# Problem Set About Unique Readability
30 August 2011

In the problems below $L$ is a signature and $X$ is a set of variables.

Problem 0.
Define a function $\lambda$ from the set of finite nonempty sequences of elements of $X \cup L$ into the integers as follows:

$$\lambda(w) = \begin{cases} -1 & \text{if } w \in X, \\ r-1 & \text{if } w \text{ is an operation symbol of rank } r, \\ \sum_{i<n} \lambda(u_i) & \text{if } w = u_0 u_1 \dots u_{n-1} \text{ where } u_i \in X \cup L \text{ and } n > 1. \end{cases}$$

Prove that $w$ is a term if and only if $\lambda(w) = -1$ and $\lambda(v) \geq 0$ for every nonempty proper initial segment $v$ of $w$.

Problem 1.
Let $w = u_0 u_1 \dots u_{n-1}$, where $u_i \in X \cup L$ for all $i < n$. Prove that if $\lambda(w) = -1$, then there is a unique cyclic variant $\hat{w} = u_i u_{i+1} \dots u_{n-1} u_0 \dots u_{i-1}$ of $w$ that is a term.

Problem 2.
Prove that if $w$ is a term and $w'$ is a proper initial segment of $w$, then $w'$ is not a term.

Problem 3.
Let $\mathbf{T}$ be the term algebra of $L$ over $X$. Prove

> If $Q$ and $P$ are operation symbols, and $P^{\mathbf{T}}(p_0, p_1, \dots, p_{n-1}) = Q_1^{\mathbf{T}}(q_0, q_1, \dots, q_{m-1})$, then $P = Q$, $n = m$, and $p_i = q_i$ for all $i < n$.

# Problem Set About the Compactness Theorem
## Due Thursday 22 September 2011

Problem 4.
Let $L$ be the signature for group theory with operation symbols $\cdot, ^{-1}$, and 1. Let $T$ be a set of $L$-sentences which includes all the group axioms (so every model of $T$ will be a group). Suppose that for each $n$, there is a model of $T$ which has no elements, other than 1, of order smaller than $n$. Prove that there is a model of $T$ such that 1 is the only element of finite order.

Problem 5.
Suppose that $G$ is a group which has elements of arbitrarily large finite order. Prove that $G$ is elementarily equivalent to a group with an element of infinite order.

Problem 6.
Let $\langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle$ be the familiar structure consisting of the natural numbers equipped with addition, multiplication, the two distinguished elements 0 and 1, and the usual order relation. Let $T$ consist of all the sentences true in $\langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle$. Prove $T$ has a model $\mathbf{M}$ with an element $\omega$ so that all the following are true in $\mathbf{M}$:

$$0 \leq \omega, 1 \leq \omega, 2 \leq \omega, \ldots.$$

Problem 7.
Let $L$ be the signature of rings. Find a set $\Sigma$ of $L$-sentences such that $\operatorname{Mod} \Sigma$ is the class of algebraically closed fields. Then prove that there is no finite set of $L$-sentences which will serve the same purpose.

Problem 8.
Let $L$ be the signature of ordered sets. Prove that there is no set $\Sigma$ of $L$-sentences such that $\operatorname{Mod} \Sigma$ is the class of all well-ordered sets.

# Second Problem Set About the Compactness Theorem
## Due Tuesday 18 October 2011

Problem 9.
Let $L$ be a signature and $\mathcal{K}$ be a class of $L$-structures. We say that $\mathcal{K}$ is *axiomatizable* provided $\mathcal{K} = \operatorname{Mod}\Sigma$ for some set $\Sigma$ and $L$-sentences. $\mathcal{K}$ is *finitely axiomatizable* provided there is a finite such $\Sigma$. Prove that $\mathcal{K}$ is finitely axiomatizable if and only if both $\mathcal{K}$ and $\{A \mid A \text{ is an } L\text{-structure and } A \notin \mathcal{K}\}$ are axiomatizable.

Problem 10.
Show that the class of fields of finite characteristic is not axiomatizable.

Problem 11.
Show that the class of fields of characteristic 0 is not finitely axiomatizable.

Problem 12.
Let $\varphi$ be any sentence in the signature of fields. Prove that if $\varphi$ is true in every field of characteristic 0, then there is a natural number $n$ so that $\varphi$ is true in every field of characteristic $p$ for all primes $p > n$.

Problem 13.
Let $L$ be a signature and for each natural number $n$ suppose that $T_n$ is a set of $L$-sentences closed with respect to logical consequence. Further, suppose that $T_0 \subset T_1 \subset T_2 \subset \ldots$ is strictly increasing. Let $T = \bigcup_{n\in\omega} T_n$. Prove that

(1) $T$ has a model.
(2) $T$ is closed under logical sonsequence.
(3) $T$ is not finitely axiomatizable.

Suppose $\mathbf{A}$ is a structure. The group $\operatorname{Aut}\mathbf{A}$ of all automorphisms of $\mathbf{A}$ partitions $A$ into orbits. [Elements $a, b \in A$ belong to the same orbit iff there is an automorphism $f$ such that $f(a) = b$.] Notice that the same applies the $n$-tuples from $A$: the group $\operatorname{Aut}\mathbf{A}$ partitions $A^n$ into orbits.

Problem 14.
Let $L$ be a countable signature and let $T$ be a complete set of $L$-sentences. Prove that $T$ is $\omega$-categorical if and only if $\operatorname{Aut}\mathbf{A}$ partitions $A^n$ into only finitely many orbits for every natural number $n$, for every countable $\mathbf{A} \models T$.

Problem 15.
Let $L$ be a countable signature and let $T$ be a complete set of $L$-sentences. Prove that $T$ is $\omega$-categorical if and only if $\operatorname{Aut}\mathbf{A}$ partitions $A^n$ into only finitely many orbits for every natural number $n$, for some countable $\mathbf{A} \models T$.

Problem 16.
Let $T$ be an elementary theory in a countable signature and suppose that $T$ is $\kappa$-categorical for some infinite cardinal $\kappa$. Let $\mathcal{K} = \{A \mid A \mod T \text{ and } A \text{ is infinite}\}$. Prove that $\mathcal{K}$ is axiomatizable and that $\operatorname{Th}\mathcal{K}$ is complete.

Problem 17.
Consider a signature provided with one 2-place relation symbol $\leq$ and a infinite list $c_0, c_1, c_2, \ldots$ of constant symbols. Let $T$ be the theory that asserts that $\leq$ is a dense linear order without endpoints and that $c_i < c_{i+1}$ for all $i \in \omega$. Prove that $T$ is a complete theory and discover (with proofs) how many countable models $T$ has, up to isomorphism.