1 Ring Theory

(84 Jan.) Show that a nonzero ideal in a principal ideal domain is maximal if and only if it is prime.

(84 Jan.) Let R be a commutative ring with identity. For $x \in R$, let $A(x) = \{r \in R : xr = 0\}$. Suppose $\theta \in R$ has the property that $A(\theta)$ is not properly contained in $A(x)$ for any $x \in R$. Prove that $A(\theta)$ is a prime ideal of R.

(84 Jan.) If F is a field, prove that $F[x]$ is a principal ideal domain.

(84 Aug.) Let R be a principal ideal domain and $0 \neq r \in R$. Let I be the ideal generated by r. Prove
i) If r is prime, then R/I is a field.
ii) If r is not prime, then R/I is not an integral domain.

(85 Jan.) Let R be a commutative ring with identity element 1 and let I be an ideal of R. Prove each of the following:
a) R is a field if and only if R has exactly two ideals.
b) R/I is a field if and only if I is a maximal proper ideal of R.

(85 Jan.) Let R be a commutative ring with identity element 1. Prove each of the following:
a) Every proper ideal of R is included in a maximal proper ideal of R.
b) R has exactly one maximal proper ideal if and only if the set of nonunit of R is an ideal of R.

(85 Aug.) Let I be an ideal of the commutative ring R. Prove that $\frac{R}{I}$ is a field if and only if I is maximal ideal of R.

(85 Aug.) Let R be the ring of formal power series $F[[x]]$, where F is any field. A typical element of R looks like $\sum_{i=0}^{\infty} \alpha_i x^i$ where $\alpha_i \in F$. The elements of R added and multiplied in the obvious manner.
a) Find all units of R.
b) Find all ideals of R.
c) Find all maximal ideals of R.

(86 Jan.) Let R be a unique factorization domain (i.e. factorial domain) and K be the quotient field of R. An element $z \in K$ is said to be integral over R if there exists a monic polynomial $f \in R[x]$ (i.e. a polynomial of the form $x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ with $a_i \in R$) such that $f(z) = 0$. Prove that if z is integral over R, then z is in R.

(86 Jan.) Let R be a commutative ring with 1. The nilradical of R is the set $N = \{r \in R | r^k = 0$ for some integer $k \geq 1\}$.

a) Prove that N is an ideal in R.

b) Let a be an element of R which is not an element of N, let S be the set $\{1, a, a^2, a^3 ...\}$, and let I be an ideal which is maximal with respect to the property that $I \cap S = \phi$ i.e. if J is an ideal of R which properly contains I then $J \cap S \neq \phi$. Prove that I is a prime ideal of R.

(86 Jan.) Let F be a field, $n \geq 2$ be an integer, and $R = M_n(F)$ be the ring of $n \times n$ matrices with entries in F.

a) Give an example of a left ideal I in R with $I \neq (0)$ and $I \neq R$.

b) Give an example of a simple left ideal I in R (i.e. $I \neq (0)$ but (0) is the only left ideal properly contained in I.) Give justification for your answer.

(86 Aug.) Prove that $y^3 + x^2y^2 + x^3y + x$ is irreducible in $R[x, y]$ if R is a unique factorization domain.

(86 Aug.) Let R be a commutative ring with 1 and let J be the intersection of all the maximal proper ideals of R. Prove that $1 + a$ is a unit of R for every $a \in J$.

(86 Aug.) Prove the Chinese Remainder Theorem: If R is any ring with 1 and I_1,I_2,...,I_n are ideals of R such that $R = I_i + I_j$ whenever $1 \leq i, j \leq n$ and $i \neq j$, then $R/I \cong R/I_1 \times R/I_2 \times \cdots \times R/I_n$, where $I = I_1 \cap I_2 \cap \cdots \cap I_n$.

(87 Jan.) Let $R = \mathbb{Z}[X]$. Give three prime ideals in R that contain the ideal $(6, 2X)$, and prove that your ideals are prime.

(87 Jan.) Let $f(x) = x^4 + 2x^3 + 10x^2 + 16x + 16$ and $g(x) = x^4 + 2x^3 + 3x^2 + 2x + 2$ in the ring $\mathbb{C}[x]$.

a) Compute the greatest common divisor of f and g, i.e. the monic generator of the ideal (f, g).

b) If f is the characteristic polynomial of a certain complex matrix A, decide whether or not $g(A)$ is singular.

(87 Jan.) Let R be a commutative ring with identity and let I and J be ideals of R. Define IJ to be the ideal generated by all products xy with $x \in I$ and $y \in J$; that is, IJ is the set of all finite sums of such products.

a) Prove that $IJ \subseteq I \cap J$.

b) Prove that $IJ = I \cap J$ if R is a principal ideal domain and $I + J = R$.

c) We say that R has the descending chain condition (DCC) if given any chain of ideals $I_1 \supseteq I_2 \supset I_3 \supseteq \cdots$ there is an integer k such that $I_k = I_{k+1} = I_{k+2} = \cdots$. If R has DCC, prove that R has only finitely many maximal ideals. (Hint: If M_1,M_2,... are distinct maximal ideals, consider the ideals $I_j = M_1 \cap \cdots \cap M_j$.)

(87 Aug.) Let $R = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. (Of course R is a subring of reals.) Let $M = \{a + b\sqrt{2} \in R : 5|a \text{ and } 5|b\}$.

a. Show that M is a maximal ideal of R. (Hint: $(a+b\sqrt{2})(a-b\sqrt{2}) = a^2 - 2b^2$). Show that if 5 divides $a^2 - 2b^2$ then 5 divides a and b.)

b. What is the order of the field R/M? Verify your answer.

(87 Aug.) Let R be a commutative ring with 1. Let $p \in R$ and suppose that the principal ideal (p) is prime. If Q is a prime ideal and $Q \subsetneqq (p)$, show that $Q \subseteq \cap_n (p^n)$

(87 Aug.) Let

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

be a commutative diagram of R-modules and R-module homomorphisms, with exact rows. Show that if α_1 is surjective and α_2 and α_4 are injective, then α_3 is injective.

(88 Aug.) Let $F = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} | a, b \in Z_3\}$.

a. Prove that F is a ring that contains $\mathbb{Z}_3$.

b. Give a basis for F as a vector space over $\mathbb{Z}_3$.

c. Show that the equation $x^2 + 1 = 0$ has a solution in F; and prove that F and $\mathbb{Z}_3[x]/(x^2 + 1)$ are isomorphic rings.

d. Prove that F is a field.

(88 Aug.) Prove that the polynomial $3X^4 + 2X^2 - X + 15$ is irreducible in both $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. (Hint: Consider the image of this polynomial in the ring $\mathbb{Z}_2[X]$.)

(88 Aug.) Let F be a field, $R = F[x]$, and M be the ideal M. If $I = (X^2)$ and $J = (X^2 - X^3)$, prove that $J \subseteq I$ in R, but $J_M = I_M$ in R_M. As usual, R_M denotes the localization $S^{-1}R$, where $S = R/M$.)

(89 Jan.) Prove that the polynomial $X^3Y + X^2Y + XY^2 + X^3 + Y$ is irreducible in $\mathbb{Z}[x, y]$.

(89 Jan.) Let R be commutative ring with 1 and let J be an ideal of R. The set $\{a \in R : a^n \in J \text{ for some } n\}$is denoted by rad$(J)$.
a. Prove that rad(J) is an ideal of R.
b. Prove that if I is a finitely generated ideal included in rad(J), then $I^m \subseteq J$ for some positive integer m.

(89 Jan.) Let R be a principal ideal domain and let I and J be ideals of R. IJ denotes the ideal of R generated by the set of the form ab where $a \in I$ and $b \in J$. Prove that if $I + J = R$, then $I \cap J = IJ$.

(89 Aug.) Let R denote an associative ring. Prove that, if $x, y \in R$ and $x - y$ is invertible, then $x(x - y)^{-1}y = y(x - y)^{-1}x$.

(90 Jan.) Let F be a field, $p(x) \in F[x]$, and $R = F[x]/(p(x))$. The *nilradical* of R is then equal to

$$\{r \in R : r^n = 0 \text{ for some integers } n \geq 1\}.$$

Fill in the blank with some property of the polynomial $p(x)$, and then prove the resulting statement: The nilradical of R is $\{0\}$ if and only if ----.

(90 Jan.) Let R be the following subring of the field of rational functions in 3 variables with complex coefficients:

$$R = \{\frac{f}{g} : f, g \in \mathbb{C}[X, Y, Z] \text{ and } g(1, 2, 3) \neq 0\}.$$

Find 3 prime ideals P_1, P_2, and P_3 in R with

$$0 \subsetneqq P_1 \subsetneqq P_2 \subsetneqq P_3 \subsetneqq R.$$

(Note: You are required to prove that your ideals are prime.)

(90 Aug.) Let R be the ring of 2×2 matrices over the field of complex numbers. Find two left ideals I_1 and I_2 of R such that
(a) I_1 and I_2 are isomorphic as left R-modules, and
(b) $I_1 \neq I_2$. Prove that I_1 and I_2 satisfy (a) and (b).

(90 Aug.) Let I be an ideal in a commutative ring R and let $\mathcal{I}$ be the set of ideals of R defined by the property that $J \in \mathcal{I}$ if and only if there exists an element a of R such that $a \notin I$ and $J = \{r \in R | ra \in I\}$. Prove that every maximal element of $\mathcal{I}$ is a prime ideal in R.

(90 Aug.) Let F be a field. Let $f_1, ..., f_r$ be polynomials in the polynomial ring $F[x]$.
(a) Find in the blank. The natural map

$$F[x] \rightarrow \frac{F[x]}{(f_1)} \oplus ... \oplus \frac{F[x]}{(f_r)}$$

is onto if and only if ______.
(b) Prove your answer to (a). (You are expected to write a complete proof. The answer "this is a well known theorem" is not acceptable.)

(91 Jan.) Prove the following form of the Chinese Remainder Theorem: Let R be a commutative ring with unit 1 and suppose that I and J are ideals of R such that $I + J = R$. Then

$$\frac{R}{I \cap J} \cong \frac{R}{I} \times \frac{R}{J}$$

(91 Jan.) Let X be a finite set and let R be the ring of all functions from X into the field R of real numbers. Prove that an ideal M is a maximal ideal of R if and only if there is an element $a \in X$ such that

$$M = \{f : f \in R \text{ and } f(a) = 0\}$$

(91 Jan.) Let D be a unique factorization domain and let I be a nonzero prime ideal of $D[x]$ minimal among all the nonzero prime ideals of $D[x]$. (That is, if $J \subseteq I$ and J is a nonzero prime ideal of $D[x]$, then $J = I$.) Prove that I is a principal ideal.

(91 Aug.) Let R be a ring and $\text{Aut}(R)$ be the group of ring automorphisms of R.
(a) Show that $\text{Aut}(R) = \{1\}$.
(b) Find $\text{Aut}(R[x])$. (Hint: It is isomorphic to a subgroup of $GL(2, R)$)

(91 Aug.) For each field F given below, factor $x^{31} - 1 \in F[x]$ into a product of irreducible polynomials and justify your answer.
(a) F is the field of complex numbers.
(b) F is the field of rational numbers.
(c) F is the field of 31 elements.
(d) F is the field of 32 elements.

(91 Aug.) Let D be a commutative domain and F be its field of fractions. The domain D is said to be *neat* if whenever $f(x)$ and $g(x)$ are monic polynomials in $F[x]$ with $f(x)g(x) \in D[x]$, then both $f(x)$ and $g(x)$ are in $D[x]$.
(a) Prove that if D is a UFD, then D is neat.
(b) Give an example of a domain D which is not neat.

(92 Jan.) Let I be the kernel of the ring homomorphism $\mathbb{Z}[x] \to \mathbb{R}$ induced by the substitution $x \mapsto 1 + \sqrt{2}$. Show that I is a principal prime ideal and find a generator for it.

(92 Jan.) Give the prime factorization of $f(x) = x^5 + 5x + 5$ in each of $Q[x]$, and $\mathbb{Z}_2[x]$.

(92 Jan.) Let D be a commutative ring. Show that if $D[x]$ is a principal ideal domain then D must be a field.

(92 Aug.) Let R be a UFD. Prove that $f(x,y,z) = x^5y^3 + x^4z^3 + x^3yz^2 + y^2z$ is irreducible in $R[x,y,z]$.

(92 Aug.) Construct a field with 8 elements.

(93 Jan.) Prove that $x^4 + xy^2 + y$ is irreducible in $\mathbb{Q}[x,y]$.

(93 Jan.) Let R be a commutative ring with unit and let n be a positive integer. Let $J, I_0, ..., I_{n-1}$ be ideals of R such that I_k is a prime ideal for all $k < n$ and $J \subseteq I_o \cup \cdots \cup I_{n-1}$. Prove that $J \subseteq I_k$ for some $k < n$.

(93 Jan.) Does there exist a polynomial $f(x) \in \mathbb{R}[x]$ such that

- $f(x) - 1$ is in the ideal $(x^2 + 2x + 1)$, and
- $f(x) - 2$ is in the ideal $(x - 1)$, and
- $f(x) + 1$ is in the ideal $(x^2 - 25)$?

If $f(x)$ exists, then find it. If $f(x)$ doesn't exist, then prove that it doesn't exist.

(93 Aug.) Let R be a commutative ring with one. Prove that every maximal ideal of R is also a prime ideal of R.

(93 Aug.)

(a) If I and J are ideals in a commutative ring R with $I + J = R$, then prove that $I \cap J = IJ$.

(b) If I, J, and K are ideals in a Principal Ideal Domain R, then prove that
$$I \cap (J + K) = (I \cap J) + (I \cap K).$$

(94 Jan.) Let R be a commutative ring with 1 and K be a maximal ideal in R. Show that R/K is a field.

(94 Jan.) Let R be a commutative ring. The nilradical of R is defined to be $N(R) = \{x \in R | x^n = 0, 0 \in N\}$

(a) Show that $N(R)$ is an ideal of R.

(b) Show that $N(R)$ is the intersection of all prime ideals of R.

(94 Aug.) Is $y^3 - x^2y^2 + x^3y + x + x^4$ irreducible in $\mathbb{Z}[x,y]$?

(94 Aug.) Fill in the blank and prove the resulting statement. If D is an integral domain, then $D[x]$ is a principal ideal domain if and only if D is ____.

(94 Aug.) Prove that there exist a polynomial $f(x) \in \mathbb{R}[x]$ such that

- $f - 1$ is in the ideal $(x^2 - 2x + 1)$, and
- $f - 2$ is in the ideal $(x + 1)$, and
- $f - 3$ is in the ideal $(x^2 - 9)$

Give an explicit construction for f.

(94 Aug.) Explain why each of the following represents or does not represent a maximal ideal in the ring $\mathbb{C}[x, y]/(y^2 - x^3 - x^2 - 4)$: $(x - 1, y + 2)$, $(x + 1, y - 2)$, $(y^2 - x^3, x^2 + 3)$.

(95 Jan.) Let R be a commutative ring. Suppose that I is an ideal of R which is contained in a prime ideal P. Prove that the collection of prime ideals containing I and contained in P has a minimal member.

(95 Jan.) Prove that $y^4 + x^2y + 4xy + x + 4y + 2$ is irreducible in $\mathbb{Q}[x, y]$. Here $\mathbb{Q}$ denotes the field of rational numbers.

(95 Jan.) Let R be a commutative ring with unit. Let I be a prime ideal of R such that R/I satisfies the descending chain condition on ideals. Prove that R/I is a field. [Hint: It is an easier but informative task to prove that every finite integral domain is a field.]

(95 Aug.) Prove that $y^3 + x^2y^2 + x^3y + x$ is irreducible in $\mathbb{Z}[x, y]$, where $\mathbb{Z}$ is the ring of integers.

(95 Aug.) Does there exist a polynomial $f(x) \in \mathbb{R}[x]$ fulfilling all of the following conditions:

- $f(x) - 1$ is in the ideal of $R[x]$ generated by $x^2 + 2x + 1$, and
- $f(x) - 2$ is in the ideal of $R[x]$ generated by $x - 3$, and
- $f(x) + 1$ is in the ideal of $R[x]$ generated by $x^2 - 16$?

(96 Jan.) Let $\mathbb{Z}_7 = \mathbb{Z}/7\mathbb{Z}$ be the integers modulo 7. Let $A = \mathbb{Z}_7[x]/(x^2 + 1)$. Prove that A is a field that is two dimensional as a vector as a vector space over $\mathbb{Z}_7$.

(96 Jan.) How many ideals are there in $\mathbb{Q}[x]/(x^2-1)$? Prove your answer is correct.

(96 Jan.) Let $f, g \in \mathbb{C}[x, y]$ be non-zero polynomials.

(a) Explain why f and g have a greatest common divisor.

(b) If h is the greatest common divisor of f and g, prove or disprove that there are polynomials $p, q \in \mathbb{C}[x, y]$ so that $h = pf + qg$. Justify your claims.

(96 Jan.) Let R be the ring of $n \times n$ matrices over the real numbers. Show R has no two sided ideals other than R and $\{0\}$

(96 Aug.) let f and g be two complex polynomials in the ring $\mathbb{C}[x, y]$. Assume that for any pair of complex numbers (a, b) either $f(a, b) \neq 0$ or $g(a, b) \neq 0$. Show there are polynomials p and q in $C[x, y]$ so that $pf + qg = 1$.

(96 Aug.) let R be the quotient ring $\mathbb{Q}[x]/(x^3 + 6x + 12)$, let $\pi : \mathbb{Q}[x] \to R$ be the natural projection, and let $\bar{x} = \pi(x)$.

a. Show that R is a field.

b. Find the dimension of R as a vector space over $\mathbb{Q}$.

c. Express $(\bar{x}^2 + 1)(\bar{x}^2 + 1)$ as polynomial of degree ≤ 2 in $\bar{x}$.

d. Express $1/\bar{x}$ as a polynomial in $\bar{x}$.

(96 Aug.) Let R be an integral domain. A non-zero prime P ideal is said to be height one if there is no prime ideal properly contained between (0) and P. If R is a UFD, prove that every height one prime ideal is a principal ideal.

(97 Jan.) Let R be the commutative integral domain that satisfies the descending chain condition. Show that R is a field.

(97 Jan.) Let R be a quotient ring $R = \mathbb{Q}[x]/(x^2+2x+2)$, let $\pi : \mathbb{Q}[x] \to R$ be the quotient map, and let $\bar{x} = \pi x$ be the image of x under the quotient map.

(a) Show that R is a field.

(b) Find the dimension of R as a vector space over $\mathbb{Q}$.

(c) Express $1/(\bar{x}^2 - \bar{x} + 4)$ as a polynomial of degree ≤ 1 in $\bar{x}$.

(97 Jan.) Show that for every polynomial $p(x) \in \mathbb{C}[x]$ of degree n there is polynomial $q(x)$ of degree $\leq n$ so that

$$(x+1)^n q(\frac{x-1}{x+1}) = p(x).$$

(97 Jan.) Let $f(x) \in Z[x]$ be a polynomial of degree $n \geq 2$ with integer coefficients. Assume that there are $2n+1$ distinct integers $k_1, k_2, ..., k_{2n+1}$ so that each $f(k_i)$ is a prime number, Show that $f(x)$ is irreducible over that rational numbers.

(97 Aug.) Let R be a ring with at least two elements so that for any $a \neq 0 \in R$ there is a unique $b \in R$ such that $aba = a$. Show that:

i) R has no zero divisors;

ii) $bab = b$;

iii) R has an identity;

iv) R is a division ring.

(97 Aug.) Determine the structure of the quotient ring $\mathbb{Z}_2[x]/(3x-2)$.

(97 Aug.) Show that two integer polynomials are relatively prime in $\mathbb{Q}[x]$ if and only if the ideal they generate in $\mathbb{Z}[x]$ contains a nonzero integer.

(98 Jan.) Let $R = \mathbb{Z}_2[x]/(x^4 + x + 1)$.

a. Prove that R is a field.

b. Determine the dimension of R as a $\mathbb{Z}_2$-vector space; and determine the order of R.

(98 Jan.) Let R be a commutative ring. The Jacobson radical $J(R)$ of R is defined to be the intersection of all maximal ideals of R.

a. Show that $a \in J(R)$ if and only if $1 - ra$ is a unit of R for all $r \in R$.

b. Let $A = I + B$ be an $n \times n$ matrix with all entries of B in $J(R)$. Prove that A is invertible.

(98 Jan.) Let $R = \mathbb{C}[x_1, ..., x_n]$ and $h = \gcd(f, g)$ for polynomials f and g in R.

a. Prove that if $n = 1$, there exist p and q in R such that $pf + qg = h$.

b. Show that the assertion of part (a) can fail if $n \geq 2$.

c. Show that the assertion of part (a) is true if the polynomials $F = f/h$ and $G = g/h$ have no common zeros, i.e., there are no points $a = (a_1, ..., a_n)$ in $\mathbb{C}^n$ such that $F(a) = G(a) = 0$.

(98 Jan.) Consider the subring $S \subset \mathbb{C}$, where $S = \mathbb{Z}[\alpha]$ and $\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{5}$.

a. Let $\phi : \mathbb{Z}[x] \to S$ be the ring homomorphism given by $\phi(x) = \alpha$ and $\phi(n) = n$ for every $n \in \mathbb{Z}$. Show that $K = \ker(\phi)$ is a principal prime ideal, and find a generator for K.

b. Prove or disprove that $S = \text{Image}(\phi)$ is a field.

(98 Aug.) Prove that $y^5 + x^2y^4 + x^3y + x$ is an irreducible element of $\mathbb{Q}[x, y]$.

(98 Aug.) An ideal I of a commutative ring R is called a radical ideal if and only if for any $r \in R$ if $r^k \in I$ for some positive integer k, then $r \in I$. A commutative ring R is said to be reduced if and only if for any $r \in R$ if $r^k = 0$ for some positive integer k, then $r = 0$. Let R be a commutative ring and I be an ideal of R. Prove that I is the radical ideal of R if and only if R/I is a reduced ring.

(98 Aug.) Prove that the additive group of all polynomials in x with integer coefficients is isomorphic to the multiplicative group of all positive rational numbers.

(98 Aug.) Let R be a commutative ring and suppose that I is maximal among all the ideals of R which are not finitely generated . Prove that I is a prime ideal.

(99 Jan.) Prove that an infinite integral domain with a finite number of units has an infinite number of maximal ideals.

(99 Jan.) Let J be a fixed ideal in the commutative ring R. For each element a of R, let $J : a$ be the ideal $\{r \in R | ra \in J\}$ in R. Prove that every maximal element in the following set of ideals

$$\{J : a | a \in R \text{ and } J : a \neq R\}$$

is a prime ideal of R.

(99 Jan.) Let R be the ring $\frac{\mathbb{R}[x,y]}{(x^2-y^2)}$. Identify four distinct prime ideals P_1, P_2, Q_1, and Q_2 of R with $P_1 \subset P_2$ and $Q_1 \subset Q_2$.

(99 Aug.) Prove that $\mathbb{Q}[x]/(x^6 + 539x^5 - 511x + 847)$ is a field.

(99 Aug.) Let R be the following subring of $\mathbb{Q}$:

$$R = \{\frac{a}{30^n}|a, n \in \mathbb{Z}\}.$$

First, prove that R is a principal ideal domain, then determine all the maximal ideals of R.

(00 Jan.) Let G be a finite group and D be a division ring. Let $f : G \to D^*$ be an nontrivial group homomorphism, where D^* is the multiplicative group $D\backslash\{0\}$. Prove that $\displaystyle\sum_{g\in G} f(g) = 0$.

(00 Jan.) Let a, b, and c be integers. The natural map

$$\mathbb{Z} \to \frac{\mathbb{Z}}{(a)} \oplus \frac{\mathbb{Z}}{(b)} \oplus \frac{\mathbb{Z}}{(c)}$$

is a surjection if and only if _____. FILL in the blank with some conditions on the integers a, b, and c and PROVE the resulting statement.

(00 Jan.) Identify six distinct proper prime ideals $P_0 \subseteq P_1 \subseteq P_2$ and $Q_0 \subseteq Q_1 \subseteq Q_2$ in the ring $\mathbb{Z}[x, y]/(x^2 - 4y^2)$.

(00 Aug.) Let $A(t), B(t) \in \mathbb{R}[t]$ with $\deg(A(t)) = \deg(B(t)) = 5$. Then show there is a nonzero polynomial $p(x, y) \in \mathbb{R}[x, y]$ so that $p(A(t), B(t)) = 0$. (That is, $p(A(t), B(t))$ is the nonzero polynomial in $\mathbb{R}[t]$.)

(00 Aug.) Let I be an ideal in the commutative ring R which is maximal among the non-finitely generated ideals of R. Prove that I is a prime ideal of R. (The hypothesis on I says that if J is an ideal of R which $I \subset J$) and $I \neq J$, then J is a finitely generated ideal of R.)

(00 Aug.) List 4 proper prime ideals of the polynomial ring $\mathbb{C}[x, y]$ which contain the ideal $(4x^2 - 13xy + 3y^2)$. Which of your ideals are maximal ideals?

(01 Aug.) Let I be the ideal of $\mathbb{C}[x, y, z]$ which consists of all polynomials $g(x, y, z)$ such that $g(1, 2, 3) = 0$. Prove that I is a maximal ideal.

(01 Jan.) Prove that the identity map is the only ring homomorphism from $\mathbb{R}$ to $\mathbb{R}$.

(01 Jan.) Suppose R is a commutative ring with 1 and an ideal I can be expressed as a product of distinct maximal ideals in two ways: $P_1 \cdots P_r$, and $Q_1 \cdots Q_s$. Prove that these factorizations are identical except for the order of the factors.

(01 Aug.) Let R be a ring (with 1, but not necessarily commutative), and let e be a nontrivial central idempotent (i.e. $e^2 = e$, $e \neq 0$, $e \neq 1$ and $er = re$ for all $r \in R$). Prove that $1 - e$ is also an idempotent, that the left ideals $R_1 = Re$ and $R_2 = R(1 - e)$ are rings "with 1", but not subrings of R, and that there is an isomorphism of rings $\phi : R \to R_1 \times R_2$. Finally, give an example of such a ring R and element e.

(01 Aug.) By adjoining a suitable elements α, produce a field that is a three-dimensional vector space over the field $F_2 = \mathbb{Z}_2$. Explain how you know that your ring is indeed a field, and in particular give the inverse of α.

(01 Aug.) Let $R = \mathbb{C}[x_1, ..., x_n]$ and h be the gcd of non-zero polynomials f and g in R. First explain why there is such a polynomial h in general.

a. If $n \geq 2$ show that h can fail to be in the ideal (f, g).
b. Let $S = \mathbb{C}[x, y]/(x^2 + y^2 - 1, 2x + y - 1)$. Find all the maximal ideals of S (and prove that you have found all of them).

(01 Aug.) A proper ideal Q in a principal ideal domain R is called **primary** if $a \notin Q$ and $ab \in Q$ implies $b^n \in Q$ for some $n \geq 1$. Prove that a non-zero ideal I is primary if and only if $I = (p^N)$ for some $N \geq 1$ and p a prime element in R.

(02 Aug.) Prove that if k is a field and $f(x)$ and $g(x)$ are relatively prime elements of $k[x]$, then the rings

$$\frac{k[x]}{(f(x)g(x))} \text{ and } \frac{k[x]}{(f(x))} \times \frac{k[x]}{(g(x))}$$

are isomorphic.

(02 Aug.) Let f and g be polynomials in an indeterminate over a ring R. Suppose that the ideal generated by the coefficients of f is R and that the ideal generated by the coefficients of g is R. Prove that the coefficients of fg also generate R.

(02 Aug.) Let R be a commutative ring and suppose that I is the maximal among all the ideals of R which are not principal. Prove that I is a prime ideal.

(03 Aug.) Show that the quotient ring $\mathbb{C}[x]/ < x^2 + 1 >$ is isomorphic to the direct product $\mathbb{C} \times \mathbb{C}$. Here $< x^2 + 1 >$ is the ideal generated by $x^2 + 1$.

(03 Aug.) Let D be a commutative ring with identity such that $D[x]$ is a principal ideal domain. Show that D is a field.

(03 Aug.)

(a) Show that $x^3y + x^3 + x^2y^2 - x^2y + xy^2 + y$ is irreducible in $\mathbb{Z}[x, y]$.

(b) Show that $5x^4 - 2x^2 + x + 15$ is irreducible in $\mathbb{Z}[x]$.

(04 Aug.) Prove the following ring isomorphism:
$\dfrac{\mathbb{C}[x, y]}{(xy)} \cong$ the subring of $\mathbb{C}[x] \times \mathbb{C}[y]$ consisting of pairs $(p(x), q(y))$ with $p(0) = q(0)$.

(04 Aug.) List three distinct prime ideals $P_0 \subset P_1 \subset P_2$ in the ring

$$\mathbb{Z}[x, y]/(9x^2 - 16y^2)$$

and prove that they are indeed prime ideals. Which of the prime ideals you listed are maximal ideals (if any)? Justify your answer.

(05 Aug.) Prove that every nontrivial commutative ring with unit has a homomorphic image which is a field.

(05 Aug.)

(a) Show that $3x^2 + 6x^2 + 12x + 14$ is irreducible in $\mathbb{Z}[x]$.

(b) Show that $x^2y^2 + xy^3 + x^2y - y^3 + x^2 + 2xy - y^2 + x + y$ is irreducible in $\mathbb{Z}[x, y]$.

(05 Aug.) Let I be the ideal of $\mathbb{Z}[x,y]$ generated by $\{7, x^2+4y^2\}$, where $\mathbb{Z}$ is the ring of integers.

(a) Prove that I is a prime ideal.

(b) Is I a maximal ideal in $\mathbb{Z}[x,y]$? If it is, prove it. Otherwise, given an example of a maximal ideal M such that $I \subsetneqq M$

(06 Jan.) Suppose that R is a commutative ring with 1 and that R has a unique maximal ideal. Prove:

(a) if u does not belong to the maximal ideal, then u is a unit of R.

(b) if $a, b \in R$ such that $a|b$ and $b|a$, then there is a unit $u \in R$ such that $b = ua$.

(06 Jan.) Let R be a unique factorization domain and let F be its field of fractions. Prove that if $\alpha \in F$ is the root of a monic polynomial with coefficients in R, then $\alpha \in R$.

(06 Jan.) Show that $x^6 + 539x^5 - 511x - 847$ is irreducible in $\mathbb{Z}[x]$.

(06 Aug.)

(a) Prove that $3x^3 + 6x^2 + 12x + 14$ is irreducible in $\mathbb{Z}[x]$.

(b) Prove that $x^2y^3+x^2y^2+x^2y-2xy^2+x+y^3+y^2-y-1$ is irreducible in $\mathbb{Z}[x,y]$.

(06 Aug.) Prove that the ring $\mathbb{Z}_{210}$ and the ring $\mathbb{Z}_{10} \times \mathbb{Z}_{21}$ are isomorphic.

(06 Aug.) Prove that in a principal ideal domain every nontrivial prime ideal is maximal. Give an example of an integral domain with a nontrivial prime ideal that is not maximal.