

EXAMINATION II
MATH 546/701I
HALLOWEEN 2001

PROBLEM 0

Let n be any natural number with $n > 1$. Prove that n is prime if and only if for every natural number m either $(n, m) = 1$ or $n \mid m$.

PROBLEM 1: CORE

Let m , and n be any integers. For any integers a and b let $a \sim b$ mean

$$\begin{aligned} a &\equiv c \pmod{m} \text{ and} \\ b &\equiv c \pmod{n}. \end{aligned}$$

for some integer c .

Prove that \sim is a symmetric relation on the set of integers.

PROBLEM 2: CORE

Let A, B , and C be sets. Let h be a function from A onto B and let g be a function from A onto C such the $\text{KER } h \subseteq \text{KER } g$.

Prove that there is a function f from B onto C .

PROBLEM 3

Let k be a positive integer and h be a homomorphism from $\langle \mathbb{Z}, +, \cdot, -, 0, 1 \rangle$ onto $\langle \mathbb{Z}_k, +_k, \cdot_k, -_k, 0, 1 \rangle$.

Let $I = \{a \in \mathbb{Z} \mid h(a) = 0\}$ Prove each of the following statements.

- $\langle a, b \rangle \in \text{KER } h$ if and only if $a - b \in I$.
- There is a positive integer a such that $I = \{an \mid n \in \mathbb{Z}\}$.

PROBLEM 4: CORE

Prove each of the following:

- Let X be a set and let $a \in X$. Define

$$H(a) = \{\sigma \mid \sigma \in \text{SYM } X \text{ and } \sigma(a) = a\}$$

. Prove that $H(a)$ is a subgroup of $\text{SYM } X$.

- Let \mathbf{G} be a group and let $a \in G$. Define $C(a) = \{g \in G \mid ga = ag\}$. Prove that $C(a)$ is a subgroup of \mathbf{G} .

PROBLEM 5.

Let \mathbf{G} be a group and let $a \in G$. Define $F_a : G \rightarrow G$ via

$$F_a(x) = axa^{-1}$$

for all $x \in G$. Prove that F_a is an isomorphism from \mathbf{G} onto \mathbf{G} .

PROBLEM 6.

Determine in each case below whether the given permutation is even or odd. Please explain your reasoning.

- $(1, 2, 3)(1, 2)$
- $(1, 2, 3, 4, 5)(1, 2, 3)(4, 5)$
- $(1, 2)(1, 3)(1, 4)(2, 5)$

PROBLEM 7: CORE

Let \mathbf{A}, \mathbf{B} , and \mathbf{C} be groups. Let h be a homomorphism from \mathbf{A} onto \mathbf{B} and let g be a homomorphism from \mathbf{A} onto \mathbf{C} such the $\text{KER } h = \text{KER } g$.

Prove that there is an isomorphism f from \mathbf{B} onto \mathbf{C} .

EXTRA CREDIT PROBLEM

Recall that a congruence relation on $\langle \mathbb{Z}, +, \cdot, -, 0, 1 \rangle$ is an equivalence relation of \mathbb{Z} which respects all the operations listed. Describe all the congruence relations on $\langle \mathbb{Z}, +, \cdot, -, 0, 1 \rangle$.