On a problem of Erdős and Lovász on coloring non-uniform hypergraphs

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Abstract

Let $f(r) = \min_H \sum_{F\in E(H)} \frac{1}{2^{|F|}}$, where H ranges over all 3-chromatic hypergraphs with minimum edge cardinality r. Erdős-Lovász (1975) conjectured $f(r) \to \infty$ as $r \to \infty$. This conjecture was proved by Beck in 1978. Here we show a new proof for this conjecture with a better lower bound:

$$f(r) \geq (\frac{1}{16} - o(1))\frac{\ln r}{\ln\ln r}.$$

1 Introduction

A hypergraph H consists of a vertex set $V(H)$ together with a family $E(H)$ of subsets of $V(H)$, which are called edges of H. A r-uniform hypergraph, or r-graph, is a hypergraph whose edges have the same cardinality r. A hypergraph H (not necessary uniform) is said to have *Property B* (or *2-colorable*) if there is a red-blue vertex-coloring of H with no monochromatic edges. Property B is named after Felix Bernstein [2] who first introduced this property in 1908. A proper k-coloring of a hypergraph H is a vertex-coloring using k-colors so that every edge has at least 2 colors. The chromatic number $\chi(H)$ is the minimum k such that H has a proper k-coloring. A hypergraph H with no Property B has chromatic number at least 3. The following problem on Property B (listed in [3]) is raised by Erdős [4] in 1963.

Problem on Property B: What is the minimum number of edges in an r-graph not having Property B?

In general, let $m_k(r)$ denote the smallest number of edges a $(k+1)$-chromatic r-graph can have. The above problem asks what $m_2(r)$ is.

The best upper bound known for $m_2(r)$ is due to Erdős [4, 5] and the best lower bound previous known is due to Radhakrishnan and Srinivasan [8].

$$0.7\sqrt{\frac{r}{\ln r}}2^r \leq m_2(r) \leq (1+\epsilon)\frac{2\ln 2}{4}r^2 2^r.$$

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To quote from Erdős and Lovász in their seminal paper [6]:

> Perhaps $r2^r$ is the correct order of magnitude of $m_2(r)$; it seems likely that
>
> $$\frac{m_2(r)}{2^r} \to \infty.$$
>
> A stronger conjecture would be: Let $E_{k=1}^m$ be a 3-chromatic (not necessarily uniform) hypergraph. Let
>
> $$f(r) = \min \sum_{k=1}^{m} \frac{1}{2^{|E_k|}},$$
>
> where the minimum is extended over all hypergraphs with $\min |E_k| = r$. We conjecture that $f(r) \to \infty$ as $r \to \infty$.

Erdős [4] showed in 1963 that $m_2(r) \geq 2^{r-1}$. Namely, color the vertices red and blue with equal probability and independently, and observe that the expected number of monochromatic edges is smaller than 1. In 1978, Beck [1] proved that $m_2(r) > r^{\frac{1}{3}-o(1)}2^r$. Spencer [9] simplified Beck's proof using probabilistic methods.

For non-uniform hypergraphs, Beck [1] proved that $f(r) \to \infty$ as $r \to \infty$ and thus settled the conjecture of Erdős and Lovász on non-uniform hypergraphs. Beck's lower bound of $f(r)$ is quite weak. Let $g_0(x) = x$, $g_k(x) = \log_2(g_{k-1}(x))$ for $k \geq 1$. For all $x > 0$, let $\log^*(x) = \min\{k \colon g_k(x) \leq 1\}$. Beck [1] proved

$$f(r) \geq \frac{\log^*(r) - 100}{7}.$$

The function $\log^*(x)$ grows very slowly since it can be viewed as the inverse function of the following tower function of height n

$$n \to \overbrace{2^{2^{\cdot^{\cdot^{2}}}}}^{n}.$$

In spite of several successful improvements of the lower bound of $m_2(r)$, the lower bound on $f(r)$ has not been improved for several decades. In this paper, we prove the following lower bound for $f(r)$.

Theorem 1 *For any $\epsilon > 0$, there is an $r_0 = r_0(\epsilon)$, for all $r > r_0$, we have*

$$f(r) \geq (\frac{1}{16} - \epsilon)\frac{\ln r}{\ln \ln r}. \tag{1}$$

An obvious upper bound for $f(r)$ is

$$f(r) \leq m_2(r)2^{-r} \leq (1+\epsilon)\frac{2\ln 2}{4}r^2.$$

The question whether $f(r) = m_2(r)2^{-r}$ remains open.

We organize the sections as follows. In section 2, we will examine twin-hypergraphs. We prove several useful lemmas and present a randomized algorithm for testing Property B of a twin-hypergraph. In section 3, we prove a theorem for non-uniform twin-hypergraphs, which implies Theorem 1.

2 Coloring twin-hypergraphs

A natural object for Property B is not a hypergraph but a pair of hypergraphs. We define a *twin-hypergraph* to be a pair of hypergraphs (H_1, H_2) with the same vertex set $V(H_1) = V(H_2)$. For any hypergraph G, we can view it as a special diagonal twin-hypergraph (G, G).

For the rest of this section, we write $H = (H_1, H_2)$ for the twin-hypergraph unless otherwise being specified. The common vertex set of H_1 and H_2 is denoted by $V(H)$.

We remark that the condition $V(H_1) = V(H_2)$ is only for convenience, but not essential. For any two hypergraphs H_1 and H_2, let $V = V(H_1) \cup V(H_2)$. By adding isolated vertices, if necessary, we can extend the hypergraph H_i (for $i = 1, 2$) over the vertex-set V so that (H_1, H_2) forms a twin-hypergraph.

A red-blue coloring C of a twin-hypergraph H is a map $C\colon V(H) \to \{red, blue\}$. For a fixed coloring C, an edge F is called *red* (or *blue*) if every vertex in F is red (or blue). A coloring C of a twin-hypergraph $H = (H_1, H_2)$ is called *proper* if H_1 has no red edge and H_2 has no blue edge. The twin-hypergraph H is called *to have Property B* (or *2-colorable*) if there exists a proper red-blue coloring.

We say $H = (H_1, H_2)$ is *trivial* if either H_1 or H_2 has no edges. In this case, H has property B since we can color all vertices in just one color.

2.1 Residue twin-hypergraphs

For a given coloring C of a twin-hypergraph H, a red (or blue) edge is an edge in which all vertices are red (or blue) respectively. Let $\mathcal{R}$ (depending on the coloring C) be the collection of red edges in H_1.

$$\mathcal{R} = \{F \in E(H_1) \mid F \text{ is red.}\}$$

Let R be the set of vertices lying in red edges of H_1.

$$R = \cup_{F \in \mathcal{R}} F.$$

Let E_R be the set of induced edges on R from $E(H_2)$ defined as:

$$E_R = \{F \cap R \mid F \in E(H_2), F \cap R \neq \emptyset, F \setminus R \text{ is blue}\}.$$

We define the *red residue* twin-hypergraph $R_C(H) = (H_1', H_2')$ as

$$V(H_1') = V(H_2') = R, \quad E(H_1') = \mathcal{R}, \quad \text{and} \quad E(H_2') = E_R.$$

Similarly, we define $\mathcal{B}$ be the collection of blue edges in H_2 and let B be the set of vertices lying in blue edges of H_2. We define E_B in the similar way:

$$E_B = \{F \cap B \mid F \in E(H_1), F \cap B \neq \emptyset, F \setminus B \text{ is red}\}.$$

Finally, we define the *blue residue* twin-hypergraph $B_C(H) = (H_1'', H_2'')$ as

$$V(H_1'') = V(H_2'') = B, \quad E(H_1'') = E_B, \quad \text{and} \quad E(H_2'') = \mathcal{B}.$$

The following lemma is the basis of the recoloring method.

Lemma 1 *For any coloring C, the twin-hypergraph H has Property B if both $R_C(H)$ and $B_C(H)$ have Property B.*

Proof: Suppose $R_C(H)$ and $B_C(H)$ have Property B. There is a proper red-blue vertex-coloring C_1 (and C_2) of $R_C(H)$ (and of $B_C(H)$ respectively). Now, we recolor the vertices in R using coloring C_1 and the vertices in B using coloring C_2. If a vertex v is neither in R nor in B, we keep the same color of v as in the coloring C. The new coloring is denoted by C'.
Claim: C' is a proper coloring of H.

We will prove the claim by contradiction. Suppose an edge F of H becomes monochromatic in C'. Note that the statement is symmetric for the red color and the blue color. Without loss of generality, we assume $F \in E(H_1)$ becomes a red edge in C'.

Case A: $F \cap B = \emptyset$. No vertex in F has been changed from blue to red. The edge F is red in C. Namely, $F \in \mathcal{R}$ and F remains red in C'. Contradiction to the assumption that C_1 is a proper coloring of $R_C(H)$.

Case B: $F \cap B \neq \emptyset$. Only vertices in B can change its color from blue to red. For each vertex v in $F \setminus B$, v is red in C. By definition, we have $F \cap B \in E_B$. Now, $F \cap B$ becomes a red edge in $B_C(H)$ after recoloring. Contradiction to the assumption that C_2 is a proper coloring of $B_C(H)$.

The proof of the claim is finished. $\square$

2.2 Lemma on reduction

A twin-hypergraph $H = (H_1, H_2)$ is called *irreducible* if

1. For any edge $F_1 \in E(H_1)$ and any vertex $v \in F_1$ there exists an edge $F_2 \in E(H_2)$ such that $F_1 \cap F_2 = \{v\}$.

2. For any edge $F_2 \in E(H_2)$ and any vertex $v \in F_2$ there exists an edge $F_1 \in E(H_1)$ such that $F_1 \cap F_2 = \{v\}$.

Otherwise, we say H is *reducible*.

A *sub-twin-hypergraph* of $H = (H_1, H_2)$ is a twin-hypergraph $H' = (H_1', H_2')$ such that H_i' is a sub-hypergraph of H_i' for $i = 1, 2$.

If $H = (H_1, H_2)$ is reducible, then there exist an edge $F \in E(H_i)$ (i is either 1 or 2) and a vertex $v \in F$ satisfying for any $F' \in E(H_{3-i})$, if $v \in F'$ then $|F \cap F'| \geq 2$. Removing F from H_i we get a twin-hypergraph with one edge less. Repeat this process until an irreducible twin-hypergraph is reached. We get a sequence of twin-hypergraphs:

$$H = H^{(0)} \supset H^{(1)} \supset \cdots \supset H^{(s)}.$$

The last twin-hypergraph $H^{(s)}$ is irreducible.

Lemma 2 *In the above process, $H^{(s)}$ is unique and does not depend on the order of edges being removed.*

Proof: We define the union of two twin-hypergraphs (H_1, H_2) and (H_1', H_2') to be a twin-hypergraph $(H_1 \cup H_1', H_2 \cup H_2')$. Let $G = (G_1, G_2)$ be the union of all irreducible sub-twin-hypergraphs. We observe that irreducibility is closed under union. G is also irreducible. In fact, it is the unique maximum irreducible sub-twin-hypergraph of H.

We will use induction to show $G \subseteq H^{(j)}$ for all $j = 0, \ldots, s$.

For $j = 0$, it is trivial since

$$G \subseteq H = H^{(0)}.$$

Now we assume that $G \subseteq H^{(j)}$. Write $G = (G_1, G_2)$ and $H^{(j)} = (H_1^{(j)}, H_2^{(j)})$. For $j+1$, $H^{(j+1)}$ is obtained by removing an edge F from $H^{(j)}$. Suppose that the removed edge F is in $H_i^{(j)}$ for some $i = 1$ or 2. We have

$$H_i^{(j+1)} = H_i^{(j)} \setminus \{F\} \quad \text{and } H_{3-i}^{(j+1)} = H_{3-i}^{(j)}.$$

By inductive hypothesis, we have

$$G_i \subseteq H_i^{(j)} \quad \text{and} \quad G_{3-i} \subseteq H_{3-i}^{(j)} = H_{3-i}^{(j+1)}.$$

Since F is removed, there is an $v \in F$ satisfying

$$|F \cap F'| \geq 2 \quad \text{for all } F' \in E(H_{3-i}^{(j)}) \text{ such that } v \in F'.$$

We have

$$|F \cap F'| \geq 2 \quad \text{for all } F' \in E(G_{3-i}) \text{ such that } v \in F'.$$

Since G is irreducible, we must have $F \notin E(G_i)$. It implies $G_i \subseteq H_i^{(j+1)}$. Together with $G_{3-i} \subseteq H_{3-i}^{(j)} = H_{3-i}^{(j+1)}$, we finish the induction step $G \subseteq H^{(j+1)}$.

We have

$$G \subseteq H^{(s)}.$$

Since $H^{(s)}$ is irreducible and G is maximum, we also have

$$H_1^{(s)} \subseteq G.$$

We get

$$H^{(s)} = G.$$

as claimed. $\square$

Remark: Such a unique H^s is called the *irreducible core* of H. It is the maximum irreducible sub-twin-hypergraph of H. Lemma 2 shows that the irreducible core can be computed by a simple deterministic greedy algorithm.

If H has Property B, then any sub-twin-hypergraph has Property B. In particular, its irreducible core has Property B. The following lemma shows the reverse statement is also true.

Lemma 3 *A twin-hypergraph H has Property B if and only if its irreducible core has Property B.*

Proof: Suppose the irreducible core $G = (G_1, G_2)$ has Property B. There is a red-blue coloring of $V(G_1)$ with no red edge in G_1 and no blue edge in G_2. We extend the coloring of $V(G_1)$ to a coloring of $V(H_1)$ in an arbitrary way. (For example, we could color every vertex not in $V(G_1)$ red.)

Recall that the irreducible core can be constructed by removing one edge at a time. We have the following chain of twin-hypergraphs.

$$H = H^{(0)} \supset H^{(1)} \supset \cdots \supset H^{(s)} = G.$$

We would like to use the induction to show $H^{(j)}$ has Property B for all $j = s, s-1, \ldots, 1, 0$. For $j = s$, it is true by the assumption.

Suppose $H^{(j)} = (H_1^{(j)}, H_2^{(j)})$ has Property B. Let C_j be a proper coloring of $H^{(j)}$. We would like to show $H^{(j-1)} = (H_1^{(j-1)}, H_2^{(j-1)})$ also has Property B.

Suppose that $F \in E(H_i^{(j-1)})$ (for some $i = 1$ or 2) is the edge being removed at step j. Without loss of generality, we assume $i = 1$. We have

$$H_1^{(j-1)} \setminus \{F\} = H_1^{(j)} \quad \text{and} \quad H_2^{(j-1)} = H_2^{(j)}.$$

If F is not completely red, then the coloring C_j is a proper coloring of $H^{(j-1)}$. Otherwise, there is a vertex $v \in F$ satisfying

$$|F \cap F'| \geq 2 \quad \text{for all } F' \in E(H_2^{(j)}) \text{ such that } v \in F'.$$

Change the color of v to blue. The action doesn't create new red edge in $H_1^{(j)}$. It can only affect edges in $H_2^{(j)}$, which contains the vertex v. For such an edge F', F' can not be completely blue since it contains another red vertex $u \in F$. The induction step is finished.

H has Property B. The proof of this lemma is finished. $\square$

2.3 Randomized testing algorithm

The following randomized algorithm tests whether a given twin-hypergraph H has a property B. We pre-assume an early termination condition, which will be specified later in the proof of Theorem 2.

Randomized Testing Algorithm:
Input: a twin-hypergraph $H = (H_1, H_2)$.
Output: "The program succeeds." or "The program fails.".

1. Randomly color each vertex of H in red with probability $\frac{1}{2}$ and in blue with probability $\frac{1}{2}$ independently. Test the early termination condition. If the early termination condition is satisfied, output "The program fails." and quit the program; Otherwise, compute two residue twin-hypergraphs $R_C(H)$ and $B_C(H)$.

2. In this step, we test whether $R_C(H)$ has Property B.

   Recall that $\mathcal{R}$ is the collection of red edges of H_1 in the coloring C. We partition $\mathcal{R}$ into many classes so that the sizes of edges in the same class varies by at most a factor of 2. This is done by introducing the *rank* of an edge. An edge F is said to have *rank* i if $r2^{i-1} \leq |F| < r2^i$. In another word, the rank of an edge F is just $\lfloor \log_2 \frac{|F|}{r} \rfloor$.

   The red edges of higher ranks are attempted to be destroyed first. To attempt to destroy all red edges of rank i, we flip the color of each vertex lying in edges of rank i independently with probability $\frac{q}{r2^{i-1}}$. Here are the details.

   Let $i_1 > i_2 > \cdots > i_s$ be the collection of ranks of edges in $\mathcal{R}$. Let $q = \Theta(\ln h)$ be a fixed parameter. Let $H^{(0)}$ be the irreducible core of $H^{C,R}$ and let $C^{(0)}$ be the coloring of $H^{(0)}$ which colors each vertex in red.

   For $j = 1, \ldots, s$, we construct a new twin-hypergraph $H^{(j)}$ and a new coloring $C^{(j)}$ from $H^{(j-1)}$ and $C^{(j-1)}$ as follows. For any vertex v, which lies in red edges of rank i_j in $C^{(j-1)}$, we recolor v into blue with probability $p_j = \frac{q}{r2^{i_j-1}}$ independently. (Note that $r2^{i_j-1}$ is the smallest possible size of an edge of rank i_j can have.) Output "The program fails." and quit if a red edge F of rank i_j survives or a new blue edge is created in $C^{(j)}$. Otherwise let $H^{(j)}$ be the irreducible core of the red residue $R_{C_j}(H^{(j-1)})$ and continue the loop.

3. Test whether $B_C(H)$ has Property B in a similar way.

4. Output "The program succeeds." and exit the algorithm.

If H has a property B, this algorithm might output "The program fails". However, if it outputs "The program succeeds.", then H must have property B. The correctness of this randomized algorithm is a direct consequence of Lemma 1 and Lemma 3.

3 Non-uniform twin-hypergraphs

It suffices to prove the following theorem.

Theorem 2 *Suppose a twin-hypergraph $H = (H_1, H_2)$ with minimum edge-cardinality r satisfies*

$$\sum_{F \in E(H_i)} \frac{1}{2^{|F|}} \leq (\frac{1}{16} - o(1)) \frac{\ln r}{\ln \ln r} \tag{2}$$

for $i = 1, 2$. Then H has property B.

Proof: For $i = 1, 2$, let $h_i = \sum_{F \in E(H_i)} \frac{1}{2^{|F|}}$ and $h = \max\{h_1, h_2\}$. Consider the randomized testing algorithm in section 2 and an early termination condition defined later. At the first step, we can define the following random variables for the random coloring C.

1. For any integer $i \geq r$, let $X^{(i)}$ be the number of pair (v, F) satisfying

   $$\text{“}v \in F, \quad F \in E(H_1), \quad |F| = i, \quad \text{and } F \setminus \{v\} \text{ is red.''}$$

   An edge $F \in E(H_1)$ with $|F| = i$ can contribute to X_i by i if F is completely red and by 1 if F has one blue vertex and $i - 1$ red vertices. In particular,

   $$\mathrm{E}(X^{(i)}) = \sum_{\substack{F \in E(H_1) \\ |F| = i}} \frac{2i}{2^i}. \tag{3}$$

   Let $X = \sum_{i \geq r} \frac{X^{(i)}}{i}$. We have

   $$\mathrm{E}(X) = \sum_{F \in E(H_1)} \frac{2}{2^{|F|}} = 2h_1. \tag{4}$$

2. For any integer $i \geq r$, let $Y^{(i)}$ be the number of pair (v, F) satisfying

   $$\text{“}v \in F, \quad F \in E(H_2), \quad |F| = i, \quad \text{and } F \setminus \{v\} \text{ is blue.''}$$

   Let $Y = \sum_{i \geq r} \frac{Y^{(i)}}{i}$. Similarly, we have

   $$\mathrm{E}(Y) = 2h_2. \tag{5}$$

For a fixed constant $M > 2$, let A be the event that

$$\text{“}X \leq 2Mh_1 \quad \text{and} \quad Y \leq 2Mh_2.\text{''}$$

Choose the early termination condition to be “the event A is not satisfied”. Using Markov's inequality, we have

$$\Pr(X > 2Mh_1) \; < \; \frac{\mathrm{E}(X)}{2Mh_1} = \frac{1}{M}, \tag{6}$$

$$\Pr(Y > 2Mh_2) \; < \; \frac{\mathrm{E}(Y)}{2Mh_2} = \frac{1}{M}. \tag{7}$$

In particular, we have

$$\Pr(A) > 1 - \frac{2}{M}.$$

The randomized algorithm outputs "The program fails." during testing $R_C(H)$ if

1. A red edge $F \in \mathcal{R}$ survives after the vertices lying in edges with the same rank of F have been recolored.
2. There is an edge $\{v_1, \ldots, v_k\}$ of $R_C(H)_2$ so that vertices $v_1, \ldots, v_k$ are sequentially recolored into blue.

When estimating the probability of first type event, we discard the probability that an red edge F is destroyed before or after edges of the same rank of F are attempted to be destroyed. The probability of first type event is at most

$$\begin{aligned}\sum_{F \in E(H_1)} \frac{1}{2^{|F|}}\left(1 - \frac{q}{r2^{\lfloor \log_2 \frac{|F|}{r} \rfloor}}\right)^{|F|} &\leq \sum_{F \in E(H_1)} \frac{1}{2^{|F|}}\left(1 - \frac{q}{|F|}\right)^{|F|} \\ &\leq \sum_{F \in E(H_1)} \frac{1}{2^{|F|}} e^{-q} \\ &= h_1 e^{-q}. \end{aligned} \tag{8}$$

If the second type event occurs, there exist an edge $F' \in E(H_2)$, and a subset $S \subset F'$, satisfying

1. Vertices in S are red in the coloring C while vertices in $F' \setminus S$ are blue.
2. All vertices in S are changed into blue eventually.
3. For each $v \in S$, there exists a red edge F_v in C so that $F_v \cap F' = \{v\}$ because of the irreducibility. Moreover, F_v survives until v is recolored into blue. In this case, we say v *blames* F_v.

Suppose v blames an edge F_v with rank i_v. The change of color of v might have occurred earlier, when red edges with higher rank $s \geq i_v$ were being recolored.

$$\begin{aligned}\Pr(v \text{ blames } F_v) &< \sum_{s=i_v}^{\infty} \frac{q}{r2^{s-1}} \\ &= \frac{4q}{r2^{i_v}} \\ &< \frac{4q}{|F_v|}. \end{aligned} \tag{9}$$

Let $\mathcal{F}_v$ be the set of red edges to whom v can blame:

$$\mathcal{F}_v = \{F \in E(H_1) | F \cap F' = \{v\}, F \setminus \{v\} \text{ is red.}\}.$$

We have

$$\begin{aligned}
\Pr(v \text{ is recolored into blue}) &\leq \sum_{F \in \mathcal{F}_v} \Pr(v \text{ blames } F) \\
&\leq \sum_{F \in \mathcal{F}_v} \frac{4q}{|F_v|}. \qquad (10)
\end{aligned}$$

During recoloring process, "a set S is completely recolored into blue" means there exists a sequence of pairs $(v_1, F_1), \ldots, (v_k, F_k)$ such that

1. The sequence $v_1, v_2, \ldots, v_k$ is the ordering of of vertices in S being recolored. Here $k = |S|$.
2. For $1 \leq i \leq k$, v_i blames F_i.

For $1 \leq i \leq j \leq k$, the vertex v_i is recolored into blue before the vertex v_j is recolored. The rank of F_i is higher than or equal to the rank of F_j. During recoloring process of a fixed rank, the events that different vertices are recolored into blue are independent. For recoloring processes of two different ranks, the recoloring process of higher rank might reduce the possible choices of red edges, whom might be blamed later during the recoloring process of lower rank. It actually reduces the probability that other vertices in S are recolored later. In other words, for any $1 \leq j \leq k$,

$$\Pr(v_j \text{ blames } F_j \mid \wedge_{1 \leq i \leq j-1}(v_i \text{ blames } F_i)) \leq \Pr(v_j \text{ blames } F_j). \qquad (11)$$

We have

$$\begin{aligned}
\Pr(\wedge_{1 \leq j \leq k}(v_j \text{ blames } F_j)) &= \prod_{j=1}^{k} \Pr(v_j \text{ blames } F_j \mid \wedge_{1 \leq i \leq j-1}(v_i \text{ blames } F_i)) \\
&\leq \prod_{j=1}^{k} \Pr(v_j \text{ blames } F_j). \qquad (12)
\end{aligned}$$

We have

$$\begin{aligned}
\Pr(S \text{ is completely recolored into blue}) &\leq \sum_{(v_1,F_1),\ldots,(v_k,F_k)} \Pr(\wedge_{1 \leq j \leq k}(v_j \text{ blames } F_j)) \\
&\leq \sum_{(v_1,F_1),\ldots,(v_k,F_k)} \prod_{j=1}^{k} \Pr(v_j \text{ blames } F_j) \\
&\leq \prod_{j=1}^{k} \sum_{(v_j,F_j)} \Pr(v_j \text{ blames } F_j) \\
&\leq \prod_{j=1}^{k} \sum_{F \in \mathcal{F}_{v_j}} \frac{4q}{|F|} \\
&= \prod_{v \in S} \sum_{F \in \mathcal{F}_v} \frac{4q}{|F|}. \qquad (13)
\end{aligned}$$

Summing up for all non-empty subset S of F', we have

$$\Pr(\text{the second type event} \mid \text{ given coloring } C) \leq \sum_{\substack{S\subset F'\\ S\neq\emptyset}} \prod_{v\in S} \sum_{F\in\mathcal{F}_v} \frac{4q}{|F|}. \tag{14}$$

We denote the above sum by Z.

$$\begin{aligned} Z &= \sum_{\substack{S\subset F'\\ S\neq\emptyset}} \prod_{v\in S} \sum_{F\in\mathcal{F}_v} \frac{4q}{|F|} \\ &= \prod_{v\in F'} \left(1 + \sum_{F\in\mathcal{F}_v} \frac{4q}{|F|}\right) - 1 \\ &\leq \prod_{v\in F'} e^{\sum_{F\in\mathcal{F}_v} \frac{4q}{|F|}} - 1 \\ &= e^{\sum_{v\in F'} \sum_{F\in\mathcal{F}_v} \frac{4q}{|F|}} - 1. \end{aligned} \tag{15}$$

Let $X^{(i)}_{F'}$ be the number of pairs (v, F) satisfying $F \in E(H_1)$, $|F| = i$, $F \cap F' = \{v\}$ and $F \setminus \{v\}$ is red. Let $X_{F'} = \sum_{i\geq r} X^{(i)}_{F'}$. We have

$$\sum_{v\in F'} \sum_{F\in\mathcal{F}_v} \frac{1}{|F|} = \sum_{i\geq r} \frac{X^{(i)}_{F'}}{i} = X_{F'}. \tag{16}$$

Combining equation (16) and inequality (15), we have

$$Z \leq e^{4qX_{F'}} - 1. \tag{17}$$

Now we will bound $X_{F'}$.

For any edge $F \in E(H_1)$ with $|F \cap F'| = 1$, the probability that all vertices in $F \setminus F'$ are red is exactly $\frac{1}{2^{|F|-1}}$.

$$\mathrm{E}(\sum_{i\geq r} X^{(i)}) = \sum_{\substack{F\in E(H_1)\\ |F\cap F'|=1}} \frac{1}{2^{|F|-1}} \leq 2h_1. \tag{18}$$

In particular, we have

$$\begin{aligned} \mathrm{E}(X_{F'}) &= \mathrm{E}(\sum_{i\geq r} \frac{X^{(i)}_{F'}}{i}) \\ &\leq \frac{1}{r}\mathrm{E}(\sum_{i\geq r} X^{(i)}_{F'}) \\ &\leq \frac{2h_1}{r}. \end{aligned} \tag{19}$$

By Markov's inequality, we have

$$\Pr(X_{F'} \geq \lambda) \leq \frac{\mathrm{E}(X_{F'})}{\lambda} \leq \frac{2h_1}{r\lambda}. \tag{20}$$

The probabilistic upper-bound is quite good for each $X_{F'}$. However, it does not holds uniformly over all F''s. Here we need a uniform bound. Namely, we wish to upper bound $\Pr(\cup_{F'\in E(H_2)}(X_{F'} \geq \lambda))$. Comparing the definition, $X^{(i)}$ only differs from $X^{(i)}_{F'}$ by dropping the requirement $F \cap F' = \{v\}$. We have $X^{(i)}_{F'} \leq X^{(i)}$ and

$$X_{F'} = \sum_{i\geq r} \frac{X^{(i)}_{F'}}{i} \leq \sum_{i\geq r} \frac{X^{(i)}}{i} = X. \tag{21}$$

We have

$$\begin{aligned}
\Pr(\cup_{F'\in E(H_2)}(X_{F'} \geq 2Mh_1)) &\leq \Pr(X \geq 2Mh_1) \\
&\leq \frac{\mathrm{E}(X)}{2Mh_1} \\
&= \frac{2h_1}{2Mh_1} \\
&= \frac{1}{M}. \qquad (22)
\end{aligned}$$

In another word, with probability at least $1 - \frac{1}{M}$, $X_{F'} \leq 2Mh_1$ holds uniformly for all $F' \in E(H_2)$.

Let $A_{F'}$ be the event that $\sum_{i\geq r} \frac{X^{(i)}_{F'}}{i} \leq 2Mh_1$. Recall A is the event that $X \leq 2Mh_1$. We have $A \subset A_{F'}$, since $X_{F'} \leq X$. The event $A_{F'}$ only depends on the coloring of vertices outside F', and is independent of the coloring of vertices inside F'. Relaxing the event A to $A_{F'}$ is one of the crucial steps toward improvement.

The random coloring C can be decomposed into two steps. First, we can color the vertices outside F'. Then color the vertices inside F'. Notice that $X^{(i)}_{F'}$ only depend on the coloring of vertices outside F'. So does the random variable Z.

Let W be the event that S are red and other vertices in F' are blue. It is clear that

$$\Pr(W) = \frac{1}{2^{|F'|}}. \tag{23}$$

This probability is the same for different choices of the subset S. The event W is independent of $A_{F'}$ and Z since W only depends on the coloring of vertices inside F'.

Thus, the probability that F' causes the failure of the algorithm is at most

$$\mathrm{E}(\mathbf{1}_{A_{F'}} Z)\frac{1}{2^{|F'|}}. \tag{24}$$

Since e^{4qx} is concave upward, the curve over any interval $[a, b]$ is blew the secant over this interval.

$$e^{4qx} \geq e^{4qa} + \frac{e^{4qb} - e^{4qa}}{b-a}(x-a). \tag{25}$$

Apply this inequality with $a = 0$, $b = 2Mh_1$, and $x = X_{F'}$. We have

$$\begin{aligned} \mathbf{1}_{A_{F'}} Z &\leq \left(\frac{e^{8Mh_1 q} - 1}{2Mh_1} \right) X_{F'} \\ &\leq \frac{e^{8Mh_1 q}}{2Mh_1} X_{F'}. \end{aligned} \tag{26}$$

The probability that the second type event occurs is at most

$$\begin{aligned} \sum_{F' \in E(H_2)} \mathrm{E}(\mathbf{1}_A Z) \frac{1}{2^{|F'|}} &\leq \sum_{F' \in E(H_2)} \frac{1}{2^{|F'|}} \mathrm{E}\left(\frac{e^{8Mh_1 q}}{2Mh_1} X_{F'}\right) \\ &= \sum_{F' \in E(H_2)} \frac{1}{2^{|F'|}} \frac{e^{8Mh_1 q}}{2Mh_1} \mathrm{E}(X_{F'}) \\ &\leq \sum_{F' \in E(H_2)} \frac{1}{2^{|F'|}} \frac{e^{8Mh_1 q}}{2Mh_1} \frac{2h_1}{r} \\ &= \frac{h_2 e^{8Mh_1 q}}{Mr}. \end{aligned} \tag{27}$$

Here we apply inequality (19).

Combine equalities (6), (7), (8), (14), (15), and (27) Replace h_1 and h_2 by their maximum h. The probability that the algorithm output "H has Property B" is at least

$$1 - \frac{2}{M} - 2he^{-q} - \frac{2he^{8Mhq}}{Mr}. \tag{28}$$

Choose $M = 2(1+\epsilon)$, $q = \ln\ln r$, and $h = \frac{1-\epsilon}{16} \frac{\ln r}{\ln\ln r}$. We observe that the above probability is

$$\frac{\epsilon}{1+\epsilon} - \frac{2h}{\ln r} - \frac{2h}{Mr^{\epsilon^2}} > 0 \tag{29}$$

for sufficiently large r.

With positive probability, the randomized algorithm will output "The program succeeds". H must have Property B. $\square$.

4 Coloring hypergraphs using k colors

For any integer k, let $f_k(r) = \min_H \sum_{F \in E(H)} \frac{1}{k^{|F|}}$, where H ranges over all $k+1$-chromatic hypergraphs with minimum edge cardinality r. Our approach works naturally for this case, we have

Theorem 3 *For any fixed $k \geq 2$ and $\epsilon > 0$, there is an $r_0 = r_0(k, \epsilon)$, for any $r \geq r_0$, we have*

$$f_k(r) \geq (\frac{k-1}{4k^2} - \epsilon)\frac{\ln r}{\ln \ln r}. \tag{30}$$

The residue twin-hypergraphs and the irreducible core can be naturally extended from twin-hypergraphs to k-tuple hypergraphs. During the recoloring process of the randomized algorithm, when a color of a vertex is to be changed, it will use one of $k-1$ other colors randomly. The analysis is very similar to the one for the 2-coloring case. We omit of the detail proof of Theorem 3.

An obvious upper bound for $f_k(r)$ is $k^{-r}m_k(r)$. Kostochka [7] pointed out $m_k(r) \leq Cr^2k^r$ can be obtained by the argument of Erdős [4, 5]. It implies

$$f_k(r) \leq Cr^2.$$

For uniform hypergraphs, Kostochka [7] proved

$$m_k(r) \geq ck^r(\frac{r}{\ln r})^{1-\frac{1}{\lfloor \log_2 r \rfloor + 1}},$$

which is significantly better for large k large. It seems to be an interesting question whether the lower bound of $f_k(r)$ in Theorem 3 can be improved for large k.

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