Problem Set 0

Due Wednesday 27 August 2014

Problem 0.
For integers a and b we use (a, b) to denote the greatest common divisor of a and b. For example, $(12, 18) = 6$. In each part below find the greatest common divisor. [Hint: Find out about Euclid's Algorithm online or in one of our resource books]

a. $(6643, 2873)$.
b. $(26460, 12600)$.
c. $(12091, 8437)$.

Problem 1.
For each of the three parts of Problem 0, find integers m and n such that $(a, b) = ma + nb$.

Problem 2.
Give a proof of the properties of the divisibility relation listed in each part below.

a. For any integers a, b, and c, if $b \mid a$, then $b \mid ac$.
b. For any integers a, b, and c, if $b \mid a$ and $c \mid b$, then $c \mid a$.

Problem 3.
Show that any nonempty set of integers that is closed under subtraction must also be closed under addition.

Problem 4 (A Challenge Problem).
Which sets of natural numbers are closed under addition?

Problem Set 1

Due 3 September 2014

Problem 5.
For each of the numbers $60, 100$, and 1575 draw the diagram of positive divisors showing the divisibility relation.

Problem 6.
Prove that $\frac{\log 2}{\log 3}$ is not a rational number.

Problem 7.
In each part below determine whether the function given is one-to-one, whether it is onto, and, in the event that it is both one-to-one and onto, describe its inverse.

(a) $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (x + y, y)$.
(b) $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (x + y, x + y)$.
(c) $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (2x + y, x + y)$.

Problem 8.
Suppose that $f : A \to B$ and $g : B \to C$ and both f and g are one-to-one and onto. Prove that $(g \circ f)^{-1}$ is a function from C to A and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Problem 9 (Challenge Problem).
Prove that, for every positive naturaly number n, there are n consecutive composite natural numbers.

Problem Set 2
Due 10 September 2014

Problem 10.
On the set $\mathbb{R}^2$ of ordered pairs of real numbers (think points in the plane) define the 2-place relation $\sim$ as follows:

$$(a, b) \sim (c, d) \text{ if and only if } a^2 + b^2 = c^2 + d^2.$$

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{R}^2$.
(b) Describe the equivalence classes with respect to $\sim$.

Problem 11.
On the set $\mathbb{Z}$ of integers define the 2-place relation $\sim$ as follows:

$$m \sim n \text{ if and only if } n \mid m^k \text{ for some positive integer } k \text{ and } m \mid n^j \text{ for some positive integer } j.$$

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{Z}$.
(b) Describe the equivalence class that contains 6 and describe the equivalence class that contains 12.
(c) Decribe the equivalence classes in general.

Problem 12.
Let X be a nonempty set and let $\sigma \in \operatorname{Sym} X$. Define the 2-place relation $\sim$ on X as follows:

$$x \sim y \text{ if and only if } \sigma^k(x) = y \text{ for some integer } k.$$

Prove that $\sim$ is an equivalence relation on X.

Problem 13.
Let X be a nonempty set. Define the 2-place relation $\sim$ on $\operatorname{Sym} X$ as follows:

$$\sigma \sim \tau \text{ if and only if } \rho^{-1} \circ \sigma \circ \rho = \tau \text{ for some permutation } \rho.$$

Prove that $\sim$ is an equivalence relation on $\operatorname{Sym} X$.

Problem Set 3
Due 17 September 2014

Problem 14.
Prove that the inverse of any isomorphism is also an isomorphism.

Problem 15.
Prove that the composition of two isomorphisms (if it is possible to form the composition) is also an isomorphism.

Problem 16.
Let $\mathbf{R}$ be any ring. Prove that the following equations must hold in $\mathbf{R}$.

(a) $x \cdot 0 = 0$.
(b) $0 \cdot x = 0$.
(c) $-(-x) = x$.
(d) $(-x) \cdot (-y) = x \cdot y$.

Problem 17.
Let I be any set and let R be the collection of all subsets of I. We impose on R the following operations:

$$\begin{aligned} A + B &:= (A \cup B) \cap (\bar{A} \cup \bar{B}) \\ -A &:= A \\ 0 &:= \emptyset \\ A \cdot B &:= A \cap B \\ 1 &:= I \end{aligned}$$

Here $\bar{A} = \{d \mid d \in I \text{ and } d \notin A\}$. The operation $+$ defined above is sometimes called the symmetric difference of the sets A and B. It may help to draw a Venn diagram of $A + B$. Prove that $\langle R, +, \cdot, -, 0, 1\rangle$, that is R equipped with the operaitons above, is a ring.

Problem Set 4
Due 24 September 2014

Problem 18.
Let $\mathbf{R}$ and $\mathbf{S}$ be rings and let h and g be homomorphisms from $\mathbf{R}$ into $\mathbf{S}$. Let $T = \{r \mid r \in R \text{ and } h(r) = g(r)\}$. Prove that T is a subring of $\mathbf{R}$.

Problem 19.
Let $R = \{m + n\sqrt{2} \mid m, n \in \mathbb{Z}\}$ and let $I = \{m + n\sqrt{2} \mid m, n \in \mathbb{Z} \text{ and } m \text{ is even}\}$.

(a) Prove that $\mathbf{R}$ is a subring of the ring of real numbers.
(b) Prove that I is an ideal of $\mathbf{R}$.

Problem 20.
Let $\mathbf{R}$ be the ring of all continuous functions from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ and let

$$I = \{f \mid f \in R \text{ and } f(\pi) = 0\}.$$

(a) Prove that I is a proper ideal of $\mathbf{R}$.
(b) Prove that if J is an ideal of $\mathbf{R}$ and $I \subseteq J \subseteq R$, then either $I = J$ or $J = R$.

Problem 21.
Let $\mathbf{D}$ be a finite integral domain. Show that every nonzero element of D has a multiplicative inverse.

Problem Set 5
Due 1 October 2014

Problem 22.
Let R be the ring of all continuous functions from the real numbers into the real numbers. Prove that R is not an integral domain.

Problem 23.
Let R be an integral domain and suppose that a is an element of R with the property that $a^2 = a$. Prove that either $a = 0$ or $a = 1$.

Problem 24.
Let R be an integral domain and suppose that a and b are nonzero elements of R. Prove that

$$a \mid b \text{ and } b \mid a \Leftrightarrow b = au \text{ for some unit } u \in R.$$

Problem 25.
Let R be an integral domain and let I and J be nontrivial ideals of R. Prove that $I \cap J$ is a nontrivial ideal of R.

Problem Set 6
Due 8 October 2014

Problem 26.
Suppose that R is a commutative ring and that u is an element of R. Prove that u is a unit of R if and only if $u \mid r$ for all $r \in R$.

Problem 27.
Let R be an integral domain and suppose that $p \in R$ has the property that $p \neq 0$ and if $p \mid ab$, then either $p \mid a$ or $p \mid b$. Prove that p is irreducible. Bonus: show that $\mathbb{Z}_6$ does not have this property.

Problem 28.
Prove that the ring $\mathbb{Z}_n$ is a field if and only if n is a prime number. Recall $\mathbb{Z}_n$ is the ring of remainders upon division by n where ring operations are defined modulo n.

Problem 29.
Let $F = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

(1) Prove F is a subring of the field $\mathbb{R}$ of real numbers.
(2) Prove that F is a field.

Problem Set 7
Due 15 October 2014

Problem 30.
Write $(4,5,6)(5,6,7)(6,7,0)(0,1,2)(1,2,3)(3,4,5)$ as a product of disjoint cycles.

Problem 31.
Let X be a set and $a \in X$. A permutation $\sigma \in \operatorname{Sym} X$ is said to **fix** a provided $\sigma(a) = a$. Let H be the collection of all permutations of X that fix the element a. Prove each of the following:

(a) $\mathbf{1}_X \in H$, where $\mathbf{1}_X$ is the identity function on X.
(b) If $\sigma \in H$, then $\sigma^{-1} \in H$.
(c) If $\sigma, \tau \in H$, then $\sigma \circ \tau \in H$.

Problem 32.
Let S_n be the set of all permutations of the set $\{0, 1, 2, \ldots, n-1\}$. Suppose that $n \geq 3$. Prove that every even permutation in S_n can be written as a product of cycles of length 3.

Problem 33.
Show that every permutation of the set $\{0, 1, 2, \ldots, n-1\}$ can be represented as a product, maybe a long one, using only the following $n-1$ transpositions

$$(0,1), (0,2), \ldots, \text{ and } (0, n-1).$$

Problem 34 (Challenge Problem).
Prove that every permutation of $\{0, 1, 2, \ldots, 6\}$ can be written as a product, maybe a long one, built up using just the transposition $(0,1)$ and the 7 cycle $(0, 1, 2, \ldots, 6)$. Is this still true if our 7-element set is replaced by one with 6-elements? What about 5-elements? What values of n work?

Problem Set 8
Due 22 October 2014

Problem 35.
Prove that $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$, for any $\sigma, \tau \in S_n$.

Problem 36.
Without writing down all 60 elements of $\mathbf{A}_5$ (the group of all even permutations on a five-element set), describe the possible shapes of the permutations (the number and length of their disjoint cycles) and how many of each type there are.

Problem 37.
Let X be an infinite set. Let H be the set of all elements σ of $\operatorname{Sym} X$ such that $\sigma(x) = x$ for all but finitely many $x \in X$. Prove that H is subgroup of $\operatorname{Sym} X$.

Problem 38 (Challenge Problem).
Prove that every group with n elements is isomorphic to a subgroup of the group of all invertible linear operators on the n-dimensional real vector space. Would something like this remain true if n were infinite? Would that even mean anything?

Problem Set 9
Due 29 October 2014

Problem 39.
Let n, a, and b be positive integers. Suppose that n and a are relatively prime and that $n \mid ab$. Prove that $n \mid b$.

Problem 40 (Core).
Let A, B, and C be sets. Let h be a function from A onto B and let g be a function from A onto C. Let $\theta_h := \{\langle a, a' \rangle \mid a, a' \in A \text{ and } h(a) = h(a')\}$. Let $\theta_g := \{\langle a, a' \rangle \mid a, a' \in A \text{ and } g(a) = g(a')\}$. Define

$$f := \{\langle h(a), g(a) \rangle \mid a \in A\}.$$

Suppose further that f is a one-to-one function from B into C.

Prove that $\theta_h = \theta_g$.

Problem 41 (Core).
Let $\mathbf{G}$ be a group and let $\mathbf{H}$ be a subgroup of $\mathbf{G}$. Let

$$N = \{g \mid g \in G \text{ such that } x^{-1}gx \in H \text{ for all } x \in G\}.$$

Prove that N is a subuniverse of $\mathbf{G}$.

Problem 42.
Problem 3

Let n and k be positive integers and h be a homomorphism from $\langle \mathbb{Z}_n, +_n, \cdot_n, -_n, 0, 1 \rangle$ onto $\langle \mathbb{Z}_k, +_k, \cdot_k, -_k, 0, 1 \rangle$. Prove each of the following statements.

a. $k \leq n$.
b. $h(k) = 0$. [Hint: $h(1 +_n \cdots +_n 1) = h(1) +_k \cdots +_k h(1)$.]
c. $h(a) = r$ for each $a \in \{0, 1, \ldots, n-1\}$, where $a = kq + r$ with $r \in \{0, 1, \ldots, k-1\}$ for some integer q.
d. $k \mid n$.

Problem Set 10
Due 5 November 2014

Problem 43.
By $\mathbb{Z}_2$ we mean the algebra $\langle\{0,1\}, \oplus, \ominus, 0\rangle$ where the two place operation $\oplus$ is given by

$$0 \oplus 0 = 0 \qquad\qquad 1 \oplus 1 = 0$$
$$0 \oplus 1 = 1 \qquad\qquad 1 \oplus 0 = 1$$

and $\ominus$ works so that $\ominus 0 = 0$ and $\ominus 1 = 1$. With these operations $\mathbb{Z}_2$ is a group (and eager students will check this). Now let $n > 2$ be a natural number and let h be the map from S_n to $\{0,1\}$ defined so that $h(\sigma) = 0$ if σ is an even permutation and $h(\sigma) = 1$ if σ is an odd permutation. Prove that h is a homomorphism from $\mathbf{S}_n$ onto $\mathbb{Z}_2$. Also discover what the kernel of h is.

Problem 44.
Let $\mathbb{C}^\times$ be the group of nonzero complex numbers, where the two place operation is complex multiplication, the one-place operation is multiplicative inverse, and the identity is 1. Likewise, let $\mathbb{R}^+$ be the group of positive real numbers with the same sorts of operations. (So we see that $\mathbb{R}^+$ is a subgroup of $\mathbb{C}^\times$.) Let ϕ be the function from $\mathbb{C}^\times$ to $\mathbb{R}^+$ defined by $\phi(a+bi) = a^2 + b^2$, for all reals a and b that are not both 0. Prove that ϕ is a homomorphism from $\mathbb{C}^\times$ onto $\mathbb{R}^+$. Also discover what the kernel of ϕ is.

Problem 45.
Let $\mathbf{G}$ be a group. Define $Z_G := \{g \mid g \in G \text{ and } gh = hg \text{ for all } h \in G\}$. Prove that $\mathbf{Z}_G$ is a normal subgroup of $\mathbf{G}$.

Problem 46.
Let $\mathbf{G}$ be a group and let $\mathbf{H}$ and $\mathbf{K}$ be normal subgroups of $\mathbf{G}$ so that $H \cap K = \{1\}$. Prove that $hk = kh$ for all $h \in H$ and all $k \in K$.

Problem Set 11
Due 12 November 2014

Problem 47.
Let $\mathbf{G}$ be the group of all 3×3 matrices with real entries that are invertible. The operations of this group are matrix multiplication, matrix inversion, and the identity matrix. Let S be the set of 3×3 matrices with real entries and with determinant 1. Let $\mathbb{R}^{\times}$ denote the group of nonzero numbers with the operations of multiplication, multiplicative inverse, and 1. Prove that $\mathbf{G}/S \cong \mathbb{R}^{\times}$. Does this hold when 3 is replaced by an arbitrary positive integer?

Problem 48.
Let $\mathbf{G}$ be a finite group and n be a natural number. Prove that if $\mathbf{G}$ has exactly one subgroup of size n, then that subgroup must be normal.

Problem 49.
Let X be an infinite set and σ be a permutation of X. We says that σ **moves only finitely many elements** provided the set $\{x \mid x \in X \text{ and } \sigma(x) \neq x\}$ is finite. Let

$$H = \{\sigma \mid \sigma \in \operatorname{Sym} X \text{ and } \sigma \text{ moves only finitely many elements}\}.$$

Prove that H is a normal subgroup of $\operatorname{Sym} X$.

Problem 50.
Let $\mathbf{G}$ be a group, let $\mathbf{H}$ be a subgroup of $\mathbf{G}$, and let $\mathbf{N}$ be a normal subgroup of $\mathbf{G}$. Prove each of the following.

(a) NH is a subgroup of $\mathbf{G}$.
(b) N is a normal subgroup of NH.
(c) $N \cap H$ is a normal subgroup of $\mathbf{H}$.
(d) $\mathbf{H}/(N \cap H) \cong NH/N$.

Problem Set 12
Due 19 November 2014

Problem 51.
Let $\mathbf{G}$ be a group and let $\mathbf{H}$ and $\mathbf{K}$ be subgroups of $\mathbf{G}$. Prove $|HK||H \cap K| = |H||K|$. [Hint: Lagrange's Theorem or its proof helps. Warning: we do not assume that either of the subgroups is normal.]

Problem 52.
Let $\mathbf{G}$ be a group and let $\mathbf{H}$ be a subgroup of $\mathbf{G}$ so that $[\mathbf{G} : \mathbf{H}] = 2$. Prove that $\mathbf{H}$ is a normal subgroup of $\mathbf{G}$.

Problem 53.
Suppose that $\mathbf{N}$ is a normal subgroup of the group $\mathbf{G}$ and the $\mathbf{N}$ is a *finite* cyclic group. Prove that every subgroup of $\mathbf{N}$ is a normal subgroup of $\mathbf{G}$.

Problem 54.
Let $\mathbf{G}$ be a group with exactly 2 subgroups. Prove that $\mathbf{G} \cong \mathbb{Z}_p$ for some prime number p.

Problem Set 13

Due 24 November 2014

Problem 55.
Let **G** be a finite group and let **H** and **K** be subgroups of **G**. Suppose that $|H|$ and $|K|$ are relatively prime. Prove that $H \cap K = \{1\}$.

Problem 56.
Let **G** be a group and let $a \in G$. Define $F_a : G \to G$ via $F_a(x) = axa^{-1}$ for all $x \in G$. Prove that F_a is an isomorphism from **G** onto **G**.

Problem 57.
Let $\mathbf{A}, \mathbf{B}$, and **C** be rings. Let h be a homomorphism from **A** onto **B** and let g be a homomorphism from **A** onto **C** such that $\ker h = \ker g$. Prove that there is an isomorphism f from **B** onto **C**.

Problem 58.
Let **G** be a group and let H, K, and N be subgroups of **G** fulfilling all the following conditions:

- $H \subseteq K$,
- $HN = KN$,
- $H \cap N = K \cap N$.

Under these stipulations, prove that $H = K$.