$\int_1^3 f(x)\ dx.$$

SOLUTION In this case, every partition has $x_0 = 1$ and $x_n = 3$. See Figure 5.2.2. The approximating sum

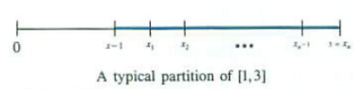


Figure 5.2.2:

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = \sum_{i-1}^{n} 4(x_i - x_{i-1})$$

since, no matter how the sampling number c_i is chosen, $f(c_i) = 4$. Now

$$\sum_{i-1}^{n} 4(x_i - x_{i-1}) = 4\sum_{i-1}^{n}(x_i - x_{i-1}) = 4 \cdot 2 = 8.$$

since the sum of the widths of the sections is the width of the interval $[1, 3]$, namely 2. All approximating sums have the same value, namely, 8. It does not matter where the c_i are picked in each section. For every partition,

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = 8.$$

Thus, as all sections are chosen smaller, the value of the sums is always 8. This must also be value of the limit:

$$\int_1^3 4\ dx = 8.$$

◇

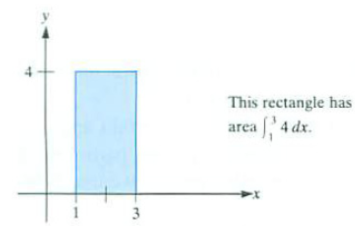


Figure 5.2.3:

We could have guessed the value of $\int_1^3 4\ dx$ by interpreting the definite integral as as area. To do so, draw a rectangle of height 4 and base coinciding with the interval $[1, 3]$. (See Figure 5.2.3.) Since the area of a rectangle is its base times its height, it follows again that $\int_1^3 4\ dx = 8$.

Similar reasoning shows that for any constant function that has the fixed value c,

$$\int_a^b c\ dx = c(b - a) \qquad (c \text{ is a constant function})$$

negative area

This result is true for any constant function: c positive, negative, or zero. When c is negative, the value of the definite integral $\int_a^b c\ dx$ will be negative (because $b > a$). Even though area is always positive, it is customary to refer to such definite integrals as **negative area**. An easy and effective way to expand our knowledge of definite intgrals is to use our knowledge of area to interpret a definite integral as the area of a region that can be found by another method.

The Definite Integral of x

Exercise 20 shows us how to find $\int_a^b x\ dx$ directly from the definition. Alternatively, let us use the "area" interpretation of the definite integral to predict the value of $\int_a^b x\ dx$.

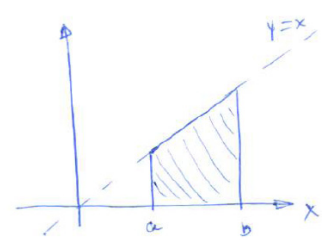


Figure 5.2.4:

When the integrand is positive ($b > a > 0$), the area in question then lies above the x-axis, as shown in Figure 5.2.4. Two copies of this region form a rectangle of width $b - a$ and height $a + b$, as shown in Figure 5.2.5. Thus, the area shown in Figure 5.2.4 is half of $(b - a)(b + a) = b^2 - a^2$. Hence,

$$\int_a^b x\ dx = \frac{b^2}{2} - \frac{a^2}{2}.$$

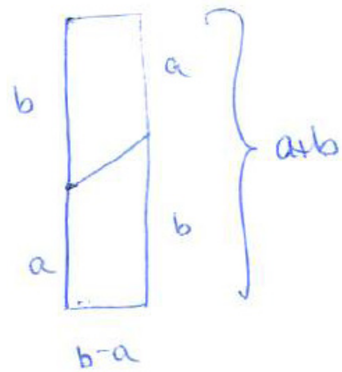


Figure 5.2.5:

This formula is true for all values of a and b, even though the argument applies only when $b > a > 0$. Exercise 48 outlines the steps needed to extend the argument to all intervals $[a, b]$.

The Definite Integral of x^2

We will find $\int_0^b x^2\ dx$ by examining the approximating sums when all the sections have the same length, as they did in Section 5.1.

Figure 5.2.6:

Pick a positive integer n and cut the interval $[0, b]$ into n sections of length $d = b/n$ as in Figure 5.2.6. Then the points of subdivision are 0, d, $2d$, ..., $(n-1)d$, and $nd = b$.

In the typical section $[(i-1)d, id]$ we pick the right-hand endpoint as the sampling point. Thus the approximating sum is

$$\sum_{i=1}^{n} (id)^2\ d = d^3 \sum_{i=1}^{n} i^2.$$

Since $d = b/n$, the over estimates can be written as

$$\sum_{i=1}^{n} (id)^2\ d = d^3 \sum_{i=1}^{n} i^2. \tag{5.7}$$

In Section 5.1 we used geometry to find that

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{3}.$$

Thus, (5.7) approaches $b^3/3$ as n increases, and we conclude that

$$\int_0^b x^2\ dx = \frac{b^3}{3}.$$

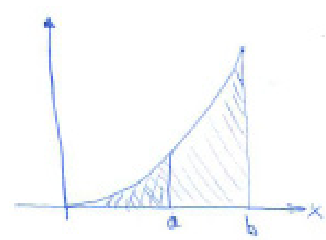


Figure 5.2.7:

Note that when $b = 3$, we have $b^3/3 = 9$, agreeing with the three examples treated in Section 5.1.

A little geometry suggests the value of $\int_z^b x^2\ dx$, for $0 \le a < b$. Interpret $\int_a^b x^2\ dx$ as the area under $y = x^2$ and above $[a, b]$. This area is equal to the area under $y = x^2$ and above $[0, b]$ minus the area under $y = x^2$ and above $[0, a]$, as shown in Figure 5.2.7. Then

$$\int_a^b x^2\ dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

The Definite Integral of 2^x

EXAMPLE 3 Use the definition of the definite integral to evaluate $\int_0^b 2^x\ dx$.

SOLUTION Divide the interval $[0, b]$ into n sections of equal length, $d = b/n$. This time we evaluate the integrand at the left-hand endpoint of each section. The approximating sum has one term from each section; the contribution from the i^{th} section is

d = width of section

$$2^{c_i} d = 2^{(i-1)d} d.$$

The total estimate is the sum

$$2^0 d + 2^d d + 2^{2d} d + \cdots + 2^{(i-1)d} d + \cdots + 2^{(n-1)d} d$$

which equals

$$d\left(1 + 2^d + (2^d)^2 + \cdots + (2^d)^i + \cdots + (2^d)^{n-1}\right). \tag{5.8}$$

The terms inside the large parentheses in (5.8) is a geometric series with n terms, whose first term is 1 and whose ratio is 2^d. Thus, its sum is

Sum of geometric series: $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{1-r^n}{1-r}$.

$$\frac{1 - (2^d)^n}{1 - 2^d}.$$

Therefore the typical over estimate is

$$\frac{d(1 - (2^d)^n)}{1 - 2^d} = \frac{d(1 - 2^{dn}}{1 - 2^d} = \frac{d(1 - 2^b)}{1 - 2^d}. \tag{5.9}$$

In the last step the fact that $nd = b$ was used. (5.9) can now be written as

$$\frac{d}{2^d - 1}\left(2^b - 1\right). \tag{5.10}$$

It still remains to take the limit as n increases without bound. To find what happens to (5.9) as $n \to \infty$, we must investigate how $\frac{d}{2^d-1}$ behaves as d

approaches 0 (from the right). Though we haven't met this quotient before, we have met its reciprocal $\frac{2^d-1}{d}$. This quotient occrs in the definition of the derivative of 2^x at $x = 0$:

$$\lim_{x\to 0} \frac{2^x - 2^0}{x} = \lim_{x\to 0} \frac{2^s - 1}{x}.$$

As we saw in Section 2.4, the derivative of 2^x is $2^x \ln(2)$. Thus $D(2^x)$ at $x = 0$ is $\ln(2)$. Hence

$$\lim_{d\to 0^+} \frac{d}{2^d - 1}(2^b - 1) = \frac{2^b - 1}{\ln(2)}.$$

We conclude that ◇

To evaluate $\int_a^b 2^x\ dx$ with $b > a \geq 0$, we reason as we did when we extended $\int_0^b x^2\ dx$ to $\int_a^b x^2\ dx$. Namely,

$$\begin{aligned} \int_a^b 26x\ dx &= \int_0^b 2^x\ dx - \int_0^a 2^x\ dx \qquad (5.11) \\ &= \frac{2^b - 1}{\ln(2)} - \frac{2^a - 1}{\ln(2)} \\ &= \frac{1}{\ln(2)}\left(2^b - 2^a\right). \end{aligned}$$

Summary

We defined the definite integral of a function $f(x)$ over the interval $[a, b]$. It is the limit of sums of the form $\sum_{i=1}^{n} f(c_i)\Delta x_i$ created from partitions of $[a, b]$. It is a purely mathematical idea. You could estimate $\int_a^b f(x)\ dx$ with your calculator – even without having any application in mind. We did this for the functions x^2 and 2^x. However, the definite integral has many applications: three of them are area under a curve, distance traveled, and volume.

The following table contains a great deal of information. Compare the first three cases with the fourth, which describes the fundamental idea of integral calculus. In this table, all the functions, whether cross-sectional length, velocity, or cross-sectional area, are denoted by the same symbol $f(x)$.

Spend some time studying this table. The concepts it summarizes will be used often.

Underlying these three applications is one purely mathematical concept, the definite integral, $\int_a^b f(x)\ dx$. The definite integral is defined as a certain limit; it is a number. It is essential to keep the definition of $\int_a^b f(x)\ dx$ clear. *It is a limit of certain sums.*

$f(x)$	$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$	$\int_a^b f(x)\ dx$
Variable length of cross section of set in plane	Approximate area of the set in the plane	The area of the set in the plane
Variable velocity	Approximation to the distance traveled	The distance traveled
Variable set of cross section of a solid	Just a sum	The limit of the sums as the $\Delta x_i \to 0$

EXERCISES for 5.2 *Key:* R–routine, M–moderate, C–challenging

1.[R] Using the formula for $\int_a^b x^2\ dx$, find the area under the curve $y = x^2$ and above the interval

(a) $[0, 5]$

(b) $[0, 4]$

(c) $[4, 5]$

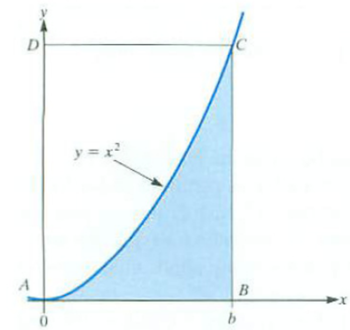


Figure 5.2.8:

2.[R] Figure 5.2.8 shows the curve $y = x^2$. What is the ratio between the shaded area under the curve and the area of the rectangle $ABCD$?

3.[M] The velocity of an automobile at time t is $v(t)$ feet per second. [Assume $v(t) \geq 0$.] The graph of v for t in $[0, 20]$ is shown in Figure 5.2.9. Explain, in complete sentences, why th shaded area under the curve equals the change in position.

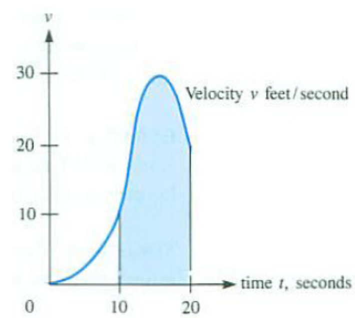


Figure 5.2.9:

4.[R]

(a) What is meant by "the definite integral of $f(x)$ from a to b, $\int_a^b f(x)\ dx$"?

(b) Give three physical examples where the definite integral arises.

5.[R] Assume $f(x)$ is decreasing for x in $[a, b]$. When you form an approximating sum for $\int_a^b f(x)\ dx$ with left-hand endpoints as sampling points, is your estimate too large or too small? Explain (in one or more complete sentneces).

6.[M] Assume that $f(x) \leq -3$ for all x in $[1, 5]$. What can be said abou the value of $\int_1^5 f(x)\ dx$? Explain, in detail, using the definition of the definite integral.

7.[M] A rocket moving with a varying speed travels $f(t)$ miles per second at time t. Let $t_0, \ldots, t_n$ be a partition of $[a, b]$, and let $T_1, \ldots, T_n$ be sampling numbers. What is the physical interpretation of each of the following quantities?

(a) $t_i - t_{i-1}$

(b) $f(T_i)$

(c) $f(T_i)(t_i - t_{i-1}$

(d) $\sum_{i=1}^n f(T_i)(t_i - t_{i-1}$

(e) $\int_a^b f(t)\ dt$

8.[M] For x in $[a, b]$, let $A(x)$ be the area of the cross section of a solid (think of slicing a potato). Let x_0, x_1, $\ldots$, x_n be a partition of $[a, b]$. Let X_1, $\ldots$, X_n be the corresponding sampling numbers. What is the geometric interpretation of each of the following quantities?

(a) $x_i - x_{i-1}$

(b) $f(X_i)$

(c) $f(X_i)(x_i - x_{i-1}$

(d) $\sum_{i=1}^{n} f(X_i)(x_i - x_{i-1}$

(e) $\int_a^b f(x)\ dx$

In Exercises 9–12 evaluate the sum **9.**[R]

(a) $\sum_{i=1}^{3} i$

(b) $\sum_{i=1}^{4} 2i$

(c) $\sum_{d=1}^{3} d^2$

10.[R]

(a) $\sum_{i=2}^{4} i^2$

(b) $\sum_{j=2}^{4} j^2$

(c) $\sum_{i=1}^{3}(i^2 + i)$

11.[R]

(a) $\sum_{i=1}^{4} 1^i$

(b) $\sum_{k=2}^{6}(-1)^k$

(c) $\sum_{j=1}^{1} 503$

12.[R]

(a) $\sum_{i=3}^{5} \frac{1}{i}$

(b) $\sum_{i=0}^{4} \cos(2\pi i)$

(c) $\sum_{i=1}^{3} 2^{-i}$

In Exercises 13–16 write each sum in Σ-notation. (Do not evaluate the sum.) **13.**[R]

(a) $1 + 2 + 2^2 + 2^3 + \cdots + 2^{100}$

(b) $x^3 + x^4 + x^5 + x^6 + x^7$

(c) $\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{102}$

14.[R]

(a) $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100}$

(b) $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11}$

(c) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{101^2}$

15.[R]

(a) $x_0^2(x_1 - x_0) + x_1^2(x_2 - x_1) + x_2^2(x_3 - x_2)$

(b) $x_1^2(x_1 - x_0) + x_2^2(x_2 - x_1) + x_3^2(x_3 - x_2)$

16.[R]

(a) $8t_0^2(t_1 - t_0) + 8t_1^2(t_2 - t_1) + \cdots + 8t_{99}^2(t_{100} - t_{99})$

(b) $8t_1^2(t_1 - t_0) + 8t_2^2(t_2 - t_1) + \cdots + 8t_n^2(t_n - t_{n-1})$

17.[R] Use the definition of definite integral to evaluate $\int_a^b e^x \, dx$.

18.[R] Use the definition of definite integral to evaluate $\int_a^b 3^x \, dx$.

Exercises 19–21 exploit "telescoping sums". Let f be a function defined at least for positive integers. A sum of the form $\sum_{i=1}^{n}(f(i+1) - f(i))$ is called telescoping. To show why, write the sum out in longhand:

$$(f(2)-f(1))+(f(3)-f(2))+(f(4)-f(3))+\cdots+(f(n)-f(n-1))+(f(n+1)-f(n)).$$

Everything cancels except $-f(1)$ and $f(n+1)$. The whole sum shrinks like a collapsible telescope, with value $f(n+1) - f(1)$. **19.**[C]

(a) Show that $\sum_{i=1}^{n}\left((i+1)^2 - i^2\right) = (n+1)^2 - 1$.

(b) From (a), show that $\sum_{i=1}^{n}(2i+1) = (n+1)^2 - 1$.

(c) From (b), show that $2\sum_{i=1}^{n} i + n = (n+1)^2 - 1$.

(d) From (c), show that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

20.[C] Exercise 19 showed that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Use this information to find $\int_0^b x \, dx$ directly from the definition of the definite integral (not by interpreting it as meaning an area).

21.[C] See Exercise 20.

(a) Starting with the telescoping sum $\sum_{i=1}^{n}\left((i+1)^3 - i^3\right)$ show that

$$3 \sim_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + n = (n+1)^3 - 1.$$

(b) Use (a) to show that $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

(c) Use (b) to show that $\int_0^b x^2 \, dx = \frac{b^3}{3}$.

22.[C] The function $f(x) = 1/x$ has a remarkable property, namely

$$\int_1^a \frac{1}{x} \, dx = \int_b^{ab} \frac{1}{x} \, dx$$

where both a and b are greater than 1. In other words, "multiplying the interval $[1, a]$ by a positive number b does not change the value of the definite integral". The following steps show why this is so

(a) Let $x_0 = 1$, x_1, x_2, ..., $x_n = a$ divide the interval $[1, a]$ into n sections. Using left endpoints write out an approximating sum to $\int_1^a \frac{1}{x}\,dx$.

(b) Let $bx_0 = b$, bx_1, bx_2, ..., $bx_n = ab$ divide the interval $[b, ab]$ into n sections. Using left endpoints write out an approximating sum to $\int_b^{ab} \frac{1}{x}\,dx$.

(c) Explain why $\int_1^a \frac{1}{x}\,dx = \int_b^{ab} \frac{1}{x}\,dx$.

23.[C] Let $L(t) = \int_1^t \frac{1}{x}\,dx$, $t > 1$.

(a) Show that $L(a) = L(ab) - L(b$.

(b) By (a), conclude that $L(ab) = L(a) + L(b)$.

(c) What familiar function has the property listed in (b)?

Gregory St. Vincent noticed the property (a) in 1647, and his friend A.A. de Sarasa saw that (b) followed. Euler, in ????, recognized that $L(x)$ is actually the logarithm of x to the base e. In short, the area under the hyperbola $y = 1/x$ and above $[1, a]$, $a > 1$, is $\ln(a)$. It can be shown that for a in $(0, 1)$, the negative of the area below that curve and below $[a, 1]$ is $\ln(a)$. (See C. H. Edwards Jr., The Historical development fo teh Calculus, pp. 154–158.)

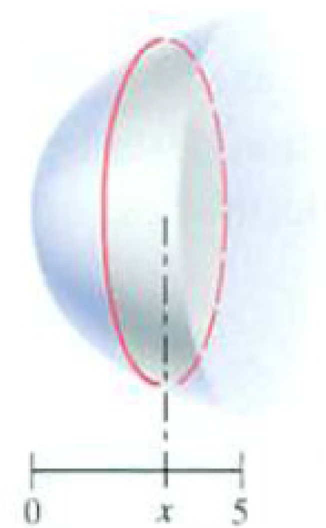


Figure 5.2.10:

24.[M]

(a) Set up an appropriate definite integral $\int_a^b f(x)\,dx$ which equals the volume of the headlight in Figure 5.2.10 whose cross section by a typical plane perpendicular to the x-axis at x is a circle whose radius is $\sqrt{x/\pi}$.

(b) Evaluate the integral found in (a).

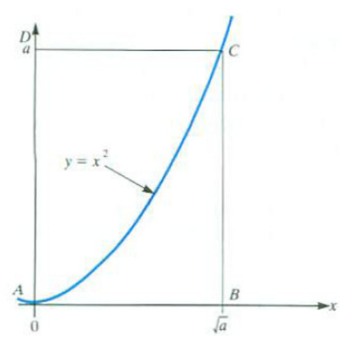


Figure 5.2.11:

See Exercise 25.

25.[M] By considering Figure 5.2.11, in particular the area of region ACD, show that $\int_0^a \sqrt{x}\,dx = \frac{2}{3}a^{3/2}$.

26.[C] Show that $\int_0^b x^{1/2}\,dx = \left(\frac{1}{n} + 1\right)^{-1} b^{\frac{1}{n}+1}$.

27.[M] Show that the volume of a right circular cone of radius a and height h is $\frac{\pi a^3 h}{3}$. Hint: First show that a cross section by a plane perpendicular to the axis of the cone and a distance x from the vertex is a circle of radius ax/h.

In Exercises 28–31 partition the interval into 4 sections of equal length. Evaluate the definite integral using sampling points chosen to be (a) the left endpoints and (b) the right endpoints. **28.**[M] $\int_1^2 (1/x^2)\ dx$.

29.[M] $\int_1^5 \ln(x)\ dx$.

30.[M] $\int_1^5 \frac{2^x}{x}\ dx$.

31.[M] $\int_0^1 \sqrt{1+x^3}\ dx$.

In Exercises 32–34 evaluate $\sum_{i=1}^n f(c_i)(x_i - x_{i-1})$ for the given function, partition, and sampling points. **32.**[R] $f(x) = \sqrt{x}$, $x_0 = 1$, $x_1 = 3$, $x_2 = 5$, $c_1 = 1$, $c_2 = 4$ $(n = 2)$

33.[R] $f(x) = \sqrt[3]{x}$, $x_0 = 0$, $x_1 = 1$, $x_2 = 4$, $x_3 = 10$, $c_1 = 0$, $c_2 = 1$, $c_3 = 8$ $(n = 3)$

34.[R] $f(x) = 1/x$, $x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$, $c_1 = 1$, $c_2 = 1.25$, $c_3 = 1.6$, $c_4 = 2$ $(n = 4)$

35.[M] Write the expression

$$c^{n-1} + c^{n-2}d + c^{n-3}d^2 + \cdots + cd^{n-2} + d^{n-1}$$

in summation notation.

<u>REVIEW</u>

In Exercises 36–43 give two antiderivatives for the given functions. **36.**[M] x^2

37.[M] $1/x^3$

38.[M] $1/x$

39.[M] $1/(2x+1)$

40.[M] 2^x

41.[M] $\sin(3x)$

42.[M] $\frac{3}{1+9x^2}$

43.[M] $\frac{4}{\sqrt{1-x^2}}$

44.[C]

(a) To estimate $\int_1^2 1/x \, dx$ divide $[1, 2]$ into n sections of equal length and use right endpoints as the sampling points.

(b) Deduce from (a) that

$$\lim_{n\to\infty} \sum_{i+1}^{2n} \frac{1}{i} = \lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \text{area under } y = 1/x \text{ and above } [1, 2].$$

45.[M] An object is moving with a velocity of t^2 feet per second at time t seconds. How far does it travel during the time interval $[2, 5]$?

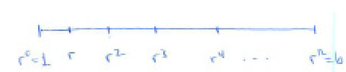


Figure 5.2.12:

46.[C] Pierre Fermat (1601–1665) found the area under $y = x^k$ and above $[1, b]$ by using approximating sums. However, he did not cut the interval $[1, b]$ into n section of equal widths. Instead, for a given positive integer n, he introduced the number r such that $r^n = b$. As n increases, r shrinks towards 1. Then he divided the interval $[0, b]$ into n sections using the number r, r^2, r^3, . . . , r^{n-1}, as shown in Figure 5.2.12. The n sections are $[1, r]$, $[r, r^2]$, . . . , $[r^{n-1}, r^n] = [r^{n-1}, b]$.

(a) Show that the width of the i^{th} section, $[r^{i-1}, r^i]$ is $r^{i-1}(r-1)$.

(b) Using the left endpoints of each section, obtain an underestimate of $\int_1^b x^2 \, dx$.

(c) Show that the estimate in (b) is equal to

$$\frac{b^3 - 1}{1 + r^2 + r^3}.$$

(d) Find $\lim_{n\to\infty} \frac{b^3-1}{1+r+r^3}$.

47.[C] Use Fermat's approach outlined in Exercise 46, but with right endpoints as the sampling points, to obtain an overestimate of the area under x^2, above $[1, b]$, and then find its limit as $n \to \infty$.

48.[M] Area is used to develop $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$ when $b > a > 0$. To see that this result is true for all values of a and b (with $b > a$) we will consider these additional cases:

Too early? Move to a later section?

(a) If $0 > b > a$, work with negative area.

(b) If $b > 0 > a$, divide into two pieces and work with signed areas.

5.3 Properties of the Antiderivative and the Definite Integral

Back in Section 2.6 we defined the antiderivative of a function $f(x)$. It is any function $F(x)$ whose derivative is $f(x)$. For instance x^3 is an antiderivative of $3x^2$. So is $x^3 + 2011$. Keep in mind that *an antiderivative is a function.*

Should we ever state that the antiderivative is a collection of functions?

In this section we discuss various properties of antiderivatives and definite integrals. These properties will be needed in Section 5.4 where we obtain a relation between antiderivatives and definite integrals. That relation will be a great time-saver in evaluating many (but not all) definite integrals.

Up to this point we have not introduced a symbol for an antiderivative of a function. We will adopt the following standard notation:

Notation: Any antiderivative of f is denoted $\int f(x)\ dx$.

warning

For instance, $x^3 = \int 3x^2\ dx$. This equation is read "x^3 is an antiderivative of $3x^2$". That means simply that "the derivative of x^3 is $3x^2$". It is true that $x^3 + 2011 = \int 3x^2\ dx$, since $x^3 + 2011$ is also an antiderivative of $3x^2$. That does *not* mean that the functions x^3 and $x^3 + 2011$ are equal. All it means is that these two functions both have derivatives that are equal to $3x^2$. The symbol $\int 3x^2\ dx$ refers to *any function* whose derivative is $3x^2$.

If $F'(x) = f(x)$ we write $F(x) = \int f(x)\ dx$. The function $f(x)$ is called the **integrand**. The function $F(x)$ is called an antiderivative of $f(x)$. The symbol for an antiderivative, $\int f(x)\ dx$, is similar to the symbol for a definite integral, $\int_a^b f(x)\ dx$, but they denote vastly different concepts. The symbol $\int f(x)\ dx$ denotes a function — any function whose derivative is $f(x)$. The symbol $\int_a^b f(x)\ dx$ denotes a number — one that is defined by a limit of certain sums. The value of the definite integral may vary as the interval $[a, b]$ change.

We apologize for the use of such similar notations, $\int f(x)\ dx$ and $\int_a^b f(x)\ dx$, for such distinct concepts. However, it is not for us to undo over three centuries of history. [Some authors have tried, but failed. For instance, one author denoted an antiderivative of $f(x)$ by the symbol $A(f(x))$ — thus $A(3x^2) = x^3$. The proposal was not accepted.] Rather, it is up to you to read the symbols $\int f(x)\ dx$ and $\int_a^b f(x)\ dx$ carefully. You distinguish between such similar-looking words as "density" and "destiny" or "nuclear" and "unclear". Be just as careful when reading mathematics.

Properties of Antiderivatives

The tables inside the covers of this book list many antiderivatives. One example is $\int \sin(x)\ dx = -\cos(x)$. Of course, $(-\cos(x)) + 17$ also is an antiderivative of $\sin(x)$. The following theorem asserts that if you have found an antiderivative $F(x)$ for a function $f(x)$, then any other antiderivative of $f(x)$ is of the form $F(x) + C$ for some constant C.

This result was anticipated way back in Section 2.6.

Theorem 5.3.1 *If F and G are both antiderivatives of f on some interval, then there is a constant C such that*

$$F(x) = G(x) + C.$$

Proof [

of Theorem 5.3.1] The functions F and G have the same derivative f. By Corollary 3.2.2 in Section 3.2, they must differ by a constant, which we denote by C. Thus $F(x) - G(x) = C$ for any x in the interval. Hence $F(x) = G(x) + C$.
•

In tables, C is usually omitted.

When writing down an antiderivative, it is best to include the constant C. (It will be needed in the study of differential equations in Section 4.2(?).) For example,

$$\begin{aligned} \int 5\ dx &= 5x + C \\ \int e^x\ dx &= e^x + C \end{aligned}$$

and
$$\int \sin(2x)\ dx = \frac{-1}{2}\cos(2x) + C.$$

Observe that

$$\frac{d}{dx}\left(\int x^3\ dx\right) = x^3$$

and
$$\frac{d}{dx}\left(\int \sin(2x)\ dx\right) = \sin(2x).$$

Are these two equations profound or trivial? Read them aloud and decide.

We know that the square of the square root of 7 is 7, again by definition.

The first says, "The derivative of an antiderivative of x^3 is x^3." *It is true simply because that is how we defined the antiderivative.* We know that

$$\frac{d}{dx}\left(\int \frac{\ln(1+x^2)}{(\sin(x))^2}\ dx\right) = \frac{\ln(1+x^2)}{(\sin(x))^2}$$

even though we cannot write out a formula for an antiderivative of $\frac{\ln(1+x^2)}{(\sin(x))^2}$. In other words, by the very definition of the antiderivative,

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Any property of derivatives gives us a corresponding property of antiderivatives. Three of the most important properties of antiderivatives are recorded in the next theorem.

Properties of antiderivatives

Theorem 5.3.2 (Properties of Antiderivatives) *Assume that f and g are functions with antiderivatives $\int f(x)\ dx$ and $\int g(x)\ dx$. Then the following hold:*

A. $\int cf(x)\ dx = c\int f(x)\ dx$ *for any constant c.*

B. $\int (f(x)+g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx$.

C. $\int (f(x)-g(x))\ dx = \int f(x)\ dx - \int g(x)\ dx$.

Proof [

of Theorem 5.3.2] (a) Before we prove that $\int cf(x)\ dx = c\int f(x)\ dx$, we stop to see what it means. This result says that "c times an antiderivative of $f(x)$ is an antiderivative of $cf(x)$". Let $F(x)$ be an antiderivative of $f(x)$. Then the equation says "c times $F(x)$ is an antiderivative of $cf(x)$". To determine if this statement is true we must differentiate $cF(x)$ and check that we get $cf(x)$. So, we compute $(cF(x))'$:

$$\begin{aligned} \tfrac{d}{dx}(cF(x)) &= c\tfrac{dF}{dx}(x) && [c \text{ is a constant}] \\ &= cf(x). && [F \text{ is antiderivative of } f] \end{aligned}$$

Thus $cF(x)$ is indeed an antiderivative of $cf(x)$. Therefore, we may write

$$cF(x) = \int cf(x)\ dx.$$

Since $F(x) = \int f(x)\ dx$, we conclude that

$$c\int f(x)\ dx = \int cf(x)\ dx.$$

(b) The proof is similar. We show that $\int f(x)\ dx + \int g(x)\ dx$ is an antiderivative of $f(x)+g(x)$. To do this we compute the derivative of the sum $\int f(x)\ dx + \int g(x)\ dx$:

$$\begin{aligned} \tfrac{d}{dx}\left(\int f(x)\ dx + \int g(x)\ dx\right) &= \tfrac{d}{dx}\left(\int f(x)\ dx\right) + \tfrac{d}{dx}\left(\int g(x)\ dx\right) && [\text{derivative of a sum}] \\ &= f(x)+g(x). && [\text{definition of antiderivatives}] \end{aligned}$$

(c) The proof is very similar to the one given for (b). •

EXAMPLE 1 Find (a) $\int 6\cos(x)\ dx$, (b) $\int (6\cos(x)+3x^2)\ dx$, and (c) $\int (6\cos(x) - \frac{5}{1+x^2}\ dx$.

SOLUTION (a) Part (a) of the theorem is used to move the "6" (a constant) past the "$\int$":

$$\int 6\cos(x)\ dx = 6\int \cos(x)\ dx = 6\sin(x) + C.$$

Notice that the "$+C$" is added as the last step in finding the antiderivative. (b)

$$\begin{aligned} \int (6\cos(x) + 3x^2)\ dx &= \int 6\cos(x)\ dx + \int 3x^2\ dx \quad \text{[part (b) of the theorem]} \\ &= 6\sin(x) + x^3 + C. \end{aligned}$$

Here, notice that separate constants are not needed for each antiderivative; again only one "$+C$" is needed for the overall antiderivative. (c)

$$\begin{aligned} \int \left(6\cos(x) - \tfrac{5}{1+x^2}\right)\ dx &= \int 6\cos(x)\ dx - \int \tfrac{5}{1+x^2}\ dx && \text{[part (c) of the theorem]} \\ &= 6\sin(x) - 5\int \tfrac{1}{1+x^2}\ dx && \text{[part (a) of the theorem]} \\ &= 6\sin 9x) - 5\arctan(x) + C && [(\arctan(x))' = \tfrac{1}{1+x^2}] \end{aligned}$$

◇

The last two parts of Theorem 5.3.2 extend to any finite number of functions. For instance,

$$\int (f(x) - g(x) + h(x))\ dx = \int f(x)\ dx - \int g(x)\ dx + \int h(x)\ dx.$$

Theorem 5.3.3 *Let a be a number other than -1. Then*

$$\int x^a\ dx = \frac{x^{a+1}}{a+1} + C.$$

Proof [
of Theorem 5.3.3]

$$\left(\frac{x^{a+1}}{a+1}\right)' = \frac{(a+1)x^{(a+1)-1}}{a+1} = x^a.$$

•

EXAMPLE 2 Find $\int \left(\frac{3}{\sqrt{1-x^2}} - \frac{2}{x} + \frac{1}{x^3}\right)\ dx$, $0 < x < 1$.
SOLUTION

If $-1 < x < 0$, replace $\ln(x)$ by $\ln|x|$.

$$\int \left(\frac{3}{\sqrt{1-x^2}} - \frac{2}{x} + \frac{1}{x^3} \right) dx = 3\int \frac{1}{\sqrt{1-x^2}}\, dx - 2\int \frac{1}{x}\, dx + \int x^{-3}\, dx \qquad (5.1)$$

$$= 3\arcsin(x) - 2\ln(x) + \frac{x^{-2}}{-2} + C \qquad (5.2)$$

$$= 3\arcsin(x) - 2\ln(x) - \frac{1}{2x^2} + C. \qquad (5.3)$$

◇

Properties of Definite Integrals

Some of the properties of definite integrals look like similar properties of antiderivatives. However, they are assertions about numbers, not about functions. In the notation for the definite integral $\int_a^b f(x)\, dx$, b is larger than a. It will be useful to be able to speak about "the definite integral from a to b" even if b is less than or equal to a. The following definitions meet this need and we will use them in the proofs of the two fundamental theorems of calculus (in Section 5.4).

> **DEFINITION** *([)*Integral from a to b, where b is less than a.] If b is less than a, then
>
> $$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx.$$

EXAMPLE 3 Compute $\int_3^0 x^2\, dx$, the integral from 3 to 0 of x^2.

SOLUTION The symbol $\int_3^0 x^2\, dx$ is defined as $-\int_0^3 x^2\, dx$. As was shown in Section 5.2, $\int_0^3 x^2\, dx = 9$. Thus

$$\int_3^0 x^2\, dx = -9.$$

◇

DEFINITION *([)*Integral from a to a.]

$$\int_a^a f(x)\, dx = 0$$

Remark The definite integral is defined with the aid of partitions. Rather than permit partitions to have sections of length 0, it is simpler just to make the preceding definition.

The point of making these two definitions is that now the symbol $\int_a^b f(x)\, dx$ is defined for any numbers a and b and any continuous function f. It is no longer necessary that a be less than b.

The definite integral has several properties, some of which we will using in this section and some in later chapters.

Move description of each property to the margin.

Theorem 5.3.4 (Properties of the Definite Intgral) *Let f and g be continuous functions, and let c be a constant. Then*

1. ***Moving a Constant Past $\int_a^b$***
   $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx.$
2. ***Definite Integral of a Sum***
   $\int_a^b (f(x)+g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx.$
3. ***Definite Integral of a Difference***
   $\int_a^b (f(x)-g(x))\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx.$
4. ***Definite Integral of a Non-Negative Function***
   If $f(x) \geq 0$ for all x in $[a,b]$, $a<b$, then
   $$\int_a^b f(x)\ dx \geq 0.$$
5. ***Definite Integrals Preserve Order***
   If $f(x) \geq g(x)$ for all x in $[a,b]$, $a<b$, then
   $$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx.$$
6. ***Sum of Definite Integral Over Adjoining Intervals***
   If a, b, and c are numbers, then
   $$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx.$$
7. ***Bounds on Definite Integrals***
   If m and M are numbers and $m \leq f(x) \leq M$ for all x between a and b, then
   $$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a) \quad \text{if } a<b$$
   and
   $$m(b-a) \geq \int_a^b f(x)\ dx \geq M(b-a) \quad \text{if } a>b$$

Proof of Property 1

Take the case $a < b$. The equation $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$ resembles part (a) of Theorem 5.3.2 about antiderivatives: $\int cf(x)\ dx = c\int f(x)\ dx$. However, its proof is quite different, since $\int_a^b cf(x)\ dx$ is defined as a limit of sums.

We have

$$\begin{aligned}
\int_a^b cf(x)\ dx &= \lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^n cf(c_i)\Delta x_i && \text{definition of definite integral} \\
&= \lim_{\text{all } \Delta x_i \to 0} c\sum_{i=1}^n f(c_i)\Delta x_i && \text{algebra} \\
&= c\lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^n f(c_i)\Delta x_i && \text{property of limits} \\
&= c\int_a^b f(x)\ dx. && \text{definition of definite integral}
\end{aligned}$$

• DOUG: Move discussion of negative area to here.

The other properties can be justified by a similar approach. However, we pause only to make them plausible. To do this, interpret each of the properties in terms of areas (assuming the integrands are positive).

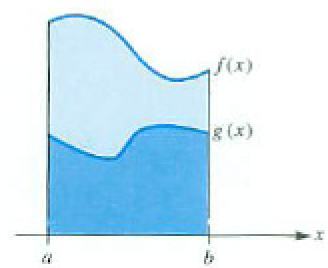


Figure 5.3.1:

Property 5 Property 5 amounts to the assertion that when the graph of $y = f(x)$ is always at least as high as the graph of $y = g(x)$, then the area of a region under the curve $y = f(x)$ is greater than or equal to the area under the curve $y = g(x)$. (See Figure 5.3.1.)

Property 6 In the case that $a < c < b$ and $f(x)$ assumes only positive values, property 6 asserts that the area of the region below the graph of $y = f(x)$ and above the interval $[a, b]$ is the sum of the areas of the regions below the graph and above the smaller intervals $[a, c]$ and $[c, b]$. Figure 5.3.2 shows that this is certainly plausible.

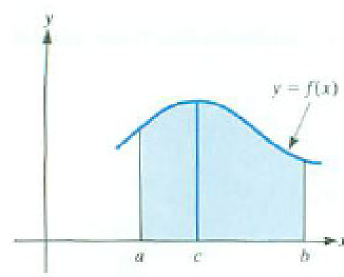


Figure 5.3.2:

Property 7 The inequalities in property 7 compare the area under the graph of $y = f(x)$ with the areas of two rectangles, one of height M and one of height m. (See Figure 5.3.3.) In the case $a < b$, the area of the larger rectangle is $M(b - a)$ and the area of the smaller rectangle is $m(b - a)$.

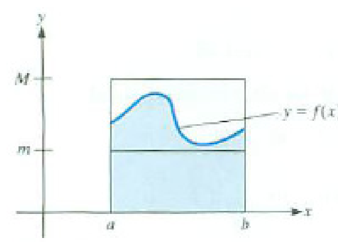


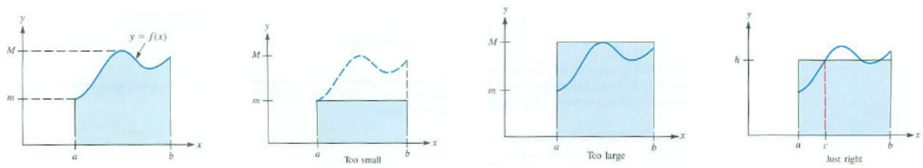
Figure 5.3.3:

The Mean-Value Theorem for Definite Integrals

The mean-value theorem for derivatives says that (under suitable hypotheses) $f(b) - f(a) = f'(c)(b - a)$ for some number c in $[a, b]$. The mean-value theorem for definite integrals has a similar flavor. First, we state it geometrically.

If $f(x)$ is positive and $a < b$, then $\int_a^b f(x)\ dx$ can be interpretted as the area of the shaded region in Figure 5.3.4(a).

Let m be the minimum and M the maximum values of $f(x)$ for x in $[a, b]$. The area of the rectangle of height M is larger than the shaded area; the area of the rectangle of height m is smaller than the shaded area. (See Figures 5.3.4(b) and (c).) Therefore, there is a rectangle whose height h is somewhere between



(a) (b) (c) (d)

Figure 5.3.4:

m and M, whose area is the same as the shaded area under the curve $y = f(x)$. (See Figure 5.3.4(d).) Hence $\int_a^b f(x)\ dx = h(b-a)$.

Now, h is a number between m and M. By the Intermediate-Value Property for continuous functions, there is a number c in $[a, b]$ such that $f(c) = h$. (See Figure 5.3.4(d).) Hence,

See Section 1.4.

$$\text{Area of shaded region under curve} = f(c)(b-a).$$

Next we state the mean-value theorem for definite integrals.

Mean-Value Theorem for Definite Integrals

Theorem 5.3.5 (Mean-Value Theorem for Definite Integrals) *Let a and b be numbers, and let f be a continuous function defined for x between a and b. Then there is a number c between a and b such that*

$$\int_a^b f(x)\ dx = f(c)(b-a).$$

Proof [

of Mean-Value Theorem for Definite Integrals] Consider the case when $a < b$. Let M be the maximum and m the minimum of $f(x)$ on $[a, b]$. By property 7,

Recall the Extreme-Value Property in Section 1.4.

$$m \leq \frac{\int_a^b f(x)\ dx}{b-a} \leq M,$$

Because f is continuous on $[a, b]$, by the Intermediate-Value Property of Section 1.4 there is a number c in $[a, b]$ such that

$$f(c) = \frac{\int_a^b f(x)\ dx}{b-a},$$

and this theorem is proved. •

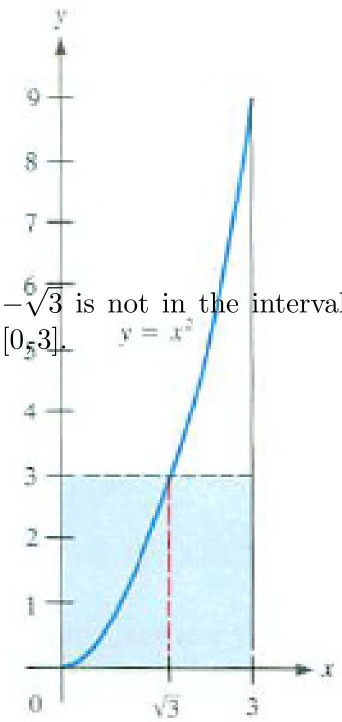
The case $b < a$ can be obtained from the case $a < b$. (see Exercise 33).

EXAMPLE 4 Verify the mean-value theorem for definite integrals when $f(x) = x^2$ and $[a, b] = [0, 3]$.

SOLUTION Since $f(x) = x^2$, we are looking for c in $[0, 3]$ such that

$$\int_0^3 x^2\ dx = 9 \quad = \quad c^2(3 - 0) \tag{5.4}$$

$$\text{or} \qquad 9 \quad = \quad 3c^2. \tag{5.5}$$



Since $c^2 = \frac{9}{3} = 3$, $c = \sqrt{3}$. (See Figure 5.3.5.) The rectangle with height $f(\sqrt{3}) = (\sqrt{3})^2 = 3$ and base $[0, 3]$ has the same area as the area of the region under the curve $y = x^2$ and above $[0, 3]$. ◇

$-\sqrt{3}$ is not in the interval $[0, 3]$.

Figure 5.3.5:

In Section 5.2 it was shown that $\int_0^3 x^2\ dx = 9$.

The Average Value of a Function

Let $f(x)$ be a continuous function defined on $[a, b]$. What shall we mean by the "average value of $f(x)$ over $[a, b]$"? We cannot add up all the values of $f(x)$ for all x's in $[a, b]$ and divide by the number of x's, since there are an infinite number of such x's. The average (or mean) of n numbers a_1, a_2, $\ldots$, a_n is their sum divided by n: $\frac{1}{n}\sum_{i=1}^{n} a_i$. For example, the average of 1, 2, and 6 is $\frac{1}{3}(1 + 2 + 6) = \frac{9}{3} = 3$.

This suggests how to define the "average value of $f(x)$ over $[a, b]$". Choose a large integer n and divide $[a, b]$ into n sections of equal length, $\Delta x = (b - a)/n$. Let the sampling points c_i be the left endpoint of each section, $c_1 = a$, $c_2 = a + Deltax$, $\ldots$, $c_n = a + (n - 1)\Delta x = b - \Delta x$. Then an estimate of the "average" would be

$$\frac{1}{n}(f(c_1) + f(c_2) + \cdots + f(c_n)). \tag{5.6}$$

Since $\Delta x = (b - a)/n$, it follows that $\frac{1}{n} = \frac{\Delta x}{b-a}$. Therefore, we may rewrite (5.6) as

$$\frac{1}{b - a}\sum_{i=1} f(c_i)\Delta x. \tag{5.7}$$

But, $\sum_{i=1} f(c_i)\Delta x$ is an estimate for $\int_a^b f(x)\ dx$. As $n \to \infty$, the average of n function values approaches $\frac{1}{b-a}\int_a^b f(x)\ dx$. This motivates the following definition:

Average Value of $f(x)$ on $[a, b]$

DEFINITION *(]*Average Value of a Function over an Interval] Let $f(x)$ be defined on the interval $[a,b]$. Assume that $\int_a^b f(x)\,dx$ exists. The **average value** or **mean value** of f on $[a,b]$ is defined to be

$$\frac{1}{b-a}\int f(x)\,dx.$$

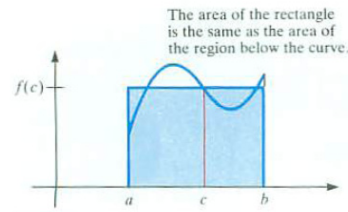


Figure 5.3.6:

Geometrically speaking [if $f(x)$ is positive], this average value is the height of the rectangle that has the same area as the area of the region under the curve $y = f(x)$, above $[a,b]$. (See Figure 5.3.6.) Observe that the average value of $f(x)$ over $[a,b]$ is between its maximum and minimum values for x in $[a,b]$. However, it is not necessarily the average of these two numbers.

EXAMPLE 5 Find the average value of 2^x over the interval $[1,3]$.
SOLUTION The average value of 2^x over $[1,3]$ be definition equals

$$\frac{1}{3-1}\int_1^3 2^x\,dx.$$

First,

$\int_a^b 2^x\,dx = \frac{1}{\ln(2)}(2^b - 2^a)$

$$\int_1^3 2^x\,dx = \frac{1}{\ln(2)}(2^3 - 2^1) \quad (5.8)$$

$$= \frac{6}{\ln(2)}. \quad (5.9)$$

Hence,

$$\text{average value of } 2^x \text{ over } [1,3] = \frac{1}{3-1}\frac{6}{\ln(2)} = \frac{3}{\ln(2)} \approx 4.2381.$$

The average of the maximum and minimum values of 2^x on $[1,3]$ is $\frac{1}{2}(2^3+2^1) = 5$.

◇

Confusing terminology

WARNING *(]*Antiderivative Terminology] We must confess that in the real world an antiderivative is most often called an "integral" or "indefinite integral". However, if you stay alert, the context will always reveal whether the word "integral" refers to an antiderivative (a function) or to a definite integral (a number). They are two wildly different beasts. Even so, the next section will show that there is a very close connection between them. This connection ties the two halves of calculus — differential and integral calculus — into one neat package.

Summary

We introduced the notation $\int f(x)\,dx$ for an **antiderivative** of $f(x)$. Using this notation we stated various properties of antiderivatives.

We defined the symbol $\int_a^b f(x)\,dx$ in the special case when $b \leq a$, and stated various properties of definite integrals.

The mean-value theorem for definite integrals asserts that $\int_a^b f(x)\,dx$ equals $f(c)$ times $(b-a)$ for at least one value of c in $[a,b]$.

The quantity $\frac{1}{b-a}\int_a^b f(x)\,dx$ is called the **average value** (or **mean value**) of $f(x)$ over $[a,b]$. It can be thought of as the height of the rectangle whose area is the same as the area of the region under the curve $y = f(x)$.

EXERCISES for 5.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1–12 evaluate each antiderivative. Do not forget to add a constant to each answer. Check each answer by differentiating it.

1.[R] $\int 5x^2\ dx$

2.[R] $\int 7/x^2\ dx$

3.[R] $\int (2x - x^3 + x^5)\ dx$

4.[R] $\int \left(6x^2 + 2x^{-1} + \frac{1}{\sqrt{x}}\right)\ dx$

5.[R]

(a) $\int e^x\ dx$

(b) $\int e^{x/3}\ dx$

6.[R]

(a) $\int \frac{1}{1+x^2}\ dx$

(b) $\int \frac{1}{1}\sqrt{1-x^2}\ dx$

7.[R]

(a) $\int \cos(x)\ dx$

(b) $\int \cos(2x)\ dx$

8.[R]

(a) $\int \sin(x)\ dx$

(b) $\int \sin(3x)\ dx$

9.[R]

(a) $\int (2\sin(x) + 3\cos(x))\ dx$

(b) $\int (\sin(2x) + \cos(3x))\ dx$

10.[R] $\int \sec(x)\tan(x)\ dx$

11.[R] $\int (\sec(x))^2\ dx$

12.[R] $\int (\csc(x))^2\ dx$

13.[R] Evaluate

(a) $\int_2^5 x^2\ dx$

(b) $\int_5^2 x^2\ dx$

(c) $\int_5^5 x^2\ dx$

14.[R] Evaluate

(a) $\int_1^2 x\ dx$

(b) $\int_2^1 x\ dx$

(c) $\int_3^3 x\ dx$

$\int_a^b x\ dx$ was discussed in Section 5.2, and also in in Exercise 20 in that section.

15.[R] Find

(a) $\int x\ dx$

(b) $\int_3^4 x\ dx$

16.[R] Find

(a) $\int 3x^2\ dx$

(b) $\int_1^4 3x^2\ dx$

17.[R] If $2 \leq f(x) \leq 3$, what can be said about $\int_1^6 f(x)\ dx$?

18.[R] If $-1 \leq f(x) \leq 4$, what can be said about $\int_{-2}^7 f(x)\ dx$?

19.[R] Write a sentence or two, in your own words, that tells what the symbols $\int f(x)\ dx$ and $\int_a^b f(x)\ dx$ mean. Include examples.

20.[R] Let $f(x)$ be a differentiable function. In this exercise you will determine if the following equation is true or false:

$$f(x) = \int \frac{df}{dx}(x)\ dx.$$

(a) Pick several functions of your choice and test if the equation is true.

(b) Determine if the equation is true. Write a brief justification for your answer.

The mean-value theorem for definite integrals asserts that if $f(x)$ is continuous, then $\int_a^b f(x)\ dx = f(c)(b-a)$ for some number c in $[a,b]$. In each of Exercises 21–24 find $f(c)$ and at least one value of c in $[a,b]$.

21.[R] $f(x) = 2x$; $[a,b] = [1,5]$

22.[R] $f(x) = 5x + 2$; $[a,b] = [1,2]$

23.[R] $f(x) = x^2$; $[a,b] = [0,4]$

24.[R] $f(x) = x^2 + x$; $[a,b] = [1,4]$

25.[R] If $\int_1^2 f(x)\ dx = 3$ and $\int_1^5 f(x)\ dx = 7$, find

(a) $\int_2^1 f(x)\ dx$

(b) $\int_2^5 f(x)\ dx$

26.[R] If $\int_1^3 f(x)\ dx = 4$ and $\int_1^5 g(x)\ dx = 5$, find

(a) $\int_1^3 2f(x) + 6g(x)\ dx$

(b) $\int_3^1 f(x) - g(x)\ dx$

27.[R] If the maximum value of $f(x)$ on $[a,b]$ is 7 and the minimum value on $[a,b]$ is 4, what can be said about

(a) $\int_a^b f(x)\ dx$?

(b) the mean value of $f(x)$ over $[a,b]$?

28.[R] Let $f(x) = c$ (constant) for all x in $[a,b]$. Find the average value of $f(x)$ on $[a,b]$.

Exercises 29–32 concern the average of a function over an interval. In each case, find the minimum, maximum, and average value of the function over the given interval.

29.[R] $f(x) = x^2$, $[2, 3]$

30.[R] $f(x) = x^2$, $[0, 5]$

31.[R] $f(x) = 2^x$, $[0, 4]$

32.[R] $f(x) = 2^x$, $[2, 4]$

33.[M] Prove the mean-value theorem for definite integrals in the case when $b < a$. HINT: Use the definition of $\int_a^b f(x)\, dx$ when $b < a$.

34.[M] Is $\int f(x)g(x)\, dx$ always equal to $\int f(x)\, dx \int g(x)\, dx$?

35.[M]

SHERMAN: You crossed out this exercise, but I think it is worthwhile.

(a) Show that $4\frac{1}{3}(\sin(x))^3$ is *not* an antiderivative of $\int (\sin(x))^2\, dx$.

(b) Use the identity $(\sin(x))^2 = \frac{1}{2}(1 - \cos(2x))$ to find an antiderivative for $\int (\sin(x))^2\, dx$.

(c) Verify your answer in (b) by differentiation.

36.[M] Use the properties of definite integrals to derive the following formula:

$$\int_a^b 2^x\, dx = \frac{2^b}{\ln(2)} - \frac{2^a}{\ln(2)}.$$

In Exercises 37–38 verify the equations quoted from a table of antiderivatives (integrals). Just differentiate each of the alleged antiderivatives and see whether you obtain the quoted integrand. (The number a is a constant in each case.)

37.[M] $\int x^2 \sin(ax)\, dx = \frac{2x}{a^2} \sin(ax) + \frac{2}{a^3} \cos(ax) - \frac{x^2}{a} \cos(ax) + C$

38.[M] $\int x(\sin(ax))^2\, dx = \frac{x^2}{4} - \frac{x}{4a} \sin(2ax) - \frac{1}{8a^2} \cos(2ax) + C$

39.[C] The average value of a certain function $f(x)$ over $[1,3]$ is 4. Over $[3,6]$ the average value of the same function is 5. What is the average value over $[1,6]$? (Explain your answer.)

40.[C]

(a) Draw the graphs of $y = (\cos(x))^2$ and $y = (\sin(x))^2$. On the basis of your picture, how do $\int_0^{\pi/2} (\cos(x))^2\, dx$ and $\int_0^{\pi/2} (\sin(x))^2\, dx$ compare?

(b) Using (a) and a trigonometric identity, show that

$$\int_0^{\pi/2} (\cos(x))^2\, dx = \frac{\pi}{4} = \int_0^{\pi/2} (\sin(x))^2\, dx.$$

(c) What is $\int_0^{\pi} (\cos(x))^2\, dx$?

SHERMAN: This problem was moved from Section 6.1. OK?

41.[M] Define $f(x) = \begin{cases} -x & 0 < x \le 1 \\ -1 & 1 < x \le 2 \\ 1 & 2 < x \le 3 \\ 4 - x & 3 < x \le 4 \end{cases}$.

(a) Sketch the graphs of $y = f(x)$ and $y = f(x)^2$ on the interval $[0, 4]$.

(b) Find the average value of f on the interval $[0, 4]$.

(c) Find the "root mean square" value of f on the interval $[0, 4]$. That is, compute $\sqrt{\frac{1}{4-0} \int_0^4 f(x)^2\, dx}$.

(d) Why is it not surprising that your answer in (b) is zero and your answer in (c) is positive?

SHERMAN: Can you construct a "he said/she said" exercise to liven up these exercises?

5.4 The Fundamental Theorem of Calculus

Introduction and Motivation

This is the most important section of the entire book.

FTC I gives a shortcut to evaluating $\int_a^b f(x)\ dx$

FTC II gives a way to evaluate $\frac{d}{dx}\left(\int_a^x f(t)\ dt\right)$

In this section, the most important in the book, we obtain two very closely related theorems. They are called the Fundamental Theorems of Calculus I and II, or simply The Fundamental Theorem of Calculus (FTC). The first part of the FTC provides a way to evaluate a definite integral if you are lucky enough to know an antiderivative of the integrand. That means that the derivative, developed in Chapter 2, has yet another application.

The second fundamental theorem tells how rapidly the value of a definite integral changes as you change the interval $[a, b]$ over which you are integrating. This part of the Fundamental Theorem is used to prove the first part of the FTC.

In Section 5.2 we found that $\int_a^b c\ dx = cb - ca$ and $\int_a^b x\ dx = \frac{b^2}{2} - \frac{a^2}{2}$ by interpreting the definite integrals as the area of a rectangle and trapezoid, respectively. In the same section we found that $\int_a^b x^2\ dx = \frac{b^3}{3} - \frac{a^3}{3}$. The reasoning was based, essentially, on the fact that congruent lopsided tents fill a cube. Finally, using the formula for the sum of a geometric series, we showed that $\int_a^b 2^x\ dx = \frac{2^b}{\ln(2)} - \frac{2^a}{\ln(2)}$.

Notice that all four results follow a similar pattern:

$$\begin{aligned}
\int_a^b c\ dx &= cb - ca \\
\int_a^b x\ dx &= \frac{b^2}{2} - \frac{a^2}{2} \\
\int_a^b x^2\ dx &= \frac{b^3}{3} - \frac{a^3}{3} \\
\int_a^b 2^x\ dx &= \frac{2^b}{\ln(2)} - \frac{2^a}{\ln(2)}
\end{aligned}$$

Motivation for the Fundamental Theorem of Calculus I

To describe the similarity in detail, compute an antiderivative of each of the four integrands:

$$\int c\, dx = cx + C \qquad \int x\, dx = \frac{x^2}{2} + C \qquad \int x^2\, dx = \frac{x^3}{3} + C \qquad \int 2^x\, dx = \frac{2^x}{\ln(2)} + C.$$

In each case the definite integral equals the difference between an antiderivative evaluated at b an at a, the two endpoints of the interval.

This suggests that maybe for any integrand $f(x)$, the following may be true: If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\ dx = F(b) - F(a). \tag{5.1}$$

If this is correct, then, instead of resorting to special tricks to evaluate a definite integral, such as cutting up a cube or summing a geometric series, we look for an antiderivative of the integrand.

We may reason using "velocity and distance" to provide further evidence for (5.1). Picture a particle moving upwards on the y-axis. At time t it is at position $F(t)$ on that line. The velocity at time t is $F'(t)$.

But we saw that the definite integral of the velocity from time a to time b changed into position covered, that is,

"the definite integral of the velocity $=$ the final position $-$ the initial position$''$

In symbols,

$$\int_a^b F'(t)\ dt = F(b) = F(a). \tag{5.2}$$

If we give $F'(t)$ the name $f(t)$, then we can restate (5.2) as: If $f(t) = F'(t)$, then

$$\int_a^b f(t)\ dt = F(b) - F(a).$$

In other words, if F is an antiderivative of f, then

$$\int_a^b f(t)\ dt = F(b) - F(a). \tag{5.3}$$

Formulas we found for the integrands c, x, x^2, and 2^x and reasoning about motion are all consistent with FTC I

Theorem 5.4.1 (Fundatmental Theorem of Calculus I) *If f is continuous on $[a, b]$ and if F is an antiderivative of f then*

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

In practical terms this theorem says, "To evaluate the definite integral of f from a to b, look for an antiderivative of f. Evaluate the antiderivative at b and subtract its value at a. This difference is the value of the definite integral you are seeking". The success of this approach hinges on finding an antiderivative of the integrand f. For many functions, it is easy to find their antiderivatives. For some it is hard, but theycan be found. For others, the antiderivatives cannot be expressed in terms of the functions met in Chapters 1 and 2, namely polynomials, quotients of polynomials, and functions built up from trigonometric, exponential, and logarithm functions and their inverses.

Some techniques for finding antiderivatives are discussed in Chapter 6.

Example 1 shows how powerful FTC I is.

EXAMPLE 1 Use the Fundamental Theorem of Calculus to evaluate $\int_0^{\pi/2} \cos(x)\, dx$.

SOLUTION Since $(\sin(x))' = cos(x)$, $\sin(x)$ is an antiderivative of $\cos(x)$. By the FTC I,

How would the evaluation be different if we used $\sin(x)+5$ as the antiderivative of $\cos(x)$?

$$\begin{aligned}\int_0^{\pi/2} \cos(x)\, dx &= \sin(\frac{\pi}{2}) - \sin(0)\\ &= 1 - 0\\ &= 1.\end{aligned}$$

This tells us tht the area under the curve $y = \cos(x)$ and above $[0, \pi/2]$, shown in Figure 5.4.1 is 1.

This result is reasonable since thearea lies inside a rectangle of area $1 \times \pi/2 = \pi/2 \approx 1.5708$ and contains a triangle of area $\frac{1}{2}\left(\frac{\pi}{2}\right)1 = \frac{\pi}{4} \approx 0.7854$. ⋄

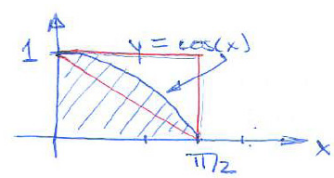


Figure 5.4.1:

Motivation for the Fundamental Theorem of Calculus II

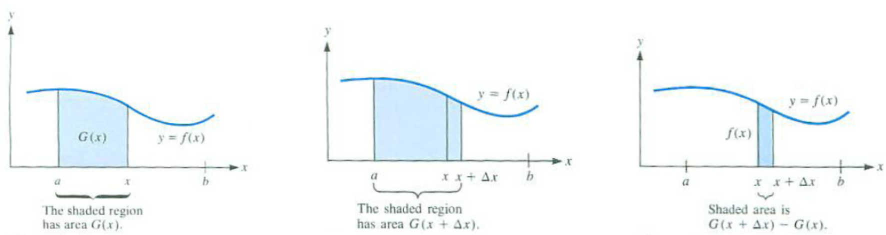
Let f be a continuous function such that $f(x)$ is positive for x in $[a, b]$. For x in $[a, b]$, let $G(x)$ be the area of teh region under the graph of f and above the interval $[a, x]$, as shown in Figure 5.4.2(a). For instance, $G(a) = 0$.

We will compute the derivative of $G(x)$, that is,

$$\lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x} = \lim_{\Delta x \to 0} \frac{G(x + \Delta x) - G(x)}{\Delta x}.$$

For simplicity, keep Δx positive. Then $G(x + \Delta x)$ is the area under the curve $y = f(x)$ above the interval $[x+\Delta x]$. If Δx is small, $G(x+\Delta x)$ is only slightly larger than $G(x)$, as shown in Figure 5.4.2(b). Then $\Delta G = G(x+\Delta x) - G(x)$ is the area of the thin shaded strip in Figure 5.4.2(c).

When Δx is small, the narrow shaded strip above $[x, x + \Delta x]$ resembles a rectangle of base Δx and height $f(x)$, with area $f(x)\Delta x$. Therefore, it seems



(a) (b) (c)

Figure 5.4.2:

reasonable that when Δx is small,

$$\frac{\Delta G}{\Delta x} \approx \frac{f(x)\Delta x}{\Delta x} = f(x).$$

In short, it seems plausible that

$$\lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x} = f(x).$$

Briefly,

$$G'(x) = f(x).$$

In words, "the derivative of the area of the region under the graph of f and above $[a, x]$ with respect to x is the value of f at x".

Now we state these observations in terms of definite integrals.

Let f be a continuous function. Let $G(x) = \int_a^x f(t)\ dt$. Then we expect that

$$\frac{d}{dx}\left(\int_a^x f(t)\ dt\right) = f(x).$$

We use t in the integrand to avoid using x to denote both an end of the interval and a variable that takes values between a and x.

This equation says that "the derivative of the definite integral of f with respect to the right end of the interval is simply f evaluated at that end". This is the substance of the Fundamental Theorem of Calculus II. It tells how rapidly the definite integral changes as we change the upper limit of integration, b.

Theorem 5.4.2 (Fundamental Theorem of Calculus II) *Let f be continuous on the interval $[a, b]$. Define*

$$G(x) = \int_a^x f(t)\ dt$$

for all $a \le x \le b$. Then G is differentiable on $[a, b]$ and its derivative is f; that is,

$$G'(x) = f(x).$$

As a consequence of FTC II, *every continuous function is the derivative of some function.*

There is a similar theorem for $H(x) = \int_x^b f(t)\ dt$. A glance at Figure 5.4.3 shows why there is s minus sign: the area in this figure shrinks as x increases.

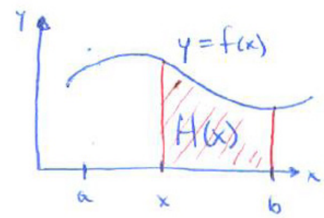


Figure 5.4.3:

See Exercise 49.

EXAMPLE 2 Give an example of an antiderivative of $\frac{\sin(x)}{x}$.

SOLUTION There are many antiderivatives of $\frac{\sin(x)}{x}$. Any two antiderivatives differ by a constant. These curves can be seen in the slope field for $y' = \frac{\sin(x)}{x}$ shown in Figure 5.4.4.

Let $G(x) = \int_1^x \frac{\sin(t)}{t}\ dt$. By FTC II, $G'(x) = \frac{\sin(x)}{x}$. The graph of $y = G(x)$ is shown in Figure 5.4.5. Notice that $G(1) = 0$. ◇

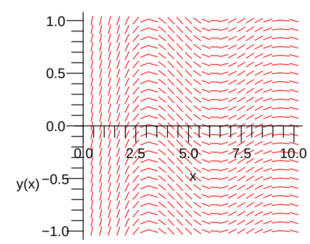


Figure 5.4.4:

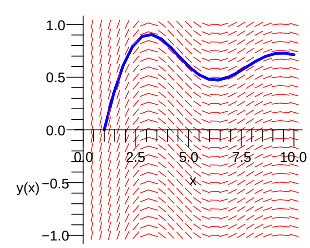


Figure 5.4.5:

Does the solution to Example 2 make you uncomfortable?

You probably expected the answer to be an explicit formula for the antiderivative expressed in terms of the familiar functions discussed in Chapters 1 and 2. Recall, from Section 2.6, that the derivative of every elementary function is an elementary function. Liouville proved that there are (many) elementary functions that do not have elementary antiderivatives. Nobody will ever find an explicit formula in terms of elementary functions for the antiderivative of $\frac{\sin(x)}{x}$. (The proof is reserved for a graduate course.)

Joseph Liouville (1809–1882) `http://en.wikipedia.org/wiki/Joseph_Liouville`

EXAMPLE 3 Give an example of an antiderivative of $\frac{\sin(\sqrt{x})}{\sqrt{x}}$.

SOLUTION This integrand appears to be more terrifying than $\frac{\sin(x)}{x}$, yet it does have an elementary antiderivative, namely $-2\cos(\sqrt{x})$. To check, we differentiate $y = -2\cos(\sqrt{x})$ by the Chain Rule. We have $y = -2\cos(u)$ where $u = \sqrt{x}$. Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -2(-\sin(u))\frac{1}{2\sqrt{x}} = \frac{\sin(\sqrt{x})}{\sqrt{x}}.$$

◇

Because the antiderivatives of $\frac{\sin(\sqrt{x})}{\sqrt{x}}$ are elementary functions, it would be easy to calculate $\int_1^2 \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$.

The constant $\frac{2}{\sqrt{\pi}}$ is introduced so that $\lim_{x\to\infty} erf(x) = 1$. See Exercise 49 in Section 5.4.

Any antiderivative of e^x is of the form $-e^x + C$, an elementary function. However, no antiderivative of e^{-x^2} is elementary. Statisticians define the **error function** to be $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$. Except that $\operatorname{erf}(0) = 0$, there is no easy way to evaluate $\operatorname{erf}(x)$. Since $\operatorname{erf}(x)$ is not elementary, it is customary to collect approximate value of the definite integral for different values of x in a table.

Algebraic Area

Is there a better location for this?

When we evaluate $\int_0^\pi \cos(x)\,dx$, we obtain $\sin(\pi) - \sin(0) = 0 - 0 = 0$. What does this say about areas? Inspection of Figure 5.4.6 shows what is happening.

For x in $[0, \pi/2]$, $\cos(x)$ is negative and the curve $y = \cos(x)$ lies *below* the x-axis. If we interpret the corresponding area as negative, then we see that it cancels with the area from 0 to $\pi/2$. So let us agree that when we say "$\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$", we mean that it represents the area between the curve and the x-axis, *with area below the x-axis taken as negative.*

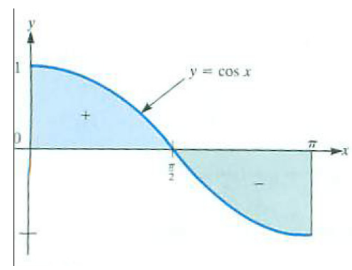


Figure 5.4.6:

EXAMPLE 4 Exaluate $\int_1^2 \frac{1}{x^2}\,dx$ by the First Fundamental Theorem of Calculus.

SOLUTION In order to apply FTC I we have to find an antiderivative of $\frac{1}{x^2}$: $\int \frac{1}{x^2}\,dx$. In Section 5.3 it was observed that

$$\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C \qquad a \neq -1.$$

An antiderivative of $\frac{1}{x^2} = x^{-2}$ is obtained when $a = -2$:

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{1}{(-2)+1}x^{(-2)+1} + C = \frac{1}{-1}x^{-1} + C = \frac{-1}{x} + C$$

By FTC II

Note that the C's cancel. We do not need the C when applying FTC II.

$$\int_1^2 \frac{1}{x^2}\,dx = \left(\frac{-1}{x} + C\right)\Big|_1^2 = \left(\frac{-1}{2} + C\right) - \left(\frac{-1}{1} + C\right) = \frac{-1}{2} - (-1) = \frac{1}{2}.$$

◇

The First Fundamental Theorem of Calculus asserts that

$$\underbrace{\int_1^2 \frac{1}{x^2}\,dx}_{\text{The definite integral: a limit of sums.}} = \underbrace{\int \frac{1}{x^2}\,dx\,\Big|_1^2}_{\text{The difference between an antiderivative evaluated at 2 and at 1}}$$

The symbols on teh right and left of the equal sign are so similar that it is tempting to think that the equation is obvious or says nothing whatsoever.

> **WARNING** *(Notation)* This compact notation is in fact a special instance of the Second Fundamental Theorem of Calculus.
>
> **Remark** Often we write $\int \frac{1}{x^2}\,dx$ as $\int \frac{dx}{x^2}$, merging the 1 with the dx. More generally, $\int \frac{f(x)}{g(x)}\,dx$ may be written as $\int \frac{f(x)\,dx}{g(x)}$.

Some Terms and Notation

The related processes of computing $\int_a^b f(x)\,dx$ and of finding an antiderivative $\int f(x)\,dx$ are both called **integrating** $f(x)$. Thus integration refers to two separate but related problems: computing a number $\int_a^b f(x)\,dx$ or finding a function $\int f(x)\,dx$. The First Fundamental Theorem of Calculus (FTC I) states that the Second Fundamental Theorem of Calculus (FTC II) may be of use in computing the definite integral $\int_a^b f(x)\,dx$.

In practice, both FTC I and FTC II are called "the Fundamental Theorem of Calculus". The context always makes it clear which one is meant.

Proofs of the Two Fundamental Theorems of Calculus

We now prove both parts of the Fundamental Theorem of Calculus — without referring to motion, area, or a few concrete examples. The proofs use only the mathematics of functions and limits. We prove FTC II first.

Proof of the Second Fundamental Theorem of Calculus

The Second Fundamental Theorem of Calculus asserts that the derivative of $G(x) = \int_a^x f(t)\,dt$ is $f(x)$. We gave a convincing argument using areas of regions. However, since definite integrals are defined in terms of approximating sums, not areas, we should include a proof that uses only properties of definite integrals.

Proof Second Fundamental Theorem of Calculus

We wish to show that $G'(x) = f(x)$. To do this we must make use of the definition of the derivative of a function.

We have

$$\begin{aligned}
G'(x) &= \lim_{\Delta x \to 0} \frac{G(x+\Delta x)-G(x)}{\Delta x} && \text{(definition of derivative)}\\
&= \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)\ dt - \int_a^x f(t)\ dt}{\Delta x} && \text{(definition of } G\text{)}\\
&= \lim_{\Delta x \to 0} \frac{\int_a^{x} f(t)\ dt + \int_x^{x+\Delta x} f(t)\ dt - \int_a^x f(t)\ dt}{\Delta x} && \text{(property 6 in Section 5.3)}\\
&= \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)\ dt}{\Delta x} && \text{(canceling)}\\
&= \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x} && \text{(MVT for Definite Integrals; } c \text{ between } x \text{ and } x+\Delta x\text{)}\\
&= \lim_{\Delta x \to 0} f(c) && \text{(canceling)}\\
&= f(x). && \text{(continuity of } f;\ c \to x \text{ as } \Delta x \to 0\text{)}
\end{aligned}$$

Hence

$$G'(x) = f(x),$$

which is the result that we set out to prove. •

Proof of the First Fundamental Theorem of Calculus

The Second Fundamental Theorem of Calculus asserts that if $F' = f$, then $\int_a^b f(x)\ dx = F(b) = F(a)$. We persuaded ourselves that this is true by thinking of f as "velocity" and F as "position", and by considering four special cases ($f(x) = c$, $f(x) = x$, $f(x) = x^2$, and $f(x) = 2^x$). We now prove the theorem, showing that it is an immediate consequence of the First Fundamental Theorem of Calculus and the fact that two antiderivatives of the same function differ by a constant.

Proof [

First Fundamental Theorem of Calculus] We are assuming that $F' = f$ and wish to show that $F(b) - F(a) = \int_a^b f(x)\ dx$. Define $G(x) = \int_a^x f(t)\ dt$. By FTC I, G is an antiderivative of f. Since F and G are both antiderivatives of f, they differ by a constant, say C. That is,

$$F(x) = G(x) + C.$$

Thus,

$$\begin{aligned} F(b) - F(a) &= (G(b) + C) - (G(a) + C) && \text{(antiderivatives differ by } C\text{)} \\ &= G(b) - G(a) && (C\text{'s cancel}) \\ &= \int_a^b f(t)\ dt - \int_a^a f(t)\ dt && \text{(definition of } G\text{)} \\ &= \int_a^b f(t)\ dt && \left(\int_a^a f(t)\ dt = 0\right) \end{aligned}$$

- SHERMAN: The property that $\int_a^a f(t)\ dt = 0$ does not have a number. Should we restructure that part of the presentation to put all properties in a common numbering system?

Summary

This section links the two basic ideas of calculus, the derivative (and antiderivative) and the definite integral.

The FTC I says that if you can find a formula for an antiderivative F of f, then you can evaluate $\int_a^b f(x)\ dx$:

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a).}$$

The FTC II says that if f is continuous then it has an antiderivative, namely $G(x) = \int_a^x f(t)\ dt$. Unfortunately, G might not be an elementary function. However, a reasonable graph of an antiderivative of f cacn be obtained from the slope field for $\frac{dy}{dx} = f(x)$.

The Second Fundamental Theorem of Calculus provided the basis for the First FTC. Therefore, from a theoretical point of view, "FTC II is more fundamental than FTC I". But, FTC I is more important from a practical point of view, for evaluating many important definite intgrals.

EXERCISES for 5.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1–2 evaluate the given expressions. **1.**[R]

(a) $x^3\big|_1^2$

(b) $x^2\big|_{-1}^2$

(c) $cos(x)|_0^\pi$

2.[R]

(a) $(x + \sec(x))|_0^{\pi/4}$

(b) $\frac{1}{x}\big|_2^3$

(c) $\sqrt{x-1}\big|_5^{10}$

3.[R] State the First Fundamental Theorem of Calculus in your own words, using as few mathematical symbols as you can.

4.[R] State the Second Fundamental Theorem of Calculus in your own words, using as few mathematical symbols as you can.

In Exercises 5–18 use the First Fundamental Theorem of Calculus to evaluate the given definite integrals.

5.[R] $\int_1^2 5x^3\ dx$

6.[R] $\int_{-1}^3 2x^4\ dx$

7.[R] $\int_1^4 (x + 5x^2)\ dx$

8.[R] $\int_1^2 (6x - 3x^2)\ dx$

9.[R] $\int_{\pi/6}^{\pi/3} 5\cos(x)\ dx$

10.[R] $\int_{\pi/4}^{3\pi/4} 3\sin(x)\ dx$

11.[R] $\int_0^{\pi/2} \sin(2x)\ dx$

12.[R] $\int_0^{\pi/6} \cos(3x)\ dx$

13.[R] $\int_4^p 5\sqrt{x}\ dx$

14.[R] $\int_1^9 \frac{1}{\sqrt{x}}\ dx$

15.[R] $\int_1^8 \sqrt[3]{x^2}\ dx$

16.[R] $\int_2^4 \frac{4}{x^3}\ dx$

17.[R] $\int_0^{`} 1 \frac{dx}{1+x^2}$

18.[R] $\int_{1/2}^2 \frac{dx}{\sqrt{1-x^2}}$

In Exercises 19–24 find the average value of the given function over the given interval.

19.[R] x^2; $[3, 5]$

20.[R] x^4; $[1, 2]$

21.[R] $\sin(x)$; $[0, \pi]$

22.[R] $\cos(x)$; $[0, \pi/2]$

23.[R] $(\sec(x))^2$; $[\pi/6, \pi/4]$

24.[R] $\sec(2x)\tan(2x)$; $[\pi/8, \pi/6]$

In Exercises 25–32 evaluate the given quantities.

25.[R] The area of the region under the curve $3x^2$ and above $[1, 4]$.

26.[R] The area of the region under the curve $1/x^2$ and above $[2, 3]$.

27.[R] The area of the region under the curve $6x^4$ and above $[-1, 1]$.

28.[R] The area of the region under the curve $\sqrt{x}$ and above $[25, 36]$.

29.[R] The distance an object travels from time $t = 1$ second to time $t = 2$ seconds, if its velocity at time t seconds is t^5 feet per second.

30.[R] The distance an object travels from time $t = 1$ second to time $t = 8$ seconds, if its velocity at time t seconds is $7\sqrt[3]{t}$ feet per second.

31.[R] The volume of a solid located between a plane at $x = 1$ and a plane located at $x = 5$ if the cross-sectional area of the intersection of the solid with the plane $x = u$ is $6u^3$ square centimeters.

32.[R] The volume of a solid located between a plane at $x = 1$ and a plane located at $x = 5$ if the cross-sectional area of the intersection of the solid with the plane $x = u$ is

$$1/u^3$$

square centimeters.

33.[R]

(a) Is $\int x^2\ dx$ a function or is it a number?

(b) Is $\int x^2\ dx\Big|_1^3$ a function or is it a number?

(c) Is $\int_1^3 x^2\ dx$ a function or is it a number?

34.[R]

(a) Which of the following expressions is defined as a limit of sums?

$$\int x^2\ dx\Bigg|_1^2 \qquad \int_1^2 x^2\ dx$$

(b) Why are the two numbers in (a) equal?

35.[R] True or false:

(a) Every elementary function has an elementary derivative.

(b) Every elementary function has an elementary antiderivative.

Explain.

36.[R] True of false:

(a) $\sin(x^2)$ has an elementary antiderivative.

(b) $\sin(x^2)$ has an antiderivative.

Explain.

37.[M] Find $\frac{dy}{dx}$ if

(a) $y = \int \sin(x^2)\ dx$

(b) $y = 3x + \int_{-2}^{3} \sin(x^2)\ dx$

(c) $y = \int_{-2}^{x} \sin(t^2)\ dt$

In Exercises 38–41 differentiate the given functions.

38.[M]

(a) $\int_1^x t^4\ dt$

(b) $\int_x^1 t^4\ dt$

39.[M]

(a) $\int_1^x \sqrt[3]{1+\sin(t)}\ dt$

(b) $\int_1^{x^2} \sqrt[3]{1+\sin(t)}\ dt$

HINT: Use the Chain Rule in (b).

40.[M] $\int_{-1}^{x} 3^{-t}\ dt$

41.[M] $\int_{2x}^{3x} t\tan(t)\ dt$ (Assume x is in the interval $(-\pi/6, \pi/6)$.) HINT: First rewrite the integral as $\int_{2x}^{0} t\tan(t)\ dt + \int_0^{3x} t\tan(t)\ dt$.

42.[M] Figure 5.4.7 shows the graph of a function $f(x)$ for x in $[1,3]$. Let $G(x) = \int_1^x f(t)\ dt$. Graph $y = G(x)$ for x in $[1,3]$ as well as you can. Explain your reasoning.

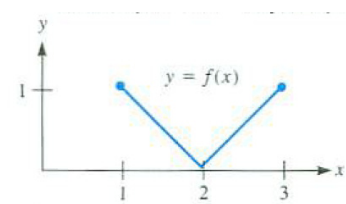
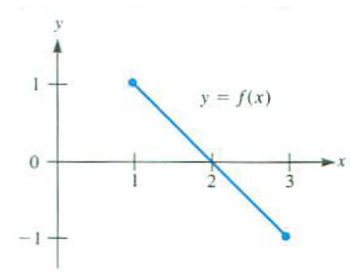


Figure 5.4.7:

43.[M] Figure 5.4.8 shows the graph of a function $f(x)$ for x in $[1,3]$. Let $G(x) = \int_1^x f(t)\ dt$. Graph $y = G(x)$ for x in $[1,3]$ as well as you can. Explain



your reasoning.

44.[C] Say that you want to find the area of a certain cross-sectional plane of a rock. One way to find it is by sawing the rock in two and measuring the area directly. But suppose you do not want to ruin the rock. However, you do have a very accurate measuring glass, as shown in Figure 5.4.9, which gives you excellent volume measurements. How could you use the glass to get a good estimate of the cross-sectional area?

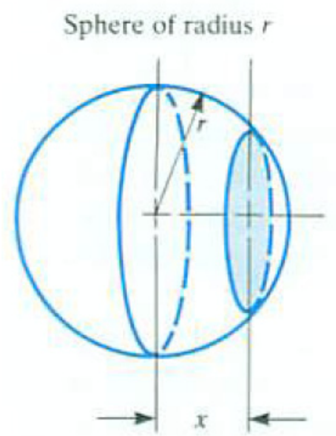


Figure 5.4.10:

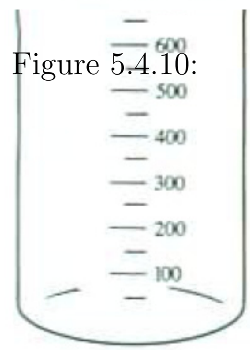


Figure 5.4.9:

45.[M] A plane at a distance x from the center of the sphere of radius r, $0 \le x \le 4$, meets the sphere in a circle. (See Figure 5.4.10.)

(a) Show that the radius of the circular intersection is $\sqrt{r^2 - x^2}$.

(b) Show that the area of the circle is $\pi r^2 - \pi x^2$.

(c) Using the FTC, find the volume of the sphere.

46.[M] Let $v(t)$ be the velocity at time t of an object moving on a straight line. The velocity may be positive or negative.

(a) What is the physical meaning of $\int_a^b v(t)\, dt$? Explain.

(b) What is the physical meaning of the slope of the graph of $y = v(t)$? Explain.

47.[M] Give an example of a function f such that $f(4) = 0$ and $f'(x) = \sqrt[3]{1 + x^2}$. Hint: Think of FTC II.

48.[C] How often should a machine be overhauled? This depends on the rate $f(t)$ at which it depreciates and the cost A of overhaul. Denote the time between overhauls by T.

(a) Explain why you would like to minimize $g(T) = \frac{1}{T}(A + \int_0^T f(t)\, dt)$.

(b) Find $\frac{dg}{dT}$.

(c) Show that when $\frac{dg}{dT} = 0$, $f(T) = g(T)$.

(d) Is this reasonable? Explain.

49.[C] Let $f(x)$ be a continuous function with positive values. Define $H(x) = \int_x^b f(t)\,dt$ for all $a \le x \le b$. Let Δx be positive.

(a) Interpreting the definite integral as an area of a region, draw the regions whose areas are $H(x)$ and $H(x + \Delta x)$.

(b) Is $H(x + \Delta x) - H(x)$ positive or negative?

(c) Draw the region whose area is related to $H(x + \Delta x) - H(x)$.

(d) When Δx is small, estimate $H(x + \Delta x) - H(x)$ in terms of the integrand f.

(e) Use (d) to evaluate the derivative $H'(x)$:

$$\frac{dH}{dx} = \lim_{\Delta x \to 0} \frac{H(x + \Delta x) - H(x)}{\Delta x}.$$

50.[C] Let R be a function with continuous second derivative R''. Assume $R(1) = 2$, $R'(1) = 6$, $R(3) = 5$, and $R'(3) = 8$. Evaluate $\int_1^3 R''(x)\,dx$. NOTE: Not all of the information provided is needed.

51.[C] Two conscientious calculus students are having an argument:

Check the wording of the work to be done by students.

She: $\int_a^b f(x)\,dx$ is a number.

He: But if I treat b as a variable, then it is a function.

She: Well, how can it be both a number and a function?

He: It depends on what "it" means.

She: You can't get out of this so easily.

Which student is correct? That is, either give two interpretations of "it" or explain why "it" has only one meaning.

52.[R]

(a) Draw the slope field for $\frac{dy}{dx} = \frac{e^{-x}}{x}$ for $x > 0$.

(b) Use (a) to sketch the graph of an antiderivative of $\frac{e^{-x}}{x}$.

(c) On the slope field, sketch the graph of $f(x) = \int_1^x \frac{e^{-t}}{t}\,dt$. (For what one value of a is $f(a)$ easy to compute?)

53.[C] The function $\frac{e^x}{x}$ does not have an elementary antiderivative. Show that $\frac{x}{e^x}$ does have an elementary antiderivative. HINT: Write $\frac{x}{e^x}$ as xe^{-x} and then experiment for a few minutes.

54.[C] Let f be a continuous function defined on $(-\infty, \infty)$. Assume that for every positive number t, the average of f over $[0, t]$ equals the average of its values at 0 and t, that is,

$$\frac{\int_0^t f(x)\ dx}{t-0} = \frac{f(0)+f(t)}{2} \tag{5.4}$$

(a) Show that f is differentiable for $t > 0$.

(b) Show that $\frac{df}{dt^2}$ exists for $t > 0$ and is 0.

(c) What functions satisfy (5.4)?

55.[C] Find all continuous functions f such that their average over $[0, t]$ always equals $f(t)$.

56.[C] Give a geometric explanation of the following properties of definite integrals:

(a) if f is an even function, then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$.

(b) if f is an odd function, then $\int_{-a}^{a} f(x)\ dx = 0$.

(c) if f is a periodic function with period p, then, for any integers m and n, $\int_{mp}^{np} f(x)\ dx = (n-m)\int_0^p f(x)\ dx$.

5.5 Estimating the Definite Integral

It is easy to evaluate $\int_0^1 x^2\sqrt{1-x^3}\,dx$ by the Fundamental Theorem of Calculus, for the integrand has an elementary antiderivative, $\frac{2}{9}(1-x^3)^{3/2}$. However, an antiderivative of $\sqrt{1-x^3}$ is not elementary so $\int_0^1 \sqrt{1-x^3}\,dx$ cannot be evaluated so easily. In this case we are compelled to estimate the value of the definite integral.

Check that $\frac{d}{dx}\frac{2}{9}(1-x^3)^{3/2} = x^2\sqrt{1-x^3}$.

Approximation by Rectangles

The definite integral $\int_a^b f(x)\,dx$ is, by definition, a limit of sums of the form

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}). \tag{5.1}$$

Any such sum is an estimate of $\int_a^b f(x)\,dx$.

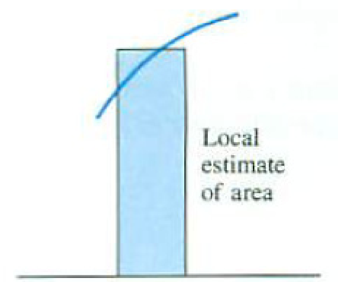


Figure 5.5.1:

In terms of area, Figure 5.5.1 shows the local approximation by a rectangle of the area under part of the curve. By summing the areas of individual rectangles, we estimate the area under the curve. Example 1 illustrates this approach, which is called the rectangular method.

In the **rectangular method**, divide the interval into sections of equal length. Then choose a sampling number c_i in the i^{th} section and form a Riemann sum. By the very definition of the definite integral, each Riemann sum approximates the value of the definite integral.

Choosing the sections to have the same length simplifies the arithmetic.

We will investigate various ways to estimate $\int_0^1 \frac{1}{1+x^2}\,dx$. We choose this particular example because we can evaluate it exactly by the FTC. This will enable us to see how accurate each estimating technique is.

$\int_0^1 \frac{dx}{1+x^2} = \arctan(x)|_0^1 = \pi/4 - 0 = \pi/4 \approx 0.785398$.

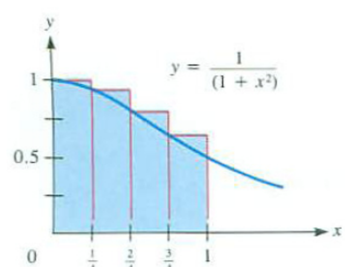


Figure 5.5.2:

EXAMPLE 1 Use four rectangles with equal lengths to estimate $\int_0^1 \frac{dx}{1+x^2}$. Use the left endpoint of each section as the sampling number to determine the height of each rectangle.

SOLUTION Since the length of $[0,1]$ is 1, each of the four sections of equal length has length $\frac{1}{4}$. See Figure 5.5.2. The sum of their areas is

$$\frac{1}{1+0^2}\cdot\frac{1}{4} + \frac{1}{1+\left(\frac{1}{4}\right)^2}\cdot\frac{1}{4} + \frac{1}{1+\left(\frac{1}{2}\right)^2}\cdot\frac{1}{4} + \frac{1}{1+\left(\frac{3}{4}\right)^2}\cdot\frac{1}{4},$$

which equals

$$\frac{1}{4}\left(1 + \frac{16}{17} + \frac{16}{20} + \frac{16}{25}\right).$$

Note that this method was used throughout Section 5.1.

This is approximately

$$\begin{aligned}\frac{1}{4}(1.0000000 + 0.9411765 + 0.8000000 + 0.6400000) &= \frac{1}{4}(3.3811765)\\ &\approx 0.845294.\end{aligned}$$

This estimate is about 0.06 larger than the exact value.

◇

Approximation by Trapezoids

In the **trapezoidal method**, trapezoids are used instead of rectangles to estimate the area under the curve. Let n be a positive integer. Divide the interval $[a, b]$ into n sections of equal length $h = (b - a)/n$ with

$$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h,\ \ldots,\ x_n = a + nh = b.$$

The sum

$$\frac{f(x_0) + f(x_1)}{2} \cdot h + \frac{f(x_1) + f(x_2)}{2} \cdot h + \cdots + \frac{f(x_{n-1}) + f(x_n)}{2} \cdot h$$

is the **trapezoidal estimate** of $\int_a^b f(x)\ dx$. It is usually written as

$$\frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right). \qquad (5.2)$$

Note that $f(x_0)$ and $f(x_n)$ have coefficient 1 while all other $f(x_i)$'s have coefficient 2. This is due to the double counting of the edges common to two trapezoids.

If $f(x)$ is a first-degree polynomial, the graph of $y = f(x)$ is a straight line and $\int_a^b f(x)\ dx$ corresponds to the area of a single trapezoid. Likewise, if $f(x)$ is a constant, then $\int_a^b f(x)\ dx$ is the area of a rectangle. In either case, the trapezoidal method gives the exact value of the definite integral. (This is true for any value of n.)

Figures 5.5.4 and 5.5.5 illustrate the trapezoidal estimate for the case $n = 4$. Notice that in Figure 5.5.4 the function is concave down and the trapezoidal estimate underestimates $\int_a^b f(x)\ dx$. On the other hand, when the curve is concave up the trapezoids overestimate, as shown in Figure 5.5.5. In both cases the trapezoids appear to give a better approximation to the value of $\int_a^b f(x)\ dx$ than the same number of rectangles. For this reason we expect the trapezoidal method to provide better estimates of a definite integral than we obtain by rectangles.

EXAMPLE 2 Use the trapezoidal method with $n = 4$ to estimate $\int_0^1 \frac{dx}{1+x^2}$.
SOLUTION In this case $a = 0$, $b = 1$, and $n = 4$, so $h = (1 - 0)/4 = \frac{1}{4}$. The four trapezoids are shown in Figure 5.5.6. The trapezoidal estimate is

$$\frac{h}{2}\left(f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + 2f(1)\right).$$

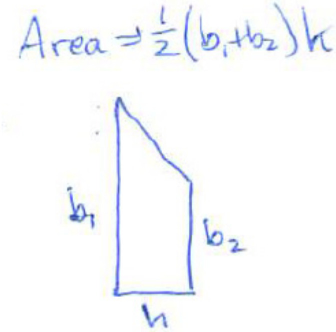


Figure 5.5.3:

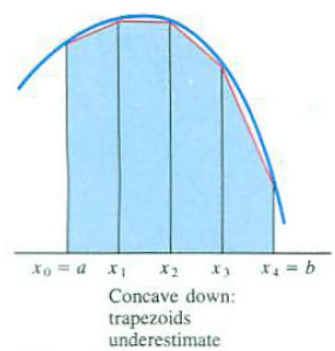


Figure 5.5.4:

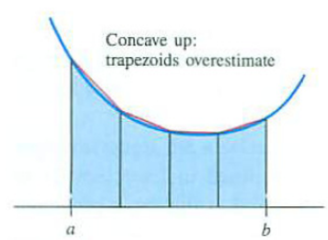


Figure 5.5.5:

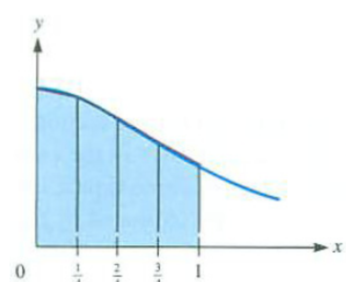


Figure 5.5.6:

x_i	$f(x_i)$	coefficient	summand	decimal form
0	$\frac{1}{1+0^2}$	1	$1 \cdot \frac{1}{1+0}$	1.0000000
$\frac{1}{4}$	$\frac{1}{1+\left(\frac{1}{4}\right)^2}$	2	$2 \cdot \frac{1}{1+\frac{1}{16}}$	1.8823529
$\frac{2}{4}$	$\frac{1}{1+\left(\frac{2}{4}\right)^2}$	2	$2 \cdot \frac{1}{1+\frac{4}{16}}$	1.6000000
$\frac{3}{4}$	$\frac{1}{1+\left(\frac{3}{4}\right)^2}$	2	$2 \cdot \frac{1}{1+\frac{9}{16}}$	1.2800000
$\frac{4}{4}$	$\frac{1}{1+\left(\frac{4}{4}\right)^2}$	1	$1 \cdot \frac{1}{1+\frac{16}{16}}$	0.5000000

Table 5.5.1:

Now $h/2 = \frac{1}{4}/2 = 1/8$. To compute the sum of the five terms involving values of $f(x) = \frac{1}{1+x^2}$, make a list as shown in Table 5.5.1. The trapezoidal sum is therefore approximately

$$\frac{1}{8}\left(\right) = \frac{1}{8}(6.2623529) \approx 0.782794.$$

Thus

$$\int_0^1 \frac{dx}{1+x^2} \approx 0.782794.$$

◇

We expect the estimate 0.782794 of Example 2 to be a better estimate than the estimate 0.845294 found in Example 1 using rectangles. Indeed it is, since $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} \approx 0.785398$.

Simpson's Method: Approximation by Parabolas

Thomas Simpson, 1710–1761, http://en.wikipedia.org/wiki/Thomas_Simpson

In the trapezoidal method a curve is approximated by chords. In Simpson's method a curve is approximated by parabolas. Given *three* points on a curve, there is a unique parabola that passes through them, as shown in Figure 5.5.7. The area under the parabola is then used to approximate the area under the curve.

The computations leading to the formula for the area under the parabola are more involved than those for the area of a trapezoid. (They are outlined in Exercises 21 to 26.) However, the final formula is fairly simple. Let the three points be $(x_1, f(x_1))$, $(x_2, f(x_2))$, $(x_3, f(x_3))$, with $x_1 < x_2 < x_3$, $x_2 - x_1 = h$, and $x_3 - x_2 = h$, as shown in Figure 5.5.8(a). The shaded area under the parabola turns out to be

$$\frac{h}{3}\left(f(x_1) + 4f(x_2) + f(x_3)\right). \tag{5.3}$$

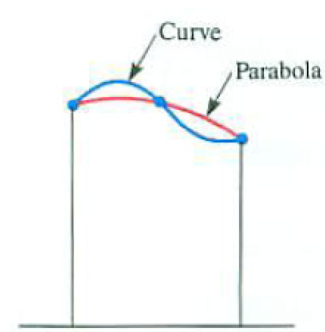
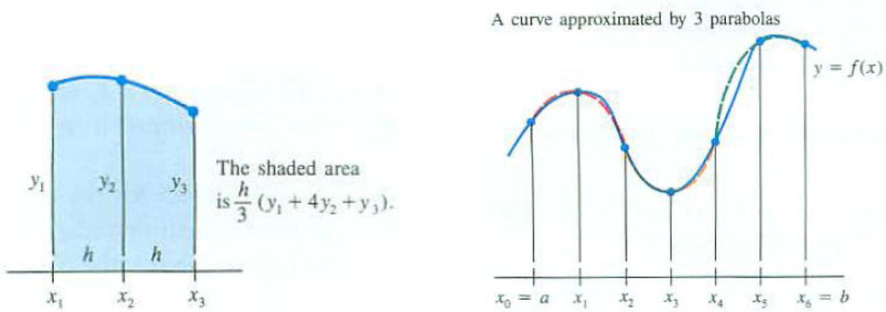


Figure 5.5.7: Curve: $y = f(x)$, Parabola: $y = Ax^2 + Bx + C$



(a) (b)

Figure 5.5.8: In (a), change y_i to $f(x_i)$. Objections?

To use (5.3) to estimate $\int_a^b f(x)\ dx$, break the interval $[a, b]$ into an even number of n of sections of equal length. Figure 5.5.8(a) illustrates this step for six sections. Formula (5.3) is applied to each of the $n/2$ pairs of adjacent sections. In the case shown in Figure 5.5.8(b), $n = 6$ and there are $\frac{6}{2} = 3$ parabolas used to approximate the curve. Then $h = (b - a)/6$ and the area under the curve is approximated by the sum of these three terms:

SHERMAN: I changed y_i to $f(x_i)$, this is how we will use it. Comments?

$$\left(\frac{(b-a)/6}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)\right) + \left(\frac{(b-a)/6}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right)\right) + \left(\frac{(b-a)/6}{3}\left(f(x_4) + 4f(x_5) + f(x_6)\right)\right),$$

which a little algebra reduces to

$$\frac{b-a}{18}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right).$$

This special case $n = 6$ shows why the coefficients, in general, are 1, 4, 2, 4, ..., 2, 4, 1, with a 1 on both ends, and 4 and 2 alternating in the middle.

In Simpson's method the interval $[a, b]$ is divided into an *even* number of sections of equal length $h = (b - a)/n$ with

$$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h,\ \cdots,\ x_n = a + nh = b.$$

Then **Simpson's estimate** of $\int_a^b f(x)\ dx$ is

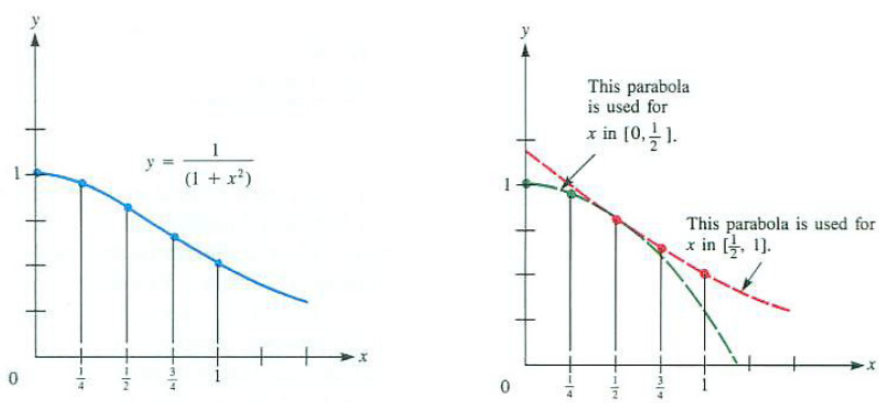
Note that $f(x_0)$ and $f(x_n)$ have coefficient 1, while the coefficients of the other terms alternate 4, 2, 4, 2, ..., 4, 2, 4.

$$\frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right). \tag{5.4}$$

EXAMPLE 3 Use Simpson's method with $n = 4$ to estimate $\int_0^1 \frac{dx}{1+x^2}$.
SOLUTION Again $n = \frac{1}{4}$. Simpson's estimate (5.4) takes the form

$$\frac{\frac{1}{4}}{3}\left(f(0) + 4f(\frac{1}{4}) + 2f(\frac{2}{4}) + 4f(\frac{3}{4}) + f(1)\right).$$

The two parabolas are shown in Figure 5.5.9(b). Note that the two parabolas



(a) (b)

Figure 5.5.9: In (b), delete the unneeded parts of the parabolas.

are virtually indistinguishable from the curve itself; this is going to be an even better approximation to the definite integral.

x_i	$f(x_i)$	coefficient	summand	decimal form
0	$\frac{1}{1+0^2}$	1	$1 \cdot \frac{1}{1+0}$	1.0000000
$\frac{1}{4}$	$\frac{1}{1+\left(\frac{1}{4}\right)^2}$	4	$4 \cdot \frac{1}{1+\frac{1}{16}}$	3.7647059
$\frac{2}{4}$	$\frac{1}{1+\left(\frac{2}{4}\right)^2}$	2	$2 \cdot \frac{1}{1+\frac{4}{16}}$	1.6000000
$\frac{3}{4}$	$\frac{1}{1+\left(\frac{3}{4}\right)^2}$	4	$4 \cdot \frac{1}{1+\frac{9}{16}}$	2.5600000
$\frac{4}{4}$	$\frac{1}{1+\left(\frac{4}{4}\right)^2}$	1	$1 \cdot \frac{1}{1+\frac{16}{16}}$	0.5000000

Table 5.5.2:

The computations are shown in Table 5.5.2. The Simpson estimate of $\int_0^1 \frac{dx}{1+x^2}$ is therefore

$$\begin{aligned}\frac{1}{12}(1 + 3.7647059 + 1.6000000 + 2.5600000 + 0.5000000) &= \frac{1}{12}(9.4247059) \\ &\approx 0.785392.\end{aligned}$$

Thus

$$\int_0^1 \frac{dx}{1+x^2} \approx 0.785392.$$

This estimate is accurate to five decimal places. ◇

Comparison of the Three Methods

We know the exact value of $\int_0^1 \frac{dx}{1+x^2}$ is $\frac{\pi}{4}$; to eight decimal places, this equals 0.78539816. Table 5.5.3 compares the estimates made in the three examples to this value.

Error = Exact − Estimate
SHERMAN: Should the error be reported as the absolute value?

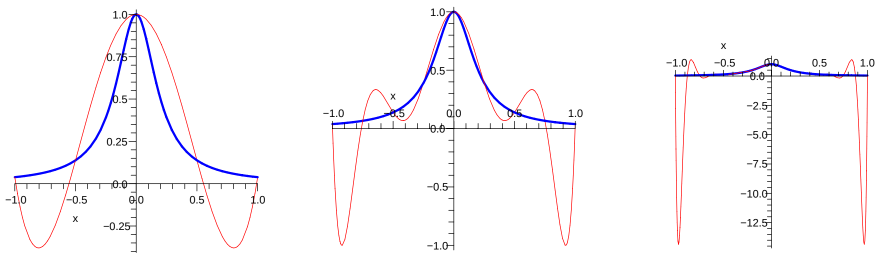
Method	Estimate	Error
Rectangles	0.845294	-0.059896
Simpson's (Parabolas)	0.785392	0.000006

Table 5.5.3:

Though each method takes about the same amount of work, the table shows that Simpson's method gives the best estimate. The trapezoidal method is next best. The rectangular method has the largest error. These results should not come as a surprise. Parabolas should fit the curve better than chords do, and chords should fit better than horizontal line segments. Note that the trapezoidal and Simpson's methods in Examples 2 and 3 used the same sampling numbers to evaluate the integrand; their only difference is in the "weights" (coefficients) given the outputs of the integrand.

Carle Runge, 1856–1927, `http://en.wikipedia.org/wiki/Carle_David_Tolm%C3%A9_Runge`

In the trapezoidal method you pass a line through two points to approximate the curve. That uses a first-degree polynomial, $Ax + B$. In Simpson's method you pass a parabola through three points, using a second-degree polynomial, $Ax^2 + Bx + C$. You would expect that as you pass higher-degree polynomials through more points on the curve you would get even better approximations. This is not always the case. For the function $f(x) = 1/(1 + 25x^2)$, defined on $[-1, 1]$, known as Runge's Counterexample, the higher-degree polynomials passing through equally-spaced points do not resemble the function. Figure 5.5.10 shows the interpolating polynomials of degree 4 (a), 8 (b), and 16 (c). Notice how the approximations improve away from the endpoints and exhibit increasingly large oscillations near the endpoints. These oscillations result in poor estimates to $\int_{-1}^1 \frac{dx}{1+25x^2}$. A Google search for "Runge's Counterexample" yields more information on this function.



(a) (b) (c)

Figure 5.5.10: NOTE: Move this figure inside boxed text.

The Error

The size of the error is closely connected to the derivatives of the integrand. For a positive number k, let M_k be the largest value of $\left|f^{(k)}(x)\right|$ for x in $[a,b]$. Table 5.5.4 lists the general upper bounds for the error when $\int_a^b f(x)\,dx$ is estimated by sections of length $h=(b-a)/n$; these results are usually developed in a course on numerical analysis.

Recall that $f^{(k)}(x)$ is the k^{th} derivative of f. For instance, $f^{(2)}(x)$ is the second derivative.

DOUG: If error is defined with absolute value, then this heading can be shortened.

Method	Bound on Absolute Value of Error
Rectangles	$M_1(b-a)h$
Trapezoids	$\frac{1}{12}M_2(b-a)h^2$
Simpson's (Parabolas)	$\frac{1}{180}M_4(b-a)h^4$

Table 5.5.4:

The coefficients in the error bounds tell us something. For instance, if $M_4=0$, then there is no error in Simpson's method. That is, if $f^{(4)}(x)=0$ for all x in $[a,b]$, then Simpson's method produces an exact answer. For in this case the error is $M_4(b-a)h^4/180=0$. As a consequence, for polynomials of at most degree 3, Simpson's approximation is exact.

But more important is the power of h that appears in the error bound. For instance, if you reduce the width of h by a factor of 10 (using 10 times as many sections) you expect the error of the rectangular method to shrink by a factor of 10, the error in the trapezoidal method to shrink by a factor of $10^2=100$, and the error in Simpson's method by a factor of $10^4=10,000$. These observations are recorded in Table 5.5.5.

SHERMAN: Midpoints are not really any more expensive to compute. Once you have $c_1=a+h/2$, then all others can be computed from $c_i=c_1+(i-1)h$ — by adding h— just as was done for the x_i.

Because the error in the rectangular method approaches 0 so slowly as $h\to 0$, we will not use it for estimating definite integrals. (Rectangular estimates using the midpoint rather that the left or right endpoint are about as accurate

Method	Reduction Factor of h	Expected Reduction Factor of Error
Rectangles	10	10
Trapezoids	10	100
Simpson's (Parabolas)	10	10,000

Table 5.5.5:

as the trapezoidal method. However, this method requires the extra arithmetic of computing the midpoint of each section.)

Reference: Handheld Calculator Evaluates Integrals, William Kahan, Hewlett-Packard Journal, vol. 31, no. 8, Aug. 1980, pp. 23–32, `http://www.cs.berkeley.edu/~wkahan/Math128/INTGTkey.pdf`.

Technology and Definite Intgrals

The trapezoidal method and Simpson's method are just two examples of what is called **numerical integration**. Such techniques are studied in detail in courses on numerical analysis. While the Fundamental Theorem of Calculus is useful for evaluating definite integrals, it applies only when an antiderivative is readily available. Numerical integration is an important tool in computing definite integrals, particularly when the FTC cannot be applied. You should always remember that numerical integration can be used to find out something about the value of a definite integral.

The design of an efficient and accurate general-purpose numerical integration algorithm is harder than it might seem. Effective algorithms typically divide the interval into unequal-length sections. The sections can be longer where the function is tame. Shorter sections are used where the function is wild: changes very rapidly or has a singularity. Large, even unbounded, intervals can lead to another set of difficulties. Some examples of challenging definite integrals include:

$$\int_0^1 \ln(x)\ dx \qquad \int_0^2 \sqrt{x(4-x)}\ dx \qquad \int_0^1 \frac{\sqrt{x}}{x-1} - \frac{1}{\ln(x)}\ dx$$
$$\int_0^1 \cos(\ln(x))\ dx \qquad \int_{-1}^1 \frac{dx}{x^2+10^{-10}} \qquad \int_0^{600\pi} \frac{(\sin(x))^2}{\sqrt{x}+\sqrt{x+\pi}}\ dx$$

See Exercises 37–44.

The HP-34C was, in 1980, the first handheld calculator to be able to perform numerical integration. Now, this is a common feature on most scientific calculators. The algorithms used vary greatly, and the details are often not corporate secrets. The basic ideas used are similar to those presented in this section and in Exercise 29.

Summary

The trapezoidal method and Simpson's method are just two examples of what is called **numerical integration**. Such techniques are studied in detail in courses on numerical analysis. While the Fundamental Theorem of Calculus is useful for evaluating definite integrals, it applies only when an antiderivative is readily available. Numerical integration is an important tool in computing definite integrals, particularly when the FTC cannot be applied. You should always remember that numerical integration can be used to find out something about the value of a definite integral.

SHERMAN: This is just the first paragraph of the Technology box. Isn't this the main point of this section? Or, would you suggest summarizing the formulas for rectangles, trapezoids, and Simpson's?

EXERCISES for 5.5 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1–8 estimate the given definite integrals by the trapezoidal method, using the given number of trapezoids.

1.[R] $\int_0^2 \frac{dx}{1+x^2}$, $n = 2$

2.[R] $\int_0^2 \frac{dx}{1+x^2}$, $n = 4$

3.[R] $\int_0^2 \sin(\sqrt{x})\ dx$, $n = 2$

4.[R] $\int_0^2 \sin(\sqrt{x})\ dx$, $n = 3$

5.[R] $\int_1^3 \frac{2^x}{x}\ dx$, $n = 3$

6.[R] $\int_1^3 \frac{2^x}{x}\ dx$, $n = 6$

7.[R] $\int_1^3 \cos(x^2)\ dx$, $n = 4$

8.[R] $\int_1^3 \cos(x^2)\ dx$, $n = 2$

In Exercises 9–12 use Simpson's method to estimate each definite integral with the given number of parabolas.

9.[R] $\int_0^1 \frac{dx}{1+x^3}$, $n = 2$

10.[R] $\int_0^1 \frac{dx}{1+x^3}$, $n = 4$

11.[R] $\int_0^1 \frac{dx}{1+x^4}$, $n = 4$

12.[R] $\int_0^1 \frac{dx}{1+x^4}$, $n = 2$

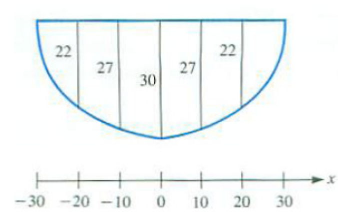


Figure 5.5.11:

13.[R] The cross-section of a ship's hull is shown in Figure 5.5.11. Estimate the area of this cross-section by

(a) the trapezoidal method

(b) Simpson's method

Dimensions are in feet. Give your answer to four decimal places.

14.[R] A ship is 120 feet long. The area of the cross-section of its hull is given at intervals in the table below:

x	0	20	40	60	80	100	120	feet
area	0	200	400	450	420	300	150	square feet

Estimate the volume of the hull in cubic feet by

(a) the trapezoidal method

(b) Simpson's method

Give you answer to four decimal places.

15.[R] A map of Lake Tahoe is shown in Figure 5.5.12. Use Simpson's method and data from the map to estimate the surface area of th lake. (Each little square is a mile on each side.)

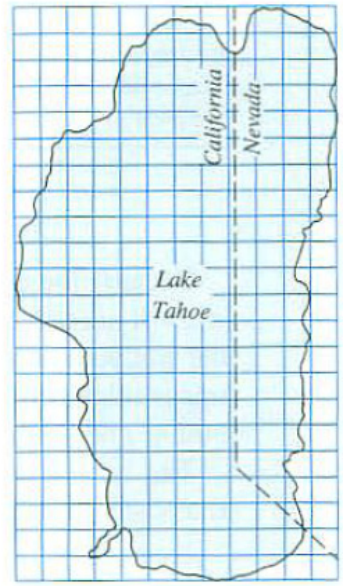


Figure 5.5.12:

16.[R] The following table gives the diameter of a tree trunk is given in feet at various heights (also measured in feet).

height	0	2	4	6	8	10	12	14	16
diameter	4.0	3.7	3.5	3.1	3.1	2.9	2.7	2.4	2.1

Use Simpson's method to estimate the volume of the tree trunk between the heights of 0 and 16 feet. Assume that each cross-section is circular.

The **right-point** estimate of $\int_a^b f(x)\,dx$ is obtained by selecting a positive integer n and dividing $[a, b]$ into n sections of equal width $h = \frac{b-a}{n}$. The points of subdivision are $x_0 < x_1 < \cdots < x_n$, with $x_0 = a$ and $x_n = b$. The right-point estimate is the approximating sum

$$h(f(x_1) + f(x_2) + \cdots + f(x_n)).$$

The **left-point** estimate is defined similarly; it is given by

$$h(f(x_0) + f(x_1) + \cdots + f(x_{n-1}))$$

and is the method used in Example 1.

17.[M] Show that for a given n the average of the left-point estimate and the right-point estimate equals the trapezoidal estimate.

18.[M] Show that if $f(a) = f(b)$, the left-point, right-point, and trapezoidal estimates for a given value of h are the same.

The next two Exercises present cases in which the bound of maximum error is actually assumed.

19.[M] Show that if the trapezoidal method with $n = 1$ is used to estimate $\int_0^1 x^2\,dx$, the error equals $\frac{(b-a)M_2h^2}{12}$, where $a = 0$, $b = 1$, $h = 1$, and M_2 is the maximum value of $\left|\frac{d^2(x^2)}{dx^2}\right|$ for x in $[0, 1]$.

20.[M] Show that if the Simpson's method with $n = 2$ is used to estimate $\int_0^1 x^4\,dx$, the error equals $\frac{(b-a)M_4h^4}{180}$, where $a = 0$, $b = 1$, $h = 1/2$, and M_4 is the maximum value of $\left|\frac{d^4(x^4)}{dx^4}\right|$ for x in $[0, 1]$.

Exercises 21–26 describe the geometric motivation of Simpson's method.

21.[M] Let $f(x) = Ax^2 + Bx + C$. Show that

$$\int_{-h}^{h} f(x)\,dx = \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right).$$

HINT: Just compute both sides.

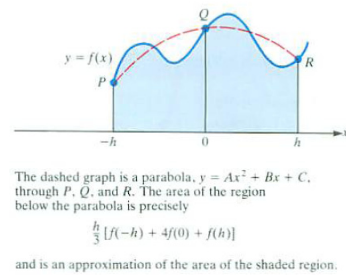


Figure 5.5.13:

22.[M] Let f be a function. Show that there is a parabola $y = Ax^2 + Bx + C$ that passes through the three points $(-h, f(= h))$, $(0, f(0))$, and $(h, f(h))$. (See Figure 5.5.13.)

23.[C] The equation in Exercise 21 is called the **prismoidal formula**. Use it to compute the volume of

(a) a sphere of radius a

(b) a right circular cone of radius a and height h.

The prisomoidal formula was known to the Greeks. Reference: `http://www.mathpages.com/home/kmath189/kmath189.htm`

24.[C] Let $f(x) = Ax^2 + Bx + C$. Show that

$$\int_{c-h}^{c+h} f(x)\,dx = \frac{h}{3}\left(f(c-h) + 4f(c) + f(c+h)\right).$$

HINT: Use the substitution $x = c + t$ to reduce this to Exercise 21.

25.[C] First, $[a, b[]$ is divided into n sections (n even), which are grouped into $n/2$ pairs of adjacent sections. Over each pair the function is approximated by the parabola that passes through the three points of the graph with x coordinates equal to those which determine the two sections of the pair. The integral of this quadratic function is used as an estimate of teh integral of f over each pair of adjacent sections. Show that when these $n/2$ separate estimates are added, Simpson's formula results.

See Figure **??**.

26.[C] Since Simpson's method was designed to be exact when $f(x) = Ax^2+Bx+C$, one would expect the error associated with it to involve $f^{(3)}(x)$. By a quirk of good fortune, Simpson's method happens to be exact even with $f(x)$ is a cubic, $Ax^3 + Bx^2 + Cx + D$. This suggests that the error involves $f^{(4)}(x)$, not $f^{(3)}(x)$.

(a) Show that if $f(x) = x^3$, then

$$\int_{-h}^{h} f(x)\ dx = \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right).$$

(b) Show that Simpson's estimate is exact for cubic polynomials.

Exercises 27 and 28 are related to the fact that Simpson's method is exact for cubic polynomials. **27.**[M] Let $g(x) = x^3 - x^2$ and $f(x) = x - x^2$.

(a) Find all three points of intersection for the graphs of $y = g(x)$ and $y = f(x)$. Call the three x values, x_1, x_2, and x_3, ordered from smallest to largest.

(b) Compute $\int_{x_1}^{x_3} g(x) - f(x)\ dx$.

28.[C] Let $g(x)$ be a cubic polynomial. Let $f(x)$ be the quadratic polynomial that interpolates g at three equally-spaced points, say x_1, x_2, and x_3, where $x_2 - x_1 = x_3 - x_2 = h$. Show that $\int_{x_1}^{x_3} g(x) - f(x)\ dx = 0$.

29.[C] There are many other methods for estimating definite integrals. Some old methods, which had been of onlyh theoretical interest because of their messy arithmetic, have, with the advent of computers, assumed pracitcal importance. This exercise illustrates the simplest of the so-called **Gaussian quadrature** formulas. For simplicity, consider only integrals over $[-1, 1]$.

(a) Show that

$$\int_{-1}^{1} f(x)\ dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

for $f(x) = 1$, x, x^2, and x^3.

(b) Let a and b be two numbers, $-1 \leq a < b \leq 1$, such that

$$\int_{-1}^{1} f(x)\ dx = f(a) + f(b)$$

for $f(x) = 1$, x, x^2, and x^3. Show that $a = \frac{-1}{\sqrt{3}}$ and $b = \frac{1}{\sqrt{3}}$

(c) Show that the Gaussian approximation

$$\int_{-1}^{1} f(x)\ dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

has no error when f is a polynomial of degree at most 3.

(d) Use the formula in (a) to estimate $\int_{-1}^{1} \frac{dx}{1+x^3}$

30.[C] Let f be a function such that $\left|f^{(2)}(x)\right| \leq 10$ and $\left|f^{(4)}(x)\right| \leq 50$ for all x in $[1, 5]$. If $\int_1^5 f(x)\ dx$ is to be estimated with an error of at most 0.01, how small must h be in

(a) the trapezoidal approximation?

(b) Simpson's approximation?

31.[C] Let T be the trapezoidal estimate of $\int_a^b f(x)\ dx$, using $x_0 = a$, x_1, …, $x_n = b$. Let M be the "midpoint estimate", $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$, where $c_i = \frac{1x_{i-1}+x_i}{2}$. Let S be Simpson's estimate using the $2n+1$ points x_0, c_1, x_1, c_1, x_2, …, c_n, x_n. Show that

$$S = \frac{2}{3}M + \frac{1}{3}T.$$

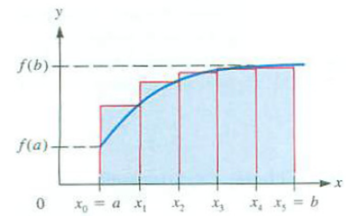


Figure 5.5.14:

32.[C] In his *Principia*, published in 1607, Newton examined the error in approximating an area by rectangles. He considered an increasing, differentiable function f defined on the interval $[a,b]$ and drew a figure similar to Figure 5.5.14. All rectangles have the same width h. Let R equal the sum of the areas of the rectangles using right endpoints and let L equal the sum of the areas of the rectangles using left endpoints. Let A be the area under the curve $y = f(x)$ and above $[a,b]$.

(a) Whay is $R - L = \frac{f(b)-f(a)}{h}$?

(b) Show that any approximating sum for A, formed with rectangles of equal width h and any sampling points, differs from A by at most $\frac{f(b)-f(a)}{h}$.

(c) Let M_1 be the maximum value of $|f'(x)|$ for x in $[a,b]$. Show that any approximating sum FOR A formed with equal widths h differs from A by at most $M_1(b-a)h$. HINT: HINT: Use the MVT for definite integrals.

(d) Newton also considered the case where the rectangle do not necessarily have the same widths. Let h be the largest of their widths. What can be said about the error?

Since the error involves the first power of h, approximation by rectangles is less efficient than approximation by trapezoids, where the error involves h^2.

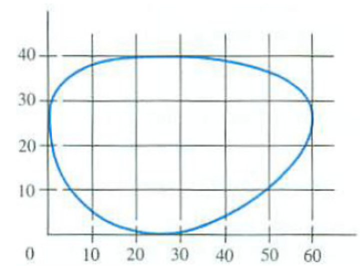


Figure 5.5.15:

33.[M] Figure 5.5.15 shows cross-sections of a pond in two directions. Use Simpson's method to estimate the area of the pond using

(a) the vertical cross-sections

(b) the horizontal cross-sections

34.[M] The table lists the values of a function f at the given points.

(a) Plot the seven points.

(b) Sketch six trapezoids that can be used to estimate $\int_1^7 f(x)\,dx$.

(c) Find the trapezoidal estimate of $\int_1^7 f(x)\,dx$.

(d) Sketch, by eye, the three parabolas that can be used in Simpson's method to estimate $\int_1^7 f(x)\,dx$.

(e) Find the Simpson's estimate to $\int_1^7 f(x)\,dx$.

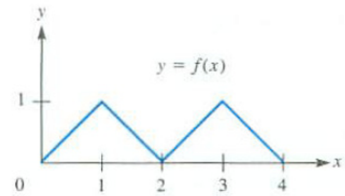


Figure 5.5.16:

35.[M] Figure 5.5.16 is the graph of a function $f(x)$. Let $g(x) = \int_0^x f(x)\,dx$.

(a) Graph $y = g(x)$ as well as you can.

(b) Graph $y = f'(x)$ as well as you can.

(c) Explain the reasoning used to find the graphs requested in (a) and (b).

36.[M] The definite integral $\int_0^1 \sqrt{x}\,dx$ gives numerical analysts a pain. The integrand is not differentiable at 0. What is worse, the derivatives (first, second, etc.) of $\sqrt{x}$ all become arbitrarily large for x near 0. It is instructive, therefore, to see how the error in Simpson's method behaves as h is made small.

(a) Use the FTC to show that $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}$.

(b) Fill in the table. (Keep at least 7 decimal places in each answer.)

h	Simpson's Method	Error
$\frac{1}{2}$		
$\frac{1}{4}$		
$\frac{1}{8}$		
$\frac{1}{16}$		
$\frac{1}{32}$		
$\frac{1}{64}$		

(c) In the typical appliation of Simpson's method, when you cut h by a factor of 2, you find that the error is cut by a factor of $2^4 = 16$. (That is, the ration of the two errors would be $\frac{1}{16} = 0.0625$.) Examine the five ratios of consecutive errors in the table.

(d) Let $E(h)$ be teh error in using Simpson's method to estimate $\int_0^1 \sqrt{x}\,dx$ with sections of length h. Assume that $E(h) = Ah^k$ for some constants k and A. Estimate K and A.

In Exercises 37–44 a definite integral and its exact value are given.

(a) Use the rectangular method to estimate this definite integral with $n = 4$, $n = 8$, and $n = 16$.

(b) Use the trapezoidal method to estimate this definite integral with $n = 4$, $n = 8$, and $n = 16$.

(c) Use Simpson's method to estimate this definite integral with $n = 4$, $n = 8$, and $n = 16$.

(d) Determine, by trial-and-error, how large n must be for the rectangular estimate to be accurate to three decimal places.

(e) Determine, by trial-and-error, how large n must be for the trapezoidal estimate to be accurate to three decimal places.

(f) Determine, by trial-and-error, how large n must be for the Simpson's estimate to be accurate to three decimal places.

37.[M] $\int_0^2 \sqrt{x(4-x)}\ dx = \pi \approx 3.141593.$

38.[M] $\int_0^4 \sqrt{x(4-x)}\ dx = 2\pi \approx 6.283185.$

39.[M] $\int_0^1 \ln(x)\ dx = -1.$

40.[M] $\int_0^1 \cos(\ln(x))\ dx = \frac{1}{2}.$

41.[M] $\int_0^1 \frac{\sqrt{x}}{x-1} - \frac{1}{\ln(x)}\ dx = 2 - 2\ln(2) - \gamma \approx 0.0364899.$

$\gamma \approx 0.577216$ is the Euler-Mascheroni constant; `http://en.wikipedia.org/wiki/Euler-Mascheroni_constant`

42.[M] $\int_0^1 \frac{2x^2}{x^2-1} - \frac{x}{\ln(x)}\ dx = 2 - 2\ln(2) - \gamma \approx 0.0364899.$

43.[M] $\int_{-1}^1 \frac{dx}{x^2+10^{-10}} = 2 \times 10^5 \arctan(10^5) \approx 3.14157 \times 10^5.$

44.[M] $\int_0^{600\pi} \frac{(\sin(x))^2}{\sqrt{x}+\sqrt{x+\pi}}\ dx \approx 21.102044.$

5.S Chapter Summary

This summary should point out the similarity between the treatments of derivatives and integrals. In both cases, we get estimates (slope of secant line and areas of rectangles) and then take a limit.

EXERCISES for 5.S *Key:* R–routine, M–moderate, C–challenging

EXERCISES for 5.5 *Key:* R–routine, M–moderate, C–challenging

45.[R] ...

Chapter 6

Applications of the Definite Integral

6.1 Computing Area by Parallel Cross-Sections

In Section 5.1 we computed the area under $y = x^2$, recognizing it as a definite integral $\int -a^b x^2\, dx$. Now we generalize the ideas behind this specific example.

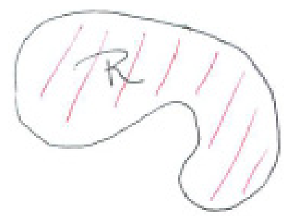


Figure 6.1.1:

Area as a Definite Integral of Cross Sections

How would we express the area of the region R shown in Figure 6.1.1 as a definite integral?

We could introduce an "x-axis", as in Figure 6.1.2.

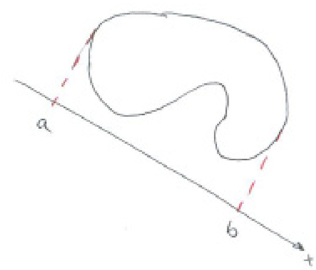


Figure 6.1.2:

SHERMAN: Do we need to be more careful with our usage of length, width, and height? We call Δx both a length and a width; and $c(x)$ is length and height.

Assume that lines perpendicular to the axis at x, $a \leq x \leq b$, intersect the region R in an interval of length $c(x)$. The interval is called the **cross section** of R at x.

We approximate R by a collection of rectangles, just as we estimated the area of the region under $y = x^2$.

Pick an integer n, and divide the interval $[a, b]$ of x into n sections. The total length of the interval $[a, b]$ is $b - a$, each section has length $\Delta x = \frac{b-a}{n}$. Then, in the i^{th} section, $i = 1, 2, \ldots, n$, we pick a "sampling point" x_i. For each of the n sections we form a rectangle of width Δx and height $c(x_i)$. These are indicated in Figure 6.1.3.

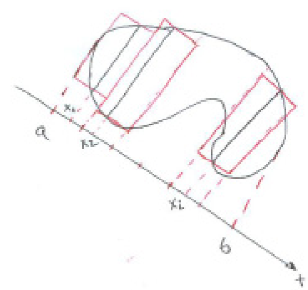


Figure 6.1.3:

Since the i^{th} rectangle has area $c(x_i)\Delta x$, the total area of the n rectangles is $\sum_{i=1}^{n} c(x_i)\Delta x$.

We assume that

$$\lim_{n\to\infty} \sum_{i=1}^{n} c(x_i)\Delta x = \text{area of region } R$$

But, by the definition of a definite integral,

$$\lim_{n\to\infty} \sum_{i=1}^{n} c(x_i)\Delta x = \int_a^b c(x)\, dx.$$

Thus,

$$\text{area of } R = \int_a^b c(x)\, dx.$$

Informally,

area of a region equals the integral of the cross-sectional length.

Note that x need not refer to the x-axis of the xy-plane; it may refer to any conveniently chosen line in the plane. It may even refer to the y-axis; in this case, the cross-sectional length would be denoted by $c(y)$.

To compute an area:

1. Find the endpoints a and b, and the cross-sectional length $c(x)$.

2. Evaluate $\int_a^b c(x)\ dx$ by the Fundamental Theorem of Calculus if the antiderivative $\int c(x)\ dx$ is elementary.

Chapter 5 showed how to accomplish Step 2. Chapter 7 presents techniques that can be used when the antiderivative is not an elementary function. The present section is concerned primarily with Step 1, how to find the cross-sectional length $c(x)$.

If the region R happens to be the region the graph of $f(x)$ and above the interval $[a, b]$, then the cross-sectional length is simply $f(x)$. We have already met this special case in Sections 5.2–5.4 with $f(x) = x^2$ and $f(x) = 2^x$.

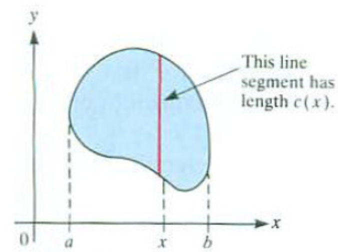


Figure 6.1.4:

EXAMPLE 1 Find the area of a disk of radius r.

SOLUTION Introduce an xy-coordinate system with its origin at the center of the disk, as in Figure 6.1.4.

The typical cross-section perpendicular to the x-axis is shown in Figure 6.1.5. The length of the cross-section AC is twice the length of BC. By the Pythagorean Theorem,

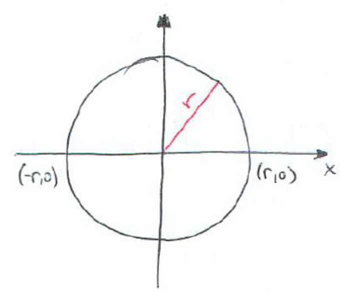


Figure 6.1.5:

$$x^2 + (BC)^2 = r^2.$$

Then

$$(BC)^2 = r^2 - x^2$$

and, because BC, a length, is positive

$$(BC) = \sqrt{r^2 - x^2}.$$

Thus,

$$\text{area of disk with radius } r = \int_{-r}^{r} 2\sqrt{r^2 - x^2}\ dx.$$

Because of symmetry,

$$\text{area of disk with radius } r = 4 \int_{0}^{r} \sqrt{r^2 - x^2}\ dx. \tag{6.1}$$

The antiderivative of $\sqrt{r^2 - x^2}$ is not an elementary function. We could use

SHERMAN: Substitution has not been introduced.

an integral table or computer, and in Chapter 7 we will learn other methods used to evaluate this integral. In Section 7.2(?) we will see that

$$\int_0^r \sqrt{r^2 - x^2}\, dx = r^2 \int_0^{\pi/2} \sin(\theta)^2\, d\theta = \frac{\pi}{4}.$$

See Exercise 42

Thus one quarter of the disk has area $r^2\left(\frac{\pi}{4}\right)$ and the whole disk therefore has area πr^2. ◇

Archimedes found the area in the next example, expressing it in terms of the area of a certain triangle (see Exercise 40). He used geometric properties of a parabola, since calculus was not invented until some 1900 years later.

Archimedes, 287BC–212BC, `http://en.wikipedia.org/wiki/Archimedes`

Reference: S. Stein: *Archimedes: What did he do besides cry Eureka?*, MAA, 1999, pages ??–??.

EXAMPLE 2 Set up a definite integral for the area of a region above the parabola $y = x^2$ and below the line through $(2,0)$ and $(0,1)$ shown in Figure 6.1.6.

SOLUTION Since the x-intercept of the line is 2 and the y-intercept is 1, an equation for the line is

$$\frac{x}{2} + \frac{y}{1} = 1.$$

Hence $y = 1 - x/2$. The length $c(x)$ of a cross-section of the region taken parallel to the y-axis is, therefore

$$c(x) = \left(1 - \frac{x}{2}\right) - x^2 = 1 - \frac{x}{2} - x^2.$$

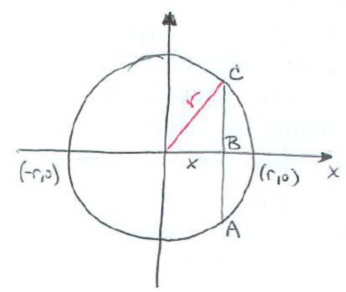


Figure 6.1.6: Appendix B describes the intercept equation of a line.

To find the interval $[a,b]$ of integration, we must find the x-coordinates of the points P and Q in Figure 6.1.6. For these values of x,

$$x^2 = 1 - \frac{x}{2},$$

so

$$2x^2 + x - 2 = 0. \tag{6.2}$$

The solutions to (6.2) are

$$x = \frac{-1 \pm \sqrt{17}}{4}.$$

Hence

$$\text{area} = \int_{(-1-\sqrt{17})/4}^{(-1+\sqrt{17})/4} \left(1 - \frac{x}{2} - x^2\right)\, dx.$$

◇

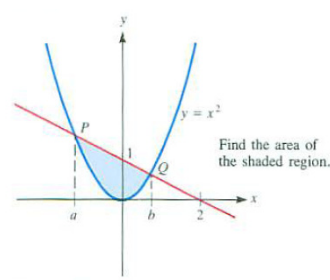


Figure 6.1.7:

EXAMPLE 3 Find the area of the region in Figure 6.1.7, bounded by $y = \arctan(x)$, $y = -2x$, and $x = 1$.

SOLUTION We will find the area two ways, first (a) with cross-sections parallel to the y-axis, then (b) with cross-sections parallel to the x-axis.

(a) The typical cross-section has length $\arctan(x) - (-2x) = \arctan(x) + 2x$. Thus the area is

$$\int_0^1 (\arctan(x) + 2x)\ dx.$$

It's easy to find $\int 2x\ dx$; it's just x^2. In Section 7.2 (or 7.5?) it is found that the integral of $\arctan(x)$ is $x\arctan(x) - \frac{1}{2}\ln(1+x^2)$. By the FTC,

$$\begin{aligned} \int_0^1 (\arctan(x) + 2x)\ dx &= \left(x\arctan(x) - \frac{1}{2}\ln(1+x^2) + x^2\right)\Bigg|_0^1 \\ &= \left(1\arctan(1) - \frac{1}{2}\ln(1+1^2) + 1^2\right) - \Big(0\arctan \\ &= \left(\frac{\pi}{4} - \frac{1}{2}\ln(2) + 1\right) - 0 \\ &= \frac{\pi}{4} + 1 - \frac{1}{2}\ln(2). \end{aligned}$$

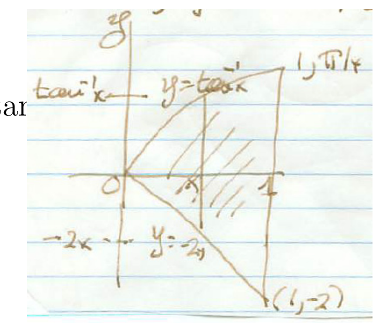


Figure 6.1.8:

(b) Now we use cross-sections parallel to the x axis, as indicated in Figure 6.1.8.

Cross-sections above the x-axis involve the curved part of the boundary, while those below the x-axis involve the slanted line.

We must find the cross-sectional length as a function of y. That means we should first find the x-coordinates of P and Q, the ends of the typical cross-section above the x-axis. The x-coordinate of Q is 1. Let the x-coordinate of P be x, then $y = \arctan(x)$, so $x = \tan(y)$. Hence $c(y) = 1 - \tan(y)$ for $y \geq 0$. The length of RS, a typical cross-section below the x-axis, is $1 -$ (x-coordinate of R). SInce R is on the line $y = -2x$, $x = -y/2$. Thus

$$c(y) = 1 - (-y/2) = 1 + y/2, \qquad \text{for } -2 \leq y \leq 0.$$

Note that the interval of integration is $[-2, \pi/4]$. Hence

$$\text{area of } R = \int_{-2}^{\pi/4} c(y)\ dy.$$

We have to break this integral into two separate definite integrals,

$$\int_0^{\pi/4} (1 - \tan(y))\ dy \text{ and } \int_{-2}^{0} \left(1 + \frac{y}{2}\right)\ dy. \tag{6.4}$$

It will be shown in Section 7.5 (see Example 4) that

sec(y) is positive

$$\int \tan(y)\ dy = \ln(\sec(y)).$$

Thus

$$\begin{aligned}\int_0^{\pi/4} (1-\tan(y))\ dy &= (y - \ln\sec(y))\Big|_0^{\pi/4} \\ &= \left(\frac{\pi}{4} - \ln(\sec(\frac{\pi}{4}))\right) - (0 - \ln(\sec(0))) \\ &= \frac{\pi}{4} - \ln(\sqrt{2}) \qquad (6.5)\end{aligned}$$

Also,

$$\begin{aligned}\int_{-2}^{0} \left(1+\frac{y}{2}\right)\ dy &= \left(y + \frac{Y^2}{4}\right)\Big|_{-2}^{0} \\ &= \left(0+\frac{0^2}{4}\right) - \left((-2) + \frac{(-2)^2}{4}\right) \\ &= 1 \qquad (6.6)\end{aligned}$$

Adding (6.5) and (6.6) gives

$$\text{area of } R = \frac{\pi}{4} - \ln(\sqrt{2}) + 1 \qquad (6.7)$$

The two answers (6.3) and (6.7) may look different but they are equal, as you may check. ◇

In this exampmle we could have simplified the solution by observing that the area below the x-axis is a triangle of area 1. But the purpose of Example 3 is to illustrate a general approach that would apply to more complicated examples.

Summary

The key idea in this section "area of a region equals integral of cross-sectional length" was alreay anticipated in Chapter 5. There we met the special case where the region is bounded by the graph of a function, the x-axis, and two lines perpendicular to the axis.

EXERCISES for 6.1 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1–6 *(a)* draw the region, *(b)* compute the lengths of vertical cross-sections ($c(x)$), and *(c)* compute the lengths of horizontal cross-sections ($c(y)$).

1.[R] The finite region bounded by $y = \sqrt{x}$ and $y = x^2$.

2.[R] The finite region bounded by $y = x^2$ and $y = x^3$.

3.[R] The finite region bounded by $y = 2s$, $y = 3x$, and $x = 1$.

4.[R] The finite region bounded by $y = x^2$, $y = 2x$, and $x = 1$.

5.[R] The triangle with vertices $(0, 0)$, $(3, 0)$, and $(0, 4)$.

6.[R] The triangle with vertices $(1, 0)$, $(3, 0)$, and $(2, 1)$.

In Exercises 7–12 find the indicated areas.

7.[R] Under $y = \sqrt{x}$ and above $[1, 2]$

8.[R] Under $y = \sin(2x)$ and above $[\pi/6, \pi/3]$

SHERMAN: Substitution is not yet known; did we do this on intuition?

9.[R] Under $y = e^(2x)$ and above $[0, 1]$

10.[R] Under $y = 1/\sqrt{1 - x^2}$ and above $[0, 1/2]$.

11.[R] Under $y = \ln(x)$ and above $[1, e]$

12.[R] Under $y = \cos(x)$ and above $[-\pi/2, \pi/2]$

In Exercises 13–20 find the indicated areas using cross-sections parallel to the x-axis.

13.[R] Between $y = x^2$ and $y = x^3$.

14.[R] Between $y = 2^x$ and $y = 2x$.

15.[R] Between $y = \arcsin(x)$ and $y = 2s/\pi$ (to the right of the y-axis).

16.[R] Between $y = 2^x$ and $y = 3^x$ (to the right of the y-axis).

17.[R] Between $y = \sin(x)$ and $y = \cos(x)$ (above $0, \pi/2]$.

18.[R] Between $y = x^3$ and $y = -x$ for x in $[1, 2]$.

19.[R] Between $y = x^3$ and $y = \sqrt[3]{2x - 1}$ for x in $[1, 2]$.

20.[R] Between $y = 1 + x$ and $y = \ln(x)$ for x in $[1, e]$.

In Exercises 21–27 *setup a definite integral* for the area of the given region. These integrals will be evaluated in Chapter 7.

21.[R] The region under the curve $y = \arctan(2x)$ and above the interval $[1/2, 1/\sqrt{3}]$.

See Exercise 74 in Section 7.6.

22.[R] The region in the first quadrant below $y = -7x + 29$ and above the portion of $y = 8/(x^2 - 8)$ that lies in the first quadrant.

See Exercise 75 in Section 7.6.

See Exercise 76 in Section 7.6.

23.[R] The region below $y = 10^x$ and above $y = \log_{10}(x)$ for x in $[1, 10]$.

24.[R] The region under the curve $y = x/(x^2+5x+6)$ and above the interval $[1, 2]$.

See Exercise 77 in Section 7.6.

25.[R] The region below $y = (2x + 1)/(x^2 + x)$ and above the interval $[2, 3]$.

See Exercise 78 in Section 7.6.

26.[R] The region bounded by $y = \tan(x)$, $y = 0$, $x = 0$, and $x = \pi/2$ by *(a)* vertical cross-sections and *(b)* horizontal cross-sections.

See Exercise 79 in Section 7.6.

27.[R] The region bounded by $y = \sin(x)$, $y = 0$, and $x = \pi/4$ (consider only $x \geq 0$) by *(a)* vertical cross-sections and *(b)* horizontal cross-sections.

See Exercise 80 in Section 7.6.

28.[R] Find a definite integral for the area of the region above the interval $[1, 2]$ and below the curve $y = x^2/(x^3 + x^2 + x + 1)$.

See Exercise 81 in Section 7.6.

29.[R]

See Exercise 82 in Section 7.6.

(a) Draw the region inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(b) Find a definite integral for the area of the ellipse in *(a)* with horizontal cross-sections.

(c) Find a definite integral for the area of the ellipse in *(a)* with vertical cross-sections.

30.[R] Cross-sections in different directions lead to different definite integrals for the same area. While both integrals must give the same area, one of the two integrals is oftentimes much easier to evaluate than the other.

(a) Identify and evaluate the easier definite integral found in Exercise 26.

(b) Identify and evaluate the easier definite integral found in Exercise 27.

31.[R] Setup the definite integral for the area $A(b)$ of the region in the first quadrant under the curve $y = e^{-x}\cos(x)^2$ and above the interval $[0, b]$.

This area is found in Exercise 83.

32.[M] Let $A(t)$ be the area of the region in the first quadrant between $y = x^2$ and $y = 2x^2$ and inside the rectangle bounded by $x = t$, $y = t^2$, and the coordinate axes. (See the shaded region in Figure 6.1.9.) If $R(t)$ is the area of the rectangle, find

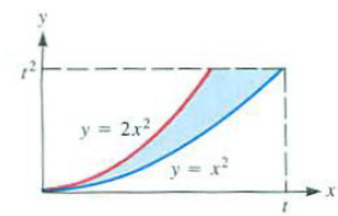


Figure 6.1.9:

(a) $\displaystyle\lim_{t\to 0} \frac{A(t)}{R(t)}$

(b) $\displaystyle\lim_{t\to\infty} \frac{A(t)}{R(t)}$

Hint: Use FTC II!

33.[M]

(a) Draw the curve $y = e^x/x$ for $x > 0$, showing any asymptotes or critical points.

(b) Find the number t such that the area below $y = e^x/x$ and above the interval $[t, t+1]$ is a minimum.

34.[M] Let R be the region bounded by $y = x^3$, $y = x + 2$, and the x-axis.

(a) Find a definite integral for the area of R. HINT: It might be necessary to specify one or both of the endpoints as solutions to an equation.

(b) Use a graph or other method to approximate the endpoints.

(c) Use the estimates in *(b)* to obtain an estimate of the area of R.

35.[M] Let R be the region between $y = 3$ and $y = e^x/x$.

(a) Graph the region R.

(b) Find a definite integral for the area of R. HINT: You will encounter an equation which cannot be solved exactly. Identify the endpoints on the graph found in *(a)*.

(c) Find approximate values for the endpoints of the definite integral for the area of R.

(d) Because the antiderivative of e^x/x is not elementary, it is still not easy to estimate the area of R. What methods do we have for estimating a definite integral?

36.[M] What fraction of the rectangle whose vertices are $(0, 0)$, $(a, 0)$, (a, a^4), and $(0, a^4)$, with a positive, is occupied by the region under the curve $y = x^4$ and above $[0, a]$?

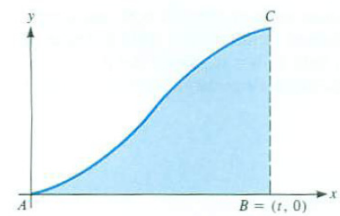


Figure 6.1.10:

37.[C] Figure 6.1.10 shows the graph of an increasing function $y = f(x)$ with $f(0) = 0$. Assume that $f'(x)$ is continuous and $f'(0) > 0$. Do not assume that $f''(x)$ exists. Investigate

$$\frac{\text{area of shaded region under the curve}}{\text{area of triangle } ABC}. \tag{6.8}$$

(a) Experiment with various functions, including some trigonometric functions and polynomials. NOTE: Make sure that $f'(0) > 0$.

(b) Make a conjecture about (6.8) and explain why it is true.

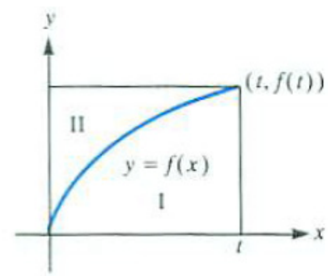


Figure 6.1.11:

38.[C] Repeat Exercise 37, but now assume that $f'(0) = 0$, f'' is continuous, and $f''(0) \neq 0$.

39.[C] Let f be an increasing function with $f(0) = 0$, and assume that it has an elementary antiderivative. Then f^{-1} is an increasing function, and $f^{-1}(0) = 0$. Prove that if f^{-1} is elementary, then it also has an elementary antiderivative. HINT: See Figure 6.1.11.

40.[C] Show that the area of the shaded region in Figure 6.1.12 is two-thirds the area of the parallelogram $ABCD$. This ia an illustration of a theorem of Archimedes concerning sectors of parabolas. He showed that the shaded area is 4/3 the area of triangle BOC.

See also Example 2.

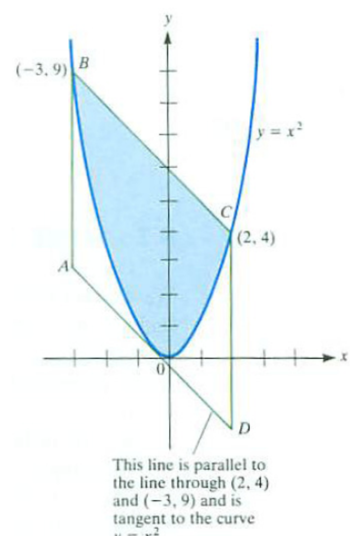


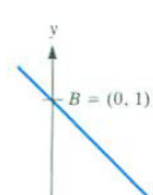
Figure 6.1.12:

41.[C] Figure 6.1.13 shows a right triangle ABC.

(a) Find equations for the lines parallel to each edge, AC, BC, and AB, that cut the triangle into two pieces of equal area.

(b) Are the three lines in *(a)* concurrent; that is, do they meet at a single point?

42.[C] Find the area of the disk using concentric circles.



6.2 Some Pointers on Drawing

None of us were born knowing how to draw solids. As we grew up, we lived in flatland: the surface of the Earth. Few high school math classes cover solid geometry, so calculus is often the first place where you have to think and sketch in terms of three dimensions. That is why we pause for a few words of advice on how to draw. Too often you cannot work a problem simply because your diagrams confuse even yourself. The following guidelines are not based on any profound artistic principles. Instead, they derive from years attempting to sketch diagrams that do more good than harm.

A Few Words of Advice

1. *Draw large.* Many students tend to draw diagrams that are so small that there is no room to place labels or to sketch cross-sections.

2. *Draw neatly.* Use a straightedge to make straight lines that are actually straight. Use a compass to make circles that look like circles. Draw each line or curve slowly. You may want to add a second (or third) color.

A jar lid or soda can will do just fine.

3. *Avoid clutter.* If you end up with too many labels or the cross-section doesn't show up well, add separate diagrams for important parts of the figure.

4. *Practice.*

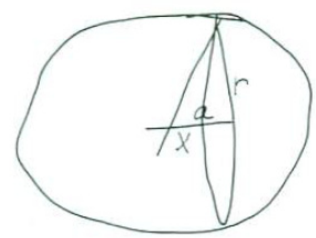


Figure 6.2.1:

EXAMPLE 1 Draw a diagram of a ball of radius a that shows the circular cross-section made by a plane at a distance x from the center of the ball. Use the diagram to help find the radius of the cross-section as a function of x.

TERRIBLE SOLUTION Is Figure 6.2.1 a potato or a ball? What segment has length r? What's x? What does the cross-section look like?

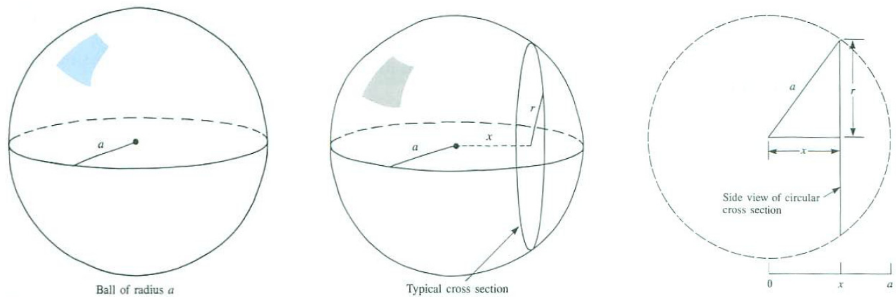
REASONABLE SOLUTION First, draw the ball carefully, as in Figure 6.2.2(a). The equator is drawn to give it perspective. Add a little shading.

Next show a typical cross-section at a distsance x from the center, as in Figure 6.2.2(b). Shading the cross-section helps, too.

To find r, the radius of the cross-section, in terms of x, sketch a companion diagram. The radius we want is part of a right triangle. In order to avoid clutter, draw only the part of interest in a convenient side view, as in Figure 6.2.4.

Inspection of the right triangle in Figure 6.2.2(c) shows that

$$r^2 + x^2 = a^2,$$



(a) (b) (c)

Figure 6.2.2: NOTE: Add shading to cross-section in (b).

hence that

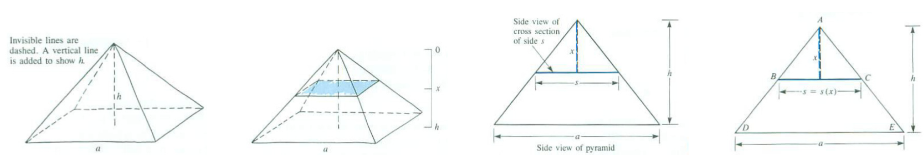
$$r = \sqrt{a^2 - x^2}.$$

◇

EXAMPLE 2 A pyramid has a square base with a side of length a. The top of the pyramid is above the center of the base at a height h. Draw the pyramid and its cross-sections by planes parallel to the base. Then find the area of the cross-sections in terms of their distance x from the top.

Figure 6.2.3:

TERRIBLE SOLUTION Figure 6.2.3 is too small; there's no room for the symbols. While it's pretty clear what side has length a, to what are the x and h attached? Also, without the hidden edges of the pyramid the shape of the base is not clear.



(a) (b) (c) (d)

Figure 6.2.4:

REASONABLE SOLUTION First draw a large pyramid with a square base, as in Figure 6.2.4(a). Note that the opposite edges of the base are drawn as parallel lines. While artists draw parallel lines as meeting in a point to enhance the sense of perspective, for our purposes it is more useful to use parallel lines to depct the lines that are parallel. Then show a typical cross-section in perspective and side views, as in Figures 6.2.4(b) and (c). Note the x-axis, which is drawn separate from the pyramid.

The use of s is recommended because it suggests its meaning - side.

As x increases, so does s, the width of the square cross-section. Thus s is a function of x, which we could call $s(x)$ (or $f(x)$, if you prefer). A glance at Figure 6.2.4(b) shows that $s(0) = 0$ and $s(h) = 1$. To find $s(x)$ for all x in $[0, h]$, use the similar triangles ABC and ADE, shown in Figure 6.2.4(c). These triangles show that

$$\frac{x}{s} = \frac{h}{a};$$

hence

$$s = \frac{ax}{h}. \tag{6.1}$$

Notice that (6.1 expresses s is a linear function of x. As a final check on (6.1), replace x by 0 and by h; we get 0 and a for the respective values s, as expected. Finally, the area A of the cross-sections is given by

$$A = s^2 = \left(\frac{ax}{h}\right)^2.$$

◇

EXAMPLE 3 A cylindrical drinking glass of height h and radius a is full of water. It is tilted until the remaining water covers exactly half the base.

A. Draw a diagram of the glass and water.

B. Show a cross-section of the water that is a triangle.

C. Find the area of the triangle in terms of the distane x of the cross-section from the axis of the glass.

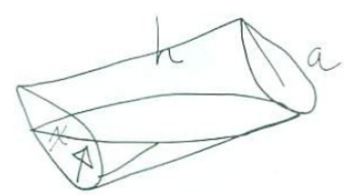


Figure 6.2.5:

TERRIBLE SOLUTION The diagram in Figure 6.2.5 is too small. It is not clear what has length a. The cross-section is unclear. What does x refer to?

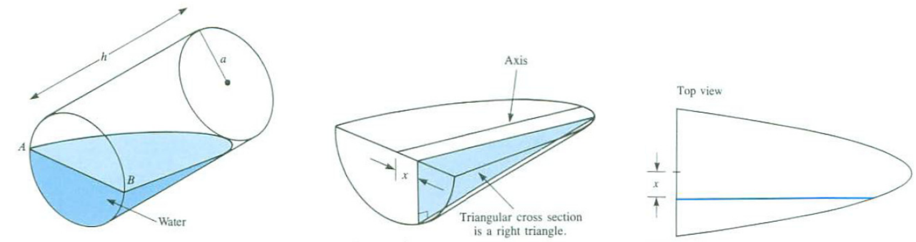
REASONABLE SOLUTION First, draw a *neat, large* diagram of a slanted cylinder, as in Figure 6.2.7. Don't put in too much detail to start. When showing the cross-section, draw only the water. Figures 6.2.6 and 6.2.7 show various views. Let u and v be the lengths of the two legs of the cross-section, as shown in Figure 6.2.7(d).

Comparing Figures 6.2.7(a) and (b), we have, by similar triangles, the relation

$$\frac{u}{a} = \frac{v}{h};$$

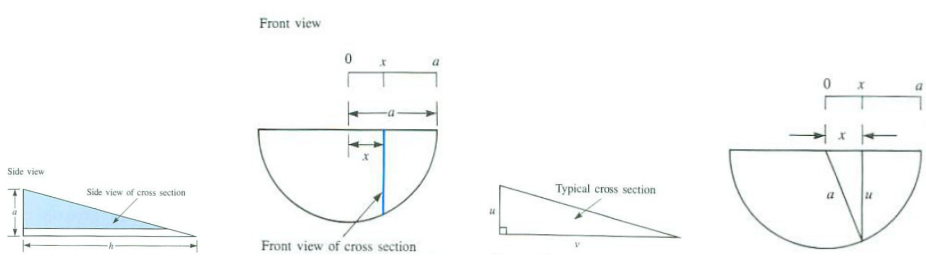
hence

$$v = \frac{h}{a}u.$$



(a) (b) (c)

Figure 6.2.6:



(a) (b) (c) (d)

Figure 6.2.7: NOTE: The triangles in (a) and (c) need to be turned upside-down.

Let $A(x)$ be the area of the cross-section at a distance x from the center of the base, as shown in Figure 6.2.6(b). If we can find u and v as functions of x, we will be able to write down a formula for $A(x) = \frac{1}{2}uv$ in terms of x.

Figure 6.2.7(b) suggests how to find u. Copy it over and draw in the necessary radius, as in Figure 6.2.7(d). By the Pythagorean Theorem,

$$u = \sqrt{a^2 - x^2}.$$

All told,

$$A(x) = \frac{1}{2}uv = \frac{1}{2}u\left(\frac{u}{a}u\right) = \frac{h}{2a}u^2 = \frac{h}{2a}(a^2 - x^2). \tag{6.2}$$

As a check, note that

$$A(a) = \frac{h}{2a}(a^2 - a^2) = 0,$$

which makes sense. Also the formula (6.2) gives

$$A(0) = \frac{h}{2a}(a^2 - 0^2) = \frac{1}{2}ah,$$

again agreeing with the geometry of, say, Figure 6.2.6(b). ◇

Summary

When you look back at these three examples, you will see that most of the work is spent on making clear diagrams. If you can't draw a straight line, use a straightedge. If you can't draw a circle, use a compass.

EXERCISES for 6.2 *Key:* R–routine, M–moderate, C–challenging

1.[R] Cross-sections of the pyramid in Example 2 are made by using planes perpendicular to the base and parallel to the edge of the base. What is the area of the cross-section made by a plane that is a distance x from the top of the pyramid? See Example 2

(a) Draw a large perspective view of the pyramid.

(b) Copy the diagram in *(a)* and show the typical cross-section shaded.

(c) Draw a side view that clearly shows the shape of the cross-section.

2.[R] Cross-sections of the water in Example 3 are made by using planes parallel to the plane that passes through the horizontal diameter of the base and the axis of the glass. What is the area of the cross-section made by a plane that is a distance x from the center of the base? See Example 3

(a) Draw a large perspective view of the water and glass.

(b) Copy the diagram in *(a)* and show the typical cross-section shaded.

(c) Draw a side view that clearly shows the shape of the cross-section.

(d) Draw a different side view.

(e) Put necessary labels, such as x, a, and h, on the diagrams, where appropriate. (You will need to introduce more labels.)

(f) Find the area of the cross-section, $A(x)$, as a function of x.

3.[R] Cross-sections of the water in Example 3 are made by using planes perpendicular to the axis of the glass. Make clear diagrams, including perspective and side views, that show the typical cross-sections. Do not find its area. See Example 3

4.[R] A cylindrical glass is full of water. The glass is tilted until the remaining water just covers the base of the glass. (Try it!) The radius of the glass is a and its height is h. Consider parallel planes such that cross-sections of the water are rectangles.

(a) Make clear diagrams that show the situation. (You may want to include a top view to show the cross-sections.)

(b) Obtain a formula for the area of the cross-sections. *Advice:* The two planes at a distance x from the axis of the glass cut out cross-sections of different areas. So introduce an x-axis with 0 at the center of the base and extending from $-a$ to a in a convenient direction.

5.[R] Repeat Exercise 4, but this time consider parallel planes such that the cross-sections are trapezoids.

6.[R] A right circular cone has a radius a and height h as shown in Figure 6.2.8. Consider cross-sections made by planes parallel to the base of the cone.

(a) Draw perspective and side views of the situation.

(b) Drawing as many diagrams as necessary, find the area of the cross-section made by a plane at a distance x from the vertex of the cone.

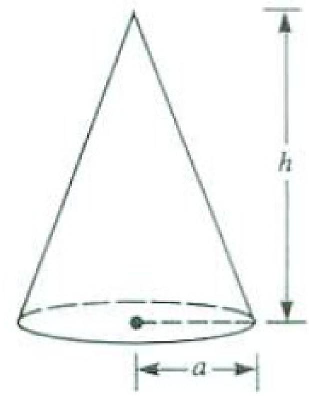


Figure 6.2.8: See Exercise 6

7.[R] Draw the typical cross-section made by a plane parallel to the axis of the cone. Draw perspective and side views of the situation, but do not find a formula for the area of the cross-section.

8.[R] Draw a cross-section of a right circular cylinder that is *(a)* a circle, *(b)* an ellipse that is not a circle, and *(c)* a rectangle.

9.[R] Figure 6.2.9 indicates an unbounded, solid right circular cone. Draw a cross-section that a bounded by *(a)* a circle, *(b)* an ellipse (but not a circle), *(c)* a parabola, and *(d)* a hyperbola.

Figure 6.2.9:

10.[R] A lumberjack saws a wedge out of a cylindrical tree of radius a. His first cut is parallel to the ground and stops at the axis of the tree. His second cut makes an angle θ with the first cut and meets it along a diameter.

(a) Draw a typical cross-section that is a triangle.

(b) Find the area of the triangle as a function of x, the distance of the plane from the axis of the tree.

(c) Draw a typical cross-section that is a rectangle.

(d) Find the area of the rectangle as a function of x, the distance of the plane from the axis of the tree.

11.[R] The plane region between the curves $y = x$ and $y = x^2$ is spun around the x-axis to produce a solid resembling the bell of a trumpet.

(a) Draw the plan region.

(b) Draw the solid region produced by spinning this region around the x-axis.

(c) Draw the typical cross-section made by a plane perpendicular to the x-axis. Show this in both perspective and side views.

(d) Find the area of the cross-section in terms of the distance x of the plane from the origin of the x-axis.

12.[R] Draw a cross-section of a solid cube that is *(a)* a square, *(b)* an equilateral triangle, *(c)* a five-sided polygon, and *(d)* a regular hexagon. HINT: For *(d)*, the vertices of the hexagon are midpoints of edges of the cube.

13.[R] Obtain a circular stick such as a broom handle or a dowel. Saw off a piece, making one cut perpendicular to the axis and the second cut at an angle to the axis. Mark on the piece you cut out the borders of cross-sections that are *(a)* rectangles and *(b)* trapezoids.

6.3 Setting Up a Definite Integral

This section presents an informal shortcut for setting up a definite integral to evaluate some quantity. First, the formal and informal approaches are contrasted in the case of setting up the definite integral for area. Then the informal approach will be illustrated as commonly applied in a variety of fields.

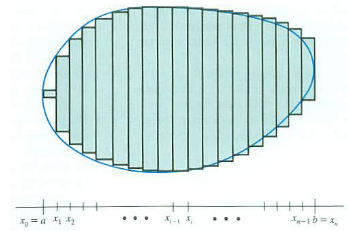


Figure 6.3.1: NOTE: Revise figure so not left-hand sum.

The Complete Approach

Recall how the formula $A = \int_a^b f(x)\ dx$ was obtained (in Section 6.1). The interval $[a, b]$ was partitioned by the numbers $x_0 < x_1 < x_2 < \cdots < x_n$ with $x_0 = a$ and $x_n = b$. A sampling number c_i was chosen in each section $[x_{i-1}, x_i]$. For convenience, all the sections are of equal length, $\Delta x = (b - z)/n$. We then form the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x \tag{6.1}$$

It equals the total area of the rectangular approximation in Figure 6.3.1.

As Δx approaches 0, the sum (6.1) approaches the area of the region under consideration. But, by the definition of the definite integral, the sum (6.1) approaches

$$\int_a^b f(x)\ dx.$$

Thus

$$\text{Area} = \int_a^b f(x)\ dx. \tag{6.2}$$

That is the complete or "formal" approach to obtain formula (6.2). Now consider the "informal" approach, which is just a shorthand for the complete approach.

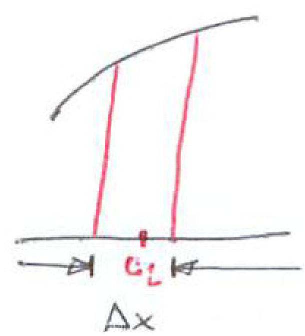


Figure 6.3.2:

The Shorthand Approach

The heart of the complete approach is the *local estimate* $f(c_i)\Delta x$, the area of a rectangle of height $f(c_i)$ and width Δx, which is shown in Figure 6.3.2.

In the shorthand approach to setting up a definite integral attention is focused on the *local approximation*. No mention is made of the partition or the sampling numbers. We illustrate this shorthand approach by obtaining formula (6.2) informally. This is *not* a new method of integration, but just a way to save time when setting up an integral - finding out the integrand and the interval of integration.

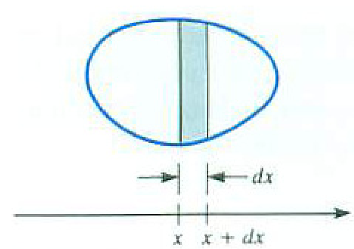


Figure 6.3.3:

For example, consider a small positive number dx. What would be a good estimate of the area of the region corresponding to the short interval $[x, x+dx]$ of width dx shown in Figure 6.3.3? The area of the rectangle of width dx and height $f(x)$ shown in Figure 6.3.4 would seem to be a plausible estimate. The area of this thin rectangle is

$$f(x)\ dx. \tag{6.3}$$

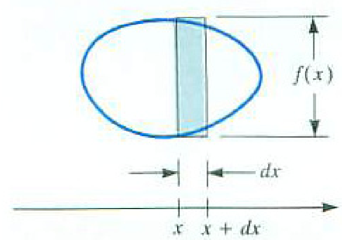


Figure 6.3.4:

Without further ado, we then write

$$\text{Area} = \int_a^b f(x)\ dx, \tag{6.4}$$

which is formula (6.2). The leap from the local approximation 6.3 to the definite integral (6.4) omits many steps of the complete approach. This informal approach is the shorthand commonly used in applications of calculus. It is the way engineers, physicists, and mathematicians set up integrals.

It should be emphasized that it is only an abbreviation of the formal approach, which deals with approximating sums.

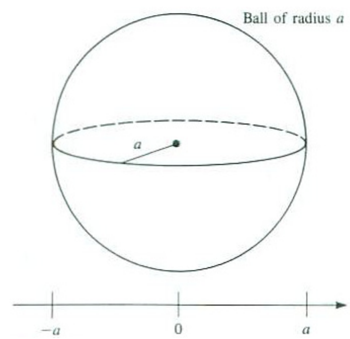


Figure 6.3.5:

The Volume of a Ball

EXAMPLE 1 Find the volume of a ball of radius a. First use the complete approach. Then use the shorthand approach.

SOLUTION Both approaches require good diagrams. In the complete approach we show an x axis, a partition into sections of equal lengths, sampling numbers, and the approximating disks. See Figures 6.3.5 and 6.3.6. The thickness of disk is Δx, as shown in the side view of Figure 6.3.7, while its radius is labeled r_i, as shown in the end view of Figure 6.3.8. The volume of this typical disk is

$$\pi r_i^2(\Delta x). \tag{6.5}$$

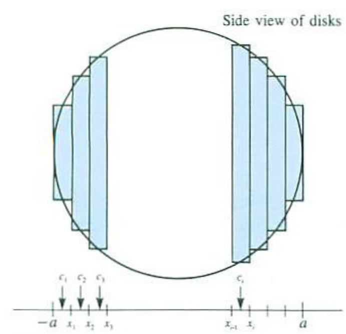


Figure 6.3.6:

All that remains is to determine r_i. Figure 6.3.9 helps us do that. By the Pythagorean Theorem,

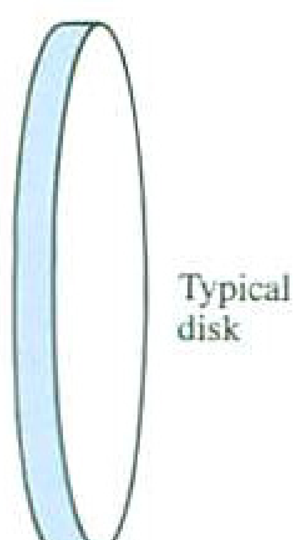
$$r_i^2 = a^2 - c_i^2. \tag{6.6}$$

Combining (6.1), (6.5), and (6.6) gives the typical estimate of the volume of a sphere of radius a:

$$\sum_{i=1}^{n} \pi(a^2 - c_i^2)\Delta x. \tag{6.7}$$

By the definition of the definite integral,

$$\lim_{\Delta x \to 0} \Sigma_{i=1}^{n} \pi(a^2 - c_i^2)\Delta x = \int_{-a}^{a} \pi(a^2 - x^2)\ dx.$$



Hence

$$\text{Volume of ball of radius } a = \int_{-a}^{a} \pi(a^2 - x^2)\ dx.$$

(By the Fundamental Theorem of Calculus, the integral equals $4\pi a^3/3$.)

Now for the shorthand approach. We draw only a short section of an x axis and label its length dx. Then we draw an approximating disk, whose radius we label r, as in Figure 6.3.10. Since the disk has a base of area πr^2 and thickness dx, its volume is $\pi r^2\ dx$. Moreover, as Figure 6.3.11 shows, $r^2 = a^2 - x^2$. Hence the local approximation is

$$\pi(a^2 - x^2)\ dx. \tag{6.8}$$

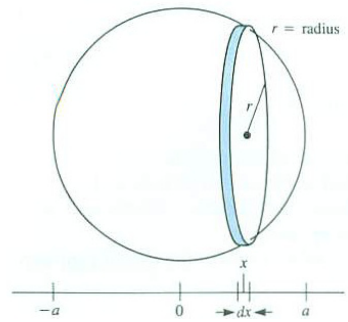


Figure 6.3.10:

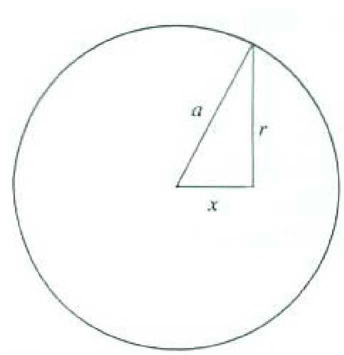


Figure 6.3.11:

Then, without further ado, without choosing any c_i showing any approximating sum, we have

$$\text{Volume of ball of radius } a = \int_{-a}^{a} \pi(a^2 - x^2)\ dx.$$

The key to this bookkeeping is the local approximation (6.8) in differential form, which gives the necessary integrand. The limits of integration are determined separately. ◇

Volcanic Ash

EXAMPLE 2 After the explosion of a volcano, ash gradually settles from the atmosphere and falls on the ground. The depth diminishes with distance from the volcano. Assume that the depth of the ash at a distance x feet from the volcano is Ae^{-kx} feet, where A and k are positive constants. Set up a definite integral for the total volume of ash that falls within a distance b of the volcano.

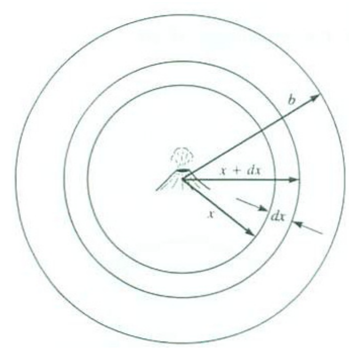


Figure 6.3.12:

SOLUTION First estimate the volume of ash that falls on a very narrow ring of width dx and inner radius x centered at the volcano. (See Fig. 12) This estimate can be made since the depth of the ash depends only on the distance from the volcano. On this ring the depth is almost constant.

The area of this ring is approximately that of a rectangle of lenght $2\pi x$ and width dx. (See Figure 6.3.13) So the area of the ring is approximately

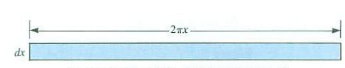


Figure 6.3.13:

$$2\pi x\ dx.$$

Exercise 4 shows that its area is $2\pi x\ dx + \pi(dx)^2$.

Although the depth of the ash on this narrow ring is not constant, it does not vary much. A good estimate of the depth throughout the ring is Ae^{-kx}. Thus the volume of the ash that falls on the typical ring of inner radius x and outer radius $x + dx$ is approximately

$$Ae^{-kx} 2\pi x \; dx \text{ cubic feet.} \tag{6.9}$$

Once we have the key local estimate (6.9), we immediately write doen the definite integral for the total volume of ash that falls within a distance b of the volcano:

$$\text{Total volume} = \int_0^b Ae^{-kx} 2\pi x \; dx.$$

(The limits of integation must be determined just as in the formal approach.) This completes the shorthand setting up the definite integral. (It could be evaluated by a technique in Chapter 7 or by Formula **??** inside the front cover of this book.) $\diamond$

Kinetic Energy

The next example of the informal approach to setting up definite integrals concerns kinetic energy. The kinetic energy associated with an object of mass m kilograms and velocity v meters per second is defined as

$$\text{Kinetic energy} = \frac{mv^2}{2} \text{joules.}$$

If the various parts of the objects are not all moving at the same speed, an integral is needed to express the total kinetic energy. We develop this integral in the next example.

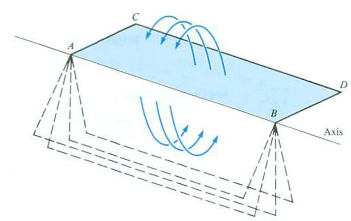


Figure 6.3.14:

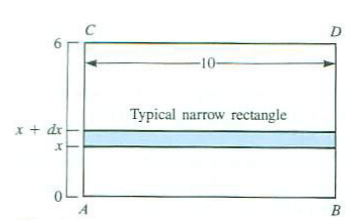


Figure 6.3.15:

EXAMPLE 3 A thin rectangular piece of sheet metal is spining around one of its longer edges 3 times per second, as shown in Figure 6.3.14. The length of its shorter edge is 6 meters and the length of its longer edge is 10 meters. The density of the sheet metal is 4 kilograms per square meter. Find the kinetic energy of the spinning rectangle.

SOLUTION The farther a mass is from the axis, the faster it moves, and therefore the larger its kinetic energy. To find the total kinetic energy of the rotating piece of sheet metal, imagine it divided into narrow rectangles of length 10 meters and width dx meters parallel to the edge AB; a typical one is shown in Figure 6.3.15. (Introduce an x axis parallel to edge AC with the orgin corresponding to A.) Since all points of the typical narrow rectangle move at roughly the same speed, we will be able to estimate its kinetic energy. That estimate will provide the key local appoximation in the informal approach to setting up a definite integral.

First of all, the mass of the typical rectangle is

$$4 \cdot 10\ dx \text{ kilograms,}$$

since its area is 10 dx square meters and the density is 4 kilograms per square meter.

Second, we must estimate its velocity. The narrow rectangle is spun 3 times per second around a circle of radius x. In 1 second each point in it covers a distance of about

$$3 \cdot 2\pi x = 6\pi x \text{ meters.}$$

Consequently, the velocity of the typical rectangle is

$$6\pi x \text{ meters per second.}$$

The local estimate of the kinetic energy associated with the typical rectangle is therefore

$$\frac{1}{2}\underbrace{40\ dx}_{\text{mass}}\ \underbrace{(6\pi x^2}_{\text{velocity squared}} \quad \text{joules}$$

or simply

The local approximation

$$720\pi^2 x^2\ dx \text{ joules.} \tag{6.10}$$

Having obtained the local estimate (6.10), we jump directly to the definite integral and conclude that

$$\text{Total energy of spinning rectangle} = \int_0^6 720\pi^2 x^2\ dx \text{ joules.}$$

◇

Summary

This section presented a shorthand approach to setting up a definite integral for a quantity Q. In this method we estimate how much the quantity Q corresponds to a very short section $[x, x + dx]$ of the x axis, say $f(x)\ dx$. Then $Q = \int_a^b f(x)\ dx$, where a and b are determined by the particular situation.

EXERCISES for 6.3 *Key:* R–routine, M–moderate, C–challenging

1.[R] In Section 5.4 we showed that if $f(t)$ is the velocity at time t of an object moving along the x axis, then $\int_a^b f(t)\ dt$ is the change in position during the time interval $[a, b]$. Develop this fact in the informal style of this section. Keep in mind that $f(t)$ may be positive or negative.

2.[R] The depth of rain at a distance r feet from the center of a storm is $g(r)$ feet.

(a) Estimate the total volume of rain that falls between a distance r feet and a distance $r+\ dr$ feet from the denter of the storm. (Assume that dr is a small positive number.)

(b) Using (a), set up a definite integral for the total volume of rain that falls between $1,000$ and $2,000$ feet from the center of the storm.

3.[R]

Consider a circular range of radius a with the home base of production at the center. Let $G(r)$ denote the density of foodstuffs (in calories per square meter) at radius r meters from the home base. Then the total number of calories produced in the range is given by what definite integral?

Using the informal approach, set ut the definite integral that appeared in the blank.

The following analysis of primitive agriculture is taken from *Is There an Optimum Level of Population?*, edited by S. Fred Singer, McGraw-Hill, New York, 1971.

4.[R] In 2 the area of the ring with inner radius x and outer radius $x + dx$ was informally estimated to be approximately $2\pi x\ dx$.

(a) Using the formula for the area of a circle, show that the area of the ring is $2\pi x\ dx + \pi(\ dx)^2$.

(b) Show that the ring has the same area as a trapezoid of height dx and bases of lengths $2\pi x$ and $2\pi(x +\ dx)$.

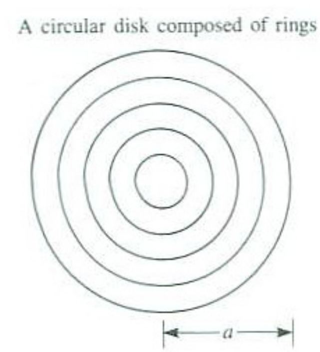


Figure 6.3.16:

5.[R] Think of a circular disk of radius a as being composed of concentric circular rings, as in Figure 6.3.16.

(a) Using the shorhand approach, set up a definite integral for the area of the disk. (Draw a good picture of the local approximation.)

(b) Evaluate the integral in *(a)*.

Exercises 6 to 8 to concern the volumes of the given solids. In each case *(a)* draw a good picture of the local approximation of widith dx, *(b)* set up the appropriate definite integral, and *(c)* evaluate the integral.

6.[R] A right circular cone of radius a and height h.

7.[R] A pyramid with a square base of side a and of height h. Its top vertex is above one corner of the base. (Use square cross sections.)

8.[R] A pyramid with a triangular base of area A and of height h. (The triangle need not be equilateral. See Figure 6.3.17.)

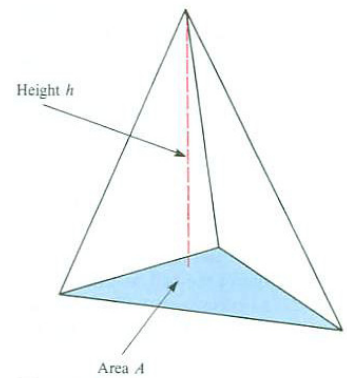


Figure 6.3.17:
Surface area of a sphere.

9.[C] Find the surface area of a sphere of radius a. HINT: Begin by estimating the area of the narrow band shown in Figure 6.3.18.

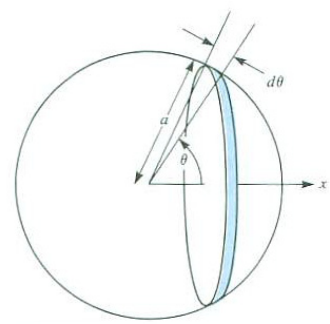


Figure 6.3.18:
The number t need not be an integer. For instance, $F(14.5)$ is the fraction who reach the age of 14 years 6 months.

10.[C] *[Actuarial tables]* Let $F(t)$ be the fraction of people born in 1900 who are alive t years later.

(a) What is $F(150)$, probably?

(b) What is $F(0)$?

(c) Sketch the general shape of the graph of $y = F(t)$.

(d) Let $f(t) = F'(t)$. (Assume F is differentiable.) Is $f(t)$ positive or negative?

(e) What fraction of the people born in 1900 die during the time interval $[t, t+dt]$? (Express your answer in terms of F.)

(f) Answer (e), but express your answer in terms of f.

(g) Evalute $\int_0^{150} f(t)\, dt$.

(h) What integral would you propose to call "the average life span of the people born in 1900"? Why?

11.[C] Let $F(t)$ be the fraction of ball bearings that wear out duing the first t hours of use. Thus $F(0) = 0$ and $F(t) \leq 1$.

(a) As t increases, what would you think happens to $F(t)$?

(b) Show that during the short interval of time $[t, t + \ dt]$, the fraction of ball bearings that wear out is approximately $F(t)\, dt$. (Assume F is differentiable.)

(c) Assume all wear out in at most $1,000$ hours. What is $F(1,000)$?

(d) Using the assumption in *(b)* and *(c)*, devise a definite integral for the average life of the ball bearings.

12.[M] At the time t hours, $0 \leq t \leq 24$, a firm uses electricity at the rate of $e(t)$ joules per hour. The rate schedule indicates that the cost per joule at time t is $c(t)$ dollars. Assume that both e and c are continuous functions.

(a) Estimate the cost of electricity consumed between times t and $t + dt$, where dt is a small positive number.

(b) Using *(a)*, set up a definite integral for the total cost of electricity for the 24-hour period.

Present Value

13.[M] The **present value** of a promise to pay one dollar t years from now if $g(t)$ dollars.

(a) What is $g(0)$?

(b) Why is it reasonable to assume that $g(t) \leq 1$ and that g is a decreasing function of t?

(c) What is the present value of a promise to pay q dollars t years from now?

(d) Assume that an investment made now will result in an income flow at the rate of $f(t)$ dollars per year t years from now. (Assume that f is a continuous function.) Estimate informally the present value of the income to be earned between time t and time $t + dt$, where dt is a small positive.

(e) On the basis of the local estimate made in (d), set up a definite integral for the present value of all the income to be earned from now to time b years in the future.

Population

14.[M] Let the number of females in a certain population in the age range from x years to $x + dx$ years, where dx is a small positive number, be approximately $f(x)\ dx$. Assume that, on average, women of age x produce $m(x)$ offspring during the year before they reach age $x + 1$ Assume that both f and m are continuous functions.

(a) What definite integral represents the number of women between ages a and b years?

(b) What definite integral represents the total number of offspring during the calendar year produced by women whose ages at the beginning of the calendar year were between a adn b years?

Exercises 15 to 20 concern **kinetic energy**. They are all based on the concept that a particle of mass M moving with velocity V has the kinetic energy $MV^2/1$. (See Example 3.) An object whose density is the same at all its points is called **homogeneous**. If the object is planar, such as a square or disk, and has mass M kilograms and area A square meters, its density is M/A kilograms per sqaure meter.

15.[M] The piece of sheet metal in Example 3 is rotated around the line midway between the edges AB and CD at the rate of 5 revolutions per second.

(a) Using the informal approach, obtain a local approximation for the kinetic energy of a narrow strip of the metal.

(b) Using (a), set up a definite integral for the kinetic energy of the piece of sheet metal.

(c) Evaluate the integral in (b).

16.[M] A circular piece of metal of radius 7 meters has a density of 3 kilograms per square meter. It rotates 5 times per second around an axis perpendicular to the circle and passing through the center of the circle.

(a) Devise a local approximation for the kinetic energy of a narrow ring in the circle.

(b) With the aid of (a), set up a definite integral for the kinetic energy of the rotating metal.

(c) Evaluate the integral in (b).

17.[M] The density of a rod x centimeters from its left end is $g(x)$ grams per centimeter. The rod has a length of b centimeters. The rod is spun around its left end 7 times per second.

(a) Estimate the mass of the rod in the section that is between x and $x + dx$ centimeters from the left end. (Assume that dx is small.)

(b) Estimate the the kinetic energy of the mass in (a).

(c) Set up a definite integral for the kinetic energy of the rotating rod.

18.[M] A homogeneous square of mass M kilograms and side a meters rotates around an edge 5 times per second.

(a) Obtain a "local estimate" of the kinetic energy. What part of the square would you use? Why? Draw it.

(b) What is the local estimate?

(c) What definite integral represents the total kinetic energy of the square?

(d) Evaluate it.

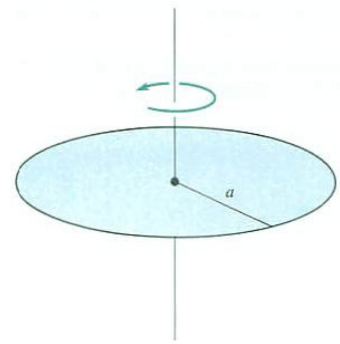


Figure 6.3.19:

19.[M] Like Exercise 18, but this time the square is spun around a line through its center and parallel to an edge.

20.[M] Like Exercise 18, for a disk of radius a and mass M spinning around a line through its center and perpendicular to it. It is spinning at the rate of ω radians per second. (See Figure 6.3.19.)

In Exercises 21 and 22 you will meet definite integrals that cannot be evaluated by the Fundamental Theorem of Calculus (since the desired antiderivative is not elementary). Use *(a)* the trapezoidal and (b) Simpson's method with six sections to estimate the definite integrals.

21.[M] A homogeneous object of mass M occupies the region under $y = e^{x^2}$ and above $[0, 1]$. It is spun at the rate of ω radians per second around the y axis. Estimate its kinetic energy.

22.[M] A homogeneous object of mass M occupies the region under $y = (\sin x)/x$ and above $[\pi/2, \pi]$. It is spun around the line $x = 1$ at the rate of ω radians per second. Estimate its kinetic energy.

In each of Exercises 23 to 26, find the kinetic energy of a planar homogeneous object that occupies the given region, has mass M, and is spun around the y axis ω radians per second.

23.[M] The region under $y = e^2$ and above the interval $[1, 2]$.

24.[M] The region under $y = \tan^{-1} x$ and above the interval $[0, 1]$.

25.[M] The region under $y = 1/(1 + x)$ and above $[2, 4]$.

26.[M] The region under $y = \sqrt{1 + x^2}$ and above $[0, 2]$.

27.[M] A solid homogeneous right circular cylinder of radius a, height h, and mass M is spun at the rate of ω radians per second around its axis. Finite kinetic energy. (Include a good picture on which your local approximation is based.)

28.[M] A solid homogeneous ball of radius a and mass M is spun at the rate of ω radians per second around a diameter. Find its kinetic energy. (Include a good picture on which your local approximation is based.)

Doug, Omit the following exercise?? too long and too obvious?

29.[C] (*Beware*) Consider the following argument: "Approximate the surface area of the sphere of radius a shown in Figure 6.3.20 as follows. To approximate the surface area between x and $x + dx$, let us try using the area of the narrow curved part of the cylinder used to approximate the volume between x and $x + dx$. (This part is shaded in Figure 6.3.20.) This local approximation can be pictured (when unrolled and laid flat) as a rectangle of width dx and length $2\pi r$. The surface area of a sphere is $\int_{-a}^{a} 2\pi r\ dx = 4\pi \int_0^a \sqrt{a^2 - x^2}\ dx$. But $\int_0^a \sqrt{a^2 - x^2}\ dx = \pi a^2/4$, since it equals the area of a quadrant of a disk. Hence the area of the sphere is $\pi^2 a^2$." This does not agree with the correct value, $4\pi a^2$, which was discovered by Archimedes in the third century B.C. What is wrong with this argument?

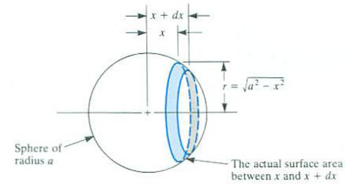


Figure 6.3.20:

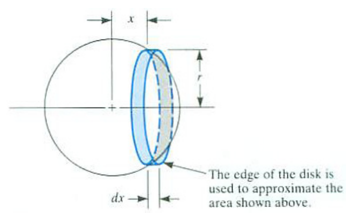


Figure 6.3.21:

30.[C] (*Poiseuille's law of blood flow* A fluid flowing through a pipe does not all move at the same velocity. The velocity of any part of the fluid depends on its distance from the center of the pipe. The fluid at the center of the pipe moves fastest, whereas the fluid near the wall of the pipe moves slowest. Assume that the velocity of the fluid at a distance x centimeters from the axis of the pipe is $g(x)$ centimeters per second.

(a) Estimate the flow of fluid (in cubic centimeters per second) through a thin ring of inner radius r and outer radius $r + dr$ centimeters centered at the axis of the pipe and perpendicular to the axis.

(b) Using *(a)*, set up a definite integral for the flow (in cubic centimeters per second) of fluid through the pipe. (Let the radius of the pipe be b centimeters.)

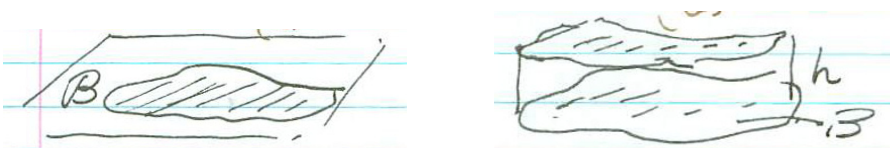
(c) Poiseuille (1797-1869), studying the flow of blood through arteries, used the function $g(r) = k(b^2 - r^2)$, where k is a constant. Show that in this case the flow of blood through an artery is proportional to the fourth power of the radius of the artery.

31.[C] The density of the earth at a distance of r miles from its center is $g(r)$ pounds per cubic mile. Set up a definite integral for the total mass of the earth. (Take the radius of the earth to be $4,000$ miles.)

6.4 Computing Volumes by Parallel Cross-Section

In Section 5.1 we computed areas by integrating lengths of cross-sections made by parallel lines. In this section we will use a similar approach, finding volumes by integrating areas of cross-sections made by parallel planes.

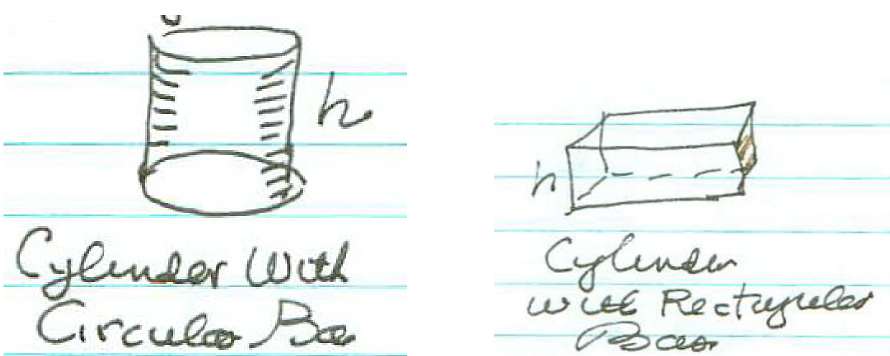
Cylinders



(a) (b)

Figure 6.4.1:

Let $\mathcal{B}$ be a region in the plane and h a positive number. The **cylinder with base $\mathcal{B}$ and height h** consists of all line segments of length h perpendicular to $\mathcal{B}$, one end of which is in $\mathcal{B}$ and the other end is on a fixed side (above or below) of $\mathcal{B}$. This typical cylinder is shown in Figure 6.4.1(b). The top of the cylinder is congruent to $\mathcal{B}$.



(a) (b)

Figure 6.4.2:

If $\mathcal{B}$ is a disk, the cynlinder is the customary circular cylinder of daily life — the can, shown in Figure 6.4.2(a). If $\mathcal{B}$ is a rectangle, the cylinder is a rectangular box, as shown in Figure 6.4.2(b).

We will make use of the formula for the volume of a cylinder:

The volume of a cylinder with base B and height h is

$$V = \text{Area of Base} \times h.$$

Volume as the Definite Integral of Cross-Sectional Area

Let's use the informal approach for setting up a definite integral to see how to use integration to calculate volumes of solids.

Consider the solid region $\mathcal{R}$ shown in Figure 6.4.3, which lies between the planes perpendicular to the x-axis at $x = a$ and at $x = b$. We use a cylinder to estimate the volume of the part of $\mathcal{R}$ that lies between two parallel planes a "small distance" Δx apart, shown in perspective in Figure 6.4.4. This thin slab is not usually a cylinder. However, we can approximate it by a cylinder. To do this, let x be, say, the left endpoint of an interval of width Δx. The plane perpendicular to the x-axis at x intersects $\mathcal{R}$ in a plane cross-section of area $A(x)$. The cylinder whose base is that cross-section and whose height is Δx is a good approximation of the part of $\mathcal{R}$. It is the this slab shown in Figure 6.4.5.

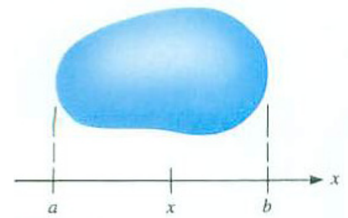


Figure 6.4.3: SHERMAN: Redraw?

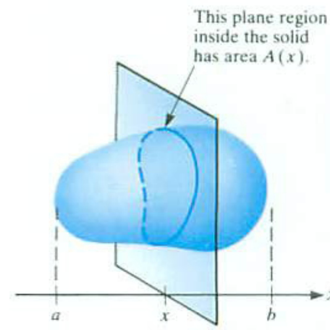


Figure 6.4.4:

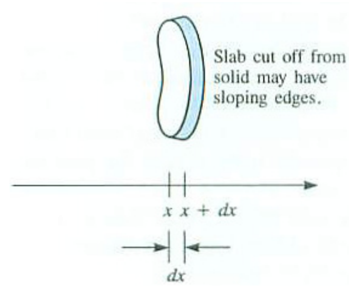


Figure 6.4.5:

We therefore have

$$\text{Local Approximation to Volume} = A(x)\Delta x.$$

Then

$$\text{Volume of Solid} = \int_a^b A(x)\ dx.$$

In short, "volume equals the integral of cross-sectional area". To apply this idea, we must compute $A(x)$. That is a where good drawings come in handy.

Given a particular solid, one just has to find a, b and the cross-sectional area $A(x)$ in order to construct a definite integral for the volume of the solid. These are the steps for finding the volume of a solid:

1. Choose a line to serve as an x-axis.

2. For each plane perpendicular to that axis, find the area of the cross-section of the solid made by the plane. Call this area $A(x)$.

3. Determine the limits of integration, a and b, for the region.

4. Evaluate the definite integral $\int_a^b A(x)\ dx$.

See Figure 6.4.3.

See Figure 6.4.4.

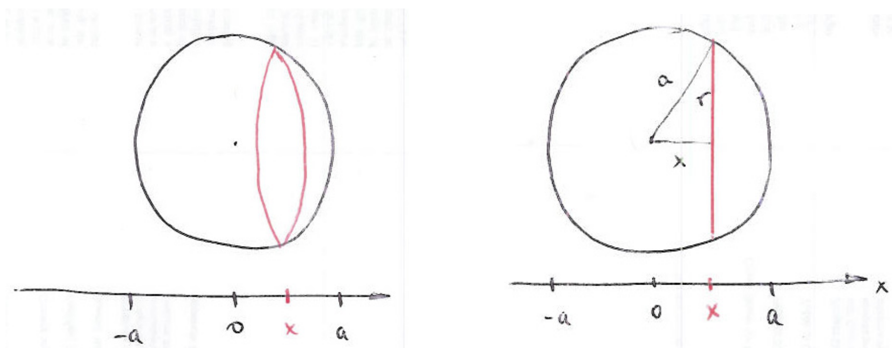
Most of the effort is usualy spent in finding the integrand $A(x)$.

In addition to the Pythagorean Theorem and the properties of similar triangles, formulas for the areas of familiar plane figures may be needed. Also keep in mind that if corresponding dimensions of similar figures have a ratio k, then their areas have the ratio k^2; that is, the area is proportional to the square of the ratios of the lengths of corresponding line segments.

Formula for the area of familiar plane regions are on the inside back cover.

EXAMPLE 1 Find the volume of a ball of radius a.

SOLUTION We sketch the typical cross-section in perspective and in side



(a) (b)

Figure 6.4.6:

view (see Figure 6.4.6(a) and (b), respectively). The cross-section is a disk of radius r, which depends on x. The area of the cross-section is πr^2. To express this area in terms of x, use the Pythagorean Theorem, which tells us that $a^2 = x^2 + r^2$, hence $r^2 = a^2 - x^2$. So we have

$$\begin{aligned}
\text{Volume} &= \int_{-a}^{a} \pi(a^2 - x^2)\,dx \\
&\underset{\text{FTC}}{=} \pi\left(a^2x - \frac{x^3}{3}\right)\Bigg|_{-a}^{a} \\
&= \pi\left(\left(a^3 - \frac{a^3}{3}\right) - \left((-a)^3 - \frac{(-a)^3}{3}\right)\right) \\
&= \frac{4\pi}{3}a^3.
\end{aligned}$$

◇

Archimedes was the first person to find the volume of a ball. He did not express the volume as a number. Rather, in the style of mathematics of the 3rd century BC, he expressed the volume in terms of the volume of a simpler object: the volume of a ball is two-thirds the volume of the smallest cylinder that contains it. That he considered this one of his greatest accomplishments is evidenced by his request that his tomb be topped with a carving of a ball within a cylinder.

See Figure 6.2.2(?) in Section 6.2.

The next example concerns the solid region discussed in Example 3 of Section 6.2.

EXAMPLE 2 A cylindrical glass of height h and radius a is full of water. It is tilted until the remaining water covers exactly half the base. Find the volume of the remaining water.

SOLUTION We use the triangular cross-section shown in Figure 6.2.7. Introduce the x-axis as in Figure 6.4.7. It was shown that the area of the cross-section at x is $\frac{h}{2a}(a^2 - x^2)$. Thus,

See Figure 6.2.7 on page 348.

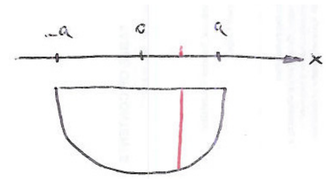


Figure 6.4.7:

$$\begin{aligned}
\text{Volume} &= \int_{-a}^{a} \frac{h}{2a}(a^2 - x^2)\, dx \\
&\underset{\text{FTC}}{=} \frac{h}{2a}\left(a^2 x - \frac{x^3}{3}\right)\Big|_{-a}^{a} \\
&= \frac{h}{2a}\left(\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right) \\
&= \frac{h}{2a}\left(\frac{4}{3}a^3\right) \\
&= \frac{2}{3}ha^2.
\end{aligned}$$

That's about 21% of the volume of the glass.

This calculation of the integral could be simplified by noting that the integrand is an even function (the volume to the right of 0 equals the volume to the left of 0). In this method we have

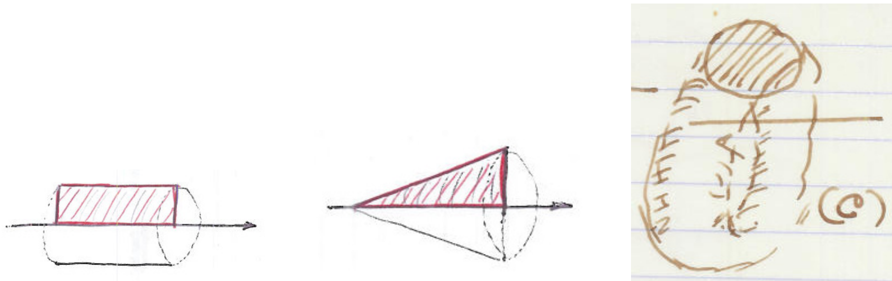
$$\begin{aligned}
\text{Volume} &= 2\int_{0}^{a} \frac{h}{2a}\left(a^2 - x^2\right)\, dx \\
&= \frac{h}{a}\left(a^2 x - \frac{x^3}{3}\right)\Big|_{0}^{a} \\
&= \frac{h}{a}\left(\left(a^3 - \frac{a^3}{3}\right) - (0 - 0)\right) \\
&= \frac{2}{3}ha^2
\end{aligned}$$

The second way, by avoiding a lot of arithmetic with negative numbers, reduces the chance of making a mistake. ◇

Solids of Revolution

The solid formed by revolving a region $\mathcal{R}$ in the plane about a liner in that plane that does not intersect the interior of $\mathcal{R}$ is called a **solid of revolution**.

Figure 6.4.8 shows three examples: *(a)* a circular cylinder obtained by revolving a rectangle about one of its edges, *(b)* a cone obtaine by revolving a



(a) (b) (c)

Figure 6.4.8:

right triangle about one of its two legs, and *(c)* a torus ("doughnut" or "ring") formed by revolving a disk about a line outside the disk.

The cross-sections by planes perpendicular to the line around which the figure is revolved is either a disk or an annulus. The latter resembles a washer, being a disk with a round hole. The cross-sections in Figure 6.4.8(a) and (b) are disks. In Figure 6.4.8(c) the cross-sections are washers. Figures 6.4.9 and 6.4.10 show that the typical cross-section is a washer.

Figure 6.4.9:

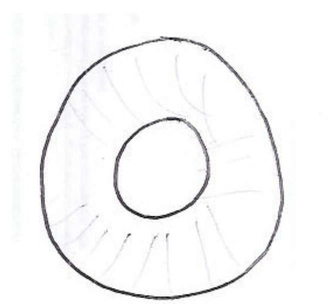


Figure 6.4.10:

EXAMPLE 3 The region under $y = e^{-x}$ and above $[1, 2]$ is revolved about the x-axis. Find the volume of the resulting solid of revolution. (See Figure 6.4.11.)

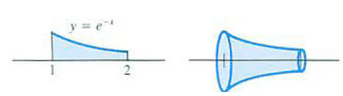


Figure 6.4.11:

SOLUTION The typical cross-section by a plane perpendicular to the x-axis is a disk of radius e^{-x}, as shown in Figure 6.4.12. The cross-sectional area is

$$\pi \left(e^{-x}\right)^2 = \pi e^{-2x}.$$

The volume of the solid is therefore

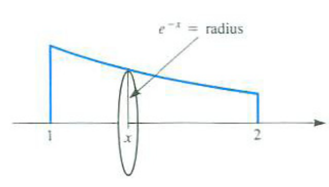
$$\int_1^2 \pi e^{-2x}\, dx.$$

Recall that $\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$, so that an antiderivative of e^{ax} is $\frac{1}{a}e^{ax}$. Hence,

$$\begin{aligned}\int_1^2 \pi e^{-2x}\, dx &= \left.\frac{\pi}{-2}e^{-2x}\right|_1^2 = \frac{\pi}{-2}\left(e^{-4} - e^{-2}\right)\\ &= \frac{\pi}{2}\left(e^{-2} - e^{-4}\right).\end{aligned}$$

◇

EXAMPLE 4 Let $\mathcal{R}$ be the region in the x-y plane under the graph of $y = \frac{e^x}{\sqrt{x}}$ and above the interval $[1, 2]$.



A. Set up a definite integral for the area of $\mathcal{R}$.

B. Set up a definite integral for the volume obtained by revolving $\mathcal{R}$ around the x-axis.

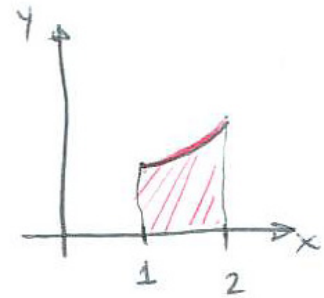


Figure 6.4.13:

SOLUTION Figure 6.4.13 shows the region $\mathcal{R}$.

A. The area of $\mathcal{R}$ is the integral of the length of the cross-section:

$$\text{Area of } \mathcal{R} = \int_1^2 \frac{e^{-x}}{\sqrt{x}}\, dx.$$

B. The volume enclosed when $\mathcal{R}$ is revolved around the x-axis is

$$\text{Volume} = \int_1^2 \pi \left(\frac{e^{-x}}{\sqrt{x}}\right)^2 dx = \pi \int_1^2 \frac{e^{-2x}}{x}\, dx. \tag{6.1}$$

See Exercise 31.

Verify that this was done in 2.6 or 5.4.

The antiderivative of $\frac{e^x}{\sqrt{x}}$ is elementary, as we saw in Section 2.6 (or Section 5.4. So we can use the FTC to evaluate the area of $\mathcal{R}$. However, as mentioned in Section 2.6, $\frac{e^x}{x}$ does not have an elementary antiderivative (see Exercise 57). To estimate (6.1) we must resort to one of the approximation techniques of Section 5.5 or turn to our trusty calculator. ◇

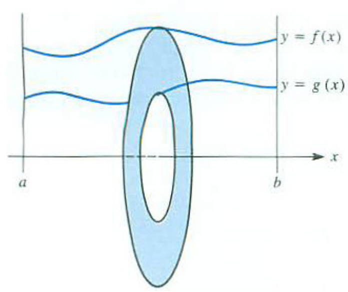


Figure 6.4.14:

Do not memorize this formula!

If the region being revolved around the x-axis is bounded by two curves $y = f(x)$ and $y = g(x)$ $f(x) \geq g(x)$, then we have to take into account the hole in the resulting solid of revolution. The cross-sections perpendicular to the x-axis are no longer disks, but "washers". Figure 6.4.14 shows that the cross-section has area

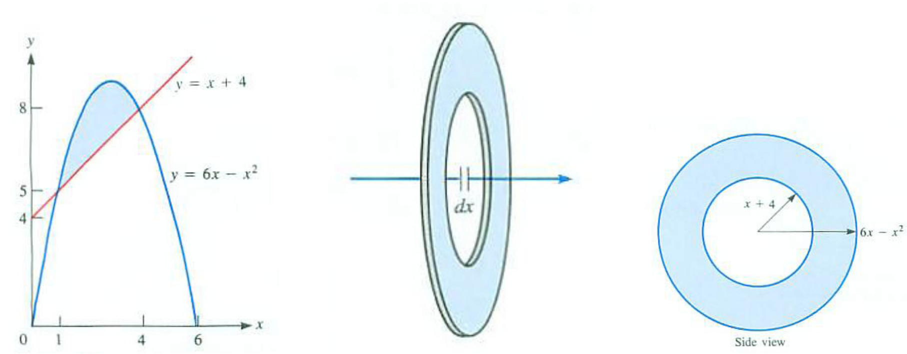
$$\pi\,(f(x))^2 - \pi\,(g(x))^2\,,$$

the difference of the areas of the disks. Hence the volume is

$$\int_a^b \pi\left((f(x))^2 - (g(x))^2\right)\, dx.$$

EXAMPLE 5 The region shown in Figure 6.4.15(a) is revolved about the x-axis to form a solid of revolution. Express the volume as a definite integral.

SOLUTION We first draw a local approximation to a thin slice of the solid



(a) (b) (c)

Figure 6.4.15:

(see Figure 6.4.15(b)). The side view in Figure 6.4.15(c) shows the area of the typical cross-section is

$$\pi\left(6x-x^2\right)^2-\pi\left(x+4\right)^2.$$

This is the integrand. Next we find the interval of integration $[a, b]$. The ends of the interval are determined by where the curves cross, that is, when

$$\begin{array}{rrcl} & x+4 & = & 6x-x^2 \\ \text{or} & x^2-5x+4 & = & 0. \\ \text{Hence} & (x-1)(x-4) & = & 0, \\ \text{so} & x=1 & \text{or} & x=4, \end{array}$$

and the volume of the solid is given by the definite integral

$$\int_1^4\left(\pi\left(6x-x^2\right)^2-\pi\left(x+4\right)^2\right)\,dx.$$

◇

Summary

The key idea in this section is that "volume is the definite intgral of cross-sectional area". To complement this idea we have to find that varying area and also the interval of integration.

EXERCISES for 6.4 *Key:* R–routine, M–moderate, C–challenging

NOTE: Need exercises that involve transcendental functions.

In Exercises 1 to 8, *(a)* draw the solid, *(b)* draw the typical cross-section, *(c)* find the area of the typical cross-section, *(d)* set up the definite integral for the volume, and *(e)* evaluate the definite integral (if possible).

1.[R] Find the volume of a cone of radius a and height h.

2.[R] The base of a solid is a disk of radius 3. Each plane perpendicular to a given diameter meets the solid in a square, one side of which is in the base of the solid. (See Figure 6.4.16.) Find its volume.

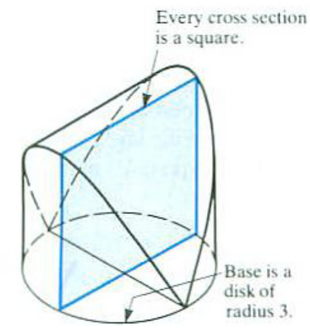


Figure 6.4.16:

3.[R] The base of a solid is the region bounded by $y = x^2$, the line $x = 1$, and the x- and y-axes. Each cross-section perpendicular to the x-axis is a square. (See Figure 6.4.17.) Find the volume of the solid.

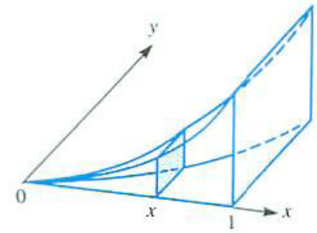


Figure 6.4.17:

4.[R] Find the volume of a pyramid with a square base of side a and height h, using square cross-sections. The top of the pyramid is above the center of the base.

5.[R] Repeat Exercise 3 except that the cross-sections are equilateral triangles.

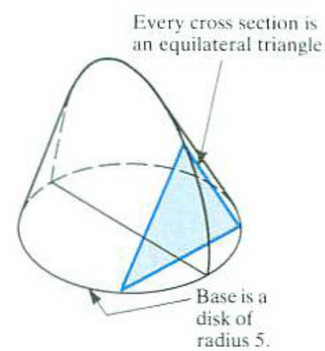


Figure 6.4.18:

6.[R] Repeat Exercise 4, but using trapezoidal cross-sections.

7.[R] Find the volume of the solid whose base is the disk of radius 5 and whose cross-sections perpendicular to the x-axis are equilateral triangles. (See Figure 6.4.18.)

8.[R] Find the volume of the pyramid shown in Figure 6.4.19 by using cross-sections perpendicular to the edge of length c.

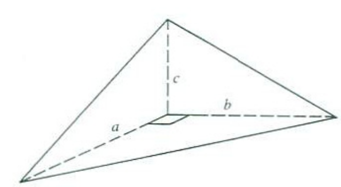


Figure 6.4.19:

In Exercises 9 to 14 set up a definite integral for the volume of the solid formed by revolving the given region R about the given axis.

9.[R] R is bounded by $y = \sqrt{x}$, $x = 1$, $x = 2$, and the x-axis, about the x-axis.

10.[R] R is bounded by $y = \frac{1}{\sqrt{1+x^2}}$, $x = 0$, $x = 1$, and the x-axis, about the x-axis.

11.[R] R is bounded by $y = x^{-1/2}$, $y = x^{-1}$, $x = 1$, and $x = 2$, about the x-axis.

12.[R] R is bounded by $y = x^2$ and $y = x^3$, about the y-axis.

DOUG: Be sure these integrals are evaluated in Chapter 7 - Examples, Exercises, Chapter Summary.

13.[R] R is bounded by $y = \tan(x)$, $y = \sin(x)$, $x = 0$, and $x = \pi/4$, about the x-axis.

14.[R] R is bounded by $y = \sec(x)$, $y = \cos(x)$, $x = \pi/6$, and $x = \pi/3$, about the x-axis.

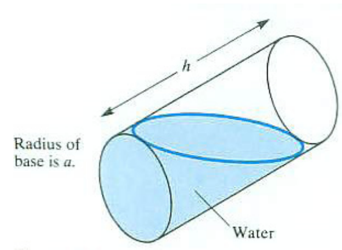


Figure 6.4.20:

15.[R] A cylindrical drinking glass of height h and radius a, full of water, is tilted until the water just covers the base. Set up a definite integral that represents the amount of water left in the glass. Use rectangular cross-sections. Refer to Figure 6.4.20 and follow the directions preceding Exercise 1.

DOUG: Be sure to evaluate these integrals in Chapter 7

DOUG: Evaluate in Chapter 7

16.[R] Repeat Exercise 15, but use trapezoidal cross-sections.

17.[R] Repeat Exercise 15 by common sense. Don't use any calculus at all.

DOUG: Evaluate in Chapter 7

18.[R] A cylindrical drinking glass of height h and radius a, full of water, is tilted until the water remaining covers half the base. Set up a definite integral for the volume of water in the glass, using triangular cross-sections.

Evaluate in Chapter 7

19.[R] Repeat Exercise 18, but use rectangular cross-sections.

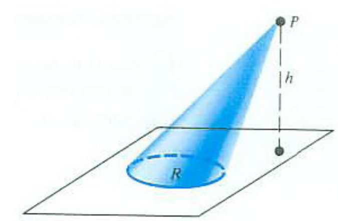


Figure 6.4.21:

20.[M] A solid is formed in the following manner. A plane region R and a point P not in the plane are given. The solid consists of all line segments joining P to points in R. If R has area A and P is a distance h from the plane R, show that the volume of the solid is $Ah/3$. (See Figure 6.4.21.)

21.[M] A drill of radius 4 inches bores a hole through a wooden sphere of radius 5 inches, passing symmetrically through the center of the sphere.

(a) Draw the part of the sphere removed by the drill.

(b) Find $A(x)$, the area of a cross-section of the region in *(a)* made by a plane perpendicular to the axis of the drill and at a distance x from the center of the sphere.

(c) Set up the definite integral for the volume of wood removed.

22.[M] What fraction of the volume of a sphere is contained between parallel planes that trisect the diameter to which they are perpendicular? (Leave your answer in terms of a definite integral.)

23.[M] The disk bounded by the circle $(x-b)^2+y^2=a^2$, where $0<a<b$, is revolved around the y-axis. Set up a definite integral for the volume of the doughnut (torus) produced.

Contributed by Archimedes.

24.[C] Set up a definite integral for the volume of one octant of the region common to two right circular cylinders of radius 1 whose axes intersect at right angles, as shown in Figure 6.4.22.

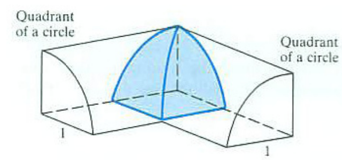


Figure 6.4.22:

25.[C] When a convex region R of area A situated to the right of the y-axis is revolved around the y-axis, the resulting solid of revolution has volume V. When R is revolved around the line $x=-k$, the volume of the resulting solid is $V*$. Express $V*$ in terms of k, A, and V.

DOUG/SHERMAN: Provide definition of convex.

26.[M] Find the volume in Example 2 using rectangular cross-sections.

Move to Chapter 7?

In Exercises 27 to 30 set up definite integrals for *(a)* the area of R, *(b)* the volume formed when R is revolved around the x-axis, and *(c)* the volume formed when R is revolved around the y-axis.

27.[M] R is the region under $y=\tan(x)$ and above the interval $[0,\pi/4]$.

28.[M] R is the region under $y=e^x$ and above the interval $[-1,1]$.

29.[M] R is the region under $y=1/\sqrt{1-x^2})$ and above the interval $[0,1/2]$.

30.[M] R is the region under $y=\sin(x)$ and above the interval $[0,\pi]$.

31.[R] The volume enclosed when the region under $y=\frac{e^x}{\sqrt{x}}$ and above the interval $[1,2]$ is revolved around the x-axis is given by

See Example 4.

$$\pi\int_1^2 \frac{e^{2x}}{x}\,dx.$$

(a) Find an antiderivative of $\frac{e^{2x}}{x}$.

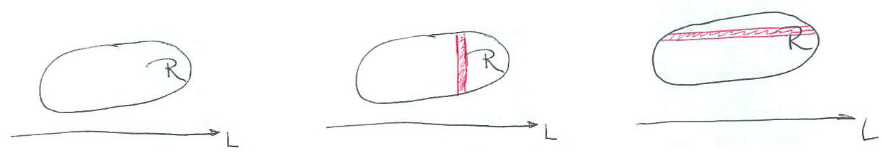
(b) Evaluate the definite integral.

32.[C] Archimedes viewed a ball as a cone whose height is the radius of the ball and whose base is the surface of the ball. On that basis he computed that the volume of the ball is one third the product of the radius and the surface area. He then gave a rigorous proof of his conjecture.

Clever Sam, inspired by this, said "I'm going to get the volume of a circular cylinder in a new way. Say its radius is r and height is h. Then I'll view it as a cylinder with a rectangular base r by h and height $2\pi r$. So the volume would be $2\pi r$ times rh, or $2\pi r^2 h$. That's twice the usual volume, so the usual volume must be wrong." Is Sam right? (Explain.)

6.5 Computing Volumes by Coaxial Shells

The key to finding the volume of a solid of revolution is to identify a cross-section whose area is known. In Section 6.4 the cross-sections were perpendicular to the axis of revolution. In this section we will see how to use cross-sections that are parallel to the axis of revolution.



(a) (b) (c)

Figure 6.5.1:

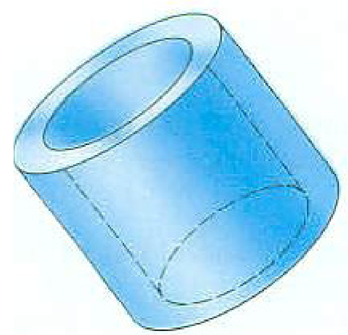


Figure 6.5.2:

Imagine revolving the planar region $\mathcal{R}$ about the line L, as in Figure 6.5.1(a). We may think of $\mathcal{R}$ as being formed from narrow strips parallel to L, as in Figure 6.5.1(b). Revolving such a piece around L produces a washer (or disk). This is the approach used in the preceding section.

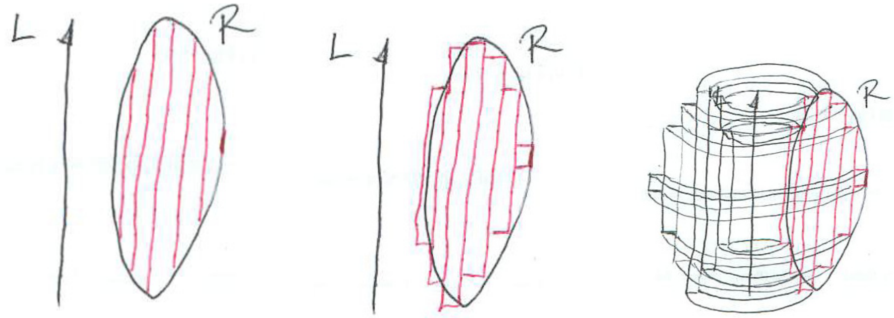
However, we can also think of $\mathcal{R}$ as being formed from narrow strips *parallel* to L, as in Figure 6.5.1(c). Revolving such a piece around L produce a solid shaped like a bracelet or part of a drinking straw, as shown, in perspective, in Figure 6.5.2. We will call such a solid a **shell**. Perhaps "tube" or "pipe" might be a better choice, but "shell" is standard in the world of calculus.

This section describes how to find the volume of a solid of revolution using shells (instead of disks).

The Shell Technique

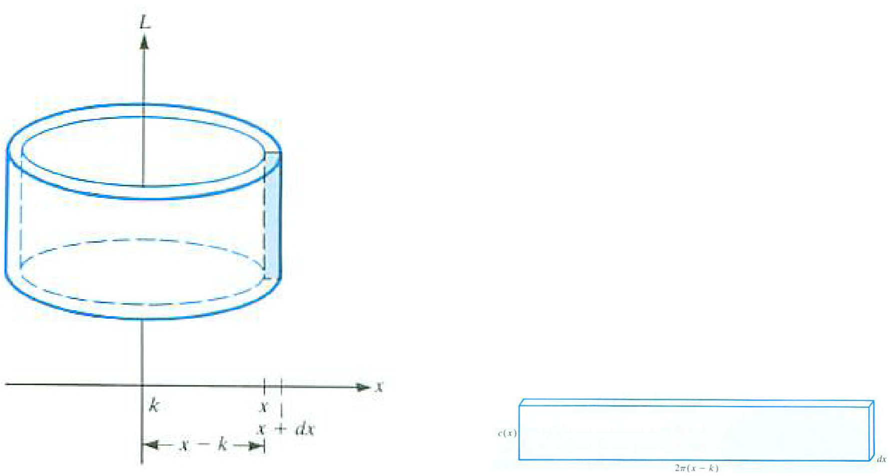
To apply the shell technique we first imagine cutting the plane region $\mathcal{R}$ in Figure 6.5.3(a) into a finite number of narrow strips by lines parallel to L. Then we approximate teh solid of revolution by a collection of tubes (like the parts of a collapsible telescope), as in Figure 6.5.3(c).

The key to the method is estimating the volume of each of these shells. Figure 6.5.4(a) shows the typical local approximation. Its height, $c(x)$, is the length of the cross-section of $\mathcal{R}$ corresponding to the value x on a line that we will call the x-axis. The radius of the shell is x. Imagine cutting the shell along a direction parallel to L and then laying it flat as though it were a carpet. When laid flat, the shell resembles a thin slab of thickness dx, width $c(x)$, and length $2\pi x$, as shown in Figure 6.5.4(b). The volume of this shell, therefore, is presumably about



(a) (b) (c)

Figure 6.5.3:



(a) (b)

Figure 6.5.4:

$$\text{Local Approximation to Volume of a Shell} = 2\pi x c(x)\ dx \qquad (6.1)$$

The exact volume of the shell is found in Exercise 21.

Note that this is only approximate; when the thin tube is unrolled in Figure 6.5.4(b), the two ends of length $c(x)$ and thickness dx are not perpendicular to the large rectangle.

With the aid of the local approximation (6.1), we then conclude that the volume of the solid of revolution is

$$\text{Volume of Solid of Revolution} = \int_a^b 2\pi x c(x)\ dx. \qquad (6.2)$$

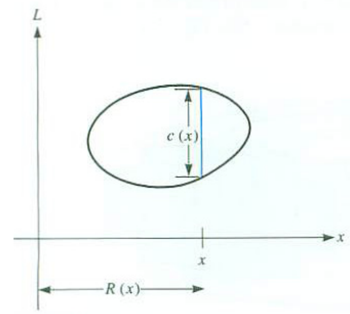


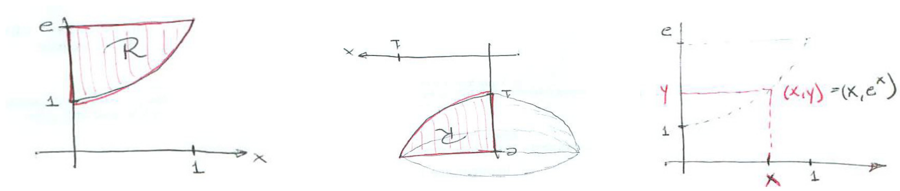
Figure 6.5.5:

This is the formula for computing the volumes by the shell technique. If x is denoted $R(x)$, the "radius of the shell" as in Figure 6.5.5, then

$$\text{Volume of Solid of Revolution} = \int_a^b 2\pi R(x)c(x)\ dx.$$

In our first example we will find the volume using cross-sections that are both perpendicular to the axis (disks) and parallel to the axis (shells).

EXAMPLE 1 The region $\mathcal{R}$ below the line $y = e$, above $y = e^x$, and to the right of the y-axis is revolved around the y-axis to produce a solid $\mathcal{S}$. Set up the definite integrals for the volume of $\mathcal{S}$ using *(a)* disks and *(b)* coaxial shells.



(a) (b) (c)

Figure 6.5.6:

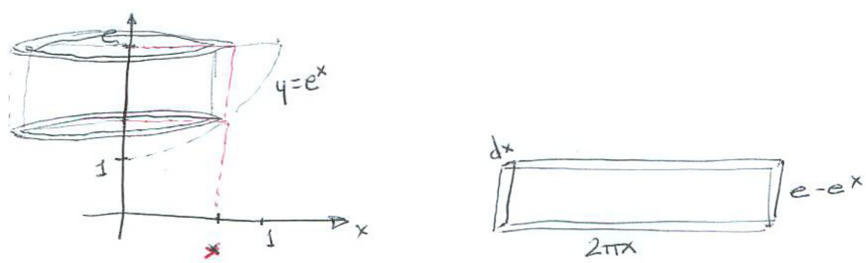
SOLUTION Figure 6.5.6(a) shows the region $\mathcal{R}$ and Figure 6.5.6(b) shows the solid $\mathcal{S}$.

(a) First, we use parallel cross-sections perpendicular to the y-axis. A typical cross-section is shown in Figure 6.5.6(c). The cross-section is a disk

whose radius is the x-coordinate of the point $(x, e^x$. Since $y = e^x$, x is $\ln(y)$. Thus the typical cross-section has area $\pi\left(\ln(y)\right)^2$. The volume is

$$\int_1^e \pi\left(\ln(y)\right)^2\ dy. \tag{6.3}$$

(b) Next we find the volume of $\mathcal{S}$ by coaxial shells. The typical shell is shown in Figure 6.5.7(a). Its radius is x. The height is $e - e^x$ and its thickness



(a) (b)

Figure 6.5.7:

is dx. Since the local approximation to the volume of the shell is

$$\underbrace{2\pi x}_{\text{circumference}}\ \underbrace{(e - e^x)}_{\text{height}}\ \underbrace{dx}_{\text{thickness}},$$

SHERMAN: I really like the local approximation language, but we need to find a way to talk about the differentials.

the volume of $\mathcal{S}$ is

$$\int_0^1 2\pi x\left(e - e^x\right)\ dx.$$

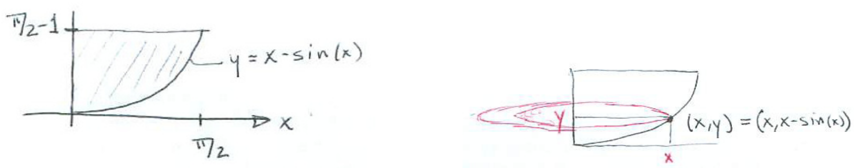
In Exercise 84 you will find that while the value of each integral is $(e-2)\pi$, the second one can be found in about half as much work. ◇

It is not unusual to find one formulation much easier than the other.

In Example 1 both methods were feasible. In the next, the shell technique is clearly preferable.

NOTE: The equation $y = x - \mathrm{e}\sin(x)$, known as **Kepler's equation**, is important in the study of the motion of planets. Here e is the eccentricity of an elliptical orbit, y is related to time, and x is related to an angle. For more information, visit `http://en.wikipedia.org/wiki/Keplerian_problem` or do a Google search for `Kepler equation`. See also Exercise 26. **EXAMPLE**

2 The region $\mathcal{R}$ bounded by the line $y = \frac{\pi}{2} - 1$, the y-axis, and the curve $y = x - \sin(x)$ is revolved around the y-axis. Attempt to set up definite integrals for the volume of this solid using *(a)* disks and *(b)* coaxial shells.



(a) (b)

Figure 6.5.8:

SOLUTION The region $\mathcal{R}$ is displayed in Figure 6.5.8(a).

(a) To use the method of parallel cross-sections you would have to find the radius of the typical disk shown in Figure 6.5.8(b). The radius for each value of y is the value of x for which $x - \sin(x) = y$. In other words, we have to express x as a function of y. This inverse function is not elementary, putting an end to our hopes of using the FTC. In fact, we will not pursue this approach further for this example.

For instance, when $y = 0$, then $x = 0$. When $y = \frac{\pi}{2} - 1$, then $x = \frac{\pi}{2}$.

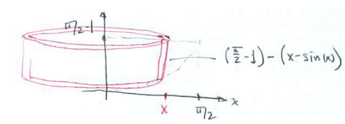


Figure 6.5.9:

(b) The shell technique goes through smoothly. The typical shell, shown in Figure 6.5.9, has radius x and height $\frac{\pi}{2} - 1 - (x - \sin(x))$. The volume of the local approximation is

$$\underbrace{2\pi x}_{\text{circumference}} \underbrace{\left(\frac{\pi}{2} - 1 - (x - \sin(x))\right)}_{\text{height}} \underbrace{dx}_{\text{thickness}} .$$

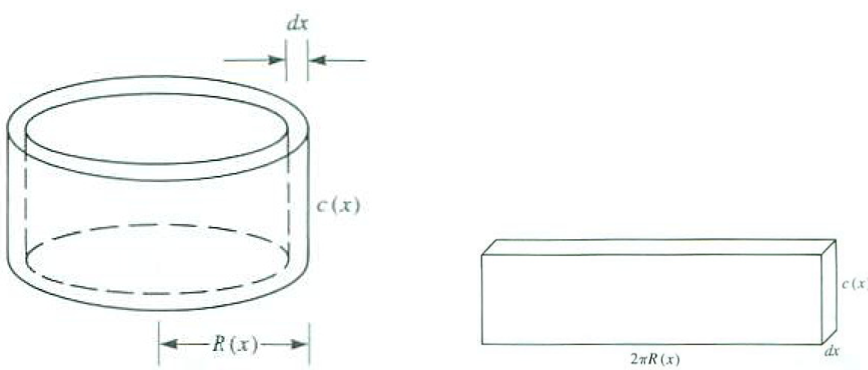
The total volume of the bowl is then given by the definite integral

$$\int_0^{\pi/2} 2\pi x \left(\frac{\pi}{2} - 1 - (x - \sin(x))\right) \, dx.$$

The value of this definite integral is found in Exercise 85. ◇

Summary

The volume of a solid of revolution may be found by approximating the solid by concentric thin shells. The volume of such a shell is aprosimately $2\pi R(x)\ c(x)\ dx$. (See Figure 6.5.8.) The shell technique is often useful when "slabs" or "washers" are difficult to set up.



(a) (b)

Figure 6.5.10:

EXERCISES for 6.5 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1–4 draw a typical approximating cylindrical shell for the solid described, and set up a definite integral for the given solid. **1.**[R] The trapezoid bounded by $y = x$, $x = 1$, $x = 2$, and the x-axis is revolved around the x-axis.

In the Chapter Review Exercises for Chapter 7 you are asked to evaluate the definite integrals found in these exercises.

2.[R] The trapezoid in Exercise 1 is revolved abou the line $x = -3$.

See Exercise 86 in Section 7.6.

3.[R] The triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ is revolved around the y-axis.

See Exercise 87 in Section 7.6.

See Exercise 88 in Section 7.6.

4.[R] The triangle in Exercise 3 is revolved abou the line x-axis.

See Exercise 89 in Section 7.6.

5.[R] Find a definite integral for the volume of the solid produced by revolving about the y-axis the finite region bounded by $y = x^2$ and $y = x^3$.

6.[R] Repeat Exercise 5, except the region is revolved around the y-axis.

7.[R] Find a definite integral for the volume of the solid produced by revolving about the x-axis the finite region bounded by $y = \sqrt{x}$ and $y = \sqrt[3]{x}$.

8.[R] Repeat Exercise 7, except the region is revolved about the y-axis.

9.[R] Find a definite integral for the volume of the right circular cone of radius a and height h by the shell method.

10.[R] Set up a definite integral for the volume of the doughnut (ring, torus) produced by revolving the disk of radius a about a line L at a distance $b > a$ from its center. (See Figure 6.5.11.)

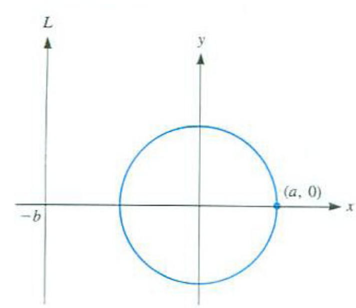


Figure 6.5.11:

11.[R] Let R be the region bounded by $y = x + x^3$, $x = 1$, $x = 2$, and the x-axis. Set up a definite integral for the volume of the solid produced by revolving R about *(a)* the x-axis and *(b)* the line $x = 3$.

12.[R] Set up a definite integral for the volume of the solid produced by revolving the region R in Exercise 11 about *(a)* the x-axis and *(b)* the line $y = -2$.

13.[R] Set up a definite integral for the volume of the solid of revolution formed by revolving the region bounded by $y = 2 + \cos(x)$, $x = \pi$, $x = 10\pi$, and the x-axis around *(a)* the y-axis and *(b)* the x-axis.

14.[R] The region below $Y = \cos(x)$, above the x-axis, and between $x = 0$ and $x = \frac{\pi}{2}$ is revolved around the x-axis. Find a definite integral for the volume of the resulting solid of revolution by *(a)* parallel cross-sections and *(b)* concentric shells.

15.[R] Let R be the region below $y = 1/\left(1+x^2\right)^2$ and above $[0, 1]$. Set up a definite integral for the volume of the solid produced by revolving R about the y-axis.

16.[R] The region between $y = e^{x^2}$, the x-axis, $x = 0$, and $x = 1$ is revolved about the y-axis.

(a) Set up a definite integral for the area of this region.

(b) Set up a definite integral for the volume of the solid produced.

It is interesting to note that the FTC is of no use in evaluating the *area* of this region.

17.[R] The region R below $y = e^x(1 + \sin(x))/x$ and above $[0, 10\pi]$ is revolved about the y-axis to produce a solid of revolution. *(a)* Find a definite integral for the volume of the solid by parallel cross-sections. *(b)* Find a definite integral for the volume of the solid by concentric shells. *(c)* Which definite integral do you think will be easier to evaluate? Why?

18.[R] Let R be the region below $y = \ln(x)$ and above $[1, e]$. Find a definite integral for the volume of the solid produced by revolving R about *(a)* the x-axis and *(b)* the y-axis.

19.[R] Let R be the region below $y = 1/\left(x^2 + 4x = 1\right)$ and above $[0, 1]$. Find a definite integral for the volume of the solid produced by revolving R about the line $x = -2$.

20.[R] Let R be the region below $y = 1/\sqrt{2 + x^2}$ and above $[\sqrt{3}, \sqrt{8}]$. Set up a definite integral for the volume of the solid produced by revolving R about the *(a)* the x-axis and (b) the y-axis.

21.[M] When we unrolled the shell as a carpet we pictured it as a rectangular solid whose faces meet at right angles. However, since the inner radius is x and the outer radius is $x + dx$ the circumference of the inside of the shell is less than the outer circumference.

(a) By viewing the shell as the difference between two circular cylinder, compute its exact volume.

(b) Show that this volume is $2\pi\left(x + \frac{dx}{2}\right)c(x)$.

This means that if we used $x+\frac{dx}{2}$ as our sampling number in the interval $[x, x+dx]$, our local approximation to the volume of the shell would be exact.

22.[C] When a region $\mathcal{R}$ in the first quadrant is revolved around the y-axis, a solid of volume 24 is produced. When $\mathcal{R}$ is revolved around the line $x = -3$, a solid of volume 82 is produced. What is the area of $\mathcal{R}$?

23.[C] Let $\mathcal{R}$ be a region in the first quadrant. When it is revolved around the x-axis, a solid of revolution is produced. When it is revolved around the y-axis, another solid of revolution is produced. Give an example of such a region $\mathcal{R}$ with the property that the volume of the first solid *cannot* be evaluated by the FTC, but the volume of the second solid can be evaluated by the FTC.

The **kinetic energy** of a particle of mass m grams moving at a velocity of v centimeters per second is $mv^2/2$ ergs. Exercises 24 and 25 ask for the kinetic energy of rotating objects. **24.**[M] A solid cylinder of radius r and height h centimeters has a uniform density of g grams per cubic centimeter. It is rotating at the rate of two revolutions per second around its axis.

(a) Find the speed of a particle at a distance x from the axis.

(b) Find a definite integral for the kinetic energy of the rotating cylinder.

25.[M] A solid ball of radius r centimeters has a uniform density of g grams per cubic centimeter. It is rotating around a diameter at the rate of three revolutions per second around its axis.

(a) Find the speed of a particle at a distance x from the diameter.

(b) Find a definite integral for the kinetic energy of the rotating ball.

SHERMAN: Include Kepler's equation in the section for Inverse Functions.

26.[M] In Example 2 the curve $y = x - \sin(x)$ appears.

(a) Show that this curve is increasing (as x increases).

(b) Graph the curve.

(c) Explain why, even though it cannot be found explicitly, you know this equation also can be solved for x as a function of y ($x = K(y)$).

(d) How are the graphs of $y = x - \sin(x)$ and $y = K(x)$ related?

6.6 Water Pressure Against a Flat Surface

This section shows how to use integration to compute the force of water against a submerged flat surface. The case of a curved surface will be treated in Section 15.6.

Introduction

Imagine the portion of the Earth's atmosphere directly above a square one inch on a side at sea level. That air forms a column some hundred miles high which weighs about 14.7 pounds per square inch (14.7 psi).

This is why astronauts wear pressurized suits.

This pressure does not crush us because the cells in our body are at the same pressure. If we were to into a vacuum, we would explode.

THe pressure inside a flat tire is 14.7 psi. When you pump up a bicyle tire so that the gauge reads 60 psi, the pressure is actually $60 + 14.7 = 74.7$ psi. The tire must be strong to avoid bursting.

Next imagine diving into a lake and descending 33 feet (10 meters). Extending that 100-mile high column into the water adds 14.7 pounds of water. The pressure is now twice 14.7 psi. The pressure is now twice 14.7, or 29.4 psi You cannot escape that pressure by turning your body, since at a given depth the pressure is the same in all directions.

If the force is the same throughout a region, then the pressure is simply "total force divided by area":

$$\text{pressure} = \frac{\text{force}}{\text{area}}.$$

Equivalently,

$$\text{force} = \text{pressure} \times \text{area}.$$

One cubic foot of water weighs 62.6 pounds, so one cubic inch weighs $\frac{62.6}{1728} =$ 0.036227 pounds and the density is 0.036227 pounds per cubic inch.

Thus, when the pressure is constant in a plane region it is easy to find the total force against it: multiply the pressure and the area of the region.

If the pressure varies in the region, we must make use of integration.

Using an Integral to Find the Force of Water

We will see how to find the total force as a flat submerged object due to the water. We will disregard the pressure due to the atmosphere.

One cubic inch of water weighs 0.036227 pounds. At a depth of h inches water therefore exerts a pressure of $0.036227h$ psi. Therefore the water exerts a force on a flat horizontal object of area $\mathcal{A}$ square inches, at a depth of h inches equal to $0.036227h\mathcal{A}$ pounds.

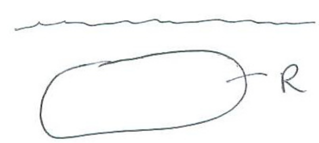
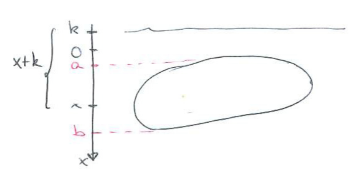


Figure 6.6.1:

Note the units: $\frac{\text{pounds}}{\text{inch}^3} \times \text{inch} = \frac{\text{pounds}}{\text{inch}^2}$.



To deal with, say, a vertical submerged surface takes more work, since the pressure is not constant. Imagine that surface $\mathcal{R}$, shown in Figure 6.6.1. Introduce a vertical x-axis, pointed down, with its origin $\mathcal{O}$, not necessarily at the water's surface. $\mathcal{R}$ lies between lines corresponding to $x = a$ and $x = b$. The depth of the water corresponding to x is not x but $x+k$. (See Figure 6.6.2.)

As usual, we will find the local approximation of the force by considering a narrow horizontal strip corresponding to the interval '$[x, x+dx]$ of the x-axis, as in Figure 6.6.3. Letting $c(x)$ denote the cross-secional length, we see that the force of the water on this strip is approximately

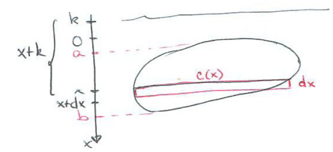
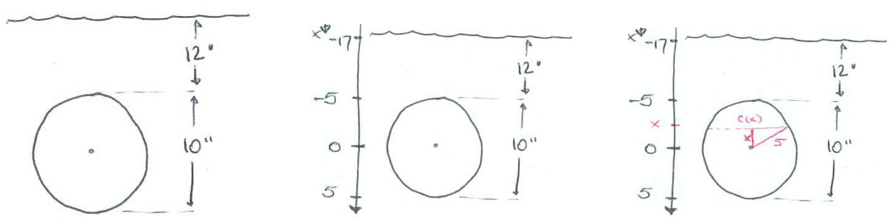


Figure 6.6.3:

$$\underbrace{(0.036227)}_{\text{density of } H_2O}\ \underbrace{(x+k)}_{\text{depth}}\ \underbrace{c(x)\ dx}_{\text{area of strip (approx)}} \quad \text{pounds.}$$

Therefore

Force against $\mathcal{R}$ is $\int_a^b (x+k)c(x)\ dx$ pounds.



(a) (b) (c)

Figure 6.6.4:

EXAMPLE 1 A circular tank is submerged in water. An end is a disk 10 inches in diameter. The top of the tank is 12 inches below the surface of the water. Express the force against one end in terms of one or more definite integrals.

SOLUTION The end of the tank is shown in Figure 6.6.4(a). Introduce a vertical x-axis with its origin $\mathcal{O}$ level with the center of the disk. (See Figure 6.6.4(b).) To find the cross-section $c(x)$ we use Figure 6.6.4(c).

This placement of $\mathcal{O}$ will make it easier to compute the cross-section lengths.

By the Pythagorean Theorem, applied to the right triangle in Figure 6.6.4(c) we have

For any number x, $|x|^2 = x^2$.

$$\left(\frac{c(x)}{2}\right)^2 + |x|^2 = 5^2.$$

Thus
$$\left(\frac{c(x)}{2}\right)^2 + x^2 = 5^2.$$

So
$$c(x) = \sqrt{100 - 4x^2}.$$

Having found the cross-section as a function of x, we still must find the depth as a function of x. To do this, inspect Figure 6.6.5.

The depth AC equals $AB + BC = 12 + (x - (-5)) = 17 + x$.

We have

$$\text{Local Estimate of Force} = \underbrace{(0.036227)(x+17)}_{\text{pressure}} \sqrt{100 - 4x^2}\ dx.$$

As a check, let $x = 0$, when the depth is clearly 17 inches.

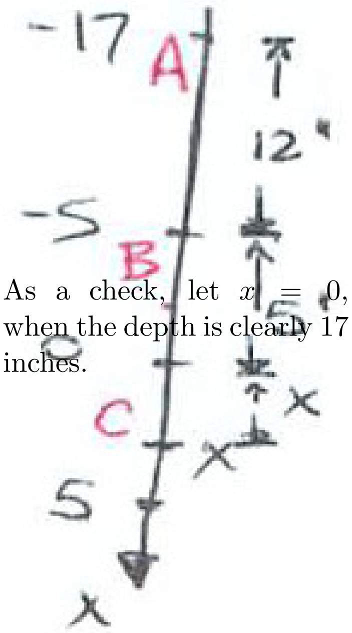


Figure 6.6.5:

From this we obtain

$$\begin{aligned}\text{Total Force} &= \int_{-5}^{5} (0.036227)(x+17)\sqrt{100-4x^2}\ dx \text{ pounds}\\ &= 0.036227 \int_{-5}^{5} x\sqrt{100-4x^2}\ dx + 0.036227 \int_{-5}^{5} 17\sqrt{100-4x^2}\ dx \text{ pounds.}\end{aligned}$$

The first integral is 0 because the integrand, $x\sqrt{100-4x^2}$, is an odd function and the interval of integration is symmetric about $x = 0$. The integrand in the second integral is even, but this does not give an explicit value for this integral. Instead, notice that $\sqrt{100-4x^2}$ is the length of the cross-section of the disk of radius 5 centered at the origin. This means

See Exercise 90 in Section 7.6.

$$\int_{-5}^{5} \sqrt{100-4x^2}\ dx = (\text{Area of Disk with Radius } r = 5) = \pi(5)^2.$$

Thus,

$$\begin{aligned}\text{Total Force} &= (0.036227)(17)(25\pi) \text{ pounds}\\ &\approx 48 \text{ pounds.}\end{aligned}$$

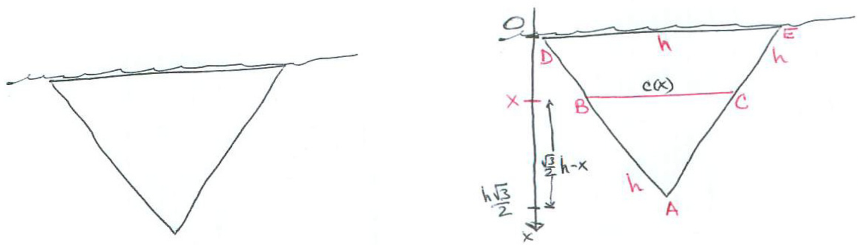
◇

EXAMPLE 2 Figure 6.6.6(a) shows a submerged equilaterial triangle of side h. Find the force of water against it.

SOLUTION In this case we place the origin of the vertical axis at the surface of the water (see Figure 6.6.6(b)). To set up an integral we must compute $c(x)$. Note $\frac{\sqrt{3}h}{2}$ is marked on the x-axis; it is the length of an altitude in the triangle.

The similar triangles ABC and ADE give us

$$\frac{c(x)}{h} = \frac{\frac{\sqrt{3}}{2}h - x}{\frac{\sqrt{3}}{2}h}.$$



(a) (b)

Figure 6.6.6:

Thus,

$$c(x) = h - \frac{2}{\sqrt{3}}x.$$

Observe that $c(0) = h$ and $c(\frac{\sqrt{3}}{2}h) = 0$ and c is linear, which agree with Figure 6.6.6(b).

The local estimate of force is therefore

$$\underbrace{0.036228x}_{\text{pressure}} \underbrace{\left(h - \frac{2}{\sqrt{3}}x\right) dx}_{\text{area}}.$$

Hence

$$\begin{aligned}
\text{Total Force} &= \int_0^{\frac{\sqrt{3}}{2}h} 0.036228x \left(h - \frac{2}{\sqrt{3}}x\right) dx \\
&= 0.036228 \int_0^{\frac{\sqrt{3}}{2}h} \left(hx - \frac{2}{\sqrt{3}}x^2\right) dx \\
&= 0.036228 \left(h\frac{x^2}{2} - \frac{2}{\sqrt{3}}\frac{x^3}{3}\right)\Bigg|_0^{\frac{\sqrt{3}}{2}h} \\
&= 0.036228\frac{h^3}{8} \textit{ pounds}.
\end{aligned}$$

◇

Summary

We introduced the notion of water presure defined as "force divided by area" or "force per unit area". If the pressure is constant over a flat region of area $\mathcal{A}$, the force is the product: pressure times area. This leads to local approximation. $p(x)c(x)\,dx$ where $p(x)$ is the pressure corresponding to x and

$c(x)$ is the length of the cross-section. The pressure $p(x)$ is 0.036227 times the depth of x on a vertical axis. In the discussoin all dimensions are in inches and force is reported in pounds.

EXERCISES for 6.6 *Key:* R–routine, M–moderate, C–challenging

A cubic inch of water weighs 0.036227 pounds. In Exercises 1–4 find definite integrals for the force of the water on the indicated surface. (All dimensions are in inches.)

1.[R]

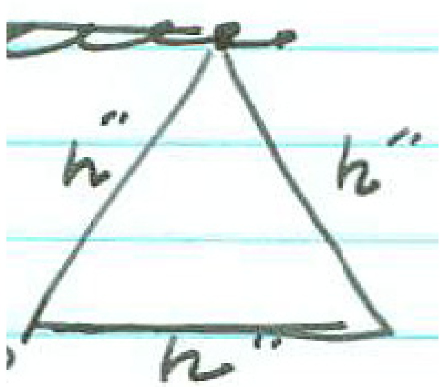


Figure 6.6.7:

2.[R]

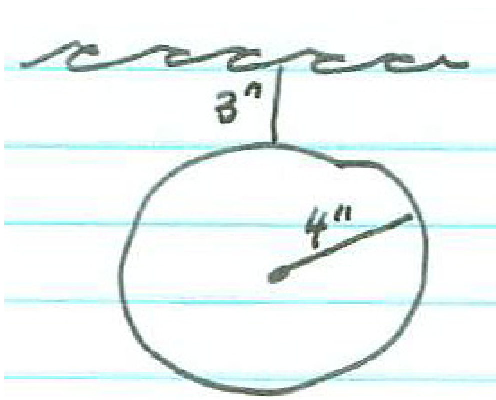


Figure 6.6.8:

3.[R]

4.[R]

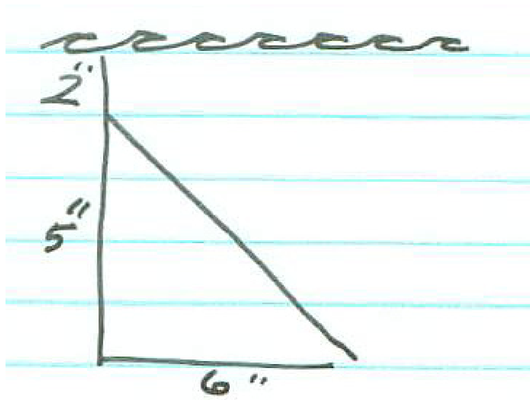


Figure 6.6.9:

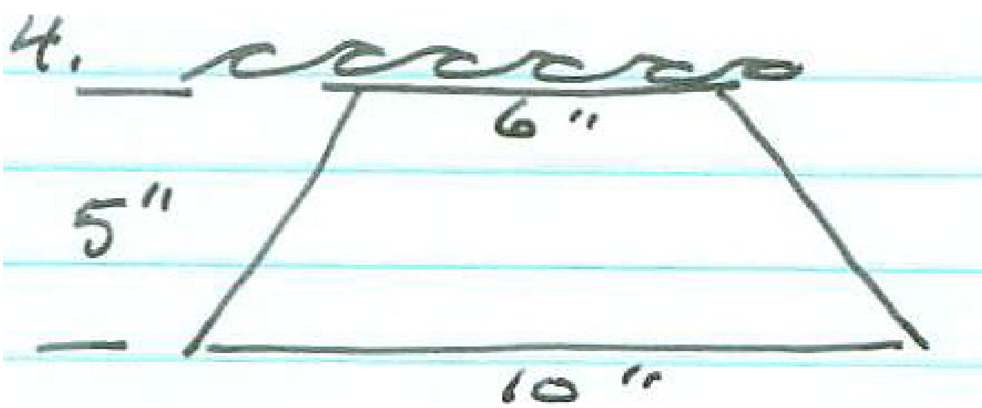


Figure 6.6.10:

In Exercises 5–6 the surface is tilted like the floor of many swimming pools. Find the force of the water against the surface. **5.**[M] The surface is an a" by b" rectangle inclined at an angle of 30° ($\pi/6$ radians) to the horizontal. The top of the surface is at a depth of k". See Figure 6.6.11.

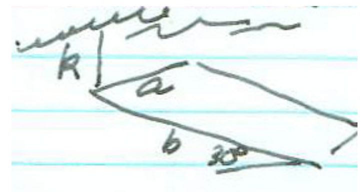


Figure 6.6.11:

6.[M] The surface is a disk of radius r" tilted at an angle of 45° ($\pi/4$ radians) to the horizontal. Its top is at the surface of the water.

7.[M] A vertical disk is totally submerged. Show that the force of the water against it is the same as the product of its area and the pressure at its center. Do not assume the disk is vertical.

SHERMAN: This problem ended with an incomplete and contradictory sentence: "It is not necessarily vertical".

8.[C] Let $\mathcal{R}$ be a convex planar region. $\mathcal{R}$ is called **centrally symmetric** if it contains a point P such that P is the midpoint of every chord of $\mathcal{R}$ that passes through P. For instance, a parallelogram is centrally symmetric. No triangle is. Now, assume that a centrally symmetric region is placed vertically in water and is completely submerged. Show that the force against it equals the product of its area and the pressure at P.

9.[C] If the region in Exercise 7 is not vertical, is the same conclusion true?

6.7 Work

In this section we treat the work accomplished by a force operating along a line, for example the work done when you stretch a spring. If the force has the *constant* value F and it operates over a distance s in the *direction* of the force, then the work W accomplished is simply

$$\text{Work} = \text{Force} \cdot \text{Distance}$$

or

$$W = F \cdot s.$$

If force is measured in newtons and distance in meters, work is measured in newton-meters or joules. For example, the force needed to lift a mass of m kilograms at the surface of the earth is about $9.8m$ newtons.

A weightlifter who raises 100 kilograms a distance of 0.5 meter accomplishes $9.8(100)(0.5) = 490$ joules of work. On the other hand, the weightlifter who just carries the barbell from one place to another in the weightlifting room, without raising or lowering it, accomplishes no work because the barbell was moved a distance zero in the direction of the force.

The Stretched Spring

As you stretch a spring (or rubber band) from its rest position, the further you stretch it the harder you have to pull. According to Hooke's law, the force you must exert is proportional to the distance that the spring is stretched, as shown in Figure 6.7.1.

Hooke's law says a spring's force is proportional to the distance it is stretched.

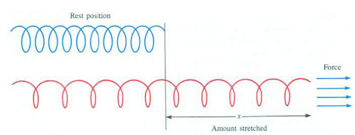


Figure 6.7.1:

Because the force is *not* constant, we cannot compute the work accomplished just by multiplying force times distance. Instead, an integration is required, as the next example illustrates.

EXAMPLE 1 A spring is stretched 0.5 meter longer than its rest length. The force required to keep it at that length is 3 newtons. Find the total work accomplished in stretching the spring 0.5 meter from its rest position.

SOLUTION Let us estimate the work involved in stretching the spring from x to $x + dx$. (See Figure 6.7.2.)

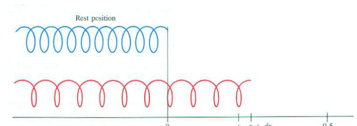


Figure 6.7.2:

The distance dx is small. As the end of the spring is stretched from x to $x + dx$, the force is almost constant. Since the force is proportional to x, it is of the form kx for some constant k. We know that $F = 3$ when $x = 0.5$, so

$$\begin{aligned} f &= kx \\ 3 &= k(0.5) \\ 6 &= k. \end{aligned}$$

The work accomplished in stretching the spring from x to $x + dx$ is then approximately

$$\underbrace{kx}_{\text{force}} \cdot \underbrace{dx}_{\text{distance}} \text{ joules.}$$

Hence the total work is

$$\begin{aligned} \int_a^b kx\ dx &= \int_0^{0.5} 6x\ dx \\ &= 3x^2\Big|_0^{0.5} \\ &= 0.75 \text{ joule.} \end{aligned}$$

◇

Work in Launching a Rocket

The force of gravity that the earth exerts on an object diminishes as the object gets further and further away from the earth. The work required to lift an object 1 foot at sea level is greater than the work required to lift the same object the same distance at the top of Mt. Everest. However, the difference in altitudes is so small in comparison to the radius of the earth that the difference in work is negligible. On the other hand, when an object is rocketed into space, the fact that the force of gravity diminishes with distance from the cener of the earth is critical.

According to Newton, the force of gravity on a given mass is proportional to the reciprocal of the square of the distance of that mass from the center of the earth. That is, there is a constant k such that the gravitational force at distance r from the center of the earth, $F(r)$, is given by

$$f(r) = \frac{k}{r^2}.$$

(See Figure 6.7.3.)

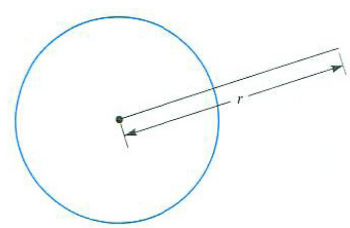


Figure 6.7.3:

The earth's surface is at the distance of $4,000$ miles from its center.

EXAMPLE 2 How much work is required to lift a 1 pound payload from the surface of the earth to the moon, which is about $240,000$ miles away?
SOLUTION The work W necessary to lift an object a distance x against a constant vertical force F is the product of force times distance:

$$W = F \cdot x.$$

Since the gravitiational pull of the earth on the payload *changes* with distance from the earth, an integral will be needed to express the total work requir3ed to lift the load.

The payload weighs 1 pound at the surface of the earth. The farther it is from the center of the earth, the less it weighs, for the force of the earth on the mass is inversely proportional to the square of the distance of the mass from the center of the earth. Thus the force on the payload is given by k/r^2 pounds, where k is a constant, which will be determined in a moment, and r is the distance in miles form the payload to the center of the earth. When $r = 4,000$ (miles), the force is 1 pound; thus

$$1 = \frac{k}{4,000^2}.$$

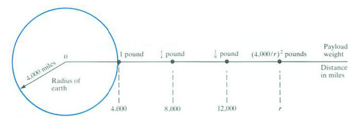


Figure 6.7.4:

From this it follows that $k = 4,000^2$, and therefore the gravitational force on a 1-pound mass is, in general, $(4,000/r)^2$ pounds. As the payload recedes from the earth, it loses weight (but not mass), as recorded in Figure 6.7.4. The work done in lifting the payload from point r to point $r + \, dr$ is approximately

$$\underbrace{\left(\frac{4,000}{r}\right)^2}_{\text{force}} \underbrace{(dr)}_{\text{distance}} \text{ miles-pounds.}$$

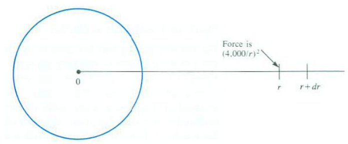


Figure 6.7.5:

(See Figure 6.7.5.)

Hence the work required to move the 1 pound mass from the surface of the earth to the moon is given by the integral

$$\begin{aligned} \int_{4,000}^{240,000} \left(\frac{4,000}{r}\right)^2 \, dr &= -\frac{4,000^2}{r}\Big|_{4,000}^{240,000} \\ &= -4,000^2\left(\frac{1}{240,000} - \frac{1}{4,000}\right) \\ &= -\frac{4,000}{60} + 4,000 \\ &\approx 3,933 \text{ miles-pounds.} \end{aligned}$$

The work is just a little less than if the payload were lifted 4,000 miles against a constant gravitational force equal to that at the surface of the earth. ◇

Summary

The work accomplished by a constant force F that moves an object a distance x in the direction of the force u the product Fx, "force times distance." The work by a variable force, moving an object over the interval $[a, b]$ is measured by an integral $\int_a^b F(x)\, dx$.

EXERCISES for 6.7 *Key:* R–routine, M–moderate, C–challenging

1.[R] A spring is stretched 0.20 meters from its rest length. The force required to keep it at that length is 5 newtons. Assuming that the force of the spring is proportional to the distance it is stretched,

(a) find the work accomplished in stretching the spring 0.20 meters from its rest length;

(b) find the work accomplished in stretching the spring 0.30 meters from its rest length.

2.[R] A spring is stretched 3 meters from its rest length. The force required to keep it at that length is 24 newtons. Assuming that the force of the spring is proportional to the distance it is stretched.

(a) find the work accomplished in stretching the spring 3 meters from its rest length;

(b) find the work accomplished in stretching the spring 4 meters from its rest length.

3.[R] Suppose a spring does not obey Hooke's law. Instead, the force it exerts when stretched x meters from its rest length is $F(x) = 3x^2$ Newtons. Find the work done in stretching the spring 0.80 meter from its rest length.

4.[R] Suppose a spring does not obey Hooke's law. Instead, the force it exerts when stretched x meters from its rest length is $F(x) = 2\sqrt{x}$ Newtons. Find the work done in stretching the spring 0.50 meter from its rest length.

5.[R] How much work is done in lifting the 1 pound payload the first $4,000$ miles of its journey to the moon?

See Example 2.

6.[R] If a mass which weighs 1 pound at the surface of the earth were launched from a position $20,000$ miles from the center of the earth, how much work would be required to send it to the moon ($240,000$ miles from the center of the earth)?

7.[R] Assume that the force of gravity oveys an inverse cub law, so that the force on a 1 pound payload a distance r miles from the center of the earth ($r \geq 4,000$) is $(4,000/r)^3$ pounds. How much work would be required to lift a 1 pound payload from the surface of the earth to the moon?

8.[R] Geologists, when considering the origin of mountain ranges, estimate the energy required to lift a mountain up from sea level. Assume that two mountains are composed of the same type of matter, which weighs k pounds per cubic foot. Both are right circular cones in which the height is equal to the radius. One mountain is twice as high as the other. The base of each is at sea level. If the work required to lift the matter in the smaller mountain above sea level is W, what is the corresponding work for the larger mountain?

See Exercise 8

9.[R] Assume that Mt. Everest has a shape of a right circular cone of height $30,000$ feet and radius $150,000$ feet, with unifrom density of 200 pounds per cubic foot.

(a) How much work was required to lift the material in Mt. Everest if it was initially all at sea level?

(b) How does this work compare with the energy of a 1 megaton H bomb? (One megaton is the energy in a million tons of TNT: about $3x10^14$ foot-pounds.)

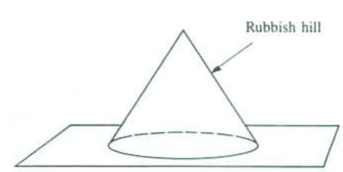


Figure 6.7.6:

10.[R] A town in a flat valley made a conical hill out of its rubbish, as shown in Figure 6.7.6. The work requireed to lift all the rubbish was W. Happy with the result, the town decided to make another hill with twice the volume, but of the same shape. How much work will be required to build this hill? Explain.

DOUG: 62.4 lbs per cubic foot of water is correct

11.[R] A container is full of water which weighs 62.4 pounds per cubic foot. All the water is pumped out of an opening at the top of the container. Develop a definite integral for the work accomplished. [The integral involves only a, b, and $A(x)$, the cross-sectional area shown in Figure 6.7.7.]

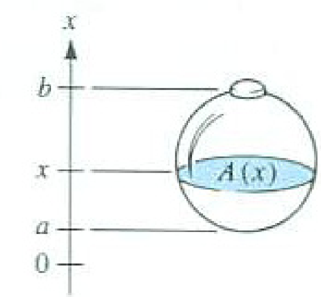


Figure 6.7.7:

SHERMAN: include transcendtal?

12.[A] horizontal tank in the form of a cylinder with base R is full of water. The cylinder has height h feet. (See Figure 6.7.8.) Develop a definite integral for the total work accomplished when all the water is pumped out an opening at the top. HINT: Express the integral in terms of a, b, $c(x)$, and h.

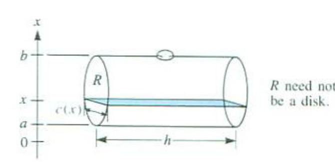


Figure 6.7.8:

Add review xrczs on D and $\int$.

6.8 Improper Integrals

This section develops the analog of a definie integral when the interval of integration is infinite or the integrand becomes arbitrarily large in the interval of integration. The definition of a definite integral does not cover these cases.

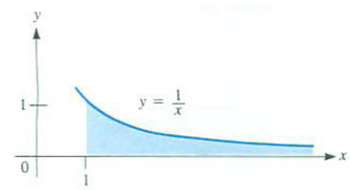


Figure 6.8.1:

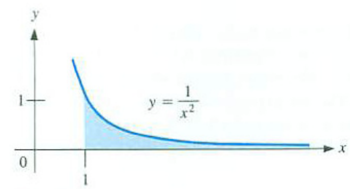


Figure 6.8.2:

Improper Integrals: Interval Unbounded

A question about areas will introduce the notion of an "improper integral". Figure 6.8.1 shows the region under $y = 1/x$ and above the interval $[1, \infty)$. Figure 6.8.2 shows the region under $y = 1/x^2$ and above the same interval.

Let us compute the areas of the two regions. We might be tempted to say that the area in Figure 6.8.1 is $\int_a^\infty f(x)\ dx$. Unfortunately, the symbol $\int_a^\infty f(x)\ dx$ has not been given any meaning so far in this book. The definition of the definite integral $\int_a^b f(x)\ dx$ involves a limit of sums of the form

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{x-1}),$$

where each $x_i - x_{i-1}$ is the length of an interval $[x_{i-1}, x_i]$. If you cut the interval $[1, \infty)$ into a finite number of intervals, then at least one section has infinite length, and such a sum is meaningless.

It does make sense, however, to find the area of that part of the region in Figure 6.8.1 from $x = 1$ to $x = b$, where $b > 1$, and find what happens to that number as $b \to \infty$. To do this, first calculate $\int_1^b (1/x)\ dx$:

$$\int_1^b \frac{dx}{x} = \ln(x)|_1^b = \ln(b) - \ln(1) = \ln(b).$$

Then

$$\lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln(b) = \infty.$$

So the area of the region in Figure 6.8.1 is infinite.

Next, examine the area of the region in Figure 6.8.2. We first find

$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x}\bigg|_1^b = -\frac{1}{b} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{b}$$

Thus,

$$\lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = 1.$$

In this case the area is finite. Though the regions in Figures 6.8.1 and 6.8.2 look a lot alike, one has an infinite area, and the other, a finite area. This contrast suggest the following definitions.

DEFINITION *()*Convergent improper integral $\int_a^\infty f(x)\,dx$. Let f be continuous for $x \geq a$. If $\lim_{b\to\infty} \int_a^b f(x)\,dx$ exists, the function f is said to have a **convergent improper integral** from a to ∞. The value of the limit is denoted by $\int_a^\infty f(x)\,dx$:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

$\int_1^\infty dx/x^2$ is convergent

We saw that $\int_1^\infty dx/x^2$ is a convergent improper integral with value 1.

DEFINITION *()*Divergent improper integral $\int_a^\infty f(x)\,dx$. Let f be a continuous function for $x \geq a$. If $\lim_{b\to\infty} \int_a^b f(x)\,dx$ does not exist, the function f is said to have a **divergent improper integral** from a to ∞.

$\int_1^\infty dx/x$ is divergent.

As we saw, $\int_1^\infty dx/x$ is a divergent improper integral.

The improper integral $\int_1^\infty dx/x$ is divergent because $\int_1^b dx/x \to \infty$ as $b \to \infty$. But an improper integral $\int_a^\infty f(x)\,dx$ can be divergent without being infinite. Consider, for instance, $\int_0^\infty \cos(x)\,dx$. We have

$$\int_0^b \cos(x)\,dx = \sin(x)|_0^b = \sin b.$$

Divergence due to integral oscillating.

As $b \to \infty$, $\sin(b)$ does not approach a limit, nor does it become arbitrarily large. As $b \to \infty$, $\sin(b)b$ just keeps going up and down in the range -1 to 1 infinitely often. Thus $\int_0^\infty \cos(x)\,dx$ is divergent.

$\int_0^\infty \cos x\,dx$ is divergent.

The improper integral $\int_{-\infty}^b f(x)\,dx$.

The improper integral $\int_{-\infty}^b f(x)\,dx$ is defined similarly:

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

If the limit exists, $\int_{-\infty}^b f(x)\,dx$ is a *convergent* improper integral. If the limit does not exist, it is a *divergent* improper integral.

The improper integral $\int_{-\infty}^\infty f(x)\,dx$

To deal with improper integrals over the entire x-axis, define

$$\int_{-\infty}^\infty f(x)\,dx$$

to be the sum

$$\int_{-\infty}^{0} f(x)\ dx + \int_{0}^{\infty} f(x)\ dx,$$

which will be called *convergent* if both

$$\int_{-\infty}^{0} f(x)\ dx \qquad \text{and} \qquad \int_{0}^{\infty} f(x)\ dx$$

are convergent. [If at least one of the two is divergent, $\int_{-\infty}^{\infty} f(x)\ dx$ will be called divergent.]

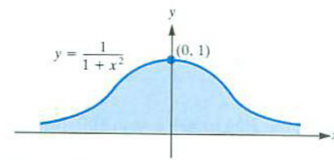


Figure 6.8.3:

EXAMPLE 1 Determine the area of the region bounded by the curve $y = 1/(1+x^2)$ and the x-axis, as indicated in Figure 6.8.3.

SOLUTION The area in question equals $\int_{-\infty}^{\infty} dx/(1+x^2)$. Now,

SHERMAN: Do we want to say the area is this integral before we know it's convergent?

$$\begin{aligned}\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} \\ &= \lim_{b\to\infty} (\tan^{-1}(b) - \tan^{-1}(0)) = \frac{\pi}{2}.\end{aligned}$$

By symmetry,

Observe that the integrand, $\frac{1}{1+x^2}$, is an even function.

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi,$$

and the area in question is π. ◇

Comparison Test for Convergence of $\int_a^\infty f(x)\ dx$, $f(x) \geq 0$

In Exercise 2 of Section 15.1 we show that $\int_0^\infty e^{-x^2}\ dx$ equals $\sqrt{\pi/2}$.

The integral $\int_0^\infty e^{-x^2}\ dx$ is important in statistics. Is it convergent or divergent? We cannot evaluate $\int_0^b e^{-x^2}\ dx$ by the Fundamental Theorem since e^{-x^2} does not have an elementary antiderivative. Even so, there is a way of showing that $\int_0^\infty e^{-x^2}\ dx$ is in fact convergent without determining its exact value. The method is described in Theorem 1.

Theorem 6.8.1 *Comparison test for convergence of improper integrals. Let $f(x)$ and $g(x)$ be continuous functions for $x \geq a$. Assume that $0 \leq f(x) \leq g(x)$ and that $\int_a^\infty g(x)\ dx$ is convergent. Then $\int_a^\infty f(x)\ dx$ is convergent and*

$$\int_a^\infty f(x)\ dx \leq \int_a^\infty g(x)\ dx.$$

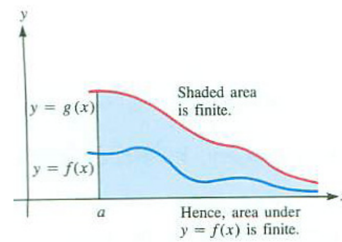


Figure 6.8.4:

A glance at Figure 6.8.4 suggests why this is true

In geometric terms, it asserts that if the area under $y = g(x)$ is finite, so is the area under $y = f(x)$. (See Figure 6.8.4.)

A similar convergence test holds for $g(x) \leq f(x) \leq 0$. If $\int_a^\infty g(x)\ dx$ converges, so does $\int_a^\infty f(x)\ dx$.

EXAMPLE 2 Show that $\int_0^\infty e^{-x^2}\ dx$ is convergent and put a bound on its value.

SOLUTION Since e^{-x^2} does not have an elementary antiderivative, we cannot evaluate $\int_0^b e^{-x^2}\ dx$ and use the result to determine its behavior as $b \to \infty$.

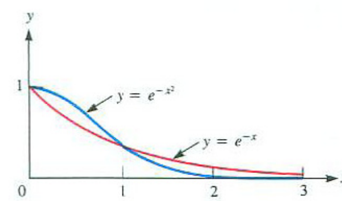


Figure 6.8.5:

However, we can compare $\int_0^\infty e^{-x^2}\ dx$ to an improper integral that we know converges.

For $x \geq 1, x^2 \geq x$; hence $e^{-x^2} \leq e^{-x}$. (See Figure 6.8.5.) Now,

$$\int_1^b e^{-x}\ dx = -e^{-x}\Big|_1^b = e^{-1} - e^{-b}.$$

Thus

$$\lim_{b\to\infty} \int_1^b e^{-x}\ dx = \frac{1}{e}$$

and $\int_1^\infty e^{-x}\ dx$ is convergent.

Since $0 < e^{-x^2} \leq e^{-x}$ for $x \geq 1$, the comparison tests tell us that $\int_1^\infty e^{-x^2}\ dx$ is convergent. Furthermore,

$$\int_1^\infty e^{-x^2}\ dx \leq \int_1^\infty e^{-x}\ dx = \frac{1}{e}.$$

Thus

$$\int_1^\infty e^{-x^2}\ dx = \int_0^1 e^{-x^2}\ dx + \int_1^\infty e^{-x^2}\ dx \leq \int_0^1 e^{-x^2}\ dx + \frac{1}{e}.$$

Since$e^{-x^2} \leq 1$ for $0 < x \leq 1$, we conclude that

$$\int_0^\infty e^{-x^2}\ dx \leq 1 + \frac{1}{e}.$$

◇

Comparison Test for Divergence of $\int_a^\infty f(x)\ dx$.

Theorem 6.8.2 *Comparison test for divergence of improper integrals. Let $f(x)$ and $g(x)$ be continuous functions for $x \geq a$. Assume that $0 \leq g(x) \leq f(x)$ and that $\int_a^\infty g(x)\ dx$ is divergent. Then $\int_a^\infty f(x)\ dx$ is also divergent.*

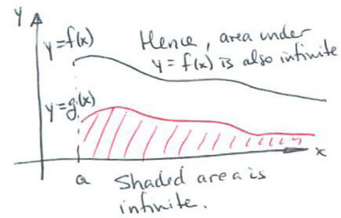


Figure 6.8.6:

A glance at Figure 6.8.6 suggests why this theorem is true. The area under the $f(x)$ is larger than the area under $g(x)$. When the area under $g(x)$ is infinite, the area under f must also be infinite.

EXAMPLE 3 Show that $\int_1^\infty (x^2+1)/x^3\ dx$ is divergent.
SOLUTION For $x > 0$,

$$\frac{x^2+1}{x^3} > \frac{x^2}{x^3} = \frac{1}{x}.$$

Since $\int_1^\infty \frac{dx}{x} = \infty$, it follows that $\int_1^\infty (x^2+1)/x^3\ dx = \infty$. ◇

Convergence of $\int_a^\infty f(x)\ dx$ When $\int_a^\infty |f(x)|\ dx$ Converges

Is $\int_0^\infty e^{-x} \sin x\ dx$ convergent or divergent? Because $\sin x$ takes on both positive and negative values, the integrand is not always positive, nor is it always negative. So we can't just compare it with $\int_0^\infty e^{-x}\ dx$.

The next theorem provides a way to establish the convergence of $\int_a^\infty f(x)\,dx$ when $f(x)$ is a function that takes on both positive and negative values. It says that if $\int_a^\infty |f(x)|\ dx$ converges, so does $\int_a^\infty f(x)\ dx$. The argument for this depends on showing that the "negative and positive parts of the function" both have convergent integrals.

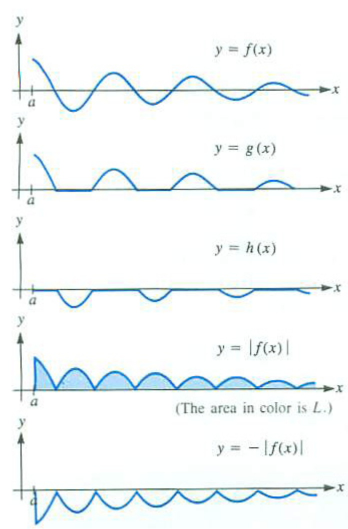


Figure 6.8.7:

Theorem 6.8.3 *The absolute-convergence test. If $f(x)$ is continuous for $x \geq a$ and $\int_a^\infty |f(x)|\ dx$ converges to the number L, then $\int_a^\infty f(x)\ dx$ is convergent and converges to a number between L and $-L$.*

Proof

We will break the function $f(x)$ into two functions that do not change sign. That will enable us to use our comparison tests. Figure 6.8.7 shows the graphs of $y = f(x)$ and four functions closely related to $f(x)$.

$$g(x) \text{ equals } \begin{cases} f(x) & \text{if } f(x) \text{ is positive} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad h(x) \text{ equals } \begin{cases} f(x) & \text{if } f(x) \text{ is negative} \\ 0 & \text{otherwise} \end{cases}$$

Note that $f(x) = g(x) + h(x)$. We will show that $\int_a^\infty g(x)\,dx$ and $\int_a^\infty h(x)\,dx$ both converge.

First, since $\int_a^\infty |f(x)|\,dx$ converges and $0 \le g(x) \le |f(x)|$, $\int_a^\infty g(x)\,dx$ converges to a number A, where

$$0 \le A \le \int_a^\infty |f(x)|\,dx = L.$$

Second, since $\int_a^\infty (-|f(x)|)\,dx$ converges and $0 \ge h(x) \ge -|f(x)|$, $\int_a^\infty h(x)\,dx$ converges to a number B, where

$$0 \ge B \ge \int_a^\infty |f(x)|\,dx = -L.$$

Thus $\int_a^\infty f(x)\,dx = \int_a^\infty (g(x)+h(x))\,dx$ converges to $A+B$, which is a number somewhere in the interval $[-L, L]$. •

EXAMPLE 4 Show that $\int_0^\infty e^{-x} \sin x\,dx$ is convergent.
SOLUTION Since $|\sin(x)| \le 1$, we have $|e^{-x}\sin(x)| \le e^{-x}$. Now, $\int_0^\infty e^{-x}\,dx$ is convergent, as we saw in Example 2. Thus $\int_0^\infty e^{-x}\sin x\,dx$ is convergent. Exercise 58 asks you to find its value. ◇

Improper Integrals: Integrand Unbounded

There is a second type of improper integral, in which the function is unbounded in an interval $[a, b]$. If $f(x)$ becomes arbitrarily large in $[a, b]$ then it is possible to have arbitrarily large approximating sums $\sum_{i=1}^n f(c_i)(x_i - x_{i-1}$ no matter how fine the partition may be by choosing a c_i that makes $f(c_i)$ large. The next example shows how to get around this difficulty.

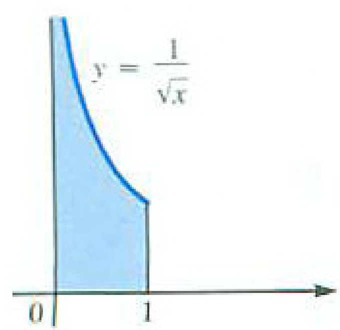


Figure 6.8.8:

Note also that the integrand is not defined at 0.

EXAMPLE 5 Determine the area of the region bounded by $y = 1/\sqrt{x}$, $x = 1$, and the coordinate axes shown in Figure 6.8.8.
SOLUTION Resist for the moment the temptation to write "Area $= \int_0^1 1/\sqrt{x}\,dx$." The integral $\int_0^1 1/\sqrt{x}\,dx$ does not exists since its integrand is unbounded in

$[0, 1]$. Instead, consider the behavior of $\int_t^1 1/\sqrt{x}\,dx$ as t approaches 0 from the right. Since

$$\int_t^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_t^1 = 2\sqrt{1} - 2\sqrt{t} = 2(1-\sqrt{t}),$$

it follows that

$$\lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}} = 2.$$

The area in question is 2.

Check to see that this is the same value for the area that can be obtained by taking horizontal cross sections and evaluating an improper integral from 0 to ∞. ◇

The reasoning in Example 5 motivates the definition of the second type of imporper integral, in which the function rather than the interval is unbounded.

Convergent and Divergent Improper Integrals $\int_a^b f(x)\,dx$.

DEFINITION *()* Let f be continuous at every number in $[a, b]$ except a. If $\lim_{t\to a^+} \int_t^b f(x)dx$ exists, the function f is said to have a **convergent improper integral** from a to b. The value of the limit is denoted $\int_a^b f(x)\,dx$.

If $\lim_{t\to a^+} \int_t^b f(x)dx$ does not exist, the function f is said to have a **divergent improper integral** from a to b; in brief, $\int_a^b f(x)\,dx$ does not exist.

In a similar manner, if f is not defined at b, define $\int_a^b f(x)\,dx$ as $\lim_{t\to a^+} \int_t^b f(x)dx$, if this limit exists.

As example 5 showed, the improper integral $\int_0^1 1/\sqrt{x}\,dx$ is convergent and has the value 2.

A “proper” integral is a definite integral.

More generally, if a function $f(x)$ is not defined at certain isolated numbers, break the domain of $f(x)$ into intervals $[a, b]$ for which $\int_a^b f(x)\,dx$ is either improper or “proper”– that is, an ordinary definite integral.

For instance, the improper integral $\int_{-\infty}^{\infty} 1/x^2\,dx$ is troublesome for four reasons: $\lim_{x\to 0^-} 1/x^2 = \infty$, $\lim_{x\to 0^+} 1/x^2 = \infty$, and the range extends infinitely to the left and also to the right. (See Figure 6.8.9.) To treat the integral, write it as the sum of four improper integrals of the two basic types:

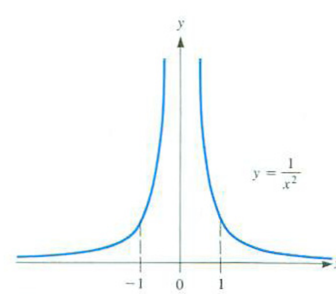


Figure 6.8.9:

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \int_{-\infty}^{-1} \frac{1}{x^2}\,dx + \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx + \int_{1}^{\infty} \frac{1}{x^2}\,dx.$$

All four of the integrals on the right have to be convergent for $\int_{-\infty}^{\infty} 1/x^2\ dx$ to be convergent. As a matter of fact, only the first and last are, so $\int_{-\infty}^{\infty} 1/x^2\ dx$ is divergent.

Summary

We introduced two types of integrals that are not definite integrals, but are defined as limits of definite integrals. The "improper integral" $\int_a^\infty f(x)\ dx$ is defined as $\lim_{b\to\infty} \int_a^b f(x)\ dx$. If $f(x)$ is continuous in $[a, b]$ except at a, then $\int_a^b f(x)\ dx$ is defined as $\lim_{t\to a^+} \int_t^b f(x)\ dx$. A similar definition holds if $f(x)$ is not defined at b. We also developed two comparison tests for converence or divergence of $\int_a^\infty f(x)\ dx$, where the integrand keeps a constant sign. In the case where the integrand $f(x)$ may have both positive and negative values, we showed that if $\int_a^\infty |f(x)|\ dx$ converges, so does $\int_a^\infty f(x)\ dx$.

EXERCISES for 6.8 *Key:* R–routine, M–moderate, C–challenging

In Exercise 1 to 22 determine whether the improper integral is convergent or divergent. Evaluate the convergent ones if possible.

1.[R] $\int_1^\infty \frac{dx}{x^3}$

2.[R] $\int_1^\infty \frac{dx}{\sqrt[3]{x}}$

3.[R] $\int_0^\infty e^{-x}\ dx$

4.[R] $\int_0^\infty \frac{dx}{x+100}$

5.[R] $\int_0^\infty \frac{x^3\ dx}{x^4+1}$

6.[R] $\int_1^\infty x^{-1.01}\ dx$

7.[R] $\int_0^\infty \frac{dx}{(x+2)^3}$

8.[R] $\int_0^\infty \sin 2x\ dx$

9.[R] $\int_1^\infty x^{-0.99}\ dx$

10.[R] $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

11.[R] $\int_0^\infty \frac{\sin x}{x^2}\ dx$

12.[R] $\int_0^\infty \frac{e^{-x}\sin x^2}{x+1}\ dx$

Exercises 13 to 18 require an integral table or techniques from Chapter 7.

13.[R] $\int_1^\infty \frac{\ln x\ dx}{x}$

14.[R] $\int_0^\infty \frac{dx}{x^2+4}$

15.[R] $\int_0^\infty \frac{x\ dx}{x^4+1}$

16.[R] $\int_0^\infty e{-}2x \sin 3x\ dx$

Observe that the integrand in Exercise 17 is undefined at both 0 and 1.

17.[R] $\int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}}$

18.[R] $\int_0^\infty\ fracdx(x+1)(x+2)(x+3)$

19.[R] $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

20.[R] $\int_0^\infty \frac{x\ dx}{\sqrt{1+x^4}}$

21.[R] $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\ dx$

22.[R] $\int_0^1 \frac{dx}{(x-1)^2}$

23.[R] Let R be the region between the curves $y = 1/x$ and $y = 1/(x+1)$ to the right of the line $x = 1$. Is the area of R finite or infinite? If it is finite, evaluate it.

24.[R] Let R be the region between the curves $y = 1/x$ and $y = 1/x^2$ to the right of $x = 1$. Is the area of R finite or infinite? If it is finite, evaluate it.

25.[R] Describe how you would go about estimating $\int_0^\infty e^{-x^2}\ dx$ with an error less than 0.02. (Do not do the arithmetic.)

26.[R] Describe how you would go about estimating $\int_0^\infty \frac{dx}{\sqrt{1+x^4}}$ with an error less than 0.01. (Do not do the arithmetic.)

27.[M] The function $f(x) = \frac{\sin(x)}{x}$ for $x \neq 0$ and $f(0) = 1$ occurs in communication theory. Show that the energy E of the signal represented by f is finite, where

$$E = \int_{-\infty}^{\infty} (f(x))^2\ dx.$$

28.[C] Plankton are small football-shaped organisms. The resistance they meet when falling through water is proportional to the integral

$$\int_0^\infty \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}},$$

where a, b, and c describe the dimensions of the plankton. Is this improper integral convergent or divergent? (Explain.)

29.[C] In R. P. Feynman, *Lectures on Physics*, Addison-Wesley, Reading, MA, 1963, appears this remark: "...the expression becomes

$$\frac{U}{V} = \frac{(kT)^4}{\hbar^3\pi^2c^3}\int_0^\infty \frac{x^3\ dx}{e^x - 1}.$$

This integral is just some number that we can get, approximately, by drawing a curve and taking the area by counting squares. It is roughly 6.5. The mathematicians among us can show that the integral is exactly $\pi^4/15$." Show at least that the integral is convergent.

30.[M] For which positive constants p is $\int_0^1 dx/x^p$ convergent? divergent?

31.[M] For which positive constants p is $\int_0^\infty dx/x^p$ convergent? divergent?

32.[M] Let $f(x)$ be a positive function and let R be the region under $y = f(x)$ and above $[1, \infty]$. Assume that the area of R is infinite. Does it follow that the volume of the solid of revolution formed by revolving R about the x axis is infinite?

DOUG: Move to Section 7.3 - Int by Parts (or Chapter 7 Review)

33.[M]

(a) Show that $\int_1^\infty (\cos x)/x^2\ dx$ is convergent.

(b) Show that $\int_1^\infty (\sin x)/x\ dx$ is convergent. Hint: Start with integration by parts.

(c) Show that $\int_0^\infty (\sin x)/x\ dx$ is convergent.

(d) Show that $\int_0^\infty \sin e^x\ dx$ is convergent.

DOUG: Move to Section 7.2 - Substitution (or Chapter 7 Review)

34.[M] Find the error in the following computations: The substitution $x = y^2$, $dx = 2y\ dy$, yields

$$\int_0^1 \frac{1}{x} = \int_0^1 \frac{2y}{y^2}\ dy = \int_0^1 \frac{2}{y}\ dy$$
$$= 2\int_0^1 \frac{1}{y}\ dy = 2\int_0^1 \frac{1}{x}\ dx.$$

Hence

$$\int_0^1 \frac{1}{x}\ dx = 2\int_1^x frac1x\ dx;$$

from which it follows that $\int_0^1 dx/x = 0$.

Laplace Transform

Let $f(t)$ be a continuous function defined for $t \geq 0$. Assume that, for certain fixed positive numbers r, $\int_0^\infty e^{-rt} f(t)\ dt$ converges and that $e^{-rt}(t) \to 0$

as $t \to \infty$. Define $P(r)$ to be $\int_0^\infty e^{-rt} f(t)\, dt$. The function P is called the **Laplace transform** of the function f. It is an important tool for solving differential equations. In Exercises 35 to 39 find the Laplace transform of the given functions.

35.[M] $f(t) = t$

36.[M] $f(t) = t^2$

37.[M] $f(t) = e^t$ (assume $r > 1$)

38.[M] $f(t) = \sin(t)$

39.[M] $f(t) = \cos(t)$

40.[M] Let f and its derivative f' both have Laplace transforms. Let P be the Laplace transform of f, and let Q be the Laplace transform of f'. Show that

$$Q(r) = -f(0) + rP(r).$$

41.[C] Let P be the Laplace transform of f. Let a be a positive constant, and let $g(t) = f(at)$. Let P be the Laplace transform of f, and let Q be the Laplace transform of g. Show that $Q(r) = (1/a)P(r/a)$.

42.[M] Assume that $f(t) = 0$ for $t < 0$ and that f has a Laplace transform. Let a be a positive constant. Define $g(t)$ to be $f(t - a)$. Show that the Laplace transform of g is e^{-ar} times the Laplace transform of f.

The graph of g is the graph of f shifted to the right by a.

43.[R]

(a) For which values of k is $\int_0^1 x^k\, dx$ improper.

(b) For which values of k is $\int_0^1 x^k$ a convergent improper integral?

(c) For which values of k is $\int_0^1 x^k$ a divergent improper integral?

44.[R]

(a) For which values of k is $\int_1^\infty x^k\, dx$ convergent?

(b) For which values of k is $\int_1^\infty x^k\, dx$ divergent?

45.[M]

(a) Sketch the graph of $y = 1/x^2$, for $x > 0$.

(b) Is the part below the graph and above $[0, 1]$ convergent to the part value the graph and above $[1, \infty]$?

46.[C]

(a) Assume that $f(x) \geq 0$ and that $\int_1^\infty f(x)\, dx$ is convergent. Show by sketchery a graph that $\lim_{x\to\infty} f(x)$ may not exist.

(b) Show that if we add the condition that f is a decreasing function, the $\lim_{x\to\infty} f(x) = 0$.

47.[R]

(a) For which values of k is $\int_0^1 x^k\, dx$ improper.

(b) For which values of k is $\int_0^1 x^k$ a convergent improper integral?

(c) For which values of k is $\int_0^1 x^k$ a divergent improper integral?

48.[R]

(a) For which values of k is $\int_1^\infty x^k\, dx$ convergent?

(b) For which values of k is $\int_1^\infty x^k\, dx$ divergent?

49.[M]

(a) Sketch the graph of $y = 1/x^2$ for $x > 0$.

(b) Is the part below the graph and above $[0, 1]$ congruent to the part below the graph and above $[1, \infty]$?

50.[C]

(a) Assume that $f(x) \geq 0$ and that $\int_1^\infty f(x)\ dx$ is convergent. Show by sketching a graph that $\lim_{x\to\infty} f(x)$ may not exist.

(b) Show that if we add the condition that f is a decreasing function, then $\lim_{x\to\infty} f(x) = 0$.

DOUG: Chapter 7 Review, or Section 7.2

51.[M] Let $\int_{-\infty}^\infty f(x)\ dx = A$.

(a) Express $\int_{-\infty}^\infty f(x+2)\ dx$ in terms of A.

(b) Express $\int_{-\infty}^\infty f(2x)\ dx$ in terms of A.

52.[C] In 1859 James Clerk Maxwell (1831-1879) worked out the distribution of speeds of molecules in a gas. Let N be the number of molecules in the gas, T the absolute temperature, m the mass of a single molecule, and k Boltzman's constant. The function $f(v)$ describes the distribution of velocities of the molecules in the sense that $f(v)\ dv$, for small dv, equals the number of molecules with speed between v and $v + dv$. Maxwell developed the formula

$$f(v) = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{\frac{-mv^2}{2kT}}.$$

(a) Without doing any calculation, determine $\int_0^\infty f(v)\ dv$. Hint: Read the physical definition of $f(v)$.

(b) In Section 15.1, we will show that $\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2}$. Compute $\int_0^\infty v^2 e^{-v^2}\ dv$. (If you could evaluate this integral, you could carry out the integration $\int f(v)\ dv$.)

Continuation of Exercise 52

53.[C] The average speed of the molecules is

$$\frac{\int_0^\infty v f(v)\ dv}{N}.$$

Show that this equals $\sqrt{8kT/\pi m} \approx 1.5958\sqrt{kT/m}$.

Continuation of Exercise 53

54.[C] The "most probable speed" occurs where $f(v)$ has a maximum. Show that this speed is $\sqrt{2kT/m} \approx 1.4142\sqrt{kT/m}$. So the most likely speed is a bit less than the average speed.

55.[C] Assume that f is continuous on $[0,\infty]$ and has period one, that is, $f(x) = f(x+1)$ for all x in $[0,\infty]$. Assume also that $\int_0^\infty e^{-x} f(x)\,dx$ is convergent. Show that

$$\int_0^\infty e^{-x} f(x)\,dx = \frac{e}{e-1}\int_0^1 e^{-x}\,dx.$$

56.[R]

(a) Show that $\int_0^\infty \frac{\sin(kx)}{x}\,dx = \int_0^\infty \frac{\sin(x)}{x}\,dx$, where k is a positive constant.

(b) Show that $\int_0^\infty \frac{\sin(x)\cos(x)}{x}\,dx = \int_0^\infty \frac{\sin(x)}{x}\,dx$.

(c) If k is negative, what is the relation between $\int_0^\infty \frac{\sin kx}{x}\,dx$ and $\int_0^\infty \frac{\sin x}{x}\,dx$.

57.[M] Evaluate $\int_0^\infty e^{-x}\sin^2(x)\,dx$.

58.[M] Evaluate $\int_0^\infty e^{-x}\sin(x)\,dx$.

59.[R] In Example 5, we calculate an area using cross-sections parallel to the y-axis. Calculate the area using cross-sections parallel to the x-axis.

SHERMAN: McQarne?

60.[M] The integral $\int_0^\infty x^{2n}e^{-kx^2}\,dx$ appears in the kinetic theory of gases. In Chapter 15, we will show that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$. With the aid of this information, evaluate

(a) $\int_0^\infty e^{-kx^2}\,dx$,

(b) $\int_0^\infty x^2e^{-kx^2}\,dx$.

61.[R] If a is a constant, show that $\int_{-\infty}^\infty e^{-(x-a)^2}\,dx = \int_{-\infty}^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx$.

DOUG: Move to Chapter 9, as example of recursively defined sequence?

$n!$ is the factorial of n, $n! = 1\cdot 2\cdot \cdots \cdot (n-1)\cdot n$

62.[C] In the study of the hydrogen atom, one meets the integral

$$\int_0^\infty r^n e^{-kr}\,dr$$

Here n is a non-negative integer and k a positive constant. Show that it equals $n!/k^{n+1}$. HINT: First find the value for $n=0$. Then use integration by parts.

63.[R] In the study of the harmonic oscillator one meets the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(1+kx^2)^3},$$

where k is a positive constant. Show it is convergent.

64.[M] In exploring why the light given off by a hot object changes color as the object cools one meets the equation

$$\int_{0}^{\infty} \frac{x^3}{e^x - 1}\, dx = \frac{\pi^4}{15}$$

Show that the integral is convergent, paying attention to x near 0 and $x \to \infty$.

DOUG: Move to Chapter 7? Trouble with $e^x - 1$.

65.[M] In the study of heat capacity of a crystal one meets

$$\int_{0}^{b} \frac{x^4 e^x}{(e^x - 1)^2}\, dx.$$

(a) Show that the integral is convergent.

(b) Is $\int_0^b \frac{xe^x}{(e^x-1)^2}\, dx$ convergent?

66.[M] Show that $\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{3/2}} = 2$.

67.[M]

(a) Show that $\int_0^{\infty} \frac{x^2}{(x^2+1)^{5/2}}$ is convergent.

(b) Show that the value of this improper integral is 1/3.

68.[M] The average distance of an electron from the nucleus of a hydrogen atom involves the integral

$$\int_{0}^{\infty} e^{-x} x^5\, dx.$$

Show that it is convergent. (Its value is $5! = 120$).

Continuation of Exercise 68.

69.[M] This exercise presents an alternate approach to evaluating the integral in Exercise 68. Express the integral as the Laplace transform of an appropriate function. Then, use a table of Laplace transforms to find the value of the integral.

70.[M] In an energy problem one meets the integral

$$\int_0^{\pi/2} \frac{\sin x}{e^x - 1}\, dx.$$

Note that the integrand is not defined at $x = 0$. Is that a big obstacle? Is this integral convergent or divergent? NOTE: Do not try to evaluate the integral.

71.[M] In the theory of probability one meets this equation

$$\int_0^{\infty} e^{-\lambda x} R(x)\, dx = \frac{1}{\lambda}\int_0^{\infty} e^{-\lambda x} R'(x)\, dx + \frac{1}{\lambda} R(0)$$

Assuming the integrals are convergent, explain how the equation is obtained.

72.[C] Here is the standard proof of the absolute convergence test. Assume that $\int_0^\infty |f(x)|\, dx$ converges. Let $g(x) = f(x) + |f(x)|$. Note that $0 \leq g(x) \leq 2|f(x)|$. Thus $int_0^\infty g(x)\, dx$ converges, that is, $\int_0^\infty (f(x) + |f(x)|)\, dx$ converges. It follows, since $f(x) = (f(x) + |f(x)|) - |f(x)|$, that $\int_0^\infty f(x)\, dx$ converges.

(a) Study this proof.

(b) State the advantages and disadvantages of each proof, the standard one and the proof in the text.

(c) Which proof do you prefer? Why?

6.S Chapter Summary

EXERCISES for 6.S *Key:* R–routine, M–moderate, C–challenging

Chapter 7

Computing Antiderivatives

As we will see, many concepts in fields as varied as physics, economics, biology, statistics, encounter definite intervals. For this reason there are many occasions when we want to evaluate a definite integral.

One approach is to use some approximation technique, such as Simpson's method, discussed in Section 5.5.

Another approach is to use a calculator that has a built-in program for estimating a definite integral to any desired degree of accuracy. (See the discussion on page **??**.)

A third approach is to use the fundamental theorem of calculus. If we can find an antiderivative $F(x)$ of the integrand $f(x)$, then $\int_a^b f(x)\ dx$ is simply $F(b) - F(a)$.

The problem of finding an antiderivative differs from that of finding a derivative in two important ways. First, the antiderivaives of some elementary functions, such as e^{x^2}, are not elementary. On the other hand, as we saw in Chapter 2, the derivatives of all elementary functions are elementary.

Second, a slight change in the form of a function can cause great change in the form of its derivative. For instance,

$$\int \frac{dx}{x^2+1} = \tan^{-1} x + C \qquad \text{while} \qquad \int \frac{x dx}{x^2+1} = \frac{1}{2}\ln(x^2+1) + C,$$

as you may check by differentiating $\tan^{-1} x$ and $\frac{1}{2} 1n(x^2+1)$. On the other hand, a slight change in the form of an elementary function produces only a slight change in the form of its derivative.

SHERMAN: This overview, with some added details, should also appear in the Chapter Summary.

There are three ways to find an antiderivative:

1. By hand, using techniques described in this chapter

2. By integral tables

3. By computer

Section 7.1 gives a few shortcuts, describes how to use integral tables, and discusses the strengths and weaknesses of computers.

Section 7.2 presents the most important techniques for find an antiderivative: "substitution". Section 7.3 describes"integration by parts", a technique that has many uses, such as in differential equations, besides finding derivatives.

Section 7.4 discusses the integration of rational functions.

Section 7.5 describes substitutions to us on some special integrands.

Section 7.6 offers an opportunity to practice the techniques when there is no clue as to which is the best to use.

7.1 Shortcuts, Integral Tables, and Technology

In this section we list antiderivatives of some common functions and some general shortcuts, then describe integral tables and the computation of antiderivatives by computers.

Some Common Integrands

Every formula for a derivative provides a corresponding formula for an antiderivative. For instance, since $(x^3/3)' = x^2$, it follows that

$$\int x^2\ dx = \frac{x^3}{3} + C.$$

The following miniature integral table lists a few formulas that should be memorized. Each can be checked by differentiating the right hand side of the equation.

$$\begin{array}{rcll}
\int x^a\ dx &=& \frac{x^{a+1}}{a+1} + C & \text{for } a \neq -1 \\
\int \frac{1}{x}\ dx &=& \ln(|x|) + C & \text{This is } \int x^a\ dx \text{ for } a = -1. \\
\int \frac{f'(x)}{f(x)}\ dx &=& \ln(|f(x)|) + C & \\
\int (f(x))^n\ f'(x)\ dx &=& \frac{(f(x))^{n+1}}{n+1} & \text{for } n \neq -1 \\
\int e^{ax}\ dx &=& \frac{e^{ax}}{a} + C & \\
\int \sin(ax)\ dx &=& \frac{-1}{a}\cos(ax) + C & \text{do not forget the } -1 \\
\int \cos(ax)\ dx &=& \frac{1}{a}\sin(ax) + C & \\
\int \frac{1}{\sqrt{a^2-x^2}}\ dx &=& \arcsin\left(\frac{x}{a}\right) + C & \\
\int \frac{1}{a^2+x^2}\ dx &=& \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C & \\
\int \frac{1}{|x|\sqrt{x^2-a^2}}\ dx &=& \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C &
\end{array}$$

Antiderivative of a polynomial

EXAMPLE 1 Find $\int (2x^4 - 3x + 2)dx$.

SOLUTION

One constant of integration is enough

$$\begin{aligned}
\int (2x^4 - 3x + 2)\ dx &= \int 2x^4\ dx - \int 3x dx + \int 2\ dx \\
&= 2\int x^4\ dx - 3\int x\ dx + 2\int 1\ dx \\
&= 2\frac{x^5}{5} - 3\frac{x^2}{2} + 2x + C
\end{aligned}$$

◇

EXAMPLE 2 Find $\int \frac{4x^3}{x^4+1}\, dx$

SOLUTION The numerator is precisely the derivative of the denominator. Hence

Antiderivative of f'/f

$$\int \frac{4x^3}{x^4+1}\, dx = \ln|x^4+1| + C$$

Since x^4+1 is always positive, the absolute-value sign is not needed, and $\int \frac{4x^3}{x^4+1}\, dx = \ln(x^4+1) + C$. ◇

EXAMPLE 3 Find $\int \sqrt{x}\, dx$.

Antiderivative of x^a

SOLUTION

$$\begin{aligned}\int \sqrt{x}\, dx &= \int x^{1/2}\, dx \\ &= \frac{x^{1/2+1}}{\frac{1}{2}+1} + C \\ &= \frac{2}{3}x^{3/2} + C \\ &= \frac{2}{3}(\sqrt{x})^3 + C.\end{aligned}$$

◇

EXAMPLE 4 Find $\int \frac{1}{x^3}\, dx$.

SOLUTION

$$\begin{aligned}\int \frac{1}{x^3}\, dx &= \int x^{-3}\, dx \\ &= \frac{x^{-3+1}}{-3+1} + C \\ &= -\frac{1}{2}x^{-2} + C \\ &= -\frac{1}{2x^2} + C.\end{aligned}$$

◇

EXAMPLE 5 Find $\int (3\cos x - 4\sin x + \frac{1}{x^2})\, dx$.

SOLUTION

$$\begin{aligned}\int (3\cos x - 4\sin x + \frac{1}{x^2})\, dx &= 3\int \cos x\, dx - 4\int \sin x\, dx + \int \frac{1}{x^2} dx \\ &= 3\sin x + 4\cos x - \frac{1}{x} + C.\end{aligned}$$

◇

EXAMPLE 6 Find $\int \frac{x}{1+x^2}\,dx$.

SOLUTION If the numerator was exactly $2x$, then the numerator would be the derivative of the denominator and we would have the case $\int (f'(x)/f(x))\,dx$. In that case, the antiderivative would be $\ln(1+x^2)$. But the numerator can be multiplied by 2 if we simultaneously divide by 2:

Multiplying the integrand by a constant

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx.$$

This step depends on the fact that a constant can be moved past the integral sign:

$$\frac{1}{2}\int 2x1+x^2\,dx = \frac{1}{2}\cdot 2\int \frac{x}{1+x^2}\,dx = \int \frac{x}{1+x^2}\,dx.$$

Thus

Since $1+x^2>0$, the absolute value is not needed in $\ln(1+x^2)$.

$$\begin{aligned}\int \frac{x}{1+x^2}\,dx &= \frac{1}{2}\int \frac{2x}{1+x^2}\,dx\\ &= \frac{1}{2}\ln(1+x^2)+C.\end{aligned}$$

◇

We present three shortcuts for evaluating some special but fairly common definite integrals.

Shortcut 1

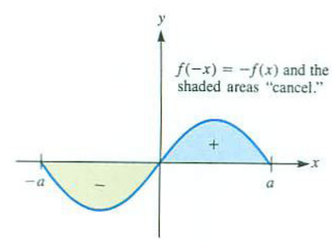


Figure 7.1.1:

Shortcut 1 If f is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Explanation Recall that for an odd function $f(-x)=-f(x)$. Figure 7.1.1 suggests why (7.1) holds. The shaded area to the left of the y-axis equals the shaded area to the right. As integrals, however, these two areas represent quantities of opposite sign: $\int_{-a}^{0} f(x)\,dx = -\int_0^a f(x)\,dx$.

Therefore, the definite integral over the entire interval is 0.

EXAMPLE 7 Find $\int_{-2}^{2} x^3\sqrt{4-x^2}\,dx$.

SOLUTION The function $f(x)=x^3\sqrt{4-x^2}$ is odd. (Check it.) By Shortcut 1,

$$\int_{-2}^{2} x^3\sqrt{4-x^2} = 0.$$

◇

Shortcut 2

Shortcut 2 $\int_0^a \sqrt{a^2 - x^2}\, dx = 1/4\pi a^2$.

Explanation The graph of $y = \sqrt{a^2 - x^2}$ is part of a circle of radius a. The definite integral $\int_0^a \sqrt{a^2 - x^2}\, dx$ is a quarter of the area of that circle. (See Figure 7.1.2.)

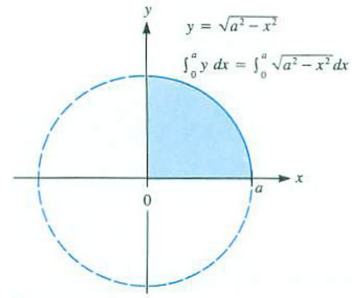


Figure 7.1.2:

EXAMPLE 8 Find $\int_0^1 \sqrt{1 - x^2}\, dx$

SOLUTION Use Shortcut 2, with $a = 1$, to get

$$\int_0^1 \sqrt{1 - x^2}\, dx = \frac{\pi}{4}.$$

◇

Shortcut 3

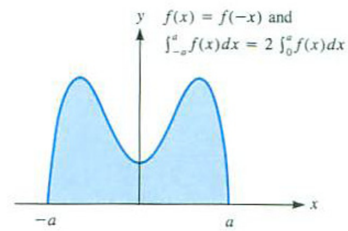


Figure 7.1.3:

Shortcut 3 If f is an even function,

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx.$$

Explanation A glance at Figure 7.1.3 suggests why this shortcut is valid.

EXAMPLE 9 Find $\int_{-1}^1 \sqrt{1 - x^2}\, dx$.

SOLUTION Since $\sqrt{1 - x^2}$ is an even function, by Shortcut 3,

$$\int_{-1}^{1} \sqrt{1 - x^2}\, dx = 2\int_0^1 \sqrt{1 - x^2}\, dx.$$

So, by Example 8,

$$\int_{-1}^{1} \sqrt{1 - x^2} dx = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}.$$

◇

Using an Integral Table

Inside the covers of this book you will find a list of antiderivatives. This list constitutes a short integral table. *Burington's Handbook of Mathematical Tables and Formulas*, 5th edition, McGraw-Hill, 1973, lists over 300 integrals in 33 pages. *CRC Standard Math Tables*, 27th edition, CRC Press, 1984, lists more than 700 integrals in almost 60 pages. There are two Wikipedia topics devoted to tables of integration: `http://en.wikipedia.org/wiki/List_of_integrals` and `http://en.wikipedia.org/wiki/Table_of_integrals`.

Often integral tables use "log" to denote "ln"; it is understood the e is the base.

The best way to use an integral table is to browse through one (buy one, check one out from the library, or navigate to an online table). Notice how the formulas are grouped. First might come the forms that everyone often uses. Then may come "forms containing $ax + b$", then "forms containing $a^2 \pm x^2$", then "forms containing ax^2+bx+c", and so on, running through many different algebraic forms. Trigonometric forms are often next, followed by sections on logarithmic and exponential functions. Our integral table from the inside front cover is similarly grouped. We use Formulas 21(?) and 22(?) from the section "expressions containing $ax + b$" in the next two examples.

EXAMPLE 10 Use the integral table to integrate

$$\int \frac{dx}{x\sqrt{3x+2}}$$

SOLUTION Search until you find Formula 21,

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln\left(\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right|\right) \qquad b > 0,$$

and replace $ax + b$ by $3x + 2$ and b by 2. Thus

$$\int \frac{dx}{x\sqrt{3x+2}} = \frac{1}{\sqrt{2}} \ln\left(\left|\frac{\sqrt{3x+2}-\sqrt{2}}{\sqrt{3x+2}+\sqrt{2}}\right|\right) + C.$$

◇

EXAMPLE 11 Use the integral table to integrate $\int \frac{dx}{x\sqrt{3x-2}}$.
SOLUTION This time we need Formula 22:

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}} \tan^{-1}\left(\sqrt{\frac{ax+b}{-b}}\right) \qquad b < 0.$$

Thus,

$$\int \frac{dx}{x\sqrt{3x-2}} = \frac{2}{\sqrt{2}} \tan^{-1}\left(\sqrt{\frac{3x-2}{2}}\right) + C$$

◇

When using a table of integrals be very cautious and keep a cool head. There is no need to make a big fuss about integral tables. Just match the patterns carefully, including any conditions on the variables and their coefficients. Note that some formulas are expressed in terms of an integral of a different integrand. In these cases you will have to search through the table more than once. (Exercises 34 and 35 illustrate this situtation.)

COMPUTERS

Using an integral table is an exercise in "pattern matching", where you hunt for the formula that fits the particular integral you have. Computers are good at pattern matching, so it is not surprising that for many years computers have been used to find antiderivatives. MACSYMA is one of the earlier computer-based mathematics programs that can perform some of the basic operations of calculus: limits, derivatives, integrals. Today, the most widely used computer algebra systems are Maple and Mathematica.

In addition to matching problems with formulas from large internal tables of integrals, these programs utilize various substitutions and computations to create integrals into forms that can be evaluated.

This technology is increasingly being found on handheld calculators. With such wide-ranging aids at our fingertips, calculus users do not need to rely as much on formal integration techniques or tables of integrals. What is essential is that they understand what an in integral is, what it can represent, and how to utilize information obtained from an integral.

EXERCISES for 7.1 *Key:* R–routine, M–moderate, C–challenging

DOUG / SHERMAN: Enough variety? Applications?

In Exercises 1 to 14 find the integrals. Use the short list at the beginning of the section.

1.[R] $\int 5x^3\ dx$

2.[R] $\int (8+11x)\ dx$

3.[R] $\int x^{1/3}\ dx$

4.[R] $\sqrt[3]{x^2}\ dx$

5.[R] $\int \frac{6\ dx}{x^2}$

6.[R] $\int \frac{dx}{x^3}$

7.[R] $\int 5e^{-2x}\ dx$

8.[R] $\int \frac{5\ dx}{1+x^2}$

9.[R] $\int \frac{6\ dx}{|x|\sqrt{x^2-1}}$

10.[R] $\int \frac{5\ dx}{\sqrt{1-x^2}}$

11.[R] $\int \frac{4x^3\ dx}{1+x^4}$

12.[R] $\int \frac{e^x\ dx}{1+e^x}$

13.[R] $\int \frac{\sin(x)\ dx}{1+\cos(x)}$

14.[R] $\int \frac{dx}{1+3x}$

In Exercises 15 to 20, change the integrand into a easier one by algebra and find the antiderivatives.

15.[R] $int \frac{1+2x}{x^2}\ dx$

Hint: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

16.[R] $\int \frac{1+2x}{1+x^2}\ dx$

17.[R] $\int (x^2+3)^2\ dx$

Hint: First multiply out the integrand.

18.[R] $\int (1+e^x)^2\ dx$

19.[R] $\int (1+3x)x^2\ dx$

20.[R] $\int \frac{1+\sqrt{x}}{x}\ dx$

In Exercises 21 to 26 use a shortcut to evaluate the integral.

21.[R] $\int_{-1}^{1} x^5\sqrt{1+x^2}\ dx$

22.[R] $\int_{-\pi/2}^{\pi/2} \sin(3x)\cos(5x)\ dx$

23.[R] $int_{-1}^{1} x^5\sqrt[4]{1-x^2}\ dx$

24.[R] $\int_{-\pi}^{\pi} \sin^3(x)\ dx$

25.[R] $\int_{-3}^{3} \sqrt{9-x^2}\ dx$

26.[R] $\int_{-3}^{3} (x^3\sqrt{9-x^2}+10\sqrt{9-x^2})\ dx$

In Exercise 27 to 32 find the antiderivative with the aid of a table of integrals. For example, the one inside the front cover will do.

27.[R]

(a) $\int \frac{dx}{(3x+2)^2}$

(b) $\int \frac{dx}{x(3x+2)}$

28.[R]

(a) $\int \frac{dx}{x\sqrt{3x+4}}$

(b) $\int \frac{dx}{x^2\sqrt{3x+4}}$

29.[R]

(a) $\int \frac{dx}{x\sqrt{3x-4}}$

(b) $\int \frac{dx}{x^2\sqrt{3x-4}}$

30.[R]

(a) $\int \frac{dx}{4x^2+9}$

(b) $\int \frac{dx}{4x^2-9}$

31.[R]

(a) $\int \frac{dx}{x^2+3x+5}$

(b) $\int \frac{dx}{x^2+2x+5}$

32.[R]

(a) $\int \frac{dx}{\sqrt{11-x^2}}$

(b) $\int \frac{dx}{\sqrt{11+x^2}}$

Done earlier?

33.[R] (A shortcut for $\int_0^{\pi/2} \sin^2(\theta)\ d\theta$)

(a) Why would you expect $\int_0^{\pi/2} \cos^2(\theta)\ d\theta$ to equal $\int_0^{\pi/2} \sin^2(\theta)\ d\theta$?

(b) Why is $\int_0^{\pi/2} \sin^2(\theta)\ d\theta + \int_0^{\pi/2} \cos^2(\theta)\ d\theta = \pi/4$.

(c) Conclude that $\int_0^{\pi/2} \sin^2(\theta)\ d\theta = \pi/4$

34.[R] Using the integral table on the inside front cover of the book, find $\int x\ dx/\sqrt{2x^2+x+5}$. [Use Formula 42(?) first, followed by Formula 40(?).]

35.[R] Using the integral table in the front of the book, find

(a) $\int \frac{dx}{\sqrt{3x^2+x+2}}$

(b) $\int \frac{dx}{\sqrt{-3x^2+x+2}}$

7.2 The Substitution Method

This section describes the substitution method that changes the form of an integrand, preferably to one that we can integrate more easily. Sometimes we can use a substitution to transform an integral not listed in an integral table to one that is listed. Several examples will illustrate the technique, which is really the chain rule in disguise. After the examples, Theorem 7.2.1 provides the basis of the substitution method. The technique uses differentials, defined in Section **??**. Recall that if $y = f(x)$, then $dy = f'(x)\ dx$.

The Substitution Method

EXAMPLE 1 Find $\int (\sin x^2) 2x\ dx$.

You could check the answer using the chain rule

SOLUTION Note that $2x$ is the derivative of x^2. Make the substitution $u + x^2$.

Then

$$du = 2x\ dx \qquad \text{and} \qquad \int (\sin(x^2))2x\ dx = \int \sin(u)\ du.$$

Now it is easy to find $\int \sin(u)\ du$:

$$\int \sin(u)\ du = -\cos(u) + C.$$

Replacing u by x^2 in $-\cos(u)$ yields $-\cos(x^2)$. Thus

$$\int (\sin(x^2))2x\ dx = -\cos(x^2) + C.$$

◇

Contrast Example 1 with $\int \sin(x^2)\ dx$, which is not elementary. The presence of $2x$, in the derivative of x^2, made it easy to find $\int (\sin(x^2))2x$.

Description of the Substitution Method

In Example 1, the integrand $f(x)$ could be written in the form

$$\text{function of } h(x) \times \text{derivative of } h(x), \tag{7.1}$$

for some function $h(x)$. To put it another way, $f(x)\ dx$ could be written as

$$\text{function of } h(x) \times \text{differential of } h(x). \tag{7.2}$$

Whenever this is the case, the substitution of u for $h(x)$ and du for $h'(x)\ dx$ transforms $\int f(x)$ to another integral, one involving u instead of x, $\int g(u)\ du$.

If you can find an antiderivative $G(u)$ of $g(u)$, replace u by $h(x)$. The resulting function, $G(h(x))$, is an antiderivative of $f(x)$. (This claim will be justified at the end of the section.)

EXAMPLE 2 Find $\int (1+x^3)^5 x^2\ dx$.

SOLUTION The derivative of $1+x^3$ is $3x^2$, which differs from the x^2 in the integrand only by the constant factor 3. So let $u = 1+x^3$. Hence

$$du = 3x^2\ dx \qquad \text{and} \qquad \frac{du}{3} = x^2\ dx. \tag{7.3}$$

Then

$$\begin{aligned} \int (1+x^3)^5 x^2\ dx &= \int u^5 \frac{du}{3} \\ &= \frac{1}{3}\int u^5\ du = \frac{1}{3}\frac{u^6}{6} + C \\ &= \frac{(1+x^3)^6}{18} + C. \end{aligned}$$

◇

If the factor x^2 were not present in the integrand in Example 2, you could compute $\int (1+x^3)^5\ dx$. In this case you would have to multiply out $(1+x^3)^5$, which would be a polynomial degree 15.

As Example 2 shows, you don't need exactly "derivative of $h(x)$" as a factor. Just "a constant times the derivative of $h(x)$" will do.

Similarly, $\int \frac{x^2}{\sqrt{1+x^3}}\ dx$ is easy (use $u = 1+x^3$), but $\int \frac{dx}{\sqrt{1+x^3}}$ is not elementary.

Substitution in a Definite Integral

The substitution technique, or "change of variables," extends to definite integrals, $\int_a^b f(x)\ dx$, with one important proviso:

When making the substitution from x to u, be sure to replace the interval $[a,b]$ by the interval whose endpoints are $u(a)$ and ub).

An example will illustrate the necessary change in the limits of integration. The technique is justified in Theorem 7.2.2.

EXAMPLE 3 Evaluate $\int_1^2 3(1+x^3)^5 x^2\ dx$.

SOLUTION Let $u = 1+x^3$. Then $du = 3x^2\ dx$. Furthermore, as x goes from 1 to 2, $u = 1+x^3$ goes from $1+1^3 = 2$ to $1+2^3 = 9$. Thus

This is the last you see of x.

$$\begin{aligned}\int_1^2 3(1+x^3)^5x^2\ dx &= \int_2^9 u^5\ du\\ &= \left.\frac{u^2}{6}\right|_2^9\\ &= \frac{9^6-2^6}{6}\end{aligned}$$

Once you make the substitution, you work only with expressions involving u. There is no need to bring back x again. ◇

The remaining examples present ideas important in Section 7.4. They also show how some formulas in integral tables are obtained.

EXAMPLE 4 Integral tables include a formula for *(a)* $\int dx/(ax+b)$ and *(b)* $\int dx/\left(((ax+b)^n\right)$, $n \neq 1$. Obtain the formulas by using the substitution $u = ax+b$.

SOLUTION *(a)* Let $u = ax+b$. Hence $du = a\ dx$ and therefore $dx = du/a$. Thus

$$\begin{aligned}\int \frac{dx}{ax+b} &= \int \frac{du/a}{u}\\ &= \frac{1}{a}\int\frac{du}{u}\\ &= \frac{1}{a}\ln(|u|)+C\\ &= \frac{1}{a}\ln(|ax+b|)+C.\end{aligned}$$

(b) The same substitution $u = ax+b$ gives

This is Formula 12(?) from the integral table.

$$\begin{aligned}\int\frac{dx}{(ax+b^n)} &= \int\frac{du/a}{u^n} = \frac{1}{a}\int u^{-n}\ du\\ &= \frac{1}{a}\frac{u^{-n+1}}{(-n+1}+C\\ &= \frac{(ax+b)^{-n+1}}{a(-n+1)}+C\\ &= \frac{1}{a(-n+1)(ax+b)^{-n-1}}+C.\end{aligned}$$

◇

EXAMPLE 5 Find $\int \frac{dx}{4x^2+9}$.

SOLUTION $\int \frac{dx}{4x^2+9}$ resembles $\int\frac{dx}{x^2+1}$. This suggests rewriting $4x^2$ as $9u^2$, so we could then factor the 9 out of $9u^2 + 9$, getting $9(u^2 + 1)$. Here are the details.

Introduce u so $4x^2 = 9u^2$. To do this let $2x = 3u$, hence $2dx = 3du$. Then $dx = (3/2)\ du$. Also, $u = (2/3)x$. With this substitution we have

$$\begin{aligned}\int \frac{dx}{4x^2+9} &= \int \frac{(3/2)\ du}{9u^2+9} \\ &= 3/2 \cdot 1/9 \int \frac{du}{u^2+1} \\ &= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + C.\end{aligned}$$

◇

The next example uses a substitution with "completing the square". To complete the square on the expression x^2+bx+c means adding and subtracting $b/2^2$ so that we have

$$x^2 + bx + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}.$$

One squares half the coefficent of b. To complete the square in ax^2+bx+c, where a is not 1, factor a out first:

$$ax^2 + bx + c = a(x^2 + b/a\ x + c/a).$$

Then complete the square in $x^2 + (b/a)x + c/a$.

EXAMPLE 6 Find $\int dx 4x^2 + 8x + 13$.
SOLUTION First complete the square in the denominator:

$$\begin{aligned}4x^2 + 8x + +13 &= 4(x^2 + 2x+) + 13 \\ &= 4(x^2 + 2x + 1^2) + 13 - 4(1^2) \\ &= 4(x+1)^2 + 9.\end{aligned}$$

We now can rewrite the integral as

$$\int \frac{dx}{4(x+1)^2+9}$$

Let $u = x + 1$, hence $du = dx$ and we have

$$\int \frac{dx}{4(x+1)^2+9} = \int \frac{du}{4u^2+9}.$$

By a remarkable piece of good luck, we found in Example 5 that

$$\int du4u^2 + 9 = \frac{1}{6}\tan^{-1}\left(\frac{2u}{3}\right) + C$$

Then

$$\int \frac{dx}{4x^2 + 8x + 9} = \frac{1}{6}\tan^{-1}\left(\frac{2(x+1)}{3}\right) + C.$$

◇

The integral

$$\int \frac{2ax + b}{ax^2 + bx + c}\, dx \tag{7.4}$$

is easy since it has the form $\int f'/f \; dx$ The integral is $\ln|az^2 + b + c| + C$. This observation is the key to treating the integral in the last example.

EXAMPLE 7 Find $\int \frac{x\;dx}{4x^2+8x+13}$.
SOLUTION No substitution comes to mind. However, if $8x + 8$, were in the numerator, we would have one easy integral, for $8x + 8$ is the derivative of the denominator.

So we will do a little algebra on x to get $8x + 8$ into the numerator.

We can write $x = \frac{1}{8}(8x + 8) - \frac{8}{8} = \frac{1}{8}(8x + 8) - 1$.

Then we have

$$\begin{aligned}
\int \frac{x\;dx}{4x^2 + 8x + 13} &= \int \frac{1/8(8x+8) - 1}{4x^2 + 8x + 13}\, dx \\
&= \frac{1}{8}\int \frac{8x+8}{4x^2 + 8x + 13} - \int \frac{dx}{4x^2 + 8x + 13} \\
&= \frac{1}{8}\ln\left(\left|4x^2 + 8x + 13\right|\right) - \frac{1}{6}\tan^{-1}\left(\frac{2(x+1)}{3}\right).
\end{aligned}$$

◇

The techniques of completing the square, substitution, and rewriting x in the numerator, illustrated in Example 6 and Example 7 show how to integrate any integrand of the form $\frac{1}{ax^2+bx+c}$ or $\frac{x}{ax^2+b+c}$.

Why Substitution Works

Theorem 7.2.1 *Assume that f and g are continuous functions and $u = h(x)$ is differentiable. Suppose that $f(x)$ can be written as $g(u)(du/dx)$ and that G is an antiderivative of g. Then $G(u(x))$ is an antiderivative of $f(x)$.*

Proof

We differentiate $G(u(x))$ and check that the result is $f(x)$, as follows:

$$\begin{aligned} \frac{d}{dx}G(u(x)) &= \frac{dG}{du}\frac{du}{dx} && \text{(Chain Rule)} \\ &= g(u)\frac{du}{dx} && \text{(by definition of } G\text{)} \\ &= f(x). && \text{(by assumption)} \end{aligned}$$

•

Warning: If x goes from a to b, $u(x)$ goes from $u(a)$ to $u(b)$. Be sure to change the limits of integration

Theorem 7.2.2 *(Substitution in a definite integral) Under the same assumptions as in Theorem 7.2.1*

$$\int_a^b f(x)\ dx = \int_{u(a)}^{u(b)}\ du. \tag{7.5}$$

Proof

Let $F(x) = G(u(x))$, where G is defined in the previous proof.

$$\begin{aligned} \int_a^b f(x)\ dx &= F(b) - F(a) && \text{(Fundamental Theorem of Calculus)} \\ &= G(u(b)) - G(u(a)) && \text{(definition of } F\text{)} \\ &= \int_{u(a)}^{u(b)} g(u)\ du && \text{(FTC, again)} \end{aligned}$$

•

Summary

This section introduced the most commonly used integration technique, "substitution", which replaces $\int f(x)dx$ by $\int g(u)\ du$ and $\int_a^b f(x)\ dx$ by $\int_{u(a)}^{u(b)} g(u)$.

It is to be hoped that the problem of finding $\int g(u)\ du$ is easier than that of finding $\int f(x)\ dx$. If it is not, try another substitution or one of the methods presented in the rest of the chapter. There is no simple routine method for antidifferentiation of elementary functions though there is an automatic procedure for differentiation of elementary functions. Practice in integration pays off in spotting which technique is most promising.

EXERCISES for 7.2 *Key:* R–routine, M–moderate, C–challenging

Too many xrcz's? Variety?

In Exercises 1 to 14 use the given substitution to find the antiderivatives or definte integrals.

1.[R] $\int (1+3x)^5 3\ dx;\qquad u = 1+3x$

2.[R] $\int e^{\sin(\theta)} \cos(\theta)\ d\theta;\qquad u = \sin\theta$

3.[R] $\int_0^1 \frac{x}{\sqrt{1+x^2}}\ dx;\qquad u = 1+x^2$

4.[R] $\int_{\sqrt{8}}^{\sqrt{15}} \sqrt{1+x^2}x\ dx;\qquad u = 1+x^2$

5.[R] $\int \sin(2x)\ dx;\qquad u = 2x$

6.[R] $\int \frac{e^{2x}}{(1+e^{2x})^2}\ dx;\qquad u = 1+e^{2x}$

7.[R] $\int_{-1}^2 e^{3x}\ dx;\qquad u = 3x$

8.[R] $\int_2^3 \frac{e^{1/x}}{x^2}\ dx;\qquad u = \frac{1}{x}$

9.[R] $\int \frac{1}{\sqrt{1-9x^2}}\ dx;\qquad u = 3x$

10.[R] $\int \frac{t\ dt}{\sqrt{2=5t^2}};\qquad u = 2-5t^2$

11.[R] $\int_{\pi/6}^{\pi/4} \tan(\theta)\sec^2(\theta)\ d\theta;\qquad u = \tan\theta$

12.[R] $\int_{\pi^2/16}^{\pi^2/4} \frac{\sin(\sqrt{x})}{\sqrt{x}}\ dx;\qquad u = \sqrt{x}$

13.[R] $\int \frac{(\ln x)^4}{x}\ dx;\qquad u = \ln x$

14.[R] $\int \frac{\sin(\ln x)}{x}\ dx;\qquad u = \ln x$

In Exercises 15 to 20 differentiate the answer in the given Example to check that it is correct.

These Exercises show the role of the Chain Rule.

15.[R] Example 2

16.[R] Example 3

17.[R] Example 4

18.[R] Example 5

19.[R] Example 6

20.[R] Example 7

In Exercises 21 to 46 use appropriate substitutions to find the antiderivatives.

21.[R] $int(1-x^2)^5 x\ dx$

22.[R] $\int \frac{x\ dx}{(x^2+1)^3}$

23.[R] $\int \sqrt[3]{1+x^2} x\ dx$

24.[R] $\int \frac{\sin(\theta)}{\cos^2(\theta)}\ d\theta$

25.[R] $\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\ dt$

26.[R] $\int e^x \sin(e^x)\ dx$

27.[R] $\int \sin(3\theta)\ d\theta$

28.[R] $\int \frac{dx}{\sqrt{2x+5}}$

29.[R] $\int (x-3)^{5/2}\ dx$

30.[R] $\int \frac{dx}{(4x+3)^3}$

31.[R] $\int \frac{2x+3}{x^2+3x+2}\ dx$

32.[R] $\int \frac{2x+3}{(x^2+3x+5)^4}\ dx$

33.[R] $\int e^{2x}\ dx$

34.[R] $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^3}$

35.[R] $\int x^4 \sin(x^5)\ dx$

36.[R] $\int \frac{\cos(\ln(x))\ dx}{x}$

37.[R] $\int \frac{x}{1+x^4}\ dx$

38.[R] $\int \frac{x^3}{1+x^4}\ dx$

39.[R] $\int \frac{x\ dx}{(1+x)^3}$

40.[R] $\int \frac{x^2\ dx}{(1+x)^3}$

41.[R] $\int \frac{\ln(3x)\ dx}{x}$

42.[R] $\int \frac{\ln(x^2)\ dx}{x}$

43.[R] $\int \frac{(\arcsin(x))^2}{\sqrt{1-x^2}}\ dx$

44.[R] $\int \frac{dx}{(\arctan(2x))(1+x^2)}$

45.[R] $\int \frac{dx}{9x^2+1}$

46.[R] $\int \frac{dx}{9x^2+25}$

In Exercises 47 and 48 complete the squares.

47.[R] $4x^2 + 6x + 11$

48.[R] $3x^2 + 5x + 12$

49.[R] Evaluate $\int \frac{dx}{x^2+2x+5}$

50.[R] Evaluate $\int \frac{dx}{2x^2+2x+5}$

51.[R] Evaluate $\int \frac{x\ dx}{x^2+2x+5}$

52.[R] Evaluate $\int \frac{x}{2x^2+2x+5}\ dx$

In Exercises 53 to 58 find the area of the region under the graph of the given function and above the given interval.

53.[R] $e^{x^3}x^2$; $[1,2]$

54.[R] $\sin^3(\theta)\cos(\theta)$; $[0,\pi/2]$

55.[R] $\frac{x^2+3}{(x+1)^4}$; $[0,1]$ HINT: Let $u = x+1$.

56.[R] $\frac{x^2-x}{(3x+1)^2}$; $[1,2]$

57.[R] $\frac{(\ln(x))^3}{x}$; $[1,e]$

58.[R] $\tan^5\theta\sec^2(\theta)$; $[0,\frac{\pi}{3}]$

In Exercises 59 and 62 use substitution to evaluate the integral.

59.[R] $\int \frac{x^2\ dx}{ax+b}$; $\quad a \neq 0$

60.[R] $\int \frac{x\ dx}{(ax+b)^2}$; $\quad a \neq 0$

61.[R] $\int \frac{x^2\ dx}{(ax+b)^2}$; $\quad a \neq 0$

62.[R] $\int x(ax+b)^n\ dx$; for (a) $n=-1$, (b) $n=-2$

63.[R] Jack (using the substitution $u = \cos(\theta)$) claims that $\int 2\cos(\theta)\sin(\theta)\ d\theta = -\cos^2(\theta)$, while Jill (using the substitution $u = \sin(\theta)$) claims that the answer is $\sin^2(\theta)$. Who is right? Explain.

64.[R] Jill says, "$\int_0^\pi \cos^2(\theta)\ d\theta$ is obviously positive."

Jack claims, "No, its zero. Just make the substitution $u = \sin(\theta)$; hence $du = \cos(\theta)\ d\theta$. Then I get

$$\begin{aligned}\int_0^\pi \cos^2(\theta)\ d\theta &= \int_0^\pi \cos(\theta)\cos(\theta)\ d\theta \\ &= \int_0^0 \sqrt{1-u^2}\ du = 0.\end{aligned}$$

Simple."

(a) Who is right? What is the mistake?

(b) Use the identity $\cos^2(\theta) = (1+\cos(2\theta))/2$ to evaluate the integral without substitution or the shortcut in Section 7.1.

65.[R] Jill asserts that $\int_{-2}^{1} 2x^2\ dx$ is obviously positive. "After all, the integrand is never negative and $-2 < 1$. It equals the area under $y = 2x^2$ and above $[-2, 1]$".

"You're wrong again," Jack replies, "It's negative. Here are my computations. Let $u = x^2$; hence $du = 2x\ dx$. Then

$$\begin{aligned}\int_{-2}^{1} 2x^2\ dx &= \int_{-2}^{1} x \cdot 2x\ dx \\ &= \int_{4}^{1} \sqrt{u}\ du = -\int_{1}^{4} \sqrt{u}\ du,\end{aligned}$$

which is obviously negative." Who is right? Explain.

66.[R] Show that if f is an odd function then $\int_{-a}^{a} f(x)\ dx = 0$. HINT: First show that $\int_{-a}^{0} f(x)\ dx = -\int_{0}^{a} f(x)\ dx$ by using the substitution $u = -x$. (Do not refer to "areas".)

67.[R] Show that if f is an even function, then $\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$. HINT: First show that $\int_{-a}^{0} f(x)\ dx = \int_{0}^{a} f(x)\ dx$ by using the substitution $u = -x$. (Do not refer to "areas".)

68.[R]

(a) Graph $y = \ln(x)/x$.

(b) Find the area under the curve in (a) and above the interval $[e, e^2]$

DOUG: Move to Chapter 9.

69.[R]

(a) Graph $y = (xe^{-x^2}$.

(b) Find the area of the region bounded by $y = xe^{-x^2}$, the line $x + y = 1$, and the x-axis.

NOTE: You will need Newton's method of estimating a solution of an equation.

70.[R] The velocity of a particle at time t seconds is $e^{-t}\sin(\pi t)$ meters per second. Find how far it travels in the first second, from time $t = 0$ to $t = 1$,

(a) using the integral table in the front of the book,

(b) using Simpson's method with $n = 4$, expressing your answer to four decimal places.

7.3 Integration by Parts

Integration by substitution, described in the previous section, is based on the chain rule. The technique called "integration by parts," is based on the product rule.

The Basis forf "Integration by Parts"

It is a tradition to use u and v instead of the expected f and g.

If u and v are differentiable functions then

$$(uv)' = u'\ v + u\ v'$$

This tells us that uv is an antiderivative of $u'\ v + u\ v''$:

$$uv = \int (u'\ v + u\ v')\ dx,$$

Then

$$uv = \int u'\ v\ dx + \int u\ v'\ dx,$$

which we rearrange as

$$\int u\ v'\ dx = u\ v - \int u'\ v\ dx \qquad (7.1)$$

This equation tells us, "if you can integrate $u'\ v$, then you can integrate $u\ v'$." Now, $u'\ v$ may look quite different from $u\ v'$; maybe $\int u'\ v\ dx$ is easier to find than $\int u\ v'\ dx$. The technique based on (7.1) is called "Integration by Parts".

SHERMAN/DOUG: Check on treatment of differentials.

Using the differentials $du = u'\ dx$ and $dv = v'\ dx$, we can replace (7.1) by the shorter version

$$\int ui\ dv = uv - \int v\ du \qquad (7.2)$$

Typical Examples

EXAMPLE 1 Find $\int xe^{3x}\ dx$.
SOLUTION Let's see what happens if we let $u = x$. Then we must have $dv = e^{3x}\ dx$, that is, $v' = e^{3x}$. Luckily, we can find v, since it equals $\int e^{3x}\ dx = e^{3x}/3$. So we have

so $du = dx$

$$\begin{aligned}\int \underbrace{x}_{u}\underbrace{e^{3x}\,dx}_{dv} &= \underbrace{x}_{u}\underbrace{\frac{e^{3x}}{3}}_{v} - \int \underbrace{\frac{e^{3x}}{3}}_{v}\underbrace{dx}_{du} \\ &= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} \\ &= e^{3x}(\frac{x}{3} - frac19)\end{aligned}$$

To check, differentiate $e^{3x}(\frac{x}{3} - frac19)$ and see if it's xe^{3x}. ⋄

Look closely at Example 1 to see why it worked. The key is that the derivative of $u = x$ is simplier than u and also we could integrate $v' = e^{3x}$ to find v.

EXAMPLE 2 Find $\int x \ln(x)\ dx$.
SOLUTION Setting $dv = \ln(x)\ dx$ is not a wise move, since $v = \int \ln(x)\ dx$ is not immediately apparent. But setting $u = \ln(x)$ is promising because $du = d(\ln(x)) = \frac{1}{x}dx$ is much easier to handle in the integrand than $\ln(x)$. This second approach goes through smoothly:

$$\begin{aligned} u &= \ln(x) \qquad & dv &= x\ dx \\ du &= \tfrac{dx}{x} & v &= \tfrac{x^2}{2}. \end{aligned}$$

(Note that we needed to find $v = \int x\ dx$.) Thus

$$\begin{aligned}\int x\ln(x)\ dx = \int \underbrace{\ln(x)}_{u}\underbrace{x\ dx}_{dv} &= \underbrace{\ln(x)}_{u}\underbrace{fracx^2 2}_{v} - \int \underbrace{fracx^2 2}_{v}\underbrace{\frac{dx}{x}}_{du} \\ &= \frac{x^2\ln(x)}{2} - \int \frac{x\ dx}{2} \\ &= \frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C.\end{aligned}$$

You may check the results by differentiation. ⋄

The key to applying integration by parts is the selection of u and dv. Usually three conditions should be met:

1. v can be found by integrating and should not be too messy.
2. du should not be messier than u.
3. $\int v\ du$ should be easier than the original $\int u\ dv$

The next example shows how to integrate any inverse trigonometric function.

EXAMPLE 3 Find $\int \tan^{-1}(x)\ dx$.

SOLUTION Recall that the derivative of $\tan^{-1}(x)$ is $1/(1+x^2)$, a much simplier function than $\tan^{-1}(x)$. This suggests the following approach to integrating the inverse tangent function:

$$\begin{aligned} u &= \tan^{-1}(x) & dv &= dx \\ du &= \tfrac{dx}{1+x^2} & v &= x \end{aligned}$$

Integrating an inverse trigonometric function by parts

$$\begin{aligned} \int \underbrace{tan^{-1}(x)}_{u}\underbrace{dx}_{dv} &= \underbrace{(tan^{-1}(x)}_{u}\underbrace{x}_{v} - \int \underbrace{x}_{v}\underbrace{\frac{dx}{1+x^2}}_{du} \\ &= x\tan^{-1}(x) - \int \frac{x}{1+x^2}\ dx. \end{aligned}$$

It is easy to compute $\int \frac{x\ dx}{1+x^2}$, since the numerator is a constant times the derivative of the denominator:

$$\int \frac{x\ dx}{1+x^2} = \frac{1}{2}\int \frac{2x}{1+x^2}\ dx = 1/2\ln(1+x^2).$$

Hence

Check by differentiation

$$\int \tan^{-1}(x)\ dx = x\tan^{-1}(x) - 1/2\ln(1+x^2) + C.$$

◇

To check that you understand the idea in Example 3, find $\int \sin^{-1}(x)\ dx$ by exactly the same method. (Compare your answer with the integral table in the front of the book.)

EXAMPLE 4 Find $\int x\sin(x)\ dx$.

SOLUTION There are two approaches. We could choose $u = \sin(x)$ and $dv = x\ dx$ or we could choose $u = x$ and $dv = \sin(x)\ dx$.

Approach 1: $u = \sin(x)$ and $dv = x\ dx$

$$\int x\sin(x)\ dx = \int \underbrace{\sin(x)}_{u}\underbrace{(x\ dx)}_{dv}.$$

The $du = \cos(x)\ dx$, which is not any worse than $u = \sin(x)$. And, since $dv = x\ dx$, $v = x^2/2$. Thus,

$$\int \underbrace{\sin(x)}_{u}\underbrace{(x\ dx)}_{dv} = \underbrace{\sin(x)}_{u}\underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v}\underbrace{\cos(x)\ dx}_{du}.$$

We have replaced the problem of finding $\int x\sin(x)\ dx$ with the harder problem of finding $1/2\int x^2\cos(x)\ dx$. That is *not* progress: we have *raised* the exponent of x in the integrand from 1 to 2.

Approach 2: $u = x$ and $dv = \sin(x)\ dx$

Let us explroe the second approach to $\int x\sin(x)\ dx$. This time let $u = x$ and $dv = \sin(x)\ dx$. Hence,

$$\begin{array}{rclcrcl} u & = & x & \quad & dv & = & sin(x)\ dx \\ du & = & dx & \quad & v & = & -\cos(x). \end{array}$$

What happens? Integration by parts goes through smoothly:

$$\begin{aligned} \int \underbrace{\sin(x)}_{u}\underbrace{(x\ dx)}_{dv} &= \underbrace{x}_{u}\underbrace{(-\cos(x))}_{v} - \int \underbrace{-\cos(x)}_{v}\underbrace{dx}_{du} \\ &= -x\cos(x) + \int \cos(x)dx \\ &= -x\cos(x) + \sin(x) + C. \end{aligned}$$

◇

EXAMPLE 5 Find $\int x^2 e^{3x}\ dx$.

SOLUTION If we let $u = x^2$, then $du = 2x\ dx$. This is good, for it *lowers* the exponent of x. Hence, try $u = x^2$ and therefore $dv = e^{3x}\ dx$:

$$\begin{array}{rclcrcl} u & = & x^2 & \quad & dv & = & e^{3x}\ dx \\ du & = & 2x\ dx & \quad & v & = & \frac{1}{3}e^{3x}. \end{array}$$

Thus

$$\begin{aligned} \int \underbrace{x^2}_{u}\underbrace{e^{3x}\ dx}_{dv} &= \underbrace{x^2}_{u}\underbrace{\frac{1}{3}e^{3x}}_{v} - \int \underbrace{\frac{1}{3}e^{3x}}_{v}\underbrace{2x\ dx}_{du} \\ &= \frac{x^2}{3}e^{3x} - \frac{2}{3}\int xe^{3x}\ dx \\ &= \frac{x^2}{3}e^{3x} - \frac{2}{3}\left(e^{3x}\left(\frac{x}{3} - \frac{1}{9}\right)\right) \qquad \text{by Example 1} \\ &= e^{3x}\left(\frac{x^2}{3} - \frac{2}{3}\left(\frac{x}{3} - \frac{1}{9} + C\right)\right) \\ &= e^{3x}\left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27} - \frac{2C}{3}\right). \end{aligned}$$

We may rename $-\frac{2C}{3}$, the arbitrary constant, as K, obtaining

$$\int x^2 e^{3x}\ dx = e^{3x}\left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right) + K.$$

◇

Example 5 generalizes.

The idea behind Example 5 applies to integrals of the form $\int P(x)g(x)\,dx$, where $P(x)$ is a polynomial and $g(x)$ is a function – such as $\sin(x)$, $\cos(x)$, or e^x – that can be repeatedly integrated. Let $u = P(x)$ and $dv = g(x)\,dx$.

Definite Integrals and Integration by Parts

Evaluation by integration by parts of a definite integral $\int_a^b f(x)\,dx$, where $f(x) = u(x)v'(x)$, takes the form

$$\int_a^b f(x)\,dx = \int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$$
$$= u(v)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)\,dx.$$

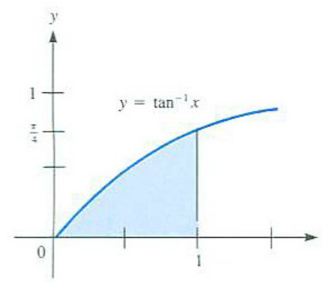


Figure 7.3.1:

EXAMPLE 6 Find the area under the curve $y = \tan^{-1}(x)$ and above [0.1]. (See Figure 7.3.1.)
SOLUTION By Section 6.1, the area is $\int_0^1 \tan^{-1}(x)\,dx$. Now, by Example 3,

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - 1/2\ln(1+x^2) + C.$$

Since only one antiderivative is needed in order to apply the Fundamental Theorem of Calculus, we may choose $C = 0$. Then

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1}(x)\Big|_0^1 - \frac{1}{2}\ln(1+x^2)\Big|_0^1$$
$$= 1\tan^{-1}(1) - 0\tan^{-1}(0) - 1/2\ln(1+1^2) + 1/2\ln(1+0^2)$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln(2) \approx 0.438824.$$

◇

Reduction Formulas

Some formulas in a table of integrals express the integral of a function that involves the n^{th} power of some expressions in terms of the integral of a function that involves a lower power of the same expression. These are **reduction formulas** or **recursion formulas**. Usually they are obtained by an integration by parts.

An example of a reduction formula is

$$\int \sin^n(x)\,dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx \qquad \text{for integer values of } n \geq 2 \tag{7.3}$$

EXAMPLE 7 Use (7.3) to evaluate $\int \sin^5(x)\ dx$.
SOLUTION In this case $n = 5$. By (7.3),

$$\int \sin^5(x)\ dx = -\frac{\sin^4(x)\cos(x)}{5} + \frac{4}{5}\int \sin^3(x)\ dx. \tag{7.4}$$

Use 7.3 again to dispose of $\int \sin^3(x)\ dx$. In this case $n = 3$:

$$\begin{aligned} \int \sin^3(x)\ dx &= -\frac{\sin^2(x)\cos(x)}{3} + \frac{2}{3}\int \sin(x)\ dx & (7.5) \\ &= -\frac{\sin^2(x)\cos(x)}{3} - \frac{2}{3}\cos(x) \qquad \Big(\text{since} \int \sin(x)\ dx\Big) & (7.6) \end{aligned}$$

Combining (7.4) and (7.5) gives

$$\int \sin^5(x)\ dx = -\frac{sin^4(x)\cos(x)}{5} + \frac{4}{5}\left(\frac{-\sin^2(x)\cos(x)}{3} - \frac{2}{3}\cos(x)\right) + C.$$

Every time you (7.3) is used, the exponent of $\sin(x)$ decreases by 2. If you keep applying (7.3), you eventually run into the exponent 1 (as we did, because n is odd) or, if n is even, into the exponent 0. ◇

The next example shows how (7.3) can be obtained by integration by parts.

EXAMPLE 8 Obtain the reduction formula (7.3).
SOLUTION First write $\int \sin^n(x)\ dx$ as $\int \sin^{n-1}(x)\sin(x)\ dx$. Then let $u = \sin^{n-1}(x)$ and $dv = \sin(x)\ dx$. Thus

$$\begin{aligned} u &= \sin^{n-1}(x) & dv &= \sin(x)\ dx \\ du &= (n-1)\ sin^{n-2}(x)\cos(x)\ dx & v &= -\cos(x). \end{aligned}$$

Integration by parts yields

$$\int \underbrace{\sin^{n-1}(x)}_{u}\underbrace{\sin(x)\ dx}_{dv} = \underbrace{(\sin^{n-1}(x))}_{u}\underbrace{(-\cos(x))}_{v} - \int \underbrace{(-\cos(x))}_{v}\underbrace{(n-1)\sin^{n-2}(x)\cos(x)\ dx}_{du}.$$

The integral on the right of the preceding equation is equal to

$$\begin{aligned} -\int (n-1)\cos^2(x)\sin^{n-2}(x)\ dx &= -(n-1)\int (1-\sin^2(x))\sin^{n-2}(x)\ dx \\ &= -(n-1)\int \sin^{n-2}(x)\ dx + (n-1)\int \sin^n(x)\ dx. \end{aligned}$$

Thus

$$\int \sin^n(x)\,dx = -\sin^{n-1}(x)\cos(x) - \left(-(n-1)\int \sin^{n-2}(x)\,dx + (n-1)\int \sin^n(x)\,dx\right)$$

or

$$\int \sin^n(x)\,dx = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx.$$

Rather than being dismayed by the reappearnce of $\int \sin^n(x)\,dx$, collect like terms:

$$n\int \sin^n(x)\,dx = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx,$$

from which (7.3) follows. ◇

The reduction formula for $\int \cos^n x\,dx$ is obtained similarly.

See Formula 50(?) in the table on the front cover, with $a = 1$.

EXAMPLE 9 Obtain the reduction formula for $\int \frac{dx}{(x^2+p)^n}$ where n is a positive integer.

SHERMAN: Originally this asked for a reduction formula for $\int \frac{dx}{(ax^2+bx+c)^n}$. But, the solution never really talked about this problem. I think it would make a good exercise to walk through the steps linking this work to the original statement for this example.

SOLUTION The only choice that comes to mind for integrals by parts is

$$\begin{aligned} u &= \tfrac{1}{(x^2+p)^n} & dv &= dx \\ du &= \tfrac{-2nx}{(x^2+p)^{n+1}} & v &= x. \end{aligned}$$

Integration by parts gives

$$\int \frac{dx}{(x^2+p)^n} = \frac{x}{(x^2+p)^{n+1}} + 2n\int \frac{x^2}{(x^2+p)^{n+1}}\,dx.$$

It looks as though we have just created a more compicated integrand. However, write x^2 as x^2+p-p. We then have

$$\int \frac{dx}{(x^2+p)^n} = \frac{x}{(x^2+p)^{n+1}} + 2n\int \frac{x^2+p}{(x^2+p)^{n+1}}\,dx - 2np\int \frac{dx}{(x^2+p)^{n+1}} \quad (7.7)$$

See also Exercises 54 and 55 in this section.

Canceling out x^2+p in the second integrand gives us an equation which could be rewritten to express $\int \frac{dx}{(x^2+p)^{n+1}}$ in terms of $\int \frac{dx}{(x^2+p)^n}$. ◇

An Unusual Example

In the next example one integration by parts appears at first to be useless, but two in succession lead to the successful evaluation of the integral.

EXAMPLE 10 Find $\int e^x\cos(x)\,dx$

SOLUTION There are two reasonable choices for applying integration by parts: $u = e^x$, $dv = \cos(x)\,dx$ and $u = \cos(x)$, $dv = e^x\,dx$. In neither case is du "simpler", but watch what happens when integration by parts is applied twice.

The second choice is explored in Exercise 45.

Following the first choice:

$$\begin{array}{rclcrcl} u & = & e^x & \quad & dv & = & \cos(x)\,dx \\ du & = & e^x\,dx & & v & = & \sin(x) \end{array}$$

Then integration by parts proceeds as follows:

$$\int \underbrace{e^x}_{u}\underbrace{\cos(x)\,dx}_{dv} = \underbrace{e^x}_{u}\underbrace{\sin(x)}_{v} - \int \underbrace{\sin(x)}_{v}\underbrace{e^x\,dx}_{du}. \tag{7.8}$$

It may seem that nothing useful has been accomplished; $\cos(x)$ is replaced by $\sin(x)$. But watch closely as the new integral is treated by an integration by parts. Capital letters U and V, instead of u and v, are used to distinguish this computation from the preceeding one.

Repeated integration by parts

$$\begin{array}{rclcrcl} U & = & e^x & \quad & dV & = & \sin(x)\,dx \\ dU & = & e^x\,dx & & V & = & -\cos(x) \end{array}$$

and so

$$\int \underbrace{e^x}_{U}\underbrace{\sin(x)\,dx}_{dV} = \underbrace{e^x}_{U}\underbrace{(-\cos(x))}_{V} - \int \underbrace{(-\cos(x))}_{V}\underbrace{e^x\,dx}_{dU} \tag{7.9}$$

$$= -e^x\cos(x) + \int e^x\cos(x)\,dx. \tag{7.10}$$

Combining (7.8) and (7.9) yields

$$\begin{aligned} \int e^x\cos(x)\,dx &= e^x\sin(x) - \left(-e^x\cos(x) + \int e^x\cos(x)\,dx\right) \\ &= e^x(\sin(x)+\cos(x)) - \int e^x\cos(x)\,dx. \end{aligned}$$

Bringing $-\int e^x\cos x\,dx$ to the left side of the equation gives

$$2\int e^x\cos(x)\,dx = e^x(\sin(x)+\cos(x)),$$

and we conclude that

$$\int e^x\cos(x)\,dx = \frac{1}{2}e^x(\sin(x) + \ cos(x)).$$

See Exercise 48.

The most general antiderivative is $\frac{1}{2}e^x(\sin(x)+\cos(x)) + C$. ◇

Summary

Integration by parts is described by the formula

$$\int u\ dv = uv - \int v\ du.$$

When you break up the original integral into the parts u and dv, try to make your choices so that

1. You can find v and it is not too messy.
2. The derivative of u is nicer than u.
3. You can integrate $\int v\ du$.

Sometimes you have to apply integration by parts more than once, for instance, in finding $\int e^x \cos(x)\ dx$. Integration by parts is also the way to develop recursion formulas, such as the one for $\int \sin^n(x)\ dx$.

EXERCISES for 7.3 *Key:* R–routine, M–moderate, C–challenging

Use integration by parts to evaluate each of the integrals in Exercises 1–20.

1.[R] $\int xe^{2x}\ dx$

2.[R] $\int(x+3)e^{-x}\ dx$

3.[R] $\int x\sin(2x)\ dx$

4.[R] $\int(x+3)\cos(2x)\ dx$

5.[R] $\int x\ln(3x)\ dx$

6.[R] $\int(2x+1)\ln(x)\ dx$

7.[R] $\int_1^2 x^2e^{-x}\ dx$

8.[R] $\int_0^1 x^2e^{2x}\ dx$

9.[R] $\int_0^1 \sin^{-1}(x)\ dx$

10.[R] $\int_0^{1/2} \tan^{-1}(2x)\ dx$

11.[R] $\int x^2\ln(x)\ dx$

12.[R] $\int x^3\ln(x)dx$

13.[R] $\int_2^3 (\ln(x))^2\ dx$

14.[R] $\int_2^3 (\ln(x))^3\ dx$

15.[R] $\int_1^e \frac{\ln(x)\ dx}{x^2}$

16.[R] $\int_e^{e^2} \frac{\ln(x)\ dx}{x^3}$

17.[R] $\int e^{3x}\cos(2x)\ dx$

18.[R] $\int e^{-2x}\sin(3x)\ dx$

19.[R] $\int \frac{\ln(1+x^2)\ dx)}{x^2}$

20.[R] $\int x \ln(x^2)\ dx$

In Exercises 21 to 24 find the integrals two ways: *(a)* by substitution, *(b)* by integration by parts.

21.[R] $\int x\sqrt{3x+7}\ dx$

22.[R] $\int \frac{x\ dx}{\sqrt{2x+7}}$

23.[R] $\int x(ax+b)^3\ dx$

24.[R] $\int \frac{x\ dx}{\sqrt[3]{ax+b}}, \qquad a \neq 0$

25.[R] Use the recursion in Example 8 to find

(a) $\int \sin^2 x\ dx$

(b) $\int \sin^4 x\ dx$

(c) $\int \sin^6 x\ dx$

26.[R] Use the recursion in Example 8 to find

(a) $\int \sin^3 x\ dx$

(b) $\int \sin^5 x\ dx$

27.[R] Explain how you would go about finding

$$\int x^{10}(\ln x)^1 8\ dx$$

(Do not say, "I'd use integral tables or a computer.") Explain why your approach would work, but include only enough calculation to convince the reader that it would succeed.

28.[(] R) Let $P(x)$ be a polynomial.

(a) Check by differentiiation that $(P(x) - P'(x) + P"(x) - \cdots)e^x$ is an antiderivative of $P(x)e^x$. (Note that the signs alternate and that the derivatives are taken to successively higher orders until they are 0.)

(b) Use (a) to find $\int (3x^3 - 2x - 2)e^x\ dx$.

(c) Apply integration by parts to $\int P(x)e^x\ dx$ to show how the formula in (a) could be obtained.

29.[R]

(a) Graph $y = e^x \sin x$ for x in $[0, \pi]$, showing extrema and inflection points.

(b) Find the area of the region below the graph and above the interval $[0, \pi]$.

30.[R]

(a) Graph $y = e^{-x} \sin x$ for x in $[0, \pi]$, showing extrema and inflection points.

(b) Find the area of the region below the graph and above the interval $[0, \pi]$.

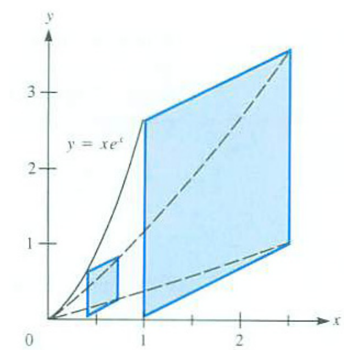


Figure 7.3.2:

31.[R] Figure 7.3.2 shows a shaded region whose cross sections by planes perpendicular to the x-axis are squares. Find its volume.

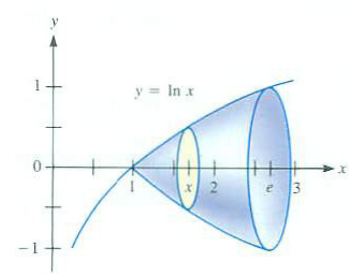


Figure 7.3.3:

32.[R] Figure 7.3.3 shows a solid whose cross sections by planes perpendicular to the x axis are circles. The solid meets the x-axis in the interval $[1.e]$. Find its volume.

In Exercises 33 to 36 find the integrals. In each case a substitution is required before integration by parts can be used. In Exercises 35 and 36 the notation $\exp(u)$ is used for e^u. This notation is often used for clarity.

33.[R] $\int \sin(\sqrt{x})\ dx$

34.[R] $\int \sin(\sqrt[3]{x})\ dx$

35.[R] $\int \exp(\sqrt{x})\ dx$

36.[R] $\int \exp(\sqrt[3]{x})\ dx$

37.[R] Given that $\int [(\sin(x))/x]\ dx$ is not elementary, deduce that $\int \cos x \ln x\ dx$ is not elementary.

38.[R] Given that $\int x \tan(x)\ dx$ is not elementary, deduce that $\int (x/\cos(x))^2\ dx$ is not elementary.

39.[R] Let I_n denote $\int_0^{\pi/2} \sin^n(\theta)\ d\theta$, where n is a nonnegative integer.

(a) Evaluate I_0 and I_1.

(b) Using the recursion in Example 8, show that

$$I_n = \frac{n-1}{n} I_{n-2}, \qquad \text{for } nge2.$$

(c) Use (b) to evaluate I_2 and I_3.

(d) Use (b) to evaluate I_4 and I_5.

(e) Explain why $I_n = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n}$ when n is odd.

(f) Explain why $I_n = \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \cdot \frac{\pi}{2}$ when n is even.

(g) Explain why $int_0^{\pi/2} \sin^n(\theta)\ d\theta = \int_0^{\pi/2} \cos^n(\theta)\ d\theta$. HINT: Use the substitution $u = \pi/2 - \theta$.

40.[R] Find $\int \ln(x+1)\ dx$ using

(a) $u = \ln(x+1)\ dx$, $dv = dx$, $v = x$

(b) $u = \ln(x+1)\ dx$, $dv = dx$, $v = x+1$

(c) Which is easier?

In Exercises 41 to 44 obtain recursion formulas for the integrals.

41.[R] $\int x^n e^{ax}\ dx$, n a positive integer, a a nonzero constant

42.[R] $\int (\ln(x))^n\ dx$, n a postive integer

43.[R] $\int x^n \sin(x)\ dx$, n a positive integer

44.[R] $\int \cos^n(ax)\ dx$, n an integer > 0

45.[R] In Example 10 $\int e^x \cos(x)\ dx$ was evaluated by applying integration by parts twice, each time differentiating an exponential and antidifferentiating a trigonometric function. What happens when integration by parts is applied (twice, if necessary) when a trigonometric function is differentiated and an exponential is antidifferentiated. That is, to get started, apply integration by parts with $u = \cos(x)$ and $dv = e^x\ dx$.

46.[M] Find $\int_{-1}^{1} x^3\sqrt{1+x^{20}}\ dx$.

47.[M] Find $\int_{-\pi/4}^{\pi/4} \tan(x)(1+\cos(x))^{3/2}\ dx$

48.[C] According to the reasoning in Example 10, it appears that $\int e^x \cos(x)\ dx$ must equal $\frac{1}{2}e^x(\sin(x)+\cos(x))$. This would contradict the fact that for any constant C, $\frac{1}{2}e^x(\sin(x)+\cos(x))+C$ is also an antiderivative of $e^x\cos(x)$. Resolve the paradox.

49.[C]

(a) What does the graph of $y = \cos(ax)$ look like when $a = 1$? $a = 3$? $a = 3$? a is very large? (a is a constant) Include graphs and a written description in your answers.

(b) Let $f(x)$ be a function with a continuous derivative. Assume that $f(x)$ is positive. What does the graph of $y = f(x)\cos(ax)$ look like when a is large? Express your response in terms of the graph of $y = f(x)$. Include a sketch of $y = f(x)\cos(ax)$ to give an idea of its shape.

(c) On the basis of (b), what do you think happens to

$$\int_0^1 f(x)\cos(ax)\ dx$$

as $a \to \infty$? Give an intuitive explanation.

(d) Use integration by parts to justify your answer in (c).

50.[M] Obtain a formula that expresses $\int x^n e^{-ax}\ dx$ in terms of $\int x^{n-1}e^{-ax}$, where n is a positive integer, a a constant.

51.[M] Find $\int x \sin(ax)\, dx$

52.[M] Let a be a constant and n a positive integer.

(a) Express $\int x^n \sin(ax)\, dx$ in terms of $\int x^{n-1} \cos(ax)\, dx$.

(b) Express $\int x^n \cos(ax)\, dx$ in terms of $\int x^{n-1} \sin(ax)\, dx$.

(c) Why do (a) and (b) enable us to find $\int x^n \sin(ax)\, dx$?

53.[M]

(a) Express $\int (\ln(x))^u\, dx$ in terms of $\int (\ln(x))^{n-1}\, dx$.

(b) Use (a) to find $\int (\ln(x))^3\, dx$

54.[C] Solve (7.7) in Example 9 to obtain the recursion for $\int \frac{dx}{(ax^2+p)^n}$. To check your answer, compare it to formula ?? in the integral table in the inside cover of this book with $a = 1$, $b = 0$, and $c = p$.

55.[M]

(a) Show how the integral $\int \frac{dx}{(ax^2+bx+c)^n}$ can be reduced to an integral of the form $\int \frac{du}{(u^2+p)^n}$.

(b) Use (a) and the recusion formula obtained in Exercise 54 to find a recursion formula for $\int \frac{dx}{(ax^2+bx+c)^n}$. (How does your answer compare with formula ?? in the integral table on the front cover of the text.?)

56.[C] If we have a recursion for $\int \frac{dx}{(ax^2+bx+c)^n}$ why don't we need one for $\int \frac{x\ dx}{(ax^2+bx+c)^n}$?

57.[R] In Example 4 in Section 6.4 it is claimed that $\frac{e^x}{x}$ does not have an elementary antiderivative. Show that

$$\int \frac{e^x}{x}\, dx = \ln(x)e^x - \int \ln(x)e^x\, dx = \frac{e^x}{x} + \int \frac{e^x}{x^2}\, dx = \int \frac{du}{\ln(u)} \quad \text{where } u = e^x.$$

HINT: Each expression can be obtained from the first by an appropriate use of integration by parts or substitution. None of these integrals is easily evaluated.

SHERMAN: This gives some indication that none of the standard techniques are useful for this integral. Did you have something in mind to actually show the antiderivative is not elementary?

7.4 Integrating Rational Functions: The Algebra

Every rational function, no matter how complicated, has an elementary integral which is the sum of some or all of these types of functions:

- rational functions (including polynomials),
- logarithm of linear or quadratic polynomials ($\ln(ax+b)$ or $\ln(ax^2+bx+c)$), and
- arctangent of linear or quadratic polynomials ($\arctan(ax+b)$ or $\arctan(ax^2+bx+c)$).

The reason is mainly algebraic. In an advanced algebra course it is proved that every rational function is the sum of much simpler rational functions, namely:

$$\text{polynomials}, \frac{k}{(ax+b)^n}, \frac{d}{(ax^2+bx+c)^n}, \text{ and} \frac{ex}{(ax^2+bx+c)^n} \tag{7.1}$$

where a, b, c, d, e, k are constants and n is a positive integer. In Sections 7.2 and 7.3 we saw how to integrate each of these types of integrands.

$$\begin{aligned}
\int \tfrac{k}{ax+b}\,dx &= \tfrac{k}{a}\ln(|ax+b|)+C \\
\int \tfrac{k}{(ax+b)^2}\,dx &= \tfrac{-k}{a}\tfrac{ax+b}{+}C \\
\\
\int \tfrac{d}{ax^2+bx+c}\,dx &= \tfrac{2d}{\sqrt{4ac-b^2}}\arctan\left(\tfrac{2ax+b}{\sqrt{4ac-b^2}}\right)+C \\
\int \tfrac{d}{(ax^2+bx+c)^2}\,dx &= \tfrac{d}{4ac-b^2}\tfrac{2ax+b}{ax^2+bx+c}+\tfrac{4ad}{(4ac-b^2)^{3/2}}\arctan\left(\tfrac{2ax+b}{\sqrt{4ac-b^2}}\right)+C \\
\\
\int \tfrac{ex}{ax^2+bx+c}\,dx &= \tfrac{e}{2a}\ln(|ax^2+bx+c|)-\tfrac{be}{a\sqrt{4ac-b^2}}\arctan\left(\tfrac{2ax+b}{\sqrt{4ac-b^2}}\right) \\
\int \tfrac{ex}{(ax^2+bx+c)^2}\,dx &= \tfrac{e}{4ac-b^2}\tfrac{bx-2c}{ax^2+bx+c}-\tfrac{2be}{(4ac-b^2)^{3/2}}\arctan\left(\tfrac{2ax+b}{\sqrt{4ac-b^2}}\right)+C
\end{aligned}$$

So this section is completely algebraic, showing how to express a rational function $f(x)$ as a sum of the functions in 7.1, called the **partial fraction decomposition** of $f(x)$. For instance, we will see how to find the decomposition

$$\frac{1}{2x^2+7x+3}=\frac{2/5}{2x+1}-\frac{1/5}{x+3}.$$

Reducible and Irreducible Polynomials

A polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where a_0 is not 0 is said to have degree n. The polynomials of degree one are all called linear; those of degree two, quadratic. A polynomial of degree zero is a constant. If all the coefficients a_i are zero, we have the zero polynomial, which is not assigned a degree.

A polynomial of degree at least one is **reducible** if it is a product of polynomials of lower degree. Otherwise, it is **irreducible**.

Recall: $a \neq 0$.

Every polynomial of degree one, $ax+b$, is clearly irreducible. A polynomial of degree two, ax^2+bx+c, is irreducible if its discriminant b^2-4ac is negative. (See Exercises 59 and 60.) However,

FACT 1: Every polynomial of degree three or higher is reducible.

This is far from obvious. For instance, x^4+1 looks like it cannot be factored, but you can check that

$$x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}+1).$$

On the other hand,

$$x^4-1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1).$$

The next non-obvious fact is that

FACT 2: Every polynomial of degree at least one is either irreducible or the product of irreducible polynomials.

The factoring of x^4+1 and x^4-1, given above, illustrate both of these possibilities.

Proper and Improper Rational Functions

In arithmetic, the rational number m/n is called proper if $|m|$ is less than $|n|$.

Let $A(x)$ and $B(x)$ be polynomials. The rational function $A(x)/B(x)$ is **proper** if the degree of $A(x)$ is less than the degree of $B(x)$. Otherwise, it is **improper**.

Every improper rational function is the sum of a polynomial and a proper rational function. The next example shows why this is true. It depends on long division.

EXAMPLE 1 Express $\frac{2x^3+1}{2x^2-x+1}$ as a polynomial plus a proper rational function.

SOLUTION We carry out long division

Keep dividing until the degree of the remainder is less than the degree of the divisor, or the remainder is 0.

$$
\begin{array}{rrrrrl}
 & & & x & +1/2 & \leftarrow \text{quotient} \\
\cline{2-5}
2x^2 - x + 1 \big) & 2x^3 & +0x^2 & +0x & +1 & \\
 & 2x^3 & -x^2 & +x & & \\
\cline{2-4}
 & & x^2 & -x & +1 & \\
 & & x^2 & -x/2 & +1/2 & \\
\cline{3-5}
 & & & -x/2 & +1/2 & \leftarrow \text{remainder}
\end{array}
$$

Thus

$$2x^3 + 1 = (2x^2 - x + 1)(x + 1/2) + (-x/2 + 1/2).$$

Division by $2x^2 - x + 1$ gives us the representation

$$\underbrace{\frac{2x^3 + 1}{2x^2 - x + 1}}_{\text{improper}} = \underbrace{x + \frac{1}{2}}_{\text{polynomial}} + \underbrace{\frac{\left(\frac{-x}{2} + \frac{1}{2}\right)}{2x^2 - x + 1}}_{\text{proper}}.$$

◇

If we must integrate a rational function we first check that it is proper. If it is improper, we carry out long division, and represent the function as the sum of a polynomial and a proper rational function. Since we already know how to integrate a polynomial we consider in the rest of this section only proper rational functions.

Partial Fractions

As mentioned in the introduction, every rational function is the sum of particularly simple rational functions, ones we know how to integrate. Here is a recipe for finding that representation for a proper rational function $A(x)/B(x)$.

1. Write $B(x)$ as a product of first-degree polynomials and irreducible second-degree polynomials.

2. If $px + q$ appears exactly n times in the factorizaiton of $B(x)$, form

   $$\frac{k_1}{px + q} + \frac{k_2}{(px + q)^2} + \cdots + \frac{k_n}{(px + q)^n},$$

   where the constants k_1, k_2, ..., k_n are to be determined later.

List summands of the form $\frac{k_i}{(px+q)^i}$.

3. If $ax^2 + bx + c$ appears exactly m times in the factorization of $B(x)$, then form the sum

   $$\frac{r_1 x + s_1}{ax^2 + bx + c} + \frac{r_2 x + s_2}{(ax^2 + bx + c)^2} + \cdots \frac{r_m x + s_m}{(ax^2 + bx + c)^m},$$

   where the constants r_1, r_2, ..., r_m and s_1, s_2, ..., s_m are to be determined later.

List summands of the form $\frac{r_j x + s_j}{(ax^2+bx+c)^j}$.

4. Find all the constants (k_i's, r_j's, and s_j's) mentioned in Steps 2 and 3 so that their sum equals $A(x)/B(x)$.

The rational functions in Steps 2 and 3 are called the **partial fractions** of $A(x)/B(x)$. This process deserves some comments.

Regarding Step 1

In practice the denominator $B(x)$ often already appears in factored form. If it does not, finding the factorization may be quite a challenge. To find first-degree factors, look for a root of $B(x) = 0$. If r is a root of $B(x)$, then $x-r$ is a factor. Divide $x-r$ into $B(x)$, getting a quotient $Q(x)$; so $B(x) = (x-r)Q(x)$. Repeat the process on $Q(x)$, continuing as long as you can find roots. Already you can see problems. Suppose you find a root numerically to several decimal places. Consequently your results of integration will be approximations. If you wanted $\int_a^b A(x)/B(x)\, dx$ it might have been simpler just to approximate the definite integral.

After finding all the linear factors "what's left" has to be the product of second-degree polynomials. If the degree of "what's left" is just two, then you are happy: you have found the complete factorization. But, if that degree is 4 or 6 or higher, you face a task best to be avoided.

Regarding Steps 2 and 3

These steps refer to the number of times a factor occurs in the denominator. If you factor $2x^2+4x+2$, you may obtain $(x+1)(2x+2)$. Note that $2x+2$ is a constant times $x+1$. The factorization may be written as $2(x+1)^2$, where $x+1$ is a repeated factor. We say that "$x+1$ appears exactly two times in the factorization of $2x^2+4x+2$. Always collect factors that are constants times each other.

Regarding Step 4

Finding the unknown constants may take a lot of work. If there are only linear factors without repetition, the method illustrated in Example 3 is quick. Clearing denominators and comparing the corresponding coefficients of the polynomials on both sides of the resulting equation always works. The number of unknown constants always equals the degree of the denominator $B(x)$. If $B(x)$ has repeated linear or second-degree factors and the degree of $B(x)$ is "large", finding the coefficients may be left to a computer program.

EXAMPLE 2 What is the form of the partial fraction representation of

$$\frac{x^{10}+x+3}{(x+1)^2(2x+2)^3(x-1)^2(x^2+x+3)^2}? \tag{7.2}$$

SOLUTION The degree of the denominator is 11 and the degree of the numerator is 10. Thus (7.2) is proper. There is no need (or possibility) of dividing the numerator by the denominator.

The factor $2x+2$ is $2(x+1)$. So $(x+1)^2(2x+x)^3$ should be written as $8(x+1)^5$. The discriminant of x^2+x+3 is $(1)^2-4(1)(3) = -11 < 0$; thus

x^2+x+3 is irreducible. Therefore the partial fraction representation of (7.2) has the form

$$\frac{1}{8}\left(\frac{k_1}{x+1}+\frac{k_2}{(x+1)^2}+\frac{k_3}{(x+1)^3}+\frac{k_4}{(x+1)^4}+\frac{k_5}{(x+1)^5}+\frac{k_6}{x-1}+\frac{r_1x+s_1}{x^2+x+3}+\frac{r_2x+s_2}{(x^2+x+3)^2}\right). \tag{7.3}$$

Note that the number of unknown constants equals the degree of the denominator in (7.2). ◇

Finding the values of the constants in Example 2 would be a major task if done by hand. Basically, it would involve solving a system of 11 linear equations for the 11 unknown constants. Fortunately, this is the type of problem that is ideally suited for solution by a computer.

Denominator Has Only Linear Factors, Each Appearing Only Once

We illustrate this case by an example.

EXAMPLE 3 Express $\frac{1}{(2x+1)(x+3)}$ in the form $\frac{k_1}{2x+1}+\frac{k_2}{x+3}$ and then find $\int \frac{dx}{(2x+1)(x+3)}$.

SOLUTION

$$\frac{1}{(2x+1)(x+3)}=\frac{k_1}{2x+1}+\frac{k_2}{x+3}. \tag{7.4}$$

To find k_1, multiply both sides of (7.4) by the denominator of the first term, $2x+1$, getting

$$\frac{1}{x+3}=k_1+\frac{k_2(2x+1)}{x+3}. \tag{7.5}$$

Equation (7.5) is valid for all values of x, in particular for the value of x that makes $2x+1=0$, namely $x=-1/2$. Evaluating (7.4) when $x=-1/2$ we get

$$\frac{1}{\left(\frac{-1}{2}\right)+3}=k_1+0.$$

We have found the exact value of k_1: $k_1=\frac{2}{5}$.

The same ideas can be used to solve for k_2: multiply both sides of (7.4) by $(x+3)$, obtaining

$$\frac{1}{2x+1}=\frac{k_1(x+3)}{2x+1}+k_2.$$

Replace x by -3, the solution to $x+3=0$, to obtain

$$\frac{1}{2(-3)+1}=0+k_2.$$

Thus $k_2 - \frac{-1}{5}$.

Since $k_1 = \frac{2}{5}$ and $k_2 = \frac{-1}{5}$, (7.4) takes the form

$$\frac{1}{(2x+1)(x+3)} = \frac{2/5}{2x+1} - \frac{1/5}{x+3}. \tag{7.6}$$

To verify that this identity holds, check it by multiplying both sides by $(2x+1)(x+3)$, getting

For a quicker, but not complete, check replace x in (7.4) by a convenient number and see if the resulting equation is correct. Try it, with, say, $x = 0$!

$$\begin{aligned} 1 &= \frac{2}{5}(x+3) - \frac{1}{5}(2x+1) \\ &= \frac{2}{5}x + \frac{6}{5} - \frac{2}{5}x - \frac{1}{5} \\ &= \frac{5}{5}. \end{aligned}$$

It checks!

Now it is easy to integrate the right-hand side of (7.4):

$$\int \frac{dx}{(2x+1)(x+3)} = \frac{2}{5}\int \frac{dx}{2x+1} - \frac{1}{5}\int \frac{dx}{x+3} \tag{7.7}$$

$$= \frac{2}{5}\frac{1}{2}\ln(|2x+1|) - \frac{1}{5}\ln(|x+3|) + C \tag{7.8}$$

Another way to solve for the unknown constants is to clear the denominator and equate coefficients of like powers of x. For instance, let us find k_1 and k_2 in (7.4) we obtain

$$1 = k_1(x+3) + k_2(x+3).$$

Collecting coefficients, we have

$$1 = (k_1 + 2k_2)x + (3k_1 + k_2). \tag{7.9}$$

Comparing coefficients on both sides of (7.9 we have

$$\begin{aligned} 0 &= k_1 + 2k_2 && \text{coefficient of } x \\ 1 &= 3k_1 + k_2 && \text{constant terms} \end{aligned}$$

There are many ways to solve these simultaneous equations. One way is to use the first equation to express k_1 in terms of k_2: $k_1 = -2k_2$. Then replace k_1 by $-2k_2$ in the second, getting

$$1 = 3(-2k_2) + k_2 = -5k_2$$

from which it is seen that $k_2 = \frac{-1}{5}$. Then $k_1 = \frac{2}{5}$.

DOUG: Further comments on CAS option; see p. 557 in Stewart.

In general, in this method the number of equations always equals the number of unknowns, hence the degree of the denominator. If that degree is large, it is not realistic to do the calculations by hand. ◇

If the denominator is just a repeated linear factor, there are two options: "clearing the denominator and equate coefficients" or "substitution". For instance, the partial fraction representation of

$$\frac{7x+6}{(x+2)^2} \tag{7.10}$$

you could let $u = x + 2$, hence $x = u - 2$. Then

$$\begin{aligned}
\frac{7x+6}{(x+2)^2} &= \frac{7(u-2)+6}{u^2} \\
&= \frac{7u}{u^2} - \frac{8}{u^2} \\
&= \frac{7u}{u^2} - \frac{8}{u^2} \\
&= \frac{7}{u} - \frac{8}{u^2} \\
&= \frac{7}{x+2} - \frac{8}{(x+2)^2}.
\end{aligned}$$

This method for representing

$$\frac{A(x)}{(ax+b)^n}$$

Binomial Theorem: $(u+v)^n = \sum_{k=0}^{n} \binom{n}{k} u^{n-k} v^k$

is practical if the degree of $A(x)$ is small. Here $u = ax + b$, hence $x = \frac{1}{a}(u - b)$. If the degree of $A(x)$ is not small, expressing a power of x, x^m, in terms of u would best be done by the Binomial Theorem.

The next example illustrates one way of dealing with a denominator that has both first and second degree factors.

EXAMPLE 4 Obtain the partial-fraction representation of $\frac{x^2}{x^4-1}$.

SOLUTION First factor the denominator: $x^4 - 1 = (x^2 + 1)(x + 1)(x - 1)$. There are constants c_1, c_2, c_3, and c_4 such that

As a check, note that there are 4 constraints to find and $x^4 - 1$ has degree 4.

$$\frac{x^2}{x^4-1} = \frac{c_1}{x+1} + \frac{c_2}{x-1} + \frac{c_3x + c_4}{x^2+1}.$$

Clear the denominator:

$$x^2 = c_1(x-1)(x^2+1) + c_2(x+1)(x^2+1) + (c_3x + c_4)(x-1)(x+1). \tag{7.11}$$

Substitute $x = 1$ and $x = -1$ into (7.11) to obtain, respectively:

$$\begin{aligned}
1 &= 0 + 4c_1 + 0 && \text{substitute } x = 1 \\
1 &= -4c_1 + 0 + 0 && \text{substitute } x = -1.
\end{aligned}$$

Already we see that $c_1 = \frac{-1}{4}$ and $c_2 = \frac{1}{4}$.

Next, substitute 0 for x in (7.11), obtaining

$$0 = -c_1 + c_2 - c_4 \quad \text{substituting 0.}$$

Setting $x = 0$ compares the constant terms on both sides of (7.11).

Hence $c_4 = \frac{1}{2}$.

We still have to find c_3. We could substitute another number, say $x = 2$, or compare coefficients in (7.11). Let us compare coefficients of just the highest degree, x^3. Without going to the bother of multiplying (7.11) out in full, we can read off the coefficient of x^3 on both sides by sight, getting

$$0 = c_1 + c_2 + c_3.$$

Since $c_1 = \frac{-1}{4}$, $c_2 = \frac{1}{4}$, if follows that $c_3 = 0$. Hence

$$\frac{x^2}{x^4-1} = \frac{\frac{-1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{2}}{x^2+1}.$$

◇

Remarks

Examples 4 used a combination of two methods: substituting convenient values of x and equating coefficients. We could have just compared coefficients. There would be an equation corresponding to each power of x up to x^3. That would give 4 equations in 4 unknowns. The Exercises sugggest how to solve such equations, if you must solve them by hand.

The constant term corresponds to the power x^0.

Summary

We described ways of integrating rational functions. The key idea is algebraic: express the function as the sum of functions that are easier to integrate.

The first step is to check that the degree of the numerator is less than the degree of the denominator. If it isn't, use long division to express the function as the sum of a polynomial and a proper rational function.

For example, TI-82, TI-83, TI-89, TI-92, HP-48GX, HP-50, and Casio FX-9860G.

DOUG: Verify result and include more comments about accuracy.

Integrand	Method of Integration
$\frac{1}{(ax+b)^n}$	substitute $u = ax + b$
$\frac{1}{ax^2+c}$, a, $c > 0$	substitute $cu^2 = ax^2$: $u = \sqrt{\frac{a}{c}}x$
$\frac{1}{ax^2+bx+c}$, $b^2 - 4ac < 0$	factor out a, complete the square, then substitute
$\frac{x}{ax^2+bx+c}$, $b^2 - 4ac < 0$	first, write x in numerator as $\frac{1}{2a}(2ax+b) - \frac{b}{2a}$, then break into two parts. (That is, get $2ax + b$ into the numerator.)
$\frac{1}{(ax^2+bx+c)^n}$ $b^2 - 4ac < 0$, $n \geq 2$	use a recursive formula from the integral tables
$\frac{x}{(ax^2+bx+c)^n}$ $b^2 - 4ac < 0$, $n \geq 2$	express in terms of the previous type by the method in Example 7.

Table 7.4.1:

THE REAL WORLD

Say that you wanted to compute the definite integral

$$\int_1^2 \frac{x+3}{x^3 + x^2 + x + 1}\ dx.$$

One way is by partial fractions, but this can be tedious. You would probably prefer to estimate the definite integral by one of the approximation techniques in Section 5.5. Alternatively, computers and many scientific calculators can be programmed to estimate a definite integral. On many graphing calculutors you would enter the integrand, the variable of integration, and the limits of integration. In a matter of seconds the TI-89 provides 0.49353 as an approximation with an error less than 0.00001.

As noted in Chapter 5, in some cases computers and calculators can even give the exact (symbolic) value of a definite integral by first finding an antiderivative. In practical applicaions, however, formal antidifferentiation is not that important. The present example could theoretically be computed by partial fractions, but modern computational tools can evaluate it accurately to as many decimal places as may be required.

EXERCISES for 7.4 *Key:* R–routine, M–moderate, C–challenging In Exercises 1 to 10 state the form of the partial fraction representation, but do not find the unknowns. NOTE: Each expression is already proper.

SHERMAN/DOUG: Check that list of exercises is adequate but not excessive.

1.[R] $\frac{3x^3+5x+2}{(x-1)(x-2)(x-3)(x-4)}$

2.[R] $\frac{x^2-5x+3}{(x+1)^2(2x+3)}$

3.[R] $\frac{2x^2+x+1}{(x+1)^3}$

4.[R] $\frac{3x}{(x+1)(2x+2)}$

5.[R] $\frac{x^2-x+3}{(x+1)(x^2+1)}$

6.[R] $\frac{2x^2+3x+5}{(x-1)(x^2+x+1)}$

7.[R] $\frac{5x^3+x^2+1}{(x^2+x+1)^2}$

8.[R] $\frac{x^3+x+1}{(x^2+x+1)^3}$

9.[R] $\frac{x^2+x+2}{x^3-x}$

10.[R] $\frac{x^2+x+2}{x^4-1}$

Exercises 11 to 14 concern improper rational functions. In each case express the given function as the sum of a polynomial and a proper rational function.

11.[R] $\frac{x^2}{x^2+x+1}$

12.[R] $\frac{x^3}{(x+1)(x+2)}$

13.[R] $\frac{x^5-2x+1}{(x+1)(x^2+1)}$

14.[R] $\frac{x^5+x}{(x+1)^2(x-2)}$

In Exercises 15 to 18 find the partial fraction representation.

15.[R] $\frac{5}{x^2-1}$

16.[R] $\frac{x+3}{(x+1)(x+2)}$

17.[R] $\frac{1}{(x-1)^2(x+2)}$

18.[R] $\frac{6x^2-2}{(x-1)(x-2)(2x-3)}$

19.[M] Show that any rational function of e^x has an elementary antiderivative.

That is, $\frac{P(e^x)}{Q(e^x)}$ where P and Q are polynomials.

20.[M] Show that $\frac{6+5e^{3x}+2e^{2x}+e^x}{5+e^{2x}+e^x}$ has an elementary antiderivative.

DOUG: Omit or save for summary?

Exercises 21 to 27 concern solving simultaneous equations. By way of illustration we solve the equations

$$\begin{aligned} 2c_1 - 3c_2 &= 5 \\ 3c_1 + 4c_2 &= 6 \end{aligned}$$

in two different ways. In one approach we solve for one of the unknowns in terms of the other unknown (using one equation). Then we substitute the results in the other equation. Thus $c_1 = (5 + 3c_2)/s$, using the first equation. Substitution in the second equation gives $3(5 + 3c_2)/2 + 4c_2 = 6$, an equation in only one unknown. Solve it for $c - 2$, then get c_1.

In another approach we multiply each equation by a constant so that the coefficients of, say, c_1 become equal. Then subtract one equation from another. Thus

$$\begin{aligned} 3(2c_1 - 3c_2) &= 3\cdot 5 \\ 2(3c_1 + 4c_2) &= 2\cdot 6 \\ \text{or}\qquad 6c_1 - 9c_2 &= 15 \\ 6c_1 + 8c_2 &= 12. \end{aligned}$$

Subtracting gives $-17c_2 = 3$, hence $c_2 = \frac{3}{17}$. Then obtain c_1 using any of the equations. Both approaches apply to three equations in three unknowns.

In Exercises 21 to 27 solve the simultaneous equations and check that your answers satisfy the equations.

21.[R]

$$\begin{aligned} 3c_1 \quad -2c_2 &= 3 \\ c_1 \quad +c_2 &= 4 \end{aligned}$$

22.[R]

$$\begin{aligned} 2c_1 \quad +5c_2 &= -3 \\ 3c_1 \quad -4c_2 &= 2 \end{aligned}$$

23.[R]

$$\begin{array}{rrcr} c_1 & +5c_2 & = & 6 \\ 2c_1 & -3c_2 & = & -2 \end{array}$$

24.[R]

$$\begin{array}{rrcr} 5c_1 & +2c_2 & = & 2 \\ -3c_1 & +4c_2 & = & 1 \end{array}$$

25.[R]

$$\begin{array}{rrrcr} c_1 & +2c_2 & +c_3 & = & 9 \\ c_1 & -c_2 & & = & -1 \\ c_1 & & +c_3 & = & 3 \end{array}$$

26.[R]

$$\begin{array}{rrrcr} 2c_1 & -c_2 & +c_3 & = & -7 \\ 3c_1 & -c_2 & -2c_3 & = & 5 \\ c_1 & +c_2 & +c_3 & = & -2 \end{array}$$

27.[R]

$$\begin{array}{rrrcr} c_1 & & +c_3 & = & 4 \\ & c_2 & -c_3 & = & -6 \\ c_1 & +c_2 & +c_3 & = & -1 \end{array}$$

28.[M] Solve Example 3 by clearing the demoninator in (7.4) to get

$$1 = k_1(x+3) + k_2(2x+1).$$

Replace x by any number you please. That gives an equation in k_1 and k_2. Then replace x by another number of your choice, to obtain a second equation in k_1 and k_2. Solve the equations.

Why are $x = -3$ and $x = -1/2$ the nicest choices?

29.[R] Express each of these polynomials as the product of first degree polynomials.

(a) $x^2 = 2x + 1$

(b) $x^2 + 5x - 3$

(c) $x^2 - 4x - 6$

(d) $2x^2 + 3x - 4$

30.[R] Which of these polynomials is irreducible:

(a) $3x^2 + 2x + 1$

(b) $2x^2 + 4x + 1$

We did not discuss the problem of factoring a polynomial $B(x)$ into linear and irreducible quadratic polynomials. Exercises 31 to 35 concern this problem when the degree of $B(x)$ is 2, 3, or 4.

31.[R] Show that when $b^2 - 4ac \geq 0$, the polynomial $ax^2 + bx + c$ is reducible. Indeed, it equals $a(x - r_1)(x - r_2)$ where r_1 and r_2 are its roots, which can be found by the quadratic formula. Factor each of these polynomials:

(a) $x^2 + 6x + 5$,

(b) $x^2 - 5$,

(c) $2x^2 + 6x + 3$.

32.[R]

(a) Show that $x^2 + 3x - 5$ is reducible.

(b) Using (a), find $\int dx/(x^2 + 3x - 5)$ by partial fractions.

(c) Find $\int dx/(x^2 + 3x - 5)$ by using an integral table.

33.[R] Let $P(x)$ be a polynomial with integer coefficients. If $P(r) = 0$, then $x - r$ is a factor of $P(x)$. You may search for a root r by Newton's method. There is an algebraic technique for determining any *rational* roots of $P(x) = 0$. Let $r = p/q$, where p and q are integers with no common divisor larger than 1. We may assume that q is positive. The rational root test asserts that if p/q is a root of $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, then q must divide a_n and p must divide a_0.

For instance, consider $P(x) = 3x^3 + x^2 + x - 2$. If $P(p/q) = 0$, then p divides -2 and q divides 3. Then p must be 1, 2, or -2 and q must be 1 or 3. There are 8 combinations of p and q to check. For example, consider $p = 1$ and $q = 1$, that is $p/q = 1$. Note that $P(1) = 3$, so $1/1$ is *not* a root. It turns out that the choice $p = 2$, $q = 3$ produces a root $\frac{2}{3}$. (Check that $P(2/3) = 0$.) Of course, a polynomial of degree greater than 1 need not have a rational root. Determine all rational roots of the following polynomials:

(a) $x^2 + x - 12$

(b) $2x^3 - 11x^2 + 17x - 6$

(c) $x^4 + x^3 + x^2 + x + 1$

(d) $3x^3 - 2x^2 - 4x - 1$

34.[R] To factor a cubic $P(x) = ax^3 + bx^2 + cx + d$ first find or estimate a root r. Then divide $x - r$ into $P(x)$, obtaining a quotient $Q(x)$, that is, a quadratic polynomial such that $P(x) = Q(x)(x-r)$. If $Q(x)$ is reducible, factoring it completes the factorization of $P(x)$. Illustrate this procedure for

(a) $4x^3 + 4x^2 - 13x - 3$

(b) $2x^3 - x^2 - x - 3$

(c) $x^3 + x + 1$

(d) $x^3 - 8$

35.[R] Compute as easily as possible.

(a) $\int \frac{x^3\,dx}{x^4+1}$

(b) $\int \frac{x\,dx}{x^4+1}$

(c) $\int \frac{dx}{x^4+1}$

36.[C] In arithmetic, the analog of the partial fraction representation is this: Every fraction can be written as the sum of an integer and fraction of the form c/p^n, where p is a prime and $|c|$ is less than p. Check whether this is true for 53/18.

DOUG: Somewhere we should do a cubic $x^3 + 2x^2 + x$ for instance or $x^3 + \cdots$ where a root is found, perhaps in xrcs's.?

37.[M]

(a) For which value of b is $3x^2 + bx + 2$ reducible?

(b) For which value of b is $3x^2 + bx - 2$ reducible?

38.[M]

(a) For which value of c is $3x^2 + 5x + c$ reducible?

(b) For which value of c is $3x^2 + 5x + c$ irreducible?

39.[C] If $ax^2 + bx + c$ is irreducible must $ax^2 - bx + c$ also be irreducible? Must $ax^2 + bx - c$?

In Exercises 40 to 49 express the rational function in terms of partial fractions.

40.[R] $\frac{5x^2-x-1}{x^2(x-1)}$

41.[R] $\frac{2x^2+3}{x(x+1)(x+2)}$

42.[R] $\frac{5x^2-2x-2}{x(x^2-1)}$

43.[R] $\frac{5x^2+9x+6}{(x+1)(x^2+2x+2)}$

44.[R] $\frac{5x^2+2x+3}{x(x^2+x+1)}$

45.[R] $\frac{x^3-3x^2+3x-3}{x^2-3x+2}$

46.[R] $\frac{3x^3+2x^2+3x+1}{x(x^2+1)}$

47.[R] $\frac{x^5+2x^4+4x^3+2x^2+x-2}{x^4-1}$

48.[R] $\frac{5x^2+6x+10}{(x-2)(x^2+3x+4)}$

49.[R] $\frac{3x^2-x-2}{(x+1)(2x^2+x+1)}$

50.[M] This exercise outlines several ways to solve a system of simultaneous equations in several unknowns. You may recall learning a way to solve such systems using a determinant. This exercise presents an alternative.

Solve for A, B, and C.

$$\begin{cases} 2A & +B & +3C & = & 13 \\ 3A & +B & +2C & = & 11 \\ A & -B & +4C & = & 11 \end{cases}$$

(a) Subtract 2 times the second equation from 3 times the first equation. This gives an equation in just B and C. Solve for B in terms of C. Substitute this result into the second and third equations, which now involve only A and C. Now, solve these equations for A and C, then find B.

(b) As in (a), except solve the equation involving B and C for C in terms of B. Substitute this result into the second and third equations and proceed as in (a).

(c) First, subtract the second equation from the first, obtaining an equation in A and C. Then proceed as in (a).

(d) First, add the third equation to the second equation. Proceed as in (a).

In short, keep your eyes open fo rsimplifications!

51.[M] John was complaining to Sally, "I found this formula in my integral tables:

$$\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab} \ln \left| \frac{a+bx}{a-bx} \right| \quad (a,\, b \text{ constants})$$

But my instructor said you won't get any logs other than logs of linear and quadratic polynomials."

Sally: "Maybe the table is wrong."

John: "I took the derivative. It's correct. Can I sue my instructor for misleading the young?"

Does John has a foundation for a case against his instructor? Explain.

Polynomials of odd degree greater than 1 are reducible.

52.[C] Explain why every polynomial of odd degree has at least one linear factor. (Therefore, every polynomial of odd degree greater than one is reducible.)

53.[C] In arithmetic every fraction can be written as an integer plus a proper fraction. For instance, $\frac{25}{3} = 8 + \frac{1}{3}$. Why?

SHERMAN: I believe these exercises are too far afield for calculus.

54.[C]

(a) In artithmetic, what is the analog of an irreducible polynomial?

(b) What is the analog of proper fractions of the partial fraction representation of proper rational rational functions? NOTE: By the way, mathematicians prove a single general theorem, which includes rational functions and rational numbers as special cases.

55.[C]

(a) In artithmetic, what is the analog of the partial fraction representation?

(b) What would it be for $\frac{34}{45}$?

56.[C] Prove that if a is a factor of the polynomial $P(x)$, then $P(a) = 0$.

Factor Theorem

57.[C] Let a be a solution of the equation $P(x) = 0$, where $P(x)$ is a polynomial. Prove that $x - a$ must be a factor of $P(x)$. HINT: When you divide $P(x)$ by $x - a$ by long division, show why the remainder is 0.

58.[C]

(a) Use the quadratic formula to find the roots of $x^2 + 7x + 9$.

(b) With the aid of the Factor Theorem (Exercise 57), write $x^2 + 7x + 9$ as the product of two linear polynomials.

(c) Check the factorization by multiplying it out.

59.[M]

The two parts show that if $b^2 - 4ac \geq 0$, then $ax^2 + bx + c$ is reducible. Compare with Exercise 60.

(a) Show that if $b^2 - 4ac > 0$, then $ax^2 + bx + c = a(x - r_1)(x - r_2)$, where r_1 and r_2 are the distinct roots of $ax^2 + bx + c$.

(b) Show that if $b^2 - 4ac = 0$, then $ax^2 + bx + c = a(x - r)(x - r)$, with r the only root of $ax^2 + bx + c - 0$.

60.[M]

(a) Show that if $ax^2 + bx + c$ is reducible, then it can be written in the form $a(x - s_1)(x - s_2)$ for some real numbers s_1 and s_2.

(b) Deduce that s_1 and s_2 are the roots of $ax^2 + bx + c = 0$.

(c) Deduce that $b^2 - 4ax \geq 0$.

From these three parts it follows that if ax^2+bx+c is reducible, then $b^2-4ac \geq 0$. Compare with Exercise 59.

7.5 Special Techniques

So far in this chapter you have met three techniques for computing integrals. The first, substitution, and the second, integration by parts, are used most often. Partial fractions applies to special rational functions and is used in solving some differential equations. In this section we compute some integrals such as $\int \sin^2(\theta)\ d\theta$, $\int \sin(mx)\cos(nx)\ dx$, and $\int \sec(\theta)\ d\theta$, which you may meet in applications. Then we describe substitutions that deal with special classes of integrands. You may or may not have occasion to use them. Regardless, you will, at least, be aware that there are such techniques.

Computing $\int \sin(mx)\sin(nx)\ dx$

m and n are constants.

The integrals $\int \sin(mx)\sin(nx)\ dx$, $\int \cos(mx)\sin(nx)\ dx$, and $\int \cos(mx)\cos(nx)\ dx$ are needed in the study of Fourier series, an important tool in such varied areas as heat, sound, and signal processing. These integrals can be computed with the aid of these identities:

$$\begin{aligned}
\sin(A)\sin(B) &= 1/2\cos(A-B) - 1/2\cos(A+B);\\
\sin(A)\cos(B) &= 1/2\sin(A+B) + 1/2\sin(A-B);\\
\cos(A)\cos(B) &= 1/2\cos(A-B) + 1/2\cos(A+B).
\end{aligned}$$

These identities can be checked using the well-known identities for $\sin(A \pm B)$ and $\cos(A \pm B)$.

EXAMPLE 1 Find $\int_0^{\pi/4} \sin(3x)\sin(2x)\ dx$.
SOLUTION

$$\begin{aligned}
\int_0^{\pi/4} \sin(3x)\sin(2x)\ dx &= \int_0^{\pi/4} (\frac{1}{2}\cos(x) - \frac{1}{2}\cos(5x)\ dx\\
&= \left.\left(\frac{1}{2}\sin(x) - \frac{1}{10}\sin(5x)\right)\right|_0^{\pi/4} |\\
&= \left(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{20}\right) - \left(\frac{0}{2} - \frac{0}{10}\right)\\
&= \frac{3\sqrt{2}}{10} \approx 0.42426.
\end{aligned}$$

◇

Computing $\int \sin^2(x)\ dx$ and $\int \cos^2(x)\ dx$

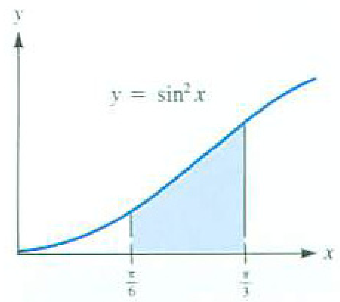


Figure 7.5.1:

These integrals can be computed with the aid of the identities

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \textit{and} \cos^2(x) = \frac{1+\cos(2x)}{2}. \tag{7.1}$$

EXAMPLE 2 Find the area of the region under the curve $y = \sin^2(x)$ and above the interval $[\pi/6, \pi/3]$ shown in Figure 7.5.1.

SOLUTION We first find an antiderivative of $\sin^2(x)$:

$$\begin{aligned}
\int \sin^2(x)\ dx &= \int \frac{1-\cos(2x)}{2}\ dx \\
&= \int \frac{dx}{2} - \int \frac{\cos(2x)}{2}\ dx \\
&= \frac{x}{2} - \frac{\sin(2x)}{4} + C.
\end{aligned}$$

Thus, by the Fundamental Theorem of Calculus, the area $\int_{\pi/6}^{\pi/3} \sin^2(x)\ dx$ equals

$$\begin{aligned}
\left(\frac{x}{2} - \frac{\sin(2x)}{4}\right)\Bigg|_{\pi/6}^{\pi/3} = \left(\frac{\pi/3}{2} - \frac{\sin(2(\pi/3))}{4}\right) - \left(\frac{\pi/6}{2} - \frac{\sin(2(\pi/6))}{4}\right) \\
&= \frac{\pi}{6} - \frac{\sin(2\pi/3)}{4} - \frac{\pi}{12} + \frac{\sin(\pi/3)}{4} \\
&= \frac{\pi}{12} - \frac{\sqrt(3)/2}{4} = \frac{\pi}{12}
\end{aligned}$$

◇

Computing $\int \sec(\theta)\ d\theta$

EXAMPLE 3 Find $\int \sec(\theta)\ d\theta$

SOLUTION

This integral is the key to Mercator maps.

$$\begin{aligned}
\int \sec(\theta\ d\theta &= \int \frac{1}{\cos(\theta)}\ d\theta \\
&= \int \frac{\cos(\theta)}{\cos^2(\theta)}\ d\theta \\
&= \int \frac{\cos(\theta)}{1-\sin^2(\theta)}\ d\theta.
\end{aligned}$$

The substitution $u = \sin(\theta)$ and $du = \cos(\theta)\ d\theta$ transforms this last integral into the integral of a rational function:

$$\begin{aligned}\int \frac{du}{1-u^2} &= \frac{1}{2}\int\left(\frac{1}{1+u}+\frac{1}{1-u}\right)\ du\\ &= \frac{1}{2}\left(\ln(1+u)-\ln(1-u)\right)+C\\ &= \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)+C.\end{aligned}$$

Since $u = \sin(\theta)$,

$$\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) = \frac{1}{2}\ln\left(\frac{1+\sin(\theta)}{1-\sin(\theta)}\right).$$

Thus,

$$\int \sec(\theta)\ d\theta = \frac{1}{2}\ln\left(\frac{1+\sin(\theta)}{1-\sin(\theta)}\right)+C. \tag{7.2}$$

◇

Most integral tables have the formula

Another formula for $\int \sec(\theta) d\theta$

$$\int \sec(\theta)\ d\theta = \ln|\sec(\theta)+\tan(\theta)|+C. \tag{7.3}$$

Exercise 34 shows that this formula agrees with (7.2).

In contrast to Example 3, $\int \sec^2(\theta)\ d\theta$ is easy, since it is simply $tan(\theta)+C$.

Computing $\int \tan(\theta)\ d\theta$

The integration of $\tan(\theta)$ is much more direct than the integration of $\sec(\theta)$.

EXAMPLE 4 Find $\int \tan(\theta)\ d\theta$.
SOLUTION

$$\begin{aligned}\int \tan(\theta)\ d\theta &= \int \tfrac{\sin(\theta)}{\cos(\theta)}\ d\theta\\ &= \int \tfrac{-du}{u} \qquad \text{(substitute } u=\cos(\theta))\\ &= -\ln(u)+C\\ &= -\ln|\cos(\theta)|+C.\end{aligned}$$

◇

Finding $\int \tan^2(\theta)\ d\theta$ is comparatively easy. Using the trigonometric identity $\tan^2(\theta) = \sec^2(\theta) - 1$, we obtain

$$\begin{aligned}\int \tan^2(\theta)\ d\theta &= \int (\sec^2(\theta)-1)\ d\theta\\ &= \tan(\theta)-\theta+C.\end{aligned}$$

The Substitution $u = \sqrt[n]{ax+b}$

The next example illustrates the use of the substitution $u = \sqrt[n]{ax+b}$. After the example we describe the integrands for which the substitution is appropriate.

EXAMPLE 5 Find $\int_4^7 x^2\sqrt{3x+4}\ dx$.
SOLUTION Let $u = \sqrt{3x+4}$, hence $u^2 = 3x+4$. Thus $x = (u^2-4)/3$ and $dx = (2u\ du)/3$. Moreover, as x goes from 4 to 7, u goes from $\sqrt{16} = 4$ to $\sqrt{25} = 5$. Thus

$$\begin{aligned}
\int_4^7 x^2\sqrt{3x+4}\ dx &= \int_4^5 \left(\frac{u^2-4}{3}\right)^2 u\frac{2u\ du}{3} \\
&= \frac{2}{27}\int_4^5 (u^2-4)^2u^2\ du \\
&= \frac{2}{27}\int_4^5 (u^6 - 8u^4 + 16u^2)\ du \\
&= \frac{2}{27}\left(\frac{u^7}{7} - \frac{8u^5}{5} + \frac{16u^3}{3}\right)\Bigg|_4^5 \\
&= \frac{2}{27}\left(\left(\frac{5^7}{7} - \frac{8\cdot 5^5}{5} + \frac{16\cdot 5^3}{3}\right) - \left(\frac{4^7}{7} - \frac{8\cdot 4^5}{5} + \frac{16\cdot 4^3}{3}\right)\right) \\
&= \frac{1214614}{2835} \approx 428.43527.
\end{aligned}$$

◇

The substitution in Example 5 illustrates the substitution $u = \sqrt[n]{ax+b}$, where the integer n is greater than or equal to 2. It may be used to integrate any "rational functions of x and $u = \sqrt[n]{ax+b}$", which we define as follows.

A **polynomial in two variables** x **and** y is a sum of terms of the form

$$a_{mn}x^my^n, \tag{7.4}$$

where m and n are nonnegative integers and a_{mn} is a real number. (The symbol a_{mn} is short for $a_{m,n}$. The comma is usually omitted.) For instance, the expression $2x^3\sqrt{2}xy^7 + xy$ is a polynomial in x and y. The quotient of two such polynomials is call a **rational function of** x **and** y.

Let $R(x,y)$ be a rational function of x and y. Let n be an integer greater than or equal to 2. Replacing y by $u = \sqrt[n]{ax+b}$ creates what is called a "rational function of x and $u = \sqrt[n]{ax+b}$", denoted $R(x, \sqrt[n]{ax+b}$. For instance, if

$$R(x, y) = \frac{x + y^2}{2x - y}. \tag{7.5}$$

then replacing y by $\sqrt[3]{4x+5}$ yields

$$R(x, \sqrt[3]{4x+5}) = \frac{x + (\sqrt[3]{4x+5})}{2x - \sqrt[3]{4x+5}}, \tag{7.6}$$

a rational function of x and $\sqrt[3]{4x+5}$. (

To integrate $R(x, \sqrt[n]{ax+b})$ To integrate $R(x, \sqrt[n]{ax+b})$, let $u = \sqrt[n]{ax+b}$. Then $u^n = ax + b, x = (u^n - b)/a$ and $dx = nu^{n-1}\, du/a$. The integrand is now a rational function of u and can be integrated by partial fractions.

There are many other substitutions, and we describe four of them. Since you may see them used occasionally, it is good to know what they are.

Three Trigonometric Substitutions

Any rational function of x and $\sqrt{a^2 - x^2}$, where a is a constant, is transformed into a rational function of $\cos(\theta)$ by the substitution $x = a\sin(\theta)$. Similar substitutions are possible in situations involving $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$. In each case, one of the trigonometric identities $1 - \sin^2(\theta) = \cos^2(\theta)$, $\tan^2(\theta) + 1$, or $\sec^2(\theta) - 1 = \tan^2(\theta)$ converts a sum or difference of squares into a perfect square.

If the integrand is a rational function of x and :

How to integrate $R(x, \sqrt{a^2 - x^2}$ $R(x, \sqrt{a^2 + x^2}$ $R(x, \sqrt{x^2 - a^2}$

Case 1 $\sqrt{a^2 - x^2}$; let $x = a\sin(\theta)$ $\quad (a > 0, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$.

Case 2 $\sqrt{a^2 + x^2}$; let $x = a\tan(\theta)$ $\quad (a > 0, -\frac{\pi}{2} < \theta < \frac{\pi}{2})$.

Case 3 $\sqrt{x^2 - a^2}$; let $x = a\sec(\theta)$ $\quad (a > 0, 0 \le \theta \le \pi, \theta \ne \frac{\pi}{2})$.

The motivation is simple. Consider Case 1, for instance. If you replace x in $\sqrt{a^2 - x^2}$ by $a\sin(\theta)$, you obtain

How to make the square root sign in $\sqrt{a^2 - x^2}$ disappear

$$\begin{aligned} \sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin(\theta))^2} &= \sqrt{a^2(1 - \sin^2(\theta)} \\ &= \sqrt{a^2\cos^2(\theta)} \\ &= a\cos(\theta). \end{aligned}$$

(Keep in mind that a and $\cos(\theta)$ are positive.) The important thing is that *the square root sign disappears.*

Case 3 raises a fine point. We have $a > 0$. However, whenever x is negative, θ is an angle in the second-quadrant, so $\tan(\theta)$ is *negative.* In that case,

If $c < 0$, $\sqrt{c^2} = -c$.

$$
\begin{aligned}
\sqrt{x^2 - a^2} &= \sqrt{(a\sec(\theta))^2 - a^2} \\
&= a\sqrt{\sec^2(\theta) - 1} \\
&= a\sqrt{\tan^2(\theta)} \\
&= a(-\tan(\theta)) \qquad \text{since } -\tan(\theta) \text{ is positive}
\end{aligned}
$$

In the Examples and Exercises involving Case 3 it will be assumed that x varies through nonnegative values, so that θ remains in the first quadrant and $\sqrt{\sec^2(\theta) - 1} = \tan(\theta)$.

Note that for $\sqrt{a^2 - x^2}$ to be meaningful, $|x|$ must be no larger than a. On the other hand, for $\sqrt{x^2 - a^2}$ to be meaningful, $|x|$ must be at least as large as a. The quantity $\sqrt{a^2 + x^2}$ is meaningful for all values of x.

EXAMPLE 6 Compute $\int \sqrt{1 + x^2}\, dx$

SOLUTION The identity $\sec(\theta) = \sqrt{1 + \tan^2(\theta)}$ suggests the substitution described in Case 2:

$$x = \tan(\theta). \tag{7.7}$$

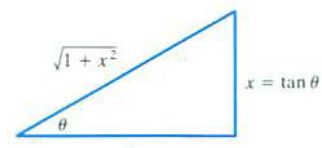


Figure 7.5.2:

Hence

$$dx = \sec^2(\theta)\, d\theta. \tag{7.8}$$

(See Figure 7.5.2 for the geometry of this substitution.)

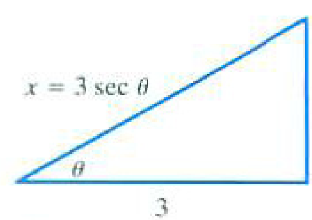
Thus $$\int \sqrt{1 + x^2}\, dx = \int \sec(\theta)\sec^2(\theta)\, d\theta = \int \sec^3(\theta)\, d\theta.$$

By Formula 55(?) from the integral table on the back cover of this text,

$$\int \sec^3(\theta)\, d\theta = \frac{\sec(\theta)\tan(\theta)}{2} + \frac{1}{2}\ln|\sec(\theta) + \tan(\theta)| + C. \tag{7.9}$$

To express the antiderivative just obtained in terms of x rather than θ, it is necessary to express $\tan\theta$ and $\sec\theta$ in terms of x. Starting with the deinition $x = \tan(\theta)$, find $\sec(\theta)$ by means of the relation $\sec(\theta) = \sqrt{1 + \tan^2(\theta)} = \sqrt{1 + x^2}$, as in Figure 7.5.2. Thus

$$\int \sqrt{1 + x^2}\, dx = \frac{x\sqrt{1 + x^2}}{2} + \frac{1}{2}\ln\left(\sqrt{1 + x^2} + x\right) + C. \tag{7.10}$$

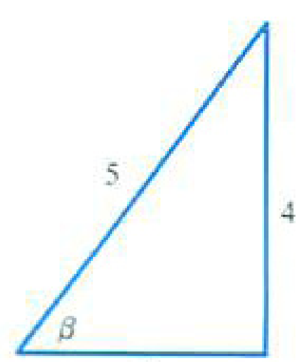


◇

Figure 7.5.3:

EXAMPLE 7 Compute $\int_4^5 \frac{dx}{\sqrt{x^2 - 9}}$.

SOLUTION Let $x = 3\sec(\theta)$; hence $dx = 3\sec(\theta)\tan(\theta)\, d\theta$. (See Fig-



ure 7.5.3.) Thus, letting $\alpha = \sec^{-1}(4/3)$ and $\beta = \sec^{-1}(5/3)$, we obtain

$$\begin{aligned}
\int_4^5 \frac{dx}{\sqrt{x^2-9}} &= \int_\alpha^\beta \frac{2\sec(\theta)\tan(\theta)\ d\theta}{\sqrt{9\sec^2(\theta)-9}} \\
&= \int_\alpha^\beta \frac{\sec(\theta)\tan(\theta)\ d\theta}{\tan(\theta)} \\
&= \int_\alpha^\beta \sec(theta)\ d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)||_\alpha^\beta \\
&= \ln\left(\tfrac{5}{3}+\tfrac{4}{3}\right) - \ln\left(\tfrac{4}{3}+\tfrac{\sqrt{7}}{3}\right) \quad \text{using Figure 7.5.4 and Figure 7.5.5 to find } \tan(\theta) \text{ at the limits of integration} \\
&= \ln(3) - \ln\left(\tfrac{4+\sqrt{7}}{3}\right) \\
&= 2\ln(3) - \ln(4+\sqrt{7}) \\
&= = \ln\left(\tfrac{9}{4+\sqrt{7}}\right) \approx 0.30325.
\end{aligned}$$

◇

A Half-Angle Substitution for $R(\cos\theta, \sin\theta$

Any rational function of $\cos(\theta)$ and $\sin(\theta)$ is transformed into a rational function of u by the substitution $u = \tan(\theta/2)$. See Exercises 50 and 51. This is sometimes useful after one of the three basic trigonometric substitutions has been used, leaving the integrand in terms of $\cos(\theta)$ and $\sin(\theta)$. The $u = \tan(\theta/2)$ substitution then yields an integral that can be done by partial fractions.

Summary

We discussed some special integrals and integration techniques. First we saw how to compute

$$\int \sin(mx)\sin(nx)\,dx, \quad \int \sin(mx)\cos(nx)\,dx, \quad \int \cos(mx)\cos(nx)\,dx, \quad \int \sin^2(x)\,dx, \quad \int \cos^2(x)\,dx, \quad \int \tag{7.11}$$

The integration of higher powers of the trionometric functions is discussed in the exercises.

We also pointed out that the substitution $u = \sqrt[n]{ax+b}$ transforms $R(x, \sqrt[n]{ax+b})$ into a rational function of u, which can be treated by partial fractions.

$R(x, \sqrt[n]{a^2-x^2})$, $R(x, \sqrt[n]{x^2-a^2})$ and $R(x, \sqrt[n]{a^2+x^2})$ can be transformed into rational functions of $\cos(\theta)$ and $\sin(\theta)$ by trigonometric substitutions. Any $R(\cos(\theta), \sin(\theta))$ can be transformed into a rational function of u by the substitution $u = \tan(\theta/2)$, which can then be integrated by partial fractions.

EXERCISES for 7.5 *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 16 are related to Examples 1 to 4. In Exercises 1 to 14 find he integrals.

1.[R] $\int \sin(5x)\sin(3x)\ dx$

2.[R] $\int \sin(5x)\cos(2x)\ dx$

3.[R] $\int \cos(3x)\sin(2x)\ dx$

4.[R] $\int \cos(2\pi x)\sin(5\pi x)\ dx$

5.[R] $\int \sin^2(3x)\ dx$

6.[R] $\int \cos^2(5x)\ dx$

7.[R] $\int (3\sin(2x) + 4\sin^2(5x))\ dx$

8.[R] $\int (5\cos(2x) + \cos^2(7x))\ dx$

9.[R] $\int (3\sin^2(\pi x) + 4\cos^2(\pi x))\ dx$

10.[R] $\int \sec(3\theta)\ d\theta$

11.[R] $\int \tan(2\theta)\ d\theta$

12.[R] $\int \sec^2(4x)\ dx$

13.[R] $\int \tan^2(5x)\ dx$

14.[R] $\int \frac{dx}{\cos^2(3x)}$

15.[R] Show that $\sin(A)\sin(B) = 1/2\cos(A-B) - 1/2\cos(A+B)$.

16.[R] Show that $\sin(A)\cos(B) = 1/2\sin(A+B) + 1/2\sin(A-B)$.

Exercises 17 to 26 concern the substitution $u = \sqrt[n]{ax+b}$. In each case evaluate the integral.

17.[R] $\int x^2\sqrt{2x+1}\ dx$

18.[R] $\int \frac{x^2\ dx}{\sqrt[3]{x+1}}$

19.[R] $\int \frac{dx}{\sqrt{x+3}}$

20.[R] $\int \frac{\sqrt{2x+1}}{x}\ dx$

21.[R] $\int x\sqrt[3]{3x+2}\ dx$

22.[R] $\int \frac{\sqrt{x}+3}{\sqrt{x}-2}\ dx$

23.[R] $\int \frac{x\ dx}{\sqrt{x}+3}$

24.[R] $\int x(3x+2)^{5/3}\ dx$

25.[R] $\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}}$ HINT: Let $u = \sqrt[6]{x}$.

26.[R] $(x+2)\sqrt[5]{x-3}\ dx$

Exercises 27 and 28 concern recursion formulas for $\tan^n(\theta)$ and $\sec^n(\theta)$.

27.[R] In the text we found $\int \tan(\theta)\ d\theta$ and $\int \tan^2(\theta)\ d\theta$ See Example 4.

(a) Obtain the recursion

$$\int \tan^n(\theta)\ d\theta = \frac{tan^{n-1}(\theta)}{n-1} - \int \tan^{n-2}(\theta)\ d\theta.$$

Begin by writing

$$\begin{aligned} \tan^n(\theta) &= \tan^{n-2}(\theta)\tan^2(\theta) \\ &= \tan^{n-2}(\theta)(\sec^2(\theta)-1). \end{aligned}$$

(b) Use the recursion to find $\int \tan^3(\theta)\ d\theta$.

(c) Find $\int \tan^4(\theta)\ d\theta$.

28.[R] In the text we found $\int \sec(\theta)\ d\theta$ and $\int \sec^2(\theta)\ d\theta$. See Example 3.

(a) Obtain the recursion

$$\int \sec^n(\theta)\ d\theta = \frac{\sec^{n-2}(\theta)\tan(\theta)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(\theta)\ d\theta.$$

Begin by writing $\sec^n(\theta) = \sec^{n-2}(\theta)\sec^2(\theta)$, and integrating by parts. After the integration, $\tan^2(\theta)$ will appear in an integrand. Write it as $\sec^2(\theta) - 1$.

(b) Evaluate $\int \sec^3(\theta)\ d\theta$.

(c) Evaluate $\int \frac{d\theta}{\cos^4(\theta)}$.

(d) Evaluate $\int \sec^2(2x)\ dx$.

29.[R] Find

(a) $\int \csc(\theta)\ d\theta$

(b) $\int \csc^2(\theta)\ d\theta$

30.[R] Find

(a) $\int \cot(\theta)\ d\theta$

(b) $\int \cot^2(\theta)\ d\theta$

31.[R] Consider $\int \sin^n(\theta)\cos^m(\theta)\ d\theta$, where m and n are nonnegative integers, m odd. To evaluate $\int \sin^n(\theta)\cos^m(\theta)\ d\theta$, write it as $\int \sin^n(\theta)\cos^{m-1}(\theta)\cos(\theta)\ d\theta$. Then rewrite $\cos^{m-1}(\theta)$ as $(1-\sin^2(\theta))^{(m-1)/2}$ and use the substitution $u = \sin(\theta)$. Using this technique, find

(a) $\int \sin^3(\theta)\cos^3(\theta)\ d\theta$

(b) $\int \sin^4(\theta)\cos(\theta)\ d\theta$

(c) $\int_0^{\pi/2} \sin^4(\theta)\cos^3(\theta)\ d\theta$

(d) $\int \cos^5(\theta)\ d\theta$.

32.[R] How would you integrate $\int \sin^n(\theta)\cos^m(\theta)\ d\theta$, where m and n are nonneg- See Exercise 31.

ative integers, n odd? Illustrate your techniques by three examples.

33.[R] The techniques in Exercises 31 and 32 apply to $\int \sin^n(\theta)\cos^m(\theta)\,d\theta$ only when at least one of m and n is odd. If both are even, first use the identities

See Exercises 31 and 32.

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \text{ and } \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}.$$

You will get a polynomial in $\cos(2\theta)$. If $\cos(2\theta)$ appears only to odd powers, the technique of Exercise 31 suffices. To treat an even power of $\cos(2\theta)$, use the identity $\cos^2(2\theta) = (1+\cos(4\theta))/2$ and continue. Using this method find

(a) $\int \cos^2(\theta)\sin^4(\theta)\,d\theta$

(b) $\int_0^{\pi/4} \cos^2(\theta)\sin^2(\theta)\,d\theta$

34.[R]

(a) Show that
$$\frac{1}{2}\ln\left(\frac{1+\sin(\alpha}{1-\sin(\alpha)}\right) = \ln\left(\tan\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)\right).$$

(b) Show that
$$\frac{1}{2}\ln\left(\frac{1+\sin(\alpha}{1-\sin(\alpha)}\right) = \ln|\sec(\alpha)+\tan(\alpha)|.$$

These identities show that Bond's conjecture for $\int \sec(\theta)\,d\theta$ agrees with the result in Example 3.

35.[R] The region R under $y = \sin(x)$ and above $[0,\pi]$ is revolved about the x axis to produce a solid S.

(a) Draw R.

(b) Draw S.

(c) Set up a definite integral for the area of R.

(d) Set up a definite integral for the volume of S.

(e) Evaluate the integrals in (c) and (d).

The theory of Fourier Series studies such representations, with n replaced by ∞.

36.[R] An arbitrary periodic sound wave can be approximated by a sum of simpler sound waves that correspond to pure pitches. This suggests representing a function $y = f(x)$ as the sum of cosine and sine functions:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx),$$

In Exercises 37 to 47 find the integrals using trigonometric substitution. (a is a positive constant.)

37.[R] $\int \sqrt{4 - x^2}\, dx$

38.[R] $\int \frac{dx}{\sqrt{9+x^2}}$

39.[R] $\int \frac{x^2\, dx}{\sqrt{x^2-9}}$

40.[R] $\int x^3\sqrt{1 - x^2}\, dx$

41.[R] $\int \frac{\sqrt{4+x^2}}{x}\, dx$

42.[R] $\int \sqrt{a^2 - x^2}\, dx$

43.[R] $\int \frac{dx}{\sqrt{a^2-x^2}}$

44.[R] $\int \sqrt{a^2 + x^2}\, dx$

45.[R] $\int \sqrt{a^2 - x^2}\, dx$

46.[R] $\int \frac{dx}{\sqrt{25x^2-16}}$

47.[R] $\int_{\sqrt{2}}^{2} \sqrt{x^2 - 1}\, dx$

48.[R] Transform the following integrals into integrals of rational functions of $\cos(\theta)$ and $\sin(\theta)$. Do *not* try to evaluate the integrals.

(a) $\int \frac{x+\sqrt{9-x^2}}{x^3}\, dx$

(b) $\int \frac{x^3\sqrt{5-x^2}}{1+\sqrt{5x^2}}\, dx$

49.[R] Transform the following integrals into integrals of rational functions of $\cos(\theta)$ and $\sin(\theta)$. Do *not* try to evaluate the integrals.

(a) $\int \frac{x^2+\sqrt{x^2-9}}{x}\, dx$

(b) $\int \frac{x^3\sqrt{5+x^2}}{x+2}]dx$

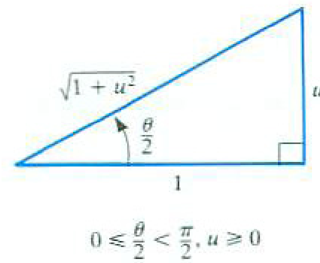


Figure 7.5.6:

Exercises7-6ex-C1 to 52 concern $\int R(\cos(\theta), \sin(\theta))\, d\theta$.

50.[R] Let $-\pi < \theta < \pi$ and $u = \tan(\theta/2)$. (See Figure 7.5.6.) The following steps show that this substitution transforms $\int R(\cos\theta, \sin\theta) d\theta$ into the integral of a rational function of u (which can be integrated by partial fractions.)

(a) Show that $\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+u^2}}$ and $\sin\frac{\theta}{2} = \frac{u}{\sqrt{1+u^2}}$.

(b) Show that $\cos\theta = \frac{1-u^2}{1+u^2}$.

(c) Show that $\sin\theta = \frac{2u}{1+u^2}$.

(d) Show that $d\theta = \frac{2du}{1+u^2}$. HINT: Note that $\theta = 2\tan^{-1} u$.

Combining (b), (c), and (d) shows that the substitution $u = \tan(\theta/2)$ transforms $\int R(\cos(\theta), \sin(\theta))\, d\theta$ into the integrals of a rational function of u.

51.[R] Using the substitution $u = \tan(\theta/2)$, transform the following integrals into integrals of rational functions. (Do not evaluate them.)

(a) $\int \frac{1+\sin(\theta)}{1+\cos^2(\theta)}\, d\theta$

(b) $\int \frac{5+\cos(\theta)}{(\sin(\theta))^2+\cos(\theta)}\, d\theta$

(c) $\int_0^{\pi/2} \frac{5\ d\theta}{2\cos(\theta)+3\sin(\theta)}$ (Be sure to transform the limits of integraton also.)

52.[R] Compute $\int_0^{\pi/2} \frac{d\theta}{4\sin(\theta)+3\cos(\theta)}$.

53.[R] Explain why any rational function of $\tan(\theta)$ and $\sec(\theta)$ has an elementary antiderivative

54.[R] Let $0 \leq \theta < \pi/2$.

(a) Show that $\int \sec(\theta)\, d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$, by differentiating $\ln|\sec(\theta) + \tan(\theta)|$.

(b) Does (a) contradict the formula given in Example 3?

55.[R] Show that any rational function of x, $\sqrt{x+a}$, $\sqrt{x+b}$ has an elementary antiderivative. HINT: Use the substitution $u = \sqrt{x+a}$.

However, it is not the case that every rational function of $\sqrt{x+a}$, $\sqrt{x+b}$, and $\sqrt{x+c}$ has an elementary antiderivative. For instance,

$$\int \frac{dx}{\sqrt{x}\sqrt{x+1}\sqrt{x-1}} = \int \frac{dx}{\sqrt{x^3-x}}$$

is not an elementary function.

56.[R] Every rational function of x and $\sqrt[n]{(ax+b)/(cx+d)}$ has an elementary antiderivative. Explain why.

7.6 What to do When Confronted with an Integral

Since the exercises in each section of this chapter focus on the techniques of that section, it is usually clear what technique to use on a give integral. But what if an integral is met "in the wild", where there is no clue how to evaluate it? This section suggests what to do in this typical situation.

The more integrals you compute, the more quickly you will be able to choose an appropriate technique. Moreover, such practice will put you at ease in using integral tables or computer software. Besides, it may be quicker just to find an integral by hand.

The following table summarizes the techniques and shortcuts emphasized in this chapter.

General	Substitution	Section 7.2
	Integration by Parts	Section 7.3
	Partial Fractions	Section 7.2 and 7.4
Special	if f is odd, then $\int_{-a}^{a} f(x)\ dx = 0$	Section 7.1
	if f is even, then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$	Section 7.1
	$\int_0^a \sqrt{a^2-x^2}\ dx = \frac{\pi a^2}{4}$	Section 7.1
	$\int \sin(mx)\sin(nx)\ dx$, etc.	Section 7.5
	$\int \sin^2(\theta)\ d\theta$, etc.	Section 7.5
	$\int \tan(\theta)\ d\theta$, $\int \sec(\theta)\ d\theta$, etc.	Section 7.5
	$\int R(x, \sqrt[n]{ax+b})\ dx$	Section 7.5
	$\int R(x, \sqrt{a^2-x^2})\ dx$, etc.	Section 7.5

Table 7.6.1:

Exercises in Section 7.5 develop other specialized techniques, but they will not be required in this section.

A few examples will illustrate how to chose a method for computing an antiderivative.

EXAMPLE 1

$$\int \frac{x\ dx}{1+x^4}$$

See Exercise 57 in Section 7.5

SOLUTION DISCUSSION Since the integrand is a rational function of x, partial fractions would work. This requires factoring x^4+1 and then representing $x/(1+x^4)$ as a sum of partial fractions. With some struggle it can be found that

$$x^4+1 = x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$$

The constants A, B, C, and D will have to be found such that

$$\frac{x}{1+x^4} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$

The method would work but would certainly be tedious.

Try another attack. The numerator x is almost the derivative of x^2. The substitution $u = x^2$ is at least worth testing:

$$u = x^2 \qquad du = 2x\ dx,$$

$$\int \frac{x\ dx}{1+x^4} = \int \frac{du/2}{1+u^2},$$

which is easy. The answer is $1/2\tan^{-1}(u) + C = 1/2\tan^{-1}(x^2) + C$. ◇

EXAMPLE 2

$$\int \frac{1+x}{1+x^2}\ dx.$$

SOLUTION DISCUSSION This is a rational function of x, but partial fractions will not help, since the integrand is already in its partial-fraction representation.

The numerator is not the derivative of the denominator, but it comes close enough to persuade us to break the integrand into summands:

$$\int \frac{1+x}{1+x^2}\ dx = \int \frac{dx}{1+x^2} + \int \frac{x\ dx}{1+x^2}.$$

Both the latter integrals can be done in your head. The first is $\tan^{-1}(x)$, and the second is $1/2\ln(1+x^2)$. So the answer is $\tan^{-1} x + 1/2\ln(1+x^2) + C$. ◇

EXAMPLE 3

$$\int \frac{e^{2x}}{1+e^x}\ dx.$$

SOLUTION DISCUSSION At first glance, this integral looks so peculiar that it may not even be elementary. However, e^x is a fairly simple function, with $d(e^x) = e^x\ dx$. This suggests trying the substitution $u = e^x$ and seeing what happens:

$$u = e^x \qquad du = e^x\ dx$$

Thus

$$dx = \frac{du}{e^x} = \frac{du}{u}.$$

It is essential to express dx completely in terms of u and du.

But what will be done to e^{2x}? Recalling that $e^{2x} = (e^x)^2$, we anticipate no problem:

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u^2}{1+u}\frac{du}{u} = \int \frac{u\,du}{1+u}.$$

which can be integrated quickly:

Long division of $u/(u+1)$ also works.

$$\begin{aligned}\int \frac{u\,du}{1+u} &= \int \frac{u+1-1}{1+u}\,du = \int \left(1 - \frac{1}{1+u}\right)\,du \\ &= u - \ln(|1+u|) + C \\ &= e^x - \ln(1+e^x) + C.\end{aligned}$$

The same substitution could have been done more elegantly:

$$\int \frac{e2x}{1+e^x}\,dx = \int \frac{e^x(e^x\,dx)}{1+e^x} = \int \frac{u\,du}{1+u}.$$

◇

EXAMPLE 4

$$\int \frac{x^3\,dx}{(1-x^2)^5}.$$

SOLUTION DISCUSSION Partial fractions would work, but the denominator, when factored, would be $(1+x)^5(1-x)^5$. There would be 10 unknown constants to find. Look for an easier approach.

Since the denominator is the obstacle, try $u = x^2$ or $u = 1 - x^2$ to see if the integrand gets simpler. Let us examine what happens in each case. Try $u = x^2$ first. Assume that we are interested only in getting an antiderivative for positive x, $x = \sqrt{u}$:

$$u = x^2 \qquad du = 2x\,dx \qquad dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}}.$$

Then

$$\int \frac{x^3\,dx}{(1-x^2)^5} = \int \frac{u^{3/2}}{(1-u)^5}\ fracdu2\sqrt{u} = \frac{1}{2}\int \frac{u\,du}{(1-u)^5}.$$

The same substitution could be carried out as follows:

$$\int \frac{x^3\,dx}{(1-x^2)^5} = \int \frac{x^2 x\,dx}{(1-x^2)^5} = \int \frac{u(du/2)}{(1-x)^5} = \frac{1}{2}\int \frac{u\,du}{(1-u)^5}.$$

The substitution $v = 1 - u$ then results in an easy integral, as the reader may check.

Observe that the two substitutions $u = x^2$ and $v = 1 - u$ are equivalent to the single substitution $v = 1 - x^2$. So, let us try the original integral with the substitution $u = 1 - x^2$, then $du = -2x\ dx$; thus

$$\int \frac{x^3\ dx}{(1-x^2)^5} = \int \frac{x^2(x\ dx)}{(1-x^2)^5} = \int \frac{(1-u)(-du/2)}{u^5} = \int \frac{1}{2}(u^{-4} - u^{-5})\ du,$$

an integral that can be computed without further substitution. So $u = 1 - x^2$ is quicker than $u = x^2$. ◇

EXAMPLE 5

$$\int x^3 e^{x^2}\ dx.$$

SOLUTION DISCUSSION Integration by parts may come to mind, since if $u = x^3$, then $du = 3x^2\ dx$ is simpler. However, dv must then be $e^{x^2}\ dx$ and force v to be nonelementary. This is a dead end.

If we can raise an exponent, we should be able to lower it.

So try integration by parts with $u = e^{x^2}$ and $dv = x^3\ dx$. What will $v\ du$ be? We have $v = x^4/4$ and $du = 2xe^{x^2}\ dx$, which is worse than the original $u\ dv$. The exponent of x has been raised by 2, from 3 to 5.

This time try $u = x^2$ and $dv = xe^{x^2}\ dx$; thus $du = 2x\ dx$ and $v = e^{x^2}/2$. Integration by parts yields

$$\begin{aligned} \int x^3 e^{x^2}\ dx &= \int \underbrace{x^2}_{u}\underbrace{xe^{x^2}\ dx}_{dv} = \underbrace{x^2}_{u}\underbrace{\frac{e^{x^2}}{2}}_{v} - \int \underbrace{\frac{e^{x^2}}{2}}_{v} 2x\ dx_{du} \\ &= \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C. \end{aligned}$$

See Exercise 71.

Another approach is to use the substitution $u = x^2$ followed by an integration by parts. ◇

EXAMPLE 6

$$\int \frac{1 - \sin(\theta)}{\theta + \cos(\theta)} d\theta.$$

See Exercise 72.

SOLUTION DISCUSSION The numerator is the derivative of the denominator, so the integral is $\ln|\theta + \cos\theta| + C$. ◇

EXAMPLE 7

$$\int \frac{1 - \sin(\theta)}{\cos(\theta)}\ d\theta.$$

SOLUTION DISCUSSION Break the integrand into two summands:

$$\begin{aligned}\int \frac{1-\sin(\theta)}{\cos(\theta)}\,d\theta &= \int\left(\frac{1}{\cos(\theta)}-\frac{\sin(\theta)}{\cos(\theta)}\right)\,d\theta \\ &= \int\left(\sec(\theta)-\tan(\theta)\right)\,d\theta \\ &= \int \sec\theta\,d\theta - \int \tan(\theta)\,d\theta \\ &= \ln|\sec(\theta)+\tan(\theta)| + \ln|\cos(\theta)| + C.\end{aligned}$$

Since $\ln(A)+\ln(B)=\ln(AB)$, the answer can be simplified to

$$\ln\left(|\sec(\theta)+\tan(\theta)|\,|\cos(\theta)|\right)+C.$$

But $\sec(\theta)\cos(\theta)=1$ and $\tan(\theta)\cos(\theta)=\sin(\theta)$. The result becomes even simpler:

$1+\sin(\theta)\geq 0$

$$\int \frac{1-sin(\theta)}{\cos(\theta)}\,d\theta = \ln\left(1+\sin(\theta)\right)+C.$$

◇

EXAMPLE 8

$$\int \frac{\ln x\ dx}{x}.$$

SOLUTION DISCUSSION Integration by parts, with $u=\ln(x)$ and $dv=dx/x$, may come to mind. In that case, $du=dx/x$ and $v=\ln(x)$; thus

$$\int \underbrace{\ln(x)}_{u}\underbrace{\frac{dx}{x}}_{dv} = \underbrace{(\ln(x))}_{u}\underbrace{(\ln(x))}_{v} - \int \underbrace{\ln(x)}_{v}\underbrace{\frac{dx}{x}}_{du}.$$

Bringing $\int \ln(x)\ dx/x$ all to one side produces the equation

$$2\int \ln(x)\frac{dx}{x} = (\ln x)^2,$$

from which it follows that

$$\int \ln(x)\frac{dx}{x} = \frac{(\ln(x))^2}{2}+C.$$

The method worked, but is not the easiest one to use. Since $1/x$ is the derivative of $\ln(x)$, we could have used the substitution $u=\ln(x)$, $du=dx/x$; thus

$$\int \frac{\ln(x)\ dx}{x} = \int u\ du = \frac{u^2}{2}+C = \frac{(\ln(x))^2}{2}+C.$$

◇

EXAMPLE 9

$$\int_0^{3/5} \sqrt{9-25x^2}\ dx.$$

SOLUTION DISCUSSION This integral reminds us of $\int_0^a \sqrt{a^2-x^2}\ dx = \pi a^2/4$, the area of a quadrant of a circle of radius a. This resemblance suggests a substitution u such that $25x^2 = 9u^2$ or $u = \frac{5}{3}x$, hence $dx = \frac{3}{5}\ du$. Then substitution gives

$$\begin{aligned}\int_0^{3/5} \sqrt{9-25x^2}\ dx &= \int_0^1 \sqrt{9-9u^2}\frac{3}{5}\ du \\ &= \frac{9}{5}\int_0^1 \sqrt{1-u^2}\ du \\ &= \frac{9}{5}\frac{\pi}{4} = \frac{9\pi}{20} \approx 1.41372.\end{aligned}$$

◇

EXAMPLE 10

$$\int \sin^5(2x)\cos(2x)\ dx.$$

SOLUTION DISCUSSION We could try integration by parts with $u = \sin^5(2x)$ and $dv = \cos(2x)\ dx$. (See Exercise 73.)

However, $\cos(2x)$ is almost the derivative of $\sin(2x)$. For this reason make the substitution

$$u = sin(2x) \qquad du = 2\cos(2x)\ dx;$$

hence

$$\cos(2x)\ dx = \frac{du}{2}.$$

Then

$$\begin{aligned}\int \sin^5(2x)\cos(2x)\ dx &= \int u^5\frac{du}{2} \\ &= \frac{1}{2}\frac{u^6}{6} + C \\ &= \frac{\sin^6(2x)}{12} + C.\end{aligned}$$

◇

EXAMPLE 11

$$\int_{-3}^{3} x^3 \cos(x)\ dx.$$

SOLUTION DISCUSSION Since the integrand is of the form $P(x)\cos(x)$, where P is a polynomial, repeated integration by parts would work. On the other hand, x^3 is an odd function and $\cos(x)$ is an even function. The integrand is therefore an odd function and the integral over $[-3, 3]$ is 0. ◇

EXAMPLE 12

$$\int \sin^2(3x)\ dx.$$

or use $\int \sin mx \sin nx\ dx$

SOLUTION DISCUSSION You could rewrite this integral as $\int \sin(3x)\sin(3x)\ dx$ and use integration by parts. However, it is easier to use the trigonometric identity $\sin^2(\theta) = (1 - \cos 2(\theta))/2$:

$$\begin{aligned}\int \sin^2(3x)\ dx &= \int \frac{1-\cos(6x)}{2}\ dx\\ &= \int \frac{dx}{2} - \int \frac{\cos(6x)}{2}\ dx\\ &= \frac{x}{2} - \frac{\sin(6x)}{12} + C.\end{aligned}$$

◇

EXAMPLE 13

$$\int_{1}^{2} \frac{x^3 - 1}{(x+2)^2}\ dx.$$

SOLUTION DISCUSSION Partial fractions would certainly work. (The first step would be division of $x^3 - 1$ by $x^2 + 4x + 4$.) However, the substitution $u = x + 2$ is easier because it makes the denominator simply u^2. We have

$$u = x + 2 \qquad du = dx \quad \text{and} \quad x = u - 2.$$

Thus

Note the new limits for u.

$$\begin{aligned}
\int_1^2 \frac{x^3-1}{(x+2)^2}\,dx &= \int_3^4 \frac{(u-2)^3-1}{u^2}\,du \\
&= \int_3^4 \frac{u^3-6u^2+12u-8-1}{u^2}\,du \\
&= \int_3^4 \left(u-6+\frac{12}{u}-\frac{9}{u^2}\right)\,du \\
&= \left.\left(\frac{u^2}{2}-6u+12\ln|u|+\frac{9}{u}\right)\right|_3^4 \\
&= \left(8-24+12\ln(4)+\frac{9}{4}\right)-\left(\frac{9}{2}-18+12\ln(3)+3\right) \\
&= -(\frac{13}{4})+12\ln(4)-12\ln(3) \approx 0.20218.
\end{aligned}$$

$\diamond$

EXERCISES for 7.6 *Key:* R–routine, M–moderate, C–challenging

All the integrals in Exercises 1 to 59 are elementary. In each case, list the technique or techniques that could be used to evaluate the integral. If there is a preferred technique, state what it is (and why). Do *not* evaluate the integrals. (For practicing in evaluating integrals, see Section 7.6, Exercises **??**–**??**.)

1.[R] $\int \frac{1+x}{x^2}\ dx$

2.[R] $\int \frac{x^2}{1+x}\ dx$

3.[R] $\int \frac{dx}{x^2+x^3}$

4.[R] $\int \frac{x+1}{x^2+x^3}\ dx$

5.[R] $\int \tan^{-1}(2x)\ dx$

6.[R] $\int \sin^{-1}(2x)\ dx$

7.[R] $\int x^{10}e^x\ dx$

8.[R] $\int \frac{ln(x)}{x^2}\ dx$

9.[R] $\int \frac{sec^2(\theta)\ d\theta}{\tan(\theta)}$

10.[R] $\int \frac{\tan(\theta)\ d\theta}{\sin^2(\theta)}$

11.[R] $\int \frac{x^3}{\sqrt[3]{x+2}}\ dx$

12.[R] $\int \frac{x^2}{\sqrt[3]{x^3+2}}\ dx$

13.[R] $\int \frac{2x+1}{(x^2+x+1)^5}\ dx$

14.[R] $\int \sqrt{\cos(\theta)}\sin(\theta)\ d\theta$

15.[R] $\int \tan^2(\theta)\ d\theta$

16.[R] $\int \frac{d\theta}{\sec^2(\theta)}$

17.[R] $\int e^{\sqrt{x}}\ dx$

18.[R] $\int \sin\sqrt{x}\ dx$

19.[R] $\int \frac{dx}{(x^2-4x+3)^2}$

20.[R] $\int \frac{x+1}{x^5}\ dx$

21.[R] $\int \frac{x^5}{x+1}\ dx$

22.[R] $\int \frac{\ln(x)}{x(1+\ln(x))}\ dx$

23.[R] $\int \frac{e^{3x}\ dx}{1+e^x+e^{2x}}$

24.[R] $\int \frac{\cos(x)\ dx}{(3+\sin(x))^2}$

25.[R] $\int \ln(e^x)\ dx$

26.[R] $\int \ln(\sqrt[3]{x}\ dx$

27.[R] $\int \frac{x^4-1}{x+2}\ dx$

28.[R] $\int \frac{x+2}{x^4-1}\ dx$

29.[R] $\int \frac{dx}{\sqrt{x}(3+\sqrt{x})^2}$

30.[R] $\int \frac{dx}{(3+\sqrt{x})^3}$

31.[R] $\int (1+\tan(\theta))^3\sec^2(\theta)\ d\theta$

32.[R] $\int \frac{e^{2x}+1}{e^x-e^{-x}}\ dx$

33.[R] $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\ dx$

34.[R] $\int \frac{(x+3)(\sqrt{x+2}+1)}{\sqrt{x+2}-1}\ dx$

35.[R] $\int \frac{(\sqrt[3]{x+2}-1\ dx}{\sqrt{x+2}+1}$

36.[R] $\int \frac{dx}{x^2-9}$

37.[R] $\int \frac{x+7}{(3x+2^{10}} \, dx$

38.[R] $\int \frac{x^3 \, dx}{(3x+2)^7}$

39.[R] $\int \frac{2^x+3^x}{4^x} \, dx$

40.[R] $\int \frac{2^x}{1+2^x} \, dx$

41.[R] $\int \frac{(x+\sin -1(x)) \, dx}{\sqrt{1-x^2}}$

42.[R] $\int \frac{x+\tan^{-1}(x)}{1+x^2} \, dx$

43.[R] $\int x^3\sqrt{1+x^2} \, dx$

44.[R] $\int x(1+x^2)^{3/2} \, dx$

45.[R] $\int \frac{x \, dx}{\sqrt{x^2-1}}$

46.[R] $\int \frac{x^3}{\sqrt{x^2-1}} \, dx$

47.[R] $\int \frac{x \, dx}{(x^2-9)^{3/2}}$

48.[R] $\int \frac{\tan^{-1}(x)}{1+x^2} \, dx$

49.[R] $\int \frac{\tan^{-1}(x)}{x^2} \, dx$

50.[R] $\int \frac{\sin(\ln(x)}{)} x \, dx$

51.[R] $\int \cos(x)\ln(\sin(x)) \, dx$

52.[R] $\int \frac{x \, dx}{\sqrt{x^2+4}}$

53.[R] $\int \frac{dx}{x^2+x+5}$

54.[R] $\int \frac{x \, dx}{x^2+x+5}$

55.[R] $\int \frac{x+3}{(x+1)^5} \, dx$

56.[R] $\int \frac{x^5+x+\sqrt{x}}{x^3}\ dx$

57.[R] $\int (x^2+9)^{10} x\ dx$

58.[R] $\int (x^2+9)^{10} x^3\ dx$

59.[R] $\int \frac{x^4\ dx}{(x+1)^2(x-2)^3}$

In Exercises 60 to 62, (a) determine which positive integers n yield integrals you can evaluate and (b) evaluate them.

60.[R] $\int \sqrt{1+x^n}\ dx$

61.[R] $\int (1+x^2)^{1/n}\ dx$

62.[R] $\int (1+x)^{1/n}\sqrt{1-x}\ dx$

63.[R] Find $\int \frac{dx}{\sqrt{x+2}-\sqrt{x-2}}$.

64.[R] Find $\int \sqrt{1-\cos(x)}\ dx$.

In Exercises 65 to 70, evaluate the integrals. Trigonometric substitution may be helpful.

65.[R] $\int \frac{x\ dx}{(\sqrt{9-x^2}^5}$

66.[R] $\int \frac{dx}{\sqrt{9-x^2}}$

67.[R] $\int \frac{dx}{x\sqrt{x^2+9}}$

68.[R] $\int \frac{x\ dx}{\sqrt{x^2+9}}$

69.[R] $\int \frac{dx}{x+\sqrt{x^2+25}}$

70.[R] $\int (x^3+x^2)\sqrt{x^2-5}\ dx$

71.[R]

(a) Evaluate $\int x^3 e^{x^2}$ using the substitution $u = x^2$ followed by an application of integration by parts.

(b) How does this approach compare with the one used in Example 5?

72.[R] In Example 6 it is found that

$$\int \frac{1-\sin(\theta)}{\theta+\cos(\theta)} d\theta = \ln|\theta+\cos\theta| + C.$$

Check this result by differentiation.

73.[R]

(a) Use integration parts to evaluate $\int \sin^5(2x)\cos(2x)\ dx$.

(b) How does this approach compare with the one used in Example 10?

7.S Chapter Summary

EXERCISES for 7.S *Key:* R–routine, M–moderate, C–challenging

74.[R] Evaluate the definite integral set up in Exercise 21 in Section 6.1.

75.[R] Evaluate the definite integral set up in Exercise 22 in Section 6.1.

76.[R] Evaluate the definite integral set up in Exercise 23 in Section 6.1.

77.[R] Evaluate the definite integral set up in Exercise 24 in Section 6.1.

78.[R] Evaluate the definite integral set up in Exercise 25 in Section 6.1.

79.[R] Evaluate the definite integral set up in Exercise 26 in Section 6.1.

80.[R] Evaluate the definite integral set up in Exercise 27 in Section 6.1.

81.[R] Evaluate the definite integral set up in Exercise 28 in Section 6.1.

82.[R] Evaluate the definite integral set up in Exercise 29 in Section 6.1.

83.[R] Evaluate the definite integral set up in Exercise 31 in Section 6.1.

84.[R] Consider the region $\mathcal{R}$ below the line $y = e$, above $y = e^x$, and to the right of the y-axis is revolved around the y-axis to form a solid $\mathcal{S}$. In Example 1 in Section 6.5 it is shown that the definite integral for the volume of $\mathcal{S}$ using disks is

$$\int_1^e \pi \left(\ln(y)\right)^2 \, dy$$

and the volume of $\mathcal{S}$ using coaxial shells is

$$\int_0^1 2\pi x \left(e - e^x\right) \, dx.$$

Evaluate each integral. Which integral is easier to evaluate?

85.[R] Consider the region $\mathcal{R}$ below the line $y = \frac{\pi}{2} - 1$, the y-axis, and the curve $y = x - \sin(x)$ is revolved around the y-axis to form a solid $\mathcal{S}$. In Example 2 in Section 6.5 it is shown that the definite integral for the volume of $\mathcal{S}$ using disks is not possible in terms of elementary functions and the volume of $\mathcal{S}$ using coaxial shells is

$$\int_0^{\pi/2} 2\pi x \left(\frac{\pi}{2} - 1 - (x - \sin(x))\right)\, dx.$$

Find the value of this integral.

86.[R] Evaluate the definite integrals set up in Exercise 1 in Section 6.5.

87.[R] Evaluate the definite integrals set up in Exercise 2 in Section 6.5.

88.[R] Evaluate the definite integrals set up in Exercise 3 in Section 6.5.

89.[R] Evaluate the definite integrals set up in Exercise 4 in Section 6.5.

90.[R] In Example 1 in Section 6.6 the total force on a submerged circular tank is found to be

At that time, the value of this integral was found using the fact that the first integral has an odd integrand over an interval symmetric about the origin and by relating the second integral to the area of a quarter circle.

(a) Evaluate the first integral using the substitution $u = 100 - 4x^2$.

(b) Evaluate the second integral using the substitution $x^2 = 25\sin^2(\theta)$.

(c) Which approach is quicker to apply?

Chapter 8

Polar Coordinates and Plane Curves

There is no calculus in Section 8.1.

This chapter presents further applications of the derivative and integral. Section 8.1 describes polar coordinates. Section 8.2 shows how to compute the area of a flat region that has a convenient descritpion in polar coordinates. Section 8.3 introduces a method of describing a curve that is especially useful in the study of motion.

Section 8.4 shows how to compute the speed of an object moving along a curved path. It also shows how to express the length of a curve as a definite integral. Section 8.5 shows how to express the area of a surface of revolution as a definite intergal. The sphere is an instance of such a surface.

Section 8.6 shows how the derivative and second derivative provides tools for measuring how curvy a curve is at each of its points. This measure, called "curvature", will be needed in Chapter 13 in the study of motion along a curve.

8.1 Polar Coordinates

Rectangular coordinates provide only one of the ways to describe points in the plane by pairs of numbers. This section describes another system call "polar coordinates".

Polar Coordinates

The rectangular coordinates x and y describe a point P in the plane as the intersection of a vertical line and a horizontal line. Polar coordinates describe a point P as the intersection of a circle and a ray from the center of that circle. They are defined as follows.

Select a point in the plane and a ray emanating from this point. The point is called the **pole**, and the ray the **polar axis**. (See Figure 8.3.1.) Measure positive angles θ counterclockwise from the polar axis and negative angles clockwise. Now let r be a number. To plot the point P that corresponds to the pair of numbers r and θ, proceed as follows:

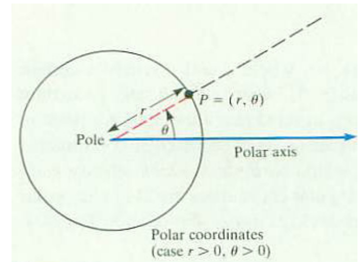


Figure 8.1.1:

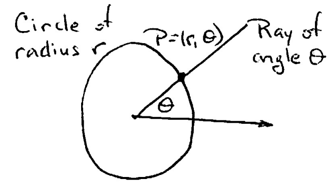


Figure 8.1.2:

- If r is positive, P is the intersection of the circle of radius r whose center is at the pole and the ray of angle θ from the pole. (See Figure 8.3.2.)
- If r is 0, P is the pole, no matter of what θ is.
- If r is negative, P is at a distance $|r|$ from the pole on the ray directly opposite the ray of angle θ, that is, on the ray of angle $\theta + \pi$.

In each case P is denoted (r, θ), and the pair r and θ are called **polar coordinates** of P. The point (r, θ) is on the circle of radius $|r|$ whose center is the pole. The pole is the midpoint of the points (r, θ) and $(-r, \theta)$. Notice that the point $(-r, \theta + \pi)$ is the same as the point $(r\theta)$. Moreover, changing the angle by 2π does not change the point; that is, $(r, \theta) = (r, \theta + 2\pi) = (r, \theta + 4\pi) = \cdot = (r, \theta + 2k\pi)$ for any integer k.

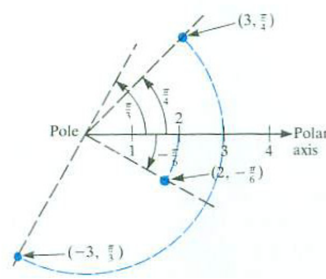


Figure 8.1.3:

EXAMPLE 1 Plot the points $(3, \pi/4)$, $(2, -\pi/6)$, $(-3, \pi/3)$ in polar coordinates.
SOLUTION

- To plot $(3, \pi/4)$, go out a distance 3 on the ray of angle $\pi/4$ (shown in Figure 8.1.3).
- To plot $(2, -\pi/6)$, go out a distance 2 on the ray of angle $-\pi/6$.
- To plot $(-3, \pi/3)$, draw the ray of angle $\pi/3$, and then go a distance 3 in the *opposite* direction from the pole. (See Figure 8.1.3.)

◇

It is customary to have the polar axis coincide with the positive x axis as in Figure 8.1.4. In that case, inspection of the diagram shows the relation between the rectangular coordinates (x, y) and the polar coordinates of the point P:

The relation between polar and rectangular coordinates.

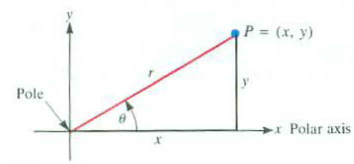


Figure 8.1.4:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$
$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

These equations hold even if r is negative. If r is positive, then $r = \sqrt{x^2+y^2}$. Furthermore, if $-\pi/2 < \theta < \pi/2$, then $\theta = \tan^{-1}(y/x)$.

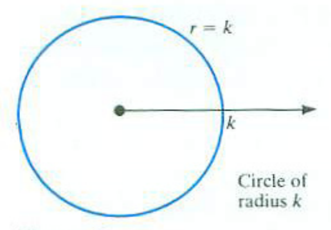


Figure 8.1.5:

Graphing $r = f(\theta)$

Just as we may graph the set of points (x, y), where x and y satisfy a certain equation, we may graph the set of points (r, θ), where r and θ satisfy a certain equation. Keep in mind that although each point in the plane is specified by a unique ordered pair (x, y) in rectangular coordinates, there are *many ordered pairs* (r, θ) *in polar coordinates which specify each point.* For instance, the point whose rectangular coordinates are $(1, 1)$ has polar coordinates $(\sqrt{2}, \pi/4)$ or $(\sqrt{2}, \pi/4 + 2\pi)$ or $(\sqrt{2}, \pi/4 + 4\pi)$ or $(-\sqrt{2}, \pi/4 + \pi)$ and so on.

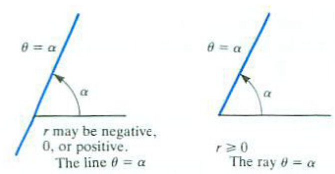


Figure 8.1.6:

The simplest equation in polar coordinates has the form $r = k$, where k is a positive constant. Its graph is the circle of radius k, centered at the pole. (See Figure 8.1.5.) The graph of $\theta = \alpha$, where α is a constant, is the line of inclination α. If we restrict r to be nonnegative, then $\theta = \alpha$ describes the ray ("half-line") of angle α. (See Figure 8.1.6.)

EXAMPLE 2 Graph $r = 1 + \cos\theta$.
SOLUTION Begin by making a table:

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	π
r	2	$1+\frac{\sqrt{2}}{2}$	1	$1-\frac{\sqrt{2}}{2}$	0	$1-\frac{\sqrt{2}}{2}$	1	$1+\frac{\sqrt{2}}{2}$	2
		≈ 1.7		≈ 0.3	$0 \approx 0.3$		≈ 1.7		

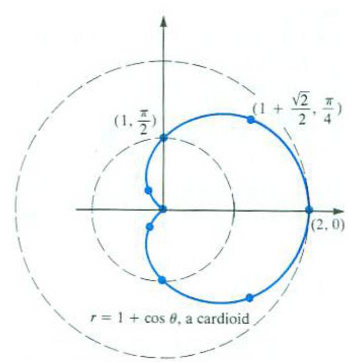


Figure 8.1.7:

Since $\cos(\theta)$ has period 2π, we consider only θ in $[0, 2\pi]$.

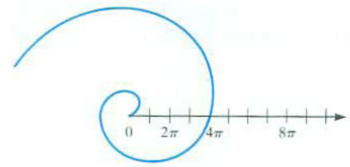
As θ goes from 0 to π, r decreases; as θ goes from π to 2π, r increases. The last point is the same as the first. The graph begins to repeat itself. This heart-shaped curve, shown in Figure 8.1.7, is called a **cardioid**. ◇

Spirals turn out to be quite easy to describe in polar coordinates. This will be illustrated by the graph of $r = 2\theta$ in the next example.

EXAMPLE 3 Graph $r = 2\theta$ for $\theta \geq 0$.
SOLUTION First make a table:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	$\cdots$
r	0	π	2π	3π	4π	5π	$\cdots$

Increasing θ by 2π does *not* produce the same value of r. As θ increaes, r increases. The graph for $\theta \geq 0$ is an endless sprial, going infinitely often around the pole. It is indicated in Figure 8.1.8. ◇



$r = 2\theta$, an archimedean spiral

Figure 8.1.8:

If a is a nonzero constant, the graph of $r = a\theta$ is called an **Archimedean spiral** for a good reason: Archimedes was the first person to study the curve, finding the area within it up to any angle and its tangent lines. The spiral with $a = 2$ is sketched in Example 3.

Polar coordinates are also convenient for describing loops arranged like the petals of a flower, as Example 4 shows.

EXAMPLE 4 Graph $r = \sin(3\theta)$.
SOLUTION Note that $\sin(3\theta)$ stays in the range -1 to 1. For instance, when $3\theta = \pi/2$, $\sin(3\theta) = \sin(\pi/2) = 1$. That tells us that when $\theta = \pi/6$, $r = \sin(3\theta) = \sin(3(\pi/6)) = \sin(\pi/2) = 1$. This case suggest that we calculate r at integer multiples of $\pi/6$, as in the following table:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
3θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{9\pi}{2}$	6π
$r = \sin(3\theta)$	0	1	0	-1	0	1	0	1	0

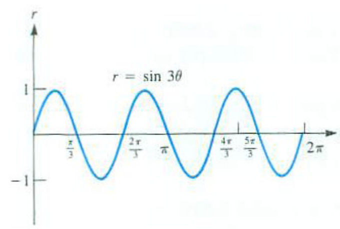
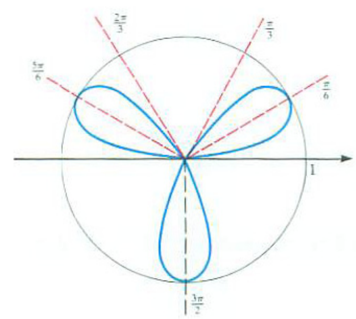


Figure 8.1.9:

The variation of r as a function of θ is shown in Figure 8.1.9. Because $\sin(\theta)$ has period 2π, $\sin(3\theta)$ has period $2\pi/3$.

As θ increases from 0 up to $\pi/3$, 3θ increases from 0 up to π. Thus r, which is $\sin(3\theta)$, goes from 0 up to 1 and then back to 0, for θ in $[0, \pi/3]$. This gives one of the three loops that make up the graph of $r = \sin(3\theta)$. For θ in $[\pi/3, 2\pi/3]$, $r = \sin(3\theta)$ is negative (or 0). This yields the lower loop in Figure 8.1.10. For θ in $[2\pi/3, \pi]$, r is again positive, and we obtain the upper left loop. Further choices of θ lead only to repetition of the loops already shown. ◇

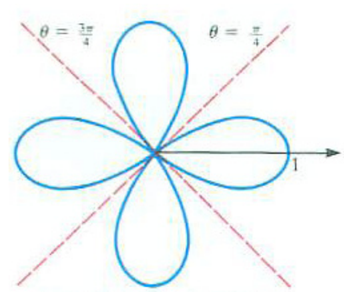


$r = \sin 3\theta$, a three-leaved rose

Figure 8.1.10:

The graph of $r = \sin(n\theta)$ or $r = \cos(n\theta)$ has n loops when n is an odd integer and $2n$ loops when n is an even integer. The next example illustrates the case when n is even.

EXAMPLE 5 Graph the four-leaved rose, $r = \cos(2\theta)$.



$r = \cos 2\theta$, a four-leaved rose

Figure 8.1.11:

SOLUTION To isolate one loop, find the two smallest nonnegative values of θ for which $\cos 2\theta = 0$. These values are the θ that satisfy $2\theta = \pi/2$ and $2\theta = 3\pi/2$; thus $\theta = \pi/4$ and $\theta = 3\pi/4$. One leaf is described by letting θ go from $\pi/4$ to $3\pi/4$. For θ in $[\pi/4, 3\pi/4]$, 2θ is in $[\pi/2, 3\pi/2]$. Since 2θ is then a second-or third-quadrant angle, $r = \cos 2\theta$ is *negative* or 0. In particular, when $\theta = \pi/2$, $\cos(2\theta)$ reaches its smallest value, -1. This loop is the bottom one in Figure 8.1.11. The other loops are obtained similarly. Of course, we could also sketch the graph by making a table of values. ◇

EXAMPLE 6 Transform the equation $y = 2$, which describes a horizontal straight line, into polar coordinates.
SOLUTION Since $y = r\sin\theta$, $r\sin\theta = 2$, or

$$r = \frac{2}{\sin(\theta)} = 2\csc(\theta).$$

This is more complicated than the original equation, but it is still sometimes useful. ◇

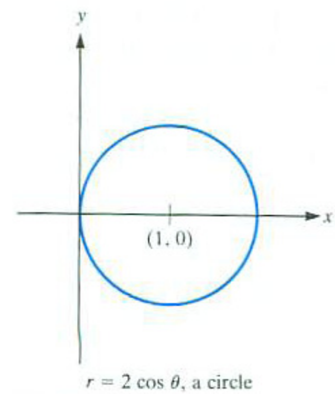


Figure 8.1.12:

EXAMPLE 7 Transform the equation $r = 2\cos(\theta)$ into rectangular coordinates and graph it.
SOLUTION Since $r^2 = x^2 + y^2$ and $r\cos\theta = x$, first multipy the equation $r = 2\cos\theta$ by r, obtaining

$$r^2 = 2r\cos(\theta)$$

Hence

$$x^2 + y^2 = 2x.$$

To graph this curve, rewrite the equation as

$$x^2 - 2x + y^2 = 0$$

and complete the square, obtaining

$$(x-1)^2 + y^2 = 1.$$

The graph is a circle of radius 1 and center at $(1, 0)$ in rectangular coordinates. It is graphed in Figure 8.1.12. ◇

Remark: The step in Example 7 where we multiply by r deserves some attention. If $r = 2\cos(\theta)$, then certainly $r^2 = 2r\cos(\theta)$. However, if $r^2 = 2r\cos(\theta)$, it does not follow that $r = 2\cos(\theta)$. We can "cancel the r" only when r is not 0. If $r = 0$, it is true that $r^2 = 2r\cos(\theta)$, but it not necessarily true that $r = 2\cos(\theta)$. Since $r = 0$ satisfies the equation $r^2 = 2r\cos\theta$, the pole is on the curve $r^2 = 2r\cos\theta$. Luckily, it is also on the original curve $r = 2\cos(\theta)$, since $\theta = \pi/2$ makes $r = 0$. Hence the graphs of $r^2 = 2r\cos(\theta)$ and $r = 2\cos(\theta)$ are the same. [However, as you may check, the graphs of $r = 2 + \cos(\theta)$ and $r^2 = r(2 + \cos(\theta))$ are *not* the same. The origin lies on the second curve, but not on the first.]

The Intersection of Two Curves

Finding the intersection of two curves in polar coordinates is complicated by the fact that a given point has many descriptions in polar coordinates. Example 8 illustrates how to find the intersection.

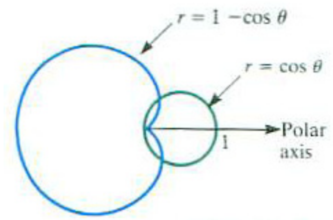


Figure 8.1.13:

EXAMPLE 8 Find the intersection of the curve $r = 1 - \cos(\theta)$ and the circle $r = \cos(\theta)$.

SOLUTION First graph the curves. The curve $r = \cos(\theta)$ is a circle half the size of the one in Example 7. The curve $r = 1-\cos(\theta)$ is shown in Figure 8.1.13. (It is a cardioid, being congruent to $r = 1 + \cos\theta$.) It appears that there are three points of intersection.

If a point of intersection is produced because the same value of θ yields the same value of r in both equations, we would have

$$1 - \cos(\theta) = \cos(\theta).$$

Hence $\cos(\theta) = \frac{1}{2}$. Thus $\theta = \pi/3$ or $\theta = -\pi/3$ (or any angle differing from these by $2n\pi$, n an integer). This gives two of the three points, but it fails to give the origin. Why?

How does the origin get to be on the circle $r = \cos(\theta)$? Because, when $\theta = \pi/2$, $r = 0$. How does it get to be on the cardioid $r = 1 - \cos(\theta)$? Because, when $\theta = 0$, $r = 0$. The origin lies on both curves, but we would not learn this by simply equating $1 = \cos(\theta)$ and $\cos(\theta)$. ⋄

When checking for the intersection of two curves, $r = f(\theta)$ and $r = g(\theta)$ in polar coordinates, examine the origin separately. The curves may also interect at other points not obtainable by setting $f(\theta) = g(\theta)$. This possibility is due to the fact that a given point P has an infinte number of descriptions in polar coordinates; that is, (r, θ) is the same as the points $(r, \theta + 2n\pi)$ and $(-r, \theta + (2n + 1)\pi)$ for any integer n. The safest procedure is to graph the two curves first and then see why they intersect at the points suggested by the graphs.

Summary

We introduced polar coordinates and showed how to graph curves given in the form $r = f(\theta)$. Some of the more common polar curves are listed below.

Equation	Curve
$r = a$, $a > 0$	circle of radius a, center at pole
$r = 1 + \cos(\theta)$	cardioid
$r = a\theta$, $a > 0$	Archimedean spiral
$r = \sin(3\theta)$	3-leafed rose (one loop symmetric about $\theta = \pi/6$)
$r = \sin(n\theta)$, n odd	n-leafed rose
$r = \sin(n\theta)$, n even	$2n$-leafed rose
$r = \cos(n\theta)$, n odd	n-leafed rose (one loop symmetric about $\theta = 0$)
$r = \cos(n\theta)$, n even	$2n$-leafed rose
$r = a\csc(\theta)$	the line $y = a$
$r = a\sec(\theta)$	the line $x = a$
$r = a\cos(\theta)$, $a > 0$	circle of radius $a/2$ through pole and $(a, 0)$
$r = a\sin(\theta)$, $a > 0$	circle of radius $a/2$ through pole and $(a, \pi/2)$

Table 8.1.1:

To find the intersection of two curves in polar coordinates, first graph them.

EXERCISES for 8.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] Plot the points whose polar coordinates are

(a) $(1, \pi/6)$

(b) $(2, \pi/3)$

(c) $(2, -\pi/3)$

(d) $(-2, \pi/3)$

(e) $(2, 7\pi/3)$

(f) $(0, \pi/4)$

2.[R] Find the rectangular coordinates of the points in Exercise 1.

3.[R] Give at least three pairs of polar coordinates (r, θ) for the point $(3, \pi/4)$,

(a) with $r > 0$,

(b) with $r < 0$.

4.[R] Find the polar coordinates $()r, \theta$ with $0 \leq \theta < 2\pi$ and r positive, for the point whose rectangular coordinates are

(a) $(\sqrt{2}, \sqrt{2})$

(b) $(-1, \sqrt{3})$

(c) $(-5, 0)$

(d) $(-\sqrt{2}, -\sqrt{2})$

(e) $(0, -3)$

(f) $(1, 1)$

In Exercises 5 to 8 transform the equation into one in rectangular coordinates. **5.**[R] $r = \sin(\theta)$

6.[R] $r = \csc(\theta)$

7.[R] $r = 3/(4\cos(\theta) + 5\sin(\theta))$

8.[(] r) $r = 4\cos(\theta) + 5\sin(\theta)$

In Exercises 9 to 12 transform the equation into one in polar coordinates.

9.[R] $x = -2$

10.[R] $y = x^2$

11.[R] $xy = 1$

12.[R] $x^2 + y^2 = 4x$

In Exercises 13 to 22 graph the given equations.

13.[R] $r = 1 + \sin\theta$

14.[R] $r = 3 + 2\cos(\theta)$

15.[R] $r = s^{-\theta/\pi}$

16.[R] $r = 4^{\theta/\pi}, \theta > 0$

17.[R] $r = \cos(3\theta)$

18.[R] $r = \sin(2\theta)$

19.[R] $r = 2$

20.[R] $r = 3$

21.[R] $r = 3\sin(\theta)$

22.[R] $r = -2\cos(\theta)$

23.[R]

(a) Graph $r = 1/\theta, \theta > 0$

(b) What happens to the y coordinate of (r, θ) as $\theta \to \infty$?

24.[R]

(a) Graph $r = 1/\sqrt{\theta}, \theta > 0$

(b) What happens to the y coordinate of (r, θ) as $\theta \to \infty$?

In Exercises 25 to 30, find the intersections of the curves.

25.[R] $r = 1 + \cos(\theta)$ and $r = \cos(\theta) - 1$

26.[R] $r = \sin(2\theta)$ and $r = 1$

27.[R] $r = \sin(3\theta)$ and $r = \cos(3\theta)$

28.[R] $r = 2\sin(2\theta)$ and $r = 1$

29.[R] $r = \sin(\theta)$ and $r = \cos(2\theta)$

30.[R] $r = \cos(\theta)$ and $r = \cos(2\theta)$

The curve $r = 1 + a\cos(\theta)$ (or $r = 1 + a\sin(\theta)$) is called a **limaçon** (pronounced lee' · ma · son). Its shape depends on the choice of the constant a. For $a = 1$ we have the cardioid of Example 2. Exercises 31 to 33 concern other choices of a.

31.[M] Graph $r = 1 + 2\cos(\theta)$. (If $|a| > 1$, then the graph of $r = 1 + a\cos\theta$ crosses itself and forms a loop.)

32.[R] Graph $r = 1 + \frac{1}{2}\cos(\theta)$.

33.[C] Consider the curve $r = 1 + a\cos(\theta)$, where $0 \leq a \leq 1$.

(a) Relative to the same polar axis, graph the curves corresponding to $a = 0$, $1/4$, $1/2$, $3/4$, 1

(b) For $a = 1/4$ the graph in (a) is convex, but not for $a = 1$. Show that for $1/2 < a \leq 1$ the curve is not convex. *Hint:* Find the points on the curve farthest to the left and compare them to the point on the curve corresponding to $\theta = \pi$.

"Convex" is defined on page **??**.

34.[M]

(a) Graph $r = 3 + \cos(\theta)$

(b) Find the point on the graph in *(a)* that has the maximum y coordinate.

35.[M] Find the y coordinate of the highest point on the right-hand leaf of the four-leaved rose $r = \cos(2\theta)$.

This curve is called a **lemniscate**.

36.[M] Graph $r^2 = \cos(2\theta)$. Note that, if $\cos(2\theta)$ is negative, r is not defined and that, if $\cos(2\theta)$ is positive, there are two values of r, $\sqrt{\cos 2\theta}$ and $-\sqrt{\cos(2\theta)}$.

37.[C] Where do the sprials $r = \theta$ and $r = 2\theta$, for $\theta \geq 0$, intersect?

In Appendix E it is shown that the graph of $r = 1/(1 + e\cos(\theta))$ is a parabola if $e = 1$, an ellipse if $0 \leq e < 1$, and a hyperbola if $e > 1$. (e here is not related to $e \approx 2.718$, Euler's constant.) Exercises 38 to 39 concern such graphs. **38.**[R]

(a) Graph $r = \frac{1}{1+\cos(\theta)}$.

(b) Find an equation in rectangular coordinates for the curve in (a).

39.[R]

(a) Graph $r = \frac{1}{1-1/2\cos(\theta)}$.

(b) Find an equation in rectangular coordinates for the curve in (a).

SHERMAN: Newton's Method has not yet been introduced.

40.[M] The spiral $r = \theta$ meets the circle $r = 2\sin(\theta)$ at a point other than the origin. Use Newton's method to estimate the coordinates of that point. (Give both the polar and rectangular coordinates of the point of intersection.)

8.2 Computing Area in Polar Coordinates

Section 5.1 showed how to compute the area of a region if the lengths of parallel cross sections are known. Sums based on estimating rectangles led to the formula

$$\text{Area} = \int_a^b c(x)\ dx$$

where $c(x)$ denotes the cross-sectional length. Now we consider quite a different situation, in which sectors of a circle, not rectangles, provide an estimate of the area.

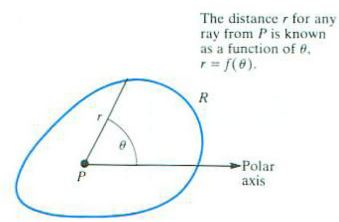


Figure 8.2.1:

Let R be a region in the plane and P a point inside it. Assume that the distance r from P to any point on the boundary of R is known as a function $r = f(\theta)$. Assume that any ray from P meets the boundary of R just once, as in Figure 8.2.1.

The cross sections made by the rays from P are *not* parallel. Instead, like spoke in a wheel, they all meet at the point P. It would be unnatural to use rectangles to estimate the area, but it is reasonable to use sectors of circles that have P as a common vertex.

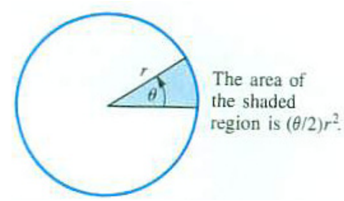


Figure 8.2.2:

Begin by recalling that in a circle of radius r a sector of central angle θ has area $(\theta/2)r^2$. (See Figure 8.2.2.) This formula plays the same role now as the formula for the area of a rectangle did in Section 5.1.

Area in Polar Coordinates

Assume $f(\theta) \geq 0$.

Let R be the region bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the curve $r = f(\theta)$, as shown in Figure 8.2.3. To obtain a **local estimate** for the area of R, consider the portion of R between the rays corresponding to the angles θ and $\theta + d\theta$, where $d\theta$ is a small positive number. (See Figure 8.2.4(a).) The area of the narrow wedge which is shaded in Figure 8.2.4(a) is approximately that of a sector of a circle of radius $r = f(\theta)$ and angle $d\theta$, shown in Figure 8.2.4(b). The area of the sector in Figure 8.2.4(b) is

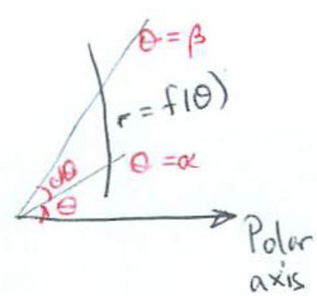


Figure 8.2.3:

$$\frac{[f(\theta)]^2 d\theta}{2}. \tag{8.1}$$

Having found the local estimate of area (8.1), we conclude that the area of R is

$$\int_\alpha^\beta \frac{(f(\theta))^2 d\theta}{2}.$$

How to find area in polar coordinates.

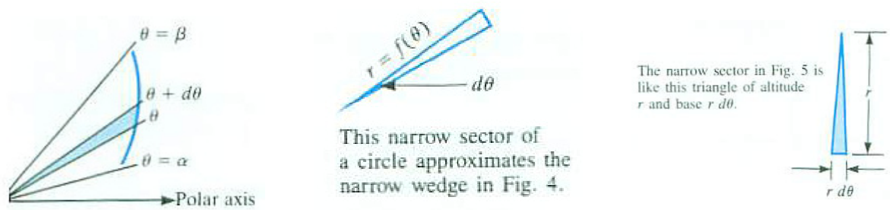
The area of the region bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the curve

$r = f(\theta)$ is

$$\int_{\alpha}^{\beta} frac(f(\theta))^2 2 \; d\theta \qquad \text{or} \qquad \int_{\alpha}^{\beta} \frac{r^2 d\theta}{2}. \tag{8.2}$$

It is assumed that no ray from the origin between α and β crosses the curve twice.

Formula 8.1 is applied in Section 13.1(?) to the motion of satellites and planets.



(a) (b) (c)

Figure 8.2.4:

Remark It may seem surprising to find $(f(\theta))^2$, not just $f(\theta)$, in the integrand. But remember that area has the dimension "length times length." Since θ, given in radians, is dimensionless, being defined as "length of circular arc divided by length of radius", $d\theta$ is also dimensionless. Hence $f(\theta)\ d\theta$, having the dimension of length, not of area, could *not* be correct. But $\frac{1}{2}(f(\theta))^2\ d\theta$, having the dimension of area (length times length), is plausible. For rectangular coordinates, in the expressions $f(x)\ dx$, both $f(x)$ and dx have the dimension of length, one along the y axis, the other along the x axis; thus $f(x)\ dx$ has the dimension of area. As an aid in remembering the area of the narrow sector in Figure 8.2.4(b), note that it resembles a triangle of height r and base $r\ d\theta$, as shown in Figure 8.2.4(c). Its area is

Memory device

$$\frac{1}{2} \cdot \underbrace{r}_{\text{height}} \cdot \underbrace{r d\theta}_{\text{base}} = \frac{r^2 d\theta}{2}.$$

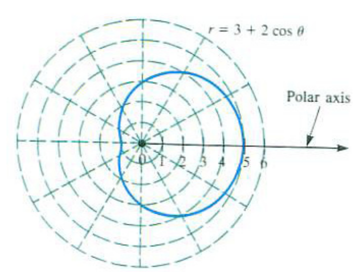


Figure 8.2.5:

EXAMPLE 1 Find the area of the region bounded by the curve $r = 3 + 2\cos\theta$, shown in Fig. 7.

SOLUTION By the formula just obtained, this area is

$$\begin{aligned}\int_0^{2\pi} \frac{1}{2}(3+2\cos(\theta))^2 d\theta &= \frac{1}{2}\int_0^{2\pi}(9+12\cos(\theta)+4\cos^2(\theta))d\theta \\ &= \frac{1}{2}(9\theta+12\sin(\theta)+2\theta+\sin(2\theta))\Big|_0^{2\pi} = 11\pi.\end{aligned}$$

◇

Sherman will make this into an exercise.

Observe than any line through the origin intersects the region of Example 1 in a segment of length 6, since $(3+2\cos\theta)+(3+2\cos(\theta+\pi)) = 6$ for any θ. Also, any line through the center of a circle of radius 3 intersects the circle in a segment of length 6. Thus two sets in the plane can have equal corresponding cross-sectional lengths through a fixed point and yet have different areas: the set in Example 1 has area 11π, while the circle or radius 3 has area 9π. *Knowing the lengths of all the cross sections of a region through a given point is not enough to determine the area of the region.*

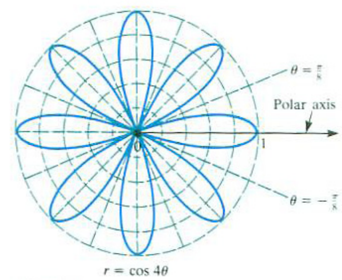


Figure 8.2.6:

EXAMPLE 2 Find the area of the region inside one of the eight loops of the eight-leaved rose $r = \cos 4\theta$.

SOLUTION To graph one of the loops, start with $\theta = 0$. For that angle, $r = \cos(4\cdot 0) = \cos 0 = 1$. The point $(r,\theta) = (1,0)$ is the outer tip of a loop. As θ increases from 0 to $\pi/8$, $\cos(4\theta)$ decreases from $\cos(0) = 1$ to $\cos(\pi/2) = 0$. One of the eight loops is therefore bounded by the rays $\theta = \pi/8$ and $\theta = -\pi/8$. It is shown in Figure 8.2.6.

The area of the loop which is bisected by the polar axis is

$$\begin{aligned}\int_{-\pi/8}^{\pi/8} \frac{r^2}{2} d\theta &= \int_{-\pi/8}^{\pi/8} \frac{\cos^2(4\theta)}{2} d\theta \\ &= \int_{-\pi/8}^{\pi/8} \frac{1+\cos(8\theta)}{4} d\theta \\ &= \left(\frac{\theta}{4}+\frac{\sin(8\theta)}{32}\right)\Big|_{-\pi/8}^{\pi/8} \\ &= \left(\frac{\pi}{32}+\frac{\sin(\pi)}{32}\right) - \left[\frac{(-\pi)}{32}+\frac{-\pi}{32}+\frac{\sin(-\pi)}{32}\right] \\ &= \frac{\pi}{16}.\end{aligned}$$

◇

The Area between Two Curves

Assume that $r = f(\theta)$ and that $r = g(\theta)$ describe two curves in polar coordinates and that $f(\theta) \geq g(\theta) \geq 0$ for θ in $[\alpha, \beta]$. Let R be the region between these two curves and the rays $\theta = \alpha$ and $\theta = \beta$, as shown in Figure 8.2.7.

The area of R is obtained by subtracting the area within the inner curve, $r = g(\theta)$, from the area within the outer curve, $r = f(\theta)$.

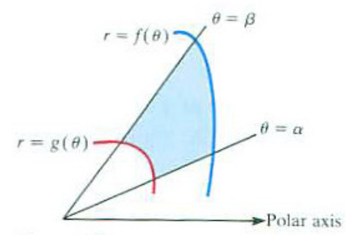


Figure 8.2.7:

EXAMPLE 3 Find the area of the top half of the region inside the cardioid $r = 1 + \cos(\theta)$ and outside the circle $r = \cos(\theta)$.

SOLUTION The region is shown in Figure 8.2.8. The top half of the circle $r = \cos\theta$ is swept out as θ goes from 0 to $\pi/2$. The top half of the cardioid is swept out as θ goes from 0 to π. The area of the top half of the cardiod is

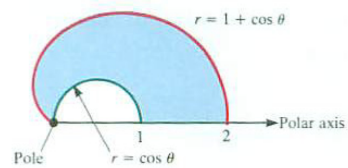


Figure 8.2.8:
We must integrate over two different intervals to find the two areas.

$$\begin{aligned}
\frac{1}{2}\int_0^{\pi}(1+\cos(\theta))^2 d\theta &= \frac{1}{2}\int_0^{\pi}\left(1+2\cos(\theta)+\cos^2(\theta)\right)d\theta \\
&= \frac{1}{2}\int_0^{\pi}\left(1+2\cos(\theta)+\frac{1+\cos(2\theta)}{2}\right)d\theta \\
&= \frac{1}{2}\int_0^{\pi}\left(\frac{3}{2}+2\cos(\theta)+\frac{\cos(2\theta)}{2}\right)d\theta \\
&= \frac{1}{2}\left(\frac{3\theta}{2}+2\sin(\theta)+\frac{\sin(2\theta)}{4}\right)\Bigg|_0^{\pi} \\
&= \frac{3\pi}{4}.
\end{aligned}$$

The area of the top half of the circle $r = \cos(\theta)$ is

$$\frac{1}{2}\int_0^{\pi/2}\cos^2\theta d\theta = \frac{\pi}{8}.$$

Thus the area in question is

$$\frac{3\pi}{4} - \frac{\pi}{8} = \frac{5\pi}{8}.$$

Or note that it is half the area of a circle of radius 1/2.

◇

Summary

In this section we determined how to find the area within a curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$. The heart of the method is the local approximation by a narrow sector of radius r and angle $d\theta$, which has area $r^2 d\theta/2$. (It resembles a triangle of height r and base $r d\theta$.) This approximation leads to the formula,

$$\text{Area} = \int_\alpha^\beta \frac{r^2}{2} d\theta.$$

It is wiser to remember the triangle than the formula because you may otherwise forget the 2 in the denominator.

EXERCISES for 8.2 *Key:* R–routine, M–moderate, C–challenging

In each of Exercise 1 to 6 ,draw the bounded retion enclosed by the indicated curve and rays and then find its area.

1.[R] $r = 2\theta$, $\alpha = 0$, $\beta = \frac{\pi}{2}$

2.[R] $r = \sqrt{\theta}$, $\alpha = 0$, $\beta = \pi$

3.[R] $r = \frac{1}{1+\theta}$, $\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{2}$

4.[R] $r = \sqrt{\sin(\theta)}$, $\alpha = 0$, $\beta = \frac{\pi}{2}$

5.[R] $r = \tan(\theta)$, $\alpha = 0$, $\beta = \frac{\pi}{4}$

6.[R] $r = \sec(\theta)$, $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{4}$

In each of Exercises 7 to 16 draw the region bounded by the indicated curve and then find its area.

7.[R] $r = 2\cos(\theta)$

8.[R] $r = e^{\theta}, 0 \leq \theta \leq 2\pi$

9.[R] Inside the cardioid $r = 3 + 3\sin(\theta)$ and outside the circle $r = 3$.

10.[R] $r = \sqrt{\cos(2\theta)}$

11.[R] One loop of $r = \sin(3\theta)$

12.[R] One loop of $r = \cos(2\theta)$

13.[R] Inside one loop of $r = 2\cos(2\theta)$ and outside $r = 1$

14.[R] Inside $r = 1 + \cos(\theta)$ and outside $r = \sin(\theta)$

15.[R] Inside $r = \sin(\theta)$ and outside $r = \cos(\theta)$

16.[R] Inside $r = 4 + \sin(\theta)$ and outside $r = 3 + \sin(\theta)$

17.[M] Sketch the graph of $r = 4 + \cos(\theta)$. Is it a circle?

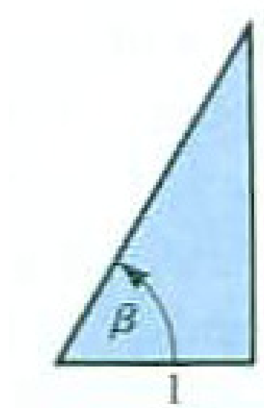


Figure 8.2.9:

18.[M]

(a) Show that the area of the triangle in Figure 8.2.9 is $\int_0^\beta \frac{1}{2}\sec^2(\theta)d\theta$.

(b) From *(a)* and the fact that the area of a triangle is $\frac{1}{2}$(base)(height), show that $\tan(\beta) = \int_0^\beta \frac{1}{2}\sec^2(\theta)d\theta$.

(c) With the aid of this equation, obtain another proof that $(\tan(x))' = \sec^2(x)$.

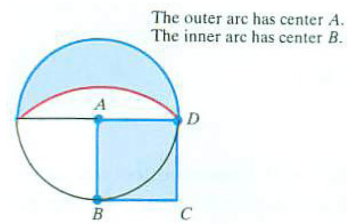


Figure 8.2.10:

19.[M] Show that the area of the shaded crescent between the two circular arcs is equal to the area of square $ABCD$. (See Figure 8.2.10.) This type of result encouraged mathematicians from the time of the Greeks to try to find a method using only straightedge and compass for constructing a square whose area equals that of a given circle. This was proved impossible at the end of the nineteenth century.

20.[C] Assume that a curve set R contours at point P such that every chord through P has length 6. Is that enough information to determine the area of R? HINT: Look at Example 1.

21.[M]

(a) Graph $r = 1/\theta$ for $0 \leq \theta \leq \pi/2$.

(b) Is the area of the region bounded by the curve drawn in *(a)* and the rays $\theta = 0$ and $\theta = \pi/2$ finite or infinte?

22.[M]

(a) Sketch the curve $r = 1/(1 + \cos(\theta))$.

(b) What is the equation of the curve in *(a)* in rectangular coordinates?

(c) Find the area of the region bounded by the curve in *(a)* and the rays $\theta = 0$ and $\theta = 3\pi/4$, using polar coordinates.

(d) Solve *(c)* using retangular coordinates and the equation in *(b)*.

23.[C] Find the area of the region bounded by $r = e^\theta, r = 2\cos(\theta)$ and $\theta = 0$. HINT: You may need to approximate a limit of integration.

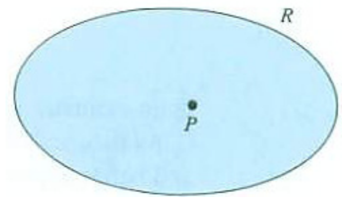


Figure 8.2.11:

24.[M] Estimate the area of the bounded region between $r = \sqrt[3]{1+\theta^2}, \theta = 0$, and $\theta = \pi/2$, using Simpson's method.

25.[C] Figure 8.2.11 shows a point P inside a convex region R.

(a) Assume that P cuts each chord through P into two intervals of equal length. Must each chord through P cut R into two regions of equal areas?

(b) Assume that each chord through P cuts R into two regions of equal areas. Must P cut each chord through P into two intervals of equal lengths?

26.[C] Let R be a region in the plane bounded by a loop whose equation is $r = f(\theta)$ in polar coordinates. Assume that every chord of R that passes through the pole has length at least 1.

(a) Draw several examples of such an R.

(b) Make a general conjecture about the area R.

(c) Prove it.

27.[C] Like Exercise 26, except that each chord through the pole has length at most 1.

8.3 Parametric Equations

Up to this point we have considered curves described in three forms: "y is a function of x", "x and y are related implicitly", and "r is a function of θ". But a curve is often described by giving both x and y as functions of a third variable. We introduce this situation as it arises naturally in the study of motion.

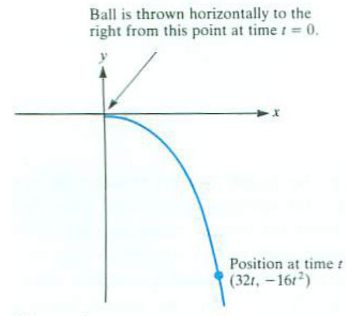


Figure 8.3.1:

Two Examples

EXAMPLE 1 If a ball is thrown horizontally with a speed of 32 feet per second, it falls in a curved path. Air resistance disregarded, its position r seconds later is given by $x = 32$, $y = -16t^2$ relative to the coordinate system in Figure 8.3.1. Here the curve is completely described, not by expressing y as a function of x, but by expressing each of x and y as functions of a third variable t. The third variable is called a **parameter**. The equations $x = 32t$, $y = 16t^2$ are called **parametric equations** for the curve.

para meaning "together," *meter* meaning "measure".

In this example it is easy to eliminate t and so find a direct relation between x and y:

$$t = \frac{x}{32}.$$

Hence

$$y = -16\left(\frac{x}{32}\right)^2 = -\frac{16}{(32)^2}x^2 = -\frac{1}{64}x^2.$$

The path of the falling ball is part of the parabola $y = -\frac{1}{64}x^2$. ◇

In Example 1 elimination of the parameter would lead to a complicated equation involving x and y. One advantage of parametric equations is that they can provide a simple description of a curve, although it may be impossible to find an equation in x and y which describes the curve.

EXAMPLE 2 As a bicycle wheel of radius a rolls along, a tack stuck in its circumference traces out a curve called a **cycloid**, which consists of a sequence of arches, one arch for each revolution of the wheel. (See Figure 8.3.2.) Find the position of the tack as a function of the angle θ through which the wheel turns.

SOLUTION The x coordinate of the tack, corresponding to θ, is

$$\overline{AF} = \overline{AB} - \overline{ED} = a\theta - a\sin(\theta),$$

and the y coordinate is

$$\overline{EF} = \overline{BC} - \overline{CD} = a - a\cos(\theta).$$

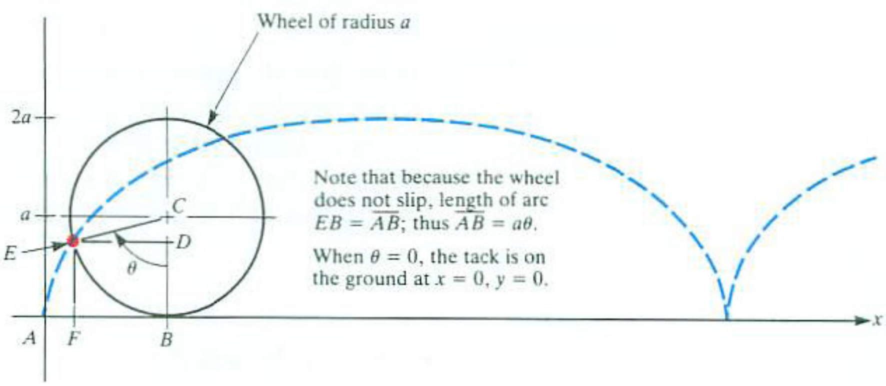


Figure 8.3.2:

Then the position of the tack, as a function of the parameter θ, is

$$\begin{cases} x = a\theta - a\sin(\theta) \\ y = a - a\cos(\theta). \end{cases}$$

In this case, eliminating θ would lead to a complicated relation between x and y. ◇

See Exercise 36.

Any curve $y = f(x)$ can be given parametrically: $x = t$, $y = f(t)$.

Any curve can be described parametrically. For instance, consider the curve $y = e^x + x$. It is perfectly legal to introduce a parameter t equal to x and write

$$\begin{cases} x = t \\ y = e^t + t. \end{cases}$$

This device may seem a bit artificial, but it will be useful in the next section in order to apply results for curves expressed by means of parametric equations to curves given in the form $y = f(x)$.

How to Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

How can we find the slope of a curve which is described parametrically by the equations

$$\begin{cases} x = g(t) \\ y = h(t)? \end{cases}$$

An often difficult, perhaps impossible, approach is to solve the equation $x = g(t)$ for t as a function of x and substitute the result into the equation $y = h(t)$, thus expressing y explicitly in terms of x; then differentiate the result to find dy/dx. Fortunately, there is a very easy way, which we will now describe. Assume that y is a differentiable function of x. Then, by the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

from which it follows that

Formula for the slope of a parameterized curve

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{8.1}$$

It is assumed that in formula (8.1) dx/dt is not 0. To obtain d^2y/dx^2 just replace y in (8.1) by dy/dx, obtaining

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

EXAMPLE 3 At what angle does the arch of the cycloid shown in Example 2 meet the x-axis at the origin?

SOLUTION The parametric equations of the cycloid are

$$x = a\theta - a\sin(\theta) \qquad \text{and} \qquad y = a - a\cos(\theta).$$

Here θ is the parameter. Then

$$\frac{dx}{d\theta} = a - a\cos(\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = a\sin(\theta).$$

Consequently,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin(\theta)}{a - a\cos(\theta)} \\ &= \frac{\sin(\theta)}{1-\cos(\theta)}. \end{aligned}$$

When $\theta = 0$, $(x, y) = (0, 0)$ and $\frac{dy}{dx}$ is not defined because $\frac{dx}{d\theta} = 0$. But, when θ is near 0, (x, y) is near the origin and the slope of the cycloid at $(0, 0$ can be found by looking at the limit of the slope, which is $\sin\theta/(1 - \cos(\theta)$, as $\theta \to 0^+$. L'Hôpital's Rule applies, and we have

$$\lim_{\theta\to 0^+} \frac{\sin(\theta)}{1-\cos(\theta)} = \lim_{\theta\to 0^+} \frac{\cos(\theta)}{\sin(\theta)} = \infty.$$

Thus the cycloid comes in vertically at the origin. ◇

EXAMPLE 4 Find d^2y/dx^2 for the cycloid of Example 2.

SOLUTION From Example 3 we know that

$$\frac{dy}{dx} = \frac{\sin(\theta)}{1-\cos(\theta)}.$$

Hence

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left(\frac{\sin(\theta)}{1-\cos(\theta)}\right)}{\frac{dx}{d\theta}}.$$

As shown in Example 3, $dx/d\theta = a - a\cos(\theta)$. Thus

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\left(\frac{(1-\cos(\theta))(\cos(\theta))-(\sin(\theta))(\sin(\theta))}{(1-\cos(\theta))^2}\right)}{a-a\cos(\theta)} && \text{Quotient Rule} \\ &= \frac{\left(\frac{\cos(\theta)-\cos^2(\theta)-\sin^2(\theta)}{(1-\cos(\theta))^2}\right)}{a-a\cos(\theta)} \\ &= \frac{\cos(\theta)-1}{a(1-\cos(\theta))^3} && (sin^2(\theta)+\cos^2(\theta)=1) \\ &= \frac{-1}{a(1-\cos(\theta))^2}. \end{aligned}$$

Since the denominator is positive (or 0), the quotient, when defined, is negative. This agrees with Figure 8.3.2, which shows each arch of the cycloid as concave down. ◇

The Rotary Engine

The **centroid** of a triangle is the point where its three altitudes meet.

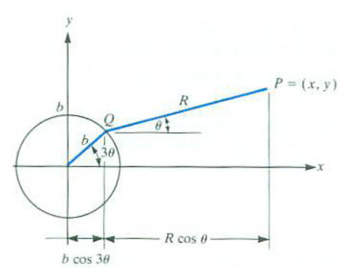


Figure 8.3.3:

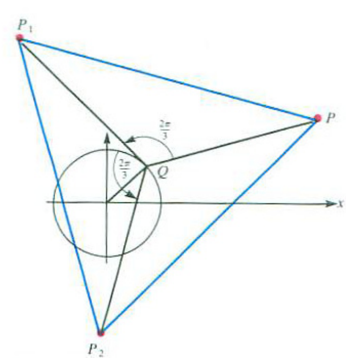


Figure 8.3.4:

The next two examples use parametric equations to describe the geometric principles of the rotary engine recognized by Felix Wankel in 1954. He found that it is possible for an equilateral triangle to revolve in a certain curve in such a way that its corners maintain contact with the curve and its centroid sweeps out a circle.

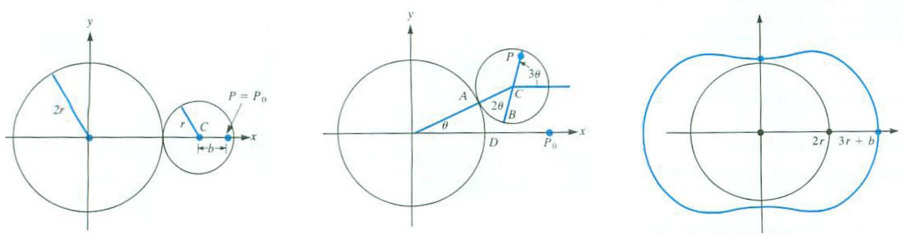
EXAMPLE 5 Let b and R be fixed positive numbers and consider the curve given parametrically by

$$x = b\cos(3\theta) + R\cos(\theta) \qquad \text{and} \qquad y = b\sin(3\theta) + R\sin(\theta).$$

Show that an equilateral triangle can revolve in this curve while its centroid describes a circle of radius b.

SOLUTION Figure 8.3.3 shows the typical point $P = (x, y)$ that corresponds to the parameter value θ. As θ increases by $2\pi/3$ from any given angle, the point Q goes once around the circle of radius b and returns to its initial position. During this revolution of Q the point P moves to a point P_1 whose angle, instead of being θ is $\theta + 2\pi/3$. Thus, if P is on the curve, so are the points P_1 and P_2 shown in Figure 8.3.4; these form an equilateral triangle. Consequently, each vertex of the equilateral triangle sweeps out the curve once, while the centroid Q goes three times around the circle of radius b. ◇

What does the curve described in Example 5 look like? Wankel graphed it without knowing that mathematicians had met it long before in a different setting, described in Example 5, which provides a way of graphing the curve.



(a) (b) (c)

Figure 8.3.5:

EXAMPLE 6 A circle of radius r rolls without slipping around a fixed circle of radius $2r$. Describe the path swept out by a point P located at a distance b from the center of the moving circle, $0 \leq b \leq r$.

SOLUTION Place the rolling circle as shown in Figure 8.3.5(a). Note that the center C of the rolling circle traces out a circle of radius $3r$. Let $R = 3r$. As the little circle rolls counterclockwise around the fixed circle without slipping, the point P traces out a path whose initial point P_0 is shown in Figure 8.3.5(a). The typical point P on the path as the circle rolls around the larger circle is shown in Figure 8.3.5(b). Since the radius of the rolling circle is half that of the fixed circle (and there is no slipping), angle ACB is 2θ. Thus the angle that CP makes with the x axis is the sum of θ and 2θ, wich is 3θ. Consequently, $P = (x, y)$ has coordinates given parametrically as

$$x = b\cos(3\theta) + R\cos(\theta) \qquad \text{and} \qquad y = b\sin(3\theta) + R\sin(\theta).$$

Thus the curve swept out by P is precisely the curve Wankel studied. Long known to mathematicians as an **epitrochoid**, it is shown in Figure 8.3.5(c). $\diamond$

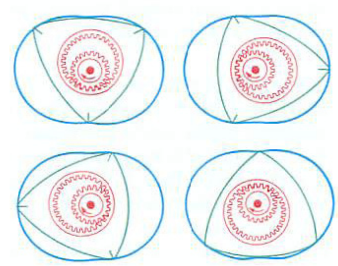


Figure 8.3.6:

In order that the moving rotor in the rotary engine can turn the drive shaft, teeth are placed in it along a circle of radius $2b$ which engage teeth in the drive shaft, which has radius b. (See Figure 8.3.6.) For each complete rotation of the rotor, the drive shaft completes three rotations. It was a Stuttgart professor, Othmar Baier, who showed that Wankel's curve was an epitrochoid. This insight was of aid in simplifying the machining of the working surface of the motor.

Summary

This section described parametric equations, where x and y are given as functions of a third variable, often time (t) or angle (θ). We also showed how to compute dy/dx and d^2y/dx^2:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

and replacing y by $\frac{dy}{dx}$,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

EXERCISES for 8.3 *Key:* R–routine, M–moderate, C–challenging

1.[R] Consider the prarmetric equations $x = 2t + 1$, $y = t - 1$.

(a) Fill in this table:

t	-2	-1	0	1	2
x					
y					

(b) Plot the five points (x, y) obtained in *(a)*.

(c) Graph the curve given by the parametric equations $x = 2t + 1$, $y = t - 1$.

(d) Eliminate t to find an equation for the graph involving only x and y.

2.[R] Consider the parametric equations $x = t + 1$, $y = t^2$.

(a) Fill in this table:

t	-2	-1	0	1	2
x					
y					

(b) Plot the five points (x, y) obtained in *(a)*.

(c) Graph the curve.

(d) Find an equation in x and y that describes the curve.

3.[R] Consider the parametric equations $x = t^2$, $y = t^2 + t$.

(a) Fill in this table:

t	-3	-2	-1	0	1	2	3
x							
y							

(b) Plot the seven points (x, y) obtained in (a).

(c) Graph the curve given by $x = t^2$, $y = t^2 + t$.

(d) Eliminate t and find an equation for the graph in terms of x and y.

4.[R] Consider the parametric equations $x = 2\cos(t)$, $y = 3\sin(t)$.

(a) Fill in this table, expressing the entries decimally:

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x									
y									

(b) Plot the eight distinct points in *(a)*.

(c) Graph the curve given by $x = 2\cos(t)$, $y = 3\sin(t)$.

(d) Using the identity $\cos^2(t) + \sin^2(t) = 1$, eliminate t.

In Exercises 5 to 8 express the curves parametrically with parameter t.

5.[R] $y = \sqrt{1 + x^3}$

6.[R] $y = \tan^{-1}(3x)$

7.[R] $r = \cos 2(\theta)$

8.[R] $r = 3 + \cos(\theta)$

In Exercises 9 to 14 find dy/dx and d^2y/dx^2 for the given curves. **9.**[R] $x = t^3 + t$, $y = t^7 + t + 1$

10.[R] $x = \sin(3t)$, $y = \cos(4t)$

11.[R] $x = 1 + \ln(t)$, $y = t\ln(t)$

12.[R] $x = e^{t^2}$, $y = \tan(t)$

13.[R] $r = \cos(3\theta)$

14.[R] $r = 2 + 3\sin(\theta)$

In Exercise 15 to 16 find the equation of the tangent line to the given curve at the given point.

15.[R] $x = t^3 + t^2$, $y = t^5 + t$; $(2, 2)$

16.[R] $x = \frac{t^2+1}{t^3+t^2+1}$, $y = \sec 3t$; $(1, 1)$

In Exercises 17 and 18 find d^2y/dx^2. **17.**[R] $x = t^3 + t + 1$, $y = t^2 + t + 2$

18.[R] $x = e^{3t} + \sin(2t)$, $y = e^{3t} + \cos(t^2)$

19.[R] For which values of t is the curve in Exercise 17 concave up? concave down?

20.[R] Let $x = t^3 + 1$ and $y = t^2 + t + 1$. For which values of t is the curve concave up? concave down?

21.[R] Find the slope of the three-leaved rose, $r = \sin(3\theta)$, at the point $(r, \theta) = (\sqrt{2}/2, \pi/12)$.

22.[R]

(a) Find the slope of the cardioid $r = 1 + \cos(\theta)$ at the point (r, θ).

(b) What happens to the slope in (a) as $\theta \to \pi^-$?

(c) What does *(b)* tell us about the graph of the cardioid? (Show it on the graph.)

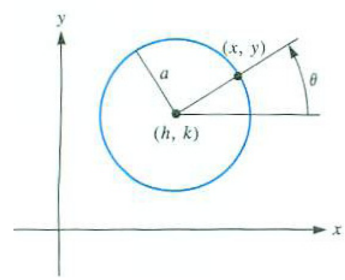


Figure 8.3.7:

23.[R] Obtain parametric equations for the circle of radius a and center (h, k), using as parameter the angle θ shown in Figure 8.3.7.

24.[R] At time $t \geq 0$ a ball is at the point $(24t, -16t^2 + 5t + 3)$.

(a) Where is it at time $t = 0$?

(b) What is its horizontal speed at that time?

(c) What is its vertical speed at that time?

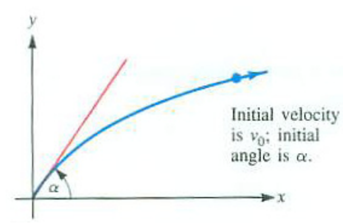


Figure 8.3.8:

25.[R] A ball is thrown at an angle α and initial velocity v_0, as sketched in Figure 8.3.8 It can be shown that if time is in seconds and distance in feet, then t seconds later the ball is at the point

$$\begin{cases} x = (v_0 \cos(\alpha))t \\ y = (v_0 \sin(\alpha))t - 16t^2. \end{cases}$$

(a) Eliminate t.

(b) In view of *(a)*, what type of curve does the ball follow?

(c) Find the coordinates of its highest point.

26.[R]

(a) The spiral $r = e^{2\theta}$ meets the ray $\theta = \alpha$ at an infinite number of points. Show that at all of these points the curve has the same slope.

(b) Show that the analog of *(a)* is not true for the spiral $r = \theta$.

27.[R] The spiral $r = \theta$, $\theta > 0$ meets the ray $\theta = \alpha$ at an infinite number of points (α, α), $(\alpha + 2\pi, \alpha)$, $(\alpha + 4\pi, \alpha)$, $\ldots$. What happens to the angle between the spiral and the ray at the point $(\alpha + 2\pi n, \alpha)$ as $n \to \infty$?

28.[M] Let a and b be positive numbers. Consider the curve given parametrically by the equations

$$x = a\cos(t) \qquad y = b\sin(t).$$

(a) Show that the curve is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(b) Find the area of the region bounded by the ellipse in *(a)* by making a substitution that expresses $4\int_0^a y\,dx$ in terms of an integral in which the variable is t and the range of integration is $[0, \pi/2]$.

29.[M] Consider the curve given parametrically by

$$x = t^2 + e^t \qquad y = t + e^t$$

for t in $[0, 1]$.

(a) Plot the points corresponding to $t = 0$, $1/2$, and 1.

(b) Find the slope of the curve at the point $(1, 1)$.

(c) Find the area of the region under the curve and above the interval $[1, e + 1]$. [See Exercise 28(b).]

30.[R] What is the slope of the cycloid in Figure 8.3.2 at the first point on it to the right of the y-axis at the height a?

31.[M] The region under the arch of the cycloid

$$\begin{cases} x &= a\theta - a\sin(\theta) \\ y &= a - a\cos(\theta) \end{cases} \quad (0 \le \theta \le 2\pi)$$

and above the x-axis is revolved around the x-axis. Find the volume of the solid of revolution produced.

32.[R] The same as the preceding exercise, except the region is revolved around the y-axis instead of the x-axis.

33.[C] l'Hôpital's rule in Section 4.5 asserts that if $\lim_{t\to 0} f(t) = 0$, $\lim_{t\to 0} g(t) = 0$, and $\lim_{t\to 0}(f'(t)/g'(t))$ exists, then $\lim_{t\to 0}(f(t)/g(t)) = \lim_{t\to 0}(f'(t)/g'(t))$. Interpret that rule in terms of the parameterized curve $x = g(t)$, $y = f(t)$. HINT: Make a sketch of the curve near $(0, 0)$ and show on it the geometric meaning of the quotients $f(t)/g(t)$ and $f'(t)/g'(t)$.

34.[M] Let a be a positive constant. Consider the curve given parametrically by the equations $x = a\cos^3(t)$, $y = a\sin^3(t)$.

(a) Sketch the curve.

(b) Find the slope of the curve at the point corresponding to the parameter value t.

35.[C] Consider a tangent line to the curve in Exercise 34 at a point P in the first quadrant. Show that the length of the segment of that line intercepted by the coordinate axes is a.

See Example 1.

36.[M] Solve the parametric equations for the cycloid, $x = a\theta - a\sin(\theta)$, $y = a - a\cos(\theta)$, for y as a function of x.

37.[C] The **Folium of Descartes** is the graph of

$$x^3 + y^3 = 3xy.$$

The graph is shown in Figure 8.3.9 It consists of a loop and two infinite pieces both asymptotic to the line $x + y + 1 = 0$. Parameterize the curve by its slope t. Thus for the point (x, y) as the curve, $y = xt$.

(a) Show that

$$x = \frac{3t}{1+t^3} \qquad \text{and} \qquad y = \frac{3t^2}{1+t^3}.$$

(b) Find the highest point on the loop.

(c) Find the point on teh loop furtherest to the right.

(d) The loop is parameterized by t in $0, \infty$. Which values of t parameterize the part in the fourth quadrant?

(e) Which values of t parameterize the part in the second quadrant?

(f) Show that the Folium of Descartes is symmetric with respect to the line $y = x$.

Visit `http://en.wikipedia.org/wiki/Folium_of_Descartes` or do a Google search of "Folium Descartes" to see its long history that goes

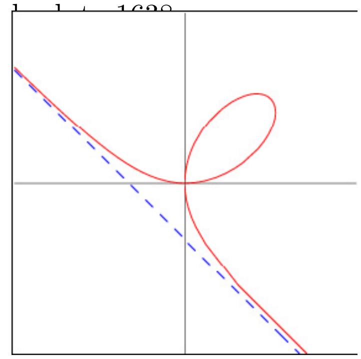


Figure 8.3.9:

8.4 Arc Length and Speed on a Curve

In Section 3.2 we studied the motion of an object moving on a line. If at time t its position is $x(t)$, then its velocity is the derivative $\frac{dx}{dt}$ and its speed is $\left|\frac{dx}{dt}\right|$. Now we will examine the velocity and speed of an object moving along a curved path.

Arc Length and Speed in Rectangular Coordinates

The path of some particle (object?) is given parametrically by

Think of t as time.

$$\begin{cases} x &= g(t) \\ y &= h(t). \end{cases}$$

A physicist might ask the following questions: How far does the particle travel from time $t = a$ to time $t = b$? What is the speed of the particle at time t?

Consider the "distance traveled" question first. In our reasoning we shall call the parameter t and think of it as time, but the results apply to any parameter, such as angle θ.

In Section 2.4 we assumed that a very short section of a curve near point P on the curve resembles a short piece of the tangent line at P. (See Figure 8.4.1.) Now we take a slightly different point of view: that a very short section of a curve between points P and Q resembles the chord joining P and Q. (See Figure 8.4.2.) Assume that $x = g(t)$ and $y = h(t)$ have continuous derivatives. Let us make a local estimate of the arc length swept out on the path during the short interval of time from t to $t + \Delta t$.

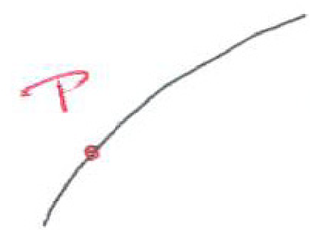


Figure 8.4.1:

Let s denote the arc length along the path. During the time Δt the x-coordinate changes by an amount Δx and the y-coordinate changes by an amount Δy, as indicated in Figure 8.4.3. Let Δs be the length of arc swept out during the time Δt and let h be the length of the hypotenuse of the right triangle with legs Δx and Δy.

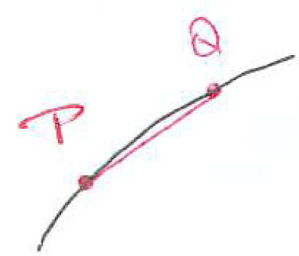


Figure 8.4.2:

By the Pythagorean Theorem, $h^2 = (\Delta x)^2 + (\Delta y)^2$. Hence

$$\frac{h^2}{(\Delta t)^2} = \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2.$$

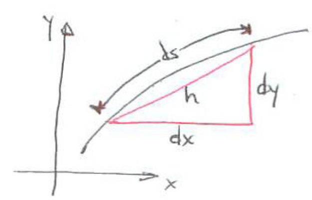


Figure 8.4.3:

Now $\lim_{\Delta t \to 0} \frac{\Delta s}{h} = 1$, since we are assuming that the chord is a good approximate of arc length "in the small". Thus

$$\begin{aligned}\lim_{\Delta t\to 0}\left(\frac{\Delta s}{\Delta t}\right)^2 &= \lim_{\Delta t\to 0}\left(\frac{\Delta s}{h}\right)^2\left(\frac{h}{\Delta t}\right)^2\\ &= \lim_{\Delta t\to 0}\left(\frac{\Delta s}{h}\right)^2\left(\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2\right)\\ &= 1\left(\left(\frac{dx}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2\right).\end{aligned}$$

We have obtained the key result for this section:

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2. \qquad (8.1)$$

This relationship holds for any parameter t. If t is, in particular, time, the formula gives the velocity $\frac{ds}{dt}$ and the speed $\left|\frac{ds}{dt}\right|$ on the curve.

Integrating the speed gives us the **arc length** covered as t varies from a to b:

Formula for arc length of a parameterized curve

$$\text{Arc Length} = \int_a^b \left|\frac{ds}{dt}\right|\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt. \qquad (8.2)$$

This formula holds for a curve given parametrically and if the derivatives dx/dt and dy/dt are continuous. If the curve is given in the form $y = f(x)$ it may be put in parametric form

$$\begin{cases} x = t\\ y = f(t).\end{cases}$$

Since $dx/dt = 1$ and $x = t$, formula (8.2) for the arc length of the curve $y = f(x)$ for x in $[a, b]$ takes the following form:

Formula for arc length of curve $y = f(x)$

$$\text{Arc Length} = \int_a^b \left|\frac{ds}{dt}\right|\,dt = \int_a^b \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}\,dx. \qquad (8.3)$$

(It is assumed that $f'(x)$ is continuous for x in $[a, b]$.)

Three examples will show how these formulas are applied. The first goes back to the year 1657, when the 20-year old Englishman, William Neil, found

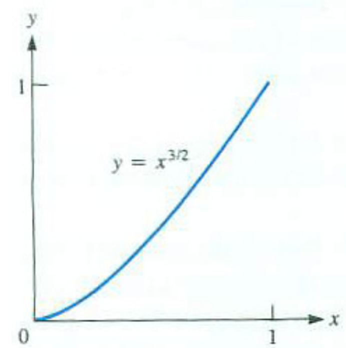


Figure 8.4.4:

the length of an arc on the graph of $y = x^{3/2}$. His method was much more complicated. Earlier in that century, Thomas Harriot had found the length of an arc of the spiral $r = e^\theta$, but his work was not widely published.

EXAMPLE 1 Find the arc length of the curve $y = x^{3/2}$ for x in $[0, 1]$. (See Figure 8.4.4.)
SOLUTION By formula (8.3),

$$\text{Arc Length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Since $y = x^{3/2}$, $dy/dx = \frac{3}{2}x^{1/2}$. Thus

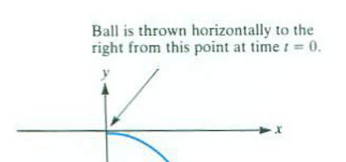
$$\begin{aligned}
\text{Arc Length} &= \int_0^1 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx \\
&= \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx \\
&= \int_1^{13/4} \sqrt{u} \cdot \frac{4}{9}\, du \qquad \text{where } u = 1 + \frac{9}{4}x,\ du = \frac{9}{4}dx \\
&= \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{13/4} \\
&= \frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1^{3/2}\right) \\
&= \frac{8}{27}\left(\frac{13^{3/2}}{8} - 1\right) \\
&= \frac{13^{3/2} - 8}{27} \approx ???.
\end{aligned}$$

◇

Incidentally, the arc length of the curved $y = x^a$ where a is a rational number, usually *cannot* be computed with the aid of the Fundamental Theorem of Calculus. The only cases in which it can be computed by the FTC are a=1 (the graph of $y = x$) and $a = 1 + \frac{1}{n}$ where n is an integer. Exercise 29 treats this question.

DOUG: This is the application that was in Section 9.3 in V. If it appears earlier in VI, some of the details can be omitted from here.

EXAMPLE 2 In Section 8.3 the parametric equations for the motion of a ball thrown horizontally with a speed of 32 feet per second were found to be



$x = 32t$, $y = -16t^2$. (See Example 1 and Figure 8.3.1.) How fast is the ball moving at time t? Find the distance s which the ball travels during the first b seconds.

SOLUTION From $x = 32t$ and $y = -16t^2$ we compute $\frac{dx}{dt} = 32$ and $\frac{dy}{dt} = -32t$. Its speed at time t is

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(32)^2 + (-32t)^2} = 32\sqrt{1+t^2} \text{ feet per second.}$$

The distance traveled is the arc length from $t = 0$ to $t = b$. By formula (8.2),

$$s = \int_0^b \sqrt{(32)^2 + (-32t)^2}\ dt = 32\int_0^b \sqrt{1+t^2}\ dt.$$

This integral can be evaluated with an integration table or with the trigonometric substitution $x = \tan(\theta)$. An antiderivative is $\frac{1}{2}\left(t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right)$ and the distance traveled is

See Formula ?? in the integral table.

$$16b\sqrt{1+b^2} + 16\ln\left(b + \sqrt{1+b^2}\right).$$

◇

EXAMPLE 3 Find the length of one arch of the cycloid found in Example 2 of Section 8.3.

SOLUTION Here the parameter is θ and $x = a\theta - a\sin(\theta)$ and $y = a - a\cos(\theta)$. To complete one arch of the cycloid, θ varies from 0 to 2π.

We compute

$$\frac{dx}{d\theta} = a - a\cos(\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = a\sin(\theta).$$

The square of the speed is

$$\begin{aligned}(a - a\cos(\theta))^2 + (a\sin(\theta))^2 &= a^2\left((1-\cos(\theta))^2 + (\sin(\theta))^2\right)\\ &= a^2\left(1 - 2\cos(\theta) + (\cos(\theta))^2 + (\sin(\theta))^2\right)\\ &= a^2\left(2 - 2\cos(\theta)\right)\\ &= 2a^2(1-\cos(\theta)).\end{aligned}$$

By formula (8.2), the

$$\begin{aligned}
\text{length of one arch} &= \int_0^{2\pi} \sqrt{2a^2(1-\cos(\theta))}\, d\theta \\
&= a\sqrt{2}\int_0^{2\pi} \sqrt{1-\cos(\theta)}\, d\theta \\
&= a\sqrt{2}\int_0^{2\pi} \sqrt{2}\sin\left(\frac{\theta}{2}\right)\, d\theta \\
&= 2a\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right)\, d\theta \\
&= 2a\left(-2\cos\left(\frac{\theta}{2}\right)\right)\Bigg|_0^{2\pi} \\
&= 2a\left(-2(-1)-(-2)(1)\right) \\
&= 8a.
\end{aligned}$$

This step is valid since $\sin(\theta/2) \geq 0$ for $0 \leq \theta \leq 2\pi$.

This means that while θ varies from 0 to 2π, a bicycle travels a distance of $2\pi a \approx 6.2832a$ and a tack in the tread of the tire travels a distance $8a$. ◇

Arc Length and Speed in Polar Coordinates

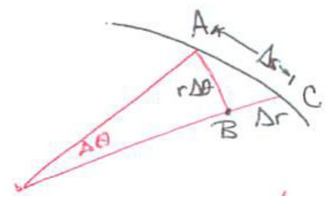


Figure 8.4.6:

So far in this section curves have been described in rectangular coordinates. Next consider a curve given in polar coordinates by the equation $r = f(\theta)$.

We will estimate the length of arc Δs corresponding to small changes $\Delta\theta$ and Δr in polar coordinates, as shown in Figure 8.4.6. The region bounded by the circular arc AB, the straight segment BC, and AC, the part of the curve, resembles a right triangle whose two legs have lengths $r\Delta\theta$ and Δr. We assume Δs is well approximated by $\sqrt{(r\Delta\theta)^2 + (\Delta r)^2}$. Thus we expect

$$\begin{aligned}
\frac{ds}{d\theta} = \lim_{\Delta\theta\to 0}\frac{\Delta s}{\Delta\theta} &= \lim_{\Delta\theta\to 0}\frac{\sqrt{(r\Delta\theta)^2+(\Delta r)^2}}{(\Delta\theta)} \\
&= \lim_{\Delta\theta\to 0}\sqrt{r^2+\left(\frac{\Delta r}{\Delta\theta}\right)^2} \\
&= \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}
\end{aligned}$$

In short

Arc length for $r = f(\theta)$.

For a curve given in polar coordinates $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \quad (8.4)$$

This derivation can also be completed directly from the formula for the case of rectangular coordinates by using $x = r\cos(\theta)$ and $y = r\sin(\theta)$. However, we prefer the geometric approach because it is *(i)* more direct, *(ii)* more intuitive, and *(iii)* easier to remember.

See Exercise 19.

HOW TO FIND THE ARC LENGTH OF $r = f(\theta)$

The length of the curve $r = f(\theta)$ for θ in $[\alpha, \beta]$ is equal to

$$\int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

or

$$\int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\theta.$$

EXAMPLE 4 Find the length of the spiral $r = e^{-3\theta}$ for θ in $[0, 2\pi]$.
SOLUTION First compute

$$r' = \frac{dr}{d\theta} = -3e^{-3\theta},$$

and then use the formula

$$\begin{aligned}
\text{Arc Length} &= \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\theta = \int_0^{2\pi} \sqrt{(e^{-3\theta})^2 + (-3e^{-3\theta})^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{e^{-6\theta} + 9e^{-6\theta}}\, d\theta = \sqrt{10}\int_0^{2\pi} \sqrt{e^{-6\theta}}\, d\theta \\
&= \sqrt{10}\int_0^{2\pi} e^{-3\theta}\, d\theta = \sqrt{10}\, \frac{e^{-3\theta}}{-3}\Big|_0^{2\pi} \\
&= \sqrt{10}\left(\frac{e^{-3\cdot 2\pi}}{-3} - \frac{e^{-3\cdot 0}}{-3}\right) = \sqrt{10}\left(\frac{e^{-6\pi}}{-3} + \frac{1}{3}\right) \\
&= \frac{\sqrt{10}}{3}\left(1 - e^{-6\pi}\right) \approx 1.054093.
\end{aligned}$$

◇

Summary

This section concerns speed along a parametric path and the length of the path. If the path is described in rectangular coordinates, then Figure 8.4.7(a) conveys the key ideas. If in polar coordinates, Figure 8.4.7(b) is the key. It is much easier to recall these diagrams than the various formulas for speed and arclength.

(a) (b)

Figure 8.4.7:

EXERCISES for 8.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 8 find the arc lengths of the given curves over the given intervals.

1.[R] $y = x^{3/2}$, x in $[1, 2]$

2.[R] $y = x^{2/3}$, x in $[0, 1]$

3.[R] $y = (e^x + e^{-x})/2$, x in $[0, b]$

4.[R] $y = x^2/2 - (\ln(x))/4$, x in $[2, 3]$

5.[R] $x = \cos^3(t)$, $y = \sin^3(t)$, t in $[0, \pi/2]$

6.[R] $r = e^\theta$, θ in $[0, 2\pi]$

7.[R] $r = 1 + \cos(\theta)$, θ in $[0, \pi]$

8.[R] $r = cos^2(\theta/2)$, θ in $[0, \pi]$

In each of Exercises 9 to 12 find the speed of the particle at time t, given the parametric description of its path.

9.[R] $x = 50t$, $y = -16t^2$

10.[R] $x = \sec(3t)$, $y = \sin^{-1}(4t)$

11.[R] $x = t + \cos(t)$, $y = 2t - \sin(t)$

12.[R] $\csc(\theta/2)$, $y = \tan^{-1}(\sqrt{t})$

13.[R]

(a) Graph $x = t^2$, $y = t$ for $0 \le t \le 3$.

(b) Estimate its arc length from $(0, 0)$ to $(9, 3)$ by an inscribed polygon whose vertices have x-coordinates 0, 1, 4, and 9.

(c) Set up a definite integral for the arc length of the curve in question.

(d) Estimate the definite integral in *(c)* by using a partition of $[0, 3]]$ into 3 sections, each of length 1, and the trapezoid method.

(e) Estimate the definite integral in *(c)* by Simpson's method with six sections.

(f) If your calculator has a program to evaluate definite integrals, use it to evaluate the definite integral in *(c)* to four decimal places.

14.[R]

(a) Graph $y = 1/x^2$ for x in $[1, 2]$.

(b) Estimate the length of the arc in *(a)* by using an inscribed polygon whose vertices at $(1, 1)$, $(\frac{5}{4}, \left(\frac{4}{5}\right)^2$, $(\frac{3}{2}, \left(\frac{2}{3}\right)^2$, and $(2, \frac{1}{4})$.

(c) Set up a definite integral for the arc length of the curve in question.

(d) Estimate the definite integral in *(c)* by the trapezoid method, using four equal length sections.

(e) Estimate the definite integral in *(c)* by Simpson's method with four sections.

(f) If your calculator has a program to evaluate definite integrals, use it to evaluate the definite integral in *(c)* to four decimal places.

15.[R] How long is the spiral $r = e^{-3\theta}$, $\theta \geq 0$?

16.[R] How long is the spiral $r = 1/theta$, $\theta \geq 2\pi$?

17.[R] Assume that a curve is described in rectangular coordinates in the form $x = f(y)$. Show that

$$\text{Arc Length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

where y ranges in the interval $[c, d]$.

See Exercise 17.

18.[R] Consider the arc length of the curve $y = x^{2/3}$ for x in the interval $[1, 8]$.

(a) Set up a definite integral for this arc length using x as the parameter.

(b) Set up a definite integral for this arc length using y as the parameter.

(c) Evaluate the easier of the two integrals found in parts *(a)* and *(b)*.

19.[R] We obtained the formula $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$ geometrically.

(a) Obtain the same result by calculus, starting with $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$, and using the relations $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

(b) Which derivation do you prefer? Why?

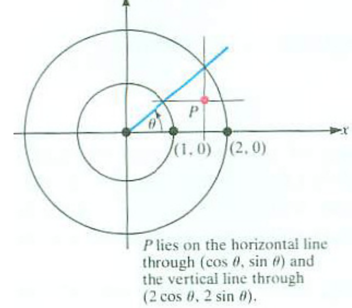


Figure 8.4.8:

20.[M] Let $P = (x, y)$ depend on θ as shown in Figure 8.4.8.

(a) Sketch the curve that P sweeps out.

(b) Show that $P = (2\cos(\theta), \sin(theta))$.

(c) Set up a definite integral for the length of the curve described in P. (Do not evaluate it.)

(d) Eliminate θ and show that P is on the ellipse

$$\frac{x^2}{4} + \frac{y^2}{1} = 1.$$

21.[M]

(a) At time t a particle has polar coordinates $r = g(t)$, $\theta = h(t)$. How fast is it moving?

(b) Use the formula in (a) to find the speed of a particle which at time t it is at the point $(r, \theta) = (e', 5t)$.

22.[M]

(a) How far does a bug travel from time $t = 1$ to time $t = 2$ if at time t it is at the point $(\cos \pi t, \sin \pi t)$?

(b) How fast is it moving at time t?

(c) Graph its path relative to an xy coordinate system. Where is it at time $t = 1$? At $t = 2$?

(d) Eliminate t to find a relation between x and y.

23.[M] Find the arc length of the archimediean spiral $r = a\theta$ for θ in $[0, 2\pi]$.

24.[M] Consider the cardioid $r = 1 + \cos\theta$ for θ in $[0, \pi]$. We may consider r as a function of θ or as a function of s, arc length along the curve, measured, say, from $(2, 0)$.

(a) Find the average of r with respect to θ.

(b) Find the average of r with respect to s. *Hint:* Express all quantities appearing in this average in terms of θ.

25.[M] Let $r = f(\theta)$ describe a curve in polar coordinates. Assume that $df/d\theta$ is continuous. Let θ be a function of time t. Let $s(t)$ be the length of the curve corresponding to the time interval $[a, t]$.

(a) What definite integral is equal to $s(t)$?

(b) What is the speed ds/dt?

26.[M] The function $r = f(\theta)$ describes, for θ in $[0, 2\pi]$, a curve in polar coordinates. Assume r' is continuous and $f(\theta) > 0$. Prove that the average of r as a function of arc length is at least as large as the quotient $2A/s$, where A is the area swept out by the radius and s is the arc length of the curve. When is the average equal to $2A/s$?

27.[M] The equations $x = \cos t$, $y = 2\sin t$, t in $[0, \pi/2]$ describes a quarter of the ellipse. Draw this arc and examine its length. That is, describe various ways of estimating the length and compare their efficiencies.

28.[C] Let $y = f(x)$ for x in $[0, 1]$ describe a curve that starts at $(0, 0)$, ends at $(1, 1)$, and lies in the square with vertices $(0, 0)$,$(1, 0)$,$(1, 1)$, and $(0, 1)$. Assume f has a continuous derivative.

(a) What can be said about the arc length of the curve? How small and how large can it be?

(b) Answer (a) if it is assumed also that $f'(x) \geq 0$ for x in $[0, 1]$.

29.[C] Consider the length of the curve $y = x^m$, where m is a rational number. Show that the fundamental theorem of calculus is of aid in computing this length only if $m = 1$ or if m is of the form $1 + 1/n$ for some integer n. *Hint:* Chebyshev proved that $\int s^p(1 + x)^q\ dx$ is elementary for rational numbers p and q only when at least one of p, q and $p + q$ is an integer.

30.[C] If one convex polygon P_1 lies inside another poligon P_2 is the perimeter of p_1 necessarily less than the perimeer of p_2? What if P_1 is not convex?

31.[M] When a curve is given in rectangular coordintes, its slope is $\frac{dy}{dx}$. To find the slope of the tangent line to the curve given in polar coordinates involves a bit more work.

Assume that $r = f(\theta)$. To begin use the relation

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}, \tag{8.5}$$

which is the Chain Rule in disguise ($\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$).

(a) Using the equations $y = r\sin(\theta)$ and $x = r\cos(\theta)$, find $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$.

(b) Show that the slope is

$$\frac{r\cos(\theta) + \frac{dr}{d\theta}\sin(\theta)}{-r\sin(\theta) + \frac{dr}{d\theta}\cos(\theta)}. \tag{8.6}$$

32.[M] Use (8.6) to find the slope of the cardioid $r = 1 + \sin(\theta)$ at $\theta = \frac{\pi}{3}$.

33.[C] One leaf of the cardioid $r = 1 + \sin(\theta)$ is traced as θ increases from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$. Find the highest point on that leaf in polar coordintes.

See Exercise 35 for the definition of $\tan(\gamma)$.

34.[C] Figure 8.4.9 shows the angle between the radius and tangent line to the curve $r = f(\theta)$. Using the fact that $\gamma - d - \omega$ and that $\tan(A - B) = \frac{(\tan(A) - \tan(B))}{1 + \tan(A)\tan(B)}$, show that $\tan(\gamma) = \frac{r}{r'}$.

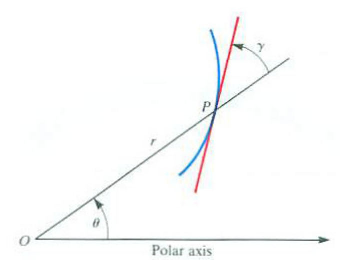


Figure 8.4.9: See Exercise 34.

35.[C] The formula $\tan(\gamma) = r/r'$ is so simple one would expect a simple geometric explanation. Use the "triangle" in Figure 8.4.6 that we used to obtain the formula for $\frac{ds}{d\theta}$ to show that $\tan(\gamma)$ should be r/r'.

36.[C] Four dogs are chasing each other counterclockwise at the same speed. Initially they are at the four vertices of a square of side a. As they chase each other, each running directly toward the dog in front, they approach the center of the square in spiral paths. How far does each dog travel?

(a) Find the equation of the spiral path each dog follows and use calculus to answer this question.

(b) Answer this question without involve calculus.

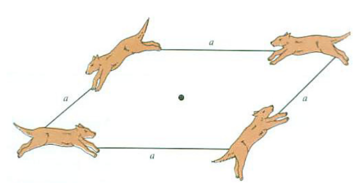


Figure 8.4.10:

37.[C] We assumed that the chord AB of a smooth curve is a good approximation of the arc $\breve{AB}$ when B is near to A. Show that the formula we obtained for arc length is consistent with this assumption. That is, if $y = f(x)$, $A = (a, f(a))$, $B = (x, f(x))$, then

$$\frac{\int_a^x \sqrt{1 + f'(t)^2}\, dt}{\sqrt{(x-a)^2 + (f(x) - f(a))^2}}$$

approaches 1 as x approaches a. HINT: *(i)* l'Hôpital's Rule is tempting but does not help us. *(ii)* For simplicity, assume $a = 0 = f(0)$.

8.5 The Area of a Surface of Revolution

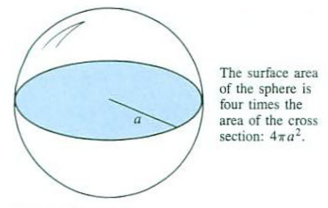


Figure 8.5.1:

In this section we develop a formula for expressing the surface area of a solid of revolution as a definite integral. In particular, we will show that the surface area of a sphere is four times the area of a cross section through its center. (See Figure 8.5.1.) This was one of the great discoveries of Archimedes in the third century B.C.

Let $y = f(x)$ have a continuous derivative for x in some interval. Assume that $f(x) \geq 0$ on this interval. When its graph is revolved about the x axis it sweeps out a surface, as shown in Figures 8.5.2 and 8.5.3.

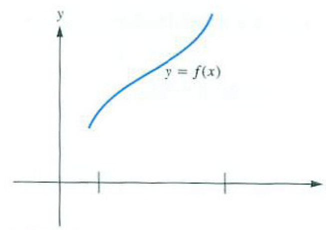


Figure 8.5.2:

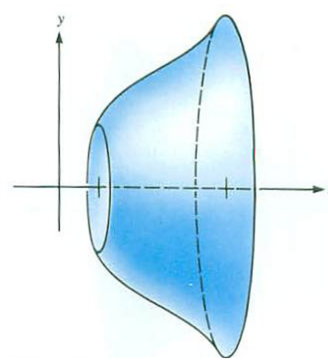
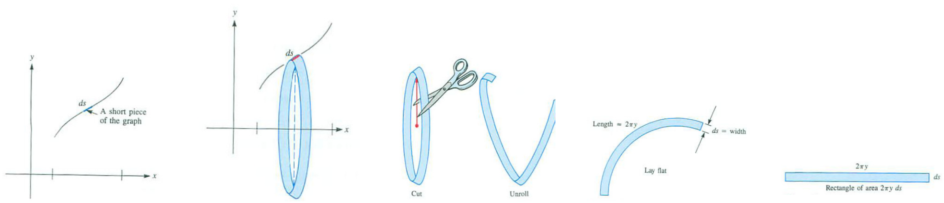


Figure 8.5.3:

To develop a definite integral for this surface area, we use the informal approach.



(a) (b) (c) (d) (e)

Figure 8.5.4:

Consider a very short section of the graph $y = f(x)$. It is almost straight. Let us approximate it by a short line segment of length ds, a very small number. When this small line segment is revolved about the x axis it sweeps out a narrow band. (See Figures 8.5.4(a) and (b).)

If we can estimate the area of this band, then we will have a local approximation of the surface area. From the local approximation we can set up a definite integral for the entire surface area.

Imagine cutting the band with scissors and laying it flat, as in Figures 8.5.4(c) and (d). It seems reasonable that the area of the flat band in Figure 8.5.4(d) is close to the area of a flat rectangle of length $2\pi y$ and width ds, as in Figure 8.5.4(e). (See Exercises 29 and 30.)

The local approximation of the surface area is therefore given by the formula

$$\text{Local approximation} = 2\pi y \, ds.$$

It leads to the formula

$$\text{Surface area} = \int_{s_0}^{s_1} 2\pi y \, ds. \tag{8.1}$$

where $[s_0, s_1]$ describes the appropriate interval on the "s-axis". Since s is a clumsy parameter, for computations we will use substitutions to change (8.1) into more convenient integrals.

Say that the section of the graph that was revolved corresponds to the interval a, b on the x-axis, as in Figure 8.5.5. Then the integral $\int_{s_0}^{s_1} 2\pi y \; ds$ becomes

$$\int_a^b 2\pi y \frac{ds}{dx} \; dx.$$

Since

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)}^{2},$$

this gives us the formula

$$\text{Surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx. \qquad (8.2)$$

(It is assumed that $y \geq 0$ and that dy/dx is continuous.)

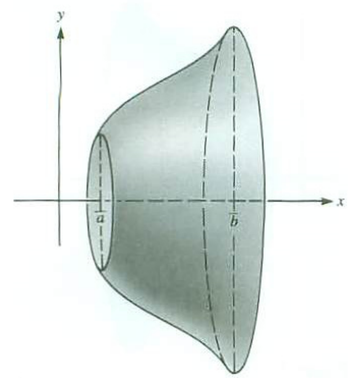


Figure 8.5.5:

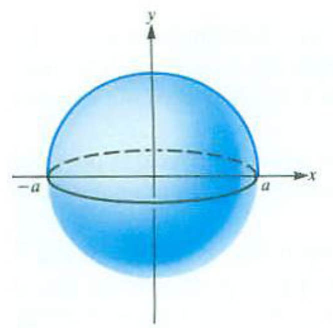


Figure 8.5.6:

EXAMPLE 1 Find the surface area of a sphere of radius a.
SOLUTION The circle of radius a has the equation $s^2 + y^2 = a^2$. The top half has the equation $y = \sqrt{a^2 - x^2}$. The sphere of radius a is formed by revolving the top half about the x axis. (See Figure 8.5.6.) We have

$$\text{Surface area of sphere} = \int_{-a}^{a} 2\pi y \frac{ds}{dx} \; dx.$$

Now $ds/dx = \sqrt{1 + (dy/dx)^2}$, where $dy/dx = -x/\sqrt{a^2 - x^2}$. Thus

$$\begin{aligned}
\text{Surface area of sphere} &= \int_{-a}^{a} 2\pi y \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2}\, dx \\
&= \int_{-a}^{a} 2\pi \sqrt{a^2 - x^2} \sqrt{1 + \frac{x^2}{a^2 - x^2}}\, dx \\
&= \int_{-a}^{a} 2\pi \sqrt{a^2 - x^2} \sqrt{\frac{a^2}{a^2 - x^2}}\, dx \\
&= \int_{-a}^{a} 2\pi a\ dx = 2\pi a x|_{-a}^{a} \\
&= 4\pi a^2.
\end{aligned}$$

The surface area of a sphere is 4 times the area of its equatorial cross section. ◇

If the graph is given parametrically, $x = g(t)$, $y = h(t)$, where g and h have continuous derivatives and $h(t) \geq 0$, then it is natural to express the integral $\int_{s_0}^{s_1} 2\pi y\ ds$ as an integral over an interval on the t-axis. If t varies in the interval $[a, b]$, then

$$\begin{aligned}
\int_{s_0}^{s_1} 2\pi y\ ds &= \int_{a}^{b} 2\pi y \frac{ds}{dt}\ dt \\
&= \int_{a}^{b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.
\end{aligned}$$

So we have

$$\text{Surface area for a curvegiven parametrically} = \int_{a}^{b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt. \tag{8.3}$$

Formula 8.2 is just the special case 8.3 when the parameter is x.

As the formulas are stated, they seem to refer only to surfaces obtained by revolving a curve about the x-axis. In fact, they refer to revolution about any line. The factor y in the integrand, $2\pi t\ ds$, is the distance from the typical point on the curve to the axis of revolution. Replace y by R (for *radius*) to free ourselves from coordinate systems. (Use capital R to aviod confusion with polar coordinates.)

The simplest way to write the formula for surface area of revolution is then

$$\text{Surface area} = \int_c^d 2\pi R\ dx,$$

where the interval $[c, d]$ refer to the parameter s. However, in practice s is seldom used as the parameter. Instead, x, y, t or θ is used and the interval of integration describes the interval through which the parameter varies.

To remember this formula, think of a narrow circular band of width ds and radius R as analogous to the rectangle shown in Figure 8.5.7.

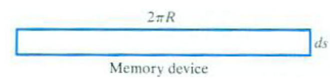


Figure 8.5.7:

EXAMPLE 2 Find the area of the surface obtained by revolving around the y-axis the part of the parabola $y = x^2$ that lies between $x = 1$ and $x = 2$. (See Figure 8.5.8.)

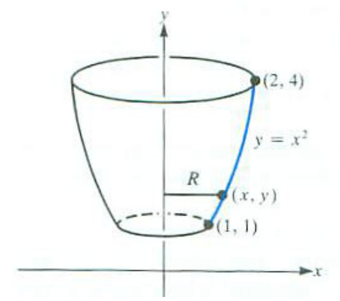


Figure 8.5.8:
R is found by inspection of a diagram.

SOLUTION The surface area is $\int_a^b 2\pi R\ ds$. Since the curve is described as a function of x, choose x as the parameter. By inspection of Figure 8.5.8, $R = x$. Next, not that

$$ds = \frac{ds}{dx}\ dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx.$$

The surface area is therefore

$$\int_1^2 2\pi x\sqrt{1 + 4x^2}\ dx.$$

To evaluate the integral, use the substitution

$$u = 1 + 4x^2 \qquad du = 8x\ dx.$$

Hence $x\ dx = du = du/8$. The new limits of integration are $u = 5$ and $u = 17$. Thus

$$\begin{aligned}\text{Surface area} &= \int_5^{17} 2\pi\sqrt{u}\frac{du}{8} = \frac{\pi}{4}\int_5^{17}\sqrt{u}\ du \\ &= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_5^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}).\end{aligned}$$

◇

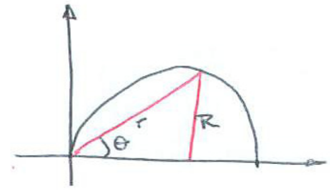


Figure 8.5.9:

EXAMPLE 3 Find the surface area when the curve $r = \cos(\theta)$, θ in $[0, \pi/2]$ is revolved around *(a)* the x-axis and *(b)* the y-axis.

SOLUTION *(a)* The curve and surface are shown in Figure 8.5.9. In this case we have $R = r\sin(\theta) = \cos(\theta)\sin(\theta)$. Also

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}^{2} = \sqrt{\left(\frac{d\cos(\theta)}{d\theta}\right)^2 + \left(\frac{d\sin(\theta)}{d\theta}\right)^2}^{2} = \sqrt{\cos^2(\theta) + \sin^2(\theta)}.$$

Then

$$\begin{aligned}
\text{Surface area} &= \int_0^{\pi/2} 2\pi R \frac{ds}{d\theta}\ d\theta \\
&= \int_0^{\pi/2} 2\pi \cos(\theta)\sin(\theta)\sqrt{\cos^2(\theta) + \sin^2(\theta)}\ d\theta \\
&= \int_0^{\pi/2} 2\pi \sin(\theta)\cos(\theta)\ d\theta \\
&= 2\pi \frac{\sin^2(\theta)}{2}\Big|_0^{\pi/2} \\
&= \pi.
\end{aligned}$$

(b) In this case $R = r\cos(\theta) = \cos^2(\theta)$. Thus

Recall the easy way to find $\int_0^{\pi/2} \cos^2(\theta)\ d\theta$ in Section 7.5(?).

$$\begin{aligned}
\text{Surface area} &= \int_0^{\pi/2} 2\pi R \frac{ds}{d\theta}\ d\theta \\
&= \int_0^{\pi/2} 2\pi \cos^2\theta \sqrt{\cos^2\theta + \sin^2\theta}\ d\theta \\
&= \int_0^{\pi/2} 2\pi \cos^2\theta\ d\theta \\
&= 2\pi \int_0^{\pi/2} \cos^2\theta\ d\theta = 2\pi(\frac{\pi}{4}) \\
&= \frac{\pi^2}{2}.
\end{aligned}$$

◇

Summary

This section developed a definite integral for the area of a surface of revolution. It rests on the local estimate of the area swept out by a short segment of length ds revolved around a line L at a distance R from the segment: $2\pi R\ ds$. We gave an informal argument for this estimate; Exercises 29 and 30 develop it formally.

When a curve is revolved about a line it sweeps out a surface. The local approximation to this area is a band whose area is proximately $2\pi R ds$, where R is the distance of a point on the curve to the line; ds is the length of a short part of the curve.

EXERCISES for 8.5 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 4 set up a definite integral for the area of the indicated surface using the suggested parameter. Show the radius R on a diagram; do *not* evaluate the definite integrals.

1.[R] The curve $y = x^3$, x in $[1, 2]$; revolved about the x-axis; parameter x.

2.[R] The curve $y = x^3$, x in $[1, 2]$; revolved about the line $y = -1$; parameter x.

3.[R] The curve $y = x^3$, x in $[1, 2]$; revolved about the y-axis; parameter y.

4.[R] The curve $y = x^3$, x in $[1, 2]$; revolved about the y-axis; parameter x.

5.[R] Find the area of the surface obtained by rotating about the x axis that part of the curve $y = e^x$ that lies above $[0, 1]$.

6.[R] Find the area of the surface formed by rotating one arch of the curve $y = \sin(x)$ about the x-axis.

7.[R] One arch of the cycloid given parametrically by the formula $x = \theta - \sin(\theta)$, $y = 1-\cos(\theta)$ is revolved around the x axis. Find the area of the surface produced.

8.[R] The curve given parametrically by $x = e^t \cos(t)$, $y = e^t \sin(t)$, t in $[0, \pi/2]$, is revolved around the x-axis. Find the area of the surface produced.

In each of Exercises 9 to 16 find the area of the surface formed by revolving the indicated curve about the indicated axis. Leave the answer as a definite integral, but indicate how it could be evaluated by the Fundamental Theorem of Calculus.

9.[R] $y = 2x^3$ for x in $[0, 1]$; about the x axis.

10.[R] $y = 1/x$ for x in $[1, 2]$; about the x axis.

11.[R] $y = x^2$ for x in $[1, 2]$; about the x axis.

12.[R] $y = x^{4/3}$ for x in $[1, 8]$; about the y axis.

13.[R] $y = x^{2/3}$ for x in $[1, 8]$; about the line $y = 1$.

14.[R] $y = x^3/6 + 1/(2x)$ for x in $[1, 3]$; about the y axis.

15.[R] $y = x^3/3 + 1/(4x)$ for x in $[1, 2]$; about the line $y = -1$.

16.[R] $y = \sqrt{1 - x^2}$ for x in $[-1, 1]$; about the line $y = -1$.

17.[M] [1] Consider the smallest tin can that contains a give sphere. (The height and diameter of the tin can equal the diameter of the sphere.)

(a) Compare the volume of the sphere with the volume of the tin can.

(b) Compare the surface area of the sphere with the area of the curved side of the can.

DOUG: In "D.E. separable" put converse of Exercise 18.

18.[M]

(a) Compute the area of the portion of a sphere of radius a that lies between two parallel planes at distances c and $c + h$ from the center of the sphere $(0 \leq c \leq c + h \leq a)$.

(b) The result in *(a)* depends only on h, not on c. What does this mean geometrically? (See Figure 8.5.10.)

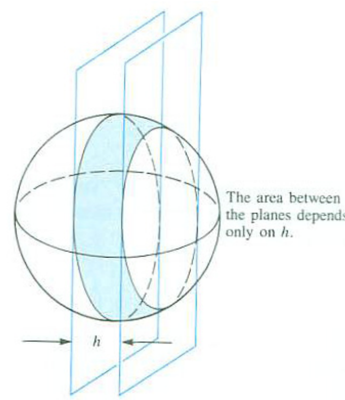


Figure 8.5.10:

[1] Archimedes, who obtained the solution about 2200 years ago, considered it his greatest accomplishment. Cicero wrote, about two centuries after Archimedes' death:

> I shall call up from the dust [the ancient equivalent of a blackboard] and his measuring-rod an obscure, insignficant person belonging to the same city [Syracuse], who lived many years after, Archimedes. When I was quaestor I tracked out his grave, which was unknown to the Syracusans (as they totally denied its existence), and found it enclosed all round and covered with brambles and thickets; for I remembered certain doggerel lines inscribed, as I had heard, upon his tomb, which stated that a sphere along with a cylinder had been set up on the top of his grave. Accordingly, after taking a good look around (for there are a great quantity of graves at Agrigentine Gate), I noticed a small column rising a little above the bushes, on which there was the figure of a sphere and a cylinder. And so I at once said to the Syracusans (I had their leading men with me) that I believed it was the very thing of which I was in search. Slaves were sent in with sickles who cleared the ground of obstacles, and when a passage to the place was opened we approached the pedestal fronting us; the epigram was traceable with about half the lines legible, as the latter portion was worn away. [Circero, *Tusculan Disputations*, vol. 23, translated by J. E. King, Loef Classical Library, Harvard Univeristy, Cambridge, 1950.]

Archimedes was killed by a Roman soldier in 212 B.C. Cicero was quaestor in 75 B.C.

In Exercises 19 and 20 estimate the surface area obtained by revolving the given arc about the given line. First, find a definite integral for the surface area. Then, use either Simpson's method with six sections or a programmable calculator or computer to approximate the value of the integral.

19.[M] $y = x^{1/4}$, x in $1, 3$, about the x-axis.

20.[M] $y = x^{1/5}$, x in $1, 3$, about the line $y = -1$.

Exercises 21 to 25 are concerned with the area of a surface obtained by revolving a curve given in polar coordinates.

21.[M] Show that the area of the surface obtained by revolving the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, around the polar axis is

$$\int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{r^2 + (r')^2}\; d\theta.$$

HINT: Use formula 8.3.

22.[M] Solve Exercise 21 by making an informal local estimate first.

23.[M] Use the formula in Exercise 21 to find the surface area of a sphere of radius a.

24.[M] Find the area of the surface formed by revolving the portion of the curve $r = 1 + \cos(\theta)$ in the first quadrant about *(a)* the x-axis, *(b)* the y-axis. HINT: The identity $1 + \cos(\theta) = 2\cos^2(\theta/2)$ may help in *(b)*.

25.[M] The curve $r = \sin 2\theta$, θ in $[0, \pi/2]$, is revolved around the polar axis. Set up an integral for the surface area.

26.[M] The portion of the curve $x^{2/3} + y^{2/3} = 1$ situated in the first quadrant is revolved around the x axis. Find the area of the surface produced.

27.[M] Although the Fundamental Theorem of Calculus is of no use in computing the perimeter of the ellipse $x^2/a^2 + y^2/b^2 = 1$, it is useful in computing the surface area of the "football" formed when the ellipse is rotated about one of its axes. Assuming that $a > b$ and that the ellipse is revolved around the x-axis, find that area. Does your answer give the correct formula for the surface area of a sphere of radius a, $4\pi a^2$? HINT: Let $b \to a^-$.

28.[M] The region bounded by $y = 1/x$ and the x-axis and situated to the right of $x = 1$ is revolved around the x-axis.

(a) Show that its volume is finite but its surface area is infinite.

(b) Does this mean that an infinite area can be painted by pouring a finite amoount of paint into this solid?

Exercises 29 and 30 develop an exact formula for the area of the surface obtained by revolving a line segment about a line that does not meet it.

29.[M] A right circular cone has slant height L and radius r, as shown in Figure 8.5.11. If this cone is cut along a line through its vertex and laid flat, it becomes a sector of a circle of radius L, as shown in Figure 8.5.12. By comparing Figure 8.5.12 to a complete disk of radius L find the area of the sector and thus the area of the cone in Figure 8.5.11.

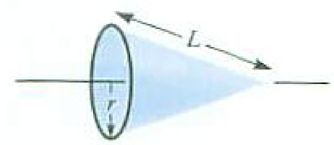


Figure 8.5.11:

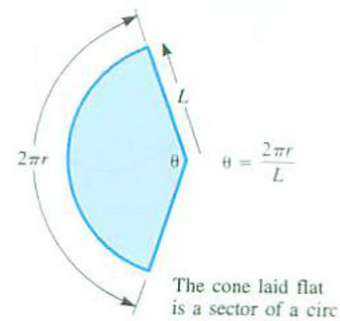


Figure 8.5.12:

30.[M] Consider a line segment of length L in the plane which does not meet a certain line in thh plane, called the axis. (See Figure 8.5.13.) When the line segment is revolved around the axis, it sweeps out a curved surface. Show that the area of this surface equals $2\pi rL$ where r is the distance from the midpoint of the line segment to the axis. The surface in Figure 8.5.4 is called a **frustum of a cone**. Follow these setps:

(a) Complete the cone by extending the frustum as shown in Figure 8.5.14. Label the radii and lengths as in that figure. Show that $\frac{r_1}{r_2} = \frac{L_1}{L_2}$, hence $r_1 L_2 = r_2 L_1$.

(b) Show that the surface area of the frustum is $\pi r_1 L_1 - \pi r_2 L_2$.

(c) Express L_1 as $L_2 + L$ and, using the result of *(a)*, show that

$$\begin{aligned} \pi r_1 L_1 - \pi r_2 L_2 &= \pi r_2 (L_1 - L_2) + \pi r_1 L \\ &= \pi r_2 L + \pi r_1 L. \end{aligned}$$

(d) Show that the surface area of the frustum is $2\pi rL$, where $r = (r_1 + r_2)/2$.

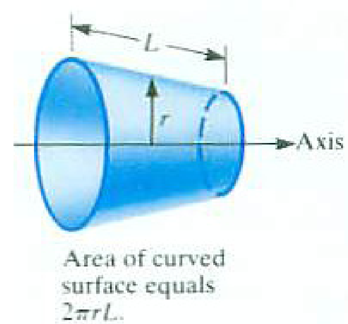


Figure 8.5.13:

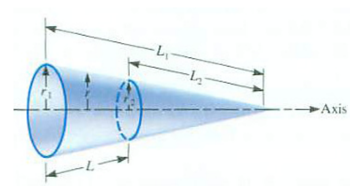


Figure 8.5.14:

31.[C] The derivative of the volume of a sphere, $4\pi r^3$ is $4\pi r^2$, its surface area. Is this simply a coincidence?

32.[C] Define the **moment of a curve around the x-axis** to be $\int_a^b y\ ds$, where a and b refer to the range of the arc length s. The **moment of the curve around**

the y-axis is defined as $\int_a^b x\ ds$. The **centroid** of the curve, $(\overline{x}, \overline{y})$, is defined by setting

$$\overline{x} = \frac{\int_a^b x\ ds}{\text{Length of curve}}$$

Find the centroid of the top half of the circle $x^2 + y^2 = a^2$.

See Exercise 32.

33.[C] Show that the area of the surface obtained by revolving about the x-axis a curve that lies above it is equal to the length of the curve times the distance that the centroid of the curve moves.

34.[C] Use Exercise 33 to find the surface area of the doughnut formed by revolving a circle of radius a around a line a distance b from its center, $b \geq a$.

35.[C] Use Exercise 33 to find the area of the curved part of a cone of radius a and height h.

See Exercise 18.
SHERMAN: Did I get close on this one?

36.[C] For some classes of functions $f(x)$ the definite integral $\int_a^b f(x)\ dx$ depends only on the width of the interval $[a, b]$, namely

$$\int_a^b f(x)\ dx = g(b-a). \tag{8.4}$$

(a) Show that every constant function satisfies (8.4).

(b) Prove that if $f(x)$ satisfies (8.4), then f must be constant.

8.6 Curvature

In this section we use calculus to obtain a measure of the "curviness" or "curvature" at points on a curve. This concept also appears in Section 13.2 in the study of motion along a curved path.

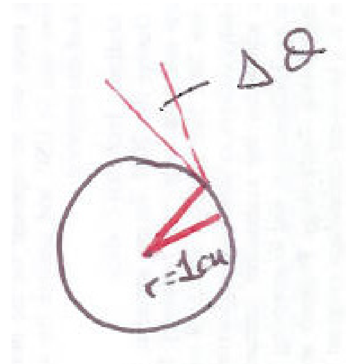


Figure 8.6.1:

Introduction

Imagine a bug crawling around a circle of radius one centimeter, as in Figure 8.6.1. As it walks a small distance, say 0.1 cm, it notices that its direction, measure by angle θ, changes. Another bug, walks around a large circle, as in Figure 8.6.2. Whenever it goes 0.1 cm, his direction, measured by angle ϕ, changes by much less. The first circle is "curvier" than the second. We will provide a measure of "curviness" or **curvature**. A straight line will have "0 curvature" everywhere. A circle of radius a will turn out to have curvature $1/a$ everywhere. For other curves, the curvature varies from point to point.

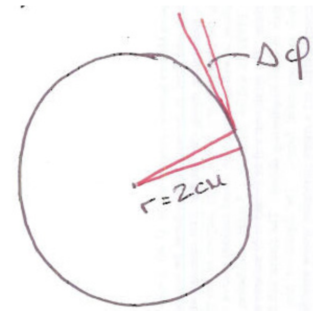


Figure 8.6.2:

Definition of Curvature

"Curvature" measures how rapidly the direction changes as we move a small distance along a curve. We have a way of assigning a numerical value to direction, namely, the angle of the tangent line. The *rate of change of this angle with respect to arc length* will be our measure of curvature.

κ is the Greek letter "kappa".

DEFINITION *(Curvature)* Assume that a curve is given parametrically, with the parameter of the typical point P being s, the distance along the curve from a fixed P_0 to P. Let ϕ be the angle between the tangent line at P and the positive part of the x-axis. The **curvature** κ at P is the absolute value of the derivative, $\frac{d\phi}{ds}$:

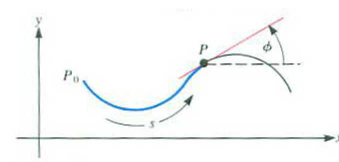


Figure 8.6.3:

$$\kappa = \left|\frac{d\phi}{ds}\right|$$

whenever the derivative exists. (See Figure 8.6.3.) NOTE: If either the angle ϕ or its derivative does not exist, then the curvature does not exist either.

Observe that a straight line has zero curvature everywhere, since ϕ is constant.

The next theorem shows that curvature of a small circle is large and the curvature of a large circle is small.

Theorem 8.6.1 *For a circle of radius a, the curvature* $\left|\frac{d\phi}{ds}\right|$ *is constant and equals* $1/a$, *the reciprocal of the radius.*

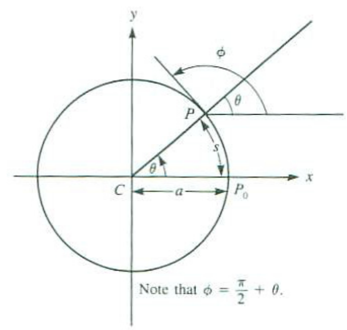


Figure 8.6.4:

Proof of Theorem 8.6.1

It is necessary to express ϕ as a function of arc length s on a circle of radius a. Refer to Figure 8.6.4. Arc length s is measured counterclockwise from the point P_0 on the x-axis. Then $\phi = \frac{\pi}{2} + \theta$, since an exterior angle of triangle PCP_0 is the sum of the two opposite angles of the triangle. (When $s = 0$, chose $\phi = \frac{\pi}{2}$.) By definition of radian measure, $s = a\theta$, so that

$$\theta = \frac{s}{a}.$$

Hence,
$$\phi = \frac{\pi}{2} + \frac{s}{a}.$$

Thus
$$\frac{d\phi}{ds} = \frac{1}{a},$$

as claimed. •

Computing Curvature

When a curve is given in the form $y = f(x)$, the curvature can be expressed in terms of the first and second derivatives, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Theorem 8.6.2 *Let arc length s be measured along the curve* $y = f(x)$ *from a fixed point* P_0. *Assume that x increases as s increases. Assume that* y' *and* y'' *are continuous. Then*

The curvature of $y = f(x)$.

$$\kappa = curvature = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}.$$

Proof of Theorem 8.6.2

By the Chain Rule,

$$\frac{d\phi}{ds} = \frac{\frac{d\phi}{dx}}{\frac{ds}{dx}}.$$

Same as $\frac{d\phi}{dx} = \frac{d\phi}{ds}\frac{ds}{dx}$.

As was shown in Section 8.3,

$$\frac{ds}{dx} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}.$$

All that remains is to express $\frac{d\phi}{dx}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Note that, in Figure 8.6.5,

$$\frac{dy}{dx} = \text{slope of tangent line to } y = f(x) = \tan(\phi). \tag{8.1}$$

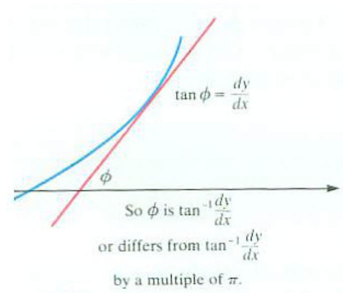


Figure 8.6.5:

We find $\frac{d\phi}{dx}$ by differentiating both sides of (8.1) with respect to x. Thus

$$\frac{d^2y}{dx^2} = \sec^2(\phi)\cdot\frac{d\phi}{dx} = \left(1+\tan^2(\phi)\right)\frac{d\phi}{dx} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)\frac{d\phi}{dx}$$

and we have

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}.$$

Consequently,

$$\frac{d\phi}{ds} = \frac{\frac{d\phi}{dx}}{\frac{ds}{dx}} = \frac{\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}},$$

and the theorem is proved. •

WARNING *()* One might have expected the curvature to be measured by the second derivative, $\frac{d^2y}{dx^2}$, since it records the rate at which the slope changes. Only when $\frac{dy}{dx} = 0$ is this expectation correct.

EXAMPLE 1 Find the curvature at a typical point (x, y) on the curve $y = x^2$.

SOLUTION In this case $\frac{dy}{dx} = 2x$ and $\frac{d^2y}{dx^2} = 2$. Thus the curvature at (x, y) is

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}} = \frac{2}{\left(1 + (2x)^2\right)^{3/2}}.$$

The maximum curvature occurs when $x = 0$. The curvatures at (x, x^2) and at $(-x, x^2)$ are equal. As $|x|$ increases, the curve becomes straighter and the curvature approaches 0. ◇

Theorem 8.6.2 holds for curves given parametrically.

Theorem 8.6.2 tells how to find the curvature if y is given as a function of x. But it holds as well when the curve is described parametrically, where x and y are functions of some parameter such as t or θ. Just use the fact that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{8.2}$$

and

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}. \tag{8.3}$$

Both of the equations (8.2) and (8.3) are just special cases of

$$\frac{df}{dx} = \frac{\frac{df}{dt}}{\frac{dx}{dt}}.$$

And this equation is just the Chain Rule in disguise,

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}.$$

In (8.2) the function f is y; in (8.3) f is $\frac{dy}{dx}$.

Example 2 illustrates the procedure.

EXAMPLE 2 The cycloid determined by a wheel of radius 1 has the parametric equations

$$x = \theta - \sin(\theta) \qquad \text{and} \qquad y = 1 - \cos(\theta),$$

as shown in Figure 8.6.6. Find the curvature at a typical point on this curve.

SOLUTION First find $\frac{dy}{dx}$ in terms of θ. We have

$$\frac{x}{\theta} = 1 - \cos(\theta) \qquad \text{and} \qquad \frac{y}{\theta} = \sin(\theta).$$

Thus

$$\frac{dy}{dx} = \frac{\sin(\theta)}{1 - \cos(\theta)}.$$

Next find $\frac{d^2y}{dx^2}$. We have

SHERMAN: Omit some of these details to create Exercise 29.

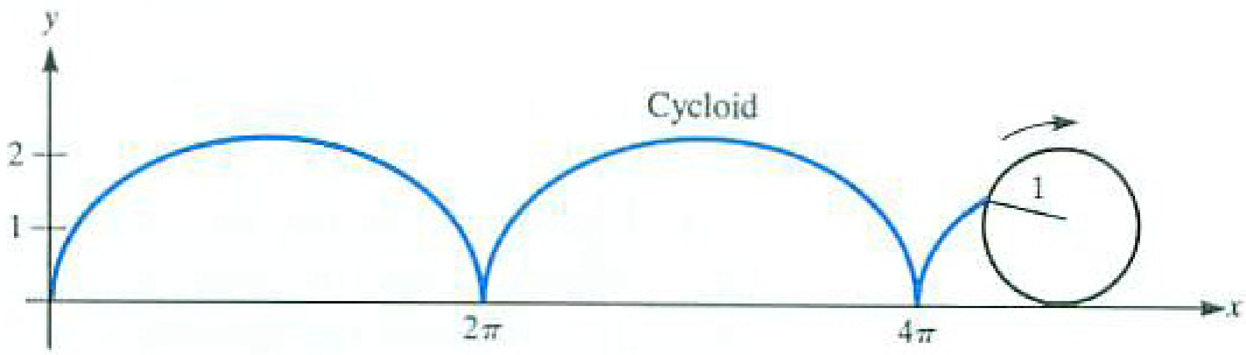


Figure 8.6.6:

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} \\
&= \frac{\frac{d}{d\theta}\left(\frac{\sin(\theta)}{1-\cos(\theta)}\right)}{1-\cos(\theta)} = \frac{\frac{(1-\cos(\theta))\cos(\theta)-\sin(\theta)\sin(\theta)}{1-\cos(\theta))^2}}{1-\cos(\theta)} \\
&= \frac{\cos(\theta)-\cos^2(\theta)-\sin^2(\theta)}{(1-\cos(\theta))^3} = \frac{\cos(\theta)-1}{(1-\cos(\theta))^3} \\
&= \frac{-1}{(1-\cos(\theta))^2}.
\end{aligned}$$

Thus the curvature is

$$\begin{aligned}
\kappa &= \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}} = \frac{\left|\frac{-1}{(1-\cos(\theta))^3}\right|}{\left(1+\left(\frac{\sin(\theta)}{1-\cos(\theta)}\right)^2\right)^{3/2}} \\
&= \frac{\frac{1}{(1-\cos(\theta))^2}}{\left(\frac{(1-\cos(\theta))^2+\sin^2(\theta)}{(1-\cos(\theta))^2}\right)^{3/2}} = \frac{1-\cos(\theta)}{\left(1-2\cos(\theta)+\cos^2(\theta)+\sin^2(\theta)\right)^{3/2}} \\
&= \frac{1-\cos(\theta)}{(2-2\cos(\theta))^(3/2)} = \frac{1}{2^{3/2}\sqrt{1-\cos(\theta)}}.
\end{aligned}$$

Since $y = 1-\cos(\theta)$ and $2^{3/2} = \sqrt{8}$, the curvature equals $1/\sqrt{8y}$. ◇

Radius of Curvature

As Theorem 8.6.1 shows, a circle with curvature κ has radus $1/\kappa$. This suggests the following definition.

A large radius of curvature implies a small curvature.

DEFINITION *(Radius of Curvature)* The **radius of curvature** of a curve at a point is the reciprocal of the curvature:

$$\text{Radius of curvature} = \frac{1}{\text{Curvature}} = \frac{1}{\kappa}.$$

As can be easily checked, the radius of curvature of a circle of radius a is, fortunately, a.

The cycloid in Example 2 has radius of curvature at the point (x, y) equal to

$$\frac{1}{1/\sqrt{8y}} = \sqrt{8y}.$$

In particular, at the top of an arch the radius of curvature is $\sqrt{8 \cdot 2} = 4$.

Keep this? Why? Why not? Boxed?

The Osculating Circle

At a given point P on a curve, the **osculating circle** at P is defined to be that circle which *(a)* passes through P, *(b)* has the same slope at P as the curve does, and *(c)* has the same second derivative there. By Theorem 8.6.2, the osculating circle and the curve have the same curvature at P. Hence the have the same radius of curvature.

For instance, consider the parabola $y = x^2$ of Example 1. When $x = 1$, the curvature is $2/5^{3/2}$ and the radius of curvature is $5^{3/2}/2 \approx 5.59017$. The osculating circle at $(1, 1)$ is shown in Figure 8.6.7.

Observe that the osculating circle in Figure 8.6.7 *crosses the parabola* as it passes through the point $(1, 1)$. Although this may be surprising, a little reflection will show why it is to be expected.

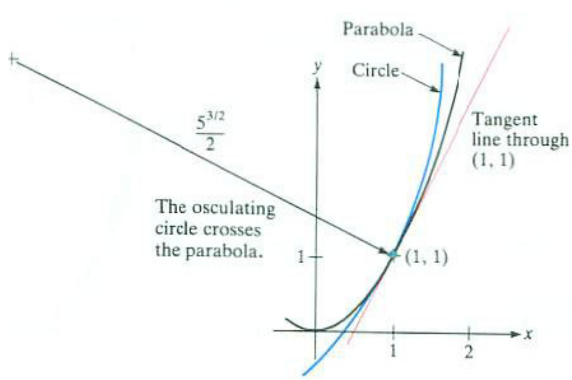


Figure 8.6.7:

Think of driving along the parabola $y = x^2$. If you start at $(1, 1)$ and drive up along the parabola, the curvature diminishes. It is smaller than that of the circle of curvature at $(1,1)$. Hence you would be turning your steering wheel to the left and would be traveling *outside* the circle of curvature at $(1,1)$. On the other hand, if you start at $(1,1)$ and move to the left on the parabola, the curvature increases and is greater than that of the osculating circle at $(1,1)$, so you would be driving *inside* the osculating circle at $(1,1)$. This informal argument shows why the osculating circle crosses the curve in general. At a point where the curvature is neither a local maximum or a local minimum, the osculating circle crosses the curve. In the case of $y = x^2$, the only osculating circle that does *not* cross the curve at its point of tangency is the one that is tangent at $(0,0)$, where the curvature is a maximum.

Summary

We defined the curvature κ of a curve as the absolute value of the rate at which the angle between the tangent line and the x-axis changes as a function of arc length; curvature equals $\left|\frac{d\phi}{ds}\right|$. If the curve is the graph of $y = f(x)$, then

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}.$$

If the curve is given in terms of a parameter t then compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ with the aid of the relation

$$\frac{d(\ )}{dx} = \frac{\frac{d(\ )}{dt}}{\frac{dx}{dt}}, \tag{8.4}$$

Equation (8.4) is our old friend, the Chain Rule; just clear the denominator.

the empty parentheses enclosing first y, then $\frac{dy}{dx}$.

"Radius of curvature" is the reciprocal of curvature.

EXERCISES for 8.6 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 6 find the curvature and radius of curvature of the given curve at the given point.

1.[R] $y = x^2$ at $(1, 1)$

2.[R] $y = \cos(x)$ at $(0, 1)$

3.[R] $y = e^{-x}$ at $(1, 1/e)$

4.[R] $y = \ln(x)$ at $(e, 1)$

5.[R] $y = \tan(x)$ at $(\frac{\pi}{4}, 1)$

6.[R] $y = \sec(2x)$ at $(\frac{\pi}{6}, 2)$

In Exercises 7 to 10 find the curvature of the given curves for the given value of the parameter.

7.[R] $\left\{ \begin{array}{lll} x & = & 2\cos(3t) \\ y & = & 2\sin(3t) \end{array} \right.$ at $t = 0$

8.[R] $\left\{ \begin{array}{lll} x & = & 1 + t^2 \\ y & = & t^3 + t^4 \end{array} \right.$ at $t = 2$

9.[R] $\left\{ \begin{array}{lll} x & = & e^{-t}\cos(t) \\ y & = & e^{-t}\sin(t) \end{array} \right.$ at $t = \frac{\pi}{6}$

10.[R] $\left\{ \begin{array}{lll} x & = & \cos^3(\theta) \\ y & = & \sin^3(\theta) \end{array} \right.$ at $\theta = \frac{\pi}{3}$

11.[R]

(a) Compute the curvature and radius of curvature for the curve $y = (e^x + e^{-x})/2$.

(b) Show that the radius of curvature at (x, y) is y^2.

12.[R] Find the radius of curvature along the curve $y = \sqrt{a^2 - x^2}$, where a is a constant. (Since the curve is part of a circle of radius a, the answer should be a.)

13.[R] For what value of x is the radius of curvature of $y = e^x$ smallest? *Hint:* How does one find the minimum of a function?

14.[R] For what value of x is the radius of curvature of $y = x^2$ smallest?

15.[R]

(a) Show that where a curve has its tangent parallel to the x axis its curvature is simply the absolute value of the second derivative d^2y/dx^2.

(b) Show that the curvature is never larger than the absolute value of d^2y/dx^2.

16.[M] An engineer lays out a railroad track as indicated in Figure 8.6.8. BC is part of a circle; AB and CD are straight and tangent to the circle. After the first train runs over the track, the engineer is fired because the curvature is not a continuous function. Why should it be?

Figure 8.6.8:

17.[M] Railroad curves are banked to reduce wear on the rails and flanges. The greater the radius of curvature, the less the curve must be banked. The best bank angle A satisfies the equation $\tan A = v^2/(32R)$, where v is speed in feet per second and R is radius of curvature in feet. A train travels in the elliptical track

$$\frac{x^2}{1000^2} + \frac{y^2}{500^2} = 1$$

at 60 miles per hour. Find the best angle A at the points $(1000, 0)$ and $(0, 500)$.

x and y are measured in feet; 60 mph=88 fps

18.[M] The flexure formula in the theory of beams asserts that the bending moment M required to bend a beam is proportional to the desired curvature, $M = c/R$, where c is a constant depending on the beam and R is the radius of curvature. A beam is bent to form the parabola $y = x^2$. What is the ratio between the moments required at *(a)* at $(0, 0)$ and *(b)* at $(2, 4)$? (See Figure 8.6.9.)

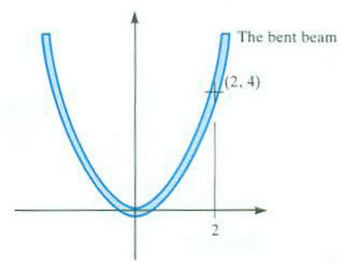


Figure 8.6.9:

Exercises 19 to 21 are related.

19.[M] Find the radius of curvature at a typical point on the ellipse

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta. \end{aligned}$$

20.[M]

(a) Show, by eliminating θ, that the curve in Exercise 19 is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(b) What is the radius of curvature of this ellipse at $(a, 0)$? at $(0, b)$?

21.[M] An ellipse has a major diameter of length 6 and a minor diameter of length 4. Draw the circles that most closely approximate this ellipse at the four points which lie at the extremities of its diameters. (See Exercises 19 and 20.)

In each of Exercises 22 to 24 a curve is given in polar coordinates. To find its curvature write it in rectangular coordinates with parameter θ, using the equations $x = r\cos\theta$ and $y = r\sin\theta$.

22.[M] Find the curvature of $r = a\cos\theta$.

23.[M] Show that at the point (r, θ) the cardioid $r = 1 + \cos\theta$ has curvature $3\sqrt{2}/(4\sqrt{r})$.

24.[M] Find the curvature of $r = \cos 2\theta$.

25.[M] If, on a curve, $dy/dx = y^3$, express the curvature in terms of y.

26.[M] As is shown in physics, the larger the radius of curvature of a turn, the faster a given car can travel around that turn. The radius of curvature required is propotional to the square of the maximum speed. Or conversely, the maximum speed around a turn is proportional to the square root of the radius of curvature. If a car moving on the path $y = x^3$ (x and y measured in miles) can go 30 miles per hour at $(1, 1)$ without sliding off, how fast can it go at $(2, 8)$?

27.[M] Find the local extrema of the curvature of

(a) $y = x + e^x$

(b) $y = e^x$

(c) $y = \sin(x)$

(d) $y = x^3$

28.[M] Sam says, "I don't like the definition of curvature. It should be the rate at which the slope changes as a function of x. That is $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, which is the second derivative, $\frac{d^2y}{dx^2}$." Give an example of a curve which would have constant curvature according to Sam's definition, but whose changing curvature is obvious to the naked eye.

29.[C] Fill in the omitted algebra in Example 2.

SHERMAN: This exercise makes sense only if some of the details in Example 2 are omitted.

The physicists show why the radius of curvature is constant, leaving it to the mathematicians to show that therefore the path is a circle.

30.[C] If a planar curve has a constant radius of curvature must it be part of a circle? That the answer is "yes" is important in experiments conducted with a cyclotron: The path of an electron entering a uniform magnetic field at right angles to the field is a circle. Here is the mathematics.

(a) Show that $\frac{ds}{d\phi} = R$, the radius of curvature.

(b) Show that $\frac{dx}{d\phi} = R\cos(\phi)$.

(c) Show that $\frac{dy}{d\phi} = R\sin(\phi)$.

(d) With the aid of *(b)* and *(c)*, find an equation for the curvature involving x and y.

HINT: For *(b)* and *(c)* draw the little triangle whose hypotenuse is like a short piece of arc length ds on the curve and whose legs are parallel to the axes.

SHERMAN: Did I get this close to what you intended?

31.[C] At the top of the cycloid in Example 2 the radius of curvature is twice the diameter of the rolling circle. What would you have guessed the radius of curvature to be at this point? Why is it not simply the diameter of the wheel, since the wheel at each moment is rotating about its point of contact with the ground?

SHERMAN: What's the answer? Some sliding?

32.[M] Assume that g and h are functions with continuous second derivatives. In addition, assume as we move along the parameterized curve $x = g(t)$, $y = h(t)$, both the arc length s from a point P_0 increases as t increases. Show that

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{\left(\dot{x}^2 + \dot{(y)}^2\right)^{3/2}}.$$

The dot notation for derivatives shortens the formula: $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $\dot{y} = \frac{dy}{dt}$, and $\ddot{y} = \frac{d^2y}{dt^2}$.

8.S Chapter Summary

EXERCISES for 8.S *Key:* R–routine, M–moderate, C–challenging