

A modular inherently nonfinitely based lattice

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This paper is dedicated to Walter Taylor.

In [5] we gave a construction of inherently nonfinitely based lattices which produced a wide variety of examples. But none of these examples was modular and we asked in Problem 1 for a modular example. Here we shall show that \mathbf{L}_∞ of Figure 1 is such an example.

Theorem 1. \mathbf{L}_∞ is an inherently nonfinitely based modular lattice.

Proof. As observed in McNulty [7], a locally finite variety \mathcal{V} of finite type is inherently nonfinitely based if and only if for infinitely many natural numbers N , there is a non-locally-finite algebra each of whose N -generated subalgebras belongs to \mathcal{V} . We prove the theorem by establishing these facts. We assume the reader is familiar with the basic facts of modular lattices; see [1], [2], [6].

Let \mathbf{B} (for bottom) be the sublattice of \mathbf{L}_∞ consisting of all elements of finite height and let \mathbf{T} consist of all elements of finite depth. Of course \mathbf{L}_∞ is the ordinal (or linear) sum $\mathbf{B} + \mathbf{T}$ of these sublattices.

Lemma 2. *The variety $\mathbf{V}(\mathbf{L}_\infty)$ generated by \mathbf{L}_∞ is locally finite.*

Proof. To prove this we need to show that for every finite n there is a bound on the size of the n -generated subalgebras of \mathbf{L}_∞ . We do this by induction on n . Suppose that x_1, \dots, x_n are elements of \mathbf{L}_∞ and let \mathbf{S} be the sublattice they generate. We may assume that all of these elements either lie in \mathbf{B} or they all lie in \mathbf{T} since otherwise \mathbf{S} is the ordinal sum of two sublattices with fewer generators. By duality we may assume they all lie in \mathbf{B} . Thus each x_k has a rank (or height) r_k .

Observe that if a and b are elements of \mathbf{B} with ranks r_a and r_b and $r_b - r_a \geq 4$ then the meet of all elements with rank at least r_b is greater than or equal to the join of all elements with rank at most r_a . So if we let r_k be the rank of x_k and (re)order the x_k 's so that $r_1 \leq r_2 \leq \dots \leq r_n$ then we may assume $r_{k+1} - r_k \leq 3$ since otherwise \mathbf{S} is a ordinal sum of sublattices with fewer generators. Thus \mathbf{S} lies

Presented by R. McKenzie.

Received August 17, 2004; accepted in final form December 20, 2005.

2000 *Mathematics Subject Classification*: 06B25, 06B05.

Key words and phrases: finitely presented lattice, covering relation, canonical form, partial lattice.

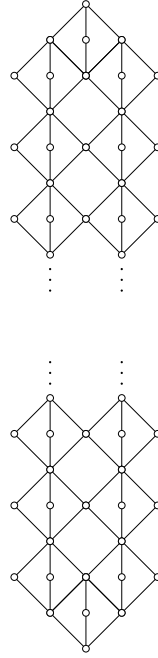


FIGURE 1. \mathbf{L}_∞ , an inherently nonfinitely based modular lattice

in an interval of \mathbf{L}_∞ which has length at most $3(n+2)$. All intervals of \mathbf{L}_∞ of fixed length have a bound on their size. Thus \mathbf{S} is of bounded size. \square

Let \mathbf{M}_3 be the five element modular, nondistributive lattice and let \mathbf{Z} be the integers as a chain. Let $\mathbf{M}_3[\mathbf{Z}]$ be the lattice of all order-preserving functions from \mathbf{Z} to \mathbf{M}_3 . If $x \in \mathbf{M}_3$ let \bar{x} denote the constant map. The following facts are easily established:

- (1) $\mathbf{M}_3[\mathbf{Z}]$ is a subdirect power of \mathbf{M}_3 and so is in the variety generated by \mathbf{M}_3 .
- (2) $x \mapsto \bar{x}$ embeds \mathbf{M}_3 in $\mathbf{M}_3[\mathbf{Z}]$.
- (3) If $v \prec u$ in \mathbf{M}_3 , then the interval $[\bar{v}, \bar{u}]$ is $\mathbf{1} + \mathbf{Z} + \mathbf{1}$ (the ordinal sum).
- (4) If a is an atom of \mathbf{M}_3 and 0 is its least element, then $\mathbf{M}_3[\mathbf{Z}]$ is generated by the constant maps and the interval $[\bar{0}, \bar{a}]$.
- (5) The (left) shift operator $\sigma [(\sigma f)(i) = f(i+1)]$ is an automorphism of $\mathbf{M}_3[\mathbf{Z}]$.

We wish do something similar with \mathbf{L}_∞ and other modular lattices. So let \mathbf{L} be a modular lattice. We start by forming \mathbf{M} , the lattice of all order-preserving maps from \mathbf{Z} into \mathbf{L} . This lattice is bigger than we want; we would like that intervals of \mathbf{L} which are chains remain chains in \mathbf{M} under the natural (diagonal) embedding

(denoted $x \mapsto \bar{x}$) of \mathbf{L} into \mathbf{M} . To do this we take the sublattice of \mathbf{M} whose universe is

$$\{x \in \mathbf{M} : \bar{v} \leq x \leq \bar{u}, \text{ for some } u, v \in \mathbf{L} \text{ with } [v, u] \text{ complemented}\}$$

The following lemma proves that this is (the universe of) a sublattice of \mathbf{M} . We denote this lattice by $\mathbf{L}[\mathbf{Z}]$.

Lemma 3. *If $[a_0, a_1]$ and $[b_0, b_1]$ are complemented intervals in a modular lattice \mathbf{L} , then $[a_0 \vee b_0, a_1 \vee b_1]$ is also complemented.*

Proof. Let $c_0 = a_0 \vee b_0$ and $c_1 = a_1 \vee b_1$. Let $x \in [c_0, c_1]$. Let $d = a_1 \vee b_0$ and $e = a_0 \vee b_1$. Since $[c_0, d]$ is a transpose of a subinterval of $[a_0, a_1]$, it is complemented, and similarly $[c_0, e]$ is complemented. Let y be a complement of $x \wedge d$ in $[c_0, d]$ and let z be a complement of $(x \vee d) \wedge e$ in $[c_0, e]$. Then $y \vee z$ is a complement of x in $[c_0, c_1]$. Indeed

$$\begin{aligned} x \wedge (y \vee z) &= x \wedge (x \vee d) \wedge (y \vee z) = x \wedge [y \vee ((x \vee d) \wedge z)] \\ &= x \wedge [y \vee ((x \vee d) \wedge e \wedge z)] = x \wedge [y \vee c_0] \\ &= x \wedge y = x \wedge d \wedge y = c_0 \\ x \vee y \vee z &= x \vee (x \wedge d) \vee y \vee z \\ &= x \vee d \vee z = x \vee d \vee e = c_1 \end{aligned}$$

□

Now we turn to constructing non-locally-finite lattices whose N -generated sublattices lie in $\mathbf{V}(\mathbf{L}_\infty)$. For n an even integer at least 4, \mathbf{B} has 5 elements of height n , only one of which is join reducible (the middle one). We let \mathbf{K}_n denote the principal ideal generated by this join reducible element. \mathbf{K}_n has two coatoms, and we let \mathbf{L}_n be the lattice obtained from \mathbf{K}_n by adding another coatom which is above the meet of these two coatoms thus forming an \mathbf{M}_3 . \mathbf{L}_{10} is diagrammed in Figure 2.

We shall modify $\mathbf{L}_n[\mathbf{Z}]$ into a lattice $\mathbf{L}_n[\mathbf{Z}]^*$ using the Hall–Dilworth gluing construction, which we now review.

If a lattice \mathbf{L} has a filter which is isomorphic to the ideal of another lattice \mathbf{K} , then we can identify each element of the filter with corresponding element of the ideal and order the elements of $L \cup K$ (with these identifications) by the transitive closure of the orders on \mathbf{L} and \mathbf{K} . The result is a lattice \mathbf{M} , and if both \mathbf{L} and \mathbf{K} are modular, then \mathbf{M} is also. This is the famous Hall–Dilworth gluing construction. Now, \mathbf{L} is an ideal of \mathbf{M} , \mathbf{K} is a filter, and $L \cup K = M$. Conversely, if a lattice \mathbf{M} is the set union of an ideal \mathbf{L} and a filter \mathbf{K} and $L \cap K \neq \emptyset$, then \mathbf{M} is the Hall–Dilworth gluing of \mathbf{L} and \mathbf{K} over their intersection using the identity map.

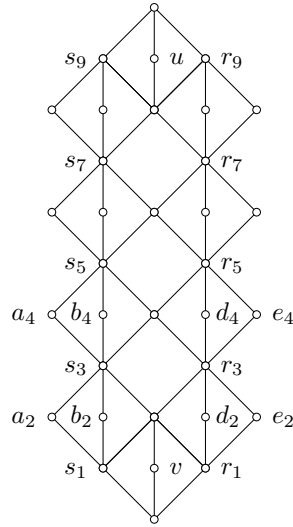


FIGURE 2. \mathbf{L}_{10}

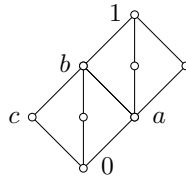


FIGURE 3. \mathbf{M}_{33}

If both \mathbf{L} and \mathbf{K} are copies of \mathbf{M}_3 and a is an atom of \mathbf{L} and b is an atom of \mathbf{K} , then we can apply this construction using the filter of \mathbf{L} generated by a and the ideal of \mathbf{K} generated by b to obtain the lattice \mathbf{M}_{33} of Figure 3.

Similarly $\mathbf{M}_{33}[\mathbf{Z}]$ is the Hall–Dilworth gluing of two copies of $\mathbf{M}_3[\mathbf{Z}]$ over the interval from \bar{a} to \bar{b} . To prove this using the remarks above one only needs to verify that every element of $\mathbf{M}_{33}[\mathbf{Z}]$ is either below \bar{b} or above \bar{a} , because the complemented intervals of \mathbf{M}_{33} are either below b or above a .

A typical element of $[\bar{a}, \bar{b}]$ is c_i , where c_i is the function $c_i(k) = a$ if $k \leq i$ and b otherwise. Instead of using the identity map when gluing the two $\mathbf{M}_3[\mathbf{Z}]$'s, we could use the shift map: $c_i \mapsto c_{i-1}$, obtaining a lattice $\mathbf{M}_{33}[\mathbf{Z}]^*$. It is easy to check that $\mathbf{M}_{33}[\mathbf{Z}]^* \cong \mathbf{M}_{33}[\mathbf{Z}]$.

In $\mathbf{L}_n - \{u, v\}$ there are two elements of each odd dimension. We can arrange these elements into two chains $s_1 < s_3 < \dots < s_{n-1}$ and $r_1 < r_3 < \dots < r_{n-1}$; see Figure 2.

We have $s_1 < r_3$, and one can verify that every element in $\mathbf{L}_n[\mathbf{Z}]$ is either in the principal ideal \mathbf{I} generated by \bar{r}_3 or in the principal filter \mathbf{F} generated by \bar{s}_1 . Thus $\mathbf{L}_n[\mathbf{Z}]$ is the Hall–Dilworth gluing of \mathbf{I} and \mathbf{F} . \mathbf{I} is isomorphic to $\mathbf{M}_{33}[\mathbf{Z}]$. Let $\mathbf{L}_n[\mathbf{Z}]^*$ be the result of gluing $\mathbf{M}_{33}[\mathbf{Z}]^*$ to \mathbf{F} using the identity map on the interval $[\bar{s}_1, \bar{r}_3]$.

Unlike the situation with $\mathbf{M}_{33}[\mathbf{Z}]^*$, $\mathbf{L}_n[\mathbf{Z}]^*$ is not isomorphic to $\mathbf{L}_n[\mathbf{Z}]$. In fact \mathbf{L}_n has the sequence of transpositions

$$\begin{array}{cccccccc}
 [0, v] & \nearrow & [a_2, s_3] & \searrow & [s_1, b_2] & \nearrow & [e_4, r_5] & \searrow & [r_3, d_4] \\
 & & \dots & \nearrow & [u, 1] & \searrow & \dots & \searrow & [r_1, d_2] & \nearrow & [e_2, r_3] & \searrow & [0, v]
 \end{array}$$

where $a_i, b_i \in [s_{i-1}, s_{i+1}]$ and $d_i, e_i \in [r_{i-1}, r_{i+1}]$ are the four irreducible elements of rank i with i even.

Of course this defines an automorphism on $[\bar{0}, \bar{v}]$ in $\mathbf{L}_n[\mathbf{Z}]^*$ but since the sequence of transpositions goes through the shifted interval, this automorphism is the shift, sending each element x with $\bar{0} < x < \bar{v}$ to its lower cover. Thus the sublattice of $\mathbf{L}_n[\mathbf{Z}]^*$ generated by \mathbf{L}_n and any element $\bar{0} < x < \bar{v}$ is infinite (in fact it is all of $\mathbf{L}_n[\mathbf{Z}]^*$, but we do not use this fact).

In order to complete the proof of Theorem 1 it suffices to show that for each N we can choose n large enough so that the N -generated sublattices of $\mathbf{L}_n[\mathbf{Z}]^*$ lie in $\mathbf{V}(\mathbf{L}_\infty) = \mathbf{V}(\mathbf{L}_\infty[\mathbf{Z}])$. As usual, we view \mathbf{L}_n as embedded in $\mathbf{L}_n[\mathbf{Z}]^*$ by the diagonal map. Every element of $x \in \mathbf{L}_n[\mathbf{Z}]^*$ lies in a uniquely determined complemented interval $[z_x, t_x]$ of \mathbf{L}_n of minimal dimension, where z_x is just the join of all elements of \mathbf{L}_n below x , and t_x is defined dually. Of course, the dimension (in \mathbf{L}_n) of $[z_x, t_x]$ is at most 2. Thus we may assign to each element x of $\mathbf{L}_n[\mathbf{Z}]^*$ a lower and upper rank (the ranks in \mathbf{L}_n of z_x and t_x), and these differ by at most 2. So if we let \mathbf{S} be an N -generated sublattice of $\mathbf{L}_n[\mathbf{Z}]^*$, then an argument similar to the proof of Lemma 2 shows that, if \mathbf{S} is linearly indecomposable, it lies in an interval $[\bar{a}, \bar{b}]$ of $\mathbf{L}_n[\mathbf{Z}]^*$ where the dimension of $[a, b]$ in \mathbf{L}_n is at most $5(N + 2)$. It follows that, for any N -generated sublattice \mathbf{S} , if $n > 5(N + 2) + 4$ then either \mathbf{S} lies in the filter $[\bar{r}_1 \vee \bar{s}_1, \bar{1}]$ or the ideal $[\bar{0}, \bar{r}_{n-1} \wedge \bar{s}_{n-1}]$ or is the linear sum of two such lattices. So it suffices to show that this filter and ideal are isomorphic to sublattices of $\mathbf{L}_\infty[\mathbf{Z}]$. For the filter this is straightforward, since the filter does not contain the shifted interval.

Let \mathbf{I} denote the ideal. To see that it is also embeddable into $\mathbf{L}_\infty[\mathbf{Z}]$ let \mathbf{P}_n be the filter $[s_1, 1]$ of \mathbf{K}_n . (Recall \mathbf{K}_n is \mathbf{L}_n with u removed.)

Now \mathbf{I} is isomorphic to $\mathbf{K}_{n-2}[\mathbf{Z}]^*$ and $\mathbf{K}_{n-2}[\mathbf{Z}]$ is naturally embeddable into $\mathbf{B}[\mathbf{Z}]$ (recall \mathbf{B} is bottom half of \mathbf{L}_∞) and so into $\mathbf{L}_\infty[\mathbf{Z}]$. Thus it suffices to show that

$\mathbf{K}_n[\mathbf{Z}] \cong \mathbf{K}_n[\mathbf{Z}]^*$. In \mathbf{P}_n every prime interval is projective with either $[r_1 \vee s_1, s_3]$ or $[r_1 \vee s_1, r_3]$, but not both. So \mathbf{P}_n is a subdirect product of two lattices, say \mathbf{Q} and \mathbf{R} , and $\mathbf{P}_n[\mathbf{Z}]$ is a subdirect product of $\mathbf{Q}[\mathbf{Z}]$ and $\mathbf{R}[\mathbf{Z}]$. So $\mathbf{P}_n[\mathbf{Z}]$ has an automorphism τ which is the shift operator on one of these factors and the identity on the other. To make τ explicit, let $x \in \mathbf{P}_n[\mathbf{Z}]$ and let z_x and t_x be the elements of \mathbf{P}_n defined above. If $z_x = t_x$ then $x \in \mathbf{P}_n$ and $\tau(x) = x$. Suppose the dimension of $[z_x, t_x]$ is 1. If $[z_x, t_x]$ projects to $[r_1 \vee s_1, r_3]$ then we apply the shift operator on $[\bar{z}_x, \bar{t}_x]$ to x ; that is, $\tau(x)$ is the unique upper cover of x in $[\bar{z}_x, \bar{t}_x]$. If $[z_x, t_x]$ projects to $[r_1 \vee s_1, s_3]$ then $\tau(x) = x$. If the dimension of $[z_x, t_x]$ is 2, then $[z_x, t_x]$ is isomorphic to either \mathbf{M}_3 or $\mathbf{2} \times \mathbf{2}$. In the former case, if the prime quotients of $[z_x, t_x] \cong \mathbf{M}_3$ project to $[r_1 \vee s_1, r_3]$, then we apply the shift operator to x , otherwise x is fixed. Finally, if $[z_x, t_x] \cong \mathbf{2} \times \mathbf{2}$ and r and s are the atoms of $[z_x, t_x]$, then $x = (x \wedge \bar{r}) \vee (x \wedge \bar{s})$ and $\tau(x) = \tau(x \wedge \bar{r}) \vee \tau(x \wedge \bar{s})$.

Now we define $\rho: \mathbf{K}_n[\mathbf{Z}] \rightarrow \mathbf{K}_n[\mathbf{Z}]^*$. If $x \geq \bar{s}_1$ we let $\rho(x) = \tau(x)$. If $x \not\geq \bar{s}_1$ then $x \leq \bar{r}_3$. If such an x is above \bar{r}_1 , then ρ applies the shift operator to x ; otherwise $\rho(x) = x$. Since $[\bar{0}, \bar{r}_3]$ in $\mathbf{K}_n[\mathbf{Z}]^*$ is obtained from gluing two copies of \mathbf{M}_3 by applying the shift operator to the top copy, ρ restricted to $[\bar{0}, \bar{r}_3]$ is an isomorphism. Finally, one checks that ρ is one-to-one and onto and that it preserves order and so is an isomorphism.

This completes the proof of Theorem 1. \square

In [5] we gave an example of a locally finite variety which is not finitely based but fails to be inherently nonfinitely based. In order to produce modular variety with these properties we use some results of [4]. There it is shown that the variety \mathcal{M}_4^∞ generated by all modular lattices of width four is finitely based and locally finite. It is also shown that \mathcal{M}_4^∞ has uncountably many subvarieties. Of course not all of these are finitely based, but being subvarieties of \mathcal{M}_4^∞ , none of them can be inherently nonfinitely based.

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