

PROBLEMS FROM LINEAR ALGEBRA

In the following \mathbb{R} denotes the field of real numbers while \mathbb{C} denotes the field of complex numbers. In general, \mathbf{U} , \mathbf{V} , and \mathbf{W} denote vector spaces. The set of all linear transformations from \mathbf{V} into \mathbf{W} is denoted by $\mathcal{L}(\mathbf{V}, \mathbf{W})$, while $\mathcal{L}(\mathbf{V})$ denotes the set of linear operators on \mathbf{V} . For a linear transformation T , the null space of T (also known as the kernel of T) is denoted by $\text{null } T$, while the range space of T (also known as the image of T), is denoted by $\text{range } T$.

PROBLEM 0.

Let \mathbf{V} be a finite-dimensional vector space and let T be a linear operator on \mathbf{V} . Suppose that T commutes with every diagonalizable linear operator on \mathbf{V} . Prove that T is a scalar multiple of the identity operator.

PROBLEM 1.

Let \mathbf{V} and \mathbf{W} be vector spaces and let T be a linear transformation from \mathbf{V} into \mathbf{W} . Suppose that \mathbf{V} is finite-dimensional. Prove $\text{rank}(T) + \text{nullity}(T) = \dim V$.

PROBLEM 2.

Let A and B be $n \times n$ matrices over a field \mathbf{F}

- (1) Prove that if A or B is nonsingular, then AB is similar to BA .
- (2) Show that there exist matrices A and B so that AB is *not* similar to BA .
- (3) What can you deduce about the eigenvalues of AB and BA . Prove your answer.



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PROBLEM 3.

Let $A = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$, where D and G are $n \times n$ matrices. If $DF = FD$ prove that $\det A = \det(DG - FE)$.

PROBLEM 4.

Let V be a finite dimensional vector space. Can V have three distinct proper subspaces W_0 , W_1 and W_2 such that $W_0 \subseteq W_1$, $W_0 + W_2 = V$, and $W_1 \cap W_2 = \{0\}$?

PROBLEM 5.

Let n be a positive integer. Define

$$G = \{A : A \text{ is an } n \times n \text{ matrix with only integer entries and } \det A \in \{-1, +1\}\},$$
$$H = \{A : A \text{ is an invertible } n \times n \text{ matrix and both } A \text{ and } A^{-1} \text{ have only integer entries}\}.$$

Prove $G = H$.

PROBLEM 6.

Let V be the vector space over \mathbb{R} of all $n \times n$ matrices with entries from \mathbb{R} .

- (1) Prove that $\{I, A, A^2, \dots, A^n\}$ is linearly dependent for all $A \in V$.
- (2) Let $A \in V$. Prove that A is invertible if and only if I belongs to the span of $\{A, A^2, \dots, A^n\}$.

PROBLEM 7.

Is every $n \times n$ matrix over the field of complex numbers similar to a matrix of the form $D + N$ where D is a diagonal matrix, $N^{n-1} = 0$, and $DN = ND$. Why?



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PROBLEM 8.

Let V and W be vector spaces and let T be a linear operator from V into W . Suppose that V is finite-dimensional. Prove $\text{rank}(T) + \text{nullity}(T) = \dim V$.

PROBLEM 9.

Let $T \in L(V, V)$, where V is a finite dimensional vector space. (For a linear operator S denote by $\mathcal{N}(S)$ the null space and by $\mathcal{R}(S)$ the range of S .)

- (1) Prove there is a least natural number k such that $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1}) = \mathcal{N}(T^{k+2}) \dots$. Use this k in the rest to this problem.
- (2) Prove that $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1}) = \mathcal{R}(T^{k+2}) \dots$.
- (3) Prove that $\mathcal{N}(T^k) \cap \mathcal{R}(T^k) = \{0\}$.
- (4) Prove that for each $\alpha \in V$ there is exactly one vector in $\alpha_1 \in \mathcal{N}(T^k)$ and exactly one vector $\alpha_2 \in \mathcal{R}(T^k)$ such that $\alpha = \alpha_1 + \alpha_2$.

PROBLEM 10.

Let \mathbf{F} be a field of characteristic 0 and let

$$W = \left\{ A = [a_{ij}] \in \mathbf{F}^{n \times n} : \text{tr}(A) = \sum_{i=1}^n a_{ii} = 0 \right\}.$$

For $i, j = 1, \dots, n$ with $i \neq j$, let E_{ij} be the $n \times n$ matrix with (i, j) -th entry 1 and all the remaining entries 0. For $i = 2, \dots, n$ let E_i be the $n \times n$ matrix with $(1, 1)$ entry -1 , (i, i) -th entry $+1$, and all remaining entries 0. Let

$$S = \{E_{ij} : i, j = 1, \dots, n \text{ and } i \neq j\} \cup \{E_i : i = 2, \dots, n\}.$$

[NOTE: You can assume, without proof, that S is a linearly independent subset of $\mathbf{F}^{n \times n}$.]

- (1) Prove that W is a subspace of $\mathbf{F}^{n \times n}$ and that $W = \text{span}(S)$. What is the dimension of W ?

(2) Suppose that f is a linear functional on $\mathbf{F}^{n \times n}$ such that

- (a) $f(AB) = f(BA)$, for all $A, B \in \mathbf{F}^{n \times n}$.
 (b) $f(I) = n$, where I is the identity matrix in $\mathbf{F}^{n \times n}$.

Prove that $f(A) = \text{tr}(A)$ for all $A \in \mathbf{F}^{n \times n}$.

PROBLEM 11.

Let V be a vector space over \mathbb{C} . Suppose that f and g are linear functionals on V such that the functional

$$h(\alpha) = f(\alpha)g(\alpha) \quad \text{for all } \alpha \in V$$

is linear. Show that either $f = 0$ or $g = 0$.

PROBLEM 12.

Let C be a 2×2 matrix over a field \mathbf{F} . Prove: There exists matrices A, B such that $C = AB - BA$ if and only if $\text{tr}(C) = 0$.

PROBLEM 13.

Prove that if A and B are $n \times n$ matrices from \mathbb{C} and $AB = BA$, then A and B have a common eigenvector.

PROBLEM 14.

Let \mathbf{F} be a field and let V be a finite dimensional vector space over \mathbf{F} . Let $T \in L(V, V)$. If c is an eigenvalue of T , then prove there is a nonzero linear functional f in $L(V, \mathbf{F})$ such that $T^*f = cf$. (Recall that $T^*f = fT$ by definition.)





PROBLEM 15.

Let \mathbf{F} be a field, $n \geq 2$ be an integer, and let V be the vector space of $n \times n$ matrices over \mathbf{F} . Let A be a fixed element of V and let $T \in L(V, V)$ be defined by $T(B) = AB$.

- (1) Prove that T and A have the same minimal polynomial.
- (2) If A is diagonalizable, prove, or disprove by counterexample, that T is diagonalizable.
- (3) Do A and T have the same characteristic polynomial? Why or why not?

PROBLEM 16.

Let M and N be 6×6 matrices over \mathbb{C} , both having minimal polynomial x^3 .

- (1) Prove that M and N are similar if and only if they have the same rank.
- (2) Give a counterexample to show that the statement is false if 6 is replaced by 7.

PROBLEM 17.

Give an example of two 4×4 matrices that are not similar but that have the same minimal polynomial.

PROBLEM 18.

Let (a_1, a_2, \dots, a_n) be a nonzero vector in the real n -dimensional space \mathbb{R}^n and let P be the hyperplane

$$\left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n a_i x_i = 0 \right\}.$$



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Find the matrix that gives the reflection across P .

PROBLEM 19.

Let V and W be finite-dimensional vector spaces and let $T : V \rightarrow W$ be a linear transformation. Prove that there exists a basis \mathcal{A} of V and a basis \mathcal{B} of W so that the matrix $[T]_{\mathcal{A},\mathcal{B}}$ has the block form $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

PROBLEM 20.

Let V be a finite-dimensional vector space and let T be a diagonalizable linear operator on V . Prove that if W is a T invariant subspace then the restriction of T to W is also diagonalizable.

PROBLEM 21.

Let T be a linear operator on a finite-dimensional vector space. Show that if T has no cyclic vector then there exists an operator U on V that commutes with T but is *not* a polynomial in T .

PROBLEM 22.

Exhibit two real matrices with no real eigenvalues which have the same characteristic polynomial and the same minimal polynomial but are not similar.

PROBLEM 23.

Let V be a vector space, not necessarily finite-dimensional. Can V have three distinct proper subspaces A , B , and C , such that $A \subset B$, $A + C = V$, and $B \cap C = \{0\}$?



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PROBLEM 24.

Compute the minimal and characteristic polynomials of the following matrix. Is it diagonalizable?

$$\begin{bmatrix} 5 & -2 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

PROBLEM 25.

- (1) Prove that if A and B are linear transformations on an n -dimensional vector space with $AB = 0$, then $r(A) + r(B) \leq n$ where $r(\cdot)$ denotes rank.
- (2) For each linear transformation A on an n -dimensional vector space, prove that there exists a linear transformation B such that $AB = 0$ and $r(A) + r(B) = n$.

PROBLEM 26.

- (1) Prove that if A is a linear transformation such that $A^2(I - A) = A(I - A)^2 = 0$, then A is a projection.
- (2) Find a non-zero linear transformation so that $A^2(I - A) = 0$ but A is *not* a projection.

PROBLEM 27.

If S is an m -dimensional vector space of an n -dimensional vector space V , prove that S° , the annihilator of S , is an $(n - m)$ -dimensional subspace of V^* .



PROBLEM 28.

Let A be the 4×4 real matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

- (1) Determine the rational canonical form of A .
- (2) Determine the Jordan canonical form of A .

PROBLEM 29.

Let T be the linear operator on \mathbb{R}^3 which is represented by

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

in the standard basis. Find matrices B and C which represent respectively, in the standard basis, a diagonalizable linear operator D and a nilpotent linear operator N such that $T = D + N$ and $DN = ND$.

PROBLEM 30.

Suppose T is a linear operator on \mathbb{R}^5 represented in some basis by a diagonal matrix with entries $-1, -1, 5, 5, 5$ on the main diagonal.

- (1) Explain why T can not have a cyclic vector.
- (2) Find k and the invariant factors $p_i = p_{\alpha_i}$ in the cyclic decomposition $\mathbb{R}^5 = \bigoplus_{i=1}^k Z(\alpha_i; T)$.
- (3) Write the rational canonical form for T .



PROBLEM 31.

Suppose that V is an n -dimensional vector space and T is a linear map on V of rank 1. Prove that T is nilpotent or diagonalizable.

PROBLEM 32.

Let M denote an $m \times n$ matrix with entries in a field. Prove that

$$\begin{aligned} & \text{the maximum number of linearly independent rows of } M \\ &= \text{the maximum number of linearly independent columns of } M \end{aligned}$$

(Do not assume that $\text{rank } M = \text{rank } M^t$.)

PROBLEM 33.

Prove the Cayley-Hamilton Theorem, using only basic properties of determinants.

PROBLEM 34.

Let V be a finite-dimensional vector space. Prove there is a linear operator T on V which is invertible if and only if the constant term in the minimal polynomial for T is non-zero.

PROBLEM 35.

- (1) Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. Find a matrix T (with entries in \mathbf{C}) such that $T^{-1}MT$ is diagonal, or prove that such a matrix does not exist.
- (2) Find a matrix whose minimal polynomial is $x^2(x-1)^2$, whose characteristic polynomial is $x^4(x-1)^3$ and whose rank is 4.



PROBLEM 36.

Suppose A and B are linear operators on the same finite-dimensional vector space V . Prove that AB and BA have the same characteristic values.

PROBLEM 37.

Let M denote an $n \times n$ matrix with entries in a field \mathbf{F} . Prove that there is an $n \times n$ matrix B with entries in \mathbf{F} so that $\det(M + tB) \neq 0$ for every non-zero $t \in \mathbf{F}$.

PROBLEM 38.

Let W_1 and W_2 be subspaces of the finite dimensional vector space V . Record and prove a formula which relates $\dim W_1$, $\dim W_2$, $\dim(W_1 + W_2)$, $\dim(W_1 \cap W_2)$.

PROBLEM 39.

Let M be a symmetric $n \times n$ matrix with real number entries. Prove that there is an $n \times n$ matrix N with real entries such that $N^3 = M$.

PROBLEM 40.

TRUE OR FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample.) If two nilpotent matrices have the same rank, the same minimal polynomial and the same characteristic polynomial, then they are similar.

PROBLEM 41.

Suppose that $T : V \rightarrow W$ is an injective linear transformation over a field \mathbf{F} . Prove that $T^* : W^* \rightarrow V^*$ is surjective. (Recall that $V^* = L(V, \mathbf{F})$ is the vector space of linear transformations from V to \mathbf{F} .)



PROBLEM 42.

If M is the $n \times n$ matrix

$$M = \begin{bmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & x \end{bmatrix}$$

then prove that $\det M = [x + (n - 1)a](x - a)^{n-1}$.

PROBLEM 43.

Suppose that T is a linear operator on a finite dimensional vector space V over a field \mathbf{F} . Prove that T has a cyclic vector if and only if

$$\{U \in L(V, V) : TU = UT\} = \{f(T) : f \in \mathbf{F}[x]\}.$$

PROBLEM 44.

Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be given by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_1, -2x_2 - x_3 - 4x_4, 4x_2 + x_3)$$

- (1) Compute the characteristic polynomial of T .
- (2) Compute the minimal polynomial of T .
- (3) The vector space \mathbb{R}^4 is the direct sum of two proper T -invariant subspaces. Exhibit a basis for one of these subspaces.

PROBLEM 45.

Let V , W , and Z be finite dimensional vector spaces over the field \mathbf{F} and let $f : V \rightarrow W$ and $g : W \rightarrow Z$ be linear transformations. Prove that

$$\text{nullity}(g \circ f) \leq \text{nullity}(f) + \text{nullity}(g)$$



PROBLEM 46.

Prove that

$$\det \begin{bmatrix} A & 0 & 0 \\ B & C & D \\ 0 & 0 & E \end{bmatrix} = \det A \det C \det E$$

where A , B , C , D and E are all square matrices.

PROBLEM 47.

Let A and B be $n \times n$ matrices with entries on the field \mathbf{F} such that $A^{n-1} \neq 0$, $B^{n-1} \neq 0$, and $A^n = B^n = 0$. Prove that A and B are similar, or show, with a counterexample, that A and B are not necessarily similar.

PROBLEM 48.

Let A and B be $n \times n$ matrices with entries from \mathbb{R} . Suppose that A and B are similar over \mathbb{C} . Prove that they are similar over \mathbb{R} .

PROBLEM 49.

Let A be an $n \times n$ with entries from the field \mathbf{F} . Suppose that $A^2 = A$. Prove that the rank of A is equal to the trace of A .

PROBLEM 50.

TRUE OR FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample.) Let W be a vector space over a field \mathbf{F} and let $\theta : V \rightarrow V'$ be a fixed surjective transformation. If $g : W \rightarrow V'$ is a linear transformation then there is linear transformation $h : W \rightarrow V$ such that $\theta \circ h = g$.



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PROBLEM 51.

Let V be a finite dimensional vector space and $A \in L(V, V)$.

- (1) Prove that there exists an integer k such that $\ker A^k = \ker A^{k+1} = \dots$
- (2) Prove that there exists an integer k such that $V = \ker A^k \oplus \text{image } A^k$.

PROBLEM 52.

Let V be the vector space of $n \times n$ matrices over a field \mathbf{F} , and let $T : V \rightarrow V^*$ be defined by $T(A)(B) = \text{tr}(A^t B)$. Prove that T is an isomorphism.

PROBLEM 53.

Let A be an $n \times n$ matrix and $A^k = 0$ for some k . Show that $\det(A + I) = 1$.

PROBLEM 54.

Let V be a finite dimensional vector space over a field F , and T a linear operator on V . Suppose the minimal and characteristic polynomials of T are the same power of an irreducible polynomial $p(x)$. Show that no non-trivial T -invariant subspace of V has a T -invariant complement.

PROBLEM 55.

Let V be the vector space of all $n \times n$ matrices over a field \mathbf{F} , and let B be a fixed $n \times n$ matrix that is not of the form cI . Define a linear operator T on V by $T(A) = AB - BA$. Exhibit a non-zero element in the kernel of the transpose of T .

PROBLEM 56.

Let V be a finite dimensional vector space over a field \mathbf{F} and suppose that S and T are triangulable operators on V . If $ST = TS$ prove that S and T have an eigenvector in common.



PROBLEM 57.

Let A be an $n \times n$ matrix over \mathbb{C} . If $\text{trace } A^i = 0$ for all $i > 0$, prove that A is nilpotent.

PROBLEM 58.

Let V be a finite dimensional vector space over a field \mathbf{F} , and let T be a linear operator on V so that $\text{rank}(T) = \text{rank}(T^2)$. Prove that V is the direct sum of the range of T and the null space of T .

PROBLEM 59.

Let V be the vector space of all $n \times n$ matrices over a field \mathbf{F} , and suppose that A is in V . Define $T : V \rightarrow V$ by $T(AB) = AB$. Prove that A and B have the same characteristic values.

PROBLEM 60.

Let A and B be $n \times n$ over the complex numbers.

- (1) Show that AB and BA have the same characteristic values.
- (2) Are AB and BA similar matrices?

PROBLEM 61.

Let V be a finite dimensional vector space over a field of characteristic 0, and T be a linear operator on V so that $\text{tr}(T^k) = 0$ for all $k \geq 1$, where $\text{tr}(\cdot)$ denotes the trace function. Prove that T is a nilpotent linear map.

PROBLEM 62.

Let $A = [a_{ij}]$ be an $n \times n$ matrix over the field of complex numbers such that

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \text{for } i = 1, \dots, n.$$



Then show that $\det A \neq 0$. (\det denotes the determinant.)

PROBLEM 63.

Let A be an $n \times n$ matrix, and let $\text{adj}(A)$ denote the adjoint of A . Prove the rank of $\text{adj}(A)$ is either 0, 1, or n .

PROBLEM 64.

Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ -3 & -3 & -5 \end{bmatrix}$$

- (1) Determine the rational canonical form of A .
- (2) Determine the Jordan canonical form of A .

PROBLEM 65.

If

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then prove that there does not exist a matrix with $N^2 = A$.

PROBLEM 66.

Let A be a real $n \times n$ matrix which is symmetric, i.e. $A^t = A$. Prove that A is diagonalizable.

PROBLEM 67.

Give an example of two nilpotent matrices A and B such that

- (1) A is not similar to B ,
- (2) A and B have the same characteristic polynomial,



- (3) A and B have the same minimal polynomial, and
 (4) A and B have the same rank.

PROBLEM 68.

Let A be an $n \times n$ matrix over a field \mathbf{F} . Show that \mathbf{F}^n is the direct sum of the null space and the range of A if and only if A and A^2 have the same rank.

PROBLEM 69.

Let A and B be $n \times n$ matrices over a field \mathbf{F} .

- (1) Show AB and BA have the same eigenvalues.
 (2) Is AB similar to BA ? (Justify your answer).

PROBLEM 70.

Given an exact sequence of finite-dimensional vector spaces

$$0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} 0$$

that is the range of T_i is equal to the null space of T_{i+1} , for all i . What is the value of $\sum_{i=1}^n (-1)^i \dim(V_i)$? (Justify your answer).

PROBLEM 71.

Let \mathbf{F} be a field with q elements and V be a n -dimensional vector space over \mathbf{F} .

- (1) Find the number of elements in V .
 (2) Find the number of bases in V .
 (3) Find the number of invertible linear operators on V .



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PROBLEM 72.

Let A and B be $n \times n$ matrices over a field \mathbf{F} . Suppose that A and B have the same trace and the same minimal polynomial of degree $n - 1$. Is A similar to B ? (Justify your answer.)

PROBLEM 73.

Let $A = [a_{ij}]$ be an $n \times n$ matrix with $a_{ij} = 1$ for all i and j . Find its characteristic and minimal polynomial.

PROBLEM 74.

Give an example of a matrix with real entries whose characteristic polynomial is $x^5 - x^4 + x^2 - 3x + 1$.

PROBLEM 75.

TRUE or FALSE. (If true prove it. If false give a counterexample.) Let A and B be $n \times n$ matrices with minimal polynomial x^4 . If $\text{rank } A = \text{rank } B$, and $\text{rank } A^2 = \text{rank } B^2$, then A and B are similar.

PROBLEM 76.

Suppose that T is a linear operator on a finite-dimensional vector space V over a field \mathbf{F} . Prove that the characteristic polynomial of T is equal to the minimal polynomial of T if and only if

$$\{U \in L(V, V) : TU = UT\} = \{f(T) : f \in \mathbf{F}[x]\}.$$

PROBLEM 77.

- (1) Prove that if A and B are 3×3 matrices over a field \mathbf{F} , a necessary and sufficient condition that A and B be similar over \mathbf{F} is that they have the same characteristic and the same minimal polynomial.
- (2) Give an example to show this is not true for 4×4 matrices.



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PROBLEM 78.

Let V be the vector space of $n \times n$ matrices over a field. Assume that f is a linear functional on V so that $f(AB) = f(BA)$ for all $A, B \in V$, and $f(I) = n$. Prove that f is the trace functional.

PROBLEM 79.

Suppose that N is a 4×4 nilpotent matrix over \mathbf{F} with minimal polynomial x^2 . What are the possible rational canonical forms for n ?

PROBLEM 80.

Let A and B be $n \times n$ matrices over a field \mathbf{F} . Prove that AB and BA have the same characteristic polynomial.

PROBLEM 81.

Suppose that \mathbf{V} is an n -dimensional vector space over \mathbf{F} , and T is a linear operator on \mathbf{V} which has n distinct characteristic values. Prove that if S is a linear operator on \mathbf{V} that commutes with T , then S is a polynomial in T .

PROBLEM 82.

Let A and B be $n \times n$ matrices over a field \mathbf{F} . Show that AB and BA have the same characteristic values in \mathbf{F} .

PROBLEM 83.

Let P and Q be real $n \times n$ matrices so that $P + Q = I$ and $\text{rank}(P) + \text{rank}(Q) = n$. Prove that P and Q are projections. (HINT: Show that if $Px = Qy$ for some vectors x and y , then $Px = Qy = 0$.)



PROBLEM 84.

Suppose that A is an $n \times n$ real, invertible matrix. Show that A^{-1} can be expressed as a polynomial in A with real coefficients and with degree at most $n - 1$.

PROBLEM 85.

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Determine the rational canonical form and the Jordan canonical form of A .

PROBLEM 86.

- (1) Give an example of two 4×4 nilpotent matrices which have the same minimal polynomial but are not similar.
- (2) Explain why 4 is the smallest value that can be chosen for the example in part (a), i.e. if $n \leq 3$, any two nilpotent matrices with the same minimal polynomial are similar.

PROBLEM 87.

This is a very basic and important fact. Let V be a finite dimensional vector space and f and g two linear functionals on V . If $\ker f = \ker g$ show g is a scalar multiple of f .

PROBLEM 88.

This problem makes explicit some facts that are used several times in solving some of the problems above.



- (1) Prove that if V is a finite dimensional vector space over the field \mathbf{F} and $T \in L(V, V)$ and V is cyclic for T that any $S \in L(V, V)$ that commutes with T is a polynomial in T . That is $ST = TS$ implies that $S = p(T)$ for some $p(x) \in \mathbf{F}[x]$. HINT: Let $\dim V = n$. Then because V is cyclic for T there is a vector $v_0 \in V$ so that $v_0, Tv_0, \dots, T^{n-1}v_0$ is a basis for V . Thus there are scalars a_0, a_1, \dots, a_{n-1} so that $Sv_0 = a_0v_0 + a_1Tv_0 + a_2T^2v_0 + \dots + a_{n-1}T^{n-1}v_0$. Then letting $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}$ we have $Sv_0 = p(T)v_0$. Now use that S commutes with T (and thus also $p(T)$) to show that $ST^i v_0 = p(T)T^i v_0$ for $i = 0, 1, \dots, n-1$. Thus the two linear maps S and $p(T)$ agree on a basis and hence are equal.
- (2) If the minimal polynomial $f(x)$ of T has $\deg f(x) = \dim V$ then V is cyclic for T . HINT: I don't know any particularly easy way to do this. The basic idea is to factor $f(x) = p_1(x)^{k_1} \dots p_l(x)^{k_l}$ into powers of primes and consider the corresponding primary decomposition $V = \ker(p_1(T)^{k_1}) \oplus \dots \oplus \ker(p_l(T)^{k_l})$ and show that if $\deg f(x) = \dim V$ then each of the primary factors $\ker(p_i(T)^{k_i})$ is cyclic (this in turn uses that each of the $\ker(p_i(T)^{k_i})$ is a sum of cyclic subspaces). Now let v_i be a cyclic for T in $\ker(p_i(T)^{k_i})$ for $i = 1, \dots, l$. Then show the vector $v_0 = v_1 + v_2 + \dots + v_l$ is cyclic for T .



Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct elements of the field \mathbf{F} . Then the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

is invertible. HINT: If A is singular then it has rank less than n and thus there is a nontrivial linear relation between the rows of A . This would in turn imply that there is a nonzero polynomial $p(x)$ of degree $\leq n - 1$ that had the n scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ as roots. But this is impossible.

PROBLEM 90.

This is another set of facts that anyone who has had a graduate linear algebra class should know. Let D be a diagonal matrix that has all its diagonal elements distinct. That is

$$D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) := \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \text{where } \lambda_i \neq \lambda_j \text{ for } i \neq j.$$

Then show

- (1) The only matrices that commute with D are diagonal matrices.
- (2) If A is any other diagonal matrix then A is a polynomial in D . That is there is a polynomial $p(x)$ so that $A = p(D)$.
- (3) If A is any matrix that commutes with D then A is a polynomial in D .



- (4) There is a cyclic vector for D . HINT: Let e_1, \dots, e_n be the standard coordinate vectors. Then as D is diagonal $De_i = \lambda_i e_i$. Let $v = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$. Then show that v is a cyclic vector for D if and only if $a_i \neq 0$ for all i (One way to do this is use the last problem). In particular $v = e_1 + e_2 + \dots + e_n$ is a cyclic vector for T .

PROBLEM 91.

This is another standard problem. Let V be a finite dimensional vector space over a field \mathbf{F} and let $T \in L(V, V)$. Let λ be an eigenvalue of T and let $V_\lambda := \{v \in V : Tv = \lambda v\}$ be the corresponding eigenspace.

- (1) Let $S \in L(V, V)$ commute with T . Then show that V_λ is invariant under S . (That is show $v \in V_\lambda$ implies $Sv \in V_\lambda$.)
- (2) Show that if A and B are $n \times n$ matrices over the complex numbers that commute, then they have a common eigenvector. HINT: As A is a complex matrix it has at least one eigenvalue λ . Let V_λ be the corresponding eigenspace. Then by what we have just done V_λ is invariant under B . But then the restriction of B to V_λ has an eigenvector in V_λ .
- (3) This is a different way of looking at Problem 90(4) above. Assume V has an basis of eigenvectors e_1, e_2, \dots, e_n of eigenvectors of T , that is $Te_i = \lambda_i e_i$. Also assume the eigenvalues are distinct: $\lambda_i \neq \lambda_j$ for $i \neq j$. Then show if S commutes with T then for some scalars c_i there holds $Se_i = c_i e_i$, and thus S is also diagonal in the basis e_1, \dots, e_n . HINT: Let $V_{\lambda_i} := \{v : Te_i = \lambda_i v\}$. Then by the assumptions V_{λ_i} is one dimensional with basis e_i . Part (1) of this problem then implies that V_{λ_i} is invariant under S . As V_{λ_i} is one dimensional this in turn implies e_i is an eigenvector of S .