

Newton's Method and `fsolve`

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Overview

The analysis of a function via calculus involves solving a variety of equations: $f'(x) = 0$ for critical points, $f''(x) = 0$ for possible inflection points. These equations are generally easy to formulate but difficult to solve. In fact, in many cases it is impossible to find exact solutions. Newton's Method is one algorithm for finding an approximate solution. More generally, Maple's `fsolve` command will be used to find approximate solutions to equations.

Maple Essentials

- The *Newton's Method* tutor in Maple 9.5 can be found under the **Tools** menu:

Tools → **Tutors** → **Calculus – Single Variable** → **Newton's Method ...**

Preparation

Review the First and Second Derivative Tests.

Introduction to Newton's Method

The `fsolve` command was introduced in the lab for Week 5 (*More Rigorous Definition of Limits*). Newton's Method is one part of the algorithm used in `fsolve`. The basic ideas behind finding an approximate solution to the equation $F(x) = 0$ are as follows:

1. pick a starting number, x_0 , near the solution to $F(x) = 0$
one possibility is to use the graph to locate a number close to the desired solution to $F(x) = 0$
2. compute x_1 , a number closer to the solution of $F(x) = 0$, as follows:
 - (a) construct the tangent line to $y = F(x)$ at $x = x_0$
 - (b) the x -intercept of the tangent line is given by $x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$
observe that the x -intercept of the tangent line is generally closer to the solution than is x_0
3. repeat this process using x_n as a new starting point and yielding x_{n+1} as an improved estimate to the solution of $F(x) = 0$; the general formula for each step in Newton's Method is $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$.
4. end the iteration, typically after a fixed number of iterations or when the estimates appear to stabilize

Activities

- Find approximations to all solutions to $x^3 - 5x = -1$.
- Use Newton's Method to estimate all critical numbers of $f(x) = x^{\sin x}$ on $[0, 10]$.
- Explain the behavior of Newton's Method for $F(x) = \frac{x}{x^2+1}$ with $x_0 = 2$, $x_0 = 1$, and $x_0 = \frac{1}{2}$.
Remember to enter the initial guess as a floating-point number.
- Find, and plot, the cubic polynomial that has a relative maximum at $(-1, 2)$ and a relative minimum at $(3, -2)$.

Using the Newton's Method Tutor

Consider the problem of finding the solutions to $x^3 - 5x = -1$. The first step is to rewrite the problem as a root-finding problem: solve $x^3 - 5x + 1 = 0$.

1. First estimates to the roots can be found from an appropriate plot. (How many real-valued solutions are there? Use the plot to estimate each of these solutions.)
2. Launch the *Newton's Method* tutor. Enter the function as $x^3-5*x+1$, the initial guess as 1, and the number of iterations as 1; press the **Display** button. This computes and shows the first iteration of Newton's Method. Change the number of iterations to 2 and press the **Display** button; this displays the first two iterations of Newton's Method. Increase the number of iterations until at least 4 digits to the right of the decimal point appear to stabilize.
3. Change the initial guess to 2 and apply Newton's Method enough times to estimate another root to at least 4 digits to the right of the decimal point.
4. Repeat the previous step with the initial guess set to -1. Note that Newton's Method does not always converge to the closest root.
5. Find an initial guess so that Newton's Method converges to the remaining root of this function.
6. Compare the estimates obtained with Newton's Method to those obtained with `fsolve`.

Implementing Newton's Method in Maple

- Basic Newton Iteration

The following command defines a new Maple command, `Newton`, that computes the next iterate in Newton's Method for solving $F(x) = 0$ with current guess $x = x_0$.

```
> Newton := (F,x0) -> eval(x-F/diff(F,x),x=x0);
```

- Step-by-Step Iteration

```
> F := x^3-5*x+1; # define function to be solved (F(x) = 0)
> Newton( F, 0.0 ); # computes x1 from x0 = 0
> Newton( F, % ); # re-execute as often as needed
```

- Fixed Number of Iterations

```
> X[0] := 2.0; # initial guess is x0 = 2
> for n from 0 to 5 do # beginning of loop (6 iterations)
>   X[n+1] := Newton( F, X[n] ); # computes xn+1 from xn
> end do; # end of loop
```

- Iterate Until Tolerance Met

```
> Digits := 20; # number of digits used in calculations
> tol := 5*10^(-9); # tolerance to get 8 decimal digits correct
> X[0] := -2.0; # initial guess is x0 = -2
> err[0] := 1; # initial error (anything > tol)
> for n from 0 to 49 while (err[n]>tol) do # loop until tolerance is met (50 iter max)
>   X[n+1] := Newton( F, X[n] ); # computes xn+1 from xn
>   err[n+1] := abs( X[n+1]-X[n] ); # update error
> end do; # end of loop
```

Assignment

The report for Project 3 (see page 3) is due at the beginning of next week's lab period.

Project 3

1. Find the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ with a relative minimum at $(-2, -2)$ and a relative maximum at $(3, 4)$.
2. Find the quintic (fifth-degree) polynomial that passes through the points $(3, 3)$ and $(-1, 3)$, has a tangent line at $x = 0$ with slope 3, a relative extreme value at $x = 4$, and an inflection point at $(1, 4)$.
3. Find the function of the form $f(x) = \frac{a+be^x}{c+de^x}$ with asymptotes $y = -2$, $y = 5$, and $x = \ln(3)$ that is increasing on its natural domain.

Your report should contain the following information for each of the three functions:

- an explanation of how the stated conditions lead to a system of equations for the unknown parameters in the function
- the solution to the system of equations and the function that meets all of the conditions
- a plot of the function on an appropriate interval that shows all of the essential qualitative features of the function

In addition, for the quintic polynomial found in 2., find:

- all zeroes of the function, found by Newton's Method (for each zero, give the initial guess and the number of iterations)
- the intervals where the function is increasing, decreasing, concave up, and concave down
- all critical points and inflection points of the function

Be creative in your presentation of these results. If, for example, you use a table, remember that the main text needs to explain what information is contained in the table. (Consult with your TA if you have any questions about your presentation.)

All floating point numbers should be reported with an accuracy of at least three digits to the right of the decimal point.

Additional Notes

Here is a nice collection of Maple tutors and maplets designed to help you develop your skills applying the First and Second Derivative Tests.

- The *GraphTerms*, *Graph_f*, *Graph_df*, and *Graph_ddf* maplets provide practice using different types of information about a function to make conclusions about properties of the function. These maplets are available from Texas A & M University at the URLs:

<http://calclabs.math.tamu.edu/maple/maplets/GraphTerms.maplet>
http://calclabs.math.tamu.edu/maple/maplets/Graph_f.maplet
http://calclabs.math.tamu.edu/maple/maplets/Graph_df.maplet
http://calclabs.math.tamu.edu/maple/maplets/Graph_ddf.maplet

- The *Curve Analysis* tutor in Maple 9.5 can be found under the **Tools** menu:

Tools → **Tutors** → **Calculus – Single Variable** → **Curve Analysis ...**

- The *FunctionAnalyzer* maplet is available from USC at the URL:

<http://www.math.sc.edu/~meade/141L-F04/maplets/CalcUSC/FunctionAnalyzer.maplet>

This maplet can also be run via MapleNet at the URL:

<http://maplenet.math.sc.edu/maplenet/141L-F04/CalcUSC/FunctionAnalyzer.html>