1 True or False Problems

(84 Jan.) Every bounded measurable function on $[a, b]$ is Riemann integrable.

(84 Jan.) If f is continuous and increasing on $[0, 1]$, then f is absolutely continuous.

(84 Jan.) If $f \in L^\infty([0, 1])$, then $||f||_\infty = \lim_{p\to\infty} ||f||_p$.

(84 Jan.) If f is measurable function on $\mathbb{R}$ and $f > 0$ a.e. then there exist $\delta > 0$ and a measurable set E with $m(E) > 0$ and $f(x) \geq \delta$ for $x \in E$.

(84 Aug.) If $f : [a, b] \to \mathbb{R}$ is increasing and continuous, $E \subseteq [a, b]$ and $m(E) = 0$, then $m(f(E)) = 0$.

(84 Aug.) If (f_n) is a sequence of measurable functions on $\mathbb{R}$ and $f_n \to f$ a.e. then $f_n \to f$ in measure.

(84 Aug.) Given $\epsilon > 0$ there exists a closed nowhere dense set E with $m(\mathbb{R} - E) \leq \epsilon$.

(87 Aug.) If $f \geq 0$ is Lebesgue integrable on $[0, 1]$ and $\int_0^1 f(x)dx > 0$, then there exist $c > 0$ and a nonempty open interval $I \subset [0, 1]$ such that $f(x) \geq c$ for all $x \in I$.

(87 Aug.) If $0 \leq f_1 \leq f_2 \leq ...$ and f_n converges to f in measure on $[0, 1]$, then $f_n \to f$ a.e. on $[0, 1]$.

(87 Aug.) If $f \in L^1(\mathbb{R}, m) \cap L^\infty(\mathbb{R}, m)$, then $f \in L^p(\mathbb{R}, m)$ for all $1 < p < \infty$.

(87 Aug.) If E is a Lebesgue measurable set in $\mathbb{R}$ such that there exists a $c \in (0, 1)$ with $m(E \cap I) \leq cm(I)$ for all intervals I, then $m(E) = 0$.

(87 Aug.) If f is a bounded Lebesgue integrable function on $[a, b]$, then f is Riemann integrable over $[a, b]$.

(88 Jan.) If f is monotone on $[a, b]$ and f' exist a.e., then $f' \in L^1([a, b])$.

(88 Jan.) If f is monotone on $[a, b]$ and $f'(x) = 0$ a.e. on (a, b), then f is a constant on $[a, b]$.

(88 Jan.) Let μ and ν be measures on a measurable space $(X, \sum)$. If $\mu \ll \nu$, then there exists $f \in L^1(X)$ such that $\mu(B) = \int_B f d\nu$, for all $B \in \sum$.

(88 Jan.) If $\{f_n\}$ is a sequence in $L^1(X, \mu)$ and $\int |f_n| d\mu \to 0$ as $n \to \infty$, then $\epsilon_n \to 0$ μ-a.e.

(88 Aug.) Let E_n be measurable sets such that $E_n \downarrow E$. Then $m(E_n) \downarrow m(E)$.

(88 Aug.) Let E_n be measurable sets such that $E_n \uparrow E$. Then $m(E_n) \uparrow m(E)$.

(88 Aug.) If f is a non-decreasing function on $[0, 1]$ then there exist $a < b$ in $[0, 1]$ such that f is continuous on (a, b).

(88 Aug.) If $|f|$ is measurable, then f is measurable.

(88 Aug.) If $f_n \to f$ in $L_1[0, 1]$, then $f_n \to f$ a.e.

(88 Aug.) If E is measurable with $m(E) < \infty$, then E is bounded.

(89 Jan.) If $f_n \to f$ a.e. on $[0, 1]$ then $f_n \to f$ in measure.

(89 Jan.) If $f_n \to f$ in measure then $f_n \to f$ a.e.

(89 Jan.) If $||f_n - f||_1 \to 0$, then $f_n \to f$ a.e.

(89 Jan.) If $f_n \to f$ a.e. then $||f_n - f||_1 \to 0$.

(89 Jan.) Let f be an integrable function on $\mathbb{R}$. If $< E_n >$ is a decreasing sequence of measurable sets and $E = \cap E_n$ then

$$\int_E f dm = \lim_{n\to\infty} \int_{E_n} f dm.$$

(89 Jan.) If f is non-decreasing and continuous on $[0, 1]$ and $f'(x) = 0$ a.e., then f is constant.

(89 Aug.) If $f_n \to f$ in $L^1(X, \mu)$, then $f_n \to f$ in measure.

(89 Aug.) If $f_n \to f$ in measure, then $f_n \to f$ in $L^1([0, 1])$.

(89 Aug.) If $f_n \to 0$ uniformly on $\mathbb{R}$ and f_n Lebesgue integrable, then $\int f_n d\lambda \to 0$.

(89 Aug.) If f_n Lebesgue measurable and $f_1 \geq f_2 \geq \cdots \geq 0$, then $\lim_{n\to\infty} \int_E f_n d\lambda = \int_E \lim_{n\to\infty} f_n d\lambda$.

(89 Aug.) Let $0 \leq f$ be Lebesgue integrable over $\mathbb{R}$. Then for all $\epsilon > 0$ there exists $E \subset \mathbb{R}$ with $\lambda(E) < \infty$ such that $\int f d\lambda \leq \int_E f d\lambda + \epsilon$.

(90 Aug.) $\lim\limits_{n\to\infty} \int_1^\infty \frac{e^{nx}}{e^{nx}x^2+1} dx = 1$.

(90 Aug.) Let $< E_n >$ be a decreasing sequence of Lebesgue measurable subsets of $\mathbb{R}$ with empty intersection. Then $\lambda(E_n) \to 0$.

(90 Aug.) Let E be a $(\lambda \times \lambda)$-measurable subset of $[0,1] \times [0,1]$ such that $\lambda\{x \in [0,1] : \lambda(E_x) \geq \frac{1}{2}\} \geq \frac{7}{8}$. Then $\lambda\{y \in [0,1] : \lambda(E^y) \geq \frac{1}{4}\} \geq \frac{1}{4}$

(91 Jan.) If f is continuous a.e. on $[a,b]$, then there exists a continuous function g on $[a,b]$ such that $f = g$ a.e.

(91 Jan.) If f is absolutely continuous on $[a,b]$ and $f' = g$ a.e., where g is continuous function on $[a,b]$, then $f' = g$ everywhere in $[a,b]$.

(91 Jan.) If $f' = 0$ a.e., then f is of bounded variation.

(91 Jan.) If f is continuous on $[0,1]$ and $\int_0^1 t^n f(t)dt = \frac{1}{n+2}$ for all $n \geq 0$, then $f(t) = t$ for all $t \in [0,1]$.

(91 Jan.) If $f_n \in L^1[0,1]$ such that $||f_n||_1 \leq 1$ and $f_n(x) \to 0$ a.e., then $\int f_n(t)g(t)dt \to 0$ for all $g \in L^\infty[0,1]$.

(91 Aug.) If $< f_n >_{n=1}^\infty$ is a sequence of measurable functions on $[0,1]$ with $0 \leq f_n \leq 1$ for all n and $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$, then $< f_n >_{n=1}^\infty$ converges to 0 in measure.

(91 Aug.)If $< f_n >_{n=1}^\infty$ is a sequence of measurable functions on $[0,1]$ with $0 \leq f_n \leq 1$ for all n and $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$, then $< f_n >_{n=1}^\infty$ converges to 0 almost everywhere.

(91 Aug.) If U is an open dense subset of $(0,1)$ then $m(U) = 1$.

(91 Aug.) There is a sequence of measurable functions $< f_n >_{n=1}^{\infty}$ with $\int_0^1 f_n(x)dx = 1$ for all n but $f_n \to 0$ almost everywhere.

(92 Aug.) If (f_n) is a sequence of nonnegative measurable functions on $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in \mathbb{R}$ and $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = 0$, then $f(x) = 0$ a.e.

(92 Aug.) If $f(z)$ is an analytic function on an open set $\mathcal{O}$ and γ is a piece-wise smooth closed curve in $\mathcal{O}$, then $\int_\gamma f(z)dz = 0$.

(92 Aug.) If $f(z)$ and $g(z)$ are analytic on an open set $\mathcal{O}$,

$$\overline{D_r(a)} = \{z \in \mathbb{C} : |z - a| \leq r\} \subset \mathcal{O},$$

for some $r > 0$, and $f(z) = g(z)$ for all $z \in T_r(a) = \{z \in \mathbb{C} : |z-a| = r\}$, then $f(z) = g(z)$ on $D_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$.

(92 Aug.) If $f(z)$ is an entire function and $|f(z)| \geq M$ for all $z \in \mathbb{C}$ (some $M > 0$) then $f(z)$ is constant.

(92 Aug.) If f is a continuous function of bounded variation on $[a, b]$ and $f'(x) = 0$ a.e. then f is constant on $[a, b]$.

(93 Jan.) If K is a compact subset of $\mathbb{R}$ and there is a continuous onto function $f : K \to [0, 1]$, then $m(K) > 0$.

(93 Jan.) If f is analytic on an open subset U of $\mathbb{C}$ and γ is a piece-wise smooth closed curve in U, then

$$\int_\gamma f(z)dz = 0$$

(93 Jan.) If $< f_n >_{n=1}^{\infty}$ is a sequence of increasing functions on $\mathbb{R}$ and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = 0$$

then $< f_n >_{n=1}^{\infty}$ converges to 0 in measure.

(93 Jan.) If f is an integrable measurable function on $[0, 1]$ then there is a subinterval $(a, b) \subset [0, 1]$ so that f is continuous on (a, b).

(94 Jan.) Let $f_n \in L_p([a,b])$ $(1 \le p < \infty)$ such that $\sum_{n=1}^{\infty} ||f_n||_p < \infty$. Then $f_n(x) \to 0$ a.e. on $[a,b]$.

(94 Jan.) Let f be a continuous function on $[0,1]$ such that $f = 0$ a.e. Then $f(x) = 0$ for all $x \in [0,1]$.

(94 Jan.) There exists a function $f(z)$ analytic in a neighborhood of 0 such that

$$f'(-\frac{1}{n}) = f'(\frac{1}{n}) = \frac{1}{n^3}.$$

(94 Jan.) Let $f : [a,b] \to \mathbb{R}$ be a function such that for all $x \in [a,b]$ there exists a $\delta > 0$ such that f is bounded on $(x-\delta, x+\delta) \cap [a,b]$. Then f is bounded on $[a,b]$.

(94 Aug.) Let f be an entire function such that $\mathrm{Re} f(z) \ge 0$ for all z. Then f is a constant.

(94 Aug.) $L_2(\mathbb{R}) \subset \{g + h : g \in L_1(\mathbb{R}), h \in L_\infty(\mathbb{R})\}$.

(94 Aug.) If $f_n \in L_1[0,1]$ and $\int_0^1 |f_n| dx \to 0$, then there exists $g \in L_1[0,1]$ such that $|f_n| \le g$.

(96 Aug.) If f is holomorphic in the region $\mathbb{C}\backslash\{0\}$ and γ is a simple closed path in $\mathbb{C}\backslash\{0\}$, then $\int_{gamma} f(z)dz = 0$.

(96 Aug.) Every uncountable set of real numbers has a non-measurable subset.

(96 Aug.) Suppose that $f : \mathbb{R} \to \mathbb{R}$ has the property that $f^{-1}(U)$ is a Borel set whenever U is an open set. Then $f^{-1}(B)$ is a Borel set whenever B is a Borel set.

(96 Aug.) Let $(X, \sum, \mu)$ be a finite measure space and let f be a non-negative measurable function such that $f(x) < \infty$ a.e. Then the measure ν defined by $\nu(E) = \int_E f du$ is σ-finite.

(97 Jan.) Suppose that f is continuous on $[a,b]$, that g is bounded on $[a,b]$ and that $g(x) = f(x)$ a.e. Then g is Riemann-integrable on $[a,b]$?

(97 Jan.) Suppose that U is a dense open subset of $\mathbb{R}$. Then U has infinite Lebesgue measure?

(97 Jan.) Suppose that $f(z)$ and $g(z)$ are holomorphic on $\mathbb{C}\backslash\{0\}$, that both have poles at $z = 0$, and that $f(1/n) = g(1/n)$ for $n \geq 1$. Then $f(z) = g(z)$ for all z?

(97 Jan.) Suppose that $f \in L^2(\mathbb{R})$ and that $||f_2|| \leq 1$. Then

$$\int_{-\infty}^{\infty} \frac{|f(x)|^{1/2}}{(1+x^2)^{3/4}} dx \leq \pi^{3/4} \ ?$$

(97 Aug.) If $< f_n >$ is a sequence of measurable functions on $[0,1]$ and $\int_{[0,1]} |f_n| dm \to 0$ then $f_n \to 0$ in measure.

(97 Aug.) If $< f_n >$ is a sequence of real valued functions in $\mathbb{R}$ and $f_n \searrow 0$ then $\int f_n dm \searrow 0$.

(97 Aug.) Let F be a function of bounded variation on $[a,b]$ such that $F'(x) = 0$ almost everywhere. Then f is a constant.

(97 Aug.) If f is an analytic function on the annulus $U = \{z \in \mathbb{C} : 1 < |z| < 4\}$, then f has an antiderivative in U.

(97 Aug.) If $f \in L^2(\mathbb{R})$, then for $a \geq 0$

$$\left| \int_0^a t f(t) dt \right| \leq \frac{a^{\frac{3}{2}}}{\sqrt{3}} ||f||_2.$$

(98 Jan.) Let f_n be measurable functions on $[0,1]$ such that $f_n \to 0$ a.e. on $[0,1]$. Then for all $\epsilon > 0$ there exist a set $X_\epsilon \subset [0,1]$ such that $m(X_\epsilon^c) < \epsilon$ and $\int_{X_\epsilon} |f_n| dx \to 0$.

(98 Jan.) Let $f \in L_2([0,1])$. Then $\int_0^1 x^2 |f(x)| dx \leq \frac{||f||_2}{\sqrt{5}}$.

(98 Jan.) Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ such that $f_n \to 0$ in measure. Then $f_n(x) \to 0$ a.e. on $[0,1]$.

(98 Jan.) Let $f(z)$ be an entire function. Then $\frac{f(z)}{z^2}$ has an antiderivative on $\mathbb{C}\backslash\{0\}$.

(98 Jan.) Let $X \subset \mathbb{R}$ be noncompact. Then there exists a continuous $g : X \to \mathbb{R}$ such that $g(X)$ is bounded, but $g(X)$ has no largest element.

(99 Aug.) If f is monotone increasing on $[a, b]$ and continuous with $f' = 0$ a.e. on $[a, b]$, then f is constant on $[a, b]$.

(99 Aug.) If f is continuous on $[0, 1]$ and absolutely continuous on $[c, 1]$ for every $c > 0$, then f is absolutely continuous on $[0, 1]$.

(99 Aug.) If f is differentiable on (a, b) then f' is continuous on (a, b).

(99 Aug.) The set of functions $f \in L^1([0, 1], \lambda)$ with $||f||_1 = 1$ sequentially compact in the norm topology of L^1.

(99 Aug.) If f is analytic in $\mathbb{C}$ satisfying $|f(z)| \leq M|z|^n$ for some constant M and all z sufficiently large, then f is a polynomial of degree less than or equal to n.

(00 Jan.) Suppose that $(f_n)_{n \geq 1}$ is a sequence of point-wise decreasing (i.e. $f_n(x) \geq f_{n+1}(x)$ for all x and for all n) nonnegative integrable functions. Then

$$\lim \int f_n d\lambda = \int (\lim f_n) d\lambda$$

(00 Jan.) If $f : [a, b] \to \mathbb{R}$ is increasing and continuous then

$$f(b) - f(a) = \int_a^b f' d\lambda$$

(00 Jan.) If A is closes and $\lambda(A) > 0$ then A contains an interval I of positive length.

(00 Jan.) There exists an entire function $f(z)$ such that $f(1/n) = (-1)^n/n^4$ for $n \geq 1$.

(00 Aug.) Every uncountable set of real numbers has a non-measurable subset.

(00 Aug.) For every $\epsilon > 0$ there exists an open dense subset $\mathcal{O}$ of $[0, 1]$ such that $m(\mathcal{O}) < \epsilon$.

(00 Aug.) If f_n are measurable on $[0, 1]$, $f_n \geq 0$, $\int_0^1 f_n = 1$, and $f_n \to f$ a.e., then $\int_0^1 f = 1$.

(00 Aug.) If f_n are measurable on $\mathbb{R}$, $0 \le f_1 \le f_2 \le ....$, $f_n \to f$ a.e., and $\int_{\mathbb{R}} f_n \to 1$, then $f \in L_1(\mathbb{R})$.

(00 Aug.) Suppose f and g are continuous on $[0,1]$, f' and g' exist a.e. on $[0,1]$, $f' = g'$ a.e., and $f(0) = g(0)$. Then $f = g$ on $[0,1]$.

(02 Aug.) There exists a compact set $K \subset [0,1]\backslash Q$ with $m(K) > \frac{1}{2}$.

(02 Aug.) If $f(x) = \sqrt{x}$, then f is uniformly continuous on $[0,\infty)$.

(02 Aug.) If $m(E) > 0$, then E has a non-empty interior.

(02 Aug.) There exists $M > 0$ such that $|\sin z| \le M$ for all $z \in \mathbb{C}$.

(02 Aug.) Let f_n be integrable functions on $\mathbb{R}$ such that $\lim_{n\to\infty} \int f_n dx = 0$. Then there exists an integrable function g such that $|f_n| \le g$ a.e. for all n.

(03 Jan.) Let $E_n \subset \mathbb{R}$ such that $\sum_{n=1}^{\infty} m^*(E_n) < \infty$. Then

$$m^*(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k) = 0$$

(03 Jan.) Let f be integrable over $[0,1]$ such that

$$|\int_0^1 f(t)g(t)dt| \le 1$$

for all continuous functions g on $[0,1]$ with $||g||_\infty \le 1$. Then

$$\int_0^1 |f(t)| \le 1.$$

(03 Jan.) If f is integrable over $[0,1]$, then f is bounded on $[0,1]$.

(03 Jan.) If f is analytic on $|z| < 1$, then there exists a $k \ge 1$ such that $|f^{(k)}(0)| < k!4^k$.

(03 Jan.) Let $|a_{nm}| \le 1$ for all $n, m \ge 1$. Then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{nm}.$$

(03 Aug.) If $(X, \sum, \mu)$ is a finite measure space, $1 \leq r \leq s < \infty$ and $f : X \to \mathbb{R}$ is a measurable function then

$$||f||_r \leq ||f||_s \mu(X)^{\frac{1}{r}-\frac{1}{s}}.$$

(03 Aug.) There exists a measurable set $A \subseteq [0,1] \times [0,1]$ such that $\lambda(A_x) = 0$ λ-a.e. and $\lambda(A^y) > 0$ λ-a.e.

(03 Aug.) The function $f(z) = z \sin \frac{1}{z}$ has a pole at 0.

(03 Aug.) There exists a region Ω of $\mathbb{C}$ which is mapped in a one-to-one way onto $\{z \in \mathbb{C}\backslash\{0\} : |z| < 1\}$ through the exponential map.

(03 Aug.) There exists an analytic function $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ such that $f(\{z \in \mathbb{C} : |z| < 1\})$ is exactly a line segment.

(04 Jan.) If $m^*(A) = 0$, then $m^*(B\backslash A) = m^*(B)$ for all subset B of $\mathbb{R}$.

(04 Jan.) Let $\mathbb{Q} = \{r_n : n = 1, 2, ...\}$. Then

$$\mathbb{R}\backslash \bigcup_{n=1}^{\infty} (r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2}) \neq \phi.$$

(04 Jan.) The function $f(z) = e^{\bar{z}}$ is analytic everywhere except at 0.

(04 Jan.) Let f be an entire function such that $\lim_{z\to\infty} |f(z)| = \infty$. Then f is a polynomial.

(04 Jan.) Let f_n be integrable function on $\mathbb{R}$ such that $\lim_{n\to\infty} \int f_n dx = 0$. Then there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and an integrable function g such that $|f_{n_k}| \leq |g|$ a.e. for all k.

(04 Aug.) If $f : [a, b] \to \mathbb{R}$ is increasing and continuous then

$$f(b) - f(a) = \int_a^b f'(x)dx$$

(04 Aug.) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $B \subseteq \mathbb{R}$ is a Borel set, then $f^{-1}(B)$ is a Borel set.

(04 Aug.) If f_n $(n \geq 1)$ and f is integrable function on $[0,1]$ and $f_n \to f$ point-wise then $\int_0^1 f_n dx \to \int_0^1 f dx$.

(04 Aug.) Suppose that U is a domain in the complex plane and that f is analytic on U. Then $\int_\gamma f(z)dz = 0$ for every simple closed smooth curve γ contained in U.

(04 Aug.) If f is an entire function and $|f(z)| \to \infty$ as $|z| \to \infty$ then f is a polynomial.

(05 Aug.) If two continuous real-valued functions defined on $\mathbb{R}$ agree everywhere on the complement of set of measure zero, then they agree everywhere.

(05 Aug.) There is no sequence $(a_{m,n})_{m,n\in\mathbb{N}}$ for real numbers such that $\sum_{n=1}^{\infty} a_{m,n} = 1$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} a_{m,n} = -1$ for all $n \in \mathbb{N}$.

(05 Aug.) Every absolutely continuous function $f : [0,1] \to \mathbb{R}$ is Lipchitz.

(05 Aug.) If $u + iv$ is analytic (where u and v are real-valued) then uv is harmonic.

(05 Aug.) Suppose that $f(z)$ is analytic inside and on a simple closed contour C (traversed once counterclockwise) which contains z_0 in its interior. Then

$$\int_C \frac{f''(z)}{(z-z_0)^2} dz = 6 \int_C \frac{f(z)}{(z-z_0)^4} dz.$$

(06 Jan.) Let $\phi : \mathbb{R} \to \mathbb{R}$ be measurable function such that $\phi(\int_0^1 f(x)dx) \leq \int_0^1 \phi(f)dx$ for all measurable bounded $f : [0,1] \to \mathbb{R}$. Then ϕ is convex.

(06 Jan.) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions and $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f_n \to f$ a.e. Then there exists a partition of $\mathbb{R}$ into disjoint measurable sets E_0, E_1, E_2,.... such that $m(E_0) = 0$ and $f_n \to f$ uniformly on E_i for all $i \geq 1$.

(06 Jan.) If $f(x) = e^{-x} \sin x$ then $\sup\{T_a^b(f) : a, b \in \mathbb{R}, a < b\} = \infty$.

(06 Jan.) Suppose that $u + iv$ is analytic on a connected open set U, where $u(x, y)$ and $v(x, y)$ are real-valued functions. Then $e^u \sin v$ is harmonic on U.

(06 Jan.) If f is analytic on a connected open set U then there exists F such that $F'(z) = f(z)$ for all $z \in U$.