Coloring Reduced Kneser Graphs

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Abstract

The vertex set of a Kneser graph $\mathsf{KG}(m, n)$ consists of all n-subsets of the set $[m] = \{0, 1, \ldots, m-1\}$. Two vertices are defined to be adjacent if they are disjoint as subsets. A subset of $[m]$ is called 2-stable if $2 \leq |a - b| \leq m - 2$ for any distinct elements a and b in that subset. The reduced Kneser graph $\mathsf{KG}_2(m, n)$ is the subgraph of $\mathsf{KG}(m, n)$ induced by vertices that are 2-stable subsets. We focus our study on the reduced Kneser graphs $\mathsf{KG}_2(2n + 2, n)$. We achieve a complete analysis of its structure. From there, we derive that the circular chromatic number of $\mathsf{KG}_2(2n + 2, n)$ is equal to its ordinary chromatic number, which is 4. A second application of the structural theorem shows that the chromatic index of $\mathsf{KG}_2(2n + 2, n)$ is equal to its maximum degree except when $n = 2$.

Keywords: circular chromatic number, chromatic index, orientation, flow ratio, Kneser graph, reduced Kneser graph

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1 Preliminaries

A graph $G(V, E)$ consists of two finite sets $V(G)$ and $E(G)$. Every element in $V(G)$ is called a vertex of G. Every element of $E(G)$ is called an edge that contains two distinct vertices, and we say that the two vertices are adjacent. The degree of a vertex v is the number of edges in G that contain v.

A subgraph of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An induced subgraph H of G is a subgraph such that, if two vertices of H are adjacent in G, then they are also adjacent in H. A graphic isomorphism is a one-to-one mapping between two graphs such that two vertices are adjacent in one graph if and only if the corresponding vertices are adjacent in the other graph.

We usually write uv to represent an edge whose endpoints are u and v. An *arc* is an edge that has an order imposed on the two endpoints. A directed graph is a graph each of whose edge is an arc. When drawing a directed graph, we draw an arrow form u to v to indicate that the order is (u, v), not (v, u). We can obtain a directed graph from a graph G by imposing an order on every edge of G, and we say that G has been endowed with an orientation D. The graph G, together with its orientation D, is also called an *oriented graph*.

To fix notation, we introduce some basic graphs that will be used in this thesis. A *path* P_n is the graph consisting of n distinct vertices $p_1, p_2, \ldots, p_n$ such that p_i is adjacent to p_{i+1} for $1 \leq i < n$. A *cycle* C_n is the graph consisting of n distinct vertices $c_1, c_2, \ldots, c_n$ such that c_i is adjacent to c_{i+1} for $1 \leq i < n$ and c_n is adjacent to c_1. We note that P_n has $n-1$ edges and C_n has n edges. The length of P_n or C_n is the number of edges. By an n-cycle we mean a cycle of length n. A *clique* or a *complete graph* K_n is the graph with n distinct vertices such that every pair of vertices are adjacent. Let E

be a subset of the set $\{1, 2, \ldots, m-1\}$ such that k belongs to E if and only if $-k \pmod m$ belongs to E. The *circulant graph* $G(m; E)$ is the graph whose vertex set is $\{v_1, v_2, \ldots, v_m\}$, and two vertices v_i and v_j are adjacent if and only if $i - j \pmod m$ belongs to E. If E contains ± 1, then the cycle $v_1 v_2 \cdots v_m v_1$ is called the *boundary* of $G(m; E)$.

The *Cartesian product* of two graphs $G_1(V, E)$ and $G_2(U, F)$ is the graph G whose vertex set is the ordinary Cartesian product $V \times U$ of the two vertex sets. Two vertices (v_i, u_k) and (v_j, u_ℓ) of G are adjacent if v_i is adjacent to v_j and $u_k = u_\ell$, or $v_i = v_j$ and u_k is adjacent to u_ℓ. We denote the Cartesian product by $G_1 \Box G_2$. The Cartesian product of two paths P_m and P_n is called a *grid graph* $G(m, n)$. We use (i, j) to represent the vertex of $G(m, n)$ whose first entry is the i-th vertex of P_m and whose second entry is the j-th vertex of P_n. The *cylinder graph* $C_m \Box P_n$ can be regarded as obtained from $G(m, n)$ by joining the vertex $(1, i)$ with the vertex (m, i) for $1 \le i \le n$. The *Möbius grid* $\mathsf{MG}(m-1, n)$ is obtained from $G(m, n)$ by identifying the vertices $(1, i)$ with the vertices $(m, n-i+1)$ for $1 \le i \le n$. We always assume $m \ge 4$ in $G(m, n)$ to avoid the apperances of loops and multiple edges in the Möbius grid. By the *boundary* of a Möbius grid $\mathsf{MG}(m-1, n)$, we mean the cycle consisting of the vertices $(1, 1), (2, 1), \ldots, (m, 1) = (1, n), (2, n), \ldots, (m, n) = (1, 1)$.

Let $[n] = \{0, 1, 2, \ldots, n-1\}$ for any positive integer n. An n-coloring of a graph G is a mapping f from $V(G)$ to $[n]$. A coloring is proper if $f(u) \ne f(v)$ whenever u is adjacent to v. The graph G is said to be n-colorable if there exists a proper n-coloring. The *chromatic number* $\chi(G)$ of G is the least number n such that G is n-colorable. We call a subset of $V(G)$ an independent set if there is no edge between any two vertices in that subset. For a proper coloring f, each of the sets $V_i = f^{-1}(i)$ for $0 \le i < n$ is an independent set, and is called a color class of f.

In 1988, Vince [10] introduced a new notion for graph coloring. What he called the star chromatic number of a graph is now widely known as the circular chromatic number. Let k and d be positive integers such that $k \geq 2d$. A (k, d)-coloring of a graph G is a function σ from $V(G)$ to $[k]$ such that any two adjacent vertices u and v satisfy $d \leq |\sigma(u) - \sigma(v)| \leq k - d$. We may define the *$k$-circular norm* of an integer x to be $|x|_k = \min\{|x|, k - |x|\}$. The number $|x - y|_k$ naturally represents the *k-circular distance* between two integers x and y. Thus, a (k, d)-coloring requires two adjacent vertices at least d units apart in the k-circular distance. A $(k, 1)$-coloring is exactly an ordinary proper coloring. The *circular chromatic number* $\chi_c(G)$ of G is defined to be the infimum of k/d such that there exists a (k, d)-coloring of G. The following two fundamental facts are well-known.

Theorem 1 *For every graph G, there exists a (k, d)-coloring of G such that k and d are relatively prime and $\chi_c(G) = k/d$.*

Theorem 2 *For any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$.*

We note that the circular chromatic number of any subgraph cannot exceed that of the original graph since every (k, d)-coloring of the graph is also a (k, d)-coloring of its subgraph. Some circular chromatic number can be readily determined. A bipartite graph has circular chromatic number 2. The complete graph K_n has circular chromatic number n. The odd cycle C_{2n+1} has circular chromatic number $2 + 1/n$. We also know from Zhu [11] that $\chi_c(G \square H) = \max\{\chi_c(G), \chi_c(H)\}$. It follows that $\chi_c(P_m \square P_n) = 2$, $\chi_c(C_{2m} \square P_n) = 2$, and $\chi_c(C_{2m+1} \square P_n) = 2 + 1/m$.

Actually, Zhu [12] provides a comprehensive survey of the recent research on circular chromatic numbers. What we want to emphasize is the fact that there are several elegant and natural equivalent ways of defining the circular chromatic number. One approach that is particular useful in our study is the one introduced in the paper by Goddyn, Tarsi, and Zhang [1].

Let G be a graph endowed with an orientation D. Let C be a cycle of the underlying graph G. When we move around C, there are two possible directions of traversal to choose. Let us fix one as the direction of traversal. An arc in C is called *forward* if its orientation under D coincides with the direction of traversal; otherwise it is called *backward*. We denote the number of forward arcs of C by C^+ and the number of backward arcs of C by C^-. The *flow ratio* $\theta(C; D)$ of C with respect to the orientation D is defined to be $\max\{C^+/C^-, C^-/C^+\}$. This quantity measures the imbalance between the arcs in opposite directions. We allow it to be infinity when all arcs are oriented along the same direction. For an oriented graph $(G; D)$, its *flow ratio* $\theta(G; D)$ is defined to be the number $\max\{\theta(C; D) \mid \ C \text{ is a cycle of } G.\}$ By convention, $\theta(G; D)$ is defined to be 1 when G has no cycles. Note that the flow ratio $\theta(G; D)$ is equal to infinity if there is a directed cycle in the oriented graph $(G; D)$.

The next theorem was first established by Goddyn, Tarsi, and Zhang using tools from matroid theory. It can be regarded as giving an equivalent definition for the circular chromatic number. Here we reproduce a direct and elementary proof in Lih, Tong, and Wang [6] without using any matroid theory. Their basic ideas come entirely from a classical result of Minty [8] which connects the chromatic number with orientations.

Theorem 3 *The circular chromatic number $\chi_c(G)$ of G satisfies the following identity.*

$$\chi_c(G) = 1 + \min\{\theta(G; D) \mid \ D \textit{ is an orientation of } G.\}$$

Proof. By our convention, the identity is satisfied when G has no cycles.

Now assume that G has at least one cycle. Let f be a (k, d)-coloring of G. We obtain an orientation D of G by orienting the edge uv from u to v when $f(u) < f(v)$. For any cycle $C : c_1c_2 \cdots c_nc_1$. We choose the direction of traversal from c_1 to c_2, and so on. Let a be the number of forward arcs and b be the number of backward arcs on C. We define the weight of an edge c_ic_{i+1} to be $f(c_{i+1}) - f(c_i)$. Let a_s be the sum of all positive weights and b_s be the absolute value of the sum of all negative weights. The sum of all weights around the cycle is zero. Hence $a_s = b_s$. The absolute value of each weight is between d and $k - d$, so we have $da \leq a_s \leq (k - d)a$, $db \leq b_s \leq (k - d)b$, and the following inequalities,

$$\frac{da}{(k-d)b} \leq \frac{a_s}{b_s} \leq \frac{(k-d)a}{db}.$$

Therefore, $d/(k - d) \leq b/a \leq (k - d)/d$. Thus, $\theta(C, D) \leq (k - d)/d$ and $\max\{\theta(C; D) \mid C \text{ is a cycle of } G.\} \leq (k - d)/d$. Hence, k/d is greater than or equal to $1 + \min\{\theta(G; D) \mid\ D \text{ is an orientation of } G.\}$

On the other hand, let D be an orientation satisfying the minimum in the theorem and C be a cycle with the maximum flow ratio $\theta(C; D)$. Write $1 + \theta(C; D)$ as k/d. Note that D has no directed cycle. Now we play a game on the graph G. We first choose an arbirary vertex v of G as the starting vertex. Suppose that, when we go from v to another vertex u, we get d points along a forward arc and lose $k - d$ points along a backward arc. The goal of the game is to maximize the points $p(u)$ collected when we reach u. Assume that we are traversing a cycle having a forward and b backward arcs. The facts $a/b \leq \theta(C; D) = (k - d)/d$ and $b/a \leq \theta(C; D) = (k - d)/d$ imply that $da - (k - d)b \leq 0$. So, in order to maximize $p(u)$, we will avoid traversing any cycle. Hence $p(u)$ has a well-defined finite value, in particular, $p(v) = 0$. Suppose that the edge xy is an arc from x to y. Then we have $p(x) + d \leq p(y) \leq p(x) + k - d$. Define a coloring f by setting $f(x) \equiv p(x) \pmod{k}$. Then f is a (k,d)-coloring of G. Therefore, $\inf\{k/d \mid \text{ There is a } (k,d)\text{-coloring of } G.\} \leq 1+\theta(G;D)$. So the theorem is proved. □

The following example is provided to illustrate Theorem 3. All possible orientations of C_5 are exhibited in Figure 1. We see that the minimum flow ratio of C_5 is $3/2$. So the circular chromatic number of C_5 is $5/2$.

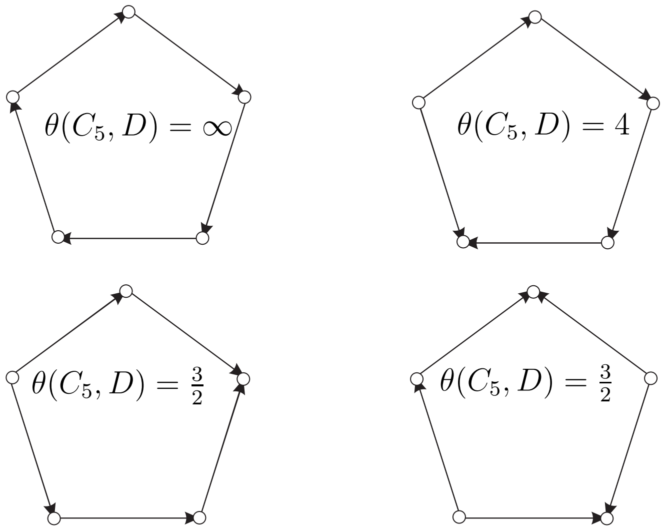

Figure 1: All possible orientations of C_5 up to isomorphism.

We conclude this section by stating the organization of this thesis as follows. In Section 1, we give necessary preliminaries and background information for our investigation. In Section 2, Kneser graphs and reduced Kneser graphs are introduced. The main problems investigated are discussed. In Section 3, we give a complete analysis of structure of the reduced Kneser graph $\mathsf{KG}_2(2n+2,n)$. In Section 4, the flow ratios of some special graphs are studied. Using results obtained in Sections 3 and 4, we establish the circular extremality of $\mathsf{KG}_2(2n+2,n)$ in Section 5. As a by-product of our structural theorem, we give an edge coloring of $\mathsf{KG}_2(2n+2,n)$ using $n+2$ colors when

$n \neq 2$ in Section 6. The final Section 7 concludes our thesis by posing a challenging open problem.

2 Kneser and Reduced Kneser Graphs

For $m \geq 2n$, the *Kneser graph* $\mathsf{KG}(m,n)$ is the graph whose vertex set is the collection of all n-subsets of $[m]$, and two vertices are adjacent if and only if they are disjoint as n-subsets. The next theorem was conjectured by Kneser [4] in 1955 and proved by Lovász [7] about twenty years later.

Theorem 4 *For every Kneser graph* $\mathsf{KG}(m,n)$*, we have* $\chi(\mathsf{KG}(m,n)) = m - 2n + 2$.

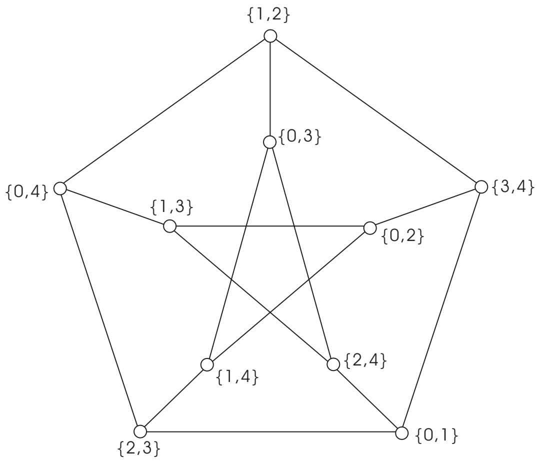

Figure 2: The Kneser graph $\mathsf{KG}(5,2)$ is the Petersen graph.

In 1997, Johnson, Holroyd, and Stahl [3] determined a few circular chromatic numbers of Kneser graphs. They obtained $\chi_c(\mathsf{KG}(2n+1,n)) = 3$, $\chi_c(\mathsf{KG}(2n+2,n)) = 4$, and $\chi_c(\mathsf{KG}(m,2)) = m-2$. A graph G is said to be *circular extremal* if $\chi_c(G) = \chi(G)$. They also proposed the following conjecture.

Conjecture 1 *Every Kneser graph* $\mathsf{KG}(m,n)$ *is circular extremal.*

A subset S of $[m]$ is said to be *2-stable* if $|a-b|_m \geq 2$ for any two distinct elements a and b of S, or equivalently, S contains no elements consecutive in the cyclic order of $[m]$. The *reduced Kneser graph* $\mathsf{KG}_2(m,n)$ is the subgraph of $\mathsf{KG}(m,n)$ induced by all vertices that are 2-stable subsets. Schrijver [9] proved that $\chi(\mathsf{KG}_2(m,n)) = \chi(\mathsf{KG}(m,n))$ and $\mathsf{KG}_2(m,n)$ is vertex color critical. Lih and Liu [5] established the following result for reduced Kneser graphs $\mathsf{KG}_2(m,2)$.

Theorem 5 *If $m \geq 4$ and $m \neq 5$, then the reduced Kneser graph $\mathsf{KG}_2(m,2)$ is circular extremal.*

It can be checked that $\mathsf{KG}_2(2n+1,n)$ is the odd cycle C_{2n+1} whose circular chromatic number is $2+1/n$. In view of the above theorem, Lih and Liu posed the following problem.

Problem 1 *Given n, does there exist $t(n)$ such that $\mathsf{KG}_2(m,n)$ is circular extremal when $m \geq t(n)$?*

Supposing that the answer to the above problem is positive, they further posed the next problem.

Problem 2 *Given n, what is the least value for $t(n)$ such that $\mathsf{KG}_2(m,n)$ is circular extremal when $m \geq t(n)$?*

Problem 1 has been answered in the positive by Hajiabolhassan and Zhu [2]. It follows from their result that Conjecture 1 holds for sufficiently large m for any given n.

Although Problem 1 has been answered in the positive, Problem 2 remains open, except we know $t(2) = 6$ and $t(n) > 2n+1$.

In the next three sections, we will establish our main result that $\mathsf{KG}_2(2n+2,n)$ is also circular extremal.

3 The Structure of $\mathsf{KG}_2(2n+2,n)$

The vertex set of the reduced Kneser graph $\mathsf{KG}_2(m,n)$ has size $\frac{m}{n}\binom{m-n-1}{n-1}$. This can be demonstrated as follows. A 2-stable n-subset of $[m]$ containing the element 0 can be determined by an integer solution to the equation $x_1+x_2+\ldots+x_n=m$, $x_i\geq 2$ for $1\leq i\leq n$, since each x_i can be thought as the circular distance among these elements. Every element will appear in $\binom{m-n-1}{n-1}$ subsets and we will count any subset n times.

We will give a complete analysis of the structure of $\mathsf{KG}_2(2n+2,n)$ when $n\geq 3$. From the foregoing formula, we know that $\mathsf{KG}_2(2n+2,n)$ has $(n+1)^2$ vertices. Our first step is to classify the vertices of $\mathsf{KG}_2(2n+2,n)$ into different kinds so that we can examine the graph more closely.

There are two kinds of solutions to the equation $x_1+x_2+\ldots+x_n=2n+2$, $x_i\geq 2$ for $1\leq i\leq n$. In the first kind, all variables equal 2, except one variable equals 4. In the second kind, all variables equal 2, except two variables equal 3. Note that each variable of the equation $x_1+x_2+\ldots+x_n=2n+2$ represents the $(2n+2)$-circular distance between two consecutive elements in a 2-stable subset. So, we may classify the vertices of $\mathsf{KG}_2(2n+2,n)$ into two kinds. In the sequel, additions or subtractions of elements in $[2n+2]$ are done modulo $2n+2$. Now, a vertex of $\mathsf{KG}_2(2n+2,n)$ is said to be of the *first kind* if it is an n-subset of the form

$$u_i=\{i,i+4,i+6,\ldots,i+2n\},$$

and to be of the *second kind* if it is an n-subset of the form

$$v_{i,j}=\{i,i+3,i+5,\ldots,j,j+3,j+5,\ldots,i+2n\}.$$

Let U be the set of all vertices of the first kind. For $i=1,2,\ldots,2n+2$, we can check that all u_i's are distinct n-subsets. So, there are $2n+2$ vertices in

U. Let V_d be the set consisting of all vertices $v_{i,j}$ satisfying $|i-j|_{2n+2} = d$. Suppose that x is vertex in V_d. If we add a fixed integer to every element of x, we still get a vertex in V_d. There are $2n+2$ vertices in V_d, except V_{n+1} has $n+1$ vertices when n is even.

Now we are going to analyze the adjacency relation among vertices of $\mathsf{KG}_2(2n+2, n)$.

Let u_i be a vertex of the first kind. It is disjoint from a 2-stable n-subset of the form

$$\{i-1, i+5, i+7, \ldots, i+2n-1\} \cup \{j\},$$

where $j \in \{i+1, i+2, i+3\}$, or a 2-stable n-subset of the form

$$\{i-1, i+5, i+7, \ldots, i+2n-1\}^* \cup \{i+1, i+3\},$$

where $\{i-1, i+5, i+7, \ldots, i+2n-1\}^*$ denotes the set $\{i-1, i+5, i+7, \ldots, i+2n-1\}$ with one element deleted. No other 2-stable n-subsets can be disjoint from u_i. Therefore, the degree of u_i is $3+(n-1) = n+2$. Among the neighbors of u_i, only $v_{i-1,i+2} = \{i-1, i+2, i+5, i+7, \ldots, i+2n-1\}$ belongs to V_3, and each of the other vertices $u_{i+1}, u_{i+3}, \ldots, u_{i+2n+1}$ belongs to U. We note that distinct vertices of U are adjacent to distinct vertices of V_3, and the vertices of U induce a subgraph of $\mathsf{KG}_2(2n+2, n)$ isomorphic to the circulant graph $G(2n+2, \{\pm 1, \pm 3, \ldots\})$.

All vertices of the second kind are partitioned into the sets V_d. When $j = i+3$, $v_{i,j}$ belongs to V_3. When $j > i+3$, every other integer from $i+3$ to j and every other integer from $j+3$ to i in the cyclic order of $[2n+2]$ must belong to $v_{i,j}$. Thus, the $(2n+2)$-circular distance between i and j is odd. Hence, the index d of V_d belongs to the set of odd integers $\{3, 5, \ldots, n \ \text{ (or } \ n+1)\}$.

For the vertex $v_{i,j} = \{i, i+3, i+5, \ldots, j, j+3, j+5, \ldots, i+2n\}$, all the possible 2-stable n-subsets disjoint from it are of the form

$$\{i+4, i+6, \ldots, j-1, j+4, j+6, \ldots, i+2n+1\} \cup \{x\} \cup \{y\},$$

where $x \in \{i+1, i+2\}$ and $y \in \{j+1, j+2\}$. So, the degree of $v_{i,j}$ is 4. Let $v_{i,j}$ belong to V_d. Then among the four neighbors of $v_{i,j}$, we see that $v_{i+1,j+1} = \{i+1, i+4, i+6, \ldots, j+1, j+4, j+6, \ldots, i+2n+1\}$ and $v_{i-1,j-1} = \{i+2, i+4, i+6, \ldots, j-1, j+2, j+4, \ldots, i+2n+1\}$ belong to V_d, $v_{i+1,j-1} = \{i+1, i+4, i+6, \ldots, j-1, j+2, j+4, \ldots, i+2n+1\}$ belongs to V_{d-2} if $d > 3$, and $v_{i-1,j+1} = \{i-1, i+2, i+4, \ldots, j-1, j+1, j+4, \ldots, i+2n-1\}$ belongs to V_{d+2}. However, when $d = 3$, one of its neighbor is in U since there is no vertex set V_1.

The two adjacent vertices $v_{i,j}$ and $v_{i+1,j+1}$ in V_d have $v_{i-1,j+1}$ and $v_{i,j+2}$ in V_{d+2} as their neighbors, respectively. We see that $v_{i-1,j+1}$ and $v_{i,j+2}$ are adjacent, too. Hence, V_d and V_{d+2} induce a subgraph which is isomorphic to $C_{2n+2} \Box P_2$ when $3 \leq d \leq n-2$. This also holds for U and V_3.

When n is odd, the neighbors of $v_{i,i+n}$ are $v_{i+1,i+n-1}, v_{i+1,i+n+1}, v_{i-1,i+n-1}$, and $v_{i-1,i+n+1}$. The last vertex also belongs to V_n since the $(2n+2)$-circular distance between $i-1$ and $i+n+1$ is n. In fact, $v_{i-1,i+n+1}$ is identical to the vertex $v_{i+n+1,i+2n+1}$. Therefore, the vertex set V_n induces a subgraph of $\mathsf{KG}_2(2n+2, n)$ isomorphic to the circulant graph $G(2n+2; \{\pm 1, \pm(n+1)\})$. However, we notice that $G(2n+2; \{\pm 1, \pm(n+1)\})$ can be obtained from the grid graph $G(n+2, 2)$ by identifying the vertex $(1,1)$ with the vertex $(n+2, 2)$ and the vertex $(1,2)$ with the vertex $(n+2, 1)$. Thus, $G(2n+2; \{\pm 1, \pm(n+1)\})$ is actually the Möbius grid $\mathsf{MG}(n+1, 2)$, and the boundary of $G(2n+2; \{\pm 1, \pm(n+1)\})$ coincides with the boundary of $\mathsf{MG}(n+1, 2)$.

When n is even, the vertex set V_{n+1} has only $n+1$ vertices. This is because we obtain the same vertex $v_{i,j}$ when $n+1$ is added to every element of a vertex $v_{i,j}$ in V_{n+1}. Every vertex in V_{n-1} has a neighbor in V_{n+1}. So every

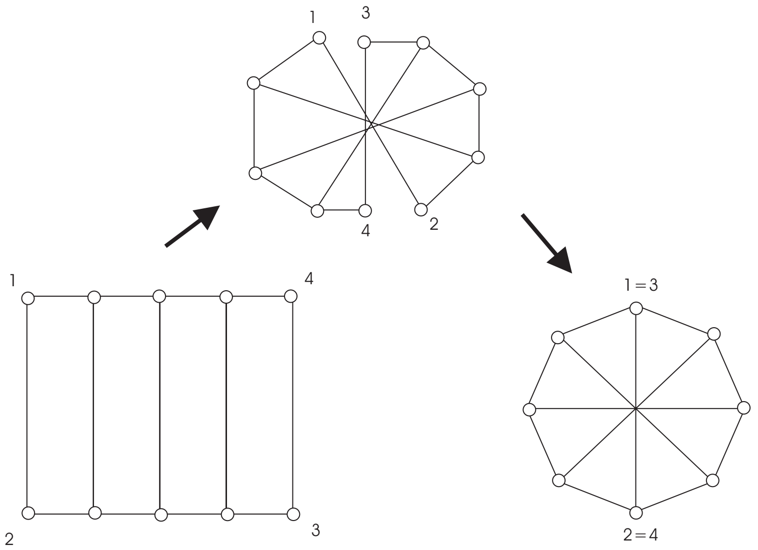

Figure 3: The circulant graph $G(2n+2;\{\pm 1, \pm(n+1)\})$ is the Möbius grid $\mathsf{MG}(n+1,2)$.

vertex in V_{n+1} has two neighbors in V_{n-1}. Since the two neighbors $v_{i-1,i+n+2}$ and $v_{i+1,i+n}$ of $v_{i,i+n+1}$ belong to V_{n-1}, these adjacencies form a grid graph when i runs through $0, 1, \ldots, n+1$. However, $v_{n+1,2n+2}$ is identical to $v_{0,n+1}$, $v_{n,2n+3}$ is identical to $v_{1,n}$, and $v_{n+2,2n+1}$ is identical to $v_{2n+1,n+2}$. When the identifications are done, we see that the subgraph of $\mathsf{KG}_2(2n+2,n)$ induced by V_{n-1} and V_{n+1} is the Möbius grid $\mathsf{MG}(n+1,3)$.

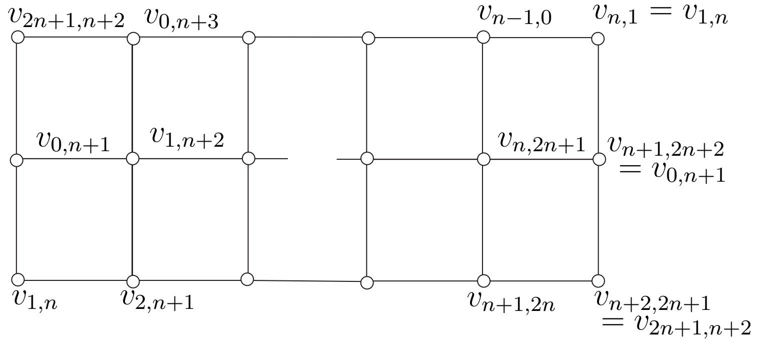

Figure 4: The subgraph induced by V_{n+1} and V_{n-1} is a Möbius grid.

Finally, we summarize our analysis in the following structural theorem. For $1 \leq j \leq b$, we use C_j^* to represent the cycle in the cylinder graph $C_a \square P_b$ induced by the vertices (i, j) for $1 \leq i \leq a$.

Theorem 6 *Let $n \geq 3$ be an integer. The reduced Kneser graph $\mathsf{KG}_2(2n+2, n)$ is obtained from the cylinder graph $C_{2n+2} \square P_{\lceil \frac{n}{2} \rceil}$ by identifying the boundary of the circulant graph $G(2n+2; \{\pm 1, \pm 3, \ldots\})$ with the first cycle C_1^* of the cylinder graph, and then by identifying the boundary of a certain Möbius grid M with the last cycle $C_{\lceil \frac{n}{2} \rceil}^*$ of the cylinder graph. When n is odd, the Möbius grid M is $\mathsf{MG}(n+1, 2)$. When n is even, the Möbius grid M is $\mathsf{MG}(n+1, 3)$.*

The construction of the Möbius grid in the above theorem only used the last cycle $C_{\lceil \frac{n}{2} \rceil}^*$ of the cylinder graph. Actually, the same method applies to the entire cylinder graph. The outcome is the Möbius grid $\mathsf{MG}(n+1, n+1)$, independent of the parity of n. Therefore, we have the following second form of our structural theorem.

Theorem 7 *Let $n \geq 3$ be an integer. The reduced Kneser graph $\mathsf{KG}_2(2n+2, n)$ is obtained by identifying the boundary of the circulant graph $G(2n+2; \{\pm 1, \pm 3, \ldots\})$ with the boundary of the Möbius grid $\mathsf{MG}(n+1, n+1)$.*

4 Flow Ratio Theorems

If the flow ratio of a 4-cycle is less than 3, then it must contain two forward and two backward arcs independent of the choice of direction of traversal. This also means that the flow ratio must be 1. In the sequel, we will use this simple fact to determine an appropriate direction of traversal so that desired results can be obtained.

When we move around a cycle, the direction of traversal is fixed once we move from a starting vertex to one of its two neighbors. Consequently, it is convenient to fix the direction of traversal for a cycle by telling which direction to move on an arbitrarily chosen pair of consecutive vertices. Let each of two cycles possess its own direction of traversal. Suppose that the two cycles share some common arcs. We say that the traversals of the two cycles are *consistent* if any arc is never forward in one traversal and backward in the other.

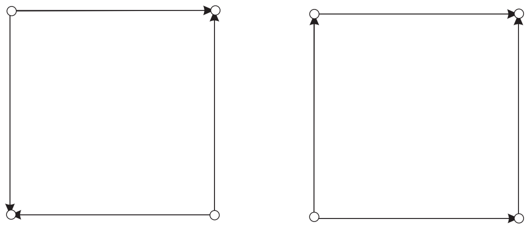

Figure 5: Orientations of C_4 with $\theta(C_4; D) = 1 < 3$.

Theorem 8 *Suppose that every* 4*-cycle of the grid graph* $G(m, n)$ *has flow ratio 1 with respect to an orientation* D*. Then* $\theta(C; D) = 1$ *for every cycle* C *of the underlying grid graph.*

Proof. For every 4-cycles $(i, j)(i+1, j)(i+1, j+1)(i, j+1)(i, j)$, $1 \leq i < m$ and $1 \leq j < n$, the direction of traversal is chosen from (i, j) to $(i+1, j)$. Then every arc shared by two such 4-cycles is forward in one cycle and backward in the other. The grid graph $G(m, n)$ can be drawn in the Euclidean plane in a canonical way. Let C be a cycle of $G(m, n)$. Then the region enclosed within C consists of a number of 4-cycles. We define the direction of traversal of C by starting from the vertex having the following properties. (1) It has the smallest first coordinate. (2) Among vertices with the smallest first coordinate, it has the smallest second coordinate. Let this vertex be (i, j). Then we move from (i, j) to $(i + 1, j)$. The traversal so defined is consistent with the traversal of every 4-cycles enclosed within C. When we count the number of forward, respectively backward, arcs of 4-cycles enclosed within C, each arc not included in C will be counted once as forward and once as backward arc. Since each 4-cycle contains exactly two forward and two backward arcs, the total number of forward arcs of 4-cycles enclosed within C is equal to the total number of backward arcs. It implies that the number of forward arcs on C is equal to the number of backward arcs on C. Therefore $\theta(C; D) = 1$. □

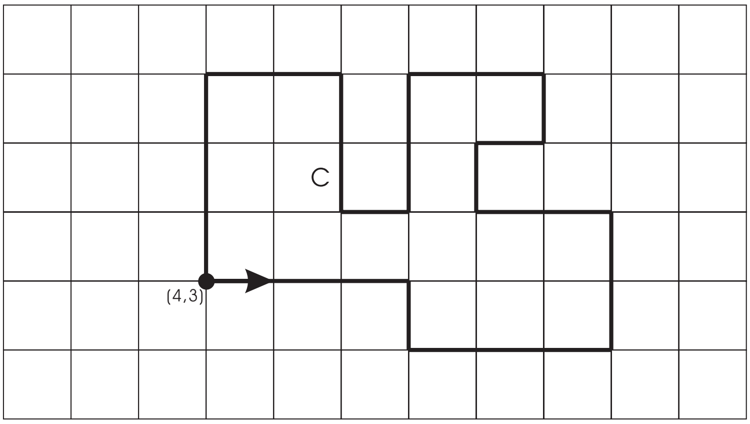

Figure 6: The direction of traversal for C is from $(4, 3)$ to $(5, 3)$.

Theorem 9 *Suppose that every* 4*-cycle of the cylinder graph* $C_m \Box P_n$ *has flow ratio 1 with respect to an orientation* D*. Then the flow ratios* $\theta(C_j^*; D)$*,* $1 \leq j \leq n$*, of all the cycles* C_j^* *are identical.*

Proof. It suffices to show the case for $n = 2$. Let $(m+1, j)$ also denote the vertex $(1, j)$. For every 4-cycle $(i,1)(i{+}1,1)(i{+}1,2)(i,2)(i,1)$, $1 \leq i \leq m$, the direction of traversal is chosen from $(i,1)$ to $(i{+}1,1)$. It follows that each arc whose endpoints are $(i,1)$ and $(i,2)$ is forward in one cycle and backward in the other. Since each 4-cycle contains exactly two forward and two backward arcs, the total number of forward arcs on all these 4-cycles is equal to the total number of backward arcs. Hence the number of forward arcs in C_1^* is equal to the number of backward arcs in C_2^*. Therefore $\theta(C_1^*; D) = \theta(C_2^*; D)$. $\square$

Recall that the *boundary* of a Möbius grid $\mathsf{MG}(m,n)$ is the cycle consisting of the vertices $(1,1), (2,1), \ldots, (m+1,1) = (1,n), (2,n), \ldots, (m+1,n) = (1,1)$.

Theorem 10 *Suppose that every* 4*-cycle of the Möbius grid* $\mathsf{MG}(m,n)$ *has flow ratio 1 with respect to an orientation* D*. Let the parities of* m *and* $n-1$ *be different. Then the flow ratio of its boundary with respect to the orientation* D *is greater than 1.*

Proof. Since the Möbius grid $\mathsf{MG}(m,n)$ is obtained from the grid graph $G(m+1,n)$ by identification, we also use (i,j)'s to represent vertices of the Möbius grid. For every 4-cycle $(i,j)(i+1,j)(i+1,j+1)(i,j+1)(i,j)$, $1 \leq i \leq m$ and $1 \leq j < n$, the direction of traversal is chosen from (i,j) to $(i+1,j)$. The orientation D of $\mathsf{MG}(m,n)$ stated in the assumption obviously induces an orientation of $G(m+1,n)$. Since the edges $(m+1,n-j)(m+1,n-j+1)$ and $(1,j)(1,j+1)$ are identified in $\mathsf{MG}(m,n)$, the traversals of the two 4-cycles $(1,j)(2,j)(2,j+1)(1,j+1)(1,j)$ and $(m,n-j)(m+1,n-j)(m+1,n-j+1)(m,n-j+1)(m,n-j)$ are consistent. Therefore our assumption implies that every 4-cycle in $G(m+1,n)$ has two forward and two backward arcs. Furthermore, if there are a forward and b backward arcs along the path $(1,1),(1,2),\ldots,(1,n)$ in $G(m+1,n)$, then there are the same numbers of forward and backward arcs along the path $(m+1,1),(m+1,2),\ldots,(m+1,n)$. Let there be c forward and d backward arcs along the path $(1,1),(2,1),(3,1),\ldots,(m,1),(m+1,1)$, and x forward and y backward arcs along the opposite path $(1,n),(2,n),(3,n),\ldots,(m,n),(m+1,n)$. Since the total number of forward arcs of $G(m+1,n)$ is equal to the total number of backward arcs, it follows that the quantities a, b, c, d, x, and y satisfy the following equations.

$$\begin{aligned} x+y &= c+d \\ c+x+2a &= d+y+2b \end{aligned}$$

Solving these equations for x and y, we get $x=d+b-a$, $y=c-b+a$.

Now we examine the flow ratio of the boundary of the Möbius grid. The direction of traversal of the boundary is chosen from $(1,1)$ to $(2,1)$. It will go through $(m+1,1)=(1,n)$, then to $(2,n)$, and so on. This traversal is not consistent with the 4-cycles containing one of the edges along the path $P:(1,n),(2,n),\ldots,(m+1,n)$. Therefore, the forward arcs of those 4-cycles on path P become backward arcs of the boundary, and vise versa. Hence, there are $y+c=2c-b+a$ forward and $x+d=2d+b-a$ backward arcs on the boundary of the Möbius grid. Since $a+b=n-1$ and $c+d=m$ and the parities of m and $n-1$ are different, it follows that the parities of $b-a$ and $c-d$ are different, too. Consequently, $x+d\neq y+c$ and the flow ratio of the boundary of the Möbius grid is greater than one. □

It is known that the circular chromatic number of a grid graph and the circular chromatic number of a cylinder graph are both less than 3. Therefore the assumptions of Theorems 8 and 9 can be satisfied. The next theorem shows that the assumption of Theorem 10 can be satisfied, too.

Theorem 11 *The circular chromatic number of a Möbius grid is determined as follows.*

(1) $\chi_c(\mathsf{MG}(m,n)) = 2$ *if the parities of* m *and* $n-1$ *are identical.*

(2) $\chi_c(\mathsf{MG}(m,n)) = 2m/(m-1)$ *if the parities of* m *and* $n-1$ *are different.*

Proof. Again, we use the vertices of $G(m+1,n)$ to represent the vertices of $\mathsf{MG}(m,n)$.

Case 1. If the parities of m and $n-1$ are identical, so are the parities of $m+1$ and n. We color every vertex (i,j) with the number $(i+j) \bmod 2$. This is a proper 2-coloring of $\mathsf{MG}(m,n)$. Hence it is a bipartite graph and has circular chromatic number 2.

Case 2.1 Assume that m is odd and $n-1$ is even.

There are $n = 2k-1$ vertices on one side of the grid graph $G(m+1,n)$, and $k = (n+1)/2$ is the center vertex. The vertices $(1,k),(2,k),\ldots,(m,k),(m+1,k) = (1,k)$ induce a cycle in $\mathsf{MG}(m,n)$ of odd length m. This cycle has circular chromatic number $m/((m-1)/2) = 2m/(m-1)$. Hence $\chi_c(\mathsf{MG}(m,n)) \geq 2m/(m-1)$. We will give $\mathsf{MG}(m,n)$ an $(m,(m-1)/2)$-coloring to derive $\chi_c(\mathsf{MG}(m,n)) = 2m/(m-1)$. Let σ be a map from $V(G(m+1,n))$ to $[m]$ defined as follows. First, we let $\sigma((i,k))$ be the number $(m-1)i/2 \bmod m$, and $\sigma((i,k \pm j))$ be the number $(m-1)(i+j)/2 \bmod m$. The difference of two adjacent vertices is $(m-1)/2$. Then we need to check two vertices of the grid graph that represent the same vertex of the Möbius grid have the same color. Since $(1,k \pm j)$ and $(m+1,k \mp j)$ are identical in $\mathsf{MG}(m,n)$ for $1 \leq j < k$, we know that $\sigma(1,k \pm j) \equiv (m-1)(1+j)/2 \pmod{m}$

and $\sigma(m+1, k \mp j) \equiv (m-1)(m+1+j)/2 \pmod{m}$. Their difference is $(m-1)[(m+1+j)-(1+j)]/2 \equiv 0 \pmod{m}$. Thus, the map σ is a $(m,(m-1)/2)$-coloring of $\mathsf{MG}(m,n)$.

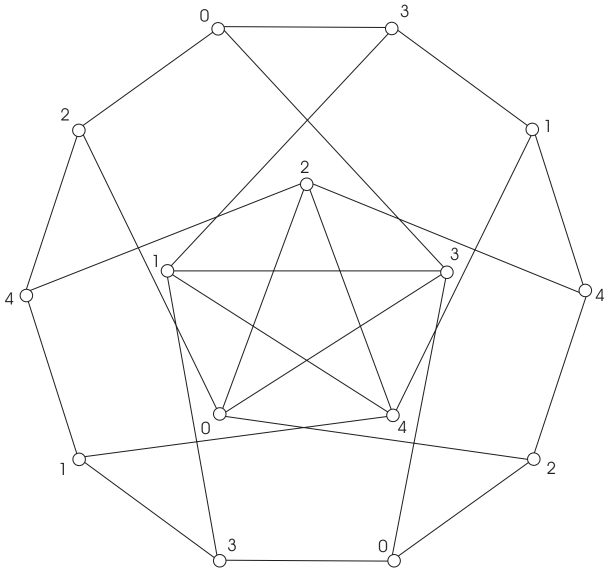

Figure 7: A $(5,2)$-coloring of $\mathsf{MG}(5,3)$.

Case 2.2 Assume that m is even and $n-1$ is odd.

Now, n is even. We give a $(2m, m-1)$-coloring to $G(m+1,n)$ and check that it is well-defined on $\mathsf{MG}(m,n)$. Let τ be a map from $V(G(m+1,n))$ to $[2m]$ defined as follows. Let $\tau((i,j))$ be the number $(i-1)(m-1) \bmod 2m$ for odd j, and $\tau((i,j))$ be the number $m+(i-1)(m-1) \bmod 2m$ for even j. The vertices $(1,j)$ and $(m+1,n+1-j)$ are identical in $\mathsf{MG}(m,n)$ for $1 \leq j \leq n$. First let j be odd. We know that $\tau((1,j)) = 0$ and $\tau((m+1,n+1-j)) \equiv m+m(m-1) \equiv m^2 \equiv 0 \pmod{2m}$ since m is even. On the other hand, if j is even, then $\tau((1,j)) = m$ and $\tau(m+1,n+1-j) \equiv m(m-1) \equiv m^2-m \equiv m \pmod{2m}$. Therefore, τ is a $(2m, m-1)$-coloring of $\mathsf{MG}(m,n)$, and hence $\chi_c(\mathsf{MG}(m,n)) \leq 2m/(m-1)$. From Theorem 3, there exists an orientation D such that $\theta(\mathsf{MG}(m,n); D) = \chi_c(\mathsf{MG}(m,n))-1 \leq (m+1)/(m-1) < 3$. Hence the flow ratio of the boundary of $\mathsf{MG}(m,n)$ is greater than 1 by Theorem 10. Since the boundary has length $2m$, it implies that its flow ratio is at least $(m+1)/(m-1)$. This implies that $\theta(\mathsf{MG}(m,n);D) \geq (m+1)/(m-1)$, and hence $\chi_c(\mathsf{MG}(m,n)) \geq 2m/(m-1)$. The theorem is thus proved. $\square$

5 Main Result

We will determine the circular chromatic number of $\mathsf{KG}_2(2n+2,n)$ in this section. The following lemmas are proved by the method established in the previous section. Recall that the vertex set of a circulant graph $G(m;E)$ is $\{v_1,v_2,\ldots,v_m\}$. If E contains ± 1, then the cycle $v_1v_2\cdots v_mv_1$ is called the *boundary* of $G(m;E)$.

Lemma 12 *Let G be the circulant graph $G(2n+2;\{\pm1,\pm3,\ldots,\pm n\})$ when n is odd; the circulant graph $G(2n+2;\{\pm1,\pm3,\ldots,\pm(n+1)\})$ when n is even. Assume that every 4-cycle of G has flow ratio 1 with respect to an orientation D. Then the boundary of G also has flow ratio 1 with respect to D.*

Proof. Since $v_{1+i}v_{2n+2-i}$ is an edge for all $0 \le i \le n$, we observe that these edges and the boundary form a grid graph $G(n+1,2)$ that is a subgraph of G. This subgraph is endowed with an orientation by the restriction of D. Every 4-cycle of $G(n+1,2)$ is a 4-cycle of G. Hence each 4-cycle of $G(n+1,2)$ has two forward and two backward arcs. It follows that the flow ratio of the boundary of the grid graph, which is the boundary of $G(2n+2;\{\pm1,\pm3,\ldots\})$, is equal to 1. □

Lemma 13 *Let G be the Möbius grid $\mathsf{MG}(n+1,2)$, where n is odd. Assume that every 4-cycle of G has flow ratio 1 with respect to an orientation D. Then the flow ratio of the boundary of G with respect to D is greater than 1.*

Proof. Since the parities of $n+1$ and 1 are different, the flow ratio of the boundary is greater than 1 by Theorem 10. □

Lemma 14 *Let G be the Möbius grid $\mathsf{MG}(n+1,3)$, where n is even. Assume that every 4-cycle of G has flow ratio 1 with respect to an orientation D. Then the flow ratio of the boundary of G with respect to D is greater than 1.*

Proof. Since the parities of $n+1$ and 2 are different, the flow ratio of the boundary is greater than 1 by Theorem 10. $\square$

Now we are ready to prove our main theorem.

Theorem 15 *The reduced Kneser graph $\mathsf{KG}_2(2n+2,n)$ is circular extremal for all n.*

Proof. We want to show that the circular chromatic number of $\mathsf{KG}_2(2n+2,n)$ is equal to 4. For $n=1$, the reduced Kneser graph is K_4 whose circular chromatic number is 4. The case for $n=2$ has already been proved in Lih and Liu [5]. When $n \geq 3$, assume that the circular chromatic number is smaller than 4. By Theorem 3, there exists an orientation D such that $\theta(\mathsf{KG}_2(2n+2,n);D) < 3$. It follows that every 4-cycle of $\mathsf{KG}_2(2n+2,n)$ has two forward and two backward arcs with respect to D. From the structure of $\mathsf{KG}_2(2n+2,n)$ given by Theorem 6 and Lemmas 12, 13, and 14, we know that the first and the last cycles of the cylinder graph $C_{2n+2}\square P_{\lceil \frac{n}{2} \rceil}$ have different flow ratios. However, they should be equal according to Theorem 9. Thus, a contradiction is obtained. $\square$

Remark. An even shorter proof of the main result can be obtained by applying the second form of our structural theorem. Since the parities of $n+1$ and n are different, the flow ratio of the boundary of the Möbius grid $\mathsf{MG}(n+1,n+1)$ is greater than 1 by Theorem 10. However, that boundary is also the boundary of the circulant graph $G(2n+2;\{\pm1,\pm3,\ldots\})$, and hence its flow ratio should be 1 by Lemma 12. Therefore, a contradiction follows from Theorem 7.

Since the reduced Kneser subgraph is a subgraph of the Kneser graph, we immediately get the following corollary.

Corollary 16 *The Kneser graph* $\mathsf{KG}(2n+2,n)$ *has circular chromatic number 4.*

This result was first established in Johnson, Holroyd, and Stahl [3]. However, our approach is completely different from theirs.

6 Edge Coloring of $\mathsf{KG}_2(2n+2, n)$

In this section, we are going to give a by-product of the structural analysis of the reduced Kneser graph $\mathsf{KG}_2(2n+2, n)$.

A proper edge coloring of a graph G is a mapping from its edge set to a set of colors such that two edges incident to a common vertex are mapped to different colors. The least number of colors required for a proper edge coloring of G is called the *chromatic index* of G and is denoted by $\chi'(G)$. The following fundamental theorem about the chromatic index was proved by Vizing.

Theorem 17 *Let $\Delta(G)$ denote the maximum degree of the graph G. The chromatic index of G is equal to either $\Delta(G)$ or $\Delta(G)+1$.*

A graph G is classified into class one if its chromatic index equal to $\Delta(G)$, otherwise it is classified into class two. A *matching* is a set of edges such that no two edges in that set are incident to a common vertex. So every color class of a proper edge coloring is a matching. A graph is called *k-regular* if every vertex has degree k.

Lemma 18 *Every k-regular graph G with odd number of vertices belongs to class two.*

Proof. Assume that G belongs to class one. Since every vertex of G has degree k, those k edges incident to a common vertex are colored with exactly k colors. Hence edges of the same color form a matching containing every vertex. This contradicts the assumption that G has an odd number of vertices. $\square$

Thus, odd cycles and complete graphs on odd number of vertices all belong to class two. The next theorem constructs a $\Delta(G)$-edge coloring of the circulant graph $G(2n+2; \{\pm 1, \pm 3, \ldots\})$.

Theorem 19 *Every circulant graph of the form $G(2n+2;\{\pm1,\pm3,\ldots\})$ belongs to class one.*

Proof. We give an $(n+1)$-edge coloring of $G(2n+2;\{\pm1,\pm3,\ldots\})$. For $0 \le j \le n$, let the edges $v_{1+j+i}v_{2n+2-j+i}$ be colored with i,where $0 \le i \le n$. $\square$

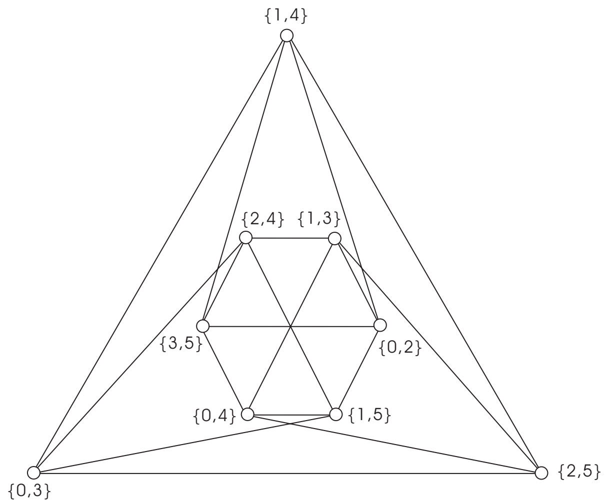

Figure 8: The reduced Kneser graph $\mathsf{KG}_2(6,2)$.

Theorem 20 *Except for $n=2$, the reduced Kneser graph $\mathsf{KG}_2(2n+2,n)$ belongs to class one.*

Proof. When $n=1$, K_4 belongs to class one. For the case $n=2$, $\mathsf{KG}_2(6,2)$ has $(2+1)^2=9$ vertices and is 4-regular. By Lemma 18, it belongs to class two. For $n \ge 3$, the maximum degree of $\mathsf{KG}_2(2n+2,n)$ is $n+2$. Since vertices in U induces the circulant graph $G(2n+2;\{\pm1,\pm3,\ldots\})$. It has an $(n+1)$-edge coloring by Theorem 19. We assign the $(n+2)$-th color to the edges between U and V_3. The remaining vertices have degree at most four. Thus, we can easily color the rest of the edges with $n+1$ colors. However, for $n = 3$, it should be noted that the remaining vertices have degree three. There is a 4-edge coloring for the rest of the edges by Theorem 17. $\square$

7 Conclusion

We know from our main theorem that the reduced Kneser graph $\mathsf{KG}_2(2n+2, n)$ is circular extremal. However, this does not solve the second problem posed by Lih and Liu. There is a rapidly growing gap between $2n+2$ and the bound calculated by Hajiabolhassan and Zhu. It is a challenging problem to determine whether $\mathsf{KG}_2(m, n)$ is circular extremal for any $m > 2n+2$. A positive answer implies that the conjecture about Kneser graphs proposed by Johnson, Holroyd, and Stahl is true. At the present stage, the structure of $\mathsf{KG}_2(m, n)$, $m \geq 2n+3$, seems too complicated to analyze.

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