EXAMINATION III
MATH 546/701I
28 NOVEMBER 2001

PROBLEM 0
Let n, a, and b be positive integers. Suppose that $(n, a) = 1$ and $n \mid ab$. Prove that $n \mid b$.

PROBLEM 1: CORE
Let m and n be any integers. Let $h : \mathbb{Z} \to \mathbb{Z}_m$ be defined so that $h(k) = r$ where r is the remainder of k upon division by m. Likewise, let $g : \mathbb{Z} \to \mathbb{Z}_n$ be defined so that $g(k) = s$ where s os the remainder of k upon division by n. Let $a, b, c \in \mathbb{Z}$ such that

$$h(a) = h(c) \text{ and}$$
$$g(b) = g(c).$$

Find an integer d such that

$$g(a) = g(d) \text{ and}$$
$$h(b) = h(d).$$

PROBLEM 2: CORE
Let A, B, and C be sets. Let h be a function from A onto B and let g be a function from A onto C. Let

$$f = \{\langle g(a), h(a)\rangle \mid a \in A\}.$$

Suppose that f is a one-to-one function from B into C.
Prove that $\operatorname{KER} g = \operatorname{KER} h$.

PROBLEM 3
Let k be a positive integer and h be a homomorphism from $\langle \mathbb{Z}, +, \cdot, -, 0, 1\rangle$ onto $\langle \mathbb{Z}_k, +_k, \cdot_k, -_k, 0, 1\rangle$.
Prove that $h(n)$ is the remainder of n upon division by k, for every integer n.

PROBLEM 4: CORE
Let $\mathbf{G}$ be a group and let $\mathbf{H}$ be a subgroup of $\mathbf{G}$. Let

$$N = \{g \in G \mid x^{-1}gx \in H \text{ for all } x \in G\}.$$

Prove that N is a subgroup of $\mathbf{G}$.

PROBLEM 5.
Let $\mathbb{R}$ be the group of real numbers with the operations of addition, negation, and zero, and let $\mathbb{C}^{\times}$ be the group of nonzero complex numbers with the operations of multiplication, multiplicative inverse, and one. Define $\Phi : \mathbb{R} \to \mathbb{C}^{\times}$ via

$$\Phi(x) = \cos x + i \sin x$$

for all $x \in \mathbb{R}$. Prove that Φ is a homomorphism from $\mathbb{R}$ into $\mathbb{C}^{\times}$.

PROBLEM 6.
Determine in each case below whether the given permutation is even or odd. Please explain your reasoning.

a. $(1, 3, 2)(1, 2)$
b. $(1, 3, 2, 4, 5)(1, 3, 2)(4, 5)$
c. $(1, 5)(1, 4)(1, 3)(2, 5)$

PROBLEM 7: CORE
Let $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ be groups. Let h be a homomorphism from $\mathbf{A}$ onto $\mathbf{B}$ and let g be a homomorphism from $\mathbf{A}$ onto $\mathbf{C}$ such the $\operatorname{KER} h \subseteq \operatorname{KER} g$.
Prove that there is a homomorphism f from $\mathbf{B}$ onto $\mathbf{C}$.

PROBLEM 8
Let $\mathbf{F}$ be a field and let $f(x)$ be any polynomial of positive degree with coefficients from $\mathbf{F}$. Prove that $f(x)$ is an irreducible member of $\mathbf{F}[x]$ if and only if for every $g(x) \in \mathbf{F}[x]$ either $(f(x), g(x)) = 1$ or $f(x) \mid g(x)$.

PROBLEM 9.
Let $\mathbf{F}$ be a field and let $a(x), b(x), c(x) \in \mathbf{F}[x]$. Suppose that

$$(x + 1) \mid (a(x) - c(x)) \text{ and}$$
$$(x - 1) \mid (b(x) - c(x)).$$

Find a polynomial $d(x) \in \mathbf{F}[x]$ so that

$$(x-1) \mid (a(x) - d(x)) \text{ and}$$
$$(x+1) \mid (b(x) - d(x)).$$

PROBELM 10.
Let $\mathbb{Q}$ be the field of rational numbers. For each $f(x) \in \mathbb{Q}[x]$ let $\varphi(f(x)) = f(2x)$. Prove that φ is an isomorphism from $\mathbb{Q}[x]$ onto $\mathbb{Q}[x]$.

PROBLEM 11.
In each part below determine whether the given polynomial is an irreducible member of $\mathbb{Q}[x]$. Be sure to explain your reasoning.

a. $x^3 + 3x^2 + 3x + 1$.

b. $x^2 - 2$.

c. $5x^3 + 18x + 12$.

EXTRA CREDIT PROBLEM
Recall that a congruence relation on $\langle \mathbb{Z}, +, \cdot, -, 0, 1 \rangle$ is an equivalence relation of $\mathbb{Z}$ which respects all the operations listed. Describe all the congruence relations on $\langle \mathbb{Z}, +, \cdot, -, 0, 1 \rangle$.