Some Individual Project Possibilities

Construction with Straight-edge and Compass

Show how to construct a square on any line segment with straight-edge and compass and prove your construction actually does the job. Can you do the same for the regular pentagon?

The Irrationality of $\sqrt{2}$ Obtained Geometrically

We saw a couple of proofs that $\sqrt{2}$ is not a rational number. These proofs depended on some understanding of how addition and multiplication of natural numbers works and they used algebraic manipulations. However, this property of $\sqrt{2}$ was first discovered by the followers of Pythagoras, who had no such notions and methods available to them.

Of course, the $\sqrt{2}$ is the length of the diagonal of a square whose sides have length 1. The followers of Pythagoras called two line segments AB and CD commensurate if there was a (perhaps very short) line segment EF so that AB could be divided into some finite number of segments each the same length as EF and that the same could be done for CD. What the Pythagoreans showed was that the diagonal of a square and the side of the square can never be commensurate. Their proof was geometric (although I think no trace of their original writings about this survive).

The task of this project is to give such a geometric proof.

Construct the Familiar Number Systems

Build the integers from the natural numbers, the rationals from the integers, and finally the real numbers from the rationals. Along the way, prove at each step that addition is commutative. If you are ambitious, you might try to see how to extend the ordering $<$ from each stage to the next and to see how the properties of these orderings differ as you go along.

The Cartesian Connection

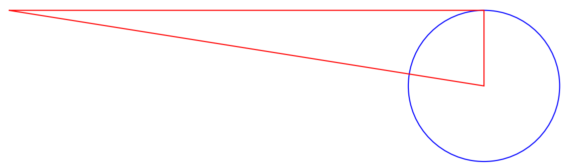
In a certain sense the algebra of the real numbers and the geometry of the Euclidean plane are really the same thing. The object of this project is to try to say exactly how this equivalence of ideas works. To succeed, you would have to writing down a list of all the basic principles that underly the algebra of real numbers (including how the ordering works) and also all the basic principles of Euclidean geometry. Then describe how one might translate back and forth between these two domains.

The Multiplication of Infinite Cardinals

Prove that for every infinite set A, we have $|A \times A| = |A|$. While this is the goal of the project, you might also provide a definition for what it means for a set to be infinite.

The Area of the Circle

In his famous work *On the measurement of the circle* Archimedes proved that the red triangle and the blue circle below have the same area,



where the red line segment at the top is tangent to the circle and has length equal to the circumference of the circle.

Give a proof of this.

Prime Numbers

In his *Elements* Euclid proved that there is no largest prime number. Give a proof of this and perhaps some other interesting facts about primes.

Comparing Cardinalities

Prove that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Finitely Branching Infinite Trees

Prove that every finitely branching infinite tree must have an infinite branch. Can you find an interesting consequence of this theorem¿

Transfinite Induction

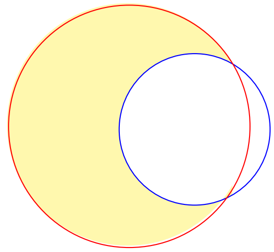
Find out about transfinite induction and well-ordered sets. Give an example of a theorem that can be proved by transfinite induction.

The Extreme Value Theorem

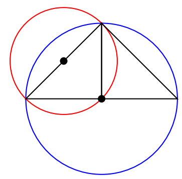
Prove the Extreme Value Theorem:
Every continuous function $f\colon [0,1] \to \mathbb{R}$ has an absolute maximum value and an absolute minimum value. Actually, I framed it here for the unit interval $0, 1]$ but it is true from any closed interval and even for other sets that are known as compact sets. To do this, you will have to refresh your understanding of what it means for a function to be continuous.

The Quadrature of a Lune

Hippocrates of Chios (not that early medical doctor) proposed a way to "square a lune" with straight-edge and compass. Hippocrates considered a very particular lune, not lunes in general. In general, a lune arises by consider two circles, as below:



The large moon-shaped region on the left with its outer edge red and its inner edge blue in a lune. Here is the lune that Hippocrates dealt with:



The two black dots are the centers of the circles and the lune is hanging off to the northwest. Notice the isosceles right triangles. Squaring the lune means constructing a square, using straight-edge and compass, which has the same area as the lune. Now Hippocrates probably did not have an altogether correct proof (ca. 440 BC). I would guess that Eudoxus obtained the first correct proof a generation later. Euclid knew one. Try your hand at a proof.