The equational complexity of Lyndon's algebra

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In memory of my friend Buddy

Abstract. The equational complexity of Lyndon's nonfinitely based 7-element algebra lies between $\frac{1}{2}n - \frac{3}{2}$ and $4n - 4$. This result is based on a new algebraic proof that Lyndon's algebra is not finitely based. We also show that the variety generated by Lyndon's algebra contains subdirectly irreducible algebras of all cardinalities except $0, 1$, and 4.

1. Introduction

In 1954 Roger Lyndon [10] published the earliest example of a finite algebra that is not finitely based. Zoltán Székely noticed that Lyndon's algebra arises from a finite automaton and can be conveniently displayed by a diagram. Figure 1 gives the diagrams of two such algebras. The elements of the algebra **L** fall into

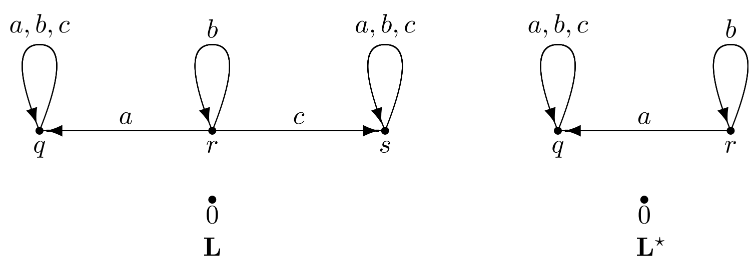

Figure 1. Two Automatic Algebras

three pairwise disjoint sets: a set $\{q, r, s\}$ of states, a set $\{a, b, c\}$ of letters, and $\{0\}$. The sole basic operation of the algebra, which is binary and written here as juxtaposition, comes from the transition function of the underlying automaton and can be read from the diagram. For example we have $q = ar$ and $r = br$. Products, like ac and rb, not associated in this way with the arrows in the diagram take the default value 0. More precisely, an groupoid **A** is **automatic** provided A is the union of three pairwise disjoint sets Σ, Q, and $\{0\}$ and the operation, written as juxtaposition, conforms to the constraint

- $ab = 0$ or
- $a \in \Sigma, b \in Q,$ and $ab \in Q,$ for all $a, b \in A$.

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The elements of Q are referred to as **states**, while the elements of Σ are referred to as **letters**.

The algebra $\mathbf{L}$ is essentially the algebra put forward in Lyndon's paper. (Actually, Lyndon's original algebra took 0—which can be defined by the term xx—as a distinguished constant and wrote the operation in the opposite order.) In 2008, Edmond W. H. Lee [9] proved that the subdirectly irreducible algebra $\mathbf{L}^\star$ generates the same variety as $\mathbf{L}$.

Lyndon's algebra has a number of interesting properties. Tamás Bajusz, George McNulty and Ágnes Szendrei [1] showed that it fails to be inherently nonfinitely based. Keith Kearnes and Ross Willard [6] observed that each automatic algebra, including $\mathbf{L}$ and $\mathbf{L}^\star$, is 2-step strongly solvable. Lee [9] proved that the variety generated by $\mathbf{L}$ is of finite height and each of its proper subvarieties is a Cross variety (and hence finitely based).

The main point of the present paper is to offer tight estimates on the equational complexity of Lyndon's algebra.

The **equational complexity** of a variety $\mathcal{V}$ of finite signature is the function $\beta_\mathcal{V}$ from the positive integers into the natural numbers so that $\beta_\mathcal{V}(n)$ is the smallest natural number ℓ such that any algebra $\mathbf{B}$ of cardinality less than n belongs to $\mathcal{V}$ if and only if each equation of length less than ℓ that is true in $\mathcal{V}$ is also true in $\mathbf{B}$. Thus the equational complexity function measures how much of the equational theory of $\mathcal{V}$ must be tested to see if $\mathbf{B}$ belongs to $\mathcal{V}$. If $\mathcal{V}$ is the variety generated by an algebra $\mathbf{A}$, we use $\beta_\mathbf{A}$ in place of $\beta_\mathcal{V}$. The equational complexity function was first introduced and investigated in Székely's dissertation [14]. Further developments were made by Gábor Kun and Vera Vertési [8], by Marcin Kozik [7], and by McNulty, Willard, and Székely [11].

Robert Cacioppo [4] observed, in essence, that the equational complexity of any finite algebra that is not finitely based but that fails to be inherently nonfinitely based (as holds for Lyndon's algebra) must eventually dominate every constant function. McNulty, Willard and Székely [11] showed that $\beta_\mathbf{L}(n)$ is dominated by $12n-20$ and left as an open problem whether $\beta_\mathbf{L}$ eventually dominates some strictly increasing linear function. That problem is solved here. We show that

$$\frac{1}{2}n - \frac{3}{2} \leq \beta_\mathbf{L}(n) \leq 4n - 4$$

for all $n \geq 5$

This paper also provides another proof that $\mathbf{L}$ is not finitely based. Lyndon's proof, which occupies less than two pages, is syntactical in nature. The proof offered here has an algebraic character. We also prove that the variety generated by $\mathbf{L}$ contains subdirectly irreducible algebras of all cardinalities (except $0, 1$, and 4). In 1976, Park [12] proved that there are arbitrarily large finite, as well as arbitrarily large infinite, subdirectly irreducible algebras in this variety. Finally, we find that Lyndon's algebra provides an example that a key hypothesis of the Shift Automorphism Theorem [2] cannot be eliminated.

2. Another proof that Lyndon's algebra is not finitely based

Let n be a natural number and $\mathcal{V}$ be a variety. We use $\mathcal{V}^{(n)}$ to denote the variety based on all the equations true in $\mathcal{V}$ such that all variables occurring in these equations are drawn from the set $\{x_0, \dots, x_{n-1}\}$. Observe that an algebra $\mathbf{B} \in \mathcal{V}^{(n)}$ if and only if every subalgebra of $\mathbf{B}$ with n or fewer generators belongs to $\mathcal{V}$. Birkhoff [3] observed that if $\mathcal{V}$ is a locally finite variety of finite signature, then $\mathcal{V}^{(n)}$ is finitely based, for every natural number n. It follows that a locally finite variety of finite signature fails to be finitely based if and only if $\mathcal{V}^{(n)} \neq \mathcal{V}$, for all natural numbers n.

Lyndon's Nonfinite Basis Theorem. *The seven element automatic algebra* $\mathbf{L}$ *is not finitely based.*

Proof. Let $\mathcal{V}$ be the variety generated by $\mathbf{L}$. To establish this theorem we will construct an algebra $\mathbf{C}_n \in \mathcal{V}^{(n)}$ so that $\mathbf{C}_n \notin \mathcal{V}$.

We render the elements of $(\mathbf{L}^\star)^n$, which are n-tuples, as words of length n. That is we use $a\ a\ b\ r\ c\ 0$ in place of $(a, a, b, r, c, 0)$. Let $\mathbf{A}_n$ be the subalgebra of $(\mathbf{L}^\star)^n$ generated by the following $n+1$ words:

$$\begin{array}{lcccccccc}
\beta_n & := & r & r & \dots & r & r & r \\
\alpha_{n-1} & := & b & b & \dots & b & b & a \\
\alpha_{n-2} & := & b & b & \dots & b & a & c \\
\alpha_{n-2} & := & b & b & \dots & a & c & c \\
 & \vdots & & & & & & \\
\alpha_2 & := & b & b & \dots & c & c & c \\
\alpha_1 & := & b & a & \dots & c & c & c \\
\alpha_0 & := & a & c & \dots & c & c & c
\end{array}$$

For $i < n$ put $\beta_i := \alpha_i \beta_{i+1}$. Hence

$$\begin{array}{llcccccc}
\beta_{n-1} & := \alpha_{n-1}\beta_n = & r & r & \dots & r & r & q \\
 & \quad\vdots & & & & & & \\
\beta_0 & := \alpha_0\beta_1 = & q & q & \dots & q & q & q
\end{array}$$

We need an additional element of $(\mathbf{L}^\star)^n$, namely

$$\gamma := \quad c \quad c \quad \dots \quad c \quad c \quad c$$

We let $\mathbf{A}'_n$ be the subalgebra of $(\mathbf{L}^\star)^n$ generated by $\beta_n, \alpha_{n-1}, \dots, \alpha_0$, and γ. So $\mathbf{A}_n$ is a subalgebra of $\mathbf{A}'_n$. Now A_n has as elements $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \beta_0, \beta_1, \dots, \beta_n$ but all the other elements of A_n have at least one entry that is 0.

Let θ be the equivalence relation on A_n which lumps into a single equivalence class all the n-tuples that have at least one entry that is 0 and that isolates into singleton classes all other n-tuples. It is easy to check that θ is a congruence relation of $\mathbf{A}_n$. Let $\mathbf{B}_n = \mathbf{A}_n/\theta$. Slightly abusing notation, we take the elements of B_n to be 0 (for the single large congruence class) as well as $\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_n$ (for the singleton congruence classes). Notice that $\mathbf{B}_n$ has $2n+2$ elements and that it is also an automatic algebra. The algebra $\mathbf{B}_6$ is displayed in Figure 2.

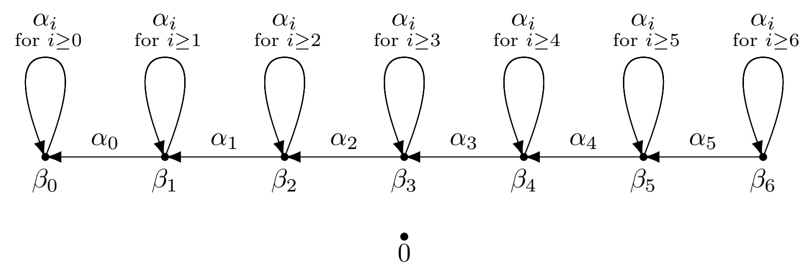

FIGURE 2. The Automatic Algebra $\mathbf{B}_6$

Most of the loops in Figure 2 have several labels; however, the loop drawn at β_6 is actually trivial in the sense that it has no label (it could be omitted, but was included for the sake of uniformity).

The algebra $\mathbf{A}'_n$ has the congruence θ' which collapses all the tuples that have at least one entry 0 into a single class and isolates the remaining elements of A'_n. Let $\mathbf{B}'_n = \mathbf{A}'_n/\theta'$. Observe that $\mathbf{B}_n$ is a subalgebra of $\mathbf{B}'_n$ and that $\mathbf{B}'_n$ has one more element than $\mathbf{B}_n$. Figure 3 displays $\mathbf{B}'_6$. The diagram of $\mathbf{B}'_6$ differs from that of $\mathbf{B}_6$

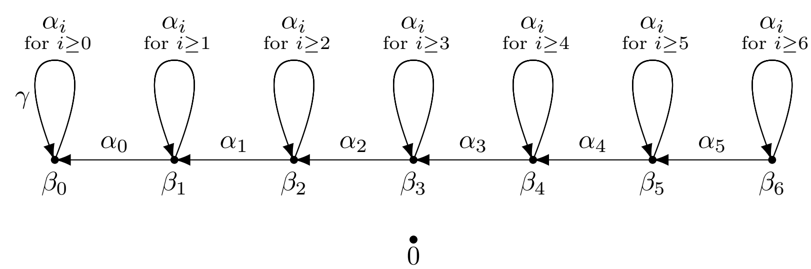

FIGURE 3. The Automatic Algebra $\mathbf{B}'_6$

only by the attachment of the label γ to the leftmost loop.

Since $\mathbf{L}^\star$ generates $\mathcal{V}$, we know that $\mathbf{B}_n, \mathbf{B}'_n \in \mathcal{V}$ for all natural numbers n.

The algebra $\mathbf{C}_n$ is like $\mathbf{B}_n$ but with one more nontrivial product: $\beta_n\beta_0 := \beta_0$. The algebra $\mathbf{C}_n$ is not an automatic algebra. We display $\mathbf{C}_6$ in Figure 4.

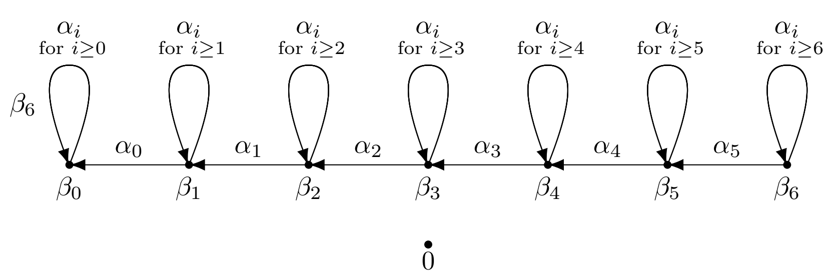

FIGURE 4. The Algebra $\mathbf{C}_6$

Claim 0. $\mathbf{C}_n \in \mathcal{V}^{(n)}$.

Proof. Consider any subalgebra $\mathbf{D}$ of $\mathbf{C}_n$ generated by any set of n or fewer elements. Now none of $\beta_n, \alpha_{n-1}, \ldots, \alpha_0$ occur as outputs of the basic operation of $\mathbf{C}_n$. So unless these elements are in the generating set of $\mathbf{D}$ they cannot be in $\mathbf{D}$. In case β_n is not in the generating set, we see that $\mathbf{D}$ is a subalgebra of $\mathbf{B}_n$. Hence $\mathbf{D} \in \mathcal{V} \subseteq \mathcal{V}^{(n)}$. In this case, $\mathbf{D} \in \mathcal{V}^{(n)}$ as desired. In case β_n is in the generating set then one of the α_i's must fail to be in the generating set and hence in $\mathbf{D}$. So suppose $\alpha_j \notin D$. Let $\mathbf{E}$ be the subalgebra of $\mathbf{C}_n$ obtained by omitting α_j. This algebra is displayed in Figure 5.

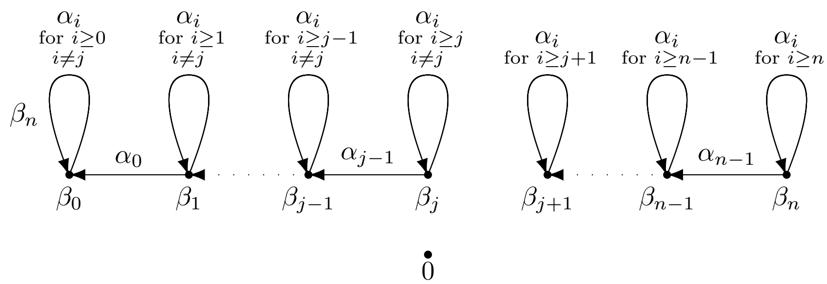

FIGURE 5. The Algebra $\mathbf{E}$

We contend that $\mathbf{E}$ is a subalgebra of the direct product of two algebras, each belonging to $\mathcal{V}$. These algebras, which depend on n, are displayed in Figure 6.

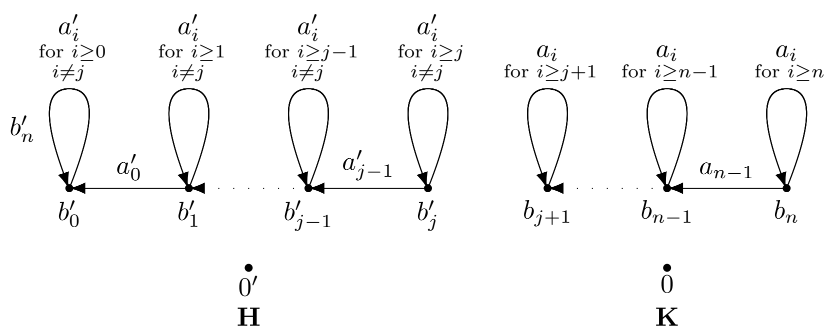

FIGURE 6. The Automatic Algebras $\mathbf{H}$ and $\mathbf{K}$

The elements of $\mathbf{K}$ are

$$a_{j+1}, \ldots, a_{n-1}, b_{j+1} \ldots, b_n \text{ and } 0,$$

while the elements of $\mathbf{H}$ are

$$a'_0, \ldots, a'_{j-1}, a'_{j+1}, \ldots, a'_{n-1}, b'_n, b'_0, \ldots, b'_j \text{ and } 0'.$$

The algebra $\mathbf{K}$ is seen to be isomorphic to a subalgebra of $\mathbf{B}_n \in \mathcal{V}$, while the algebra $\mathbf{H}$ is isomorphic to a subalgebra of $\mathbf{B}'_n \in \mathcal{V}$. So $\mathbf{H}, \mathbf{K} \in \mathcal{V}$. The map that embeds $\mathbf{E}$ into $\mathbf{H} \times \mathbf{K}$ is given below:

$$\begin{aligned}
\alpha_i &\mapsto (a'_i, 0) \text{ for } i < j \\
\alpha_i &\mapsto (a'_i, a_i) \text{ for } i > j \\
\beta_i &\mapsto (b'_i, 0) \text{ for } i \leq j \\
\beta_i &\mapsto (0, b_i) \text{ for } n > i > j \\
\beta_n &\mapsto (b'_n, b_n) \\
0 &\mapsto (0', 0)
\end{aligned}$$

In this way we see that $\mathbf{E} \in \mathcal{V}$ and therefore every subalgebra of $\mathbf{C}_n$ with n or fewer generators also belongs to $\mathcal{V}$. This means that $\mathbf{C}_n \in \mathcal{V}^{(n)}$, as the claim requires. □

Claim 1. *$\mathbf{C}_n \notin \mathcal{V}$.*

Proof. For each natural number n, let ϵ_{n+1} be the following equation:

$$y(x_0(x_1 \ldots (x_{n-1}y) \ldots)) \approx yy.$$

Each of these equations is easily seen to be true in every automatic algebra. In particular, each ϵ_{n+1} is true in $\mathbf{L}$ and hence in the variety $\mathcal{V}$. On the other hand, ϵ_{n+1} fails in $\mathbf{C}_n$ under the assignment that gives the value β_n to the variable y and, for each $i < n$, the value α_i to the variable x_i. Therefore $\mathbf{C}_n \notin \mathcal{V}$. □

The two claims constitute this proof of Lyndon's Nonfinite Basis Theorem. □

Lyndon's original proof also used the equations ϵ_{n+1} to separate $\mathcal{V}^{(n)}$ from $\mathcal{V}$.

3. Estimating the equational complexity of Lyndon's algebra

Main Theorem. *Let $\mathbf{L}$ be Lyndon's seven element nonfinitely based algebra. Then $\frac{1}{2}n - \frac{3}{2} \leq \beta_{\mathbf{L}}(n) \leq 4n - 4$ for all $n \geq 5$.*

Proof. The following lemma was proven in [11, Lemma 6].

Lemma A. *Let $\mathcal{V}$ be a variety of finite signature and let d and f be functions on the positive real numbers that assign integers to integers, with d strictly increasing and continuous and f monotonically increasing. Suppose that for each sufficiently large natural number n there is an algebra $\mathbf{C}_n$ with all the following properties*

(a) $|C_n| \leq d(n)$,
(b) $\mathbf{C}_n \in \mathcal{V}^{(f(n))}$, *and*
(c) $\mathbf{C}_n \notin \mathcal{V}$.

The $\beta_{\mathcal{V}}(m)$ eventually dominates $f(d^{-1}(m-1)-1)+1$.

Applying this lemma with $\mathcal{V}$ being the variety generated by $\mathbf{L}$, with $\mathbf{C}_n$ the algebra described in the proof of the previous section, with $d(n) = 2n+2$, and with $f(n) = n$, we conclude that $\frac{1}{2}n - \frac{3}{2} \leq \beta_{\mathbf{L}}(n)$ for all large enough n. Actually, this inequality holds for all $n \geq 5$, as an examination of the proof of Lemma A given in [11] reveals.

The next lemma was proven by Lyndon in [10].

Lyndon's Lemma. *Let $\mathcal{V}$ be the variety generated by $\mathbf{L}$ and let $n \geq 3$ be a natural number. The following equations constitute a base for $\mathcal{V}^{(n)}$:*

$$(xy)z \approx xx \qquad x(yy) \approx zz$$

$$y(x_0(x_1 \dots (x_{m-2}y)\dots)) \approx yy \tag{ϵ_m}$$

$$x_{m-2}(x_0(\dots(x_{m-2}x_{m-1})\dots)) \approx x_0(\dots(x_{m-2}x_{m-1})\dots) \tag{δ_m}$$

where $m \leq n$.

The longest equation in Lyndon's base for $\mathcal{V}^{(n-1)}$ is δ_{n-1}, which has length $4n-4$. Consider an algebra $\mathbf{B}$ of size less than n. Evidently, $\mathbf{B} \in \mathcal{V}$ if and only if $\mathbf{B} \in \mathcal{V}^{(n-1)}$. This requires checking only the equations in Lyndon's base, the longest having length $4n-4$. So $\beta_{\mathbf{L}}(n) \leq 4n-4$ for all $n \geq 3$. Both of the required inequalities hold. □

4. Cardinalities of subdirectly irreducible algebras in $\mathcal{V}$

At Oberwolfach in 1976 Bjarni Jónsson speculated that if $\mathcal{V}$ is a finitely generated variety of finite signature with a finite residual bound, then $\mathcal{V}$ is finitely based. Also in 1976, Robert E. Park, in his dissertation [12] written at UCLA under the direction of Kirby Baker, framed the same statement as a conjecture. It is still open. Framed contrapositively we obtain

The Jónsson-Park Problem

> Does every finitely generated nonfinitely based variety of finite signature have arbitrarily large finite subdirectly irreducible algebras?

(The alternative of an infinite subdirectly irreducible algebra can be omitted, in view of a theorem of Robert W. Quackenbush [13]—cf. Wiesław Dziobiak [5].)

In essence, only five finite but nonfinitely based algebras (including $\mathbf{L}$) were known at the time Park was writing his dissertation, the fifth supplied by Park himself. In the last chapter of his dissertation he demonstrated that each of these five generate varieties with arbitrarily large finite subdirectly irreducible algebras. For the variety $\mathcal{V}$ generated by Lyndon's algebra $\mathbf{L}$ and each cardinal $\kappa \geq 2$, Park's examples can be described as $(\mathbf{L}^\star)^\kappa/\theta$ where θ is the congruence that isolates each κ-tuple of letters and each κ-tuple of states but collapses all other κ-tuples into a single congruence class. Park's algebras have cardinality $1 + 2^\kappa + 3^\kappa$ for every cardinal $\kappa \geq 2$. The algebras $\mathbf{B}_n \in \mathcal{V}$ described in Section 2 are easily seen to be subdirectly irreducible (with critical pair $(\beta_0, 0)$) when $n \geq 2$. These algebras have cardinality $2n + 2$ and so differ from those constructed by Park.

There is a third way to construct subdirectly irreducible algebras in $\mathcal{V}$. For every cardinal $\kappa \geq 2$, we provide the algebra $\mathbf{S}_\kappa \in \mathcal{V}$. This algebra has $2\kappa + 2$ elements and is subdirectly irreducible. Figure 7 displays $\mathbf{S}_5$. For these algebras the pair

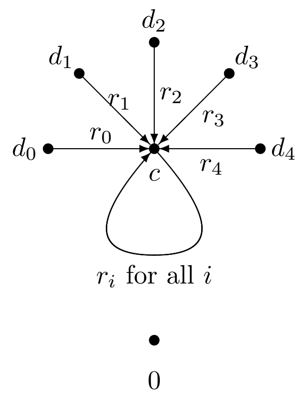

FIGURE 7. The Subdirectly Irreducible Automatic Algebra $\mathbf{S}_5$

$(c, 0)$ is critical. To see that these algebras are in $\mathcal{V}$, in $(\mathbf{L}^\star)^\kappa$ for each $\alpha \in \kappa$ put

$$\begin{array}{lllllllllll} d_\alpha & := & \dots & q & q & q & r & q & q & \dots & r \text{ at position } \alpha \\ r_\alpha & := & \dots & c & c & c & a & c & c & \dots & a \text{ at position } \alpha \\ c & := & \dots & q & q & q & q & q & q & \dots & \end{array}$$

Let $\mathbf{T}_\kappa$ be the subalgebra generated by these elements. The equivalence relation θ_κ which places in one class all the tuples with at least one entry 0 and isolates all the other tuples is a congruence of $\mathbf{T}_\kappa$. We take $\mathbf{S}_\kappa$ to be $\mathbf{T}_\kappa/\theta_\kappa$.

To obtain finite subdirectly irreducible algebras of odd cardinality we modify $\mathbf{T}$ by including the tuple

$$s \;:=\; \dots \; c \; c \; c \; c \; \dots$$

In this way, we obtain subdirectly irreducible algebras in $\mathcal{V}$ of every cardinality $\kappa \geq 6$. The algebra $\mathbf{L}^\star$ has subdirectly irreducible subalgebras of cardinalities $2, 3,$ and 5 which are displayed in Figure 8.

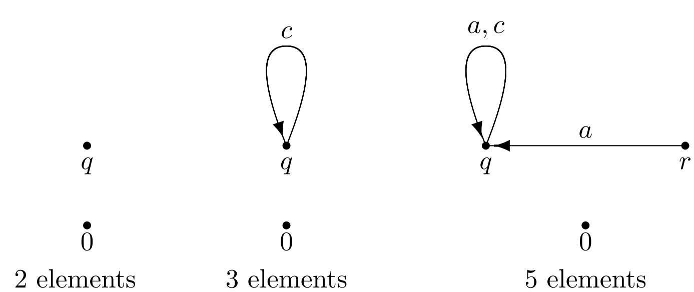

FIGURE 8. Three Small Subdirectly Irreducible Algebras

Up to isomorphism, $\mathcal{V}$ has just five algebras of cardinality 4 and none of them is subdirectly irreducible. These algebras, which are subalgebras of $\mathbf{L}$, are displayed in Figure 9

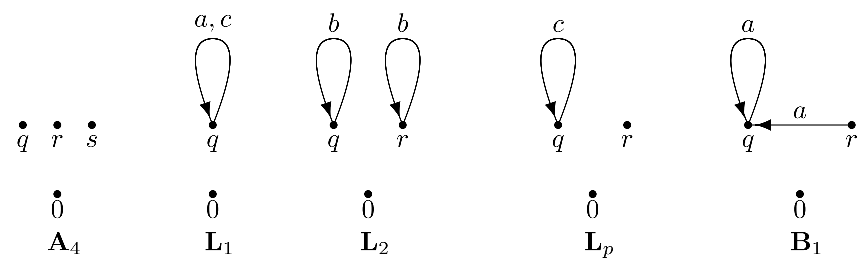

FIGURE 9. The 4-Element Algebras in Lyndon's Variety

To exclude the other possibilities, suppose the 4 elements are $0, a, b$, and c. For all $x, y, z \in \{0, a, b, c\}$, we have $0x = x0 = 0, xx = 0, (xy)z = 0, y(xy) = 0$, and $x(xy) = xy$ because these equations are true in $\mathbf{L}$.

We can suppose that some product is not 0, since if all products are 0 we get $\mathbf{A}_4$. It is harmless to suppose that $ab \neq 0$. It follows that a cannot be a product, since if $a = uv$, the $0 \neq ab = (uv)b = 0$. So there are two cases: $ab = c$ and $ab = b$.

First, suppose $ab = c$. Now $0 = (ab)x = cx$ for all x. Also $ac = a(ab) = ab = c$. Finally, $bc = b(ab) = 0$. This leaves only the product ba unsettled. If $ba = 0$, we get the algebra $\mathbf{B}_1$. We already know it is impossible that $ba = a$. It is also impossible that $ba = b$ since this leads to $b = ba = (ba)a = 0$. Finally, it is also impossible that $ba = c$ since this leads to $c = ac = a(ba) = 0$.

Second, suppose that $ab = b$. Observe $0 = (ab)x = bx$ for all x. We know $ca = a$ is impossible and $ca = b$ is impossible since $0 = a(ca) = ab = b$ and $ca = c$ is impossible since $c = ca = (ca)a = 0$. Therefore $ca = 0$. In view of the first supposition, we can assume that $ac \neq b$. Because $ac = a$ is impossible, there are only two subcases: $ac = c$ and $ac = 0$. If $ac = c$, then $0 = (ac)x = cx$ for all x and the resulting algebra is $\mathbf{L}_2$. So consider the remaining alternative $ac = 0$. The only product left to consider is cb. It is impossible that $cb = c$ since then $c = cb = (cb)b = 0$ and we know $cb = a$ is impossible. In case $cb = b$ we obtain the algebra $\mathbf{L}_1$ and in case $cb = 0$ we obtain the algebra $\mathbf{L}_p$.

So we find that $\mathcal{V}$ has subdirectly irreducible algebras of every cardinality except $0, 1$, and 4. Ralph Freese has confirmed that $\mathcal{V}$ has no subdirectly irreducible algebras of cardinality 4 with the help of the Universal Algebra Calculator. This was accomplished by examining all 25 meet irreducible congruences of the algebra $\mathcal{V}$-freely generated by three elements. The Universal Algebra Calculator, originated by Matthew Valeriote, is largely the work of Ralph Freese and Emil Kiss. It is available online at `http://www.uacalc.org/`.

5. The Shift Automorphism Theorem and Lyndon's algebra

An algebra $\mathbf{A}$ of finite signature is **inherently nonfinitely based** provided $\mathbf{A}$ belongs to a locally finite variety but to no finitely based locally finite variety. The theorem below can be used to show that a wide assortment of algebras are inherently nonfinitely based. It was published in 1989 by Kirby Baker, George McNulty, and Heinrich Werner [2]. An apparently stronger version can be found in [11]. An element $0 \in A$ is **absorbing** provided 0 is the output of any basic operation of $\mathbf{A}$ whenever 0 is among its inputs. All the elements of A other than absorbing elements are said to be **proper**. A tuple of elements is **proper** if all of its entries are proper. A basic r-ary operation F of $\mathbf{A}$ is a set of $r+1$-tuples.

The Shift Automorphism Theorem. *Let $\mathbf{A}$ be an infinite locally finite algebra with only finitely many fundamental operations, with an absorbing element 0, and with an automorphism σ such that*

(a) $\{0\}$ *is the only σ-orbit of $\mathbf{A}$ that is finite;*
(b) *the set of all proper tuples in F is partitioned by σ into only finitely many orbits, for each fundamental operation F of $\mathbf{A}$;*
(c) $f(a) = \sigma(a)$ *for some proper element a of A and some nonconstant unary polynomial function f of $\mathbf{A}$.*

Then $\mathbf{A}$ is inherently nonfinitely based.

We know that the variety $\mathcal{V}$ generated by Lyndon's algebra fails to be inherently nonfinitely based. So no algebra in $\mathcal{V}$ can fulfill all the hypotheses of the Shift Automorphism Theorem. We construct an algebra $\mathbf{B}_{\mathbb{Z}} \in \mathcal{V}$ which fulfills all the conditions except for (b). This provides an example to show that the hypothesis (b) cannot be eliminated from the theorem (nor can it we replaced by the weaker condition that there be only finitely many σ-orbits).

The algebra $\mathbf{B}_{\mathbb{Z}}$ is displayed in Figure 10

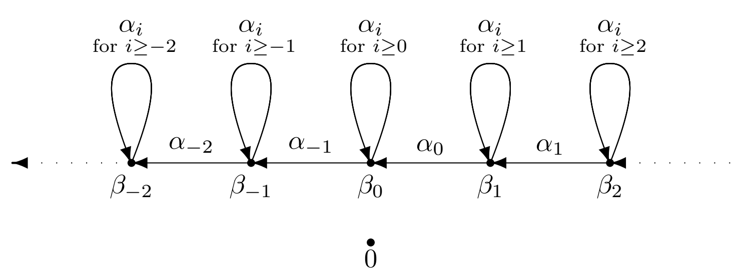

FIGURE 10. The Automatic Algebra $\mathbf{B}_{\mathbb{Z}}$

The algebra $\mathbf{B}_{\mathbb{Z}}$ is a homomorphic image of a subalgebra of $(\mathbf{L}^\star)^{\mathbb{Z}}$. The construction proceeds much as the construction of $\mathbf{B}_n$ in Section 2. For each integer k put

$$\begin{array}{llllllllll} \beta_k & := & \dots & r & r & r & q & q & q & \dots \quad \text{with the leftmost } q \text{ in the } k^{\text{th}} \text{ position} \\ \alpha_k & := & \dots & b & b & b & a & c & c & \dots \quad \text{with the } a \text{ in the } k^{\text{th}} \text{ position} \end{array}$$

Let $\mathbf{G}$ be the subalgebra of $(\mathbf{L}^\star)^\mathbb{Z}$ generated by $\{\alpha_k \mid k \in \mathbb{Z}\} \cup \{\beta_k \mid k \in \mathbb{Z}\}$. The equivalence relation φ on G which puts all $\mathbb{Z}$-tuples with at least one entry 0 into one big equivalence class and isolates all the other tuples of G is a congruence of $\mathbf{G}$. The algebra $\mathbf{B}_\mathbb{Z}$ is $\mathbf{G}/\varphi$. The map σ that sends each β_{k+1} to β_k, that sends each α_{k+1} to α_k and that fixes 0 is easily seen to be an automorphism of $\mathbf{B}_\mathbb{Z}$. This automorphism partitions $B_\mathbb{Z}$ into three orbits: the trivial orbit $\{0\}$, two infinite orbits $\{\alpha_k \mid k \in \mathbb{Z}\}$ and $\{\beta_k \mid k \in \mathbb{Z}\}$. Condition (a) of the Shift Automorphism Theorem is fulfilled. Observe that $\sigma(\beta_1) = \beta_0 = \alpha_0\beta_1$. Hence the polynomial function $f(x)$ and the proper element a required by hypothesis (c) of the Shift Automorphism Theorem can be taken as $\alpha_0 x$ and β_1 respectively. However, hypothesis (b) fails: the triples $(\alpha_k, \beta_0, \beta_0)$ for $k \geq 0$ must all lie in distinct orbits.

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