Explicit Construction of Small Folkman Graphs

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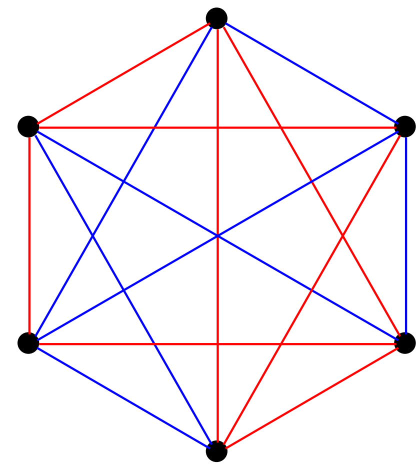
The 22nd Clemson Mini-Conference
on Discrete Mathematics and Algorithms

Ramsey number $R(3,3) = 6$

- If edges of K_6 are 2-colored then there exists a monochromatic triangle.



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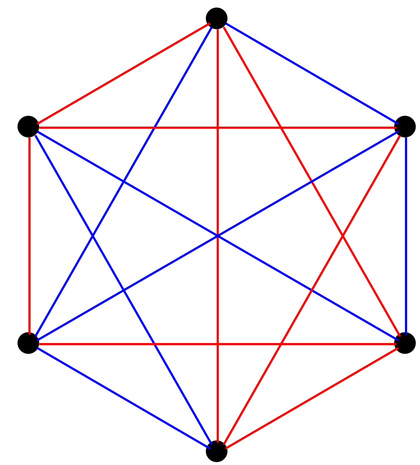
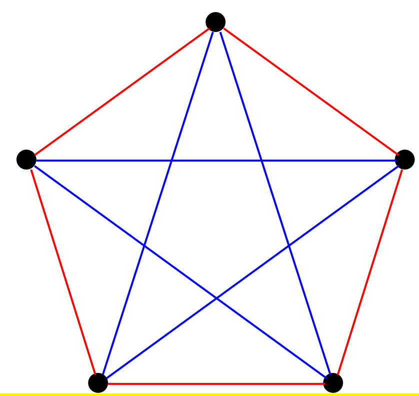
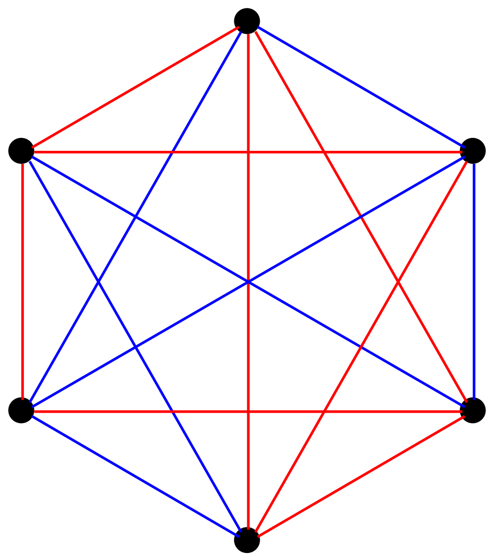


- There exists a 2-coloring of edges of K_5 with no monochromatic triangle.



Rado's arrow notation

$G \longrightarrow (H)$: if the edges of G are 2-colored then there exists a monochromatic subgraph of G isomorphic to H.



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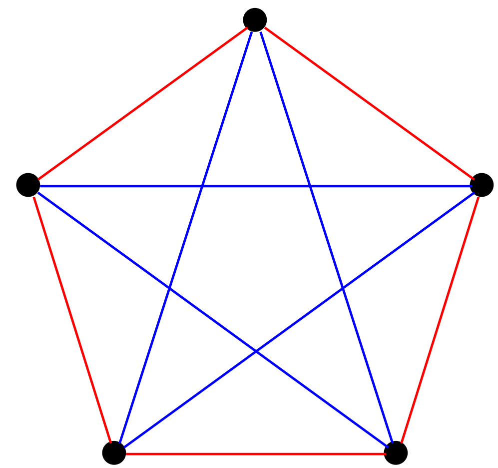

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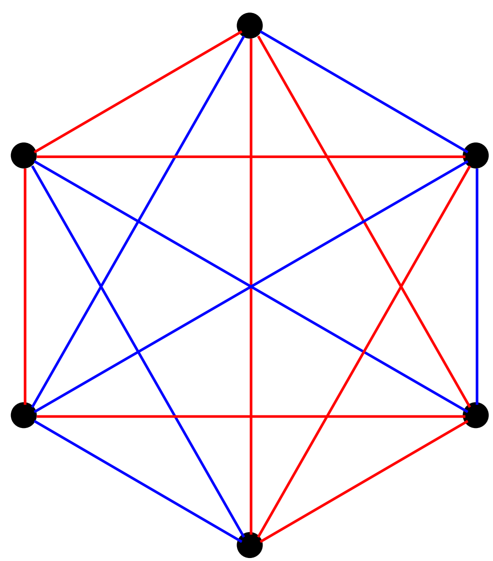

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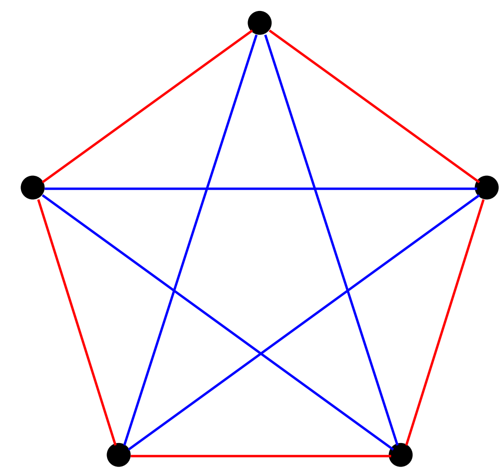

$K_5 \not\rightarrow (K_3)$

Fact: If $K_6 \subset G$, then $G \rightarrow (K_3)$.

A question of Erdős and Hajnal

Is there a K_6-free graph G with $G \longrightarrow (K_3)$?

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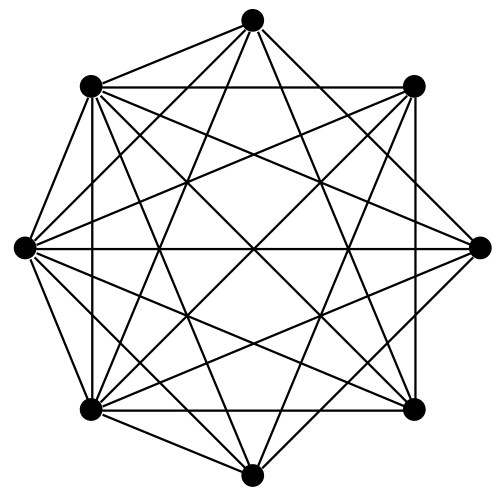
Graham (1968): Yes!



$K_8 \setminus C_5$

Graham's graph $K_8 \setminus C_5 = K_3 * C_5$

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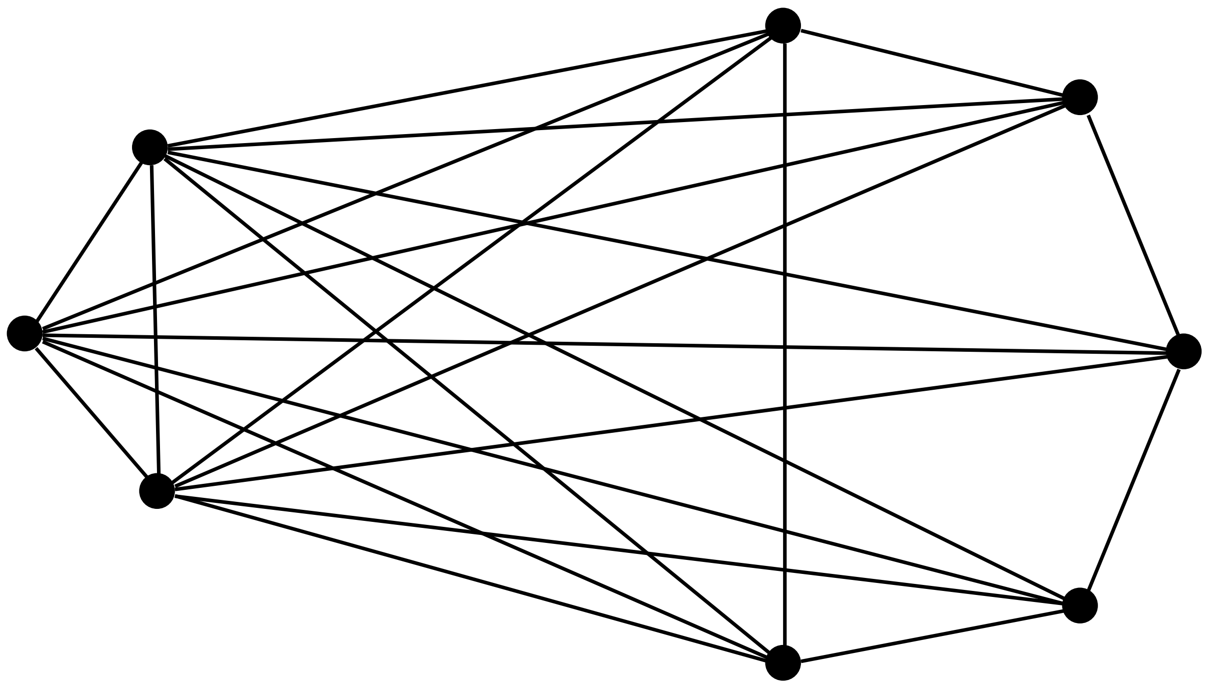

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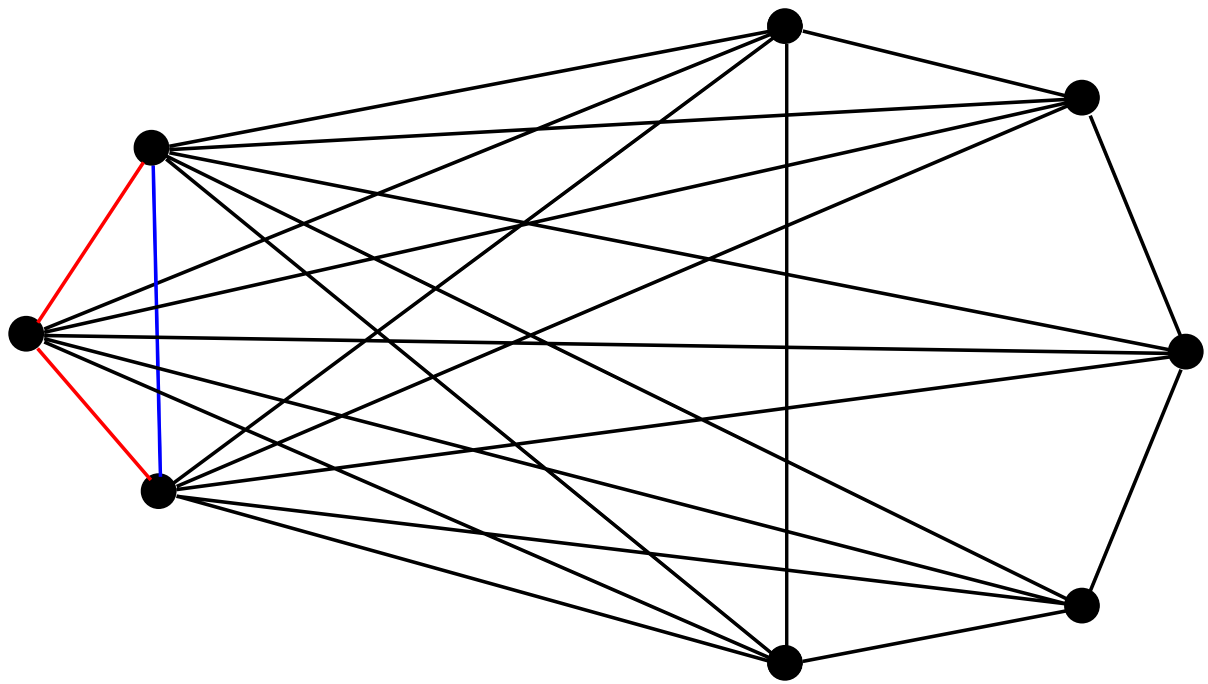

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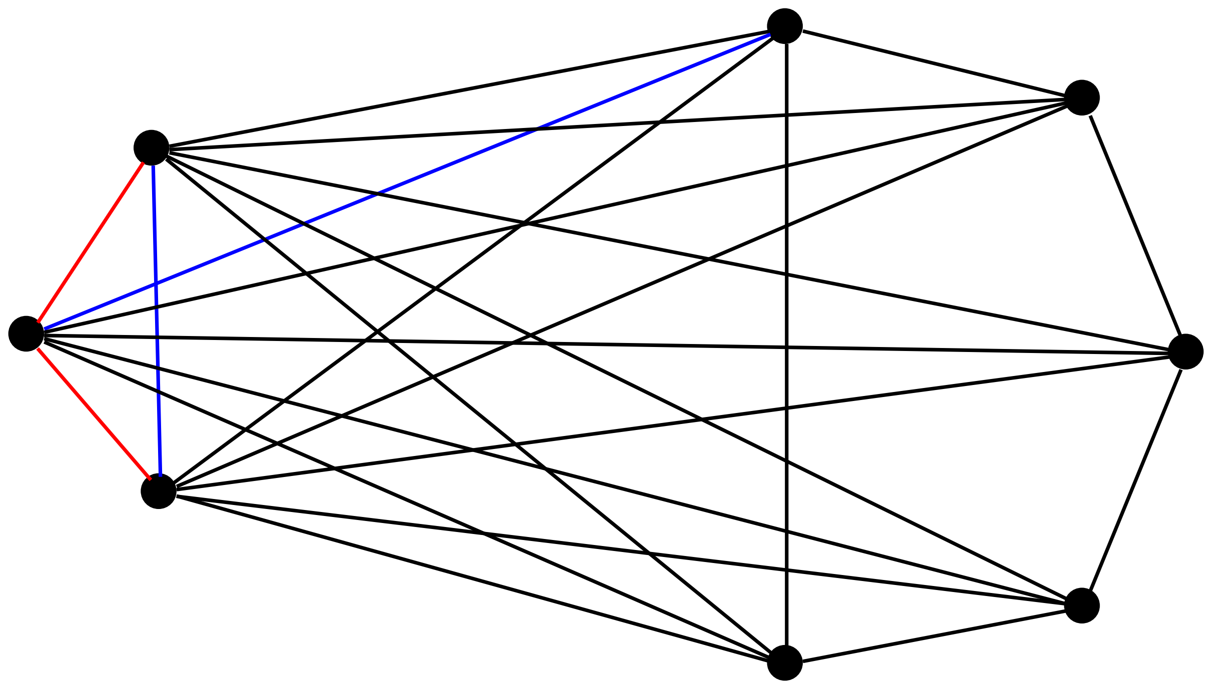

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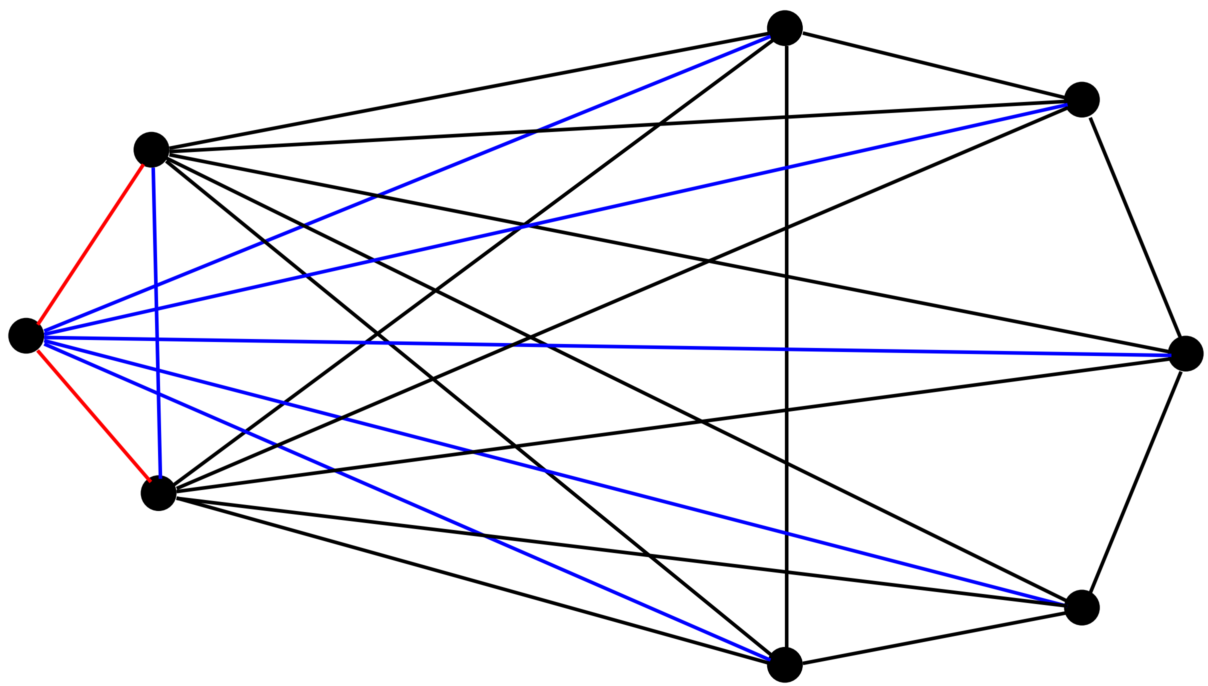

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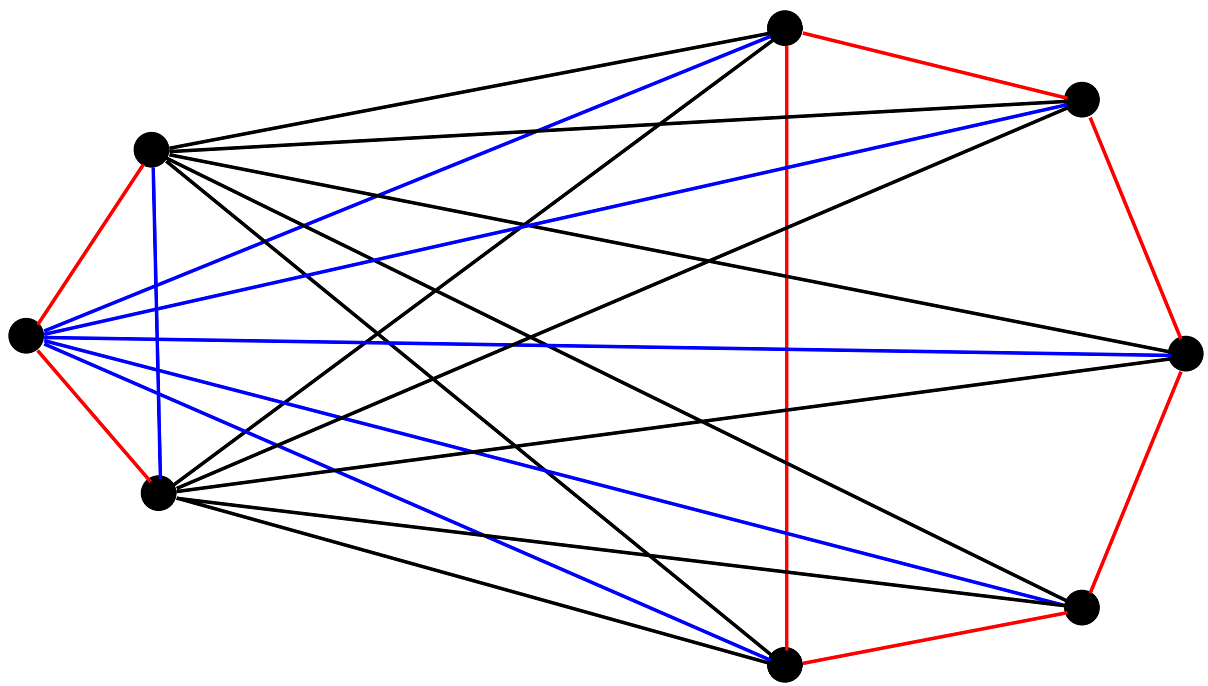

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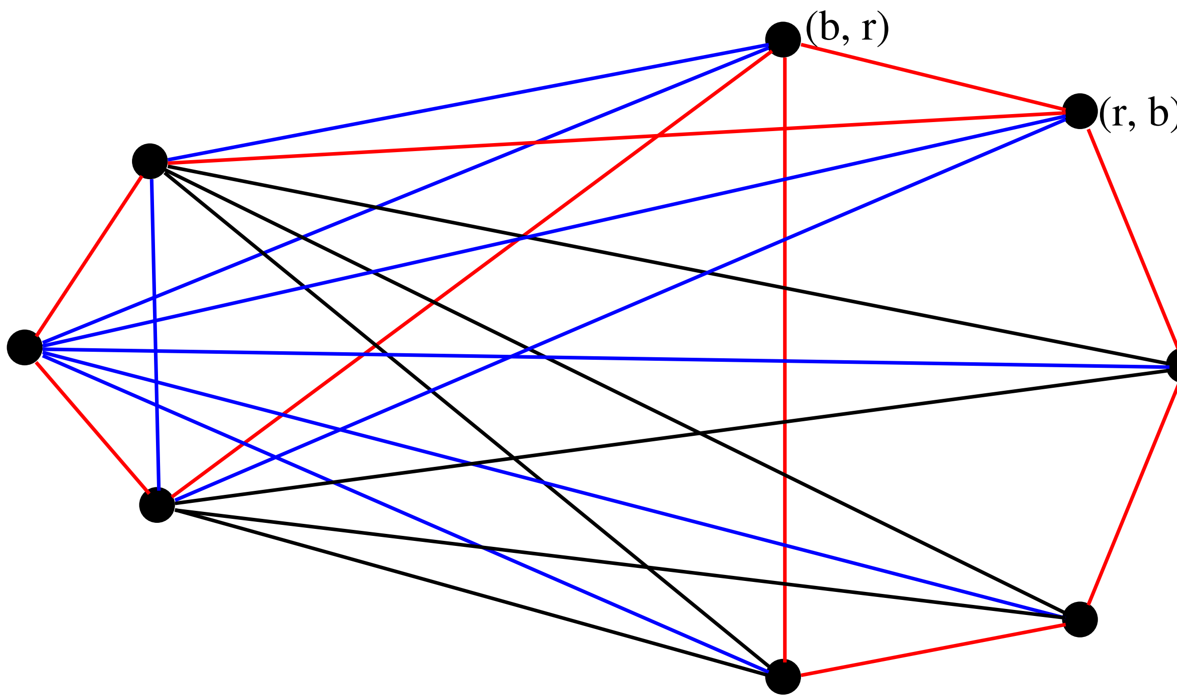

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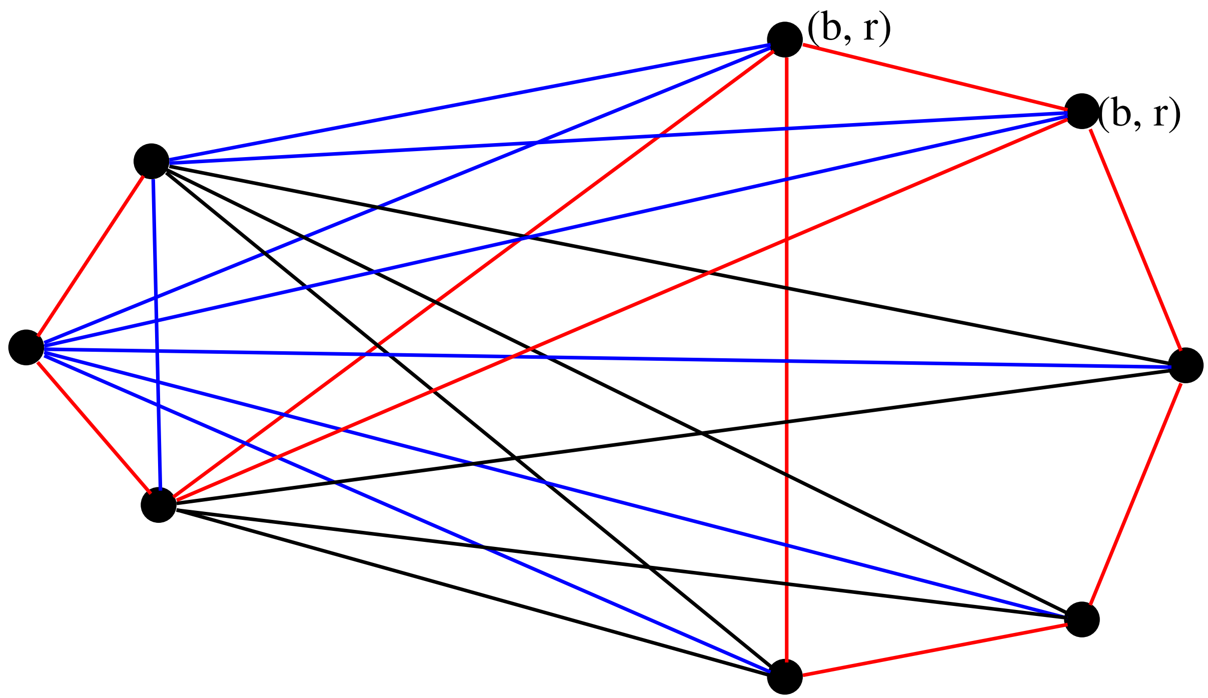

Label the vertices of C_5 by either (r, b) or (b, r).

A red triangle is unavoidable since $\chi(C_5) = 3$.

K_5-free graphs G with $G \to (K_3)$

Year	Authors	$\|G\|$
1969	Schäuble	42
1971	Graham, Spencer	23
1973	Irving	18
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In 1998, Piwakowski, Radziszowski and Urbański used a computer-aided exhaustive search to rule out all possible graphs on less than 15 vertices.

General results

Folkman's theorem (1970): *For any* $k_2 > k_1 \geq 3$*, there exists a* K_{k_2}*-free graph* G *with* $G \rightarrow (K_{k_1})$.

These graphs are called Folkman Graphs.

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Nešetřil-Rödl's theorem (1976): *For* $p \geq 2$ *and any* $k_2 > k_1 \geq 3$, *there exists a* K_{k_2}*-free graph* G *with* $G \to (K_{k_1})_p$.

Here $G \to (H)_p$: if the edges of G are p-colored then there exists a monochromatic subgraph of G isomorphic to H.

$f(p, k_1, k_2)$

Let $f(p, k_1, k_2)$ denote the smallest integer n such that there exists a K_{k_2}-free graph G on n vertices with $G \rightarrow (K_{k_1})_p$.

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$$f(2, 3, 6) = 8.$$

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- What about $f(2, 3, 4)$?

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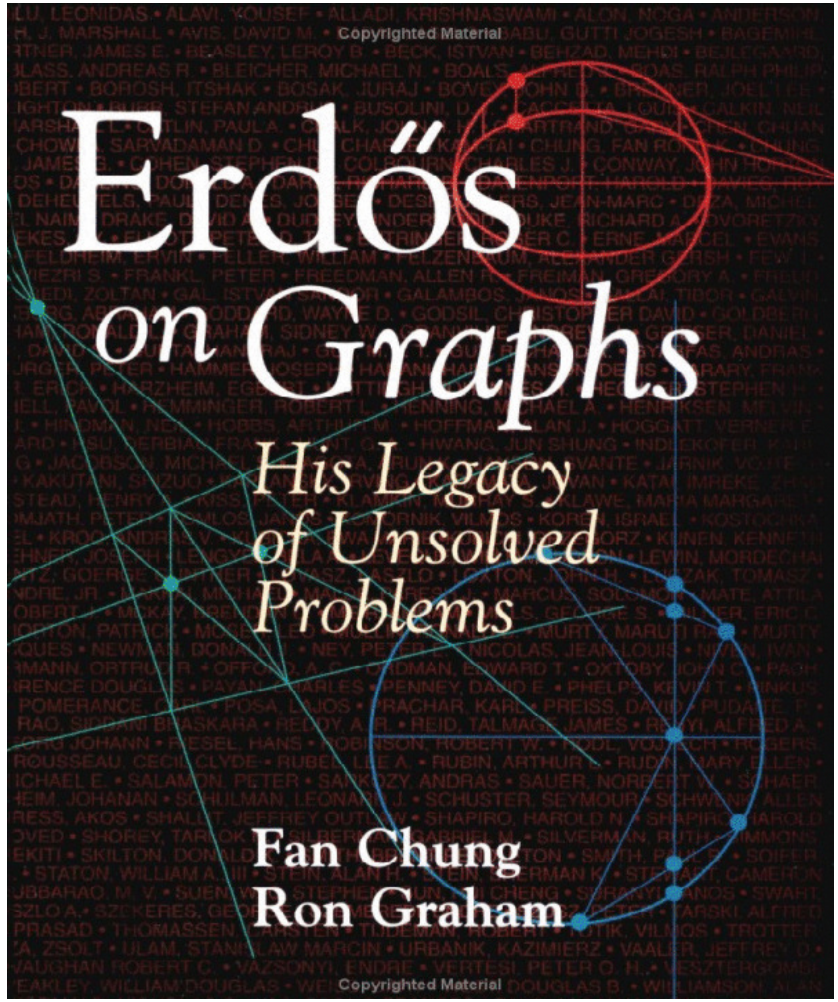
- **Erdős** re-set a prize of $100 for the new challenge

$$f(2,3,4) \leq 10^{6}.$$

The most wanted Folkman Graph



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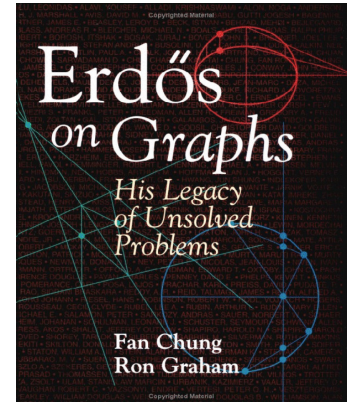

Problem on triangle-free subgraphs in graphs containing no K_4 $100
(proposed by Erdős)[48]
Let $f(p, k_1, k_2)$ denote the smallest integer n such that there is a graph G with n vertices satisfying the properties:
(1) any edge coloring in p colors contains a monochromatic K_{k_1};
(2) G contains no K_{k_2}.
Prove or disprove:

$$f(2,3,4) < 10^6.$$

Difficulty

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- For moderate n, Folkman graphs are very rare among all K_4-free graphs on n vertices.
- Probabilistic methods are generally good choices for asymptotic results. However, it is not good for moderate size n.

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- Localization and δ-fairness.
- Circulant graphs and $L(m, s)$.

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We claim the reward by proving

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Recent update: Dudek and Rödl (2008) proved $f(2,3,4) \leq 941$.

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For vertex transitive graph G, all G_v's are isomorphic.

Spectral lemma

- H: a graph on n vertices
- A: the adjacency matrix of H
- $\mathbf{d} = (d_1, d_2, \ldots, d_n)$: degrees of H
- $\mathrm{Vol}(S) = \sum_{v \in S} d_v$: the volume of S
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Lemma (Lu) *If the smallest eigenvalue of* $M = A - \frac{1}{\mathrm{Vol}(H)}\mathbf{d} \cdot \mathbf{d}'$ *is greater than* $-2\delta\bar{d}$, *then* H *is* δ*-fair.*

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Similar results hold for A and L. However, they are weaker than using M in experiments.

Corollary

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Proof: We can replace M by A in the previous lemma.

- $\mathbf{1}$ is an eigenvector of A with respect to d.
- M is the projection of A to the hyperspace $\mathbf{1}^{\perp}$.
- M and A have the same smallest eigenvalues.

The proof of the Lemma

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- We observe $M\mathbf{1} = 0$.
- For each $t \in (0,1)$, let $\alpha(t) = (1-t)\mathbf{1}_X - t\mathbf{1}_Y$. We have

$$\alpha(t)' \cdot M \cdot \alpha(t) = -e(X,Y) + \frac{1}{\mathrm{Vol}(H)}\mathrm{Vol}(X)\mathrm{Vol}(Y).$$

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Let ρ be the smallest eigenvalue of M. We have

$$e(X,Y) - \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(H)} \leq -\alpha(t)' \cdot M \cdot \alpha(t) \leq -\rho \|\alpha_t\|^2.$$

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$$\begin{aligned} e(X,Y) &\leq \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(H)} + \rho\frac{|X||Y|}{n}. \\ &\leq \frac{\mathrm{Vol}(H)}{4} - \rho\frac{n}{4} \\ &< (\frac{1}{2}+\delta)|E(H)|, \text{ since } \rho > -2\delta\bar{d}. \qquad \square \end{aligned}$$

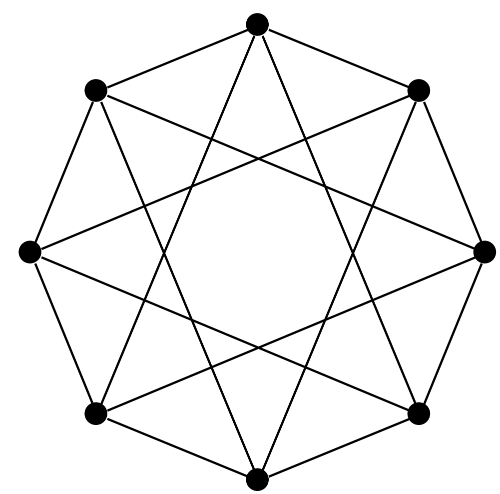
Circulant graphs

- $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$
- S: a subset of $\mathbb{Z}_n$ satisfying $-S = S$ and $0 \notin S$.

We define a circulant graph H by

- $V(H) = \mathbb{Z}_n$
- $E(H) = \{xy \mid x - y \in S\}$.

Example: A circulant graph with $n = 8$ and $S = \{\pm 1, \pm 3\}$.



Spectrum of circulant graphs

Lemma: *The eigenvalues of the adjacency matrix for the circulant graph generated by* $S \subset \mathbb{Z}_n$ *are*

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Proof: Note $A = g(J)$, where

$$g(x) = \sum_{s \in S} x^s.$$

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Proof continues...

Let $\phi = e^{\frac{2\pi\sqrt{-1}}{n}}$ denote the primitive n-th unit root.
J has eigenvalues

$$1, \phi, \phi^2, \ldots, \phi^{n-1}.$$

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Thus, the eigenvalues of $A = g(J)$ are

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For $i = 0, 1, 2, \ldots, n-1$, we have

$$g(\phi^i) = \Re(g(\phi^i)) = \sum_{s \in S} \cos \frac{2\pi i s}{n}.$$

$\square$

Graph $L(m, s)$

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Proposition: The local graph G_v of $L(m, s)$ is also a circulant graph.

Algorithm

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- If the ratio the smallest eigenvalue verse the largest eigenvalue of G_v is less than $-\frac{1}{3}$, reject it and try a new graph $L(m, s)$.
- Output a Folkman graph $L(m, s)$.

Computational results

$L(m,s)$	σ
$L(127,5)$	$-0.6363\cdots$
$L(761,3)$	$-0.5613\cdots$
$L(785,53)$	$-0.5404\cdots$
$L(941,12)$	$-0.5376\cdots$
$L(1777,53)$	$-0.5216\cdots$
$L(1801,125)$	$-0.4912\cdots$
$L(2641,2)$	$-0.4275\cdots$
$L(9697,4)$	$-0.3307\cdots$
$L(30193,53)$	$-0.3094\cdots$
$L(33121,2)$	$-0.2665\cdots$
$L(57401,7)$	$-0.3289\cdots$

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Computational results

$L(m,s)$	σ
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$L(761,3)$	$-0.5613\cdots$
$L(785,53)$	$-0.5404\cdots$
$L(941,12)$	$-0.5376\cdots$
$L(1777,53)$	$-0.5216\cdots$
$L(1801,125)$	$-0.4912\cdots$
$L(2641,2)$	$-0.4275\cdots$
$L(9697,4)$	$-0.3307\cdots$
$L(30193,53)$	$-0.3094\cdots$
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- Graphs in black are good candidates.

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- A new challenge: prove or disprove

$$f(2, 3, 4) \leq 100.$$