SAMPLE FINAL EXAMINATION
MATH 241 SECTION H02
SPRING SEMESTER 2020

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PROBLEM 0.
Let $\mathbf{u} = \langle 1,2,3\rangle, \mathbf{v} = \langle 3,2,1\rangle,$ and $\mathbf{w} = \langle -1,0,1\rangle$. Perform the calculation in each part below.

a. $2\mathbf{u} - 3\mathbf{v}$.
b. $\mathbf{u}\cdot\mathbf{v}$
c. $|\mathbf{w}|$
d. $\mathbf{u}\times\mathbf{v}$

PROBLEM 1.
In each part below let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in three dimensional space.

a. Explain why $|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2$.
b. Explain why $(2\mathbf{u}+3\mathbf{v})\cdot(\mathbf{u}\times\mathbf{v}) = 0$

PROBLEM 2 (CORE).

a. Suppose C is the curve on the plane described by $\mathbf{r}(t) = \sin t\mathbf{i} + 2\cos t\mathbf{j}$. Then the point $\langle\sqrt{2})/2, \sqrt{2}\rangle$ lies on the curve C. (What is an appropriate choice for t?). Find an equation that describes the line tangent to C at $\langle\sqrt{2})/2, \sqrt{2}\rangle$.
b. Find an equation that describes the plane which is determined by the points $\langle 0,2,1\rangle, \langle 0,2,-1\rangle,$ and $\langle 2,0,0\rangle$.

PROBLEM 3 (CORE).
Let $\mathbf{F}(t) = \sin t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}, \mathbf{G}(t) = t^2\mathbf{i} + \cos t\mathbf{j} + e^t\mathbf{k},$ and $h(t) = 4t$. Calculate the derivative each function below.

a. $h(t)\mathbf{F}(t)$.
b. $\mathbf{F}(h(t))\cdot\mathbf{G}(t)$.
c. $\mathbf{F}(t)\times\mathbf{G}(t)$.

PROBLEM 4 (CORE).
In each part below find an equation for the plane tangent to the surface it describes at the point given.

a. $x^2 + y^2 + z^2 = 3$ at $\langle 1,1,1\rangle$.
b. $z = \ln(x^2+y^2)$ at $\langle 1,0,0\rangle$.

SOLUTION
For part (a) let $f(x,y,z) = x^2 + y^2 + z^2$. Then $\nabla f(x,y.z) = \langle 2x,2y,2z\rangle$. So $\nabla f(1,1,1) = \langle 2,2,2\rangle$. Now observe that the point $\langle x,y,z\rangle$ lies on the tangent plane if and only if $\nabla f(1,1,1)\cdot(\langle x,y,z\rangle - \langle 1,1,1\rangle) = 0$; that is if and only if

$$\langle 2,2,2\rangle\cdot\langle x-1,y-1,z-1\rangle = 0.$$

So

$$2(x-1) + 2(y-1) + 2(z-1) = 0$$

is an equation for the tangent plane. This simplifies to

$$2x + 2y + 2z = 6.$$

For part (b) let $g(x, y, z) = \ln(x^2 + y^2) - z$. Then $\nabla g(x, y, z) = \langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}, -1 \rangle$. So $\nabla g(1, 0, 0) = \langle 2, 0, -1 \rangle$. As in part (a) this gives an equation for the tangent plane as follows:

$$\begin{aligned}
\nabla g(1, 0, -1) \cdot (\langle x, y, z \rangle - \langle 1, 0, 0 \rangle &= 0 \\
\langle 2, 0, -1 \rangle \cdot \langle x - 1, y, z \rangle &= 0 \\
2(x - 1) + 0y - z &= 0 \\
2x - z &= 2
\end{aligned}$$

S $x = 1$ is an equation for the tangent plain.

PROBLEM 5.

Use the Chain Rule to complete each part below.

a. Calculate $\dfrac{dw}{dt}$ at $t = 0$ where $w = x^2 + y^2, x = \cos t + \sin t,$ and $y = \cos t - \sin t$.

b. Express $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ as functions of r and θ where $z = 4e^x \ln y, x = \ln(r \cos\theta),$ and $y = r \sin\theta$.

SOLUTION

For part (a).

The Chain Rule tells of that

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Since $w = x^2 + y^2$ we see that $\frac{\partial w}{\partial x} = 2x$ and $\frac{\partial w}{\partial y} = 2y$. Given also the definitions of x and y above, we have

$$\frac{dx}{dt} = -\sin t + \cos t \text{ and } \frac{dy}{dt} = -\sin t - \cos t.$$

Putting all these together, we get

$$\begin{aligned}
\frac{dw}{dt} &= (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t) \\
\frac{dw}{dt} &= (2(\cos t + \sin t))(-\sin t + \cos t) + (2(\cos t - \sin t))(-\sin t - \cos t).
\end{aligned}$$

When we evaluate this at $t = 0$, we obtain

$$\frac{dw}{dt}|_{t=0} = (2)(1) + (2)(-1) = 0.$$

For part (b):

It helps at the outset to compute a number of partial derivatives and express them in terms of r and θ.

$$\begin{aligned}\frac{\partial z}{\partial x} &= 4e^x \ln y \\ &= 4e^{\ln(r\cos\theta)}\ln(r\sin\theta) \\ &= 4r\cos\theta \ln r\sin\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{4e^x}{y} \\ &= \frac{4e^{\ln r\cos\theta}}{r\sin\theta} \\ &= \frac{4r\cos\theta}{r\sin\theta} \\ &= 4\cot\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial x}{\partial r} &= \frac{\cos\theta}{r\cos\theta} \\ &= \frac{1}{r}\end{aligned}$$

$$\begin{aligned}\frac{\partial x}{\partial \theta} &= \frac{-r\sin\theta}{r\cos\theta} \\ &= -\tan\theta\end{aligned}$$

$$\frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

We have two tasks. Here is the first.

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= (4r\cos\theta\ln r\sin\theta)\frac{1}{r} + 4\cot\theta\sin\theta \\ &= 4\cos\theta\ln r\sin\theta + 4\cos\theta = 4\cos\theta(1+\ln r\sin\theta)\end{aligned}$$

And here is the second.

$$\begin{aligned}\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\ &= (4r\cos\theta\ln r\sin\theta)(-\tan\theta) + (4\cot\theta)(r\cos\theta) \\ &= -4r\sin\theta\cot\theta\ln r\sin\theta + 4r\cot\theta\cos\theta.\end{aligned}$$

PROBLEM 6.

Complete each part below.

a. Calculate the gradient of $f(x,y,z)=x^2+y^2-2z^2+z\ln x$ at $\langle 1,1,1\rangle$.

b. Calculate the derivative of $f(x,y,z)=3e^x\cos yz$ at $\langle 0,0,0\rangle$ in the direction of $\mathbf{v}=2\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

SOLUTION

For part (a):

$$\nabla f(x,y,z)=\langle 2x+\frac{z}{x},2y,-4z+\ln x\rangle.$$

Evaulating this at $\langle 1,1,1\rangle$ we get

$$\nabla f(1,1,1)=\langle 3,2,-4\rangle.$$

For part (b):

First, let us calculate the unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$. We see $|\mathbf{v}|=\sqrt{2^2+1^2+(-2)^2}=3$. So we get

$$\mathbf{u}=\frac{1}{3}\langle 2,1,-2\rangle=\langle\frac{2}{3},\frac{1}{3},\frac{-2}{3}\rangle.$$

We know that

$$\frac{df(0,0,0)}{d\mathbf{u}}=\nabla f(0,0,0)\cdot\mathbf{u}.$$

Calculating $\nabla f(x,y,z)$ we get $\langle 3e^x\cos xy,-3ze^z\sin yz\rangle,-3ye^x\sin yz$. Evaluating this at $\langle 0,0,0\rangle$ we find $\nabla f(0,0,0)=\langle 3,0,0\rangle$. Now plug this into the formula above for the directional derivative.

$$\frac{df(0,0,0)}{d\mathbf{u}}=\langle 3,0,0\rangle\cdot\langle\frac{2}{3},\frac{1}{3},\frac{-2}{3}\rangle=2.$$

PROBLEM 7 (CORE).

In each part below find all the local critical points. For each extreme point, determine whether it is a local minimum, a local maximum, or a saddle point.

a. $f(x,y)=xy-x^2-y^2-2x-2y+4$

b. $g(x,y)=\frac{1}{x}+xy+\frac{1}{y}$

SOLUTION

For part (a):

First we find the critical points. To do this we need that first partial derivatives.

$$f_x(x,y)=y-2x-2$$
$$f_y(x,y)=x-2y-2$$

Setting these to 0 we find

$$y=2(x+1)$$
$$x=2(y+1)$$

Subsituting the first into the second we get

$$
\begin{aligned}
x &= 2((2(x+1))+1) = 4x+6 \\
-6 &= 3x \\
-2 &= x
\end{aligned}
$$

But we know $y = 2x + 2$, so

$$-2 = y$$

So there is just one critical point and it is $\langle -2, -2 \rangle$.

To figure out what sort of critical point it is we need more partial derivatives.

$$
\begin{aligned}
f_{xx}(x,y) &= -2 \\
f_{yy}(x,y) &= -2 \\
f_{xy}(x,y) &= 1 \\
D(x,y) &= (-2)(-2) - 1 = 3 > 0
\end{aligned}
$$

Since $D(-2,-2) > 0$ and $f_{xx}(-2,-2) = -2 < 0$, we conclude that $\langle -2, -2 \rangle$ is a local maximum.

For part (b) we do a similar analysis.

$$
\begin{aligned}
g_x(x,y) &= -\frac{1}{x^2} + y \\
g_y(x,y) &= -\frac{1}{y^2} + x
\end{aligned}
$$

Setting these to 0 we find

$$
\begin{aligned}
y &= \frac{1}{x^2} \\
x &= \frac{1}{y^2}
\end{aligned}
$$

Substituting the first into the second we get

$$
\begin{aligned}
x &= \frac{1}{(\frac{1}{x^2})^2} = x^4 \\
0 &= x^4 - x = x(x^3 - 1)
\end{aligned}
$$

But $x = 0$, regardless of the value of y is not in the domain of g. So we can cancel an x

$$
\begin{aligned}
0 &= x^3 - 1 \\
1 &= x \\
1 &= y
\end{aligned}
$$

So again there is jst one critical point and it is $\langle 1, 1 \rangle$.

$$
\begin{aligned}
g_{xx}(x,y) &= \frac{2}{x^3} \\
g_{yy}(x,y) &= \frac{2}{y^3} \\
g_{xy}(x,y) &= 1 \\
D(x,y) &= \frac{4}{x^3y^3} - 1
\end{aligned}
$$

Since $D(1,1) = 4 - 1 = 3 > 0$ and $g_{xx}(1,1) = 2 > 0$, we conclude that $\langle 1,1 \rangle$ is a local minimum.

PROBLEM 8 (CORE).
Do each part below.

a. Evaluate $\iint\limits_D x + 2y\,dA$ where D is the region bounded by the parabolas described by $y = 2x^2$ and $y = 1 + x^2$.
b. Find the volume of the solid bounded by the plane described by $z = 0$ and the paraboloid described by $z = 1 - x^2 - y^2$.

PROBLEM 9 (CORE).
Do each part below.

a. Evaluate $\int\limits_C 2x + 9z\,ds$ where C is the curve parameterized by $x = t, y = t^2$, and $z = t^3$ for $0 \le t \le 1$.
b. Evaluate $\int_C e^x \sin y\,dx + e^x \cos y\,dy$, where C is any curve connecting $(0,0)$ to $(1, \pi/2)$.

PROBLEM 10.
Evaluate the integral in each part below.

a. $\oint_C 3y - e^{\sin x}\,dx + (7x + \sqrt{y^4+1})\,dy$ where C is the circle described by $x^2 + y^2 = 9$.
b. Let $\mathbf{F} = (x^2+y^2)\mathbf{i} + 2xy\mathbf{j}$ and let C be the boundary of the unit square with vertices at $(0,0), (1,0), (1,1)$, and $(0,1)$. Evaluate $\oint_C \mathbf{F} \cdot \mathbf{T}\,ds$.

PROBLEM 11.
Find the surface area of that part of the cylinder described by $y^2 + z^2 = 9$ that is directly over the rectangle in the XY-plane with vertices $(0,0), (2,0), (2,3)$, and $(0,3)$.